Mathematisches Forschungsinstitut Oberwolfach

Report No. 26/2025

DOI: 10.4171/OWR/2025/26

Statistical Physics Out of Equilibrium: Quantitative Results and Universality

Organized by
Ana Patrícia Carvalho Gonçalves, Lisboa
Hugo Duminil-Copin, Genève/Bures-sur-Yvette
Grégory Miermont, Lyon

26 May – 30 May 2025

Abstract. Over the last 30 years, there has been spectacular progress in deriving the well-known hydrodynamic limits from stochastic interacting particle systems, as well as characterizing the fluctuations of locally conserved quantities around this limit. Many interesting results on the aforementioned topic have been derived from stochastic integrability, an approach relying on very specific combinatorial and algebraic properties of the underlying dynamics which allow deriving several scaling limits. However, a microscopic change on the dynamics can dramatically impact the macroscopic level, in the sense that scaling limits are no longer tractable by this methodology. Moreover, microscopic perturbations can lead to evolution equations with a variety of behaviours and at the critical parameter of the underlying dynamics, several universal anomalous laws can emerge, both in hydrodynamics and in fluctuations. More generally, understanding critical points where physical systems undergo phase transitions, and establishing that the phenomenology is described by universal mathematical objects that do not depend on the specific properties of the underlying microscopic dynamics, is a cornerstone of modern probability and mathematical physics, both from a pure and an applied point of view.

Mathematics Subject Classification (2020): 60Fxx, 60G60, 60Hxx, 60Jxx, 60K35, 81Pxx, 81Txx, 82Cxx, 82Dxx.

License: Unless otherwise noted, the content of this report is licensed under CC BY SA 4.0.

Introduction by the Organizers

The workshop Statistical Physics Out of Equilibrium: Quantitative Results and Universality, organized by Hugo Duminil-Coupin (Paris), Grégory Miermont (Lyon) and Patrícia Gonçalves (Lisbon), was well attended with 51 participants (including 3 online participants), and with broad geographic representation from several continents. The core topic of the workshop focused on recent progress in particle systems, considered under the prism of their hydrodynamics limits, convergence to equilibrium, integrable and geometric properties. Faithful to the general spirit of the event, we however kept a large span of topics in modern probability theory, and also had a number of presentations on other aspects of statistical physics, quantum systems, Markov chain convergence and random geometry, with an emphasis on the question of universality that is an ubiquitous concept in physics.

The week began with a thematic morning on integrable probability, featuring talks by Tomohiro Sasamoto, Makiko Sasada, and Alessandra Occelli. Tomohiro Sasamoto presented the latest findings of his research group – which includes Cristian Giardinà, Hayate Suda, and Kirone Mallick – on large deviations for interacting particle systems with large spin. The aim of the talk was the derivation of a new type of large deviation principle based on the Feng-Kurtz framework. Sasamoto's talk was followed by an inspiring presentation by Makiko Sasada, who discussed the independence-preserving property of certain bijections. Sasada reviewed recent results obtained with her research group – David Croydon, Ryosuke Uozumi, Hiroki Kondo, and Sachiko Nakajima – on the use of this property to study stationary solutions of stochastic integrable models and invariant measures of discrete (deterministic) integrable systems, as well as the connections between the independence-preserving property of F and integrability itself. The next talk was given by Alessandra Occelli, who discussed hard-edge fluctuations and large deviations of the Muttalib–Borodin ensemble. The presentation concluded with the derivation of a large deviation principle for the empirical measure of the plane partition, and the characterization of the corresponding equilibrium measure via a Riemann-Hilbert problem associated with the minimization of the rate function.

The afternoon focused on criticality around the KPZ equation. The first talk was given by Nikolaos Zygouras, who presented recent work (in collaboration with Francesco Caravenna and Rongfeng Sun) on the critical 2d stochastic heat flow, which is a non-trivial solution of the SHE in the critical dimension d=2 at the transition point, but it is neither Gaussian nor a Gaussian multiplicative chaos. Zygouras reviewed the phase transition of the 2d SHE and explained the construction and properties of the critical stochastic heat flow. The talk was followed by Pedro Cardoso, who presented the derivation of the regional fractional stochastic Burgers equation from interacting particle systems. The underlying model was the boundary-driven exclusion process with long jumps and asymmetric jump rates, depending on a parameter that leads the system to stationary solutions of either the Ornstein–Uhlenbeck equation or energy solutions of the stochastic Burgers equation. The afternoon session concluded with an inspiring talk by Giuseppe

Cannizzaro on the super-diffusive central limit theorem (CLT) for the SBE at the critical dimension. In collaboration with Fabio Toninelli and Quentin Moulard, they have identified the exact large-time asymptotic behavior of the diffusion matrix and proved that the solution of the SBE satisfies a CLT under a logarithmic correction scaling.

The Tuesday morning session began with a talk by Tadahisa Funaki on quantitative results for the hydrodynamic limits of non-gradient Glauber–Kawasaki dynamics. The aim of the talk was to discuss the derivation of interface motion from a non-gradient type Glauber–Kawasaki dynamics and convergence rates were obtained using quantitative homogenization methods, and results on the fluctuations of the interface were also presented.

Alessandra Faggionato then presented her recent results, partly with Ivailo Hartarsky, linking the percolative properties of a vast class of spatial random network models with the non-degeneracy of the effective homogenized matrix of the random walk in the large scale limit.

The next talk was from Michael Aizenman who sketched an innovative route to entanglement in quantum spin chains by drawing on deep parallels with classical statistical mechanics systems, such as loop models and percolation.

The afternoon session was once again dedicated to critical phenomena in interface growth and the KPZ equation, featuring talks by Duncan Dauvergne, Guillaume Barraquand, and Ofer Busani. Dauvergne presented a talk on a joint project with Lingfu Zhang, focused on the characterization of the directed landscape via the KPZ fixed point. The talk aimed to give a full characterization of the directed landscape based on these KPZ fixed point marginals. The talk was followed by Guillaume Barraquand's talk, who spoke about KPZ growth with open boundary conditions and he presented a general framework that applies to various models and compared it with the classical Matrix Product Ansatz approach. The session concluded with a talk by Ofer Busani on global solutions of the KPZ fixed point at exceptional slopes, based on joint work with Sudeshna Bhattacharjee and Evan Sorensen, where he provided a complete characterization of the global solutions of the KPZ fixed point, almost surely in all directions, and showed that in exceptional directions, there exist infinitely many global solutions.

On Tuesday evening, we had a 1-hour night session with quick presentations by younger participants (William Fleurat, Julian Kern, Maria Chiara Ricciuti, Beatriz Salvador, Leander Schnee) followed by a short open problem session.

Wednesday morning was devoted to another aspect of 2D random geometry, with talks by Jason Miller, Armand Riera and Ewain Gwynne. Miller reported on remarkable progress, obtained in numerous collaborations of his research group, on the chemical distance structure and diffusions on conformal loop ensembles (CLS), in the so-called dense regime. This requires a multitude of techniques inspired from first-passage percolation, random media, Dirichlet forms. Armand Riera discussed recent results with Curien and Miermont on the scaling limit of random planar maps with a Boltzmann distribution, when the faces have a heavy-tailed degree distribution. The possible scaling limits, which are random metric spaces called

stable *gaskets/carpets*, present a structure that is analogous to the CLE, and in particular, they possess two *dense/dilute* phases as to whether the boundary of the space may or may not have self-intersections. Indeed, these spaces are expected to represent a "quantum" version of CLEs, in the sense that CLEs with distances measured using an independent Gaussian multiplicative chaos instead of a Euclidean chemical distance should coincide with these stable random fractals. To conclude the morning session, Ewain Gwynne presented recent progress on supercritical Liouville Quantum Gravity (LQG) metrics, which form a family of non-compact random metrics with infinitely many topological ends, and that are conjectured to be relevant in the context of conformal field theories with central charge c, which arises in particular when one aims at describing statistical physics models with matter field in $\mathbb{R}^d$. Among others, Ewain also gave a possible explanation of the counterintuitive statement that can sometimes be found in the physics literature that such matter fields present a *branched polymer phase*, by showing that natural models of random maps expected to converge to supercritical LQG, but *conditioned on the small-probability event that they are finite*, converge to the Brownian continuum random tree.

The rain left us with a lot of time for discussions in the Wednesday afternoon, after which Scott Sheffield presented in a night session his views on Yang-Mills theory. Scott Sheffield delivered an engaging evening overview of the latest breakthroughs in lattice Yang-Mills theory. He illustrated how these discrete probabilistic frameworks provide fresh insight into their elusive continuum counterparts in gauge theory.

The morning of Thursday started with the talk of Barbara Dembin who reported striking advances obtained with Dor Elboim and Ron Peled on rigorous results for minimal surfaces in random environments firmly anchoring several physics based predictions. The next speaker was Xin Sun that unveiled a suite of novel exact formulae for the scaling limit of two-dimensional critical Bernoulli percolation most notably, precise annular crossing estimates. Achieving what was once thought infeasible, these results synthesize sophisticated tools from conformal probability theory, particularly Liouville quantum gravity (LQG) and conformal loop ensembles (CLE). Finally, Allan Sly presented cutting-edge findings on Glauber dynamics in Ising and Potts models initialized with an excess of plus spins. He showed that under these conditions, the system exhibits rapid mixing in time $O(\log(n))$.

Thursday afternoon's session began with a talk by Dominik Schmid on quantitative results for the open Asymmetric Simple Exclusion Process (ASEP), where he presented a simple characterization of the Bryc polynomials, which enabled sharp approximation results in the shock region of the open ASEP, while in the fan region, the results relied on a recent generalization of the Ennola–Derrida representation of the classical Matrix Product Ansatz method. The next speaker was Milton Jara, who gave an expository talk on the non-equilibrium stationary state (NESS) of the KPZ equation. The underlying discrete model considered was the WASEP with speed change, for which a novel methodology was proposed to show that the density fluctuations of the NESS in boundary-driven, weakly asymmetric

interacting particle systems are tight, and that any limit point is an energy solution of the KPZ equation. The session ended with a talk by Gunter Schütz on a reverse duality property for the ASEP with open boundaries. A duality relationship was established between the ASEP with non-conservative open boundary conditions and an asymmetric exclusion process with particle-dependent hopping rates and conservative reflecting boundaries and this allows relating the measures of the dual processes. The reverse duality also sheds light on certain properties of the stationary matrix product measures of the open ASEP that are represented using finite-dimensional matrices.

The last day of the event started with a talk from Patrik Ferrari about decoupling and decay of the two-point functions in a two-species TASEP on the full line starting from a translation-invariant stationary measure. The main result was about the asymptotic decoupling of the marginal height profiles along characteristic lines and the decay of the mixed correlations in the large-time limit, thus confirming predictions of the nonlinear fluctuating hydrodynamics theory. The talk was followed by another talk of Simone Warzel. Together with her coauthors, they have adapted techniques developed by Stroock–Zegarlinski and Martinelli for Glauber dynamics. Their work establishes a modified log Sobolev inequality governing the quantum Markovian dynamics that underlie convergence to equilibrium in the Kitaev toric code model. The workshop ended with a beautiful and inspiring presentation by Justin Salez, who reported on his work aiming at giving a full characterization of the cutoff phenomenon for Markov chains, meaning that the convergence to equilibrium occurs abruptly. After introducing the useful concept of *varentropy*, which allows to formulate a simple sufficient condition for cutoff, and explained how an assumption of positive curvature in the Bakry-Emery sense allows one to make this condition effective. Justin concluded by predicting that the cutoff phenomenon for negatively curved chains should be of a different nature.

Workshop: Statistical Physics Out of Equilibrium: Quantitative Results and Universality

Table of Contents

Tomohiro Sasamoto (joint with Cristian Giardinà, Kirone Mallick, Hiroki Moriya, Hayate Suda)
Large deviation for large spin for interacting particle systems 1341

Makiko Sasada (joint with David Croydon, Hiroki Kondo, Sachiko Nakajima, Ryosuke Uozumi)
Independence preserving property and integrable systems 1342

Alessandra Occelli (joint with Dan Betea, and with Jonathan Husson and Guido Mazzuca)
Discrete and Continuous Muttalib–Borodin Processes: old and new results 1344

Nikos Zygouras (joint with Francesco Caravenna and Rongfeng Sun)
Exploring the Critical 2*d Stochastic Heat Flow* 1346

Pedro Cardoso (joint with Patrícia Gonçalves)
Regional Fractional Stochastic Burgers from random interactions 1347

Giuseppe Cannizzaro (joint with Quentin Moulard, Fabio L. Toninelli)
Superdiffusive Central Limit Theorem for the critical Stochastic Burgers Equation 1350

Tadahisa Funaki
Quantitative hydrodynamic limit for non-gradient Glauber-Kawasaki dynamics 1352

Alessandra Faggionato
Transport and crossings 1353

Michael Aizenman (joint with Simone Warzel)
Quantum stoquastic tensor networks from a stat mech perspective 1354

Duncan Dauvergne (joint with Lingfu Zhang)
Characterization of the directed landscape from the KPZ fixed point 1355

Guillaume Barraquand
KPZ growth with open boundary conditions 1357

Ofer Busani (joint with Sudeshna Bhattacharjee, Evan Sorensen)
On the global solutions of the KPZ fixed point 1359

Jason Miller (joint with Valeria Ambrosio, Irina Dankovic, Maarten Markering, Yizheng Yuan)
The scaling limit of simple random walk and the intrinsic metric on 2D critical percolation clusters 1359

Armand Riera (joint with Nicolas Curien, Grégory Miermont)
The scaling limit of random planar maps with large faces 1360

Ewain Gwynne
Liouville quantum gravity with central charge in $(1, 25)$ 1365

Scott Sheffield (joint with Sky Cao, Ron Nissim, Joshua Pfeffer, Minjae Park, Pu Yu)
Yang-Mills theory and random surfaces 1366

Barbara Dembin (joint with Dor Elboim, Ron Peled)
Minimal surfaces in random environments 1366

Xin Sun
Application of Liouville quantum gravity in 2D percolation 1369

Allan Sly (joint with Reza Gheissari, Youngtak Sohn)
Rapid phase ordering for Ising and Potts dynamics on random regular graphs 1372

Dominik Schmid (joint with Zongrui Yang)
Quantitative results for the open asymmetric simple exclusion process 1374

Milton Jara (joint with Juan Arroyave)
NESS for KPZ 1375

Gunter M. Schütz (joint with V. Belitsky, N.P.N. Ngoc)
Reverse duality and applications 1376

Patrik L. Ferrari (joint with Sabrina Gernholt)
Decoupling and decay of two-point functions in a two-species TASEP 1378

Simone Warzel (joint with Angela Capel Cuevas, Cambyse Rouze, Sebastian Stengele)
Rapid thermalization of toric codes through a log Sobolev inequalities 1381

Justin Salez (joint with Francesco Pedrotti)
An invitation to the cutoff phenomenon 1381

Abstracts

Large deviation for large spin for interacting particle systems

Tomohiro Sasamoto
(joint work with Cristian Giardinà, Kirone Mallick, Hiroki Moriya, Hayate Suda)

The large deviation principle for symmetric simple exclusion process (SEP) had been established by Kipnis, Olla, Varadhan in 1989 [1]. A somewhat different formulation, known as the macroscopic fluctuation theory (MFT), was initiated and developed by Jona-Lasinio et al in 2000's [2] for more generic systems. The basic equations of the theory, MFT equations, are coupled nonlinear partial differential equations and had resisted exact analysis except for stationary situation. A few years ago we have found that a novel generalization of the Cole-Hopf transformation maps the MFT equations for SEP to the classically integrable Ablowitz-Kaup-Newell-Segur(AKNS) system. This allows us to solve the equations exactly in time dependent regime by adapting standard ideas of inverse scattering method [3, 4]. In this presentation we consider a lattice analogue of the large deviation and its solution [5].

In recent years several interacting particle systems which have a parameter called a "spin" have been introduced and studied. They include the partial exclusion process, inclusion process and the harmonic model. For this class of models, we propose a new type of large deviation for large spin. We first explain the basic formulation based on the scheme of Feng-Kurtz and calculate the associated Hamiltonian for a few examples. Next we explain a proof of the large deviation principle (LDP) for the case of partial exclusion on a finite lattice. We also show how one can calculate the rate function exactly by mapping to a classical integrable system on a lattice. If time allows we also discuss connections to the macroscopic fluctuation theory (MFT).

At the moment we have been able to prove the LDP for models on a finite lattice. It would be interesting to prove the LDP for models on infinite lattice. Then it is a challenging question to ask if the large deviation for large spin leads to better understanding of hydrodynamic LDP and MFT. Finally understanding integrable structure of the MFT equations for various models with various geometry is a very interesting question in another direction.

References

[1] C. Kipnis, S. Olla, and S.R.S. Varadhan, *Hydrodynamics and large deviations for simple exclusion processes*, Comm. Pure Appl. Math., **42** (1989), 115-137.

[2] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim,*Macroscopic fluctuation theory*, Rev. Mod. Phys., **87** (2015), 593-636.

[3] K. Mallick, H. Moriya, and T. Sasamoto, *Exact solution of the macroscopic fluctuation theory for the symmetric exclusion process*, Phys. Rev. Lett. **118** (2022), 160601.

[4] K. Mallick, H. Moriya, and T. Sasamoto, *Exact solution of the macroscopic fluctuation theory for the symmetric exclusion process*, JSTAT **2024** (2024), 074001.

[5] C. Giardinà and T. Sasamoto, *Large spin large deviations for interacting particle systems*; C. Giardinà, K. Mallick, T. Sasamoto, and H. Suda, *Exact solution of discrete macroscopic fluctuation theory for an integrable spin system*, in preparation.

Independence preserving property and integrable systems

Makiko Sasada

(joint work with David Croydon, Hiroki Kondo, Sachiko Nakajima, Ryosuke Uozumi)

The well-known **Kac–Bernstein theorem** asserts that if X and Y are independent real-valued random variables, and moreover $X+Y$ and $X-Y$ are independent as well, then X and Y must follow normal distributions with the same variance. This result implies that the bijection $F(x,y) = (x+y, x-y)$ leads to a class of quadruples of non-degenerate probability measures $(\mu, \nu, \tilde{\mu}, \tilde{\nu})$ such that

$$F(\mu \times \nu) = \tilde{\mu} \times \tilde{\nu}.$$

When a bijection F satisfies this condition, it is said to have the *independence preserving property* (abbreviated as the *IP property*).

Previous research has primarily focused on examining the IP property for specific, explicit bijections F, which has led to characterizations of notable probability distributions including the normal, gamma, exponential, inverse-Gaussian, and beta distributions. When F takes a particular form, this property is also referred to as the *Matsumoto–Yor property*.

The IP properties of the maps $F_G(x,y) := (x+y, \frac{x}{y})$ characterizing the gamma distribution and $F_B(x,y) := (1-xy, \frac{1-x}{1-xy})$ characterizing the beta distribution played an essential role in the paper [1] in characterizing the stationary distribution of the $1+1$-dimensional lattice polymer model. By applying their method, in [2] we investigated the stationary distribution of zero-temperature version of $1+1$-dimensional polymer lattice models, and in particular, we discovered a new stationary distribution. In this analysis, we considered the zero-temperature limits—or in other words, the *ultra-discretizations*—of F_G and F_B, and utilized the IP property of the resulting bijections.

Although the above results focused on stochastic models, it has become evident that the same methodology can be extended to deterministic models as well. In particular, in our paper [3], we introduced a class of deterministic models on the $1+1$ dimensional lattice, called locally-defined dynamics. We showed that a model in this class has a stationary distribution that is independent and identically distributed (i.i.d.) if and only if the bijection F governing the local evolution satisfies the IP property. Moreover, we demonstrated that this stationary distribution satisfies the Burke property. This class of models includes important integrable systems such as the discrete KdV equation, the discrete Toda equation, the ultra-discrete KdV equation, and the ultra-discrete Toda equation. Using the aforementioned theorem, we also showed that each of these four models indeed has a class of i.i.d. stationary distributions.

From the last results, we observed that four well-known examples of bijections F defining integrable systems possess the IP property. This naturally raises the question of whether there is an intrinsic connection between the IP property of the map F and the integrability of the locally-defined dynamics given by F. In particular, since it is known that the class of bijections F defining the discrete KdV equation also satisfies the Yang–Baxter equation, in [5] we conducted the first study on the relationship between the Yang–Baxter equation and the IP property. As a result, we newly found that some quadrirational Yang–Baxter maps possess the IP property. Furthermore, we discovered that most known maps with the IP property can be derived–through limiting procedures, change of variables or specifying parameters–from these quadrirational Yang–Baxter maps.

As a continuation of this work, in [4] we considered ultra-discrete analogues of quadrirational Yang–Baxter maps, and showed that they also satisfy the Yang–Baxter equation and possess the IP property. In this study, we introduced the concept of ultra-discretization of probability measures. A similar approach was also employed in [2] to investigate the zero-temperature limit of the stationary distribution in $1+1$-dimensional lattice polymer models.

Thus, through the IP property, various similarities between probabilistic integrable models and deterministic integrable systems have been revealed. However, as a fundamental open problem, it remains completely unresolved whether there is any intrinsic connection between the IP property and integrability, and whether there exists any direct and universal relationship between probabilistic integrable models and deterministic integrable systems. To clarify these questions, further research focusing on various concrete examples is expected to advance in the future.

References

[1] H. Chaumont and C. Noack, *Characterizing stationary* $1+1$ *dimensional lattice polymer models*, Electron. J. Probab. **23** (2018), 1–19.

[2] D. A. Croydon and M. Sasada, *On the stationary solutions of random polymer models and their zero-temperature limits*, J. Stat. Phys. **188-3** (2022), Paper No. 23, 32.

[3] D. A. Croydon and M. Sasada, *Detailed balance and invariant measures for discrete KdV- and Toda-type systems*, arXiv:2007.06203

[4] H. Kondo, S. Nakajima, and M. Sasada, *Ultra-discretization of Yang-Baxter maps, probability distributions and independence preserving property* , arXiv:2504.21359

[5] M. Sasada and R. Uozumi, *Yang-Baxter maps and independence preserving property*, Electron. J. Probab., **29** (2024), 1–21.

Discrete and Continuous Muttalib–Borodin Processes: old and new results

Alessandra Occelli

(joint work with Dan Betea, and with Jonathan Husson and Guido Mazzuca)

A plane partition Λ is a matrix $(\Lambda_{i,j})_{1\le i\le M, 1\le j\le N}$ of non-negative integers satisfying the conditions:

$$\Lambda_{i,j} \ge \Lambda_{i,j+1} \quad \text{and} \quad \Lambda_{i,j} \ge \Lambda_{i+1,j}.$$

This arrangement can be visualised as stacks of cubes in a three-dimensional space, where the array corresponds to the number of cubes placed at each coordinate point of an $M \times N$ rectangular base. Plane partitions have applications in combinatorics, statistical mechanics, and representation theory. Plane partitions are equivalently described by a sequence of interlacing integer partitions:

$$\{\lambda^{(t)}\}_{t=-M+1}^{N-1} : \quad \lambda^{(t-1)} \prec \lambda^{(t)}, \text{as } t \ge 0, \quad \lambda^{(t)} \prec \lambda^{(t-1)}, \text{as } t < 0,$$

where the interlacing condition $\lambda^{(s)} \prec \lambda^{(t)}$ means $\quad \lambda_1^{(s)} \ge \lambda_1^{(t)} \ge \lambda_2^{(s)} \ge \lambda_2^{(t)} \ge \cdots$. This representation connects plane partitions with lozenge tilings, Schur functions, and determinantal point processes. It can also be visualised as a particle process on $\{-M+1, \dots, 0, \dots, N-1\} \times \mathbb{N}$ by the shift $\ell_i^{(t)} = \lambda_i^{(t)} + M - i$. Given real parameters $a, q > 0$ and $\eta, \theta \ge 0$, we consider the following weight associated with a plane partition Λ:

$$\mathbb{P}(\Lambda) \propto \left(aq^{\frac{\eta+\theta}{2}}\right)^{\text{CentralVol}} q^{\eta\cdot\text{LeftVol}+\theta\cdot\text{RightVol}}, \tag{1}$$

where LeftVol, CentralVol, and RightVol represent the volumes of cubes in different regions of the plane partition. Under this measure each time slice of a plane partition/particle process corresponds to a discrete *Muttalib–Borodin like* ensemble (a biorthogonal ensemble [3]), described by

$$\mathbb{P}(\ell^{(t)} = \ell) = \frac{1}{Z_d} \prod_{1\le i<j\le L_t} (q^{\eta\ell_j} - q^{\eta\ell_i})(q^{\theta\ell_j} - q^{\theta\ell_i}) \prod_{1\le i\le L_t} w_d(\ell_i) \tag{2}$$

where $Z_d = \prod_{1\le i\le M} \prod_{1\le j\le N} (1 - aq^{\eta(i-\frac{1}{2})} q^{\theta(j-\frac{1}{2})})^{-1}$ is the partition function and $w_d(l_i)$ represent discrete weights derived from the volume contributions of the partitions, with

$$w_d(x) = \begin{cases} a^x (Q\tilde{Q})^{\frac{x}{2}} Q^{|t|x} (\tilde{Q}^{x-|t|+1}; \tilde{Q})_{N-(M-|t|)} & \text{if } t \le 0, \\ a^x (Q\tilde{Q})^{\frac{x}{2}} \tilde{Q}^{tx} (\tilde{Q}^{x+1}; \tilde{Q})_{N-t-M} & \text{if } t > 0 \text{ and } N - t \ge M, \\ a^x (Q\tilde{Q})^{\frac{x}{2}} \tilde{Q}^{tx} (Q^{x+N-t-M+1}; Q)_{M-(N-t)} & \text{if } t > 0 \text{ and } N - t < M. \end{cases} \tag{3}$$

The Muttalib–Borodin ensemble (MBE) generalises β-ensembles by introducing an additional interaction parameter $\theta > 0$: the interacting potential $\Delta(x)^\beta$ is replaced by $\Delta(x^\eta)\Delta(x^\theta)$. One can think of it as a system with two-particle interactions, one between type x_i's, one between type x_i^θ's. The interaction term distinguishes

MBEs from classical β-ensembles, making them suitable for modeling disordered conductors [6].

We study the ensemble in the limit

$$q = e^{-\epsilon}, \qquad a = e^{-\alpha\epsilon}, \qquad \lambda_i(t) = -\frac{\ln x_i(t)}{\epsilon}, \qquad \epsilon \to 0+, \tag{4}$$

which gives rise to a space-continuous process $(x(t))_t$ supported in $[0, 1]$.

First we connect the discrete and continous ensembles to models of inhomogeneous last passage percolation with geometric/power law weights and show that the last passage time (the total weight of the heaviest directed path) is in distribution equal the peak of the plane partition, or the position of the left-most particle $x_1^{(0)}$. Its fluctuations, of order $N \log N$ are described by a Fredholm determinant of an explicit kernel operator: in the case $\eta = \theta$ corresponds to the continous Bessel kernel in exponential coordinate, which shows interpolating regimes between Gumbel and Tracy–Widom laws [2].

Secondly, we prove a large deviation principle (LDP) for the empirical measure associated to the process $(x^{(t)})_t$: each time slice satisfies an LDP with speed N^2 and a good rate function that encodes both the biorthogonal interaction and the external potential [5]. This result generalizes classical LDPs for Wigner and Wishart ensembles [1], which provide explicit variational characterizations of the limit shapes.

Finally, we characterise the limit density profile: if the large deviation rate function has a unique minimizer, the LDP gives us a law of large numbers for the empirical measure. The equilibrium measure minimizing the rate function can be characterized via Riemann–Hilbert analysis, a powerful technique in integrable probability and random matrix theory [4].

References

[1] G. Ben Arous and A. Guionnet, *Large deviations for Wigner's law and Voiculescu's non-commutative entropy*, Probab. Theory Related Fields **108** (1997), 517–542.

[2] D. Betea and A. Occelli, *Discrete and continuous Muttalib–Borodin processes: The hard edge*, Ann. Inst. Henri Poincaré Comb. Phys. Interact. (2024).

[3] A. Borodin, *Biorthogonal ensembles*, Nuclear Phys. B **536** (1998), 704–732.

[4] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, Courant Lecture Notes, 1999.

[5] J. Husson, G. Mazzuca, and A. Occelli, *Discrete and Continuous Muttalib–Borodin process: Large deviations and Riemann–Hilbert analysis*, arXiv:2505.23164v1, 2025.

[6] K. A. Muttalib, *Random matrix models with additional interactions*, J. Phys. A: Math. Gen. **28** (1995), L159–L164.

[7] D. Wang, *Muttalib–Borodin ensembles in random matrix theory*, arXiv:1502.07147.

Exploring the Critical 2*d* Stochastic Heat Flow

Nikos Zygouras
(joint work with Francesco Caravenna and Rongfeng Sun)

The Critical 2*d* Stochastic Heat Flow (SHF) was constructed in [4]. It is a non-trivial, random object that describes a solution of the two-dimensional Stochastic Heat Equation (SHE) at the critical dimension 2 and at a critical temperature. It also represents the scaling limit of models in the universality of SHE, such as the *Directed Polymer Model* that we will discuss later.

The stochastic heat equation is

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta\, u(t,x)\, \xi(t,x)\,, \qquad t>0,\ x\in\mathbb{R}^2, \tag{1}$$

where $\xi(t,x)$ denotes space-time white noise. Dimension 2 is critical for the SHE as it is the dimension where the singularity of the noise matches the smoothing effect of the Laplacian and thus cannot be treated perturbatively.

The singularity of the noise and the nonlinear operation (multiplication) between the noise and the solution in the right-hand side makes the equation ill-defined. To make sense of it, a suitable approximation is necessary. One approach is to mollify the noise with a smooth mollifier $j^\epsilon(x) = \epsilon^{-2} j(x/\epsilon)$ and consider the limit of the corresponding, approximate solution u_ϵ as the mollification tends to 0. Alternatively, one can consider a descretisation provided by the partition function of the *Directed Polymer Model* $Z_N(x,y) := \mathsf{E}_x\big[e^{\sum_{n=1}^N (\beta\omega(n,S_n)-\lambda(\beta))}\,;\,\mathbf{1}_{\{S_N=y\}}\big]$, where $(S_n)_{n\geq 1}$ is a simple two-dimensional random walk, with expectation represented as E_x, when the walk starts from $x\in\mathbb{Z}^2$, and $(\omega(n,x)_{n\geq 1, x\in\mathbb{Z}^2})$ is a family of i.i.d. random variables with log-moment generating function $\lambda(\beta)$.

Either working directly with the continuous equation or its discrete approximation, the temperature β needs to be finely tuned in order for a non-trivial limit to exist. Presented in the case of the directed polymer for simplicity, the tuning is $\beta = \beta_N = \hat\beta\sqrt{\pi/\log N}$.

A phase transition was observed in [2]: For $\hat\beta < 1$ the one-point fluctuations are identified in the limit to a log-normal distribution and the field, suitably centered and scaled by $\sqrt{\log N}$ converges to a gaussian, log-correlated field, which falls in the Edwards-Wilkinson universality class. On the other hand, within a critical window $\hat\beta = 1 + \frac{\theta}{\sqrt{\log N}}$, with $\theta\in\mathbb{R}$, correlations emerge and the field converges without any rescaling to the *Critical 2d Stochastic Heat Flow*. Bertini-Cancrini [1] studied the asymptotics of the correlations of the SHE at this critical window. Combined with higher moment estimates [3, 9] this implies that any limit would be log-correlated. Thus, the unique limit, which is the SHF, is a log-correlated field.

The SHF is a *flow* in the sense that it is a continuous, measure valued process [10], singular with respect to the Lebesgue measure [7], which satisfies a Chapman-Kolmogorov type relation [8]. A characterisation of the SHF as the unique stochastic flow which is continuous in time, has independent increments and has prescribed first four moments was provided in [10].

In terms of spatial regularity, it was shown in [7] that the SHF is a C^{0-} field, that is, it just fails to be a function. Moreover it was shown in [5] that it forms neither a Gaussian nor a Gaussian Multiplicative Chaos (exponential of Gaussian) process, thus, marking a distinct universality class.

There are many interesting questions around the features and universality of the SHF and we refer the reader to [6] for a broad overview and a list of questions.

References

[1] L. Bertini, N. Cancrini, *The two-dimensional stochastic heat equation: renormalizing a multiplicative noise.* J. Phys. A: Math. Gen. **31** (1998) 615.

[2] F. Caravenna, R. Sun, and N. Zygouras, *Universality in marginally relevant disordered systems*, Ann. Appl. Prob., **27**, No. 5, (2017) 3050–3112

[3] F. Caravenna, R. Sun, and N. Zygouras, *On the moments of the* $2+1$*-dimensional directed polymer and stochastic heat equation in the critical window.* Comm. Math. Phys., **372** (2) (2019) 385–440,

[4] F. Caravenna, R. Sun, and N. Zygouras, *The Critical* 2*d Stochastic Heat Flow*, Inventiones Math. 233 **32** (2023), 325–460.

[5] F. Caravenna, R. Sun, and N. Zygouras, *The critical 2d Stochastic Heat Flow is not a Gaussian Multiplicative Chaos*, Ann. Prob., **51** (6) (2023) 2265–2300.

[6] F. Caravenna, R. Sun, and N. Zygouras, *The Critical 2d Stochastic Heat Flow and Related Models* arXiv:2412.10311 (2024)

[7] F. Caravenna, R. Sun, and N. Zygouras, *Singularity and regularity of the critical 2D Stochastic Heat Flow* arXiv:2504.06128 (2025)

[8] J. Clark and B. Mian, *Continuum polymer measures corresponding to the critical 2d stochastic heat flow*, arXiv:2409.01510, (2024).

[9] Y. Gu, J. Quastel, and L. C. Tsai. *Moments of the 2D SHE at criticality.* Probab. Math. Phys., **2** (2021) 179–219,

[10] L. C. Tsai, *Stochastic heat flow by moments,* arXiv:2410.14657, (2024).

Regional Fractional Stochastic Burgers from random interactions

Pedro Cardoso

(joint work with Patrícia Gonçalves)

Over the last years there has been a tremendous development in the field of singular stochastic partial differential equations (SPDEs). One remarkable example is the KPZ equation, which was proposed in 86' by Kardar, Parisi and Zhang in [1] as a universal law ruling the evolution of the profile of a randomly growing interface, and can be described as follows. For $t \geq 0$ and $x \in \mathbb{R}$, if $h(t,x)$ denotes the height of that interface at time t and position x, then the KPZ equation reads as

$$\partial_t h(t,x) = A\partial_x^2 h(t,x)dt + B[\partial_x h(t,x)]^2 dt + \sqrt{C}\mathcal{W}_t, \tag{1}$$

where A, B and C are constants that depend on the thermodynamical quantities of the interface (see for example [2]) and $\mathcal{W}_t$ is a space-time white noise. There is another singular SPDE related to the KPZ equation, that is known as the Stochastic Burgers equation (SBE). Its solution can be obtained, at least formally,

from the solution h of the KPZ equation by taking its space derivative, namely, defining $Y_t = \partial_x h_t$. In this case, Y_t solves the following equation

$$dY_t = A\partial_x^2 Y_t dt + B\partial_x Y_t^2 dt + \sqrt{C}\partial_x \mathcal{W}_t. \tag{2}$$

We stress that both (1) and (2) are singular due to the fact that they contain a non-linear term, and their solutions are not functions, but rather random distributions. For instance, the second term on the right-hand side of (1) is quadratic, but defining the product of random distributions in a precise way is often nontrivial. Some advances in this direction were achieved by Gubinelli and Perkwoski, who developed the theory of paracontrolled in [3]. There, it was possible to show the well-posedness not only for (1) but also for a multitude of SPDEs with more general characteristic operators. We highlight that the notion of solutions to the KPZ equation used by Gubinelli and Perkwoski is based on the definition of energy solutions, which was first cooked up by Gonçalves and Jara in [4]; this was done by looking at the fluctuations of a collection of weakly asymmetric exclusion processes in a one-dimensional lattice and at equilibrium.

An energy solution of a SPDE such as the KPZ equation is a random distribution Y_t, which is continuous in time and satisfies a martingale problem that contains an integral term corresponding to the non-linear term of the equation.

This notion of solution is well adapted to the derivation of fluctuations from microscopic random systems; they are governed by energy solutions to the KPZ equation for various models described in many recent works. This has provided advances for proving the weak KPZ universality conjecture, which states that the KPZ equation (or its companion, the SBE) is an universal law ruling the fluctuations of several random growth interfaces close to a stationary state.

A natural question that arose afterwards is regarding the derivation of other singular SPDEs from scaling limits of random microscopic systems. This was done in [5], where it was obtained an energy solution to a fractional stochastic Burgers equation (or its companion the fractional KPZ equation) from an exclusion dynamics that allows long jumps. There, the characteristic operators of the equation replace the Laplacian and the derivative operators (which are present in the classical KPZ equation) by their fractional versions. Since the particles were evolving in the set of integers $\mathbb{Z}$, the SPDEs in [5] were stated without boundary conditions. This was not the case in [6], where it was possible to produce energy solutions to the KPZ/SBE equation with Neumann/Dirichlet boundary conditions.

It is worth to note that by taking $B = 0$ in (2), one obtains the Ornstein-Uhlenbeck equation. This corresponds to a Gaussian process that is typically obtained in the fluctuations of a variety of microscopic systems whose asymmetry is missing (such as the symmetric simple exclusion process), or negligible. Some different notions of solutions for this process were also described in [5, 6].

In this talk an energy solution to the KPZ/SBE equation with characteristic operators given in terms of fractional operators such as the regional fractional Laplacian, which was not present previously in the literature. As this operator is non-local and it may a priori not be well-defined at the boundary, some Neumann/Dirichlet boundary conditions are required for our test functions, depending

on the values of some parameters ruling the dynamics. And similarly as in other previous works, when the asymmetry of our model is not strong enough we obtain solutions to the Ornstein-Uhlenbeck equation, as expected. All of this is done by applying a scaling limit to a particular interacting particle system.

We consider the model introduced in [7, 8] which consists of an exclusion process evolving on the one-dimensional discrete set of points $\Lambda_n := \{1, \cdots, n-1\}$ that we call *bulk*. The transition probability depends on the size of the jump but, and contrarily to the setting of [7, 8], it may not be symmetric; its moments are regulated by a parameter $\gamma \in (0, 2)$. Moreover at each site $x, y \in \mathbb{Z}$ satisfying $x \leq 0$ or $y \geq n$ we add a reservoir that can inject or remove particles in the system. As in [7, 8], the symmetric part of the dynamics is tuned by parameters $\alpha > 0$ and $\beta \in \mathbb{R}$ in the boundary, but not in the bulk. On the other hand, the antisymmetric part of the dynamics is tuned in *both* the boundary and the bulk by parameters $\alpha_a > 0$ and $\beta_a \geq 0$. Depending on the range of the aforementioned parameters, we conclude that the fluctuations at equilibrium of our models are either given in terms of the Ornstein-Uhlenbeck process or energy solutions to the SBE. We do not present the analogous results for the KPZ equation but we observe that one could follow the same strategy as in [4] and restate our results for the KPZ instead of the SBE. In order to do so, it is only necessary to replace the density fluctuation by the height density field, as an object of study. We leave this to the interested reader.

We observe that the recent article [9] not only proves the existence and uniqueness to energy solutions of the equation that we derive, but also for a much more general version of this equation, in which the noise is more irregular and presents bilinear nonlinearities. Their approach is not based on Fourier series expansion and this allows treating the uniqueness problem for a much larger class of characteristic operators, as the regional fractional Laplacian, a wide range of domains and boundary conditions. We believe that the results in [9] can be applied in many more contexts, aside from the ones of our work.

At this point it is worth to make a few comments about possible extensions of our model. We note that it should be possible to extend our results to more microscopic dynamics, with corresponding macroscopic operators given in terms of other kernels. A particular feature of our dynamics is the exclusion rule, which allows at most one particle per site. This restriction is absolutely not mandatory, but has the advantage of avoiding additional technical issues. Taking everything into account, our work is still a significant contribution to the literature, due to the fact that this is the first article regarding the derivation of SPDE written in terms of the regional fractional Laplacian in bounded domains from particle systems.

Lastly we remark that the Ornstein-Uhlenbeck equation and the SBE here obtained (or their analogous counterparts) should also arise as as the scaling limit of equilibrium fluctuations in the case of multi-component models. In this direction, we refer the reader to the articles of [10, 11] where it is conjectured that the fluctuations for such models are described by a number of limit SPDEs. However, very

little has been done in a rigorous way to prove those conjectures. Quite possibly the results in [9] are an important step in that direction.

References

[1] M. Kardar, G. Parisi, and Y. C. Zhang, *Dynamic Scaling of Growing Interfaces*, Phys. Rev. Lett. **56, n. 9** (1986), 889-892.

[2] P. Gonçalves and M. Jara, *The Einstein relation for the KPZ equation*, J. Stat. Phys. **158, n. 6** (2015), 1262–1270.

[3] M. Gubinelli and N. Perkowski, *Energy solutions of KPZ are unique*, J. Amer. Math. Soc. **31, n. 2** (2018), 427–471.

[4] P. Gonçalves and M. Jara, *Nonlinear fluctuations of weakly asymmetric interacting particle systems*, Arch. Ration. Mech. Anal. **212, n. 2** (2014), 597–644.

[5] P. Gonçalves and M. Jara, *Density fluctuations for exclusion processes with long jumps*, Probab. Theory Related Fields **170, n. 1 - 2** (2018), 311–362.

[6] P. Gonçalves, N. Perkowski, and M. Simon, M., *Derivation of the stochastic Burgers equation with Dirichlet boundary conditions from the WASEP*, Ann. H. Lebesgue **3** (2020), 87–167.

[7] C. Bernardin, P. Gonçalves, and B. Jiménez-Oviedo, *A microscopic model for a one parameter class of fractional Laplacians with Dirichlet boundary conditions*, Arch. Ration. Mech. Anal. **239, n. 1** (2021), 1–48.

[8] C. Bernardin, C. Cardoso P. Gonçalves, and S. Scotta, *Hydrodynamic limit for a boundary driven super-diffusive symmetric exclusion*, Stochastic Process. Appl. **165** (2023), 43–95.

[9] L. Grafner, N. Perkwoski, and S. Popat, *Energy solutions of singular SPDEs on Hilbert spaces with applications to domains with boundary conditions*, arXiv preprint arXiv:2411.07680 (2024).

[10] H. Spohn, *Nonlinear fluctuating hydrodynamics for anharmonic chains*, J. Stat. Phys. **154, n. 5** (2014), 1191–1227.

[11] V. Popkov, J. Schmidt, G. M. Schütz, *Universality classes in two-component driven diffusive systems*, J. Stat. Phys. **160, n. 4** (2015), 835–860.

Superdiffusive Central Limit Theorem for the critical Stochastic Burgers Equation

Giuseppe Cannizzaro

(joint work with Quentin Moulard, Fabio L. Toninelli)

The Stochastic Burgers Equation (SBE) is the singular, non-linear Stochastic Partial Differential Equation (SPDE) given by

$$\partial_t \eta = \tfrac{1}{2}\Delta\eta + \lambda(\mathfrak{w}\cdot\nabla)\eta^2 + \nabla\cdot\vec{\xi} \tag{1}$$

where $\vec{\xi} = (\xi_1, \dots, \xi_d)$ is a d-dimensional vector-valued space-time white noise, and $\lambda > 0$ is the so-called *coupling constant*. Here, $\mathfrak{w}$ is a unit vector in $\mathbb{R}^d$, but, by rotational invariance, the specific choice of $\mathfrak{w}$ is irrelevant.

The SBE was introduced in [3] as an effective continuous model for the mesoscopic fluctuations of driven diffusive systems with a single conserved scalar quantity, such as the Asymmetric Simple Exclusion Process (ASEP), the Zero-Range Process with asymmetric rates and other driven, conservative, interacting particle systems. In fact, the three terms at the right-hand side of (1) represent respectively a diffusive term, the non-linear effect of the driving field acting in direction $\mathfrak{w}$, and a microscopic noise of conservative type. On the basis of the so-called

mode-coupling theory, [3] predicts the large-scale behaviour of SBE in any dimension $d \geq 1$: in the subcritical dimension $d = 1$, the authors conjecture strong superdiffusion with a $t^{1/3}$ divergence of the diffusion coefficient for large time t; in the critical dimension $d = 2$, logarithmic enhancement of diffusion of order $(\log t)^{2/3}$; and, in the super-critical dimensions $d \geq 3$, classical diffusive behaviour. Let us point out that in $d = 1$, the SBE is nothing but the spatial derivative of the KPZ equation which, by now, is mathematically well-understood (see [11], for a review): not only the $t^{1/3}$ behaviour was confirmed [2], but the solution was shown to have a non-Gaussian and universal fixed point (the "KPZ fixed point" [10]), that also describes the large-scale behaviour of a large class of one-dimensional driven diffusive processes such as ASEP [12] and directed polymers in random environment [13]. In dimension $d \geq 3$, the series of works [9, 6, 8] has confirmed the conjectured normal diffusion of ASEP and the convergence of the particle density fluctuations to a linear Stochastic Heat Equation at large scales. A counterpart on the SPDE side is the recent [4], where the same result is obtained for the SBE in $d \geq 3$. As for the critical dimension $d = 2$, the breakthrough work [14] proved $(\log t)^{2/3}$ diffusivity for ASEP, while in [7] the analogous was shown for the SBE. Let us stress though that in both cases, the upper and lower bounds on the diffusivity feature *diverging subleading corrections* and their statement is phrased in the *Tauberian sense*, i.e. in terms of Laplace transform.

The problem when analysing the SPDE (1) is that it is severely ill-defined due to the presence of the non-linearity. Since dimension $d = 2$ is critical, it falls outside of the scope of any of the path-wise theories mentioned above which successfully addressed subcritical equations and, in fact, a local solution theory is not even expected. Our main results are the following. First, we determine the *sharp large-time asymptotic behaviour of the diffusion matrix*, including the explicit constant prefactor, thus significantly improving over both [14, 7] and fully addressing the long-standing conjecture of [3]. Second, we move well beyond that and obtain the (Gaussian) *large-scale fixed point of the regularised SBE*, when space-time is rescaled in a suitable, logarithmically superdiffusive way.

In addition to being the first such result for critical SPDEs, our work opens intriguing perspectives for other driven diffusive systems, including interacting particle systems. In broader terms, the paper [5] can be seen as an out-of-equilibrium counterpart of the celebrated work [1] that proves that the large-scale limit of the Ising and Φ^4_d equilibrium measures, in the critical dimension $d = 4$, is Gaussian, despite the presence of logarithmic corrections to the mean field critical exponents.

References

[1] M. Aizenman and H. Duminil-Copin, *Marginal triviality of the scaling limits of critical 4D Ising and ϕ^4_4 models*, Ann. Math. (2) **194** (2021), 163–235.

[2] M. Balázs, J. Quastel, and T. Seppäläinen, *Fluctuation exponent of the KPZ/stochastic Burgers equation*, J. Am. Math. Soc. **24** (2011), 683–708.

[3] H. van Beijeren, R. Kutner, and H. Spohn, *Excess noise for driven diffusive systems*, Phys. Rev. Lett. **54** (1985), 2026–2029.

[4] G. Cannizzaro, M. Gubinelli, and F. Toninelli, *Gaussian fluctuations for the stochastic Burgers equation in dimension $d \geq 2$*, Commun. Math. Phys. **405** (2024), 60.

[5] G. Cannizzaro, Q. Moulard, and F. Toninelli, *Superdiffusive central limit theorem for the stochastic Burgers equation at the critical dimension*, arXiv preprint 2501.00344 (2025), available at `https://arxiv.org/abs/2501.00344`.
[6] C. Chang, C. Landim, and S. Olla, *Equilibrium fluctuations of asymmetric simple exclusion processes in dimension $d \geq 3$*, Probab. Theory Relat. Fields **119** (2001), 381–409.
[7] D. De Gaspari and L. Haunschmid-Sibitz, *$(\log t)^{2/3}$-superdiffusivity for the 2d stochastic Burgers equation*, Electron. J. Probab. **29** (2024), 1–34.
[8] C. Landim, S. Olla, and H. T. Yau, *Diffusive behaviour of the equilibrium fluctuations in the asymmetric exclusion processes*, Adv. Stud. Pure Math. **39** (2004), 307–324.
[9] C. Landim and H. T. Yau, *Fluctuation-dissipation equation of asymmetric simple exclusion processes*, Probab. Theory Relat. Fields **108** (1997), 321–356.
[10] K. Matetski, J. Quastel, and D. Remenik, *The KPZ fixed point*, Acta Math. **227** (2021), 115–203.
[11] J. Quastel and H. Spohn, *The one-dimensional KPZ equation and its universality class*, J. Stat. Phys. **160** (2015), 965–984.
[12] J. Quastel and S. Sarkar, *Convergence of exclusion processes and the KPZ equation to the KPZ fixed point*, J. Am. Math. Soc. **36** (2023), 251–289.
[13] B. Virág, *The heat and the landscape I*, arXiv preprint 2008.07241 (2020), available at `https://arxiv.org/abs/2008.07241`.
[14] H. T. Yau, *$(\log t)^{2/3}$ law of the two-dimensional asymmetric simple exclusion process*, Ann. Math. (2) **159** (2004), 377–405.

Quantitative hydrodynamic limit for non-gradient Glauber-Kawasaki dynamics

Tadahisa Funaki

Let us consider the Glauber-Kawasaki dynamics on a d-dimensional periodic lattice of large size N, i.e., the particle system of interacting random-walks subject to the exclusion rule (Kawasaki part) with the creation and annihilation of particles (Glauber part) whose rates are set to favor two levels of particle density, called sparse and dense. We then studied the limit of our dynamics under the hydrodynamic scaling, i.e., $1/N$ in space and a diffusive scaling N^2 for the Kawasaki part and another scaling $K = K(N)$, which diverges slower, for the Glauber part in time. In the limit as $N \to \infty$, the particles autonomously undergo phase separation into sparse or dense phases at the microscopic level, and an interface separating two regions forms at the macroscopic level and evolves under an anisotropic curvature flow; see [1, 2, 3].

We also discussed the fluctuation of interfaces under a simple situation that the Kawasaki part is a simple exclusion and the interface is flat and immobile; see [4, 5]. Furthermore, we discussed the hydrodynamic limit for the conservative dynamic $P(\phi)_2$ model on a two-dimensional continuous torus of size N.

References

[1] T. Funaki, *Interface motion from Glauber-Kawasaki dynamics of non-gradient type*, arXiv:2404.18364v2
[2] T. Funaki, C. Gu, and H. Wang, *Quantitative homogenization and hydrodynamic limit of non-gradient exclusion process*, arXiv:2404.12234

[3] T. Funaki and H. Park, *Motion of sharp interface of Allen-Cahn equation with anisotropic nonlinear diffusion*, arXiv:2403.01732
[4] T. Funaki, C. Landim, and S. Sethuraman, *Linear fluctuation of interfaces in Glauber-Kawasaki dynamics*, arXiv:2412.04015
[5] T. Funaki, *Stochastic PDE approach to fluctuating interfaces*, arXiv:2412.00708

Transport and crossings

Alessandra Faggionato

The effective homogenized matrix D enters in the large scale limit of transport in disordered media, modeled e.g. via random resistor networks or interacting particle systems on random graphs. The non-degeneracy of D implies that the medium is a conductor and, on the other hand, is implied by the existence of enough long crossings.

In the first part of the talk we recall several large scale limits for random walks, interacting particle systems and random resistor networks on a very large class of random graphs with random conductances (cf. [1, 2, 3]). The resulting limiting behavior is described in terms of the matrix D, which could also be non-degenerate.

In the second part of the talk we present lower bounds (obtained in collaboration with Ivailo Hartarsky) for the statistics of left-right (LR) crossings leading to the non-degeneracy of D for a very large class of random graphs built on a Poisson point process with independent marks [4]. We briefly recall these results on crossings.

Given $\rho > 0$, a probability measure ν on a Polish space $\mathbb{M}$ and a symmetric connection function $\varphi : (\mathbb{R}^d \times \mathbb{M})^2 \to [0,1]$, we consider the homogeneous Poisson point process with intensity ρ on $\mathbb{R}^d$ and mark its points x by i.i.d. random variables m_x with common law ν. The random connection model $\mathrm{RCM}(\rho, \nu, \varphi)$ is then the random graph with vertex set the realization of the Poisson point process and edge set obtained by inserting an edge between vertices x, y with probability $\varphi\big((x, m_x), (y, m_y)\big)$. A special case is given by the so–called generalized Boolean models (as the Poisson-Boolean model and the Mott variable range hopping graph with cutoff) where φ is the characteristic function of some set $\mathcal{S}$.

We assume the following for a triple (λ, ν, φ):

(A1) φ is stationary, i.e. $\varphi\big((x,m),(x',m')\big) = \varphi\big((0,m),(x'-x,m')\big)$;
(A2) $\exists\ \ell_* > 0$ such that $\varphi\big((x,m),(x',m')\big) = 0$ if $|x - x'|_\infty \geq \ell_*$;
(A3) $\varphi((0,m),(x,m')) = \varphi((0,m),(y,m'))$ if
- y is obtained from x by a permutation of the coordinates of x,
- or y is obtained from x by flipping the sign of one coordinate of x.

(A4) for all $\rho \geq \lambda$ the graph $\mathrm{RCM}(\rho, \nu, \varphi)$ has at most one unbounded connected component a.s.;
(A5) the graph $\mathrm{RCM}(\lambda, \nu, \varphi)$ has an unbounded connected component a.s..

Under (A1),...,(A5) in [4] we have proved the following. For any $\rho > \lambda$ there exist $c_1, c_2 > 0$ such that, for all $\ell \geq 1$, it holds

$$\mathcal{P}\left(\mathcal{N}_\ell \geq c_1 \ell^{d-1}\right) \geq 1 - \exp\left(-c_2\, \ell^{d-1}\right) ,$$

where $\mathcal{N}_\ell$ denotes the maximal number of vertex-disjoint LR crossings of the box $[-\ell, \ell]^d$ included in the unique infinite cluster of $\mathrm{RCM}(\rho, \nu, \varphi)$.

As a consequence we have also shown that, by considering the unique infinite cluster of $\mathrm{RCM}(\rho, \nu, \varphi)$ with unit conductances, for any $\rho > \lambda$ the corresponding effective homogenized matrix D is strictly positive definite. By monotonicity this result can be extended to more models.

For other results, including the discussion of the Boolean model without (A2) and the applications to Mott's law we refer to [4].

References

[1] A. Faggionato, *Hydrodynamic limit of simple exclusion processes in symmetric random environments via duality and homogenization*. Probab. Theory Relat. Fields. **184** (2022), 1093–1137.
[2] A. Faggionato, *Stochastic homogenization of random walks on point processes*, Ann. Inst. Henri Poincaré Probab. Stat. **59** (2023), 662–705.
[3] A. Faggionato, *Scaling limit of the directional conductivity of random resistor networks on point processes*. Ann. Inst. Henri Poincaré Probab. Stat. **61** (2025), 1487–1521.
[4] A. Faggionato and I. Hartarsky. *Crossings and diffusion in Poisson driven marked random connection models*. Forthcoming.

Quantum stoquastic tensor networks from a stat mech perspective

Michael Aizenman
(joint work with Simone Warzel)

Tensor networks form platforms for quantum computations, and appear naturally also in the representation of thermal and ground states of many body quantum systems. The talk focused on correlations and entanglement in a class of such networks through methods that emerge from analogies with systems of classical stat mech.

The presentation was organized as follows.

I. The entropy of the restriction of a pure quantum state to a subsystem is a measure of the entanglement between the system's two components.

II. After explaining the concepts, the talk focused on conditions implying an area- type bound on the entanglement in pure states of quantum lattice models.

III. The general criterion was demonstrated to be applicable to the ground states of the quantum Ising model on $\mathbb{Z}^d$, for which it implies that an area-type entanglement upper bound holds in all dimensions up to the model's quantum phase transition.

References

[1] A. Aizenman and S. Warzel, *Entanglement Entropy Bounds for Pure States of Rapid Decorrelation*, Comm. Math. Phys.**32** (2025), 406:71.
DOI: https://doi.org/10.1007/s00220-025-05324-3 and references therein.

Characterization of the directed landscape from the KPZ fixed point

Duncan Dauvergne
(joint work with Lingfu Zhang)

The KPZ (Kardar-Parisi-Zhang) universality class is a loose term for a collection of two-dimensional random metric and polymer models built from an i.i.d. field, together with random interface growth models and particle systems that are often built from these structures. The directed landscape is the scaling limit of random metrics in this class. The KPZ fixed point is the scaling limit for random growth models in this class, and arises as a marginal of the directed landscape. In the present work [4], we give a characterization of the directed landscape from its KPZ fixed point marginals. For a large range of models, this reduces the problem of proving convergence to the directed landscape to proving convergence to the KPZ fixed point.

The KPZ fixed point is a Markov process on the space of upper semicontinuous functions $\mathrm{UC} = \{h : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}\}$, introduced by Matetski, Quastel, and Remenik [5] as the scaling limit of TASEP. It has been shown to be the scaling limit of a handful of other integrable growth models, e.g. Brownian TASEP [6], ASEP [1], and the KPZ equation [9]. We write $H(x, t; f)$ for the value of the KPZ fixed point at time $t \geq 0$ and location $x \in \mathbb{R}$ started from an initial condition f. For functions $f, g \in \mathrm{UC}$, the transition probability

$$\mathbb{P}(H(x, t; f) \leq -g(x) \quad \text{for all } x \in \mathbb{R})$$

can be expressed as a Fredholm determinant. It is this determinantal formula which was used in [5] to define the KPZ fixed point.

The directed landscape is a random continuous function $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$, where $\mathbb{R}^4_\uparrow = \{(x, s; y, t) \in \mathbb{R}^4 : s < t\}$. It was constructed by Dauvergne, Ortmann, and Virág [2] as the scaling limit of Brownian last passage percolation, and has since been shown to be the scaling limit of a handful of other integrable metric and polymer models in an i.i.d. field, e.g. exponential/Poisson last passage percolation [3], coloured ASEP and the stochastic six vertex model [1], and the continuum directed random polymer [7]. As with its prelimits, the directed landscape value $\mathcal{L}(x, s; y, t)$ is best thought of as defining a distance between points (x, s) and (y, t) in $\mathbb{R}^2$. It is directed, in the sense that distances are not defined if the points (x, s) and (y, t) are reversed, and real-valued rather than positive. However, it still satisfies a (reverse) triangle inequality: for any $o = (x, s)$, $p = (y, r)$, $q = (z, t) \in \mathbb{R}^2$ with $s < r < t$, we have

$$\mathcal{L}(o; p) + \mathcal{L}(p; q) \leq \mathcal{L}(o; q), \tag{1}$$

and so we may meaningfully think of it as a metric. It also has an independent increment property which allows us to view it as maximizing path length through an (ill-defined) independent noise field: for any disjoint time intervals $[s_i, t_i], i = 1, \ldots, k$,

$$\text{the functions } \mathcal{L}(\cdot, s_i; \cdot, t_i), i = 1, \ldots, k \text{ are independent.} \tag{2}$$

Unlike the KPZ fixed point, there are not exact formulas for all statistics in the directed landscape. Instead, the construction proceeds by studying a scaling limit of the one aspect of the classical Robinson-Schensted-Knuth (RSK) correspondence.

The connection between the directed landscape and the KPZ fixed point was demonstrated in [3, 6]. For any fixed $f \in \mathrm{UC}$ and $s \in \mathbb{R}$, as functions of $(x,t) \in \mathbb{R} \times (0,\infty)$ we have the following equality in law:

$$H(x,t;f) \stackrel{d}{=} \sup_{y\in\mathbb{R}} f(y) + \mathcal{L}(y,s;x,s+t). \tag{3}$$

That is, the directed landscape can be viewed as a coupling of the KPZ fixed point simultaneously from all initial conditions and started at all times.

The upshot of this relationship is that in any metric or polymer model where we have convergence to the directed landscape, we immediately have convergence of an associated growth model to the KPZ fixed point. Is the opposite true? If we have a collection of growth models converging to the KPZ fixed point coupled together in a natural way, can we say that the limiting coupling comes from the directed landscape? In other words, can we characterize the directed landscape from its KPZ fixed point marginals? Our work [4] offers an answer to these questions with the following theorem.

Theorem 1. Suppose that $\mathcal{L} : \mathbb{R}^4_{\uparrow} \to \mathbb{R}$ is a random continuous function satisfying the triangle inequality (1), the independent increment property (2), and the KPZ fixed point marginal property (3). Then $\mathcal{L}$ is a directed landscape.

Theorem 1 has a rephrasing in terms of monotone, shift commutative couplings of the KPZ fixed point, which is useful for models in which a metric structure is less obvious. We can also weaken various conditions in the theorem. For example, the initial function $\mathcal{L}$ could instead be defined only on a countable dense set, and we can ask that the marginal property (3) only holds for narrow wedge initial conditions. As a consequence of Theorem 1, we can quickly show that many models which converge to the KPZ fixed point also converge to the directed landscape. We can recover known examples this way without passing through RSK or similar machinery, and also cover a handful of new models without any RSK-type combinatorics: the Brownian/random walk web distances from [8], the particle coupling of ASEP, and a family of *exotic* couplings of ASEP and TASEP that lack integrability.

References

[1] A. Aggarwal, I. Corwin, and M. Hegde, *Scaling limit of the colored ASEP and stochastic six-vertex models*, arXiv preprint *arXiv:2403.01341* (2024).

[2] D. Dauvergne, J. Ortmann, and B. Virág. *The directed landscape*, Acta Mathematica **229** (2022), no. 2, 201–285.

[3] D. Dauvergne and B. Virág, *The scaling limit of the longest increasing subsequence*, arXiv preprint *arXiv:2104.08210* (2021).

[4] D. Dauvergne and L. Zhang, *Characterization of the directed landscape from the KPZ fixed point*, arXiv preprint *arXiv:2412.13032* (2024).

[5] K. Matetski, J. Quastel, and D. Remenik. *The KPZ fixed point*, Acta Mathematica, **227** (2021), no. 1 115–203.
[6] M. Nica, J. Quastel, and D. Remenik. *One-sided reflected Brownian motions and the KPZ fixed point*, Forum of Mathematics, Sigma, **8**:e63 (2020), 1–16.
[7] X. Wu. *The KPZ equation and the directed landscape*, arXiv preprint *arXiv:2301.00547* (2023).
[8] B. Vető and B. Virág, *The geometry of coalescing random walks, the Brownian web distance and KPZ universality*, arXiv preprint *arXiv:2305.15246* (2023).
[9] B. Virág. *The heat and the landscape I*, arXiv preprint *arXiv:2008.07241* (2020).

KPZ growth with open boundary conditions

Guillaume Barraquand

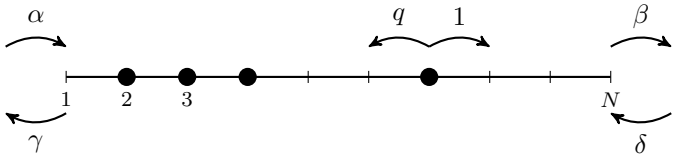
Stationary measures of a variety of growth models in the Kardar-Parisi-Zhang class can be described in terms of couples of interacting random walks or Brownian motions, called two-layer Gibbs measures. A prototypical example is the open KPZ equation: the stochastic PDE

$$\partial_t h(t,x) = \tfrac{1}{2}\partial_{xx} h(t,x) + \tfrac{1}{2}\big(\partial_x h(t,x)\big)^2 + \xi(t,x)$$

for $x \in [0,L]$, with boundary conditions

$$\partial_x h(t,x)\Big|_{x=0} = u, \quad \partial_x h(t,x)\Big|_{x=L} = -v$$

where $u, v \in \mathbb{R}$ are boundary parameters. It can be viewed as a continuous scaling limit [8] of ASEP's height function with appropriately scaled asymmetry $q = e^{-1/\sqrt{N}}$ and boundary parameters so that density is $1/2 + O(1/\sqrt{N})$.



The stationary measure of open ASEP is characterized through the Matrix Product Ansatz introduced in [9] and further exploited in [10, 6]. Taking a scaling limit, [7] obtained stationary measures for the open KPZ equation. The following characterization was obtained in [5] and [3]. When $u + v > 0$, the open KPZ stationary measure can be written as

$$h^{\mathrm{stat}}_{u,v}(x) = \Lambda_1(x) - \Lambda_1(0)$$

where Λ_1, Λ_2 are continuous processes distributed as

$$\mathbb{P}^{u,v}_{\mathrm{KPZ}}(\Lambda_1,\Lambda_2) = \frac{1}{\mathcal{Z}_{u,v}(L)} F_{u,v}(\Lambda_1,\Lambda_2)\ \mathbb{P}_{\mathrm{Brown}}(\Lambda_1)\mathbb{P}_{\mathrm{Brown}}(\Lambda_2)$$

where $\mathbb{P}_{\text{Brown}}$ is the Brownian measure with free endpoints, and the functional $F_{u,v}$ is

$$F_{u,v}(\Lambda_1, \Lambda_2) = \exp\left(- \int_0^L e^{-(\Lambda_1(s)-\Lambda_2(s))} ds \right) e^{-u(\Lambda_1(0)-\Lambda_2(0))} e^{-v(\Lambda_1(L)-\Lambda_2(L))}.$$

After presenting this result, the talk mentioned a number of further directions:

- An alternative description from [3] involving the Pitman transform, expected to be valid for all $u, v \in \mathbb{R}$;
- A formula for $\mathbb{E}[h(t, 0)]$ when starting from the stationary initial condition [1];
- Some non-rigorous results about fluctuations and large deviations [4].

Then, the rest of the talk was devoted to explaining why this structure involving a couple of two interacting Brownian motions is natural. In general, for solvable discrete models, there exist analogous results where interacting random walks are defined in a Gibbsian way [2]. The Boltzman weights used to define them originate in branching rules satisfied by families of symmetric functions in the Macdonald hierarchy (Schur polynomials, Hall-Littlewood functions, (q)-Whittaker...). This framework applies to various models: interacting particle systems between boundary reservoirs, directed polymer models in a strip, last passage percolation, the stochastic six-vertex model. To conclude, we also compared the outcome of this method with the traditional approach via Matrix Product Ansatz.

References

[1] G. Barraquand, Integral formulas for two-layer Schur and Whittaker processes. *Bull. Soc. Math. France*, 2025.

[2] G. Barraquand, I. Corwin, and Z. Yang, Stationary measures for integrable polymers on a strip. *Inventiones Mathematicae*, Volume 237, pages 1567–1641, 2024.

[3] G. Barraquand and P. Le Doussal, Steady state of the KPZ equation on an interval and Liouville quantum mechanics. *Europhys. Lett.*, 137(6):61003, 2022.

[4] G. Barraquand and P. Le Doussal, Large time cumulants of the KPZ equation on an interval. *preprint* arXiv:2504.18292, 2025.

[5] W. Bryc, A. Kuznetsov, Y. Wang, and J. Wesołowski, Markov processes related to the stationary measure for the open KPZ equation. *Probab. Theor. Rel. Fields*, 185(1-2):353–389, 2023.

[6] W. Bryc and J. Wesołowski, Asymmetric simple exclusion process with open boundaries and quadratic harnesses. *J. Stat. Phys.*, 167(2):383–415, 2017.

[7] I. Corwin and A. Knizel, Stationary measure for the open KPZ equation. *Comm. Pure Appl. Math.*, 77(4):2183–2267, 2024.

[8] I. Corwin and H. Shen, Open ASEP in the weakly asymmetric regime. *Comm. Pure Appl. Math.*, 71:2065–2128, 2018.

[9] B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier, Exact solution of a 1D asymmetric exclusion model using a matrix formulation. *J. Phys. A: Math. Gen.*, 26(7):1493, 1993.

[10] M. Uchiyama, T. Sasamoto, and M. Wadati, Asymmetric simple exclusion process with open boundaries and Askey–Wilson polynomials. *J. Phys. A: Math. Gen.*, 37(18):4985, apr 2004.

On the global solutions of the KPZ fixed point

Ofer Busani
(joint work with Sudeshna Bhattacharjee, Evan Sorensen)

The KPZ fixed-point is a random process appearing as the universal scaling limit of $1+1$ random growth models. As the slope of an initial condition is a conserved quantity for this system, it is natural to ask, for a fixed realization of the system, how many eternal solutions (bi-infinite in time) of a fixed slope exist? It is known that for a fixed realization, there is a 'good' set of slopes for which 'one force one solution' principle holds i.e. there exists a unique global solution to the KPZ fixed point with a prescribed slope. Outside the good set not much is known other than that for 'bad' slopes uniqueness fails. In particular, the number and nature of such 'bad slope' global solutions is unknown.

In this work we completely characterize all global solutions to the KPZ fixed point with a prescribed slope. In particular we show that for bad slopes there exists infinitely many global solutions.

The scaling limit of simple random walk and the intrinsic metric on 2D critical percolation clusters

Jason Miller
(joint work with Valeria Ambrosio, Irina Dankovic, Maarten Markering, Yizheng Yuan)

The conformal loop ensemble (CLE_κ, $\kappa \in (8/3, 8)$) is the canonical one-parameter family of conformally invariant probability measures on non-crossing loops in a simply connected domain $D \subseteq \mathbf{C}$ [3, 4]. Each loop looks locally like one of Schramm's SLE_κ curves [2], meaning that the loops are simple and do not intersect each other or the domain boundary for $\kappa \in (8/3, 4]$ and intersect themselves, each other, and the domain boundary for $\kappa \in (4, 8)$. The CLE_κ are important because they arise as the scaling limit of a variety of discrete lattice models, including critical percolation ($\kappa = 6$) [5].

In this work, we focus on the case that $\kappa \in (4, 8)$. The *gasket* of such a CLE_κ is the set of points which are not surrounded by one of the loops. The reason for this terminology is that the CLE_κ gasket is like a random analog of the Sierpinski gasket. Jointly with Valeria Ambrosio and Yizheng Yuan, we show that the CLE_κ gasket can be associated with:

- a canonical *geodesic metric* (the *CLE_κ metric*) and
- a canonical *diffusion* (the *CLE_κ Brownian motion*).

We conjecture that the CLE_κ metric describes the scaling limit of the intrinsic (a.k.a. chemical distance) metric of any discrete model which converges to a CLE_κ in the scaling limit. We also conjecture that the CLE_κ Brownian motion describes the scaling limit of simple random walk on the clusters of any discrete model which converges to a CLE_κ in the scaling limit. Jointly with Irina Dankovic, Maarten Markering, and Yizheng Yuan, we prove these conjectures in the case of

critical percolation on the triangular lattice. This work in particular is aimed at addressing the "Ant in the Labyrinth" problem (i.e., the study of simple random walk on critical percolation clusters) put forward by de Gennes [1].

References

[1] P. G. de Gennes, *La percolation: un concept unificateur*, La recherche **72** (1976), 919–927.
[2] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288.
[3] S. Sheffield, *Exploration trees and conformal loop ensembles*, Duke **147** (2009), 79–129.
[4] S. Sheffield and W. Werner, *Conformal loop ensembles: the Markovian characterization and the loop-soup construction*, Ann. of Math. **176** (2012), 1827–1917.
[5] S. Smirnov, *Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Ser. I Math. **333** (2001), 239–244.

The scaling limit of random planar maps with large faces

Armand Riera
(joint work with Nicolas Curien, Grégory Miermont)

In [2], we prove that large bipartite Boltzmann planar maps in the heavy-tailed regime $\alpha \in (1, 2)$ admit a universal scaling limit that we construct explicitly. Specifically, after rescaling graph distances by $n^{-1/(2\alpha)}$, these random maps converge in law – in the Gromov-Hausdorff-Prokhorov sense – to a one-parameter family of fractal metric spaces $\mathcal{S}_\alpha$.

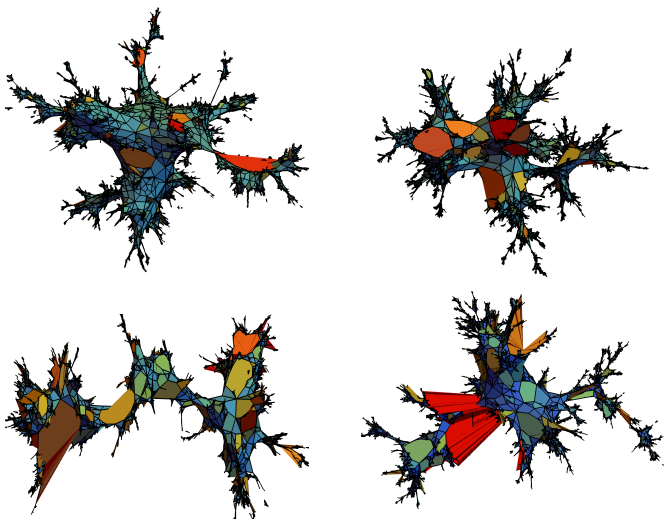


Figure 1. Simulations of Boltzmann random planar maps with $\alpha \in \{1.8, 1.6, 1.5, 1.4\}$, from top left to bottom right.

Discret model. A planar map is a proper embedding of a finite multigraph in the two-dimensional sphere such that the connected components of the complement are simply connected. Embeddings are considered up to orientation-preserving homeomorphism. As usual, all maps are *rooted* (one corner is distinguished) and *marked* (one vertex is distinguished). In the bipartite Boltzmann model, each face of degree (or perimeter) $2k$ carries weight $q_k \geq 0$, so that a map M with root half-edge $\vec{e}$ and mark ρ has total weight

$$w_{\mathbf{q}}(\mathsf{M}, \vec{e}, \rho) = \prod_{f \in \mathrm{Faces}(\mathsf{M})} q_{\deg(f)/2},$$

where $\deg(f)$ stands for the degree of f. When the tail of a typical face degree is in domain of attraction of a α-stable law, we say that $\mathbf{q}$ is *non-generic of exponent* α. Such maps arise naturally as gaskets of critical $O(N)$ loop models, percolation clusters, or other statistical-mechanics models, see Figure 2.

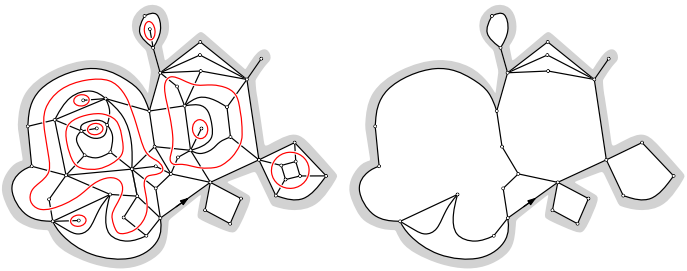


FIGURE 2. On the left, a (rigid) loop configuration on a quadrangulation with a boundary. On the right, its gasket obtained by removing the loops and the faces inside them. At criticality, under natural multiplicative measures, the gasket is a non-generic bipartite Boltzmann planar map in the heavy-tailed regime.

Main Theorem. Let $(\mathfrak{M}_n, \vec{e}_n, \rho_n)$ be a α-non-generic $\mathbf{q}$-Boltzmann map, conditioned to have n vertices (with root half-edge $\vec{e}_n$ and mark ρ_n). Denote the graph distance by d_n^{gr} and write vol_n for the uniform measure on the vertices of $\mathfrak{M}_n$. The main result of [2] established that there exists an explicit constant $\mathsf{s}_{\mathbf{q}} > 0$ (depending only on $\mathbf{q}$) such that

$$\Big(V(\mathfrak{M}_n),\, (\mathsf{s}_{\mathbf{q}}\, n)^{-1/(2\alpha)}\, d_n^{\mathrm{gr}},\, \rho_n, \mathrm{vol}_n\Big) \xrightarrow{(d)} \big(\mathcal{S}_\alpha,\, D_\alpha^\star,\, \rho_\alpha^\star,\, \mathrm{Vol}_\alpha\big),$$

where $V(\mathfrak{M}_n)$ stands for the vertices of $\mathfrak{M}_n$, and $\big(\mathcal{S}_\alpha, D_\alpha^\star, \rho_\alpha^\star, \mathrm{Vol}_\alpha\big)$ is a random compact metric space with distinguished point $\rho_\alpha^\star$ and Borel measure Vol_α. We call it the *α-stable carpet* if $\alpha \geq 3/2$, and the *α-stable gasket* if $\alpha < 3/2$.

Continuum construction. The limit space is obtained by first letting X be the normalized excursion of a spectrally positive α-stable Lévy process over $[0,1]$. Identifying pairs of times according to the jump structure of X yields the *stable*

looptree $\mathcal{L}$. Next, on each loop corresponding to a jump of size Δ, one grafts an independent Brownian bridge of length Δ, producing a continuous Gaussian label process Z, which can also be interpreted as the Brownian motion indexed by $\mathcal{L}$. Finally, one defines the intrinsic pseudo-metric:

$$D^\star_\alpha(s,t) = \inf \sum_{k=1}^{p} \big(Z_{s_k} + Z_{t_k} - 2\max\big(\min_{[s_k,t_k]} Z, \min_{[t_k,s_k]} Z\big)\big),$$

where the infimum is over all integers $p \geq 1$ and sequences $(s_k, t_k)_{1\leq k\leq p}$ such that $(s_1, t_p) = (s, t)$ and, for each $1 \leq k < p$, the times t_k and s_{k+1} project to the same point in $\mathcal{L}$. The label process Z realizes its infimum at a unique time $t^\star$. The quotient $[0,1]/\{D^\star_\alpha = 0\}$[1] endowed with $D^\star_\alpha$, the equivalent class of $t^\star$, and the pushforward of Lebesgue measure is the space $(\mathcal{S}_\alpha, D^\star_\alpha, \rho^\star_\alpha, \mathrm{Vol}_\alpha)$. The loops of $\mathcal{L}$ become the boundaries of the faces of $\mathcal{S}_\alpha$.

Main steps of the proof. Tightness in Gromov–Hausdorff–Prokhorov follows from the Bouttier–Di Francesco–Guitter bijection and the contour–label convergence of Le Gall–Miermont [3]. Furthermore, it follows that any subsequential limit can be written in the form

$$\big([0,1]/\{D=0\},\, D,\, \rho,\, \mathrm{Vol}\big),$$

where D is a random pseudo-distance, ρ a distinguished point, and Vol the image of the Lebesgue measure. By construction, we can couple D and $D^\star_\alpha$ in such a way that, almost surely, $D \leq D^\star_\alpha$ and so that they share a family of geodesics to $\rho^\star_\alpha$ called *simple geodesics*, see Figure 3. The term "simple geodesics" reflects the fact that they can be read from the construction of $\mathcal{S}_\alpha$ in a simple way.

The goal is then to prove that $D = D^\star_\alpha$ using this family of simple geodesics. To this end, one starts establishing that

$$D(s,t) = 0 \iff D^\star_\alpha(s,t) = 0$$

by exploiting the continuous "faces" inherited from $\mathcal{L}$, which implies that the two spaces have the same topology and volume measure, and that ρ coincides with $\rho^\star_\alpha$, which is then a uniform point with respect to the volume measure.

Next, via an adaptation of Miermont's two-point bijection with delays, one shows that all geodesics (for D and $D^\star_\alpha$) to $\rho^\star_\alpha$ are simple geodesics. A final surgery argument – approximating arbitrary geodesics by concatenations of simple geodesics – yields $D = D^\star_\alpha$ everywhere. This analysis relies on studying the process (X, Z) using the theory of stochastic processes, and on developing an analogue of the Brownian snake theory in this setting.

Topological Phases and Outlook. The topology exhibits a phase transition at $\alpha = 3/2$. In the *dilute phase*, $\alpha \geq 3/2$, face boundaries are disjoint Jordan loops, and Moore–Whyburn's theorems imply that $\mathcal{S}_\alpha$ is homeomorphic to the Sierpiński carpet. In the *dense phase*, $\alpha < 3/2$, loops intersect in fractal patterns, yielding a conjecturally random homeomorphism type.

[1] If $d : [0,1] \to \mathbb{R}^2_+$ is a pseudo-distance we write $[0,1]/\{d=0\}$ for the quotient space obtained using the equivalence relation $s \sim_d t$ if and only if $d(s,t) = 0$.

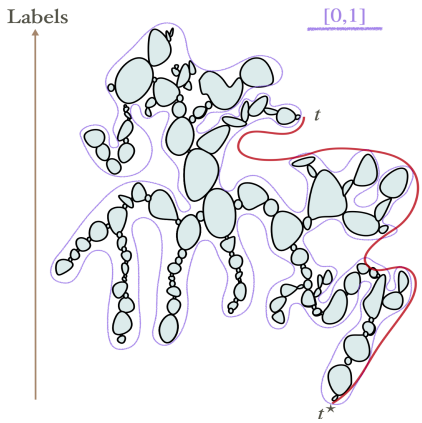


FIGURE 3. Label–cactus representation of the looptree: the vertical axis is the label value. The looptree is equipped with a clockwise exploration $p\colon [0,1] \to \mathcal{L}$ (violet). The simple geodesic associated with t, starts at $p(t)$ and then follows successive record minima (red) until $p(t^\star)$, which becomes $\rho^\star$ in the quotient.

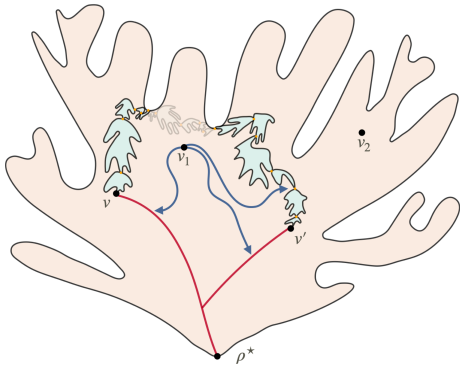


FIGURE 4. Illustration for $D(s,t) = 0 \iff D^\star_\alpha(s,t) = 0$. The "faces" of $\mathcal{S}_\alpha$ (blue holes) prevent extra identifications: any two distinct points are separated by a sequence of faces and simple geodesics (red).

Natural extensions of this work include scaling limits of non-bipartite and higher-genus maps, statistical-physics decorations (percolation, Ising), and connections with Liouville quantum gravity decorated with conformal loop ensembles. A "stable geometry" theory parallel to Brownian geometry is currently under development, which we hope will enrich the interplay between probability, fractal topology, and mathematical physics. We also expect to directly connect this "stable geometry" with the recent theory of self-similar Markov trees [1] (see

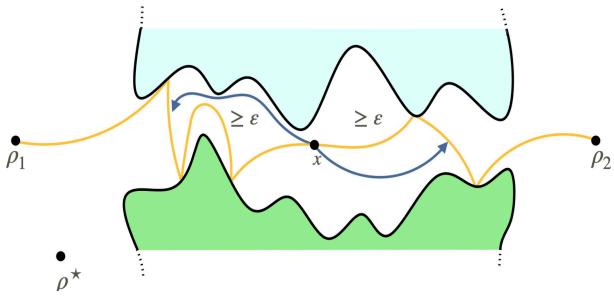


FIGURE 5. Illustration in yellow of a geodesic between two typical (for Vol) points ρ_1 and ρ_2. The geodesic bounces off the boundaries of faces. This phenomenon allows us to trap neighborhoods of the geodesic so that any geodesic from these neighborhoods towards $\rho^\star$ will intersect the yellow geodesic. This makes it possible to recover portions of the geodesic between ρ_1 and ρ_2 by using simple geodesics of the same length for D and $D^\star_\alpha$. The rest of the geodesic is controlled via stretched-exponential volume-of-balls estimates.

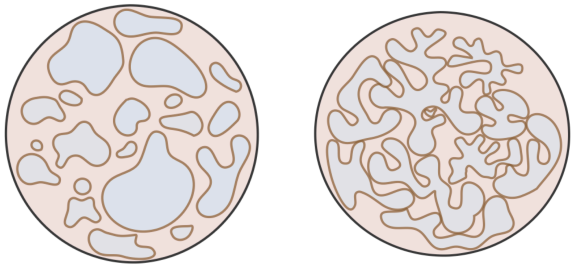


FIGURE 6. Illustration of the behavior of the faces in the dilute (left) and dense (right) phases.

Section 8.2 therein for the discrete counterpart and [4, 5] for similar results for Liouville quantum gravity).[2]

REFERENCES

[1] J. Bertoin, N. Curien, and A. Riera, *Self-similar Markov trees and scaling limits*, `arXiv:2407.07888`, (2025).
[2] N. Curien, G. Miermont, and A. Riera, *The scaling limit of planar maps with large faces*, `arXiv:2501.18566`, (2025).
[3] J.-F. Le Gall and G. Miermont, *Scaling limits of random planar maps with large faces.* The Annals of Probability, 1–69, (2011).

[2]All results presented here are from [2]. Figures 1, 4, and 5 are variations of figures in [2], and Figure 2 is taken from [1].

[4] J. Miller, S. Sheffield, and W. Werner, *Non-simple conformal loop ensembles on Liouville quantum gravity and the law of CLE percolation interfaces*, Probability Theory and Related Fields, 669–710, (2021).
[5] J. Miller, S. Sheffield, and W. Werner, *Simple conformal loop ensembles on Liouville quantum gravity*, The Annals of Probability, 905–949, (2022).

Liouville quantum gravity with central charge in $(1, 25)$

Ewain Gwynne

A **Liouville quantum gravity (LQG) surface** with the disk topology, with central charge $c > 1$, is the random surface with Riemannian metric tensor g on the unit disk $\mathbb{D}$, where g is sampled from "uniform measure on Riemannian metric tensors, weighted by $(\det \Delta_g)^{-(26-c)/2}$" and Δ_g denotes the Laplace-Beltrami operator.

The above definition does not make literal sense. In the **subcritical** or **weakly coupled** phase, which corresponds to $c > 25$, as well as the critical case $c = 25$, there has been enormous progress in recent years on making rigorous sense of LQG surfaces and relating them to other objects. Some highlights include connections between LQG surface and Schramm-Loewner evolution (SLE) and random planar maps, the construction of the LQG metric, and the rigorous formulation of Liouville conformal field theory.

The **supercritical** or **strongly coupled** phase of LQG, which corresponds to $c \in (1, 25)$, is much more mysterious than the subcritical phase, even at a physics level of rigor. However, this phase is potentially even more interesting than the subcritical phase from the perspective of bosonic string theory and, more speculatively, Yang-Mills theory. In my talk, I gave an overview of some recent mathematical developments concerning supercritical LQG:

- The construction of a metric associated with supercritical LQG (joint with Jian Ding).
- A coupling of supercritical LQG with the conformal loop ensemble (CLE_4) for each $c \in (1, 25)$. This coupling has similar Markovian and solvability properties as couplings of subcritical LQG with SLE. In contrast to the subcritical case, the LQG surface and the CLE_4 are neither independent from each other nor determine each other; rather, the coupling depends on c (based on joint work with Morris Ang).
- A class of combinatorially defined loop-decorated random planar maps whose scaling limit is conjectured to be supercritical LQG coupled to CLE_4 (based on joint work with Morris Ang).
- A proof that it is not possible to associate a locally finite area measure with supercritical LQG (based on joint work with Manan Bhatia and Jinwoo Sung).
- A rigorous version, in the setting of random planar maps, of the heuristic that supercritical LQG surfaces "conditioned to be finite" should correspond to "branched polymers", i.e., they should look like the continuum random tree (based on joint work with Manan Bhatia and Jinwoo Sung).

- A way of defining the Tutte embedding of random planar maps with finite boundary but infinitely many ends, including ones conjectured to converge to supercritical LQG (based on joint with with Jinwoo Sung).

Yang-Mills theory and random surfaces

Scott Sheffield

(joint work with Sky Cao, Ron Nissim, Joshua Pfeffer, Minjae Park, Pu Yu)

The Clay Institute famously offered one million dollars to anyone who could mathematically construct and understand a certain continuum version of "Yang-Mills gauge theory". This theory is the basis of the standard model of physics, and the heart of the problem is to understand the so-called "Wilson loop expectations". Following recent work with Sky Cao and Minjae Park, I will explain how the "Wilson loop expectations" in a lattice Yang-Mills model are equivalent to "insertion costs" of loops in a related random-closed-surface-ensemble model. In a sense, these results allow us to convert one famously hard problem into another presumably hard problem. But the new problem is all about random surfaces and random permutations, and it has a lot of relationships with and similarities to other problems we understand (think domino tilings, random planar maps, Young tableaux and symmetric group representation theory, and the Weingarten calculus). It gives us some intuition for *why* certain things should be true like the "area law" or "exponential correlation decay" (what physicists call "quark confinement" or "mass gap") even if we can't prove all of them yet. I'll also explain more recent work with Cao and Nissim that applies these ideas to extend the regime of parameters for which the area law can be proved.

Minimal surfaces in random environments

Barbara Dembin

(joint work with Dor Elboim, Ron Peled)

Let $d, n \geq 1$ be integers. The surface is modeled by a function $\varphi : \mathbb{Z}^d \to \mathbb{R}^n$, defined on the cubic lattice $\mathbb{Z}^d$ and having n components. Given an *environment* $\eta : \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$, later taken to be random and termed the *disorder*, the *formal Hamiltonian* for φ is

$$H^\eta(\varphi) := \frac{1}{2} \sum_{u \sim v} \|\varphi_u - \varphi_v\|^2 + \sum_v \eta_{v,\varphi_v}, \tag{1}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$ and $u \sim v$ indicates that $u, v \in \mathbb{Z}^d$ are adjacent. Our goal is to study the minimizers of H^η in finite domains with

prescribed boundary values. Given a finite $\Lambda \subset \mathbb{Z}^d$ and a function $\tau : \mathbb{Z}^d \to \mathbb{R}^n$, the *finite-volume Hamiltonian* in Λ is given by

$$H^{\eta,\Lambda}(\varphi) := \frac{1}{2} \sum_{\substack{u \sim v \\ \{u,v\} \cap \Lambda \neq \emptyset}} \|\varphi_u - \varphi_v\|^2 + \sum_{v \in \Lambda} \eta_{v,\varphi_v},$$

and the *configuration space with boundary values τ outside Λ* is given by

$$\Omega^{\Lambda,\tau} := \{\varphi : \mathbb{Z}^d \to \mathbb{R}^n \colon \varphi_v = \tau_v \text{ for } v \in \mathbb{Z}^d \setminus \Lambda\}.$$

We write $\varphi^{\eta,\Lambda,\tau}$ for the *ground configuration* of the finite-volume model, i.e., for the $\varphi \in \Omega^{\Lambda,\tau}$ which minimizes $H^{\eta,\Lambda}$ (if existing). We let

$$\mathrm{GE}^{\eta,\Lambda,\tau} := H^{\eta,\Lambda}(\varphi^{\eta,\Lambda,\tau})$$

be the *ground energy*. In the minimization process, there is a trade-off between the elastic energy, which favors flatness of the surface, and the noise contribution, which drives the surface to delocalize in order to exploit favorable fluctuations. A key question is whether the surface *delocalizes* depending on the dimension, and how the vertical fluctuations (i.e., the typical height of the surface) relate to the fluctuations of the ground energy. In two dimensions, scaling relations are known for models such as first and last passage percolation [3, 1, 2], but no analogous results have been rigorously established for minimal surface models (see [5] for physics prediction).

This model is very general and, depending on the choice of disorder η, can serve as a toy model for various physical systems. For instance, when η is a mollified white noise, the model can approximate the domain wall in the random bond Ising model at zero temperature under Dobrushin boundary conditions. In earlier work (see [4]), we derived, for the mollified white noise, a version of the scaling relation $\chi = 2\xi + d - 2$, connecting the ground energy fluctuation exponent χ and the vertical fluctuation exponent ξ, along with bounds on the vertical fluctuations.

In this talk, we focus on the case where $(\eta_{v,.})_{v \in \mathbb{Z}^d}$ are independent fractional Brownian motions with Hurst parameter $H \in (0,1)$. The case $H = 1/2$ corresponds to standard two-sided Brownian motion and is particularly relevant as a simplified model for domain walls in the random field Ising model.

Main Results

In dimensions $d \in \{1,2,3\}$, we show that the surface delocalizes polynomially and that the scaling relation holds. More precisely, there exist constants $C, c > 0$, depending only on H and n, such that for all $t > 0$ and all integers $L \geq 1$:

- **Height fluctuations:**

$$ce^{-Ct^{4-2H}} \leq \mathbb{P}\left(\max_{v \in \Lambda_L} \|\varphi_v^{\eta,\Lambda_L}\| \geq tL^{\frac{4-d}{4-2H}}\right) \leq Ce^{-ct^{4-2H}}.$$

- **Ground energy fluctuations:**

$$ce^{-Ct^{2-H}} \leq \mathbb{P}\left(|\mathrm{GE}^{\eta,\Lambda_L} - \mathbb{E}[\mathrm{GE}^{\eta,\Lambda_L}]| \geq tL^{\frac{4-d}{2-H}+d-2}\right) \leq Ce^{-ct^{2-H}}.$$

In dimension $d = 4$, which we identify as the *critical dimension*, the height fluctuations are bounded between $(\log\log L)^{1/(4-2H)}$ and $(\log L)^{5/(4-2H)}$.

For dimensions $d \geq 5$, the surface remains essentially flat, and the standard deviation of the ground energy scales like $L^{d/2}$.

Remark. Just prior to the first appearance of this paper, Otto, Palmieri and Wagner published the work [6] which studies the model (1) in the $1+1$-dimensional case with Brownian disorder. Our results overlap for the upper-bound for the height fluctuations ($d = n = 1$ and $H = 1/2$).

Proof ideas

Our approach hinges on the following identity, which plays a central role in the analysis. For any functions $s, \varphi : \mathbb{Z}^d \to \mathbb{R}^n$, we have

$$H^{\eta^s,\Lambda}(\varphi + s) - H^{\eta,\Lambda}(\varphi) = (\varphi, -\Delta_\Lambda s) + \frac{1}{2}\|\nabla s\|_\Lambda^2 - \sum_{v\in\Lambda} \eta_{v,-s_v},$$

where the *shifted and recentered disorder* η^s is defined by

$$\eta^s_{v,t} := \eta_{v,t-s_v} - \eta_{v,-s_v}.$$

This identity enables to quantify the cost of vertically shifting the surface. Delocalization occurs when the energetic gain to be at a higher height exceeds the shift cost. The scaling relation naturally emerges when these two effects are balanced. Unlike in our previous work [4], here the gain in energy is dominated by the additive term $\sum_{v\in\Lambda} \eta_{v,-s_v}$, which can be easily controlled. This enables us to precisely identify the scaling exponents.

References

[1] A. Auffinger, and M. Damron, *A simplified proof of the relation between scaling exponents in first-passage percolation,* The Annals Of Probability. **42**, 1197-1211 (2014)

[2] R. Basu, V. Sidoravicius, and A. Sly, *Rotationally invariant first passage percolation: Concentration and scaling relations,* ArXiv Preprint ArXiv:2312.14143. (2023)

[3] S. Chatterjee, *The universal relation between scaling exponents in first-passage percolation,* Annals Of Mathematics. pp. 663-697 (2013)

[4] B. Dembin, D. Elboim, D. Hadas, and R. Peled, *Minimal surfaces in random environment,* ArXiv Preprint ArXiv:2401.06768. (2024)

[5] D. Huse, C. Henley, D. Fisher, *Huse, Henley, and Fisher respond,* Physical Review Letters. **55**, 2924 (1985)

[6] F. Otto, M. Palmieri and C. Wagner, *On minimizing curves in a Brownian potential,* ArXiv Preprint ArXiv:2503.12471. (2025)

Application of Liouville quantum gravity in 2D percolation

Xin Sun

Two-dimensional (2D) percolation at criticality can be studied by various techniques related to conformal invariance. The most famous example is Cardy's formula for rectangular crossing [6]. Viewing the crossing probability as a four-point correlation function of a conformal field theory (CFT), Cardy's predicted an exact formula as the solution to a second order differential equation. This was later proved by Smirnov for site percolation on the triangular lattice [27]. Schramm-Loewner evolution (SLE) [26] is an approach to scaling limit of 2D lattice models that is complementary to CFT. In particular, the SLE_6 curves describe the scaling limit of 2D percolation. Therefore the exact formulae for 2D percolation predicted by CFT can be formulated as exact statements about SLE_6. For example, the aforementioned Cardy's formula for SLE_6 can be proved by considering a proper martingale observable. Furthermore, the SLE connection was used by Lawler, Schramm, Smirnov, Werner [28, 16] to rigorously prove conjectural formulae for the one-arm exponent and the polychromatic arm exponents for 2D percolation.

The main purpose of this talk is to explain how the method of Liouville quantum gravity (LQG) can be used to obtain exact results for 2D percolation that is hard to assess via pure SLE method based on martingale observable. As an illustration, we consider the crossing probability for critical percolation on an annulus in the continuum limit. By conformal invariance, the probability is a function of the ratio between the inner radius r and the outer radius R of the annulus. In [8], Cardy predicted a formula for $\mathbb{P}[\tau]$:

$$\mathbb{P}[\tau] = \sqrt{\frac{3}{2}} \cdot \frac{\eta(6i\tau)\eta\left(\frac{3}{2}i\tau\right)}{\eta(2i\tau)\eta(3i\tau)}, \quad \text{with} \quad \tau = \frac{1}{2\pi}\log(\frac{R}{r}) \tag{1}$$

and $\eta(z) = e^{\frac{i\pi z}{12}} \prod_{n=1}^{\infty}(1 - e^{2ni\pi z})$ is the *Dedekind eta function*.

Cardy's prediction is based on a non-rigorous Coulomb gas method. In a recent work [29] with Shengjing Xu and Zijie Zhuang, we rigorously proved (1). This formula is not easy to asscess with the martingale observable method, since the differential equation involved is a PDE from which (1) is not clear [14].

Our derivation of (1) is based on LQG on the annulus. Recall that the scaling limit of uniformly sampled random triangulation is the Brownian surface of certain topology [17, 18, 4]. For Brownian sphere and disk, Miller and Sheffield [19, 20, 21] showed that the random geometry can be produced by the LQG geometry with parameter $\gamma = \sqrt{8/3}$ induced by a variant of Gaussian free field (GFF). It was further showed in [3] that the GFF variant is given by Liouville CFT [9]. Altogether these confirm a picture from Polyakov's seminal paper [24]. For the Brownian annulus, there is an additional randomness coming from the conformal structure of the surface. Namely the conformal modulus of the Brownian annulus is random. The exact law of of this random modulus was predicted in physics, and rigorous proved in our recent work with Ang and Remy [2].

The approach in [2] is based on the integrability of Liouville CFT and the mating-of-trees framework of SLE/LQG coupling [12]. The paper gives a general

method to compute the random modulus for natural LQG surfaces. Consider the scaling limit of the uniform triangulation of annular topology, decorated with critical percolation, restricted to the event that the annulus crossing event occurs. The additional crossing constraint does not change the local geometry of the surface, but the exact law of the random modulus differs from that of the Brownian annulus. By conformal invariance, the Radon-Nikodym derivative between the new and old random moduli is exactly $P[\tau]$. This gives (1).

This is an example of a wave of applications of LQG in obtaining exact results for 2D percolation. Another example is the evaluation of the backbone exponent [23], which was previously unknown in physics. The idea of using quantum gravity to the study the continuum limit of 2D lattice models is not new. For example, Duplantier [10] famously used the Knizhnik-Polyakov-Zamolodchikov (KPZ) relation to give a convincing derivation for Mandelbrot's conjecture that the frontier of planar Brownian motion has dimension 4/3. The first mathematical version of the KPZ relation was established by Duplantier and Sheffield [11], marking the beginning of LQG in probability. The key novelty in our recent applications is the crucial usage of the integrability of Liouville CFT. For example, the derivation of the backbone exponent relies on the boundary analog of the DOZZ formula [15], which was obtained by Remy and Zhu [25]. The derivation of (1) relies on the exact formula for the annulus partition function of Liouville CFT on the annulus obtained by Wu [30].

Our proof method for (1) is quite general. In the original [2], we used it to derived the annulus partition for conformal loop ensemble with $\kappa \in (\frac{8}{3}, 4)$. In [29], in addition to (1), we derived a formula for the probability that there exists two paths of opposite colors crossing the annulus, and a formula for the probability that there exists two disjoint paths of the same color. The former formula was originally conjectured by Cardy [7], whose leading asymptotic is given by the polychromatic two-arm exponent. The latter probability corresponds to the monochromatic two-arm exponent, namely the backbone exponent. Our latter formula is again new to physicists. In a forthcoming work, with Cai, Fu, and Xie, we will give an exact formula for the disconnection probability for a Brownian excursion that crosses an annulus.

We conclude our talk with a list of future directions and open questions.

(1) So far we have only rigorously derived the random modulus for the Brownian annulus. In fact, there is a conjectural formula for Brownian surface of general topology, based on bosonic string theory [13]. We are working with Baverez, Jego, and Wu towards proving this conjecture. We expect that this can help obtaining exact results for percolation on general Riemann surfaces.
(2) All of the aforementioned exact results for 2D percolation except for the backbone exponent and its corresponding annular crossing probability have a CFT interpretation. It would be extremely interesting to find such interpretations for the backbone related quantities. In fact, the formula

we obtained for the annulus crossing probability in [29] is quite suggestive; see Section 1.1 there for more discussion.

(3) We are still far from a complete understanding of the CFT behind 2D percolation; see the most recent progress in math [1, 5], and in physics [22], respectively. We expect that understanding the Coulomb gas method rigorously will play a key role.

(4) Our discussion is not specific to 2D percolation. All existing applications and further questions discussed here makes sense for other 2D lattice models with conformally invariant scaling limit, such as random cluster model and self avoiding walk.

References

[1] M. Ang, G. Cai X. Sun, and B. Wu, *Integrability of Conformal Loop Ensemble: Imaginary DOZZ*, arXiv e-prints (2024).

[2] M. Ang, G. Remy, and X. Sun, *The moduli of annuli in random conformal geometry*, arXiv e-prints (2022), to appear at Annales Scientifiques de l'ENS.

[3] J. Aru, Juhan, Y. Huang, and X. Sun, *Two perspectives of the 2D unit area quantum sphere and their equivalence*, Comm. Math. Phys. **356** (2017), 261–283.

[4] J. Bettinelli and G. Miermont, *Compact Brownian surfaces II. Orientable surfaces*, arXiv e-prints (2022).

[5] F. Camia and Y. Feng, *Conformally covariant probabilities, operator product expansions, and logarithmic correlations in two-dimensional critical percolation*, arXiv e-prints (2024).

[6] J. Cardy, *Critical percolation in finite geometries*, Journal of Physics. A. **25** (1992), 201–205.

[7] J. Cardy, *Crossing formulae for critical percolation in an annulus*, Journal of Physics. A. **35** (2002), L565–L572.

[8] J. Cardy, *The* O(n) *model on the annulus*, J. Stat. Phys. **125** (2006), 1–21.

[9] F. David, A. Kupiainen, R. Rhodes, and V. Vargas, *Liouville quantum gravity and KPZ*, Invent. Math. **185** (2011), 333–393.

[10] B. Duplantier, *Random walks and quantum gravity in two dimensions*, Phys. Rev. Let. **81** (1998), 5489–5492.

[11] B. Duplantier, and S. Sheffield, *Liouville quantum gravity on the Riemann sphere*, Communications in Mathematical Physics. **342** (2011), 869–907.

[12] B. Duplantier, J. Miller, and S. Sheffield, *Liouville quantum gravity as a mating of trees*, Astérisque. **427** (2021).

[13] C. Guillarmou, R. Rhodes, and V. Vargas, *Polyakov's formulation of* 2*d bosonic string theory*, Publ. Math. Inst. Hautes Études Sci. **130** (2019), 111–185.

[14] J. Dubédat, *Critical percolation in annuli and* SLE_6, Comm. Math. Phys. **245** (2004), 627–637.

[15] A. Kupiainen, R. Rhodes, and V. Vargas, *ntegrability of Liouville theory: proof of the DOZZ Formula*, Annals of Mathematics. **191** (2020), 81–166.

[16] G. F: Lawler, O. Schramm, and W. Werner, *One-arm exponent for critical 2D percolation*, Electron. J. Probab. **7** (2002), no. 2, 13.

[17] J.-F. Le Gall, *Uniqueness and universality of the Brownian map*, Ann. Probab. **41** (2012), 2880–2960.

[18] G. Miermont, *The Brownian map is the scaling limit of uniform random plane quadrangulations*, Acta Math. **210** (2013), 319–401.

[19] J. Miller and S. Sheffield, *Liouville quantum gravity and the Brownian map I: the* $\mathrm{QLE}(8/3, 0)$ *metric*, Invent. Math. **219** (2020), 75–152.

[20] J. Miller and S. Sheffield, *Liouville quantum gravity and the Brownian map II: Geodesics and continuity of the embedding*, Ann. Probab. **49** (2021), 2732–2829.

[21] J. Miller and S. Sheffield, *Liouville quantum gravity and the Brownian map III: the conformal structure is determined*, Probab. Theory Related Fields. **179** (2021), 1183–1211.
[22] R. Nivesvivat, S. Ribault, and J. L. Jacobsen, *Critical loop models are exactly solvable*, arXiv e-prints (2023).
[23] P. Nolin, W. Qian, X. Sun, and Z. Zhuang, *Backbone exponent for two-dimensional percolation*, arXiv e-prints (2023).
[24] A. Polyakov, *Quantum geometry of bosonic strings*, Physics Letters B. **103** (1981), 207–210.
[25] G. Remy and T. Zhu, *Integrability of boundary Liouville conformal field theory*, Communications in Mathematical Physics. **395** (2022), 179–268
[26] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288.
[27] S. Smirnov, *Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 239–244.
[28] S. Smirnov and W. Werner, *Critical exponents for two-dimensional percolation*, Math. Res. Lett. **8** (2001), 729–744.
[29] X. Sun, S. Xu, and Z. Zhuang, *Annulus crossing formulae for critical planar percolation*, arXiv e-prints (2024).
[30] B. Wu, *Conformal Bootstrap on the Annulus in Liouville CFT*, arXiv e-prints (2022).

Rapid phase ordering for Ising and Potts dynamics on random regular graphs

Allan Sly
(joint work with Reza Gheissari, Youngtak Sohn)

The out-of-equilibrium dynamics of spin systems like the Ising and Potts models are very well-studied in a variety of fields. One of their notable features is that at low-temperatures, the dynamics are slow to equilibrate due to a bottleneck between configurations that are mostly in one state, versus mostly in another. A central question is in what sense this is the "only" obstruction to fast relaxation of the low-temperature dynamics. This question can be posed in the following form: if you initialize with a small bias towards one ground state, does the low-temperature dynamics converge rapidly to the restriction of the Gibbs distribution to the corresponding phase?

In the mathematical physics literature, such questions have been studied numerically for a long time, with extensive predictions for the convergence of the Ising dynamics from random initializations, in which it is expected that the magnetization diffuses away from 0 and picks one phase to dominate, at which point quasi-equilibration within that phase is rather fast. The Ising and Potts models on non-trivial geometries serve as the natural baselines with which to put these predictions on a rigorous footing.

Mathematically, a major obstruction to answering these kinds of questions is the fact that many of the tools for bounding mixing times (e.g., functional inequalities like spectral gap and bottleneck sets, and the more modern tools of spectral independence and localization schemes) are useful for controlling mixing from worst-case initialization using local decay of correlation inputs, but in contexts where the worst-case mixing time is exponentially slow due to strong local

correlations, showing fast mixing from classes of "nice" initializations can be very challenging.

In this talk, we discuss the low (but positive) temperature Ising and Potts Glauber dynamics on random d-regular graphs initialized from biased initializations and demonstrate that they quasi-equilibrate in optimal $O(\log n)$ time to the corresponding metastable distribution, i.e., the Potts distribution conditioned on having plurality in state-1. Notably, the methods are general enough to apply to a wide range of ferromagnetic models on treelike expander graphs.

We consider the Glauber dynamics of the Ising model on graphs G drawn from the uniform distribution over d-regular graphs on n vertices, which we denote by $G \sim \mathcal{G}_d(n)$. The Ising model on $G \sim \mathcal{G}_d(n)$ undergoes a phase transition at $\beta_c(d) := \tanh^{-1}(1/(d-1))$ (which goes to zero as $1/d$ as $d \to \infty$). While for high temperatures $\beta < \beta_c(d)$, the worst-case mixing time (the time to be at total-variation distance at most $1/4$ to the stationary distribution from a worst-case initialization) is an optimal $O(\log n)$, for $\beta > \beta_c(d)$, the worst-case mixing time is slow, $\exp(\Theta(n))$, because of a bottleneck between configurations with a majority plus,

$$\Omega^+ = \{\sigma : m(\sigma) \geq 0\}\,, \qquad \text{where} \qquad m(\sigma) = \frac{1}{n}\sum_{v \in V} \sigma_v\,,$$

and ones with a majority minus, Ω^-. This in particular means that if we initialize from the *plus phase*, by which we mean the Gibbs distribution conditioned on being in Ω^+, i.e., $\pi^+ = \pi(\cdot \mid \Omega^+)$, the time for the Markov chain to hit Ω^- is exponentially long.

Our main theorem is that at all sufficiently low temperatures, the Ising Glauber dynamics initialized from a configuration X_0 with a bias to the plus phase, i.e., $m(X_0) \geq \epsilon$, (quasi-)equilibrates to π^+ in $O(n \log n)$ time steps.

Theorem: For every $d \geq 7$, there exists constants $C(\epsilon, \beta, d) > 0$, $\epsilon_0(d) \in (0,1)$ and $\beta_0 < \infty$ with $\epsilon_0(d) \asymp_d \frac{1}{\log d}$ and $\beta_0 \asymp_d \frac{\log d}{d}$, such that for every $\epsilon \in (\epsilon_0, 1]$ and every $\beta > \beta_0$, if $G \sim \mathcal{G}_d(n)$, the following holds with probability $1 - o(1)$. If X_0 has $m(X_0) \geq \epsilon$, then

$$\|\mathbb{P}_{X_0}(X_t \in \cdot) - \pi_G(\cdot \mid \Omega^+)\|_{tv} \leq n^{-10}\,, \qquad \text{for all} \qquad C \log n \leq t \leq e^{n/C}\,.$$

By symmetry, we have the same bound on the distance to $\pi_G^- = \pi_G(\cdot \mid \Omega^-)$ if X_0 has $m(X_0) \leq -\epsilon$.

Let us make some comments on the different parameter requirements in the theorem. Firstly, we expect the analogous theorem to hold for any $d \geq 3$, but the condition $d \geq 7$ is a barrier to our proof method, and getting down to $d \geq 3$ would require some new ideas to more carefully handle short cycles in G that are all-minus by chance. We leave this to future investigation.

In the other direction, we can say the following about the large-d asymptotics of ε_0, β_0 and C. The requirement $\beta_0 \geq C\frac{\log d}{d}$ is the threshold below which under π^+, the set of minus spins, has a giant component, preventing use of the locally treelike geometry to confine minus regions of the stationary distribution. The rate

of decay of the minimal bias, $\varepsilon_0 \asymp_d \frac{1}{\log d}$, can be improved to $\epsilon_0 \asymp_d \frac{1}{\sqrt{d}}$ at the expense of $\beta_0 \asymp_d \frac{1}{\sqrt{d}}$, but we note that for any fixed d, for sufficiently small ε, there may exist configurations with magnetizations near ε that in the $\beta \to \infty$ limit trap the dynamics. Therefore, the treatment at very small ϵ would need to subtly address the ability of positive temperature dynamics to escape even though zero-temperature dynamics may not.

References

[1] R. Gheissari, A. Sly, and Y. Sohn, *Rapid phase ordering for Ising and Potts dynamics on random regular graphs*, arXiv:2505.15783 (2025).

Quantitative results for the open asymmetric simple exclusion process

Dominik Schmid

(joint work with Zongrui Yang)

The open asymmetric simple exclusion process (open ASEP) is among the best studied examples of an interacting particle system. It can intuitively be described as follows. Consider a segment of length $N \in \mathbb{N}$ and bias parameter $q \in [0, 1)$. Each site of the segment is either occupied by a particle, or left vacant. The particles perform independent random walks with jumps to the right at rate 1, and to the left at rate q. However, a jump is performed if and only if the target is a vacant site. This exclusion rule ensures that each site is occupied by at most one particle at a time. In addition, for some fixed $\alpha, \beta, \gamma, \delta \geq 0$, we let particles enter at site 1 at rate α, exit at site N at rate β, exit at site 1 at rate γ, and enter at site N at rate δ, respectively, subject to the exclusion constraint.

The stationary distribution of the open ASEP is intensively studied over the past decades. Depending on the choice of boundary parameters, it is well-known that the open ASEP can be partitioned into three phases: The high density phase, the low density phase, and the maximal current phase. Furthermore, the phase diagram can be partitioned in the so-called fan and region. In joint work with Yang, we show that the stationary distribution of the open ASEP can in all three phases be well-approximated in total-variation distance by certain simple random measures. This supersedes earlier joint work with Nestoridi when $\gamma = \delta = 0$ in the fan region, using a Motzkin path representation [4].

Our arguments rely on several new developments in the past two years. In a recent breakthrough, Barraquand, Corwin and Yang establish a two-layer Gibbs structure for the invariant measure of a class of integrable models on a strip [1]. Very recently, Bryc showed that the stationary measure of the open ASEP has a two-layer representation [2]. His arguments involve a specific polynomial, to which we refer as the Bryc polynomial. We provide a simple characterization of the Bryc polynomial to achieve sharp approximation results in the shock region of the open ASEP. For the fan region, we rely on a recent generalization of the Enaux-Derrida representation of the matrix product ansatz, established by Hegde and Yang in [3].

References

[1] G. Barraquand, I. Corwin, and Z. Yang, *Stationary measures for integrable polymers on a strip*, Inventiones mathematicae 237.3 (2024): 1567-1641.

[2] W. Bryc, *Stationary distribution of open asymmetric simple exclusion processes on an interval as a marginal of a two-layer ensemble*, arXiv preprint arXiv:2408.06535 (2024).

[3] M. Hegde and Z. Yang, *Large deviation principle for the stationary measures of open asymmetric simple exclusion processes*, arXiv preprint arXiv:2412.12026 (2024).

[4] E. Nestoridi and D. Schmid, *Approximating the stationary distribution of the ASEP with open boundaries*, Communications in Mathematical Physics 405.8 (2024): 176.

NESS for KPZ

Milton Jara

(joint work with Juan Arroyave)

Let $\Lambda_N := \{1, \dots, N-1\}$ be the discrete interval and let $\Omega_N := \{0,1\}^{\Lambda_N}$ be the state space of an interacting particle system to be described below. We denote by $\eta = (\eta_x; x \in \Lambda_N)$ the elements of Ω_N. We will extend η to the interval $\{-1, 0, \dots, n, n+1\}$ by taking $\eta_{-1} = \eta_0 = \rho^A$ and $\eta_n = \eta_{n+1} = \rho^B$, with $\rho^A, \rho^B \in [0,1]$. For $x, y \in \Lambda_N$ and $\eta \in \Omega_N$, let $\eta^{x,y} \in \Omega_N$ be obtained from η by interchanging the values of η_x and η_y. If y does not belong to Λ_N, we define $\eta^{x,y}$ by exchanging η_x with $1 - \eta_x$. Consider the linear operator L_N given by

$$L_N f(\eta) = N^2 \sum_{x=0}^{n-1} c_{x,x+1}(\eta)(p_N \eta_x (1-\eta_{x+1}) + q_N \eta_{x+1}(1-\eta_x))(f(\eta^{x,x+1}) - f(\eta))$$

for every $f : \Omega_N \to \mathbb{R}$ and every $\eta \in \Omega_N$, where

$$c_{x,x+1}(\eta) := 1 + a(\eta_{x-1} + \eta_{x+2}),$$

$p_N := 1 + \frac{v}{\sqrt{N}}$, $q_N := 1 - \frac{v}{\sqrt{N}}$, and $a > -1/2, v \neq 0$ are fixed parameters. The operator L_N is the generator of a Markov chain $(\eta(t); t \geq 0)$ known in the literature as the *speed-change WASEP* on the interval with reservoirs of densities ρ^A, ρ^B. For $a = 0$, this model corresponds to the WASEP analyzed in [3], [2], [1]. In these references, the authors identified the limit of the density fluctuations under the stationary measure of the process $(\eta(t); t \geq 0)$ as the stationary measure of the KPZ equation with Dirichlet boundary conditions. They identified a two-parameter family of solutions, which correspond to the choice of boundary values as perturbations of the triple point of the dynamics:

$$\rho^A := \rho_* + \frac{A}{\sqrt{N}}, \quad \rho^B := \rho_* + \frac{B}{\sqrt{N}}.$$

In the case $a = 0$, $\rho_* = \frac{1}{2}$. For $a \neq 0$, we identify the triple point as the unique solution of

$$2a\rho_*(1-\rho_*) + (1 + 2a\rho_*)(1 - 2\rho_*) = 0$$

We report the following result:

Theorem 1. *For every fixed a, v there exists $\epsilon_0(a, v) > 0$ such that whenever $|A| + |B| \leq \epsilon_0$, the density fluctuation field around the triple point associated to $(\eta(t); t \geq 0)$ is tight and every limit point is an energy solution of the KPZ equation on the interval $[0, 1]$ with Dirichlet conditions. Conditioned on the uniqueness of such energy solutions, the fluctuations of the density under the invariant measure of the process $(\eta(t); t \geq 0)$ are given by the process constructed in* [3], [2], [1].

The proof of this theorem is based on a new methodology that we call *quantitative hydrodynamics*, which applies for a general class of weakly asymmetric systems, and could be of independent interest.

References

[1] G. Barraquand and P. Le Doussal, *Steady state of the KPZ equation on an interval and Liouville quantum mechanics*, Europhysics Letters **137** (6), 2022, 61003.

[2] W. Bryc, Y. Wang, and J. Wesołowski, *From the asymmetric simple exclusion processes to the stationary measures of the KPZ fixed point on an interval*, Ann. Inst. H. Poincaré Probab. Statist. **59**, (4), 2023, 2257–2284.

[3] I. Corwin and A. Knizel, *Stationary measure for the open KPZ equation*, Communications on Pure and Applied Mathematics, **77** (4), 2023, 2183–2267.

Reverse duality and applications

Gunter M. Schütz

(joint work with V. Belitsky, N.P.N. Ngoc)

Consider two Markov processes $\boldsymbol{\eta}(t)$ and $\boldsymbol{\xi}(t)$ with (in general different) countable state spaces Ω and Ξ and intensity matrices Q and W whose matrix elements $Q_{\boldsymbol{\eta}\boldsymbol{\eta}'} = q(\boldsymbol{\eta} \to \boldsymbol{\eta}')$, $W_{\boldsymbol{\xi}\boldsymbol{\xi}'} = w(\boldsymbol{\xi} \to \boldsymbol{\xi}')$ are the transition rates define the two processes. Reverse duality [6] is a relationship between two processes which for countable state space is defined by a duality matrix R with matrix elements $R(\boldsymbol{\xi}, \boldsymbol{\eta}) = R_{\boldsymbol{\xi},\boldsymbol{\eta}}$ as the intertwining relation

$$RQ = W^T R$$

If $\mu^{\boldsymbol{\xi}}(\boldsymbol{\eta}) := R(\boldsymbol{\xi}, \boldsymbol{\eta})$ defines a family of probability measures on Ω indexed by $\boldsymbol{\xi} \in \Xi$, then reverse duality yields *time-dependent measures* $\mu_t^{\boldsymbol{\xi}} := \mu^{\boldsymbol{\xi}} S_t$ of one process in terms of the dual as asserted in the following theorem [1].

Theorem: For countable state spaces Ω and Ξ let $\mu^{\boldsymbol{\xi}}$ be a family of probability measures on Ω indexed by $\boldsymbol{\xi} \in \Xi$ and let $\boldsymbol{\eta}(t)$ and $\boldsymbol{\xi}(t)$ be Markov chains with transition functions denoted by $p_t(\cdot, \cdot)$ and $\tilde{p}_t(\cdot, \cdot)$ respectively. The following two assertions are equivalent:

(1) $\boldsymbol{\eta}(t)$ and $\boldsymbol{\xi}(t)$ are reversely dual w.r.t. the duality function $R(\boldsymbol{\xi}, \omega) = \mu^{\boldsymbol{\xi}}(\omega)$.

(2) For an initial measure $\mu_0^{\boldsymbol{\xi}} = \mu^{\boldsymbol{\xi}}$ the time evolution of this measure under the semigroup $S(t)$ is given by

$$\mu_t^{\boldsymbol{\xi}} = \sum_{\boldsymbol{\xi}' \in \Xi} \tilde{p}_t(\boldsymbol{\xi}', \boldsymbol{\xi}) \mu^{\boldsymbol{\xi}'}$$

for all $\boldsymbol{\xi} \in \Xi$.

Thus the time evolution of the initial measure $\mu_0^{\boldsymbol{\xi}}$ under the semigroup S_t associated with the Markov chain $\boldsymbol{\eta}(t)$ is fully encoded in the transition probability of the dual process. For sufficiently simple dual processes this allows for obtaining detailed microscopic information about the time-dependent measures $\mu_t^{\boldsymbol{\xi}}$.

In [6] we prove a duality between the asymmetric simple exclusion process (ASEP) with non-conservative open boundary conditions and an asymmetric exclusion process with particle-dependent hopping rates and conservative reflecting boundaries. This is a reverse duality in the sense that the duality function relates the measures of the dual processes rather than expectations. Specifically, for a certain parameter manifold of the boundary parameters of the open ASEP this duality expresses the time evolution of a family of shock product measures with N microscopic shocks in terms of the time evolution of N particles in the dual process. The reverse duality also elucidates some so far poorly understood properties of the stationary matrix product measures of the open ASEP given by finite-dimensional matrices.

In [1] we introduce the headway exclusion process which is an exclusion process with N particles on the one-dimensional discrete torus with L sites with spatially asymmetric nearest-neighbor jumps and the jump rates that depend only on the distance to the next particle in the direction of the jump and not on N and L. For a finitely many particles that jump unidirectionally on the integer lattice it is proved by reverse duality that a certain family of non-stationary measures with a microscopic shock and antishock evolves into a convex combination of such measures with weights given by random walk transition probabilities. On macroscopic scale this domain random walk is a travelling wave phenomenon tantamount to phase separation with a stable shock and a stable antishock.

It turns out that also various earlier results on the time evolution of probability measures of Markov chains can be understood in terms of reverse duality. In particular, when in the branching coalescing random walk (BCRW) [2] defined in a finite integer lattice the jump rate and the coalescence rate are equal and specific open boundary conditions are considered, then the time-evolution of a microscopic Fisher wave proved in [4] is an instance of reverse duality of the BCRW with a biased random walk.

As is the case for conventional duality (defined for a duality matrix D by the intertwining relation $QD = DW^T$) reverse self-duality (where $Q = W$) is intimately related to non-Abelian symmetries of the intensity matrix [5, 3]. An intriguing open problem is the role of such symmetries for the ASEP with open boundaries.

References

[1] V. Belitsky, N.P.N. Ngoc, and G.M. Schütz, *Asymmetric exclusion process with long-range interactions*, arXiv:2409.05017 [math.PR].

[2] R. Durrett, *Lecture Notes on Particle Systems and Percolation*, Wadsworth and Brooks Cole, Pacific Grove CA, 1988.

[3] C. Giardinà, J. Kurchan, F. Redig, and K. Vafayi, *Duality and Hidden Symmetries in Interacting Particle Systems,* J. Stat. Phys. **135** (2009), 25–55.

[4] K. Krebs, F.H. Jafarpour, and G.M. Schütz, *Microscopic structure of travelling wave solutions in a class of stochastic interacting particle systems*, New J. Phys. **5** (2003), 145.1-145.14.

[5] G. M. Schütz, and S. Sandow, *Non-abelian symmetries of stochastic processes: derivation of correlation functions for random vertex models and disordered interacting many-particle systems,* Phys. Rev. E **49** (1994), 2726–2744.

[6] G.M. Schütz, *A reverse duality for the ASEP with open boundaries,* J. Phys. A: Math. Theor. **56** (2023), 274001.

Decoupling and decay of two-point functions in a two-species TASEP

Patrik L. Ferrari

(joint work with Sabrina Gernholt)

We consider a special case of an n-component lattice gas models on $\mathbb{Z}$. These consist of particles of types $\alpha \in \{1, \dots, n\}$ with at most one particle per site evolving with nearest-neighbor jumps. Let $\eta_\alpha(j,t) = 1$ if an α-particle is at site j at time t and $\eta_\alpha(j,t) = 0$ otherwise. We consider the case where the jumps rates are local and translation-invariant. Assume that for any fixed $\rho = (\rho_1, \dots, \rho_n)$ with $\rho_i \in [0,1]$ and $\sum_{i=1}^n \rho_i \le 1$, there exists an ergodic, translation-invariant stationary measure μ_ρ with $\mathbb{E}_{\mu_\rho}(\eta_\alpha)(j,t) = \rho_\alpha$.

The main observable we study is the two-point function $S = (S_{\alpha,\beta})_{1\le\alpha,\beta\le n}$, with

$$S_{\alpha,\beta}(j,t) = \mathbb{E}_{\mu_\rho}(\eta_\alpha(j,t)\eta_\beta(0,0)) - \rho_\alpha\rho_\beta. \tag{1}$$

The *nonlinear fluctuating hydrodynamic theory* (NLFH) gives a prediction on the large time behavior of the two-point function (see [7, 9, 11, 12, 13, 14] for related papers). Denote by $\mathsf{j}_\alpha(\rho)$ be the expected infinitesimal current of α-particles under μ_ρ. Let $C = C^T = \sum_j S(j,t) = \sum_j S(j,0)$ be the susceptibility matrix and define the matrix A with components $A_{\alpha,\beta}(\rho) = \frac{\partial}{\partial\rho_\beta}\mathsf{j}_\alpha(\rho)$. It is known that $AC = CA^T$ and assume $C > 0$ to avoid having components that do not evolve over time.

In order to see something meaningful one needs to consider appropriate linear combinations of the types of particles, the so-called *normal modes*. More precisely, one needs to find a matrix R such that

$$RAR^{-1} = \mathrm{diag}(v_1, \dots, v_n) \quad \text{and} \quad RCR^T = \mathbf{1}. \tag{2}$$

Then the normal modes are given by $\xi = R\eta$ and v_α is the speed of propagation of the mode ξ_α. The two-point function of the normal modes is given by

$$S^{\#}_{\alpha,\beta}(j,t) = (RSR^T)_{\alpha,\beta}(j,t) = \mathbb{E}_{\mu_\rho}(\xi_\alpha(j,t)\xi_\beta(0,0)) - (R\rho)_\alpha(R\rho)_\beta. \tag{3}$$

The prediction (as stated in [7]) is the following: assume that $v_1, \dots, v_n$ are all distinct, then there exists some (explicit) constants $\lambda_1, \dots, \lambda_n$ such that

$$(\lambda_\alpha t)^{2/3} S^{\#}_{\beta,\gamma}(v_\alpha t + w(\lambda_\alpha t)^{2/3}, t) \simeq \delta_{\beta,\alpha}\delta_{\gamma,\alpha} f_{\mathrm{KPZ}}(w) \tag{4}$$

as $t \to \infty$. Here, the scaling function f_{KPZ} is the one of one-dimensional system [10] and it is given by

$$f_{\mathrm{KPZ}}(w) = \frac{1}{4}\frac{d^2}{dw^2}\int_{\mathbb{R}} s^2 dF_{\mathrm{BR},w}(s), \tag{5}$$

where $F_{\mathrm{BR},w}$ is the Baik-Rains distribution with parameter w [4].

In our work [5] we consider a model with $n = 2$: a two-species totally asymmetric simple exclusion process with first class particles η_1 and second class particles η_2. The normal modes are given by

$$\xi_1(j,t) = \frac{\eta_1(j,t)}{\sqrt{\rho_1(1-\rho_1)}}, \quad \xi_2(j,t) = \frac{\eta_1(j,t) + \eta_2(j,t)}{\sqrt{(\rho_1+\rho_2)(1-\rho_1-\rho_2)}}, \tag{6}$$

and the speeds are $v_1 = 1 - 2\rho_1$, $v_2 = 1 - 2(\rho_1 + \rho_2)$.

Our main result is the following.

Theorem 1. *Given a speed v, define*

$$\mathcal{S}^{\#}_v(\phi) = \lim_{t\to\infty} t^{-2/3} \sum_{w \in t^{-2/3}\mathbb{Z}} \phi(w) t^{2/3} S^{\#}(vt + wt^{2/3}, t) \tag{7}$$

for ϕ smooth test functions with compact support. Then we have the following cases:
(a) if $v \notin \{v_1, v_2\}$,

$$\mathcal{S}^{\#}_v(\phi) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

(b) if $v = v_1$, then

$$\mathcal{S}^{\#}_v(\phi) = \begin{pmatrix} \chi_1 \int_{\mathbb{R}} \phi(w)\lambda_1^{-2/3} f_{\mathrm{KPZ}}(\lambda_1^{-2/3} w)dw & 0 \\ 0 & 0 \end{pmatrix}$$

with $\chi_1 = \rho_1(1-\rho_1)$ and $\lambda_1 = 2\sqrt{2\chi_1}$,
(c) if $v = v_2$, then

$$\mathcal{S}^{\#}_v(\phi) = \begin{pmatrix} 0 & 0 \\ 0 & \chi_2 \int_{\mathbb{R}} \phi(w)\lambda_2^{-2/3} f_{\mathrm{KPZ}}(\lambda_2^{-2/3} w)dw \end{pmatrix}$$

with $\chi_2 = (\rho_1+\rho_2)(1-\rho_1-\rho_2)$ and $\lambda_2 = 2\sqrt{2\chi_2}$.

The weak convergence of the diagonal terms was shown in [3] building on [8, 10]. In our paper [5] we prove that the off-diagonal terms vanishes in the large time limit. A similar result for ASEP under double scaling limit has been obtained in [1].

In order to prove our result, we derived a new identity [5, Proposition 1.2]

$$S^{\#}_{1,2}(\tilde{x}+i,\tilde{t}) + S^{\#}_{2,1}(x+i,t) = \frac{1}{4}\Delta\mathrm{Cov}\left(h^1(\tilde{x}+i,\tilde{t}), h^{1+2}(x+i,t)\right), \tag{8}$$

where h^1 (resp. h^{1+2}) is the standard height function for first (resp. first plus second) class particles and Δ is the discrete Laplacian. Then the main steps are the following:

(a) Suppose that $\mathrm{Supp}(\phi) \subset [-L, L]$ and take $x = v_2 t$, $i = wt^{2/3}$, $\tilde{x} = 2Lt^{2/3}$ and $\tilde{t} = 0$. Using the properties of the stationary measure of the multi-species TASEP, we first get an a-priori bound on $S^{\#}_{1,2}(\tilde{x} + i, \tilde{t})$, so that we need to consider only the second term.

(b) We perform summation by parts to move the discrete Laplacian to the test function ϕ: it remains to control $\mathrm{Cov}\left(h^1(\tilde{x} + i, \tilde{t}), h^{1+2}(x + i, t)\right)$.

(c) Using the queuing representation of the stationary measure [2, 6], we see that $h^1((2L + w)t^{2/3}, 0) = \tilde{h}^1((2L + w)t^{2/3}, 0) + R((2L + w)t^{2/3})$ where $\tilde{h}^1$ is independent of h^{1+2} (thus the covariance is zero) and the remainder term $|R| \ll t^{1/3}$. By Cauchy-Schwarz one finally control the covariance between R and h^{1+2}.

References

[1] A. Aggarwal, I. Corwin, and M. Hegde, *Scaling limit of the colored ASEP and stochastic six-vertex models*, arXiv:2403.01341v2 (2024).

[2] O. Angel, *The stationary measure of a 2-type totally asymmetric exclusion process*, J. Comb. Theory Ser. A **113** (2006), 625–635.

[3] J. Baik, P.L. Ferrari, and S. Péché, *Convergence of the two-point function of the stationary TASEP*, Singular Phenomena and Scaling in Mathematical Models, Springer, 2014, pp. 91–110.

[4] J. Baik and E.M. Rains, *Limiting distributions for a polynuclear growth model with external sources*, J. Stat. Phys. **100** (2000), 523–542.

[5] P.L. Ferrari and S. Gerholt, *Decoupling and decay of two-point functions in a two-species TASEP*, preprint, arXiv:2504.00765 (2025).

[6] P.A. Ferrari and J.B. Martin, *Stationary distribution of multi-type totally asymmetric exclusion processes*, Ann. Probab. **35** (2007), 807–832.

[7] P.L. Ferrari, T. Sasamoto, and H. Spohn, *Coupled Kardar-Parisi-Zhang equations in one dimension*, J. Stat. Phys. **153** (2013), 377–399.

[8] P.L. Ferrari and H. Spohn, *Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process*, Comm. Math. Phys. **265** (2006), 1–44.

[9] C.B. Mendl and H. Spohn, *Dynamic correlators of Fermi-Pasta-Ulam chains and nonlinear fluctuating hydrodynamics*, Phys. Rev. Lett. **111** (2013), 230601.

[10] M. Prähofer and H. Spohn, *Exact scaling function for one-dimensional stationary KPZ growth*, J. Stat. Phys. **115** (2004), 255–279.

[11] H. Spohn, *Nonlinear fluctuating hydrodynamics for anharmonic chains*, J. Stat. Phys. **154** (2014), 1191–1227.

[12] H. Spohn, *Fluctuating hydrodynamics approach to equilibrium time correlations for anharmonic chains*, Thermal transport in low dimensions: from statistical physics to nanoscale heat transfer, in Lect. Notes Phys. (2016), 107–158.

[13] H. Spohn and G. Stoltz, *Nonlinear fluctuating hydrodynamics in one dimension: the case of two conserved fields*, J. Stat. Phys. **160** (2015), 861–884.

[14] H. van Beijeren, *Exact results for anomalous transport in one-dimensional hamiltonian systems*, Phys. Rev. Lett. **108** (2012), 180601.

Rapid thermalization of toric codes through a log Sobolev inequalities

Simone Warzel

(joint work with Angela Capel Cuevas, Cambyse Rouze, Sebastian Stengele)

Toric codes are prominent examples of quantum Hamiltonians that promise to encode quantum information stably in their ground states. Their Hamiltonian structure is simple because they are composed of a sum of local commuting many-body interactions. In contact with a heat bath, the code's quantum state will eventually converge to a Gibbs state. In quantum mechanics, a standard Markovian model of a heat bath is a Davis quantum Markovian semigroup [3, 6]. This dynamics is the analogue of a Gauber dynamics for classical systems. As in the classical case, the mixing time of the Markovian quantum dynamics is a central quantity of interest.

In this talk, I sketched the argument for establishing a bound on the mixing time in the high-temperature regime, which is logarithmically in the system's size. The high-temperature regime is thereby characterized by conditions of Dobrushin-Shlosman type [4]. This improves on earlier bounds, which were polynomial in the system's size and proven relying on the gap of the quantum dynamics' generator [1]. It puts this class of quantum models on par with the classical results on Glauber dynamics (see [5] and references therein). The technical core of the proof is a modified logarithmic Sobolev inequality for the quantum relative entropy, which we establish [2].

References

[1] R. Alicki, M. Fannes, and M. Horodecki, *On thermalization in Kitaev's 2D model*. Journal of Physics A: Mathematical and Theoretical, 42(6):065303 (2009).

[2] A. Capel Cuevas, C. Rouze, S. Stengele, and S. Warzel, *Rapid thermalization of quantum codes*, in preparation 2025.

[3] E. B. Davies. *Generators of dynamical semigroups*. J. Funct. Anal., 34(3):421–432, 1979.

[4] R. L. Dobrushin and S. B. Shlosman, *Complete analytical interactions: Constructive description*. J. Stat. Phys., 46:983–1014 (1987).

[5] F. Martinelli. *Lectures on Glauber dynamics for discrete spin models*. Volume 1717 of Bernard P. (eds) Lectures on Probability Theory and Statistics. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1999.

[6] M. J. Kastoryano and F. G. S. L. Brandao, *Quantum Gibbs Samplers: The commuting case*. Communications in Mathematical Physics, 344 (3), 915–957 (2016).

An invitation to the cutoff phenomenon

Justin Salez

(joint work with Francesco Pedrotti)

The cutoff phenomenon is an abrupt transition from out of equilibrium to equilibrium undergone by certain Markov processes in the limit where the size of the state space tends to infinity. Discovered four decades ago in the context of card shuffling, this surprising phenomenon has since then been observed in a variety of models, from random walks on groups or complex networks to interacting particle

systems. It is now believed to be universal among fast-mixing high-dimensional processes. Yet, current proofs are heavily model-dependent, and identifying the general conditions that trigger a cutoff remains one of the biggest challenges in the quantitative analysis of finite Markov chains. In this talk, I will provide a self-contained introduction to this fascinating question, and then describe a recent partial answer for non-negatively curved processes developed in [3, 1, 2], and based on a new information-theoretic quantity called varentropy.

References

[1] J. Salez, *Cutoff for non-negatively curved diffusions,* arXiv preprint arXiv:2501.01304 (2025).
[2] F. Pedrotti and J. Salez, *A new cutoff criterion for non-negatively curved chains,* arXiv preprint arXiv:2501.13079 (2025).
[3] J. Salez, *Cutoff for non-negatively curved Markov chains* J. Eur. Math. Soc. (JEMS), 26(11):4375–4392, 2024.

Reporter: Chiara Franceschini

Participants

Prof. Dr. Michael Aizenman
Departments of Physics and Mathematics
Princeton University
Fine Hall
Princeton NJ 08544
UNITED STATES

Dr. Guillaume Barraquand
Laboratoire de Physique de l'École Normale Supérieure
24, rue Lhomond
75231 Paris Cedex 05
FRANCE

Dr. Oriane Blondel
Institut Camille Jordan
CNRS
Université Claude Bernard Lyon I
43, Boulevard du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Wlodzimierz Bryc
Dept. of Mathematical Sciences
University of Cincinnati
2600 Clifton Ave.
P.O. Box 210025
Cincinnati OH 45221-0025
UNITED STATES

Dr. Ofer Busani
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings, Mayfield Road
Edinburgh EH9 3JZ
UNITED KINGDOM

Dr. Giuseppe Cannizzaro
Department of Statistics
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Dr. Pedro Cardoso
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Gioia Carinci
Department of Mathematics
University of Modena and Reggio Emilia
Via Campi 213/B
41125 Modena
ITALY

Prof. Dr. Ana Patrícia Carvalho Gonçalves
Instituto Superior Técnico
Universidade de Lisboa
Avenida Rovisco Pais, 1
Lisboa 1049-001
PORTUGAL

Prof. Dr. Nicolas Curien
Institut de Mathématiques
Université Paris Saclay
Batiment 307
91400 Orsay
FRANCE

Dr. Duncan Dauvergne
Department of Mathematics
University of Toronto
40 St. George Street
Toronto ON M5S 2E4
CANADA

Barbara Dembin
CNRS, Université de Strasbourg
67084 Strasbourg
FRANCE

Prof. Dr. Hugo Duminil-Copin
Institut des Hautes Etudes Scientifiques (IHES), Le Bois-Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Dr. Clément Erignoux
INRIA Lyon & Institut Camille Jordan
Université Claude Bernard Lyon 1
56 Bd Niels Bohr
69100 Villeurbanne Cedex
FRANCE

Prof. Dr. Alessandra Faggionato
Department of Mathematics G. Castelnuovo
Sapienza University of Rome
Piazzale Aldo Moro, 2
00185 Roma
ITALY

Prof. Dr. Patrik L. Ferrari
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. William Fleurat
Bâtiment 307
Université Paris-Saclay
L.M.O. Laboratory
Rue Michel Magat
91405 Orsay Cedex
FRANCE

Dr. Chiara Franceschini
Dip. di Matematica
Universita degli Studi di Modena e Reggio Emilia
Via Giuseppe Campi 213
Modena 41125
ITALY

Prof. Dr. Tadahisa Funaki
Beijing Institute of Mathematical Sciences and Applications (BIMSA)
No. 544 Hefangkou, Huairou District
101408 Beijing
CHINA

Prof. Dr. Christophe Garban
Institut Camille Jordan
Université de Lyon I
43 Blvd. du 11 Novembre 1918
69622 Lyon Cedex
FRANCE

Prof. Dr. Ewain Gwynne
University of Chicago
University Avenue
60637 Chicago
UNITED STATES

Prof. Dr. Nina Holden
Courant Institute of Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
UNITED STATES

Dr. Milton Jara
Instituto de Matematica Pura e Aplicada
IMPA
Jardim Botanico
Estrada Dona Castorina, 110
Rio de Janeiro RJ 22460-320
BRAZIL

Julian Kern
WIAS Berlin
Mohrenstr. 39
10117 Berlin
GERMANY

Dr. Alisa Knizel
Department of Mathematics
Barnard College
Columbia University
New York, NY 10027
UNITED STATES

Prof. Dr. Gady Kozma
Faculty of Mathematics and Computer Science
The Weizmann Institute of Science
P.O. Box 26
Rehovot 76100
ISRAEL

Dr. Piet Lammers
LPSM, Sorbonne Université
Campus Pierre et Marie Curie
4, place Jussieu
75005 Paris
FRANCE

Prof. Dr. Jean-Francois Le Gall
Département de Mathématiques
Université Paris-Saclay, Centre d'Orsay
Bât. 307
91405 Orsay Cedex
FRANCE

Prof. Dr. Grégory Miermont
Unité de mathématiques pures et appliquées
École normale supérieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Jason P. Miller
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Alessandra Occelli
LAREMA
Dept. de Mathématiques
Faculté des Sciences
Université d'Angers
2, Boulevard Lavoisier
49045 Angers Cedex
FRANCE

Prof. Dr. Stefano Olla
CEREMADE
Université Paris Dauphine - PSL
and IUF
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

Prof. Dr. Nicolas Perkowski
Fachbereich Mathematik & Informatik
Freie Universität Berlin
Arnimallee 7
14195 Berlin
GERMANY

Prof. Dr. Daniel Remenik
Depto. de Ingenieria Matematica
Universidad de Chile
Beauchef 851 Edificio Norte, Piso 5
8370456 Santiago
CHILE

Maria Chiara Ricciuti
Imperial College London
Department of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Armand Riera
LPSM
Sorbonne Université
4 Place Jussieu
75252 Paris Cedex
FRANCE

Prof. Dr. Justin Salez
Université Paris Dauphine -
CEREMADE
Place du Maréchal de Lattre de Tassigny
F-75775 Paris Cedex 16
FRANCE

Dr. Beatriz Salvador
Department of Mathematics
University of Lisboa
R. Ernesto Vasconcelos B1
C1-Piso 3
1700 Lisboa Codex
PORTUGAL

Prof. Dr. Makiko Sasada
Graduate School of Mathematical
Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Tomohiro Sasamoto
Institute of Science Tokyo
2-12-1 Ookayama, Meguro
152-8551 Tokyo
JAPAN

Prof. Dr. Dominik Schmid
Institut für Mathematik
Universität Augsburg
86135 Augsburg
GERMANY

Leander Schnee
Institut für Mathematik
Freie Universität Berlin
Arnimallee 6
14195 Berlin
GERMANY

Prof. Dr. Gunter M. Schütz
Forschungszentrum Jülich GmbH
IAS 2
52425 Jülich
GERMANY

Dr. Florian Schweiger
Section de Mathématiques
Université de Genève
rue du Conseil-Général 7-9
1205 Genève
SWITZERLAND

Prof. Dr. Scott Sheffield
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
02139 Cambridge
UNITED STATES

Prof. Dr. Allan Sly
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Xin Sun
Beijing International Center for
Mathematical Research (BICMR)
Beijing University
Beijing 100871
CHINA

Prof. Dr. Vincent Tassion
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Bálint Tóth
Alfréd Rényi Institute of Mathematics
Budapest
and
School of Mathematics, University of Bristol
Reáltanoda u. 13-15, H-1053
P.O. Box H-1053
Budapest H-1364
HUNGARY

Prof. Dr. Simone Warzel
Department Mathematik und
Department Physik
Technische Universität München
Boltzmannstr. 3
85748 Garching bei München
GERMANY

Prof. Dr. Nikolaos Zygouras
Department of Mathematics
University of Warwick
Coventry CV4 7AL
UNITED KINGDOM