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The q -Deformed Random-to-Random
Family in the Hecke Algebra

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The q -deformed random-to-random family in the Hecke algebra

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The q -deformed random-to-random family in the Hecke algebra.

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Brief abstract. We generalize Reiner–Saliola–Welker’s well-known but mysterious family of k -random-to-random shuffles from Markov chains on symmetric groups to Markov chains on the Type- A Iwahori–Hecke algebras. We prove that the family of operators pairwise commutes and has eigenvalues that are polynomials in q with non-negative integer coefficients. Our work generalizes work of Reiner–Saliola–Welker and Lafrenière for the symmetric group, and simplifies all known proofs in this case.

Keywords: Hecke algebra, Iwahori–Hecke algebra, symmetric group algebra, random-to-random shuffle, card shuffling, Young–Jucys–Murphy elements, Specht modules, Young tableaux, permutations, algebraic combinatorics, discrete Markov chains, representation theory, q -deformations.

Mathematics Subject Classifications (MSC 2020): 20C08, 20C30, 60J10, 05E10.

Cover image: a horizontal strip $\lambda \setminus \mu$ (gray) with a desarrangement tableau of shape μ (black on white).

THE q -DEFORMED RANDOM-TO-RANDOM FAMILY IN THE HECKE ALGEBRA

SARAH BRAUNER, PATRICIA COMMINS, DARIJ GRINBERG, FRANCO SALIOLA

ABSTRACT. We generalize Reiner–Saliola–Welker’s well-known but mysterious family of k -random-to-random shuffles from Markov chains on symmetric groups to Markov chains on the Type- A Iwahori–Hecke algebras. We prove that the family of operators pairwise commutes and has eigenvalues that are polynomials in q with non-negative integer coefficients. Our work generalizes work of Reiner–Saliola–Welker and Lafrenière for the symmetric group, and simplifies all known proofs in this case.

1. INTRODUCTION

We introduce the k -random-to-random shuffling operators $\mathcal{R}_{n,k}(q)$, which are elements of the Type- A Iwahori–Hecke algebra $\mathcal{H}_n(q)$, and show that

- (1) they commute and are generically diagonalizable, and
- (2) their eigenvalues are polynomials in q with non-negative integer coefficients.

These elements encode Markov chains on $\mathcal{H}_n(q)$. The case $k = 1$ is the *random-to-random* shuffling operator defined and studied in [1]. The general- k case was studied for $q = 1$ (that is, in the symmetric group algebra $\mathfrak{S}_n$) by Reiner, Saliola and Welker [18, Theorem 1.1], Dieker and Saliola [10], and Lafrenière [14]. Our work generalizes their results to arbitrary q , provides simpler proofs than all previously published proofs of the $q = 1$ case, and resolves a Conjecture of Lafrenière about the second largest eigenvalue of each $\mathcal{R}_{n,k}(q)$.

Notation. To simplify notation, we omit “(q)” for nonspecific q , writing $\mathcal{H}_n$ for $\mathcal{H}_n(q)$ and $\mathcal{R}_{n,k}$ for $\mathcal{R}_{n,k}(q)$, and so on. The specialization to the $q = 1$ case will be denoted, for example, by $\mathcal{R}_{n,k}(1)$.

History of the problem. This work arose from a long quest of studying Markov chains — also known as “random shuffles” — on the symmetric groups $\mathfrak{S}_n$. More precisely, we are interested in those Markov chains that arise from right multiplication by elements of the group algebra $\mathbb{R}[\mathfrak{S}_n]$, or more generally its q -deformation the Hecke algebra $\mathcal{H}_n := \mathcal{H}_n(q)$.

One of the oldest such Markov chains is the *Tsetlin library*, which — in one of its forms — is given by right multiplication in $\mathbb{R}[\mathfrak{S}_n]$ by the element

$$\mathcal{B}_n^*(1) = \sum_{i=1}^n s_i s_{i+1} \cdots s_{n-2} s_{n-1}.$$

In probabilistic language, if we view permutations in one-line notation as decks of n cards labelled $1, 2, \dots, n$, then the Tsetlin library corresponds to picking a card at random and moving it to the bottom of the deck, where the bottom card of the deck is the n -th position. Thus, we call $\mathcal{B}_n^*(1)$ the *random-to-bottom shuffling element*. (In the literature, $\mathcal{B}_n^*(1)$ is sometimes referred to as “random-to-top” by considering the card in position n to be the top card of the deck. This is the case for instance in [10].)

The *random-to-random shuffling element* is obtained by composing $\mathcal{B}_n^*(1)$ with the *bottom-to-random shuffling element*:

$$\mathcal{R}_n(1) = \mathcal{B}_n^*(1) \mathcal{B}_n(1) \quad \text{and} \quad \mathcal{B}_n(1) = \sum_{i=1}^n s_{n-1} s_{n-2} \cdots s_{i+1} s_i.$$

This element was introduced by Diaconis and Saloff-Coste in [9, p. 2147].

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Key words and phrases. Hecke algebra, Iwahori–Hecke algebra, symmetric group algebra, random-to-random shuffle, card shuffling, Young–Jucys–Murphy elements, Specht modules, Young tableaux, permutations, algebraic combinatorics, discrete Markov chains, representation theory, q -deformations.

Reiner, Saliola and Welker introduced in their monograph [18] a family of elements they called *k-random-to-random shuffling elements* $\mathcal{R}_{n,k}(1)$ in the group algebra of $\mathfrak{S}_n$, which contained $\mathcal{R}_n(1) =: \mathcal{R}_{n,1}(1)$.

Informally, each element $\mathcal{R}_{n,k}(1)$ can be understood as a (rescaled) *shuffling* process as follows. Associate a deck of n cards with a word w of length n with letters $1, 2, \dots, n$ (i.e., a permutation in one-line notation). Right multiplication by $\mathcal{R}_{n,k}(1)$ acts on w by randomly selecting k letters $w_{i_1}, \dots, w_{i_k}$ and placing them in new random positions. More precisely,

$$\mathcal{R}_{n,k}(1) = \frac{1}{k!} \mathcal{B}_{n,k}^*(1) \mathcal{B}_{n,k}(1) \in \mathbb{Z}[\mathfrak{S}_n],$$

where the actions of $\mathcal{B}_{n,k}^*(1)$ and $\mathcal{B}_{n,k}(1)$ are defined as follows (cf. Definition 2.1):

- (1) $\mathcal{B}_{n,k}^*(1)$ selects k letters $w_{i_1}, \dots, w_{i_k}$ from a word w and moves them to the end of the word;
- (2) $\mathcal{B}_{n,k}(1)$ moves the last k letters in a word u to new positions, chosen with uniform probability.

Example 1.1. Consider $n = 4$ and $k = 2$. Using one-line notation, $\mathcal{B}_{4,2}^*(1)$ is

$$\mathcal{B}_{4,2}^*(1) = 34\textcolor{red}{12} + 34\textcolor{red}{21} + 24\textcolor{red}{13} + 24\textcolor{red}{31} + 23\textcolor{red}{14} + 23\textcolor{red}{41} + 14\textcolor{red}{23} + 14\textcolor{red}{32} + 13\textcolor{red}{24} + 13\textcolor{red}{42} + 12\textcolor{red}{34} + 12\textcolor{red}{43},$$

which is the sum of all permutations that can be obtained from the identity permutation 1234 by moving two letters to the end of the word; the red color indicates the moved letters. Similarly,

$$\mathcal{B}_{4,2}(1) = 12\textcolor{red}{34} + 12\textcolor{red}{43} + 13\textcolor{red}{24} + 13\textcolor{red}{42} + 14\textcolor{red}{23} + 14\textcolor{red}{32} + \textcolor{red}{31}24 + \textcolor{red}{31}42 + \textcolor{red}{34}12 + \textcolor{red}{41}23 + \textcolor{red}{41}32 + \textcolor{red}{43}12$$

is the sum of all permutations that can be obtained from 1234 by moving its last two letters.

Since $\mathcal{R}_{4,2}(1) = \frac{1}{2!} \mathcal{B}_{4,2}^*(1) \mathcal{B}_{4,2}(1)$, the coefficient of 2314 in $\mathcal{R}_{4,2}(1)$ is $8/2!$, since there are eight ways (shown below) to obtain 2314 by first moving two letters from 1234 to the end of the word, and then moving those letters to new positions.

$$\begin{array}{ll} 1234 \rightarrow 23\textcolor{red}{14} \rightarrow 2314 & \textcolor{red}{1}234 \rightarrow 23\textcolor{red}{41} \rightarrow 2314 \\ \textcolor{red}{1}234 \rightarrow 24\textcolor{red}{13} \rightarrow 2314 & \textcolor{red}{1}234 \rightarrow 24\textcolor{red}{31} \rightarrow 2314 \\ 1234 \rightarrow 34\textcolor{red}{12} \rightarrow \textcolor{red}{2}314 & 1234 \rightarrow 34\textcolor{red}{21} \rightarrow \textcolor{red}{2}314 \\ \textcolor{red}{1}234 \rightarrow 14\textcolor{red}{23} \rightarrow \textcolor{red}{2}314 & \textcolor{red}{1}234 \rightarrow 14\textcolor{red}{32} \rightarrow \textcolor{red}{2}314 \end{array}$$

In fact, the family $\mathcal{R}_{n,k}(1)$ are examples of a more general construction introduced by Reiner–Saliola–Welker called *symmetrized shuffling operators*. An important tool in their study is the theory of random walks on hyperplane arrangements initiated by Bidigare, Hanlon and Rockmore (BHR) in their seminal work [3] and subsequently developed by Brown and Diaconis [5, 4].

The symmetrized shuffling operators are obtained by composing a BHR random walk with its transpose. The term “symmetrized” refers to the fact that these operators can be realized as a product of a real matrix with its transpose, and hence are always symmetric. Reiner–Saliola–Welker introduced one such operator ν_λ for every integer partition $\lambda \vdash n$, each corresponding to a Markov chain on the symmetric group $\mathfrak{S}_n$.

The monograph [18] posed several questions and puzzles about the symmetrized shuffling operators; some of them remain open to this day. For instance, Reiner–Saliola–Welker identified just *two* families of partitions giving rise to pairwise commuting elements in $\mathbb{R}[\mathfrak{S}_n]$:

- (1) the family $\{\nu_\lambda\}$ with $\lambda = (n - k, 1^k)$ for $1 \leq k \leq n - 1$, and
- (2) the family $\{\nu_\lambda\}$ with $\lambda = (2^k, 1^{n-2k})$ for $1 \leq k \leq \lfloor n/2 \rfloor$.

The first family is precisely the *k-random-to-random shuffling elements*:

$$\mathcal{R}_{n,k}(1) = \nu_{(n-k, 1^k)}$$

which are the focus of our study—and generalization—here. A forthcoming paper will resolve conjectures regarding the second family. It is suspected, but still unproved ([18, Conjecture 1.7]), that there are no further nontrivial commutativities between symmetrized shuffling operators.

Extending a conjecture for $\mathcal{R}_n(1)$ of Uyemura-Reyes [23], Reiner–Saliola–Welker conjectured that both families of operators had eigenvalues in $\mathbb{Z}_{\geq 0}$, and were able to prove it for the second family (albeit without giving a manifestly positive formula).

The spectrum of $\mathcal{R}_{n,k}(1)$ remained elusive for nearly two decades, until Dieker and Saliola [10] proved that $\mathcal{R}_{n,1}(1)$ had non-negative integer eigenvalues using the representation theory of the symmetric group. In her thesis, Lafrenière [14] built on their results to resolve the full conjecture for $\mathcal{R}_{n,k}(1)$, and provided

a second proof that these elements pairwise commute. The proofs of these results were quite technical and specific to the symmetric group, and many enigmatic features of the story endured.

One intriguing mystery was the emergence of the *content* of a partition in the Dieker–Saliola formula for the eigenvalues of $\mathcal{R}_{n,1}(1)$, which hinted at a link to the *Young–Jucys–Murphy elements* for the symmetric group. This connection was elucidated in recent work by the first and second authors, along with Axelrod–Freed, Chiang and Lang [1], who introduced a q -deformed version of $\mathcal{R}_{n,1}(1)$ in the Hecke algebra $\mathcal{H}_n$ and proved that its eigenvalues are in $\mathbb{Z}_{\geq 0}[q]$. Importantly, the proof established the connection to the Young–Jucys–Murphy elements of $\mathcal{H}_n$, which opened the door to a variety of techniques from representation theory.

Main results. Here, we extend and deepen the perspective introduced in [1]. We generalize the random-to-random shuffling elements $\mathcal{R}_{n,k}(1)$ in $\mathbb{R}[\mathfrak{S}_n]$ to the q -deformed *random-to-random shuffling elements* $\mathcal{R}_{n,k}$ in $\mathcal{H}_n$. Each $\mathcal{R}_{n,k}$ is defined as a product

$$\mathcal{R}_{n,k} := \frac{1}{[k]!_q} \mathcal{B}_{n,k}^* \mathcal{B}_{n,k},$$

where $\mathcal{B}_{n,k}$ is a sum of elements in $\mathcal{H}_n$ indexed by minimal right coset representatives of the parabolic Young subgroup $\mathfrak{S}_{(n-k, 1^k)}$, and $[k]_q$ and $[k]!_q$ are the q -integers and the q -factorials defined by

$$[k]_q = 1 + q + \cdots + q^{k-1} \quad \text{and} \quad [k]!_q = [1]_q [2]_q \cdots [k]_q.$$

This deformation is motivated in part by recent interest in Markov chains on Hecke algebras; for example, Bufetov showed that many interacting particle systems arising in statistical mechanics can be understood as random walks on $\mathcal{H}_n$ [6]. His work implies that questions of convergence of Markov chains are well-posed; from this perspective, it is of significant interest to understand the eigenvalues of such Markov chains.

On the algebra-combinatorial side, our study of the family $\{\mathcal{R}_{n,k}\}$ will uncover new properties of the Hecke algebra, while also shedding light on the original mysterious family $\{\mathcal{R}_{n,k}(1)\}$ in $\mathbb{R}[\mathfrak{S}_n]$. In particular, specializing our arguments to $q = 1$, we recover the work of Reiner–Saliola–Welker [18] (item (i) below), Dieker–Saliola [10] (item (iii) for $k = 1$) and Lafrenière [14] (items (i) and (iii)), each time obtaining a simpler proof than previously known. Our main results are summarized in the following theorem:

Theorem A. Let $n \geq 0$. Then the following holds for $\{\mathcal{R}_{n,k}\}_{k \geq 0}$:

- (i) (Theorem 4.1) The elements $\mathcal{R}_{n,k}$ pairwise commute;
- (ii) (Proposition 5.5 (d)) When $q \in \mathbb{R}_{>0}$, the elements $\mathcal{R}_{n,k}$ (acting from the right on $\mathcal{H}_n$) are diagonalizable with a common eigenbasis defined in Theorem 5.13 and Corollary 5.14, and in fact, there is a containment of images

$$\mathcal{H}_n = \text{im}(\mathcal{R}_{n,0}) \supseteq \text{im}(\mathcal{R}_{n,1}) \supseteq \text{im}(\mathcal{R}_{n,2}) \supseteq \cdots$$

- (iii) (Theorem 5.17) For any $q \in \mathbb{C}$, the eigenvalues of $\mathcal{R}_{n,k}$ are in $\mathbb{Z}_{\geq 0}[q]$.

Theorem A is proved by discovering a remarkable recursion relating the elements $\mathcal{B}_{n,k}$; working modulo the kernel of $\mathcal{B}_{n,k}^*$ yields a recursive relationship between the $\mathcal{R}_{n,k}$. In particular, letting J_n be the n -th *Young–Jucys–Murphy element* of $\mathcal{H}_n$ (defined in equation (2.37)), we prove the following.

Theorem B (Theorem 3.6). For any $1 \leq k \leq n$, we have

$$\mathcal{B}_n \mathcal{R}_{n,k} = \left(q^k \mathcal{R}_{n-1,k} + \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{R}_{n-1,k-1} \right) \mathcal{B}_n.$$

This recursion is the first key ingredient in proving each item in Theorem A, yielding in particular item (i) without too much trouble. The remaining parts require a second ingredient, which is the representation theory of $\mathcal{H}_n$; we use this theory to inductively construct a shared eigenbasis for the $\mathcal{R}_{n,k}$ (for $q \in \mathbb{R}_{>0}$; such an eigenbasis does not always exist for negative or complex q).

Recall that when $\mathcal{H}_n$ is semisimple, the irreducible representations of $\mathcal{H}_n$ — like those of the symmetric group — can be described as Specht modules S^λ for λ an integer partition of n . We show that the eigenvalues of each $\mathcal{R}_{n,k}$ acting on S^λ are parametrized by horizontal strips $\lambda \setminus \mu$ with a certain restriction on μ (in terms of so-called *desarrangement tableaux*). The eigenvalue indexed by $\lambda \setminus \mu$ is recursively constructed from the (smaller) Specht module S^μ by a “lifting” procedure.

In general, a closed form expression for the eigenvalues of $\mathcal{R}_{n,k}$, as well as a computation of precise multiplicities, is complicated. We give an explicit general formula for the eigenvalues in equation (5.8),

which we use to prove Theorem A(iii). We show the general expression simplifies nicely in the following special cases, which proves and generalizes a conjecture of Lafrenière about the second-largest eigenvalue [14, Conjecture 62] when $q = 1$.

Theorem C. For $0 \leq j, \ell, k \leq n$, the eigenvalue $\mathcal{E}_{\lambda \setminus \mu}(k)$ for partitions $\lambda = (n - \ell, 1^\ell)$ and $\mu = (j, 1^j)$ is given by

$$\mathcal{E}_{(n-\ell, 1^\ell) \setminus (j, 1^j)}(k) = [k]!_q \begin{bmatrix} n-j-\ell \\ k \end{bmatrix}_q \begin{bmatrix} n+j \\ k \end{bmatrix}_q. \quad (1.1)$$

When $q \in \mathbb{R}_{>0}$, the case $\ell = j = 0$ yields the largest eigenvalue of $\mathcal{R}_{n,k}$:

$$\mathcal{E}_{(n) \setminus \emptyset}(k) = [k]!_q \begin{bmatrix} n \\ k \end{bmatrix}_q^2. \quad (1.2)$$

When $q \in \mathbb{R}_{>0}$, the case $\ell = j = 1$ yields the second-largest eigenvalue of $\mathcal{R}_{n,k}$:

$$\mathcal{E}_{(n-1, 1) \setminus (1, 1)}(k) = [k]!_q \begin{bmatrix} n-2 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q.$$

This theorem is detailed in Propositions 5.22, 5.23, 5.24, 5.25. We further show that the case $\ell = 1$ and $j \in [n-1]$ above describes all of the eigenvalues of $\mathcal{R}_{n,k}$ acting on the Specht module $S^{(n-1, 1)}$. The case $\ell = j = 0$ yields the unique eigenvalue from the $\mathcal{R}_{n,k}$ action on $S^{(n)}$. More generally, the formula in (1.1) describes some of the eigenvalues obtained from the $\mathcal{R}_{n,k}$ action on $S^{(n-\ell, 1^\ell)}$.

We note that our proof of Theorem A item (i) is self-contained except for some elementary computations in $\mathcal{H}_n$ that we recall from [1, Proposition 3.10, Lemma 3.12, (5.2)]. The proofs of items (ii) and (iii), on the other hand, rely on deeper results from [1] and prior work on Hecke algebras. However, a weaker version of item (iii), saying only that the eigenvalues are in $\mathbb{Z}[q]$ (rather than $\mathbb{Z}_{\geq 0}[q]$), is reproved using relatively elementary tools in Appendix C. We suspect that the latter tools might be reusable in other settings when an integrality of eigenvalues is to be shown (e.g., in algebraic graph theory or representation theory).

The remainder of the paper proceeds as follows:

- In Section 2, we review the definition of the Hecke algebra $\mathcal{H}_n$, define the elements $\mathcal{B}_{n,k}$, $\mathcal{B}_{n,k}^*$ and $\mathcal{R}_{n,k}$, and introduce the Young–Jucys–Murphy elements J_n .
- Section 3 introduces and subsequently proves our main recursion (Theorem B / Theorem 3.6). This lays the groundwork for the proofs in the remaining sections.
- We then prove the commutativity of the $\mathcal{R}_{n,k}$ in Section 4 (Theorem 4.1). This proves Theorem A(i).
- In Section 5, we build on results from [1] to compute an eigenbasis for the $\mathcal{R}_{n,k}$ ’s when q is a positive real number. This allows us to prove Theorem A(ii). We also find explicit expressions for the eigenvalues of the $\mathcal{R}_{n,k}$ ’s, which help prove Theorem A(iii). Finally, we prove Theorem C.
- Appendix A proves folklore theorems used throughout the paper. Appendix B gives another proof of Theorem A(i). Appendix C develops a theory of split elements that can be used to prove the integrality of eigenvalues of linear operators recursively, and provides an alternative proof of Theorem A(iii) without the positivity. Finally, for the curious reader, the polynomial expansion of the nonzero eigenvalues of $\mathcal{R}_{n,k}$ for $3 \leq n \leq 5$ is included in Appendix D.

2. THE q -DEFORMED k -RANDOM-TO-RANDOM SHUFFLES

2.1. The Hecke algebra $\mathcal{H}_n$. We study the Hecke algebras of the symmetric groups $\mathfrak{S}_n$ over some arbitrary field $\mathbf{k}$. We briefly recall their definition.

Let n be a nonnegative integer. The symmetric group $\mathfrak{S}_n$ is the group of all permutations of the set $[n] := \{1, 2, \dots, n\}$, and is generated by the simple transpositions $s_i = (i, i+1)$ for all $i \in [n-1]$. We will use cycle notation for permutations unless declared otherwise. Permutations are multiplied by the rule $(\alpha\beta)(i) = \alpha(\beta(i))$ for all $\alpha, \beta \in \mathfrak{S}_n$ and $i \in [n]$.

We fix $q \in \mathbf{k}$. The *Hecke algebra* $\mathcal{H}_n$ (often denoted as $\mathcal{H}_n(q)$, but we omit the mention of q) is the (unital, associative) $\mathbf{k}$ -algebra generated by elements $T_1, T_2, \dots, T_{n-1}$ subject to the relations

$$T_i^2 = (q-1)T_i + q \quad \text{for all } i \in [n-1]; \quad (2.1)$$

$$T_i T_j = T_j T_i \quad \text{whenever } |i-j| > 1; \quad (2.2)$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for all } i \in [n-2]. \quad (2.3)$$

As a $\mathbf{k}$ -vector space, $\mathcal{H}_n$ has a basis $(T_w)_{w \in \mathfrak{S}_n}$, defined by the formula

$$T_w = T_{i_1} T_{i_2} \cdots T_{i_k} \quad (2.4)$$

if $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression¹ for w . In particular, $T_{\text{id}} = 1$ and $T_{s_i} = T_i$ for each $i \in [n-1]$. We refer to [15] and [13, Section 7.4] for more in-depth introductions to Hecke algebras.

There is a unique algebra anti-automorphism of $\mathcal{H}_n$ that sends each basis vector T_w to $T_{w^{-1}}$ and thus, in particular, preserves all the generators $T_1, T_2, \dots, T_{n-1}$. Following [17, Lemma 2.3], we shall denote this anti-automorphism by $a \mapsto a^*$. Moreover, the map $a \mapsto a^*$ is an involution, i.e., we have $a^{**} = a$ for each $a \in \mathcal{H}_n$.

In what follows, we will often assume that q is an indeterminate². Many of our proofs will require that q is nonzero and not a root of unity of order $\leq n$. However, we can always specialize our indeterminate q to any specific value in $\mathbf{k}$, and all our results will remain true upon this specialization as long as they are well-defined (e.g., the denominators don't become zero). In particular, for the specialization $q = 1$, the basis elements T_w of $\mathcal{H}_n$ become the permutations $w \in \mathfrak{S}_n$ in the group algebra $\mathbf{k}[\mathfrak{S}_n]$. When we specialize an element $a \in \mathcal{H}_n$ by setting $q = 1$, we will denote the resulting element by $a(1)$, to distinguish it from the general case.

We will use the q -integers $[k]_q := q^0 + q^1 + \cdots + q^{k-1}$ and the q -factorials $[k]!_q = [1]_q [2]_q \cdots [k]_q$ for all $k \in \mathbb{N}$. These are elements of $\mathbf{k}$.

We implicitly consider the Hecke algebras of all symmetric groups to form a chain of subalgebras:

$$\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots$$

by identifying homonymous generators T_i of different Hecke algebras.

2.2. The shuffles. In this work, we will focus on a series of elements of these Hecke algebras, the simplest of which we define next:

Definition 2.1. For any $n \geq 0$, we define the elements

$$\mathcal{B}_n = \sum_{i=1}^n T_{n-1} T_{n-2} \cdots T_i \quad \text{and}$$

$$\mathcal{B}_n^* = (\mathcal{B}_n)^* = \sum_{i=1}^n T_i T_{i+1} \cdots T_{n-1}$$

in the Hecke algebra $\mathcal{H}_n$. We understand $\mathcal{B}_n$ and $\mathcal{B}_n^*$ to be 0 (like any empty sum) whenever $n \leq 0$.

Example 2.2. Suppose $n \geq 4$. Using cycle notation for permutations, we have:

$$\begin{aligned} \mathcal{B}_4 &= 1 + T_3 + T_3 T_2 + T_3 T_2 T_1 = T_{\text{id}} + T_{(4,3)} + T_{(4,3,2)} + T_{(4,3,2,1)}; \\ \mathcal{B}_4^* &= 1 + T_3 + T_2 T_3 + T_1 T_2 T_3 = T_{\text{id}} + T_{(3,4)} + T_{(2,3,4)} + T_{(1,2,3,4)}. \end{aligned}$$

Note that our notations agree with those in [1], but we omit the “ (q) ” from notations like “ $\mathcal{H}_n(q)$ ” and “ $\mathcal{B}_n(q)$ ”, and we abbreviate T_{s_i} by T_i .

¹A *reduced expression* for a permutation $w \in \mathfrak{S}_n$ means a way of writing w as a product of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$ with minimum k . This minimum k is called the (*Coxeter*) *length* of w , and equals the number of inversions of w .

²That is, $\mathbf{k}$ will be a field of rational functions $\mathbf{m}(q)$ over a smaller field $\mathbf{m}$.

Definition 2.3. For any $n, k \geq 0$, we define the following elements of $\mathcal{H}_n$:

$$\mathcal{B}_{n,k} := \mathcal{B}_{n-(k-1)} \cdots \mathcal{B}_{n-1} \mathcal{B}_n \quad \text{and} \quad (2.5)$$

$$\mathcal{B}_{n,k}^* := (\mathcal{B}_{n,k})^* = \mathcal{B}_n^* \mathcal{B}_{n-1}^* \cdots \mathcal{B}_{n-(k-1)}^* \quad \text{and} \quad (2.6)$$

$$\mathcal{R}_{n,k} := \frac{1}{[k]!_q} \mathcal{B}_{n,k}^* \mathcal{B}_{n,k}. \quad (2.7)$$

We will soon see that $\mathcal{B}_{n,k}^* \mathcal{B}_{n,k}$ is a multiple of $[k]!_q$ (Lemma 2.10), which implies that $\mathcal{R}_{n,k}$ is well-defined for any value of q , even if the denominator in (2.7) might vanish; in this case, we expand $\mathcal{R}_{n,k}$ for an indeterminate q and then substitute our desired value of q .

The elements $\mathcal{B}_{n,k}$, $\mathcal{B}_{n,k}^*$ and $\mathcal{R}_{n,k}$ are called (respectively) the *k-bottom-to-random shuffle*, the *k-random-to-bottom shuffle* and the *k-random-to-random shuffle* in the Hecke algebra $\mathcal{H}_n$ (although the names are swapped in some of the literature). The *k-top-to-random shuffle* and *k-random-to-top shuffle* are obtained from $\mathcal{B}_{n,k}$ and $\mathcal{B}_{n,k}^*$ by applying the involutive algebra endomorphism of $\mathcal{H}_n$ that sends each T_i to T_{n-i} .

Both $\mathcal{B}_{n,0}$ and $\mathcal{B}_{n,0}^*$ equal 1, being empty products. Hence, $\mathcal{R}_{n,0} = 1$ as well. Note that $\mathcal{B}_{n,1} = \mathcal{B}_n$ and $\mathcal{B}_{n,1}^* = \mathcal{B}_n^*$. On the other hand, recall that $\mathcal{B}_m$ and $\mathcal{B}_m^*$ are 0 whenever $m \leq 0$; thus, Definition 2.3 yields that

$$\mathcal{B}_{n,k} = \mathcal{B}_{n,k}^* = 0 \quad \text{for all } k > n, \quad (2.8)$$

and consequently

$$\mathcal{R}_{n,k} = 0 \quad \text{for all } k > n. \quad (2.9)$$

Note that $\mathcal{B}_{n,k}$ is the $\mathcal{B}_{n,k}(q)$ from [1], whereas $\mathcal{R}_{n,1}$ is the $\mathcal{R}_n(q)$ from [1].

Example 2.4. For $n = 3$, we have (using cycle notation for permutations)

$$\mathcal{B}_{3,0} = \mathcal{B}_{3,0}^* = \mathcal{R}_{3,0} = 1$$

and

$$\begin{aligned} \mathcal{B}_{3,1} &= \mathcal{B}_3 = 1 + T_2 + T_2 T_1 = T_{\text{id}} + T_{(3,2)} + T_{(3,2,1)}; \\ \mathcal{B}_{3,1}^* &= \mathcal{B}_3^* = 1 + T_2 + T_1 T_2 = T_{\text{id}} + T_{(2,3)} + T_{(1,2,3)}; \\ \mathcal{R}_{3,1} &= \frac{1}{[1]!_q} \mathcal{B}_{3,1}^* \mathcal{B}_{3,1} = [3]_q T_{\text{id}} + q[2]_q T_{(1,2)} + [2]_q T_{(2,3)} + qT_{(1,2,3)} + qT_{(1,3,2)} + (q-1)T_{(1,3)} \end{aligned}$$

and

$$\mathcal{B}_{3,2} = \mathcal{B}_2 \mathcal{B}_3 = \sum_{w \in \mathfrak{S}_3} T_w; \quad \mathcal{B}_{3,2}^* = \mathcal{B}_3^* \mathcal{B}_2^* = \sum_{w \in \mathfrak{S}_3} T_w; \quad \mathcal{R}_{3,2} = [3]_q \sum_{w \in \mathfrak{S}_3} T_w$$

and

$$\mathcal{B}_{3,3} = \mathcal{B}_{3,3}^* = \sum_{w \in \mathfrak{S}_3} T_w; \quad \mathcal{R}_{3,3} = \sum_{w \in \mathfrak{S}_3} T_w.$$

Note that the $(q-1)T_{(1,3)}$ term in $\mathcal{R}_{3,1}$ is invisible for $q = 1$.

Example 2.5. For $n = 4$ and $k = 2$, we have

$$\begin{aligned} \mathcal{R}_{4,2} &= (q^4 + q^3 + 2q^2 + q + 1) T_{\text{id}} + (q^3 + 2q^2 + q + 1) T_{(3,4)} + (q^4 + q^3 + q^2 + q + 1) T_{(2,3)} \\ &\quad + (q^3 + q^2 + q + 1) T_{(2,3,4)} + (q^3 + q^2 + q + 1) T_{(2,4,3)} + (q^3 + q + 1) T_{(2,4)} \\ &\quad + (q^4 + q^3 + 2q^2 + q) T_{(1,2)} + (q^3 + 2q^2 + q) T_{(1,2)(3,4)} + (q^4 + q^3 + q^2 + q) T_{(1,2,3)} \\ &\quad + (q^3 + q^2 + q) T_{(1,2,3,4)} + (q^3 + q^2 + q) T_{(1,2,4,3)} + (q^3 + q) T_{(1,2,4)} \\ &\quad + (q^4 + q^3 + q^2 + q) T_{(1,3,2)} + (q^3 + q^2 + q) T_{(1,3,4,2)} + (q^4 + q^3 + q^2 + q - 1) T_{(1,3)} \\ &\quad + (q^3 + q^2 + q - 1) T_{(1,3,4)} + (q^3 + q) T_{(1,3)(2,4)} + (q^3 + q - 1) T_{(1,3,2,4)} \\ &\quad + (q^3 + q^2 + q) T_{(1,4,3,2)} + (q^3 + q) T_{(1,4,2)} + (q^3 + q^2 + q - 1) T_{(1,4,3)} \\ &\quad + (q^3 + q - 1) T_{(1,4)} + (q^3 + q - 1) T_{(1,4,2,3)} + (q^3 + q - 2) T_{(1,4)(2,3)}. \end{aligned}$$

Note that all 24 basis elements T_w of $\mathcal{H}_n$ appear in this sum. However, the coefficient $q^3 + q - 2$ in front of $T_{(1,4)(2,3)}$ vanishes at $q = 1$, so that $\mathcal{R}_{4,2}(1)$ contains only 23 permutations $w \in \mathfrak{S}_n$.

2.3. Alternative formulas for the shuffles. The elements $\mathcal{B}_n$ and $\mathcal{B}_n^*$ can also be rewritten in more combinatorial terms:

Proposition 2.6. *Let $n \geq 0$. Then,*

$$\mathcal{B}_n = \sum_{i=1}^n T_{(n, n-1, \dots, i)} = \sum_{\substack{w \in \mathfrak{S}_n; \\ w^{-1}(1) < w^{-1}(2) < \dots < w^{-1}(n-1)}} T_w; \quad (2.10)$$

$$\mathcal{B}_n^* = \sum_{i=1}^n T_{(i, i+1, \dots, n)} = \sum_{\substack{w \in \mathfrak{S}_n; \\ w(1) < w(2) < \dots < w(n-1)}} T_w, \quad (2.11)$$

where $(n, n-1, \dots, i)$ and $(i, i+1, \dots, n)$ are understood as cycles.

Proof. For each $i \in \{1, 2, \dots, n\}$, we have $T_{n-1}T_{n-2} \cdots T_i = T_{(n, n-1, \dots, i)}$, since $s_{n-1}s_{n-2} \cdots s_i$ is a reduced expression for the cycle $(n, n-1, \dots, i)$. Thus, the first equality sign in (2.10) is just a restatement of the definition of $\mathcal{B}_n$.

The second equality sign in (2.10) follows from the combinatorial observation that the permutations $w \in \mathfrak{S}_n$ satisfying $w^{-1}(1) < w^{-1}(2) < \dots < w^{-1}(n-1)$ are precisely the cycles $(n, n-1, \dots, i)$ for $i \in \{1, 2, \dots, n\}$ (because each of the former permutations w is uniquely determined by the value $w^{-1}(n)$, and if we denote this value by i , then our w is the cycle $(n, n-1, \dots, i)$). Thus, (2.10) is proved.

The proof of (2.11) is similar (here, the permutations $w \in \mathfrak{S}_n$ satisfying $w(1) < w(2) < \dots < w(n-1)$ are classified by the value $w(n)$). $\square$

Remark 2.7. More generally, it can be shown that

$$\mathcal{B}_{n,k} = \sum_{\substack{w \in \mathfrak{S}_n; \\ w^{-1}(1) < w^{-1}(2) < \dots < w^{-1}(n-k)}} T_w; \quad (2.12)$$

$$\mathcal{B}_{n,k}^* = \sum_{\substack{w \in \mathfrak{S}_n; \\ w(1) < w(2) < \dots < w(n-k)}} T_w \quad (2.13)$$

for all $n \geq k$. Indeed, the former equality is [1, Lemma 3.12] (applied to $j = n-k$), whereas the latter follows from it using (2.15). For $q = 1$, this implies that the element $\mathcal{B}_{n,k}^*(1)$ lies in the subalgebra of the symmetric group algebra known as *Solomon's descent algebra* [20]. However, the descent algebra does not seem to generalize to the Hecke algebra with general q (the relevant basis elements no longer span a subalgebra, even for $n = 3$), which renders any use of the descent algebra a dead end at our level of generality.

Before we move on to more substantial statements, recall from Section 2.1 that $a \mapsto a^*$ denotes the unique algebra anti-automorphism of $\mathcal{H}_n$ that sends each basis vector T_w to $T_{w^{-1}}$. This notation extends our existing notations: In fact, from Definition 2.1 and Definition 2.3, we have

$$(\mathcal{B}_n)^* = \mathcal{B}_n^* \quad \text{and} \quad (2.14)$$

$$(\mathcal{B}_{n,k})^* = \mathcal{B}_{n,k}^* \quad (2.15)$$

for all nonnegative n and k . Since $a \mapsto a^*$ is an involution, from (2.14) and (2.15), we obtain

$$(\mathcal{B}_n^*)^* = \mathcal{B}_n \quad \text{and} \quad (2.16)$$

$$(\mathcal{B}_{n,k}^*)^* = \mathcal{B}_{n,k} \quad (2.17)$$

for all $n, k \geq 0$. As a consequence, the anti-automorphism $a \mapsto a^*$ preserves the k -random-to-random shuffles:

Lemma 2.8. *Let $n, k \geq 0$. Then,*

$$(\mathcal{R}_{n,k})^* = \mathcal{R}_{n,k}. \quad (2.18)$$

Proof. Because of (2.7), we just need to show that $(\mathcal{B}_{n,k}^* \mathcal{B}_{n,k})^* = \mathcal{B}_{n,k}^* \mathcal{B}_{n,k}$. But this follows from

$$\begin{aligned} (\mathcal{B}_{n,k}^* \mathcal{B}_{n,k})^* &= (\mathcal{B}_{n,k})^* (\mathcal{B}_{n,k}^*)^* \quad (\text{since } a \mapsto a^* \text{ is an anti-morphism}) \\ &= \mathcal{B}_{n,k}^* \mathcal{B}_{n,k} \quad (\text{by (2.15) and (2.17)}). \end{aligned} \quad \square$$

When $q = 1$, the elements $\mathcal{R}_{n,n-k} \in \mathcal{H}_n$ become the elements $\nu_{(k,1^{n-k})} \in \mathbf{k}[\mathfrak{S}_n]$ from [18, Section 1.2]. Indeed, this is a consequence (Proposition 2.11) of a factorization (2.20) of $\mathcal{R}_{n,k}$ that holds for all q . To state the latter, we need some more notations.

For each subset I of $[n]$, let $u_I \in \mathfrak{S}_n$ be the unique permutation that

- (1) is strictly increasing on I and on $[n] \setminus I$, and
- (2) satisfies $u_I(I) = \{1, 2, \dots, |I|\}$.

Thus, u_I sends the elements of I in increasing order to the numbers $1, 2, \dots, |I|$, and sends the remaining elements of $[n]$ in increasing order to the remaining numbers $|I| + 1, |I| + 2, \dots, n$.

Given any $i \in \{0, 1, \dots, n\}$, we write $\mathfrak{S}_{[n]-[i]}$ for the subgroup of $\mathfrak{S}_n$ consisting of all permutations $w \in \mathfrak{S}_n$ that fix the numbers $1, 2, \dots, i$. This subgroup is generated by $s_{i+1}, s_{i+2}, \dots, s_{n-1}$, and is isomorphic to $\mathfrak{S}_{n-i}$.

Proposition 2.9. *Let $n \geq k$, and set*

$$\mathcal{X}_{n,k} := \sum_{\substack{I \subseteq [n]; \\ |I|=n-k}} T_{u_I} \quad \text{and} \quad \mathcal{M}_{n,k} := \sum_{w \in \mathfrak{S}_{[n]-[n-k]}} T_w.$$

Then,

$$\mathcal{B}_{n,k} = \mathcal{M}_{n,k} \mathcal{X}_{n,k}; \tag{2.19}$$

$$\mathcal{R}_{n,k} = \mathcal{X}_{n,k}^* \mathcal{M}_{n,k} \mathcal{X}_{n,k}; \tag{2.20}$$

$$\mathcal{M}_{n,k}^2 = [k]!_q \mathcal{M}_{n,k}; \tag{2.21}$$

$$\mathcal{M}_{n,k}^* = \mathcal{M}_{n,k}. \tag{2.22}$$

Furthermore, if j is a positive integer such that $n - k < j < n$, then

$$T_j \mathcal{M}_{n,k} = q \mathcal{M}_{n,k}; \tag{2.23}$$

$$T_j \mathcal{B}_{n,k} = q \mathcal{B}_{n,k}. \tag{2.24}$$

Proof. We recall some notations from [1]. A *composition* of n means a finite tuple of positive integers whose sum is n . We shall use exponential notation, so that $(1^{n-k}, k)$ means the composition of n that consists of $n - k$ many 1s followed by a k .

For a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$ of n , consider the Young subgroup $\mathfrak{S}_\alpha$ of $\mathfrak{S}_n$ consisting of the permutations that permute the smallest α_1 elements among themselves, permute the next-smallest α_2 elements among themselves, and so on. Thus, $\mathfrak{S}_{(1^{n-k}, k)} = \mathfrak{S}_{[n]-[n-k]}$.

Furthermore, for a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$ of n , we let X_α denote the set of minimal-length right coset representatives of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$. Explicitly, these representatives are the permutations that (when written in one-line notation) contain the smallest α_1 elements in increasing order from left to right, contain the next-smallest α_2 elements in increasing order from left to right, and so on. Thus, $X_{(n-k, k)}$ consists of the permutations $w \in \mathfrak{S}_n$ that contain the numbers $1, 2, \dots, n - k$ in increasing order from left to right, and contain the numbers $n - k + 1, n - k + 2, \dots, n$ in increasing order from left to right. These permutations are just the u_I for the $n - k$ -element subsets I of $[n]$.

For any composition α of n , we define the elements

$$x_\alpha := \sum_{w \in X_\alpha} T_w \quad \text{and} \quad m_\alpha := \sum_{w \in \mathfrak{S}_\alpha} T_w \quad \text{in } \mathcal{H}_n.$$

It is shown in [1, Proposition 3.10] (for $j = n - k$) that

$$\mathcal{B}_{n,k} = m_{(1^{n-k}, k)} x_{(n-k, k)}. \tag{2.25}$$

However, $\mathfrak{S}_{(1^{n-k}, k)} = \mathfrak{S}_{[n]-[n-k]}$ shows that $m_{(1^{n-k}, k)} = \mathcal{M}_{n,k}$, whereas our above description of $X_{(n-k, k)}$ shows that $x_{(n-k, k)} = \mathcal{X}_{n,k}$. Hence, (2.25) can be rewritten as

$$\mathcal{B}_{n,k} = \mathcal{M}_{n,k} \mathcal{X}_{n,k}.$$

This proves (2.19).

Now let us show (2.22). The set $\mathfrak{S}_{[n]-[n-k]}$ is a subgroup of $\mathfrak{S}_n$, and thus is closed under taking inverses (i.e., we have $w^{-1} \in \mathfrak{S}_{[n]-[n-k]}$ for each $w \in \mathfrak{S}_{[n]-[n-k]}$). Thus, the map $a \mapsto a^*$ permutes the addends of the sum $\mathcal{M}_{n,k} = \sum_{w \in \mathfrak{S}_{[n]-[n-k]}} T_w$. Hence, $\mathcal{M}_{n,k}^* = \mathcal{M}_{n,k}$. This proves (2.22).

We let $\ell(w)$ denote the (Coxeter) length of any permutation $w \in \mathfrak{S}_n$, that is, the number of inversions of w .

Next, we shall prove (2.23). For this, we fix a positive integer j such that $n - k < j < n$. Then, $s_j \in \mathfrak{S}_{[n]-[n-k]} = \mathfrak{S}_{(1^{n-k}, k)}$. A permutation $w \in \mathfrak{S}_{[n]-[n-k]}$ will be called *southern* if it satisfies $w^{-1}(j) < w^{-1}(j+1)$, and *northern* otherwise. If w is a southern permutation, then $\ell(s_j w) = \ell(w) + 1$ and thus $T_{s_j w} = T_j T_w$, so that

$$T_w + T_{s_j w} = T_w + T_j T_w = (1 + T_j) T_w. \quad (2.26)$$

Moreover, the map $w \mapsto s_j w$ is a bijection from the set of all southern permutations in $\mathfrak{S}_{[n]-[n-k]}$ to the set of all northern permutations in $\mathfrak{S}_{[n]-[n-k]}$. Hence,

$$\sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is northern}}} T_w = \sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is southern}}} T_{s_j w}. \quad (2.27)$$

Now, the sum $\mathcal{M}_{n,k} = \sum_{w \in \mathfrak{S}_{[n]-[n-k]}} T_w$ can be split into southern and northern permutations. Thus we find

$$\begin{aligned} \mathcal{M}_{n,k} &= \sum_{w \in \mathfrak{S}_{[n]-[n-k]}} T_w = \sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is southern}}} T_w + \sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is northern}}} T_w \\ &= \sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is southern}}} T_w + \sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is southern}}} T_{s_j w} \quad (\text{by (2.27)}) \\ &= \sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is southern}}} (T_w + T_{s_j w}) = \sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is southern}}} (1 + T_j) T_w \end{aligned}$$

(by (2.26)). Left multiplication by T_j transforms this into

$$T_j \mathcal{M}_{n,k} = \sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is southern}}} \underbrace{T_j (1 + T_j) T_w}_{=q(1+T_j)} = q \underbrace{\sum_{\substack{w \in \mathfrak{S}_{[n]-[n-k]}; \\ w \text{ is southern}}} (1 + T_j) T_w}_{=\mathcal{M}_{n,k}} = q \mathcal{M}_{n,k}.$$

Thus, (2.23) is proved.

Equation (2.24) follows from (2.19) and (2.23): if j is a positive integer such that $n - k < j < n$, then

$$T_j \mathcal{B}_{n,k} = T_j \mathcal{M}_{n,k} \mathcal{X}_{n,k} = q \mathcal{M}_{n,k} \mathcal{X}_{n,k} = q \mathcal{B}_{n,k}.$$

By repeatedly applying (2.23), we easily see that

$$T_w \mathcal{M}_{n,k} = q^{\ell(w)} \mathcal{M}_{n,k} \quad (2.28)$$

for any $w \in \mathfrak{S}_{[n]-[n-k]}$ (since w has a reduced expression containing only s_j with $n - k < j < n$). Since $\mathcal{M}_{n,k} = \sum_{w \in \mathfrak{S}_{[n]-[n-k]}} T_w$, we now have

$$\begin{aligned} \mathcal{M}_{n,k} \mathcal{M}_{n,k} &= \sum_{w \in \mathfrak{S}_{[n]-[n-k]}} T_w \mathcal{M}_{n,k} \\ &= \sum_{w \in \mathfrak{S}_{[n]-[n-k]}} q^{\ell(w)} \mathcal{M}_{n,k} \quad (\text{by (2.28)}) \\ &= \left(\sum_{w \in \mathfrak{S}_{[n]-[n-k]}} q^{\ell(w)} \right) \mathcal{M}_{n,k}. \end{aligned} \quad (2.29)$$

However, the subgroup $\mathfrak{S}_{[n]-[n-k]}$ of $\mathfrak{S}_n$ is isomorphic to the symmetric group $\mathfrak{S}_k$ (via the isomorphism that sends each generator s_i to s_{n-i}), and this isomorphism preserves the length of a permutation. Hence,

$$\sum_{w \in \mathfrak{S}_{[n]-[n-k]}} q^{\ell(w)} = \sum_{w \in \mathfrak{S}_k} q^{\ell(w)} = [k]!_q \quad (2.30)$$

by a well-known identity ([19, Theorem 3.2.1]). Thus, (2.29) rewrites as $\mathcal{M}_{n,k} \mathcal{M}_{n,k} = [k]!_q \mathcal{M}_{n,k}$. In other words, $\mathcal{M}_{n,k}^2 = [k]!_q \mathcal{M}_{n,k}$. This proves (2.21).

The two equalities (2.21) and (2.22) entail

$$\mathcal{M}_{n,k}^* \mathcal{M}_{n,k} = \mathcal{M}_{n,k} \mathcal{M}_{n,k} = \mathcal{M}_{n,k}^2 = [k]!_q \mathcal{M}_{n,k}.$$

Now, (2.19) yields

$$\begin{aligned} \mathcal{B}_{n,k}^* \mathcal{B}_{n,k} &= (\mathcal{M}_{n,k} \mathcal{X}_{n,k})^* \mathcal{M}_{n,k} \mathcal{X}_{n,k} = \mathcal{X}_{n,k}^* \underbrace{\mathcal{M}_{n,k}^* \mathcal{M}_{n,k}}_{=[k]!_q \mathcal{M}_{n,k}} \mathcal{X}_{n,k} \\ &= [k]!_q \mathcal{X}_{n,k}^* \mathcal{M}_{n,k} \mathcal{X}_{n,k}. \end{aligned} \quad (2.31)$$

Dividing this by $[k]!_q$, we obtain

$$\mathcal{R}_{n,k} = \mathcal{X}_{n,k}^* \mathcal{M}_{n,k} \mathcal{X}_{n,k},$$

because of (2.7). This proves (2.20). $\square$

We can now prove something we have claimed right after Definition 2.3:

Lemma 2.10. *Let $n, k \geq 0$. Then, $\mathcal{R}_{n,k}$ is well-defined for all values of q , since the product $\mathcal{B}_{n,k}^* \mathcal{B}_{n,k}$ is a multiple of $[k]!_q$.*

Proof. If $k > n$, then the claim follows from $\mathcal{B}_{n,k} = 0$. If $k \leq n$, then it follows from (2.31). $\square$

The proof shows that $\mathcal{R}_{n,k}$ is actually a linear combination of basis elements T_w , with each coefficient being a universal polynomial in q with integer coefficients. Hence, any identity between $\mathcal{R}_{n,k}$'s that we prove for an infinite set of q 's will automatically be true for every q . In particular, we can thus assume the invertibility of $[n]!_q$ and of q in our proofs without having to require it in our claims (unless $[n]!_q$ or q appear as denominators in said claim).

As another byproduct of Proposition 2.9, let us show that the $q = 1$ specialization of $\mathcal{R}_{n,n-k}$ agrees with the element $\nu_{(k, 1^{n-k})} \in \mathbf{k}[\mathfrak{S}_n]$ from [18, Section 1.2]:

Proposition 2.11. *Assume that $q = 1$. Let $n \geq k \geq 0$. For each $w \in \mathfrak{S}_n$, let $\text{noninv}_k w$ denote the number of all k -element subsets of $[n]$ on which w is strictly increasing (i.e., the number of choices of $1 \leq i_1 < i_2 < \dots < i_k \leq n$ satisfying $w(i_1) < w(i_2) < \dots < w(i_k)$). Then,*

$$\mathcal{R}_{n,n-k}(1) = \sum_{w \in \mathfrak{S}_n} (\text{noninv}_k w) w \in \mathbf{k}[\mathfrak{S}_n].$$

Proof. Applying (2.20) to $n - k$ instead of k , we obtain

$$\mathcal{R}_{n,n-k} = \mathcal{X}_{n,n-k}^* \mathcal{M}_{n,n-k} \mathcal{X}_{n,n-k} = \left(\sum_{\substack{I \subseteq [n]; \\ |I|=k}} T_{u_I} \right)^* \left(\sum_{w \in \mathfrak{S}_{[n]-[k]}} T_w \right) \left(\sum_{\substack{I \subseteq [n]; \\ |I|=k}} T_{u_I} \right)$$

by the definitions of $\mathcal{X}_{n,n-k}$ and $\mathcal{M}_{n,n-k}$. In the specialization $q = 1$, the element T_w maps to $w \in \mathfrak{S}_n$, so the above simplifies to

$$\begin{aligned} \mathcal{R}_{n,n-k}(1) &= \left(\sum_{\substack{I \subseteq [n]; \\ |I|=k}} u_I \right)^* \left(\sum_{w \in \mathfrak{S}_{[n]-[k]}} w \right) \left(\sum_{\substack{I \subseteq [n]; \\ |I|=k}} u_I \right) \\ &= \sum_{\substack{I \subseteq [n]; \\ |I|=k}} \sum_{w \in \mathfrak{S}_{[n]-[k]}} \sum_{\substack{J \subseteq [n]; \\ |J|=k}} u_I^{-1} w u_J \quad (\text{since } u_I^* = u_I^{-1}) \\ &= \sum_{\substack{J \subseteq [n]; \\ |J|=k}} \sum_{\substack{I \subseteq [n]; \\ |I|=k}} \left(\sum_{w \in \mathfrak{S}_{[n]-[k]}} u_I^{-1} w u_J \right). \end{aligned}$$

The sum inside the parentheses here is the sum of all permutations in $\mathfrak{S}_n$ that send the k elements of J to the k elements of I in increasing order (because u_J sends the k elements of J to the k numbers $1, 2, \dots, k$ in increasing order, then $w \in \mathfrak{S}_{[n]-[k]}$ leaves these k numbers $1, 2, \dots, k$ unchanged, and finally u_I^{-1} sends

the numbers $1, 2, \dots, k$ to the k elements of I in increasing order). In other words, it is the sum of all permutations $w \in \mathfrak{S}_n$ that are strictly increasing on J and satisfy $w(J) = I$. Hence, the above equality can be rewritten as

$$\begin{aligned} \mathcal{R}_{n,n-k}(1) &= \sum_{\substack{J \subseteq [n]; \\ |J|=k}} \sum_{\substack{I \subseteq [n]; \\ |I|=k}} \left(\sum_{\substack{w \in \mathfrak{S}_n; \\ w \text{ is increasing on } J; \\ w(J)=I}} w \right) = \sum_{\substack{J \subseteq [n]; \\ |J|=k}} \sum_{\substack{w \in \mathfrak{S}_n; \\ w \text{ is increasing on } J}} w \\ &= \sum_{w \in \mathfrak{S}_n} \sum_{\substack{J \subseteq [n]; \\ |J|=k; \\ w \text{ is increasing on } J}} w = \sum_{w \in \mathfrak{S}_n} \underbrace{(\text{noninv}_k w) w}_{=(\text{noninv}_k w) w}. \end{aligned} \quad \square$$

Remark 2.12. If w is the permutation $w_0 \in \mathfrak{S}_n$ that sends $1, 2, \dots, n$ to $n, n-1, \dots, 1$, then $\text{noninv}_k w = 0$ for all $k > 1$. Hence, w_0 does not appear on the right hand side of Proposition 2.11 for $k > 1$. However, T_{w_0} can well appear in $\mathcal{R}_{n,n-k}$, as we saw in Example 2.5. This shows that Proposition 2.11 is likely a special feature of the $q = 1$ case.

2.4. Recursions. The following recursions for our shuffles $\mathcal{B}_{n,k}$, $\mathcal{B}_{n,k}^*$ and $\mathcal{R}_{n,k}$ follow easily from their definitions.

Proposition 2.13. *Let n and k be positive. Then,*

$$\mathcal{B}_{n,k} = \mathcal{B}_{n-1,k-1} \mathcal{B}_n; \quad (2.32)$$

$$\mathcal{B}_{n,k}^* = \mathcal{B}_n^* \mathcal{B}_{n-1,k-1}^*; \quad (2.33)$$

$$\mathcal{B}_{n,k} = \mathcal{B}_{n-k+1} \mathcal{B}_{n,k-1}; \quad (2.34)$$

$$\mathcal{B}_{n,k}^* = \mathcal{B}_{n,k-1}^* \mathcal{B}_{n-k+1}^*; \quad (2.35)$$

$$[k]_q \mathcal{R}_{n,k} = \mathcal{B}_n^* \mathcal{R}_{n-1,k-1} \mathcal{B}_n. \quad (2.36)$$

Proof. First, we prove (2.32). By definition, we have $\mathcal{B}_{n,k} = \mathcal{B}_{n-(k-1)} \cdots \mathcal{B}_{n-1} \mathcal{B}_n$. Likewise, $\mathcal{B}_{n-1,k-1} = \mathcal{B}_{n-(k-1)} \cdots \mathcal{B}_{n-2} \mathcal{B}_{n-1}$, since $(n-1) - ((k-1) - 1) = n - (k-1)$. Thus,

$$\mathcal{B}_{n,k} = \mathcal{B}_{n-(k-1)} \cdots \mathcal{B}_{n-1} \mathcal{B}_n = (\mathcal{B}_{n-(k-1)} \cdots \mathcal{B}_{n-2} \mathcal{B}_{n-1}) \mathcal{B}_n = \mathcal{B}_{n-1,k-1} \mathcal{B}_n.$$

This proves (2.32). The proof of (2.33) is analogous, and the proofs of (2.34) and (2.35) are similar.

Finally, to prove (2.36), we observe that

$$\begin{aligned} \mathcal{B}_n^* \mathcal{R}_{n-1,k-1} \mathcal{B}_n &= \mathcal{B}_n^* \left(\frac{1}{[k-1]_q!} \mathcal{B}_{n-1,k-1}^* \mathcal{B}_{n-1,k-1} \right) \mathcal{B}_n \quad (\text{by (2.7)}) \\ &= \frac{1}{[k-1]_q!} (\mathcal{B}_n^* \mathcal{B}_{n-1,k-1}^*) (\mathcal{B}_{n-1,k-1} \mathcal{B}_n) \\ &= \frac{1}{[k-1]_q!} \mathcal{B}_{n,k}^* \mathcal{B}_{n,k} \quad (\text{by (2.33) and (2.32)}) \\ &= \frac{[k]_q}{[k]_q!} \mathcal{B}_{n,k}^* \mathcal{B}_{n,k} \quad \left(\text{since } [k]_q! = [k-1]_q! [k]_q \right) \\ &= [k]_q \mathcal{R}_{n,k} \quad (\text{by (2.7)}). \end{aligned} \quad \square$$

2.5. Enter the Young–Jucys–Murphy elements. Assume that $q \neq 0$. Recall the *Young–Jucys–Murphy elements* $J_1, J_2, J_3, \dots$ in the respective Hecke algebras $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \dots$ defined by

$$J_k = \sum_{i=1}^{k-1} q^{i-k} T_{(i,k)} \quad \text{for all } k \geq 1, \quad (2.37)$$

where (i, k) denotes the transposition that swaps i with k . For instance, $J_3 = q^{-2} T_{(1,3)} + q^{-1} T_{(2,3)}$. Note that $J_1 = 0$.

The anti-automorphism $a \mapsto a^*$ of the Hecke algebra $\mathcal{H}_n$ preserves T_w for each transposition w (since $w^{-1} = w$), and thus also preserves the Young–Jucys–Murphy elements, i.e., we have

$$(J_k)^* = J_k \quad \text{for all } k \geq 1. \quad (2.38)$$

We will now state two well-known properties of Young–Jucys–Murphy elements that will aid our computations later on. While they both appear in the literature (e.g., [15, Proposition 3.26] or [16, (4.1)]), we nevertheless include their proofs in Appendix A to keep this work self-contained.

Lemma 2.14. *Let $n > 1$. Then,*

$$T_{n-1}J_{n-1} = J_n(T_{n-1} - q + 1) - 1.$$

Lemma 2.15. *Let $n, i > 0$ with $i \notin \{n-1, n\}$. Then, T_i commutes with J_n .*

Lemma 2.16. *Assume that $n > 0$. Then, J_n commutes with any element of $\mathcal{H}_{n-1}$.*

Proof. The algebra $\mathcal{H}_{n-1}$ is generated by the elements $T_1, T_2, \dots, T_{n-2}$. But Lemma 2.15 shows that all these elements commute with J_n . $\square$

Lemma 2.16 easily yields the well-known fact that the Young–Jucys–Murphy elements $J_1, J_2, J_3, \dots$ all commute.

Lemma 2.17. *Let $n > 1$. Then,*

$$\mathcal{B}_n \mathcal{B}_n^* = [n]_q + q^n J_n + \mathcal{B}_{n-1}^* T_{n-1} \mathcal{B}_{n-1}.$$

Proof. See [1, Equation (5.2) in the proof of Theorem 5.2]. $\square$

3. A RECURSION

We shall next work our way towards a crucial recursive formula expressing $\mathcal{B}_n \mathcal{R}_{n,k}$ in terms of $\mathcal{R}_{n-1,k} \mathcal{B}_n$, $\mathcal{R}_{n-1,k-1} \mathcal{B}_n$ and J_n (Theorem 3.6).

For each $1 \leq i \leq n$, we define the element

$$\lambda_{n,i} := (T_{n-1} \cdots T_{n-i+1}) \left([n-i+1]_q + q^{n-i+1} J_{n-i+1} \right)$$

of $\mathcal{H}_n$. These elements clearly satisfy the formula

$$\lambda_{n,1} = [n]_q + q^n J_n \quad (3.1)$$

and the recursion

$$\lambda_{n,i} = T_{n-1} \lambda_{n-1,i-1} \quad \text{for all } 1 < i \leq n. \quad (3.2)$$

Furthermore, for any $0 \leq k \leq n$, we define

$$\gamma_{n,k} := \sum_{i=1}^k \lambda_{n,i} = \sum_{i=1}^k T_{n-1} \cdots T_{n-i+1} \left([n-i+1]_q + q^{n-i+1} J_{n-i+1} \right).$$

Thus, $\gamma_{n,0} = 0$ and $\gamma_{n,1} = \lambda_{n,1} = [n]_q + q^n J_n$. The elements $\gamma_{n,k}$ furthermore satisfy the following recursion:

Lemma 3.1. *For any $1 \leq k \leq n$, we have*

$$\gamma_{n,k} = [n]_q + q^n J_n + T_{n-1} \gamma_{n-1,k-1}. \quad (3.3)$$

Proof. The definition of $\gamma_{n,k}$ yields

$$\begin{aligned} \gamma_{n,k} &= \sum_{i=1}^k \lambda_{n,i} = \lambda_{n,1} + \sum_{i=2}^k \lambda_{n,i} = \lambda_{n,1} + \sum_{i=2}^k T_{n-1} \lambda_{n-1,i-1} \quad (\text{by (3.2)}) \\ &= \lambda_{n,1} + T_{n-1} \sum_{i=2}^k \lambda_{n-1,i-1} = \lambda_{n,1} + T_{n-1} \sum_{i=1}^{k-1} \lambda_{n-1,i} \end{aligned} \quad (3.4)$$

(here, we reindexed the sum by substituting i for $i-1$). However, by the definition of $\gamma_{n-1,k-1}$, we have

$$\sum_{i=1}^{k-1} \lambda_{n-1,i} = \gamma_{n-1,k-1}$$

and we also know that $\lambda_{n,1} = [n]_q + q^n J_n$. Thus, we can rewrite (3.4) as $\gamma_{n,k} = [n]_q + q^n J_n + T_{n-1} \gamma_{n-1,k-1}$. $\square$

Finally, we set

$$\Lambda_{n,k} := \mathcal{B}_{n-1,k-1}^* \gamma_{n,k} \quad (3.5)$$

for each $1 \leq k \leq n$. We extend this to $k = 0$ by setting $\Lambda_{n,0} := 0$. Note that

$$\Lambda_{n,1} = \mathcal{B}_{n-1,0}^* \gamma_{n,1} = \gamma_{n,1} = [n]_q + q^n J_n. \quad (3.6)$$

Proposition 3.2. *For any $1 \leq k \leq n$, we have*

$$\mathcal{B}_n \mathcal{B}_{n,k}^* = \mathcal{B}_{n-1,k}^* \left(T_{n-1} T_{n-2} \cdots T_{n-k} \right) \mathcal{B}_{n-k} + \Lambda_{n,k}.$$

Here, we understand T_0 to mean 0.

Proof. We proceed by induction on k . As a base case, when $k = 1$, we have

$$\begin{aligned} \mathcal{B}_n \mathcal{B}_n^* &= \mathcal{B}_{n-1}^* T_{n-1} \mathcal{B}_{n-1} + [n]_q + q^n J_n \quad (\text{by Lemma 2.17}) \\ &= \mathcal{B}_{n-1}^* T_{n-1} \mathcal{B}_{n-1} + \Lambda_{n,1} \quad (\text{by (3.6)}). \end{aligned} \quad (3.7)$$

(This also holds for $n = 1$, since we set $T_0 = 0$.)

Now, let $k > 1$, and assume the proposition holds for $k - 1$. Then, the induction hypothesis (applied to $n - 1$ and $k - 1$ instead of n and k) yields

$$\begin{aligned} \mathcal{B}_{n-1} \mathcal{B}_{n-1,k-1}^* &= \mathcal{B}_{n-2,k-1}^* \left(T_{n-2} T_{n-3} \cdots T_{n-k} \right) \mathcal{B}_{n-k} + \Lambda_{n-1,k-1} \end{aligned} \quad (3.8)$$

since $(n - 1) - (k - 1) = n - k$. Furthermore, since $\mathcal{B}_{n-1,k-1}^* \in \mathcal{H}_{n-1}$, Lemma 2.16 shows that J_n commutes with $\mathcal{B}_{n-1,k-1}^*$. Thus, $[n]_q + q^n J_n$ commutes with $\mathcal{B}_{n-1,k-1}^*$ as well. In view of (3.6), we can restate this as follows: $\Lambda_{n,1}$ commutes with $\mathcal{B}_{n-1,k-1}^*$. But (2.33) yields

$$\begin{aligned} \mathcal{B}_n \mathcal{B}_{n,k}^* &= (\mathcal{B}_n \mathcal{B}_n^*) \mathcal{B}_{n-1,k-1}^* \\ &= (\mathcal{B}_{n-1}^* T_{n-1} \mathcal{B}_{n-1} + \Lambda_{n,1}) \mathcal{B}_{n-1,k-1}^* \quad (\text{by (3.7)}) \\ &= \mathcal{B}_{n-1}^* T_{n-1} (\mathcal{B}_{n-1} \mathcal{B}_{n-1,k-1}^*) + \Lambda_{n,1} \mathcal{B}_{n-1,k-1}^* \\ &= \mathcal{B}_{n-1}^* T_{n-1} (\mathcal{B}_{n-1} \mathcal{B}_{n-1,k-1}^*) + \mathcal{B}_{n-1,k-1}^* \Lambda_{n,1} \quad (\text{since } \Lambda_{n,1} \text{ commutes with } \mathcal{B}_{n-1,k-1}^*) \\ &= \mathcal{B}_{n-1}^* T_{n-1} \left(\mathcal{B}_{n-2,k-1}^* (T_{n-2} T_{n-3} \cdots T_{n-k}) \mathcal{B}_{n-k} + \Lambda_{n-1,k-1} \right) + \mathcal{B}_{n-1,k-1}^* \Lambda_{n,1} \quad (\text{by (3.8)}) \\ &= \underbrace{\mathcal{B}_{n-1}^* T_{n-1} \mathcal{B}_{n-2,k-1}^* (T_{n-2} T_{n-3} \cdots T_{n-k}) \mathcal{B}_{n-k}}_{=: \alpha} + \underbrace{\mathcal{B}_{n-1}^* T_{n-1} \Lambda_{n-1,k-1} + \mathcal{B}_{n-1,k-1}^* \Lambda_{n,1}}_{=: \beta}. \end{aligned}$$

Note that T_{n-1} commutes with $\mathcal{B}_{n-2,k-1}^*$ since $\mathcal{B}_{n-2,k-1}^* \in \mathcal{H}_{n-2}$. With α, β defined above, we thus have

$$\begin{aligned} \alpha &= \mathcal{B}_{n-1}^* \mathcal{B}_{n-2,k-1}^* (T_{n-1} T_{n-2} T_{n-3} \cdots T_{n-k}) \mathcal{B}_{n-k} \\ &= \mathcal{B}_{n-1,k}^* (T_{n-1} T_{n-2} \cdots T_{n-k}) \mathcal{B}_{n-k} \quad (\text{by (2.33)}). \end{aligned}$$

Hence, to complete the proof, it suffices to prove that $\beta = \Lambda_{n,k}$. But the definition of $\Lambda_{n-1,k-1}$ yields

$$\begin{aligned} \beta &= \mathcal{B}_{n-1}^* T_{n-1} \mathcal{B}_{n-2,k-2}^* \gamma_{n-1,k-1} + \mathcal{B}_{n-1,k-1}^* \Lambda_{n,1} \\ &= \mathcal{B}_{n-1}^* \mathcal{B}_{n-2,k-2}^* T_{n-1} \gamma_{n-1,k-1} + \mathcal{B}_{n-1,k-1}^* \Lambda_{n,1} \quad (\text{since } T_{n-1} \text{ commutes with } \mathcal{B}_{n-2,k-2}^*) \\ &= \mathcal{B}_{n-1,k-1}^* T_{n-1} \gamma_{n-1,k-1} + \mathcal{B}_{n-1,k-1}^* \Lambda_{n,1} \quad (\text{by (2.33)}) \\ &= \mathcal{B}_{n-1,k-1}^* (\Lambda_{n,1} + T_{n-1} \gamma_{n-1,k-1}) \\ &= \mathcal{B}_{n-1,k-1}^* \left([n]_q + q^n J_n + T_{n-1} \gamma_{n-1,k-1} \right) \quad (\text{since } \Lambda_{n,1} = [n]_q + q^n J_n) \\ &= \mathcal{B}_{n-1,k-1}^* \gamma_{n,k} \quad (\text{by (3.3)}) \\ &= \Lambda_{n,k}, \end{aligned}$$

completing the proof. $\square$

Proposition 3.3. *Let $0 \leq k \leq n$. There exists $I_{n,k} \in \mathcal{H}_n$ such that*

$$\gamma_{n,k} = [k]_q \left([n-k+1]_q + q^{n-k+1} J_n \right) + I_{n,k} \quad \text{and} \quad I_{n,k} \mathcal{B}_{n,k} = 0.$$

Proof. We let $\ker(\mathcal{B}_{n,k})$ be the set of all $h \in \mathcal{H}_n$ that satisfy $h \mathcal{B}_{n,k} = 0$. This is a left ideal of $\mathcal{H}_n$. Furthermore, set

$$\rho_{n,k} := [k]_q \left([n-k+1]_q + q^{n-k+1} J_n \right).$$

We must show that $\gamma_{n,k} \equiv \rho_{n,k} \pmod{\ker(\mathcal{B}_{n,k})}$.

We proceed by induction on k , where the $k=0$ case boils down to $0=0$, and the $k=1$ case follows from $\gamma_{n,1} = [n]_q + q^n J_n = \rho_{n,1}$.

Now assume $k > 1$ (whence $n > 1$ as well). By our induction hypothesis,

$$\gamma_{n-1,k-1} \equiv \rho_{n-1,k-1} \pmod{\ker(\mathcal{B}_{n-1,k-1})}.$$

Since $\ker(\mathcal{B}_{n-1,k-1}) \subseteq \ker(\mathcal{B}_{n,k})$ (a consequence of (2.32)), this entails

$$\gamma_{n-1,k-1} \equiv \rho_{n-1,k-1} \pmod{\ker(\mathcal{B}_{n,k})}.$$

In view of this, (3.3) becomes

$$\gamma_{n,k} \equiv [n]_q + q^n J_n + T_{n-1} \rho_{n-1,k-1} \pmod{\ker(\mathcal{B}_{n,k})}. \quad (3.9)$$

We rewrite the term $T_{n-1} \rho_{n-1,k-1}$ as follows:

$$\begin{aligned} T_{n-1} \rho_{n-1,k-1} &= T_{n-1} [k-1]_q \left([n-k+1]_q + q^{n-k+1} J_{n-1} \right) \quad (\text{by definition}) \\ &= [k-1]_q [n-k+1]_q T_{n-1} + q^{n-k+1} [k-1]_q T_{n-1} J_{n-1} \\ &\quad (\text{by distributivity}) \\ &= [k-1]_q [n-k+1]_q T_{n-1} + q^{n-k+1} [k-1]_q \left(J_n (T_{n-1} - q + 1) - 1 \right) \\ &\quad (\text{by Lemma 2.14}) \\ &\equiv [k-1]_q [n-k+1]_q q + q^{n-k+1} [k-1]_q \left(J_n (q - q + 1) - 1 \right) \\ &\quad (\text{since (2.24) shows that } T_{n-1} \equiv q \pmod{\ker(\mathcal{B}_{n,k})}) \\ &= q [k-1]_q [n-k+1]_q + q^{n-k+1} [k-1]_q (J_n - 1) \pmod{\ker(\mathcal{B}_{n,k})}. \end{aligned}$$

We substitute this into (3.9) to obtain

$$\begin{aligned} \gamma_{n,k} &\equiv [n]_q + q^n J_n + q [k-1]_q [n-k+1]_q + q^{n-k+1} [k-1]_q (J_n - 1) \\ &= \left([n]_q - q^{n-k+1} [k-1]_q \right) + q [k-1]_q [n-k+1]_q + q^{n-k+1} (q^{k-1} + [k-1]_q) J_n \\ &= [n-k+1]_q + q [k-1]_q [n-k+1]_q + q^{n-k+1} [k]_q J_n \quad (\text{by (3.10) and (3.11)}) \\ &= (1 + q [k-1]_q) [n-k+1]_q + q^{n-k+1} [k]_q J_n \\ &= [k]_q [n-k+1]_q + q^{n-k+1} [k]_q J_n \quad (\text{by (3.12)}) \\ &= [k]_q \left([n-k+1]_q + q^{n-k+1} J_n \right) \pmod{\ker(\mathcal{B}_{n,k})}, \end{aligned}$$

where we used the q -integer identities

$$[n]_q - q^{n-k+1} [k-1]_q = [n-k+1]_q, \quad (3.10)$$

$$q^{k-1} + [k-1]_q = [k]_q, \quad (3.11)$$

$$1 + q [k-1]_q = [k]_q. \quad (3.12)$$

In other words, $\gamma_{n,k} \equiv \rho_{n,k} \pmod{\ker(\mathcal{B}_{n,k})}$. This completes the induction step, and with it the proof of the proposition. $\square$

We obtain the following natural corollary.

Corollary 3.4. *Let $1 \leq k \leq n$. Then,*

$$\Lambda_{n,k} \mathcal{B}_{n,k} = [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n-1,k-1}^* \mathcal{B}_{n,k}.$$

Proof. Lemma 2.16 shows that J_n commutes with $\mathcal{B}_{n-1,k-1}^*$ since $\mathcal{B}_{n-1,k-1}^* \in \mathcal{H}_{n-1}$. Hence,

$$[k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right)$$

commutes with $\mathcal{B}_{n-1,k-1}^*$ as well.

By Proposition 3.3, there exists $I_{n,k} \in \mathcal{H}_n$ such that

$$\gamma_{n,k} = [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) + I_{n,k} \quad \text{and} \quad I_{n,k} \mathcal{B}_{n,k} = 0.$$

Multiplying this by $\mathcal{B}_{n,k}$ from the right, we obtain

$$\gamma_{n,k} \mathcal{B}_{n,k} = [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n,k}. \quad (3.13)$$

Recalling the definition of $\Lambda_{n,k}$, we have

$$\begin{aligned} \Lambda_{n,k} \mathcal{B}_{n,k} &= \mathcal{B}_{n-1,k-1}^* \gamma_{n,k} \mathcal{B}_{n,k} \\ &= \mathcal{B}_{n-1,k-1}^* [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n,k} \quad (\text{by (3.13)}) \\ &= [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n-1,k-1}^* \mathcal{B}_{n,k} \end{aligned}$$

since $[k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right)$ commutes with $\mathcal{B}_{n-1,k-1}^*$. □

Lemma 3.5. *For any $1 \leq k \leq n$, we have*

$$\mathcal{B}_n \mathcal{B}_{n,k}^* \mathcal{B}_{n,k} = q^k \mathcal{B}_{n-1,k}^* \mathcal{B}_{n,k+1} + [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n-1,k-1}^* \mathcal{B}_{n,k}.$$

Proof. By Proposition 3.2, we have that

$$\mathcal{B}_n \mathcal{B}_{n,k}^* = \mathcal{B}_{n-1,k}^* \left(T_{n-1} T_{n-2} \cdots T_{n-k} \right) \mathcal{B}_{n-k} + \Lambda_{n,k}.$$

Multiply both sides by $\mathcal{B}_{n,k}$ from the right. The first term on the right-hand side becomes

$$\begin{aligned} &\mathcal{B}_{n-1,k}^* \left(T_{n-1} T_{n-2} \cdots T_{n-k} \right) \mathcal{B}_{n-k} \mathcal{B}_{n,k} \\ &= \mathcal{B}_{n-1,k}^* \left(T_{n-1} T_{n-2} \cdots T_{n-k} \right) \mathcal{B}_{n,k+1} \quad (\text{by (2.34)}) \\ &= q^k \mathcal{B}_{n-1,k}^* \mathcal{B}_{n,k+1}, \end{aligned}$$

since (2.24) yields $\left(T_{n-1} T_{n-2} \cdots T_{n-k} \right) \mathcal{B}_{n,k+1} = q^k \mathcal{B}_{n,k+1}$ (this holds in the border case $k = n$ as well, since (2.8) entails $\mathcal{B}_{n,k+1} = 0$ in this case). On the other hand, Corollary 3.4 implies that

$$\Lambda_{n,k} \mathcal{B}_{n,k} = [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n-1,k-1}^* \mathcal{B}_{n,k}.$$

Combining these terms completes the proof. □

We are now ready to prove a recursion that will help us understand $\mathcal{R}_{n,k}$.

Theorem 3.6. *For any $1 \leq k \leq n$, we have*

$$\mathcal{B}_n \mathcal{R}_{n,k} = \left(q^k \mathcal{R}_{n-1,k} + \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{R}_{n-1,k-1} \right) \mathcal{B}_n.$$

Proof. By Lemma 3.5, we have

$$\begin{aligned}
\mathcal{B}_n \mathcal{B}_{n,k}^* \mathcal{B}_{n,k} &= q^k \mathcal{B}_{n-1,k}^* \mathcal{B}_{n,k+1} + [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n-1,k-1}^* \mathcal{B}_{n,k} \\
&= q^k \mathcal{B}_{n-1,k}^* \mathcal{B}_{n-1,k} \mathcal{B}_n + [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n-1,k-1}^* \mathcal{B}_{n-1,k-1} \mathcal{B}_n \\
&\quad (\text{since } \mathcal{B}_{n,k+1} = \mathcal{B}_{n-1,k} \mathcal{B}_n \text{ and } \mathcal{B}_{n,k} = \mathcal{B}_{n-1,k-1} \mathcal{B}_n) \\
&= \left(q^k \mathcal{B}_{n-1,k}^* \mathcal{B}_{n-1,k} + [k]_q \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{B}_{n-1,k-1}^* \mathcal{B}_{n-1,k-1} \right) \mathcal{B}_n.
\end{aligned}$$

Now divide both sides by $[k]_q!$. In view of the fact that $\mathcal{R}_{n,k} = \frac{1}{[k]_q!} \mathcal{B}_{n,k}^* \mathcal{B}_{n,k}$ and the analogous identities for smaller subscripts, this becomes precisely the claim of the theorem. $\square$

Remark 3.7. Proposition 3.3 and Corollary 3.4 can be made stronger by replacing each appearance of $\mathcal{B}_{n,k}$ by $\mathcal{M}_{n,k}$ (see Proposition 2.9 and recall (2.19)). However, we will have no need for these stronger variants, so we leave the proofs to the interested reader (the main new ingredient is the easy fact that $\mathcal{M}_{n,k} \in \mathcal{M}_{n-1,k-1} \mathcal{H}_n$).

4. COMMUTATIVITY OF THE k -RANDOM-TO-RANDOM SHUFFLES

We are now ready to prove our first main result:

Theorem 4.1. *Let $n \geq 0$. The elements $\mathcal{R}_{n,k}$ for all $k \geq 0$ commute.*

We shall give one proof of this theorem here and another in Appendix B.

Proof of Theorem 4.1. We induct on n . The base case ($n = 0$) is obvious, so we proceed to the induction step.

Let $n > 0$ and $i, j \geq 0$. We must show that $\mathcal{R}_{n,i} \mathcal{R}_{n,j} = \mathcal{R}_{n,j} \mathcal{R}_{n,i}$. This is obvious if $i = 0$ (since $\mathcal{R}_{n,i} = 1$ in this case) and also obvious if $i > n$ (since (2.9) yields that $\mathcal{R}_{n,i} = 0$ in this case). Hence, we assume without loss of generality that $1 \leq i \leq n$. For similar reasons, we assume that $1 \leq j \leq n$.

The element $\mathcal{R}_{n-1,j-1} \in \mathcal{H}_{n-1}$ commutes with $\mathcal{R}_{n-1,i}$ and $\mathcal{R}_{n-1,i-1}$ by the induction hypothesis, and also commutes with J_n since any element of $\mathcal{H}_{n-1}$ commutes with J_n by Lemma 2.16. The element $\mathcal{R}_{n-1,i-1} \in \mathcal{H}_{n-1}$ also commutes with J_n for a similar reason.

Theorem 3.6 yields

$$\mathcal{B}_n \mathcal{R}_{n,i} = \left(q^i \mathcal{R}_{n-1,i} + \left([n+1-i]_q + q^{n+1-i} J_n \right) \mathcal{R}_{n-1,i-1} \right) \mathcal{B}_n.$$

Setting

$$\mathcal{W}_{n,i} := q^i \mathcal{R}_{n-1,i} + \left([n+1-i]_q + q^{n+1-i} J_n \right) \mathcal{R}_{n-1,i-1}, \quad (4.1)$$

we can rewrite this as

$$\mathcal{B}_n \mathcal{R}_{n,i} = \mathcal{W}_{n,i} \mathcal{B}_n. \quad (4.2)$$

Recall that $\mathcal{R}_{n-1,j-1}$ commutes with $\mathcal{R}_{n-1,i}$ and $\mathcal{R}_{n-1,i-1}$ and with J_n . Thus, $\mathcal{R}_{n-1,j-1}$ commutes with $\mathcal{W}_{n,i}$ as well (since $\mathcal{W}_{n,i}$ is a polynomial in $\mathcal{R}_{n-1,i}$, $\mathcal{R}_{n-1,i-1}$ and J_n).

Applying the anti-involution $a \mapsto a^*$ to the equality (4.1), we obtain

$$\begin{aligned}
(\mathcal{W}_{n,i})^* &= q^i (\mathcal{R}_{n-1,i})^* + (\mathcal{R}_{n-1,i-1})^* \left([n+1-i]_q + q^{n+1-i} (J_n)^* \right) \\
&\quad (\text{since an anti-involution reverses all products}) \\
&= q^i \mathcal{R}_{n-1,i} + \mathcal{R}_{n-1,i-1} \left([n+1-i]_q + q^{n+1-i} J_n \right) \quad (\text{by (2.18) and (2.38)}) \\
&= q^i \mathcal{R}_{n-1,i} + \left([n+1-i]_q + q^{n+1-i} J_n \right) \mathcal{R}_{n-1,i-1} \\
&\quad (\text{since } \mathcal{R}_{n-1,i-1} \text{ commutes with } J_n) \\
&= \mathcal{W}_{n,i} \quad (\text{by (4.1)}). \quad (4.3)
\end{aligned}$$

But (2.36) yields $[j]_q \mathcal{R}_{n,j} = \mathcal{B}_n^* \mathcal{R}_{n-1,j-1} \mathcal{B}_n$, so that

$$\mathcal{R}_{n,j} = \frac{1}{[j]_q} \mathcal{B}_n^* \mathcal{R}_{n-1,j-1} \mathcal{B}_n.$$

Thus,

$$\begin{aligned}\mathcal{R}_{n,j} \mathcal{R}_{n,i} &= \frac{1}{[j]_q} \mathcal{B}_n^* \mathcal{R}_{n-1,j-1} \mathcal{B}_n \mathcal{R}_{n,i} \\ &= \frac{1}{[j]_q} \mathcal{B}_n^* \mathcal{R}_{n-1,j-1} \mathcal{W}_{n,i} \mathcal{B}_n\end{aligned}\tag{4.4}$$

(by (4.2)). Applying the anti-involution $a \mapsto a^*$ to both sides of this equality (recalling that an anti-involution reverses all products), we find

$$\begin{aligned}(\mathcal{R}_{n,i})^* (\mathcal{R}_{n,j})^* &= \frac{1}{[j]_q} (\mathcal{B}_n)^* (\mathcal{W}_{n,i})^* (\mathcal{R}_{n-1,j-1})^* (\mathcal{B}_n^*)^* \\ &= \frac{1}{[j]_q} \mathcal{B}_n^* \mathcal{W}_{n,i} \mathcal{R}_{n-1,j-1} \mathcal{B}_n \quad (\text{by (2.14), (4.3), (2.18) and (2.16)}) \\ &= \frac{1}{[j]_q} \mathcal{B}_n^* \mathcal{R}_{n-1,j-1} \mathcal{W}_{n,i} \mathcal{B}_n\end{aligned}$$

(since $\mathcal{R}_{n-1,j-1}$ commutes with $\mathcal{W}_{n,i}$). Comparing this with (4.4), we obtain

$$\mathcal{R}_{n,j} \mathcal{R}_{n,i} = (\mathcal{R}_{n,i})^* (\mathcal{R}_{n,j})^* = \mathcal{R}_{n,i} \mathcal{R}_{n,j} \quad (\text{by (2.18)}).$$

In other words, $\mathcal{R}_{n,i} \mathcal{R}_{n,j} = \mathcal{R}_{n,j} \mathcal{R}_{n,i}$. Thus, the induction is complete. $\square$

5. COMPUTING THE EIGENVALUES

We shall now study the actions of the elements $\mathcal{R}_{n,k} \in \mathcal{H}_n$ on the irreducible $\mathcal{H}_n$ -modules in the case where $\mathcal{H}_n$ is semisimple. We will rely on the eigenbasis $\mathfrak{B}_\lambda$ of $\mathcal{R}_{n,1}$ constructed in [1, Theorem 7.2] in the case when $q \in \mathbb{R}_{>0}$ (in other cases, there might be no eigenbasis, as the operator can fail to be semisimple). We will show that $\mathfrak{B}_\lambda$ is in fact an eigenbasis for **all** the operators $\mathcal{R}_{n,k}$ (Theorem 5.13), and compute the eigenvalues. We will then generalize the latter to the case of arbitrary q (Theorem 5.15), and combine the different Specht modules to describe the eigenvalues of $\mathcal{R}_{n,k}$ on the whole Hecke algebra $\mathcal{H}_n$ (Theorem 5.16).

In this section, we assume that q is neither 0 nor a primitive k -th root of unity for any $k \in \{2, 3, \dots, n\}$. Moreover, we assume that if $q = 1$, then $n!$ is invertible in $\mathbf{k}$ (that is, we don't have $1 \leq \text{char } \mathbf{k} \leq n$). Thus, the Hecke algebra $\mathcal{H}_n$ is semisimple, and its representation theory is well-understood (cf. [15, Chapter 3]): For each partition λ of n , there is a simple right $\mathcal{H}_n$ -module S^λ called the *Specht module*, and these Specht modules constitute all the simple right $\mathcal{H}_n$ -modules up to isomorphism. The regular right $\mathcal{H}_n$ -module $\mathcal{H}_n$ can be decomposed as

$$\mathcal{H}_n \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}, \tag{5.1}$$

where f^λ denotes the number of standard Young tableaux of shape λ . See also [11, Theorem 5.2] for a proof in the case when $\mathbf{k}$ is a field of characteristic 0 and q is either an indeterminate or a positive real. (This case is sufficient for our needs, since our results — that is, Theorems 5.15, 5.16 and 5.17 — can be automatically extended from it to the general case.)

5.1. Notation. We follow the definitions and notations in [1] for Specht modules and related concepts, which we briefly recall here.

5.1.1. (Skew) partitions and tableaux.

- A *partition* is a finite weakly decreasing tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers. We identify it with the infinite sequence $(\lambda_1, \lambda_2, \dots, \lambda_k, 0, 0, 0, \dots)$. The notation “ $\lambda \vdash n$ ” is shorthand for “ λ is a partition of n ”, which means that $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$.
- The *Young diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a table with rows of lengths $\lambda_1, \lambda_2, \dots, \lambda_k$. These rows are listed from the top (English notation) and left-aligned. Formally, this Young diagram is the set of all pairs (i, j) of positive integers satisfying $i \leq k$ and $j \leq \lambda_i$. These pairs are called the *cells* (or *boxes*) of the diagram.

- Two partitions μ and λ are said to satisfy $\mu \subseteq \lambda$ if and only if the Young diagram of μ is contained in that of λ .
- The *content* (or *diagonal index*) of a cell (i, j) is defined to be the integer $j - i$.
- If λ is a partition, then a (*Young*) *tableau* of shape λ means a filling of the Young diagram of λ with positive integers (one positive integer per cell). Formally, this means a function from the Young diagram of λ to $\{1, 2, 3, \dots\}$.
- A Young tableau $\mathfrak{t}$ with n cells is said to be *standard* if its entries are $1, 2, \dots, n$ and increase weakly left-to-right along rows and top-to-bottom down columns. The set of all standard tableaux of shape λ is denoted $\text{SYT}(\lambda)$.
- If $\mathfrak{t}$ is a standard tableau of shape $\lambda \vdash n$, and if $m \in \{0, 1, \dots, n\}$, then the *restriction* $\mathfrak{t}|_m$ of $\mathfrak{t}$ to m is the tableau defined by removing all entries larger than m from $\mathfrak{t}$ (along with the cells that they occupy). This is again a standard tableau, but now of a shape μ for some partition $\mu \vdash m$.
- A *skew partition* is a pair (λ, μ) of two partitions satisfying $\mu \subseteq \lambda$. It is stylized $\lambda \setminus \mu$. Its (*skew Young*) *diagram* is the set of all cells of the Young diagram of λ that do not belong to the Young diagram of μ .
- A *skew tableau* of shape $\lambda \setminus \mu$ (where $\lambda \setminus \mu$ is a skew diagram) is a filling of this skew Young diagram. Standard skew tableaux are defined in the obvious way, and are filled with the integers $j + 1, j + 2, \dots, n$ if $\mu \vdash j$ and $\lambda \vdash n$.
- A skew partition $\lambda \setminus \mu$ is said to be a *horizontal strip* if its skew diagram has no two cells in the same column.
- Given two partitions λ and μ , we write “ $\mu \leq \lambda$ ” meaning “the Young diagram of μ is the Young diagram of λ with one cell removed”. In other words, it means that $\mu \subseteq \lambda$ and the skew diagram $\lambda \setminus \mu$ consists of a single box.

5.1.2. Hecke algebra.

- Each partition λ of n determines an idempotent $p_\lambda \in \mathcal{H}_n$, called the *isotypic projector*. Each standard tableau $\mathfrak{t} \in \text{SYT}(\lambda)$ determines an idempotent $p_{\mathfrak{t}} \in \mathcal{H}_n$, called the *Young idempotent* (or *seminormal idempotent*). For details, see [1, Section 2.2.1].
- Every partition λ of n corresponds to a right $\mathcal{H}_n$ -module W^λ , called the (*word*) *Young module*. As a vector space, it is the span of all words of length n and content λ (that is, with λ_i letters i for each positive integer i). The $\mathcal{H}_n$ -action on W^λ is specified in [1, (2.9)].
- To every partition λ of n corresponds a simple right $\mathcal{H}_n$ -module, called the *Specht module* and denoted by S^λ . It is defined as the submodule of the word module W^λ spanned by the *Young seminormal units* $w_{\mathfrak{t}} = \text{word}(\mathfrak{t}) p_{\mathfrak{t}}$. See [1, Section 2.2.2].
- Let $\lambda \setminus \mu$ be a skew partition. Let $c_1, c_2, \dots, c_k$ be the cells of its Young diagram, listed row by row from top to bottom, where the cells in each given row are listed from left to right. Let $r_1, r_2, \dots, r_k$ be the numbers of the rows in which the respective cells $c_1, c_2, \dots, c_k$ lie. The linear map $\Phi_{\lambda \setminus \mu} : W^\mu \rightarrow W^\lambda$ sends each word $w \in W^\mu$ to the concatenation $w r_1 r_2 \dots r_k$. We write this map $\Phi_{\lambda \setminus \mu}$ on the right of its argument – i.e., we write $a \Phi_{\lambda \setminus \mu}$ rather than $\Phi_{\lambda \setminus \mu}(a)$.
- Furthermore, we define the skew tableau $\mathfrak{t}^{\lambda \setminus \mu}$ of shape $\lambda \setminus \mu$ by filling the numbers $j + 1, j + 2, \dots, n$ (where $\lambda \vdash n$ and $\mu \vdash j$) into the respective cells $c_1, c_2, \dots, c_k$. (See Example 5.7 for an illustration.)
- Finally, we define $p_{\lambda \setminus \mu} \in \mathcal{H}_{|\lambda|}$ to be the element $p_{\mathfrak{t}^{\lambda \setminus \mu}}$ from [1, Definition 2.14].

5.1.3. q -integers.

- We extend our definition of the q -integers $[k]_q$ to the case of negative k by setting

$$[k]_q = (1 - q^k) / (1 - q).$$

For negative k , this is no longer a polynomial, but rather a Laurent polynomial in q , and explicitly equals $-q^{-1} - q^{-2} - \dots - q^k$.

These “negative” q -integers will not end up in our explicit formula for the eigenvalues of $\mathcal{R}_{n,k}$, but they can appear in some of the intermediate results below, such as Lemma 5.1.

5.2. Eigenvector construction: adding a single box. As in [1], [10], and [14], the heart of our eigenvector construction involves building Specht modules S^λ from $\lambda' \triangleleft \lambda$. Lemma 5.1 will be an important tool in this regard.

Lemma 5.1. *Let $\lambda' \vdash n-1$ and $\lambda \vdash n$ be such that $\lambda' \triangleleft \lambda$. Let $c_{\lambda \setminus \lambda'}$ be the content of the box in $\lambda \setminus \lambda'$, and r be its row. Let Φ_r be the operator $\Phi_{\lambda \setminus \lambda'}$ (which simply appends the letter r to the end of each word). Let $v \in S^{\lambda'}$. Then,*

$$v \Phi_r p_\lambda J_n = [c_{\lambda \setminus \lambda'}]_q v \Phi_r p_\lambda.$$

Proof. Recall that the Specht module $S^{\lambda'}$ has a basis consisting of the $w_{\mathbf{t}'}$, where $\mathbf{t}'$ ranges over all standard Young tableaux of shape λ' . By linearity, we can, without loss of generality, assume that v is one of these basis vectors. Thus, $v = w_{\mathbf{t}'}$ for some $\mathbf{t}' \in \text{SYT}(\lambda')$. Consider this $\mathbf{t}'$. Extend this standard tableau $\mathbf{t}'$ to a standard tableau $\mathbf{t}$ of shape λ by putting the number n into the remaining box of λ (the one that is not in λ'). Then, [1, Proposition 3.8] yields

$$w_{\mathbf{t}} = w_{\mathbf{t}'} \Phi_r p_\lambda, \tag{5.2}$$

whereas [1, Theorem 2.22 (2)] yields

$$w_{\mathbf{t}} J_n = [c_{\lambda \setminus \lambda'}]_q w_{\mathbf{t}}. \tag{5.3}$$

Using (5.2), we can rewrite (5.3) as

$$w_{\mathbf{t}'} \Phi_r p_\lambda J_n = [c_{\lambda \setminus \lambda'}]_q w_{\mathbf{t}'} \Phi_r p_\lambda.$$

Since $v = w_{\mathbf{t}'}$, this is precisely the claim of Lemma 5.1. $\square$

We next use Theorem 3.6 to show how one obtains an eigenvector for $\mathcal{R}_{n,k}$ from a joint eigenvector of $\mathcal{R}_{n-1,k}$ and $\mathcal{R}_{n-1,k-1}$.

Proposition 5.2. *Let $\lambda' \vdash n-1$ and $\lambda \vdash n$ be such that $\lambda' \triangleleft \lambda$. Let $c_{\lambda \setminus \lambda'}$ be the content of the box in $\lambda \setminus \lambda'$, and r be its row. Finally, let $1 \leq k \leq n$.*

If $v \in S^{\lambda'}$ is an eigenvector³ of $\mathcal{R}_{n-1,k}$ with eigenvalue ε_k and of $\mathcal{R}_{n-1,k-1}$ with eigenvalue ε_{k-1} , then

$$v \Phi_r \mathcal{B}_n p_\lambda$$

is in S^λ and is an eigenvector⁴ for $\mathcal{R}_{n,k}$ with eigenvalue

$$q^k \varepsilon_k + [n+1-k+c_{\lambda \setminus \lambda'}]_q \varepsilon_{k-1}.$$

Proof. To see that $v \Phi_r \mathcal{B}_n p_\lambda \in S^\lambda$, see [1, Theorem 5.4]. It remains to prove the eigenvector claim.

Recall that p_λ is central in $\mathcal{H}_n$, hence commutes with both $\mathcal{B}_n$ and $\mathcal{R}_{n,k}$. Thus,

$$\begin{aligned} (v \Phi_r \mathcal{B}_n p_\lambda) \mathcal{R}_{n,k} &= v \Phi_r (\mathcal{B}_n \mathcal{R}_{n,k}) p_\lambda \\ &= v \Phi_r \left(q^k \mathcal{R}_{n-1,k} + ([n+1-k]_q + q^{n+1-k} J_n) \mathcal{R}_{n-1,k-1} \right) \mathcal{B}_n p_\lambda \end{aligned}$$

by Theorem 3.6. We need to show that this equals

$$(q^k \varepsilon_k + [n+1-k+c_{\lambda \setminus \lambda'}]_q \varepsilon_{k-1}) v \Phi_r \mathcal{B}_n p_\lambda.$$

Since p_λ commutes with $\mathcal{B}_n$, this is tantamount to showing that

$$\begin{aligned} v \Phi_r \left(q^k \mathcal{R}_{n-1,k} + ([n+1-k]_q + q^{n+1-k} J_n) \mathcal{R}_{n-1,k-1} \right) p_\lambda \mathcal{B}_n \\ = (q^k \varepsilon_k + [n+1-k+c_{\lambda \setminus \lambda'}]_q \varepsilon_{k-1}) v \Phi_r p_\lambda \mathcal{B}_n. \end{aligned}$$

³We do not require eigenvectors to be nonzero. Thus, the zero vector is an eigenvector of any linear map with any eigenvalue. As a consequence, when we say that some vector is an eigenvector of some operator with eigenvalue z , we are not actually saying that z is an eigenvalue of the operator; this conclusion can only be drawn if our vector is nonzero.

⁴As we said, we allow eigenvectors to be zero. This is a real possibility here, even when $v \neq 0$.

It will suffice to show that

$$\begin{aligned} v \Phi_r \left(q^k \mathcal{R}_{n-1,k} + ([n+1-k]_q + q^{n+1-k} J_n) \mathcal{R}_{n-1,k-1} \right) p_\lambda \\ = (q^k \varepsilon_k + [n+1-k + c_{\lambda \setminus \lambda'}]_q \varepsilon_{k-1}) v \Phi_r p_\lambda, \end{aligned} \quad (5.4)$$

because this equality will yield our claim upon right multiplication with $\mathcal{B}_n$.

To prove (5.4), we recall that $\mathcal{R}_{n-1,k-1}$ commutes with J_n (by Lemma 2.16), and furthermore that $\mathcal{R}_{n-1,k-1}$ and $\mathcal{R}_{n-1,k}$ commute with the operator Φ_r . Hence, distributivity yields

$$\begin{aligned} v \Phi_r \left(q^k \mathcal{R}_{n-1,k} + ([n+1-k]_q + q^{n+1-k} J_n) \mathcal{R}_{n-1,k-1} \right) p_\lambda \\ = q^k v \mathcal{R}_{n-1,k} \Phi_r p_\lambda + [n+1-k]_q v \mathcal{R}_{n-1,k-1} \Phi_r p_\lambda \\ + q^{n+1-k} v \Phi_r J_n \mathcal{R}_{n-1,k-1} p_\lambda \\ = q^k \varepsilon_k v \Phi_r p_\lambda + [n+1-k]_q \varepsilon_{k-1} v \Phi_r p_\lambda \\ + q^{n+1-k} v \Phi_r J_n \mathcal{R}_{n-1,k-1} p_\lambda, \end{aligned} \quad (5.5)$$

since the eigenvector condition on v yields $v \mathcal{R}_{n-1,k} = \varepsilon_k v$ and $v \mathcal{R}_{n-1,k-1} = \varepsilon_{k-1} v$. But

$$\begin{aligned} v \Phi_r J_n \mathcal{R}_{n-1,k-1} p_\lambda &= v \Phi_r p_\lambda J_n \mathcal{R}_{n-1,k-1} \quad (\text{since } p_\lambda \text{ is central}) \\ &= [c_{\lambda \setminus \lambda'}]_q v \Phi_r p_\lambda \mathcal{R}_{n-1,k-1} \quad (\text{by Lemma 5.1}) \\ &= [c_{\lambda \setminus \lambda'}]_q v \mathcal{R}_{n-1,k-1} \Phi_r p_\lambda \\ &\quad (\text{since } p_\lambda \text{ is central, and } \mathcal{R}_{n-1,k-1} \text{ commutes with } \Phi_r) \\ &= [c_{\lambda \setminus \lambda'}]_q \varepsilon_{k-1} v \Phi_r p_\lambda \quad (\text{since } v \mathcal{R}_{n-1,k-1} = \varepsilon_{k-1} v). \end{aligned}$$

Hence, (5.5) rewrites as

$$\begin{aligned} v \Phi_r \left(q^k \mathcal{R}_{n-1,k} + ([n+1-k]_q + q^{n+1-k} J_n) \mathcal{R}_{n-1,k-1} \right) p_\lambda \\ = q^k \varepsilon_k v \Phi_r p_\lambda + [n+1-k]_q \varepsilon_{k-1} v \Phi_r p_\lambda + q^{n+1-k} [c_{\lambda \setminus \lambda'}]_q \varepsilon_{k-1} v \Phi_r p_\lambda \\ = (q^k \varepsilon_k + ([n+1-k]_q + q^{n+1-k} [c_{\lambda \setminus \lambda'}]_q) \varepsilon_{k-1}) v \Phi_r p_\lambda \\ = (q^k \varepsilon_k + [n+1-k + c_{\lambda \setminus \lambda'}]_q \varepsilon_{k-1}) v \Phi_r p_\lambda, \end{aligned}$$

since

$$[n+1-k]_q + q^{n+1-k} [c_{\lambda \setminus \lambda'}]_q = [n+1-k + c_{\lambda \setminus \lambda'}]_q.$$

This proves (5.4), thus completing the proof. $\square$

5.3. Nested kernels and images. Recall that elements of $\mathcal{H}_n$ act from the right on $\mathcal{H}_n$ by multiplication. Thus, for any $x \in \mathcal{H}_n$, we let

- $\ker x$ denote the kernel of the right action of x on $\mathcal{H}_n$, that is, the set of all $y \in \mathcal{H}_n$ such that $yx = 0$;
- $\text{im } x$ denote the image of this right action, i.e., the left ideal $\mathcal{H}_n x$ of $\mathcal{H}_n$.

In the “positive case” (the case when q is a positive real), the kernels and images of $\mathcal{B}_{n,k}$, $\mathcal{B}_{n,k}^*$ and $\mathcal{R}_{n,k}$ are related due to the following lemma:

Lemma 5.3. *Assume that $\mathbf{k} \subseteq \mathbb{R}$ and $q \in \mathbb{R}_{>0}$. Let $x \in \mathcal{H}_n$ be arbitrary. Then,*

$$\ker(x^*x) = \ker(x^*) \quad \text{and} \quad \text{im}(x^*x) = \text{im}(x).$$

Proof. From [1, Lemma 6.1 (1)], we know that

$$\text{if } a \in \mathcal{H}_n \text{ satisfies } a^*a = 0, \text{ then } a = 0. \quad (5.6)$$

This innocent-looking fact will be crucial to the present proof.

Clearly, $\ker(x^*) \subseteq \ker(x^*x)$. Let us now prove the reverse inclusion. Indeed, let $v \in \ker(x^*x)$. Then, $vx^*x = 0$, so that $(xv^*)^*(xv^*) = \underbrace{v^{**}}_{=v} x^* x v^* = \underbrace{v x^* x}_{=0} v = 0$. Hence, (5.6) (applied to $a = xv^*$) yields $xv^* = 0$.

Hence, $(xv^*)^* = 0^* = 0$. In view of $(xv^*)^* = v^{**}x^* = vx^*$, this rewrites as $vx^* = 0$. That is, $v \in \ker(x^*)$. Since we have proved this for each $v \in \ker(x^*x)$, we thus obtain $\ker(x^*x) \subseteq \ker(x^*)$. Combined with $\ker(x^*) \subseteq \ker(x^*x)$, this entails $\ker(x^*x) = \ker(x^*)$. Thus, the first half of Lemma 5.3 is proved.

It remains to show that $\text{im}(x^*x) = \text{im}(x)$. Since $\text{im}(x^*x) \subseteq \text{im}(x)$ is obvious, it will suffice to show that $\dim(\text{im}(x^*x)) \geq \dim(\text{im}(x))$.

Assume the contrary. Thus, $\dim(\text{im}(x^*x)) < \dim(\text{im}(x))$, so that $\dim(\text{im}(x)) > \dim(\text{im}(x^*x)) = \dim(\mathcal{H}_n) - \dim(\ker(x^*x))$ by the rank-nullity theorem. Hence, $\dim(\text{im}(x)) + \dim(\ker(x^*x)) > \dim(\mathcal{H}_n)$. Therefore, the subspaces $\text{im}(x)$ and $\ker(x^*x)$ of $\mathcal{H}_n$ have a nonzero vector u in common. Consider this u . Thus, $u = wx$ for some $w \in \mathcal{H}_n$ (since $u \in \text{im}(x)$) and $ux^* = 0$ (since $u \in \ker(x^*x) = \ker(x^*)$). From $u = wx$, we obtain $u^* = (wx)^* = x^*w^*$, so that $uu^* = \underbrace{ux^*}_{=0}w^* = 0$.

Set $a := u^*$. Then, $a^* = u^{**} = u$ and thus $a^*a = uu^* = 0$. Hence, (5.6) yields $a = 0$. Consequently, $a^* = 0$, so that $u = a^* = 0$, which contradicts u being nonzero. This contradiction shows that our assumption was false, and the proof of the lemma is complete. $\square$

Proposition 5.4. *Assume that $\mathbf{k} \subseteq \mathbb{R}$ and $q \in \mathbb{R}_{>0}$. Then, each $k \geq 0$ satisfies*

$$\ker \mathcal{R}_{n,k} = \ker \mathcal{B}_{n,k}^* \quad \text{and} \quad \text{im } \mathcal{R}_{n,k} = \text{im } \mathcal{B}_{n,k}.$$

Proof. Let $k \geq 0$. Recall that $\mathcal{R}_{n,k}$ equals $\mathcal{B}_{n,k}^* \mathcal{B}_{n,k}$ up to a nonzero scalar factor (by (2.7), since $q > 0$ entails $[k]!_q \neq 0$). Hence, $\ker \mathcal{R}_{n,k} = \ker(\mathcal{B}_{n,k}^* \mathcal{B}_{n,k}) = \ker \mathcal{B}_{n,k}^*$ (by the first equality in Lemma 5.3, applied to $x = \mathcal{B}_{n,k}$) and $\text{im } \mathcal{R}_{n,k} = \text{im}(\mathcal{B}_{n,k}^* \mathcal{B}_{n,k}) = \text{im } \mathcal{B}_{n,k}$ (by the second equality in Lemma 5.3, applied to $x = \mathcal{B}_{n,k}$). This proves the proposition. $\square$

The kernels and the images of the $\mathcal{R}_{n,k}$ exhibit a nesting behavior when q is a positive real. The part about the kernels will be used in the computation of eigenvalues later on (proof of Corollary 5.9).

Proposition 5.5 (Nested kernels and images). *Let $q \in \mathbf{k}$ be arbitrary.*

(a) *We have*

$$\ker \mathcal{B}_{n,0}^* \subseteq \ker \mathcal{B}_{n,1}^* \subseteq \ker \mathcal{B}_{n,2}^* \subseteq \cdots.$$

(b) *If $\mathbf{k} \subseteq \mathbb{R}$, and if $q \in \mathbb{R}_{>0}$, then*

$$\ker \mathcal{R}_{n,0} \subseteq \ker \mathcal{R}_{n,1} \subseteq \ker \mathcal{R}_{n,2} \subseteq \cdots.$$

(c) *We have*

$$\text{im } \mathcal{B}_{n,0} \supseteq \text{im } \mathcal{B}_{n,1} \supseteq \text{im } \mathcal{B}_{n,2} \supseteq \cdots.$$

(d) *If $\mathbf{k} \subseteq \mathbb{R}$, and if $q \in \mathbb{R}_{>0}$, then*

$$\text{im } \mathcal{R}_{n,0} \supseteq \text{im } \mathcal{R}_{n,1} \supseteq \text{im } \mathcal{R}_{n,2} \supseteq \cdots.$$

Proof. (a) For each $k > 0$, we have $\ker \mathcal{B}_{n,k-1}^* \subseteq \ker \mathcal{B}_{n,k}^*$ by (2.35). This proves part (a).

(b) This follows from part (a), since (under the assumptions $\mathbf{k} \subseteq \mathbb{R}$ and $q \in \mathbb{R}_{>0}$) we have $\ker \mathcal{R}_{n,k} = \ker \mathcal{B}_{n,k}^*$ for each $k \geq 0$ by Proposition 5.4.

(c) For each $k > 0$, we have $\text{im } \mathcal{B}_{n,k-1} \supseteq \text{im } \mathcal{B}_{n,k}$ by (2.34).

(d) This follows from part (c), since (under the assumptions $\mathbf{k} \subseteq \mathbb{R}$ and $q \in \mathbb{R}_{>0}$) we have $\text{im } \mathcal{R}_{n,k} = \text{im } \mathcal{B}_{n,k}$ for each $k \geq 0$ by Proposition 5.4. $\square$

5.4. The numbers $\mathfrak{d}$. We will now define a statistic on skew partitions that depends on both $\lambda \setminus \mu$ and two integers ℓ and i . These will be used to describe the eigenvalues of $\mathcal{R}_{n,k}$.

Definition 5.6. *Let $\lambda \setminus \mu$ be a skew partition with $\mu \vdash j$ and $\lambda \vdash n$.*

- *For any $j < \ell \leq n$, we let $\mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell}$ denote the content of the cell of $\mathfrak{t}^{\lambda \setminus \mu}$ that contains the entry ℓ .*
- *For any $j < \ell \leq n$ and $i \in \mathbb{Z}$, we define a constant $\mathfrak{d}_{\ell, i, \lambda \setminus \mu} \in \mathbf{k}$ by*

$$\mathfrak{d}_{\ell, i, \lambda \setminus \mu} := [\ell + 1 - i + \mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell}]_q.$$

Example 5.7. Let $\mu = (3, 1) \vdash j = 4$ and $\lambda = (5, 3, 1) \vdash n = 9$. Then,

$$t^{\lambda \setminus \mu} = \begin{array}{|c|c|c|c|c|} \hline & & & 5 & 6 \\ \hline & 7 & 8 & & \\ \hline 9 & & & & \\ \hline \end{array}$$

Thus, the contents $c_{t^{\lambda \setminus \mu}, \ell}$ for $\ell = 5, 6, 7, 8, 9$ are $3, 4, 0, 1, -2$, respectively. Hence, the respective constants $d_{\ell, i, \lambda \setminus \mu}$ are

$$\begin{aligned} \ell = 5 : & \quad [5 + 1 - i + 3]_q = [9 - i]_q, \\ \ell = 6 : & \quad [6 + 1 - i + 4]_q = [11 - i]_q, \\ \ell = 7 : & \quad [7 + 1 - i + 0]_q = [8 - i]_q, \\ \ell = 8 : & \quad [8 + 1 - i + 1]_q = [10 - i]_q, \\ \ell = 9 : & \quad [9 + 1 - i + (-2)]_q = [8 - i]_q. \end{aligned}$$

5.5. Eigenvector construction: adding multiple boxes. We now show how an eigenvector of $\mathcal{R}_{n,k}$ can be obtained from an eigenvector of smaller $\mathcal{R}_{j,k-i}$ for $j, i \leq k$. This is similar to the procedures in [1] (for $k = 1$) and [10] (for $k = 1$ and $q = 1$).

Lemma 5.8. *Let $\mu \vdash j$ and let $v \in S^\mu$ be an eigenvector of the operators $\{\mathcal{R}_{j,i} : 0 \leq i \leq k\}$ with eigenvalues $\{\varepsilon_i : 0 \leq i \leq k\}$, respectively. Fix $\lambda \vdash n$ with $\mu \subseteq \lambda$.*

Then,

$$v \Phi_{\lambda \setminus \mu} p_{t^{\lambda \setminus \mu}} \mathcal{B}_{n, n-j} \in S^\lambda$$

and is an eigenvector for $\mathcal{R}_{n,k}$ with eigenvalue

$$\sum_{i=0}^{n-j} q^{(n-j-i)k} \varepsilon_{k-i} \sum_{j < (\ell_1 < \ell_2 < \dots < \ell_i) \leq n} \prod_{m=1}^i q^{-(\ell_m - (j+m))} d_{\ell_m, k+m-i, \lambda \setminus \mu}.$$

(When $i = 0$, we interpret $\ell_1 < \ell_2 < \dots < \ell_i$ to be an empty chain of length zero, so that the $i = 0$ summand is $q^{(n-j)k} \varepsilon_k$. By convention, we also let $\varepsilon_i = 0$ for $i < 0$.)

Proof. We prove this by induction on $n - j$. The base case $n - j = 0$ is straightforward.

Let $n > j$ and let $\mu = \mu^{(0)} \triangleleft \mu^{(1)} \triangleleft \dots \triangleleft \mu^{(n-j)} = \lambda$ be the path in Young's lattice encoding the skew standard tableau $t^{\lambda \setminus \mu}$, in the sense that $\mu^{(i)}$ is obtained from $\mu^{(i-1)}$ by adding a box in the position of $i + j$ in $t^{\lambda \setminus \mu}$.

For $0 \leq i \leq k$, define

$$\nu_i := q^i \varepsilon_i + d_{j+1, i, \lambda \setminus \mu} \varepsilon_{i-1}. \quad (5.7)$$

By Proposition 5.2, the vector $v \Phi_{\mu^{(1)} \setminus \mu} \mathcal{B}_{j+1} p_{\mu^{(1)}}$ is in $S^{\mu^{(1)}}$ and is an eigenvector of the operators $\{\mathcal{R}_{j+1, i} : 1 \leq i \leq j+1\}$ with eigenvalues $\{\nu_i : 1 \leq i \leq j+1\}$. Additionally, given our convention that $\varepsilon_\ell = 0$ for $\ell < 0$ and the fact that $\mathcal{R}_{i,0}$ is the identity operator for all i , we can extend our claim to encompass zero and negative values for i ; namely, $v \Phi_{\mu^{(1)} \setminus \mu} \mathcal{B}_{j+1} p_{\mu^{(1)}}$ is an eigenvector of the operators $\{\mathcal{R}_{j+1, i} : i \leq j+1\}$ with eigenvalues $\{\nu_i : i \leq j+1\}$.

Hence, by the induction hypothesis,

$$(v \Phi_{\mu^{(1)} \setminus \mu} \mathcal{B}_{j+1} p_{\mu^{(1)}}) \Phi_{\lambda \setminus \mu^{(1)}} p_{t^{\lambda \setminus \mu^{(1)}}} \mathcal{B}_{n, n-j-1} \in S^\lambda$$

is an eigenvector for $\mathcal{R}_{n,k}$ with eigenvalue

$$\begin{aligned}
& \sum_{i=0}^{n-j-1} q^{(n-j-1-i)k} \nu_{k-i} \sum_{j+1 < (\ell_1 < \ell_2 < \dots < \ell_i) \leq n} \prod_{m=1}^i q^{-(\ell_m - (j+1+m))} \mathfrak{d}_{\ell_m, k+m-i, \lambda \setminus \mu} \\
&= \sum_{i=0}^{n-j-1} q^{(n-j-1-i)k} (q^{k-i} \varepsilon_{k-i} + \mathfrak{d}_{j+1, k-i, \lambda \setminus \mu} \varepsilon_{k-(i+1)}) \sum_{j+1 < (\ell_1 < \ell_2 < \dots < \ell_i) \leq n} \prod_{m=1}^i q^{-(\ell_m - (j+1+m))} \mathfrak{d}_{\ell_m, k+m-i, \lambda \setminus \mu} \\
&\quad \left(\text{since } \nu_{k-i} = q^{k-i} \varepsilon_{k-i} + \mathfrak{d}_{j+1, k-i, \lambda \setminus \mu} \varepsilon_{k-(i+1)} \right) \\
&= \sum_{i=0}^{n-j} \varepsilon_{k-i} \left(q^{(n-j-1-i)k} q^{k-i} \sum_{j+1 < (\ell_1 < \ell_2 < \dots < \ell_i) \leq n} \prod_{m=1}^i q^{-(\ell_m - (j+1+m))} \mathfrak{d}_{\ell_m, k+m-i, \lambda \setminus \mu} \right. \\
&\quad \left. + q^{(n-j-1-(i-1))k} \mathfrak{d}_{j+1, k-(i-1), \lambda \setminus \mu} \sum_{j+1 < (\ell_1 < \ell_2 < \dots < \ell_{i-1}) \leq n} \prod_{m=1}^{i-1} q^{-(\ell_m - (j+1+m))} \mathfrak{d}_{\ell_m, k+m-(i-1), \lambda \setminus \mu} \right) \\
&= \sum_{i=0}^{n-j} q^{(n-j-i)k} \varepsilon_{k-i} \left(\sum_{j+1 < (\ell_1 < \ell_2 < \dots < \ell_i) \leq n} q^{-i} \prod_{m=1}^i q^{-(\ell_m - (j+1+m))} \mathfrak{d}_{\ell_m, k+m-i, \lambda \setminus \mu} \right. \\
&\quad \left. + \sum_{j+1 < (\ell_1 < \ell_2 < \dots < \ell_{i-1}) \leq n} \mathfrak{d}_{j+1, k-(i-1), \lambda \setminus \mu} \prod_{m=1}^{i-1} q^{-(\ell_m - (j+1+m))} \mathfrak{d}_{\ell_m, k+m-(i-1), \lambda \setminus \mu} \right).
\end{aligned}$$

The claim then follows by observing that

$$\begin{aligned}
& \sum_{j+1 < (\ell_1 < \ell_2 < \dots < \ell_i) \leq n} q^{-i} \prod_{m=1}^i q^{-(\ell_m - (j+1+m))} \mathfrak{d}_{\ell_m, k+m-i, \lambda \setminus \mu} \\
&\quad + \sum_{j+1 < (\ell_1 < \ell_2 < \dots < \ell_{i-1}) \leq n} \mathfrak{d}_{j+1, k-(i-1), \lambda \setminus \mu} \prod_{m=1}^{i-1} q^{-(\ell_m - (j+1+m))} \mathfrak{d}_{\ell_m, k+m-(i-1), \lambda \setminus \mu} \\
&= \sum_{j < (\ell_1 < \ell_2 < \dots < \ell_i) \leq n} \prod_{m=1}^i q^{-(\ell_m - (j+m))} \mathfrak{d}_{\ell_m, k+m-i, \lambda \setminus \mu}
\end{aligned}$$

and

$$\begin{aligned}
& (v \Phi_{\mu^{(1)} \setminus \mu} \mathcal{B}_{j+1} p_{\mu^{(1)}}) \Phi_{\lambda \setminus \mu^{(1)}} p_{\mathfrak{t}^{\lambda \setminus \mu^{(1)}}} \mathcal{B}_{n, n-j-1} \\
&= (v \Phi_{\mu^{(1)} \setminus \mu} p_{\mu^{(1)}} \mathcal{B}_{j+1}) \Phi_{\lambda \setminus \mu^{(1)}} p_{\mathfrak{t}^{\lambda \setminus \mu^{(1)}}} \mathcal{B}_{n, n-j-1} \\
&\quad (\text{since } p_{\mu^{(1)}} \text{ is central in } \mathcal{H}_{j+1}) \\
&= v \Phi_{\mu^{(1)} \setminus \mu} \Phi_{\lambda \setminus \mu^{(1)}} p_{\mu^{(1)}} \mathcal{B}_{j+1} p_{\mathfrak{t}^{\lambda \setminus \mu^{(1)}}} \mathcal{B}_{n, n-j-1} \\
&\quad (\text{since the } \Phi \text{ operators commute with the Hecke algebra actions}) \\
&= v \Phi_{\lambda \setminus \mu} p_{\mu^{(1)}} \mathcal{B}_{j+1} p_{\mathfrak{t}^{\lambda \setminus \mu^{(1)}}} \mathcal{B}_{n, n-j-1} \\
&\quad (\text{since } \Phi_{\mu^{(1)} \setminus \mu} \Phi_{\lambda \setminus \mu^{(1)}} = \Phi_{\lambda \setminus \mu}) \\
&= v \Phi_{\lambda \setminus \mu} p_{\mu^{(1)}} p_{\mathfrak{t}^{\lambda \setminus \mu^{(1)}}} \mathcal{B}_{j+1} \mathcal{B}_{n, n-j-1} \\
&\quad \left(\begin{array}{l} \text{since } p_{\mathfrak{t}^{\lambda \setminus \mu^{(1)}}} \text{ commutes with all elements of } \mathcal{H}_{j+1} \\ \text{(because it is a product of central elements of } \mathcal{H}_{j+2}, \mathcal{H}_{j+3}, \dots, \mathcal{H}_n) \end{array} \right) \\
&= v \Phi_{\lambda \setminus \mu} p_{\mathfrak{t}^{\lambda \setminus \mu}} \mathcal{B}_{n, n-j} \\
&\quad (\text{since } p_{\mu^{(1)}} p_{\mathfrak{t}^{\lambda \setminus \mu^{(1)}}} = p_{\mathfrak{t}^{\lambda \setminus \mu}} \text{ and } \mathcal{B}_{j+1} \mathcal{B}_{n, n-j-1} = \mathcal{B}_{n, n-j}). \quad \square
\end{aligned}$$

When the eigenvector we are lifting is in the kernel of $\mathcal{R}_{n,1}$, we can simplify our eigenvalue formula from Lemma 5.8.

Corollary 5.9. *Let $\mu \vdash j$. Assume $q \in \mathbb{R}_{>0}$, and let $v \in S^\mu$ be in $\ker \mathcal{R}_{j,1}$. Fix $\lambda \vdash n$ and recall the constants $\mathfrak{d}_{\ell,i,\lambda \setminus \mu} \in \mathbf{k}$ defined in Definition 5.6. Then,*

$$v \Phi_{\lambda \setminus \mu} p_{\mathfrak{t}^{\lambda \setminus \mu}} \mathcal{B}_{n,n-j} \in S^\lambda$$

and is an eigenvector for $\mathcal{R}_{n,k}$ with eigenvalue

$$\mathcal{E}_{\lambda \setminus \mu}(k) := q^{nk - \binom{k}{2}} \sum_{j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n} \prod_{m=1}^k q^{-\ell_m} \mathfrak{d}_{\ell_m, m, \lambda \setminus \mu}.$$

(Note that this eigenvalue is 0 when $k > n - j$, since the sum is empty.)

Proof. By Proposition 5.5 (b), we have $v \in \ker \mathcal{R}_{j,1} \subseteq \ker \mathcal{R}_{j,i}$ for all $i \geq 1$. Thus, v is an eigenvector of the operators $\mathcal{R}_{j,1}, \mathcal{R}_{j,2}, \dots, \mathcal{R}_{j,k}$ with eigenvalue 0 each, and of course an eigenvector of the operator $\mathcal{R}_{j,0}$ with eigenvalue 1 (since $\mathcal{R}_{j,0} = 1$). Thus, by Lemma 5.8 (applied to $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_k = 0$ and $\varepsilon_0 = 0$), the vector $v \Phi_{\lambda \setminus \mu} p_{\mathfrak{t}^{\lambda \setminus \mu}} \mathcal{B}_{n,n-j} \in S^\lambda$ and is an eigenvector for $\mathcal{R}_{n,k}$ with eigenvalue

$$\begin{aligned} & q^{(n-j-k)k} \sum_{j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n} \prod_{m=1}^k q^{-(\ell_m - (j+m))} \mathfrak{d}_{\ell_m, m, \lambda \setminus \mu} \\ &= q^{nk - \binom{k}{2}} \sum_{j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n} \prod_{m=1}^k q^{-\ell_m} \mathfrak{d}_{\ell_m, m, \lambda \setminus \mu}. \end{aligned} \quad \square$$

Example 5.10. When $k = 1$, Corollary 5.9 recovers the eigenvalues of $\mathcal{R}_{n,1}$ found in [1]. Specifically, when $k = 1$, we can rewrite

$$q^{nk - \binom{k}{2}} \sum_{j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n} \prod_{m=1}^k q^{-\ell_m} \mathfrak{d}_{\ell_m, m, \lambda \setminus \mu} = q^n \sum_{\ell=j+1}^n q^{-\ell} \mathfrak{d}_{\ell,1,\lambda \setminus \mu} = \sum_{\ell=j+1}^n q^{n-\ell} [\ell + c_{\mathfrak{t}^{\lambda \setminus \mu}, \ell}]_q,$$

which agrees with the eigenvalues of $\mathcal{R}_{n,1}$ as written in [1, Equation (7.2)].

For more examples of the formula in Corollary 5.9, see Section 5.8.

Let us give the eigenvalues in Corollary 5.9 a name: For any horizontal strip $\lambda \setminus \mu$ with $\lambda \vdash n$ and $\mu \vdash j$, and for any $k \geq 0$, we set

$$\mathcal{E}_{\lambda \setminus \mu}(k) := q^{nk - \binom{k}{2}} \sum_{j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n} \prod_{m=1}^k q^{-\ell_m} \mathfrak{d}_{\ell_m, m, \lambda \setminus \mu}, \quad (5.8)$$

where the $\mathfrak{d}_{\ell,i,\lambda \setminus \mu}$ are given in Definition 5.6.

For future reference, we record the recursion for the eigenvalues $\mathcal{E}_{\lambda \setminus \mu}(k)$ that we implicitly showed in the proof of Lemma 5.8:

Proposition 5.11. *Let $n, k \geq 0$. Let $\lambda \setminus \mu$ be a horizontal strip with $\lambda \vdash n$. Then:*

- (a) *If $k = 0$, then $\mathcal{E}_{\lambda \setminus \mu}(k) = 1$.*
- (b) *If $k > 0$ and $\mu = \lambda$, then $\mathcal{E}_{\lambda \setminus \mu}(k) = 0$.*
- (c) *If $k > 0$ and $\mu \neq \lambda$, then*

$$\mathcal{E}_{\lambda \setminus \mu}(k) = q^k \mathcal{E}_{\lambda' \setminus \mu}(k) + [n + 1 - k + c_{\lambda \setminus \lambda'}]_q \mathcal{E}_{\lambda' \setminus \mu}(k - 1),$$

where $\lambda' \vdash n - 1$ is the partition obtained from λ by removing the box containing n in $\mathfrak{t}^{\lambda \setminus \mu}$, and where $c_{\lambda \setminus \lambda'}$ is the content of this box.

Proof. (a) If $k = 0$, then the sum on the right hand side of (5.8) has only one addend, and this addend equals 1.

(b) If $k > 0$ and $\mu = \lambda$, then the sum on the right hand side of (5.8) is empty, since $j = n$.

(c) Assume that $k > 0$ and $\mu \neq \lambda$. Then, $j < n$ in (5.8). Each k -tuple $j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n$ satisfies either $\ell_k \leq n - 1$ or $\ell_k = n$. Thus, the sum on the right hand side of (5.8) can be split into two subsums. The first simplifies to $q^k \mathcal{E}_{\lambda' \setminus \mu}(k)$, while the latter rewrites as $[n + 1 - k + c_{\lambda \setminus \lambda'}]_q \mathcal{E}_{\lambda' \setminus \mu}(k - 1)$ (since $\mathfrak{d}_{n,k,\lambda \setminus \mu} = [n + 1 - k + c_{\lambda \setminus \lambda'}]_q$). $\square$

We also note the following nonnegativity property, which will later imply the nonnegativity of all eigenvalues of $\mathcal{R}_{n,k}$ for $q > 0$ (but is more general, since only some of the $\mathcal{E}_{\lambda \setminus \mu}(k)$ appear as eigenvalues with nonzero multiplicities):

Theorem 5.12. *Let $n, k \geq 0$. Let $\lambda \setminus \mu$ be a horizontal strip. Then, $\mathcal{E}_{\lambda \setminus \mu}(k)$ is a polynomial in q with nonnegative integer coefficients.*

Proof. Let $j = |\mu|$, so that $\mu \vdash j$. Then, the equality (5.8) can be rewritten as

$$\mathcal{E}_{\lambda \setminus \mu}(k) = \sum_{j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n} q^{nk - \binom{k}{2}} \prod_{m=1}^k q^{-\ell_m} \mathfrak{d}_{\ell_m, m, \lambda \setminus \mu}. \quad (5.9)$$

Hence, it remains to show

- (A) that the negative powers $q^{-\ell_m}$ in the product in (5.9) get cancelled by the $q^{nk - \binom{k}{2}}$, and
- (B) that each $\mathfrak{d}_{\ell_m, m, \lambda \setminus \mu}$ in (5.9) is a positive q -integer (i.e., has the form $[u]_q$ for some $u > 0$).

Claim (A) is easy: The condition $j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n$ under the sum entails that $\ell_1 + \ell_2 + \dots + \ell_k \leq n + (n-1) + \dots + (n-k+1) = nk - \binom{k}{2}$, which ensures that the $q^{nk - \binom{k}{2}}$ term cancels all the negative powers $q^{-\ell_m}$ in the product.

It remains to prove Claim (B). Fix a sequence $j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n$ and an $m \in \{1, 2, \dots, k\}$. We must prove that $\mathfrak{d}_{\ell_m, m, \lambda \setminus \mu}$ is a positive q -integer. By Definition 5.6, we have $\mathfrak{d}_{\ell_m, m, \lambda \setminus \mu} = [\ell_m + 1 - m + \mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell_m}]_q$. Hence, it remains to show that $\ell_m + 1 - m + \mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell_m} > 0$. But from $j < (\ell_1 < \ell_2 < \dots < \ell_k)$, we obtain $\ell_m \geq j + m$, so that $\ell_m + 1 - m \geq j + 1$. Moreover, $\mu \vdash j$ shows that the partition μ has at most j rows. Thus, λ has at most $j + 1$ rows, since $\lambda \setminus \mu$ is a horizontal strip. Therefore, any cell of λ has content $\geq 1 - (j + 1) = -j$. In particular, this shows that $\mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell_m} \geq -j$. Therefore,

$$\underbrace{\ell_m + 1 - m}_{\geq j+1} + \underbrace{\mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell_m}}_{\geq -j} \geq j + 1 + (-j) = 1 > 0, \quad (5.10)$$

which concludes the proof of Claim (B). Thus, the theorem is proved. $\square$

5.6. The eigenbasis. We will now use our eigenvector construction, as well as results in [1], to produce an *eigenbasis* for $\mathcal{R}_{n,k}$ whenever $q \in \mathbb{R}_{>0}$.

Theorem 5.13. *Assume that $\mathbf{k} \subseteq \mathbb{R}$ and $q \in \mathbb{R}_{>0}$.*

Let $\lambda \vdash n$. Fix a basis κ_μ of the vector space $\ker(\mathcal{R}_{|\mu|,1} |_{S^\mu})$ for each partition $\mu \subseteq \lambda$. For any partition $\mu \subseteq \lambda$ and any $u \in S^\mu$, we set

$$y_{\mu(u)}^\lambda := u \Phi_{\lambda \setminus \mu} \mathcal{B}_{n, n-|\mu|} p_\lambda.$$

Then, the set

$$\mathfrak{B}_\lambda := \{y_{\mu(u)}^\lambda : u \in \kappa_\mu \text{ and } \lambda \setminus \mu \text{ is a horizontal strip}\}$$

is a basis of S^λ , and every $y_{\mu(u)}^\lambda \in \mathfrak{B}_\lambda$ is an eigenvector of each $\mathcal{R}_{n,k}$ with eigenvalue $\mathcal{E}_{\lambda \setminus \mu}(k)$ defined in (5.8).

Proof. The first claim (i.e., that $\mathfrak{B}_\lambda$ is a basis) is [1, Theorem 7.2].

The second claim (i.e., that each $y_{\mu(u)}^\lambda$ is an eigenvector of $\mathcal{R}_{n,k}$ with eigenvalue as in (5.8)) follows from Corollary 5.9 (applied to $v = u \in \kappa_\mu = \ker(\mathcal{R}_{|\mu|,1} |_{S^\mu})$), since each $u \in S^\mu$ satisfies

$$\begin{aligned} y_{\mu(u)}^\lambda &= u \Phi_{\lambda \setminus \mu} \mathcal{B}_{n, n-|\mu|} p_\lambda \\ &= u \Phi_{\lambda \setminus \mu} p_\lambda \mathcal{B}_{n, n-|\mu|} \quad (\text{since } p_\lambda \text{ is central}) \\ &= u \Phi_{\lambda \setminus \mu} p_{\mathfrak{t}^{\lambda \setminus \mu}} \mathcal{B}_{n, n-|\mu|} \end{aligned}$$

because [1, Lemma 6.12] yields $u \Phi_{\lambda \setminus \mu} p_\lambda = u \Phi_{\lambda \setminus \mu} p_{\mathfrak{t}^{\lambda \setminus \mu}}$. $\square$

Note that by base change, we can generalize Theorem 5.13 from the case $\mathbf{k} \subseteq \mathbb{R}$ to the case when $\mathbf{k}$ is any field of characteristic 0, as long as we still require that $q \in \mathbb{R}_{>0}$ (embedded in $\mathbf{k}$ by a field morphism). We will have no need for this generality, though.

For the sake of reference, we also mention an obvious consequence of Theorem 5.13 and (5.1):

Corollary 5.14. *Assume that $\mathbf{k} \subseteq \mathbb{R}$ and $q \in \mathbb{R}_{>0}$. Then, there is a basis of $\mathcal{H}_n$ that is an eigenbasis for all the operators $\mathcal{R}_{n,0}, \mathcal{R}_{n,1}, \mathcal{R}_{n,2}, \dots$*

5.7. The eigenvalues of $\mathcal{R}_{n,k}$. For the rest of this section, we no longer make any assumptions on q , and allow q to be any scalar, including the “degenerate” cases $q = 0$ and $[n]!_q = 0$.

We are now ready to describe the spectrum of the k -random-to-random shuffles $\mathcal{R}_{n,k}$ acting on the Specht modules S^λ .

To do so, we need the following definitions. If $\mathbf{t}$ is a standard tableau with n entries, then the *descents* of $\mathbf{t}$ are the numbers $i \in [n-1]$ that appear further north (and thus weakly east) than $i+1$ in $\mathbf{t}$. The set of all descents of $\mathbf{t}$ is denoted by $\text{Des}(\mathbf{t})$. A standard tableau $\mathbf{t}$ with n entries is said to be a *desarrangement tableau* if the smallest element of $[n] \setminus \text{Des}(\mathbf{t})$ is even. For instance, here are the five standard tableaux $\mathbf{t}$ of shape $(2, 2, 1)$ with their descent sets $\text{Des}(\mathbf{t})$:

t	<table><tr><td>1</td><td>4</td></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td></td></tr></table>	1	4	2	5	3		<table><tr><td>1</td><td>3</td></tr><tr><td>2</td><td>5</td></tr><tr><td>4</td><td></td></tr></table>	1	3	2	5	4		<table><tr><td>1</td><td>3</td></tr><tr><td>2</td><td>4</td></tr><tr><td>5</td><td></td></tr></table>	1	3	2	4	5		<table><tr><td>1</td><td>2</td></tr><tr><td>3</td><td>5</td></tr><tr><td>4</td><td></td></tr></table>	1	2	3	5	4		<table><tr><td>1</td><td>2</td></tr><tr><td>3</td><td>4</td></tr><tr><td>5</td><td></td></tr></table>	1	2	3	4	5	
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dt?	no	yes	yes	no	no																														

(where “dt” stands for “desarrangement tableau”).

Theorem 5.15. *Let $n, k \geq 0$. Let $\lambda \vdash n$. Consider the linear endomorphism of the Specht module S^λ given by the action of the element $\mathcal{R}_{n,k}$. The characteristic polynomial of this endomorphism⁵ is*

$$\prod_{j=0}^n \prod_{\substack{\lambda \setminus \mu \text{ is a} \\ \text{horizontal strip;} \\ |\mu|=j}} (x - \mathcal{E}_{\lambda \setminus \mu}(k))^{d^\mu},$$

where d^μ is the number of desarrangement tableaux of shape μ . Here, the $\mathcal{E}_{\lambda \setminus \mu}(k)$ are as in (5.8).

Proof. We proceed similarly to the proof of [1, Lemma 2.5 (1)]. The claim we are trying to prove is a family of intricate polynomial identities in q . This is because each Specht module S^λ has a basis defined uniformly for all $q \in \mathbf{k}$ —see [11, Theorem 5.6]—and therefore the action of $\mathcal{R}_{n,k}$ on S^λ can be described by a matrix whose entries are rational functions in q ; moreover, Theorem 5.12 shows that the $\mathcal{E}_{\lambda \setminus \mu}(k)$ are polynomials in q as well. Thus, we can assume without loss of generality that $\mathbf{k} = \mathbb{R}$ and $q \in \mathbb{R}_{>0}$, since two polynomials in $\mathbb{Z}[q]$ that are equal for all $q \in \mathbb{R}_{>0}$ must be identical. Under this assumption, the set $\mathfrak{B}_\lambda$ constructed in Theorem 5.13 is a basis of S^λ , and is in fact an eigenbasis for $\mathcal{R}_{n,k}$, with eigenvalues as given in Theorem 5.13. Thus, the characteristic polynomial of $\mathcal{R}_{n,k}$ (or, more precisely, of the action of $\mathcal{R}_{n,k}$ on S^λ) is the product $\prod (x - \varepsilon)$, where ε ranges over these eigenvalues. This gives precisely the formula claimed in Theorem 5.15, once we remember that the size of the basis κ_μ is $\dim \ker(\mathcal{R}_{|\mu|,1}|_{S^\mu}) = d^\mu$ by [1, Proposition 7.1]. $\square$

Since the whole Hecke algebra $\mathcal{H}_n$ decomposes as a direct sum of Specht modules, we can now also describe the spectrum of $\mathcal{R}_{n,k}$ acting on the whole of $\mathcal{H}_n$:

Theorem 5.16. *Let $n, k \geq 0$. Then, the characteristic polynomial of the element $\mathcal{R}_{n,k}$ (acting on $\mathcal{H}_n$ by right multiplication) is*

$$\prod_{j=0}^n \prod_{\substack{\lambda \setminus \mu \text{ is a} \\ \text{horizontal strip;} \\ |\lambda|=n; |\mu|=j}} (x - \mathcal{E}_{\lambda \setminus \mu}(k))^{d^\mu f^\lambda},$$

where f^λ is the number of standard tableaux of shape λ , where d^μ is the number of desarrangement tableaux of shape μ . Here, the $\mathcal{E}_{\lambda \setminus \mu}(k)$ are as in (5.8).

Proof. If q is neither zero nor a root of unity, then this follows from Theorem 5.15, due to the decomposition (5.1). The general case follows from this by a similar polynomiality argument as in Theorem 5.15. $\square$

⁵We are using x for the indeterminate.

Theorem 5.17. *Let $n, k \geq 0$. Then, all eigenvalues of the element $\mathcal{R}_{n,k}$ acting on $\mathcal{H}_n$ by right multiplication (or on a Specht module S^λ) are polynomials in q with nonnegative integer coefficients.*

Proof. By Theorem 5.15 and Theorem 5.16, all these eigenvalues have the form $\mathcal{E}_{\lambda \setminus \mu}(k)$. Hence, the claim follows from Theorem 5.12. $\square$

5.8. Tables of eigenvalues. We shall now use Theorem 5.16 and (5.8) to explicitly determine the eigenvalues $\mathcal{E}_{\lambda \setminus \mu}(k)$ of $\mathcal{R}_{n,k}$ for some small values of n .

Example 5.18. The eigenvalues and multiplicities of $\mathcal{R}_{n,k}$ for $n = 3$ and $k \in \{1, 2, 3\}$ are shown in Table 1.

Shape $\lambda \setminus \mu$	Multiplicity of $\mathcal{E}_{\lambda \setminus \mu}(k)$	$\mathcal{E}_{\lambda \setminus \mu}(1)$	$\mathcal{E}_{\lambda \setminus \mu}(2)$	$\mathcal{E}_{\lambda \setminus \mu}(3)$
$(3) \setminus \emptyset$	1	$[3]_q^2$	$[2]!_q \cdot [3]_q^2$	$[3]!_q$
$(2, 1) \setminus (2, 1)$	2	0	0	0
$(2, 1) \setminus (1, 1)$	2	$[4]_q$	0	0
$(1, 1, 1) \setminus (1, 1)$	1	$[1]_q$	0	0

TABLE 1. The eigenvalues of $\mathcal{R}_{n,k}$ when $n = 3$ for $k \in \{1, 2, 3\}$.

We explain each entry in Table 1.

- $\lambda \setminus \mu = (3) \setminus \emptyset$: This eigenvalue has multiplicity $d^\emptyset f^{(3)} = 1 \cdot 1 = 1$. The relevant values of $\mathfrak{d}_{\ell, m, \lambda \setminus \mu}$ are given by the following table:

	$\ell = 1$	$\ell = 2$	$\ell = 3$
$m = 1$	$[1]_q$	$[3]_q$	$[5]_q$
$m = 2$		$[2]_q$	$[4]_q$
$m = 3$			$[3]_q$

Hence, the eigenvalue $\mathcal{E}_{(3) \setminus \emptyset}(k)$ of $\mathcal{R}_{n,k}$ indexed by $(3) \setminus \emptyset$ is:

k	$\mathcal{E}_{(3) \setminus \emptyset}(k)$	Simplified expression
1	$q^3 \left(\underbrace{q^{-1}[1]_q}_{\ell_1=1} + \underbrace{q^{-2}[3]_q}_{\ell_1=2} + \underbrace{q^{-3}[5]_q}_{\ell_1=3} \right)$	$[3]_q^2$
2	$q^5 \left(\underbrace{q^{-3}[1]_q[2]_q}_{\ell_1, \ell_2=1,2} + \underbrace{q^{-4}[1]_q[4]_q}_{\ell_1, \ell_2=1,3} + \underbrace{q^{-5}[3]_q[4]_q}_{\ell_1, \ell_2=2,3} \right)$	$[2]!_q \cdot [3]_q^2$
3	$q^6 \left(\underbrace{q^{-6}[1]_q[2]_q[3]_q}_{\ell_1, \ell_2, \ell_3=1,2,3} \right)$	$[3]!_q$

- $\lambda \setminus \mu = (2, 1) \setminus (1, 1)$: This eigenvalue has multiplicity $d^{(1,1)} f^{(2,1)} = 1 \cdot 2 = 2$. The only relevant value of $\mathfrak{d}_{\ell, m, \lambda \setminus \mu}$ is

$$\mathfrak{d}_{3,1,\lambda \setminus \mu} = [4]_q.$$

Note that the sum in the eigenvalue formula is empty unless $k = 1$. Hence, the eigenvalue $\mathcal{E}_{(2,1) \setminus (1,1)}(k)$ of $\mathcal{R}_{n,k}$ indexed by $(2, 1) \setminus (1, 1)$ is:

$$\mathcal{E}_{(2,1) \setminus (1,1)}(k) = \begin{cases} q^3 \left(\underbrace{q^{-3}[4]_q}_{\ell_1=3} \right) = [4]_q & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- $\lambda \setminus \mu = (2, 1) \setminus (2, 1)$: This eigenvalue has multiplicity $d^{(2,1)} \cdot f^{(2,1)} = 1 \cdot 2 = 2$. Since the interval $(3, 3)$ is empty, the sum in the eigenvalue formula is always empty and so the eigenvalue is

$$\mathcal{E}_{(2,1) \setminus (2,1)}(k) = 0 \quad \text{for all } k.$$

- $\lambda \setminus \mu = (1, 1, 1) \setminus (1, 1)$: This eigenvalue has multiplicity $d^{(1,1,1)} \cdot f^{(1,1,1)} = 1 \cdot 1 = 1$. The only relevant value of $\mathfrak{d}_{\ell, m, \lambda \setminus \mu}$ is

$$\mathfrak{d}_{3,1,\lambda \setminus \mu} = 1.$$

Note that the sum in the eigenvalue formula is empty unless $k = 1$. Hence, the eigenvalue is

$$\mathcal{E}_{(1,1,1) \setminus (1,1)}(k) = \begin{cases} q^3 \left(\underbrace{q^{-3}[1]_q}_{\ell_1=3} \right) = [1]_q & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.19. We use Theorem 5.16 and (5.8) to compute the eigenvalues of $\mathcal{R}_{4,k}$ in Table 2.

Shape $\lambda \setminus \mu$	Mult. of $\mathcal{E}_{\lambda \setminus \mu}(k)$	$\mathcal{E}_{\lambda \setminus \mu}(1)$	$\mathcal{E}_{\lambda \setminus \mu}(2)$	$\mathcal{E}_{\lambda \setminus \mu}(3)$	$\mathcal{E}_{\lambda \setminus \mu}(4)$
$(4) \setminus \emptyset$	1	$[4]_q^2$	$[4]_q^2 \cdot [2]_q!$	$[4]_q^2 \cdot [3]_q!$	$[4]_q!$
$(3, 1) \setminus (3, 1)$	3	0	0	0	0
$(3, 1) \setminus (2, 1)$	3	$[6]_q$	0	0	0
$(3, 1) \setminus (1, 1)$	3	$q[4]_q + [6]_q$	$[4]_q[5]_q$	0	0
$(2, 2) \setminus (2, 2)$	2	0	0	0	0
$(2, 2) \setminus (2, 1)$	2	$[4]_q$	0	0	0
$(2, 1, 1) \setminus (2, 1, 1)$	3	0	0	0	0
$(2, 1, 1) \setminus (2, 1)$	3	$[2]_q$	0	0	0
$(2, 1, 1) \setminus (1, 1)$	3	$q[4]_q + [2]_q$	$[4]_q[1]_q$	0	0
$(1, 1, 1, 1) \setminus (1, 1, 1, 1)$	1	0	0	0	0

TABLE 2. Eigenvalues for $\mathcal{R}_{n,k}$ when $n = 4$ and $k \in \{1, 2, 3, 4\}$.

Example 5.20. For $n = 5$ and $\lambda \setminus \mu = (2, 2, 1) \setminus (2, 1)$, Theorem 5.15 yields that $\mathcal{R}_{5,1}$ has an eigenvalue

$$q^5 \sum_{3 < \ell_1 \leq 5} q^{-\ell_1} \mathfrak{d}_{\ell_1, 1, \lambda \setminus \mu} = q^5 (q^{-4} \mathfrak{d}_{4,1,\lambda \setminus \mu} + q^{-5} \mathfrak{d}_{5,1,\lambda \setminus \mu}) = q[4]_q + [3]_q = q^4 + q^3 + 2q^2 + 2q + 1$$

(with multiplicity $d^{(2,1)} = 1$ on S^λ). As a polynomial in q , this has a non-abelian Galois group over $\mathbb{Q}$ and two roots outside of the unit circle. Thus, it is not a product of quotients of q -integers, showing that the eigenvalues of $\mathcal{R}_{n,k}$ do not always have expressions as products of q -integers. The same behavior appears for $\lambda \setminus \mu = (3, 2) \setminus (2, 1)$ and for $\lambda \setminus \mu = (4, 1, 1) \setminus (1, 1)$.

Example 5.21. We use Theorem 5.16 and (5.8) to compute the eigenvalues of $\mathcal{R}_{5,k}$ in Table 3.

Shape $\lambda \setminus \mu$	Mult.	$\mathcal{E}_{\lambda \setminus \mu}(1)$	$\mathcal{E}_{\lambda \setminus \mu}(2)$	$\mathcal{E}_{\lambda \setminus \mu}(3)$	$\mathcal{E}_{\lambda \setminus \mu}(4)$	$\mathcal{E}_{\lambda \setminus \mu}(5)$
$(5) \setminus \emptyset$	1	$[5]_q^2$	$[2]_q! [5]_q^2$	$[3]_q! [5]_q^2$	$[4]_q! [5]_q^2$	$[5]_q!$
$(4, 1) \setminus (4, 1)$	4	0	0	0	0	0
$(4, 1) \setminus (3, 1)$	4	$[8]_q$	0	0	0	0
$(4, 1) \setminus (2, 1)$	4	$q[6]_q + [8]_q$	$[6]_q[7]_q$	0	0	0
$(4, 1) \setminus (1^2)$	4	$q^2[4]_q + q[6]_q + [8]_q$	$q^2[4]_q[5]_q + q[4]_q[7]_q + [6]_q[7]_q$	$[4]_q[5]_q[6]_q$	0	0
$(3, 2) \setminus (3, 2)$	10	0	0	0	0	0
$(3, 2) \setminus (2, 2)$	5	$[7]_q$	0	0	0	0
$(3, 2) \setminus (3, 1)$	5	$[5]_q$	0	0	0	0
$(3, 2) \setminus (2, 1)$	5	$q[6]_q + [5]_q$	$[6]_q[4]_q$	0	0	0
$(3, 1^2) \setminus (3, 1^2)$	12	0	0	0	0	0
$(3, 1^2) \setminus (2, 1^2)$	6	$[7]_q$	0	0	0	0
$(3, 1^2) \setminus (3, 1)$	6	$[3]_q$	0	0	0	0
$(3, 1^2) \setminus (2, 1)$	6	$q[6]_q + [3]_q$	$[6]_q[2]_q$	0	0	0
$(3, 1^2) \setminus (1^2)$	6	$q^2[4]_q + q[6]_q + [3]_q$	$q^2[4]_q[5]_q + q[4]_q[2]_q + [6]_q[2]_q$	$[4]_q[5]_q[1]_q$	0	0
$(2^2, 1) \setminus (2^2, 1)$	10	0	0	0	0	0
$(2^2, 1) \setminus (2, 2)$	5	$[3]_q$	0	0	0	0
$(2^2, 1) \setminus (2, 1^2)$	5	$[5]_q$	0	0	0	0
$(2^2, 1) \setminus (2, 1)$	5	$q[4]_q + [3]_q$	$[4]_q[2]_q$	0	0	0
$(2, 1^3) \setminus (2, 1^3)$	8	0	0	0	0	0
$(2, 1^3) \setminus (1^4)$	4	$[6]_q$	0	0	0	0
$(2, 1^3) \setminus (2, 1^2)$	4	$[2]_q$	0	0	0	0
$(1^5) \setminus (1^4)$	1	$[1]_q$	0	0	0	0

TABLE 3. Eigenvalues of $\mathcal{R}_{n,k}$ for $n = 5$ and $k \in \{1, 2, 3, 4, 5\}$.

5.9. Product formulas for horizontal strips from hooks. The formula (5.8) for the eigenvalues $\mathcal{E}_{\lambda \setminus \mu}(k)$ in Theorem 5.15 does not appear to succumb to simplification in general. However, when the shape λ is one of the partitions (n) and $(n-1, 1)$, the eigenvalues can be computed explicitly in terms of q -binomial coefficients, with no need for sums. This leads to the formulas in Theorem C, which we shall prove now (in Propositions 5.22, 5.23, 5.24, 5.25).

We recall that q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad \text{for all } n, k \in \mathbb{N},$$

where we understand $[m]_q!$ to be ∞ for all negative m (and any fraction with ∞ in its denominator is 0).

We begin with describing the action of $\mathcal{R}_{n,k}$ on the Specht module $S^{(n)}$. We recall that this Specht module is 1-dimensional, and is the q -analogue of the trivial representation of $\mathfrak{S}_n$. The action of $\mathcal{H}_n$ on it is given by the rule $vT_w = q^{\ell(w)}v$ for each $w \in \mathfrak{S}_n$ and each $v \in S^{(n)}$ (see, e.g., [1, Example 2.24]). Thus, the following should come as no surprise:

Proposition 5.22. *Let $k \geq 0$. The only eigenvalue of $\mathcal{R}_{n,k}$ acting on the Specht module $S^{(n)}$ is*

$$\mathcal{E}_{(n) \setminus \emptyset}(k) = [k]!_q \left[\begin{matrix} n \\ k \end{matrix} \right]_q^2.$$

Proof. We read off the eigenvalues from the characteristic polynomial in Theorem 5.15. The partitions μ such that $(n) \setminus \mu$ is a horizontal strip are precisely the partitions (j) with $0 \leq j \leq n$. But the only one of them that furthermore satisfies $d^\mu \neq 0$ is the empty partition $(0) = \emptyset$. Hence, the only eigenvalue is $\mathcal{E}_{(n) \setminus \emptyset}(k) = \mathcal{E}_{(n) \setminus \emptyset}(k)$. It remains to show that it equals $[k]!_q \left[\begin{matrix} n \\ k \end{matrix} \right]_q^2$. This can be done by induction on n using Proposition 5.11, but is probably easier to do directly by seeing how $\mathcal{R}_{n,k}$ acts on $S^{(n)}$: Pick any nonzero vector $v \in S^{(n)}$. Recall that the action of $\mathcal{H}_n$ on it is given by the rule $vT_w = q^{\ell(w)}v$ for each $w \in \mathfrak{S}_n$. In particular,

$$vT_i = qv \quad \text{for each } i \in [n-1]. \quad (5.11)$$

Thus, for each $m \in \{0, 1, \dots, n\}$, we have

$$v\mathcal{B}_m = v \sum_{i=1}^m T_{m-1}T_{m-2} \cdots T_i = \sum_{i=1}^m \underbrace{vT_{m-1}T_{m-2} \cdots T_i}_{=q^{m-i}v} = \sum_{i=1}^m q^{m-i}v = [m]_q v.$$

(by (5.11), applied many times)

Hence, easily,

$$v\mathcal{B}_{n,k} = [n]_q [n-1]_q \cdots [n-k+1]_q v,$$

and similarly

$$v\mathcal{B}_{n,k}^* = [n]_q [n-1]_q \cdots [n-k+1]_q v.$$

Thus, by the definition of $\mathcal{R}_{n,k}$, we easily get

$$v\mathcal{R}_{n,k} = \frac{1}{[k]!_q} [n]_q [n-1]_q \cdots [n-k+1]_q \cdot [n]_q [n-1]_q \cdots [n-k+1]_q v = [k]!_q \left[\begin{matrix} n \\ k \end{matrix} \right]_q^2 v.$$

This shows that the only eigenvalue of $\mathcal{R}_{n,k}$ on $S^{(n)}$ is $[k]!_q \left[\begin{matrix} n \\ k \end{matrix} \right]_q^2$. Hence, $\mathcal{E}_{(n) \setminus \emptyset}(k)$ must be $[k]!_q \left[\begin{matrix} n \\ k \end{matrix} \right]_q^2$, qed. $\square$

The next-simplest shape is the partition $(n-1, 1)$ (for $n \geq 2$). Its Specht module $S^{(n-1,1)}$ is the q -analogue of the reflection representation of $\mathfrak{S}_n$; in particular, it has dimension $n-1$. Generalizing Lafrenière's [14, Théorème 88] to arbitrary values of q , we describe the eigenvalues of $\mathcal{R}_{n,k}$ on it as follows:⁶

Proposition 5.23. *Let $k \geq 0$. The eigenvalues of $\mathcal{R}_{n,k}$ acting on the Specht module $S^{(n-1,1)}$ are the numbers*

$$\mathcal{E}_{(n-1,1) \setminus (j,1)}(k) = [k]!_q \left[\begin{matrix} n-j-1 \\ k \end{matrix} \right]_q \left[\begin{matrix} n+j \\ k \end{matrix} \right]_q \quad \text{for all } j \in [n-1].$$

Each appears with multiplicity 1, unless two of these numbers happen to coincide.

Proof. Theorem 5.15 tells us that the eigenvalues are the numbers $\mathcal{E}_{(n-1,1) \setminus \mu}(k)$ for the partitions μ such that $(n-1,1) \setminus \mu$ is a horizontal strip and such that $d^\mu \neq 0$. It is easy to see that these partitions μ are precisely the $(j,1)$ for $j \in [n-1]$ (indeed, any partition μ of length 1 would have $d^\mu = 0$, whereas the empty partition $\emptyset$ fails the “horizontal strip” test). Thus, there is one eigenvalue $\mathcal{E}_{(n-1,1) \setminus (j,1)}(k)$ for each of these partitions $(j,1)$. The multiplicity of this eigenvalue is $d^\mu = d^{(j,1)} = 1$, since there is only one desarrangement tableau of shape $(j,1)$ (the standard tableau with the entry 2 in box $(2,1)$).

It remains to show that each $j \in [n-1]$ satisfies

$$\mathcal{E}_{(n-1,1) \setminus (j,1)}(k) = [k]!_q \left[\begin{matrix} n-j-1 \\ k \end{matrix} \right]_q \left[\begin{matrix} n+j \\ k \end{matrix} \right]_q. \quad (5.12)$$

To prove (5.12), we fix j and induct on $n \geq j+1$.

The *base case* $n = j+1$ is easy: In this case, $n-1 = j$, so that the partitions $(n-1,1)$ and $(j,1)$ are equal. If $k = 0$, then Proposition 5.11 (a) yields $\mathcal{E}_{(n-1,1) \setminus (j,1)}(k) = 1$, and thus (5.12) boils down to $1 = 1$. Thus, we can restrict ourselves to the case $k > 0$. Hence, Proposition 5.11 (b) shows that the left hand side

⁶Our j corresponds to Lafrenière's $n+1-i$.

of (5.12) equals 0 (since the partitions $(n-1, 1)$ and $(j, 1)$ are equal). The same holds for the right hand side of (5.12), since the $\begin{bmatrix} n-j-1 \\ k \end{bmatrix}_q$ factor simplifies to $\begin{bmatrix} 0 \\ k \end{bmatrix}_q$ in view of $n = j + 1$.

For the *induction step*, we fix $n > j + 1$. Again, we assume that $k > 0$, since the $k = 0$ case is trivial. Then, Proposition 5.11 (c) (applied to $\lambda = (n-1, 1)$ and $\mu = (j, 1)$) yields

$$\mathcal{E}_{(n-1,1) \setminus (j,1)}(k) = q^k \mathcal{E}_{(n-2,1) \setminus (j,1)}(k) + [n+1-k+(n-2)]_q \mathcal{E}_{(n-2,1) \setminus (j,1)}(k-1)$$

(since $\lambda' = (n-2, 1)$ and $c_{\lambda \setminus \lambda'} = n-2$). This is a recurrence for the left hand side of (5.12). Thus, it suffices to show that the right hand side of (5.12) satisfies the same recurrence, as we will then be able to conclude (by the induction hypothesis) that the two sides are equal. In other words, it suffices to show that

$$\begin{aligned} & [k]!_q \begin{bmatrix} n-j-1 \\ k \end{bmatrix}_q \begin{bmatrix} n+j \\ k \end{bmatrix}_q \\ &= q^k [k]!_q \begin{bmatrix} n-1-j-1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1+j \\ k \end{bmatrix}_q + [n+1-k+(n-2)]_q [k-1]!_q \begin{bmatrix} n-1-j-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n-1+j \\ k-1 \end{bmatrix}_q. \end{aligned}$$

Upon using the q -factorial formula $\begin{bmatrix} u \\ v \end{bmatrix}_q = \frac{[u]!_q}{[v]!_q [u-v]!_q}$, this simplifies to the identity

$$[n-j-1]_q [n+j]_q = q^k [n-j-1-k]_q [n+j-k]_q + [2n-1-k]_q [k]_q,$$

which is easily verified by hand. This completes the induction step, and thus (5.12) is proven, and with it the proposition. $\square$

Propositions 5.22 and 5.23 might suggest that the eigenvalues of $\mathcal{R}_{n,k}$ should be describable as products and fractions of q -integers for other shapes λ as well. But, as Example 5.20 shows, this is too much to hope for, even for hook-shaped partitions λ . However, in the case of hook-shaped λ , we can at least describe some of the eigenvalues in this form:

Proposition 5.24. *Let $k \geq 0$ and $\ell \in \{0, 1, \dots, n-1\}$. Then,*

$$\mathcal{E}_{(n-\ell, 1^\ell) \setminus (j, 1^\ell)}(k) = [k]!_q \begin{bmatrix} n-j-\ell \\ k \end{bmatrix}_q \begin{bmatrix} n+j \\ k \end{bmatrix}_q \quad \text{for all } j \in [n-\ell].$$

Here, we are using exponential notation, so that 1^ℓ means $\underbrace{1, 1, \dots, 1}_{\ell \text{ times}}$.

Proof. This is analogous to the above proof of (5.12): Fix j and induct on $n \geq j + \ell$. The induction step boils down to checking that

$$[n-j-\ell]_q [n+j]_q = q^k [n-j-\ell-k]_q [n+j-k]_q + [2n-\ell-k]_q [k]_q. \quad \square$$

The eigenvalues covered by Propositions 5.22 and 5.24 are merely the tip of the iceberg. But, as the following proposition shows, they include the two largest eigenvalues of $\mathcal{R}_{n,k}$ when q is a positive real⁷:

Proposition 5.25. *Assume that $\mathbf{k} \subseteq \mathbb{R}$ and $q \in \mathbb{R}_{>0}$. Then, the largest two eigenvalues of $\mathcal{R}_{n,k}$ acting on $\mathcal{H}_n$ are $\mathcal{E}_{(n) \setminus \emptyset}(k)$ and $\mathcal{E}_{(n-1,1) \setminus (1,1)}(k)$ (in this order). More concretely:*

- (a) *For any horizontal strip $\lambda \setminus \mu$ with $\lambda \vdash n$, we have $\mathcal{E}_{\lambda \setminus \mu}(k) \leq \mathcal{E}_{(n) \setminus \emptyset}(k)$.*
- (b) *For any horizontal strip $\lambda \setminus \mu \neq (n) \setminus \emptyset$ with $\lambda \vdash n$ and $d^\mu \neq 0$, we have $\mathcal{E}_{\lambda \setminus \mu}(k) \leq \mathcal{E}_{(n-1,1) \setminus (1,1)}(k)$. (Here, d^μ denotes the number of desarrangement tableaux of shape μ .)*

Proof. Theorem 5.16 shows that the eigenvalues of $\mathcal{R}_{n,k}$ are the $\mathcal{E}_{\lambda \setminus \mu}(k)$ for the horizontal strips $\lambda \setminus \mu$ with $\lambda \vdash n$ and $d^\mu \neq 0$. Hence, it suffices to prove parts (a) and (b).

We shall only prove part (b), since the proof of (a) is analogous but easier.

We fix μ and proceed by induction on $n = |\lambda| \geq |\mu|$. The base case is $\lambda = \mu$, which follows from Theorem 5.12 and Proposition 5.11 (a) and (b). For the induction step, we assume that $\lambda \neq \mu$, and we define the partition $\lambda' \vdash n-1$ as in Proposition 5.11 (c). We furthermore assume that $k > 0$, since the $k = 0$ case is clear from Proposition 5.11 (a). Moreover, we assume that $k \leq n$, since otherwise all relevant eigenvalues are 0 (since $\mathcal{R}_{n,k} = 0$ for $k > n$).

⁷For $\mathcal{R}_{n,1}$, this was already proved in [1, Corollaries 7.9 and 7.12].

From $\lambda \setminus \mu \neq (n) \setminus \emptyset$, we know $\lambda' \setminus \mu \neq (n-1) \setminus \emptyset$. (Indeed, this is clear if $\mu \neq \emptyset$. If $\mu = \emptyset$, then $\lambda \setminus \mu \neq (n) \setminus \emptyset$ shows that $\lambda \neq (n)$, and furthermore the fact that $\lambda \setminus \mu$ is a horizontal strip ensures that $\lambda \neq (n-1, 1)$; but these two facts ensure that $\lambda' \neq (n-1)$ (since the only partitions λ that would satisfy $\lambda' = (n-1)$ are (n) and $(n-1, 1)$) and thus $\lambda' \setminus \mu \neq (n-1) \setminus \emptyset$ again.) Hence, we can apply the induction hypothesis to $\lambda' \setminus \mu$ instead of $\lambda \setminus \mu$, and conclude that $\mathcal{E}_{\lambda' \setminus \mu}(k) \leq \mathcal{E}_{(n-2, 1) \setminus (1, 1)}(k)$ and (by applying it to $k-1$ instead of k) also $\mathcal{E}_{\lambda' \setminus \mu}(k-1) \leq \mathcal{E}_{(n-2, 1) \setminus (1, 1)}(k-1)$. Also, $\mathcal{E}_{(n-2, 1) \setminus (1, 1)}(k-1) \geq 0$ by Theorem 5.12.

Next we shall show that $c_{\lambda \setminus \lambda'} \leq n-2$. In fact, it suffices to prove that the content of each cell of λ is $\leq n-2$ (this is sufficient, since $c_{\lambda \setminus \lambda'}$ is the content of the single cell in $\lambda \setminus \lambda'$). To prove this, assume the contrary. Then, λ has a cell with content at least $n-1$. Since $\lambda \vdash n$, this cell can only be $(1, n)$, and thus we conclude that $\lambda = (n)$. Thus, $\mu = (j)$ for some $j \leq n$ (since $\lambda \setminus \mu$ is a skew diagram). Combining this with $d^\mu \neq 0$, we obtain $\mu = \emptyset$ (since the only desarrangement tableau with only one row is the empty tableau of shape $\emptyset$). Thus, $\lambda \setminus \mu = (n) \setminus \emptyset$, contradicting $\lambda \setminus \mu \neq (n) \setminus \emptyset$. Hence, our assumption was wrong, so that $c_{\lambda \setminus \lambda'} \leq n-2$ is proved.

We now have $c_{\lambda \setminus \lambda'} \leq n-2 = c_{(n-1, 1) \setminus (n-1, 1)'}$ (since the skew diagram $(n-1, 1) \setminus (n-1, 1)'$ consists of the single cell $(1, n-1)$). Since $q > 0$, this entails $[n+1-k+c_{\lambda \setminus \lambda'}]_q \leq [n+1-k+c_{(n-1, 1) \setminus (n-1, 1)'}]_q$. Also, $[n+1-k+c_{(n-1, 1) \setminus (n-1, 1)'}]_q \geq 0$, since $n+1-k+c_{(n-1, 1) \setminus (n-1, 1)'} = n+1-k+(n-2) = 2n-1-k \geq 0$ (because $k \leq n$).

Now, Proposition 5.11 (c) yields

$$\begin{aligned} \mathcal{E}_{\lambda \setminus \mu}(k) &= q^k \underbrace{\mathcal{E}_{\lambda' \setminus \mu}(k)}_{\leq \mathcal{E}_{(n-2, 1) \setminus (1, 1)}(k)} + \underbrace{[n+1-k+c_{\lambda \setminus \lambda'}]_q}_{\leq [n+1-k+c_{(n-1, 1) \setminus (n-1, 1)'}]_q} \underbrace{\mathcal{E}_{\lambda' \setminus \mu}(k-1)}_{\leq \mathcal{E}_{(n-2, 1) \setminus (1, 1)}(k-1)} \\ &\leq q^k \mathcal{E}_{(n-2, 1) \setminus (1, 1)}(k) + [n+1-k+c_{(n-1, 1) \setminus (n-1, 1)'}]_q \mathcal{E}_{(n-2, 1) \setminus (1, 1)}(k-1) \\ &= \mathcal{E}_{(n-1, 1) \setminus (1, 1)}(k) \quad (\text{again by Proposition 5.11 (c), now applied to } (n-1, 1) \setminus (1, 1)). \end{aligned}$$

(Note that we have tacitly used $\mathcal{E}_{(n-2, 1) \setminus (1, 1)}(k-1) \geq 0$ and $[n+1-k+c_{(n-1, 1) \setminus (n-1, 1)'}]_q \geq 0$ here as we multiplied two inequalities.) This completes the induction step, and thus the proof of Proposition 5.25. $\square$

With this proposition, the proof of Theorem C is complete.

5.10. More about the eigenvalues. Some more properties of the polynomials $\mathcal{E}_{\lambda \setminus \mu}(k)$ are not hard to show:

Proposition 5.26. *Let q be an indeterminate. Let $k \geq 0$, and let $\lambda \setminus \mu$ be a horizontal strip with $\lambda \vdash n$.*

- (a) *If $k > |\lambda \setminus \mu|$, then $\mathcal{E}_{\lambda \setminus \mu}(k)$ is identically zero.*
- (b) *Now assume that $k \leq |\lambda \setminus \mu|$. Then, $\mathcal{E}_{\lambda \setminus \mu}(k)$ is a monic polynomial in q of degree $k(n-k) + w_k(\lambda \setminus \mu)$, where $w_k(\lambda \setminus \mu)$ is the sum of the contents of the rightmost k boxes of $\lambda \setminus \mu$.*

Proof. (a) The equality (5.9) expresses $\mathcal{E}_{\lambda \setminus \mu}(k)$ as a sum. If $k > n-j$, then this sum is empty, since there are no k distinct numbers $\ell_1 < \ell_2 < \dots < \ell_k$ between j and n in this case. Hence, $\mathcal{E}_{\lambda \setminus \mu}(k)$ is identically zero if $k > n-j$. Since $|\lambda \setminus \mu| = n-j$, this is precisely the claim of Proposition 5.26 (a).

(b) The equality (5.9) expresses $\mathcal{E}_{\lambda \setminus \mu}(k)$ as a sum. As we saw in the proof of Theorem 5.12, all addends of this sum are polynomials with nonnegative integer coefficients. Moreover, all these polynomials are monic, since any positive q -integer (i.e., any q -integer $[u]_q$ with $u > 0$) is a monic polynomial in q (namely, it is $1+q+q^2+\dots+q^{u-1}$). Thus, it suffices to show that the largest degree of these addends is $k(n-k)+w_k(\lambda \setminus \mu)$, and is attained for only one addend.

To show this, we observe that each $\mathfrak{d}_{\ell_m, m, \lambda \setminus \mu} = [\ell_m + 1 - m + \mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell_m}]_q$ appearing in (5.9) is a positive q -integer (since (5.10) says that $\ell_m + 1 - m + \mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell_m} > 0$), and thus is a monic polynomial of degree $\ell_m + 1 - m + \mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell_m} - 1$ (since any positive q -integer $[u]_q$ is a monic polynomial in q of degree $u-1$).

Therefore, any addend of the sum in (5.9) is a polynomial in q of degree

$$nk - \binom{k}{2} + \sum_{m=1}^k (-\ell_m + (\ell_m + 1 - m + \mathbf{c}_{\mathfrak{t}^{\lambda \setminus \mu}, \ell_m} - 1)) \quad \text{for some } j < (\ell_1 < \ell_2 < \dots < \ell_k) \leq n.$$

This formula for the degree can easily be simplified to

$$nk - \binom{k}{2} + \sum_{m=1}^k (-m + c_{t^{\lambda \setminus \mu}, \ell_m}) = \underbrace{nk - \binom{k}{2} + \sum_{m=1}^k (-m)}_{=nk - k^2 = k(n-k)} + \sum_{m=1}^k c_{t^{\lambda \setminus \mu}, \ell_m} = k(n-k) + \sum_{m=1}^k c_{t^{\lambda \setminus \mu}, \ell_m}.$$

This is clearly maximized for exactly one choice of indices $\ell_1, \ell_2, \dots, \ell_k$: namely, for the choice in which the boxes of $t^{\lambda \setminus \mu}$ containing the entries $\ell_1, \ell_2, \dots, \ell_k$ are the k boxes of $\lambda \setminus \mu$ that have the largest contents. Since the contents of the boxes of $\lambda \setminus \mu$ strictly increase from left to right (because $\lambda \setminus \mu$ is a horizontal strip), these are precisely the k rightmost boxes of $\lambda \setminus \mu$. In this case, the sum $\sum_{m=1}^k c_{t^{\lambda \setminus \mu}, \ell_m}$ becomes precisely $w_k(\lambda \setminus \mu)$, and thus the above formula for the degree of the addend becomes $k(n-k) + w_k(\lambda \setminus \mu)$. Thus, the latter degree is the degree of the whole polynomial $\mathcal{E}_{\lambda \setminus \mu}(k)$. Furthermore, the polynomial is monic, since this degree is attained for only one choice of $\ell_1, \ell_2, \dots, \ell_k$. $\square$

The next proposition is concerned with the particular case when the horizontal strip $\lambda \setminus \mu$ is supported on a single row. This case makes up for most examples for low n .

Proposition 5.27. *Let $\lambda \setminus \mu$ be a horizontal strip whose boxes all lie in the same row. Let $k \geq 0$. Then, $\mathcal{E}_{\lambda \setminus \mu}(k)$ (as a polynomial in q) is palindromic (i.e., its sequence of coefficients reads the same forward and backward) and unimodal (i.e., its coefficients first weakly increase and then weakly decrease).*

Proof idea. The degree of the polynomial can be obtained from Proposition 5.26. Now, induct on n using Proposition 5.11 (c) and the fact that palindromic unimodal polynomials are closed under multiplication ([21, Proposition 1]). $\square$

Question 5.28. Are all the eigenvalues $\mathcal{E}_{\lambda \setminus \mu}(k)$ unimodal as polynomials in q ? This has been verified using SageMath for all $n \leq 28$. (Note that, as we saw in Example 5.20, these polynomials are generally not palindromic. Nor are they generally log-concave, as we see for $\mathcal{E}_{(2,1,1) \setminus (1,1)}(1)$.)

Question 5.29. Assume that q is generic (e.g., an indeterminate). Then, the dimension of the $\mathbf{k}$ -subalgebra of $\mathcal{H}_n$ generated by $\mathcal{R}_{n,1}, \mathcal{R}_{n,2}, \dots, \mathcal{R}_{n,n}$ equals the total number of distinct “eigenvalue packets” (i.e., tuples $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ of eigenvalues of $(\mathcal{R}_{n,1}, \mathcal{R}_{n,2}, \dots, \mathcal{R}_{n,n})$ acting on a joint eigenvector in $\mathcal{H}_n$) (since the bases $\mathfrak{B}_\lambda$ of the Specht modules S^λ are joint eigenbases for this subalgebra, and can be combined into a joint eigenbasis of the whole $\mathcal{H}_n$). By Theorem 5.13, this number is bounded from above by the number of horizontal strips $\lambda \setminus \mu$ with $d^\mu \neq 0$. Yet this bound is not sharp. Distinct horizontal strips often lead to the same eigenvalue packet, even discounting the obvious case when $\lambda = \mu$. Can the number be computed?

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APPENDIX A. PROOFS OF SOME FOLKLORE LEMMAS

Proof of Lemma 2.14. The definition of J_n yields

$$\begin{aligned}
 J_n &= \sum_{i=1}^{n-1} q^{i-n} T_{(i,n)} = \underbrace{q^{(n-1)-n}}_{=q^{-1}} \underbrace{T_{(n-1,n)}}_{=T_{n-1}} + \sum_{i=1}^{n-2} \underbrace{q^{i-n}}_{=q^{-1}q^{i-(n-1)}} T_{(i,n)} \\
 &= q^{-1}T_{n-1} + q^{-1} \sum_{i=1}^{n-2} q^{i-(n-1)} T_{(i,n)}. \tag{A.1}
 \end{aligned}$$

Let $i \in [n-2]$. It is known that $s_{n-1}s_{n-2} \cdots s_i \cdots s_{n-2}s_{n-1}$ (with the subscripts decreasing from $n-1$ to i and then rising again to $n-1$) is a reduced expression for the transposition (i, n) . Hence,

$$T_{(i,n)} = T_{n-1}T_{n-2} \cdots T_i \cdots T_{n-2}T_{n-1}. \tag{A.2}$$

Similarly,

$$T_{(i,n-1)} = T_{n-2}T_{n-3} \cdots T_i \cdots T_{n-3}T_{n-2}. \tag{A.3}$$

Now, (A.2) can be rewritten as

$$T_{(i,n)} = T_{n-1} \underbrace{T_{n-2}T_{n-3} \cdots T_i \cdots T_{n-3}T_{n-2}}_{\substack{=T_{(i,n-1)} \\ \text{(by (A.3))}}} T_{n-1} = T_{n-1}T_{(i,n-1)}T_{n-1}.$$

We have proved this equality for all $i \in [n-2]$. Multiplying it by $q^{i-(n-1)}$ and summing over all $i \in [n-2]$, we obtain

$$\begin{aligned} \sum_{i=1}^{n-2} q^{i-(n-1)} T_{(i,n)} &= \sum_{i=1}^{n-2} q^{i-(n-1)} T_{n-1} T_{(i,n-1)} T_{n-1} \\ &= T_{n-1} \underbrace{\sum_{i=1}^{n-2} q^{i-(n-1)} T_{(i,n-1)}}_{=J_{n-1}} T_{n-1} \\ &\quad \text{(by the definition of } J_{n-1}\text{)} \\ &= T_{n-1} J_{n-1} T_{n-1}. \end{aligned}$$

Thus, we can rewrite (A.1) as

$$J_n = q^{-1} T_{n-1} + q^{-1} T_{n-1} J_{n-1} T_{n-1} = q^{-1} (1 + T_{n-1} J_{n-1}) T_{n-1}.$$

Multiplying this equality by $T_{n-1} - q + 1$ from the right, we find

$$\begin{aligned} J_n (T_{n-1} - q + 1) &= q^{-1} (1 + T_{n-1} J_{n-1}) \underbrace{T_{n-1} (T_{n-1} - q + 1)}_{=T_{n-1}^2 - (q-1)T_{n-1} = q} \\ &\quad \text{(since (2.1) yields } T_{n-1}^2 = (q-1)T_{n-1} + q\text{)} \\ &= q^{-1} (1 + T_{n-1} J_{n-1}) q = 1 + T_{n-1} J_{n-1}. \end{aligned}$$

Thus, $T_{n-1} J_{n-1} = J_n (T_{n-1} - q + 1) - 1$. This proves Lemma 2.14. $\square$

For our next proof, we need a minor lemma:

Lemma A.1. *Let $n, i > 0$ and $a \in \mathcal{H}_n$. Then, T_i commutes with $T_i a T_i + qa$.*

Proof. We have

$$\begin{aligned} T_i (T_i a T_i + qa) &= \underbrace{T_i^2}_{=(q-1)T_i + q \text{ (by (2.1))}} a T_i + q T_i a = (q-1) T_i a T_i + qa T_i + q T_i a. \end{aligned}$$

The same computation (just with the order of factors reversed) yields

$$(T_i a T_i + qa) T_i = (q-1) T_i a T_i + q T_i a + qa T_i.$$

The right hand sides of these two equalities are clearly equal. Thus, so are the left hand sides. In other words, $T_i (T_i a T_i + qa) = (T_i a T_i + qa) T_i$. This proves Lemma A.1. $\square$

Proof of Lemma 2.15. We must prove that T_i commutes with J_n . If $i > n$, then T_i commutes with all the generators $T_1, T_2, \dots, T_{n-1}$ of $\mathcal{H}_n$ (by (2.2)), and thus commutes with all elements of $\mathcal{H}_n$, including J_n . Hence, without loss of generality, we assume that $i \leq n$. Since $i \notin \{n-1, n\}$, this entails $i \leq n-2$. The definition of J_n yields

$$\begin{aligned} J_n &= \sum_{j=1}^{n-1} q^{j-n} T_{(j,n)} = \sum_{j \in [n-1] \setminus \{i, i+1\}} q^{j-n} T_{(j,n)} + \underbrace{q^{i-n} T_{(i,n)} + q^{(i+1)-n} T_{(i+1,n)}}_{=q^{i-n} (T_{(i,n)} + q T_{(i+1,n)})} \\ &= \sum_{j \in [n-1] \setminus \{i, i+1\}} q^{j-n} T_{(j,n)} + q^{i-n} (T_{(i,n)} + q T_{(i+1,n)}). \end{aligned}$$

Hence, in order to show that T_i commutes with J_n , it will suffice to show that

$$T_i \text{ commutes with } T_{(j,n)} \text{ for each } j \in [n-1] \setminus \{i, i+1\} \quad (\text{A.4})$$

and that

$$T_i \text{ commutes with } T_{(i,n)} + q T_{(i+1,n)}. \quad (\text{A.5})$$

Let us first verify (A.4). Indeed, let $j \in [n-1] \setminus \{i, i+1\}$. Then, the transpositions $s_i = (i, i+1)$ and (j, n) are disjoint (since $i \leq n-2$) and thus commute: i.e., we have $s_i (j, n) = (j, n) s_i$. Hence, $T_{s_i(j,n)} = T_{(j,n)} s_i$. Moreover, $(j, n)(i) < (j, n)(i+1)$ (since this is just saying $i < i+1$). Hence, $T_{(j,n)s_i} = T_{(j,n)} T_i$ (since $T_{ws_i} = T_w T_i$ for any $w \in \mathfrak{S}_n$ satisfying $w(i) < w(i+1)$). Furthermore, $(j, n)^{-1}(i) < (j, n)^{-1}(i+1)$ (again

since $i < i+1$), so that $T_{s_i(j,n)} = T_i T_{(j,n)}$ (since $T_{s_i w} = T_i T_w$ for any $w \in \mathfrak{S}_n$ satisfying $w^{-1}(i) < w^{-1}(i+1)$). Thus, $T_i T_{(j,n)} = T_{s_i(j,n)} = T_{(j,n)s_i} = T_{(j,n)} T_i$. In other words, T_i commutes with $T_{(j,n)}$. This proves (A.4).

It remains to verify (A.5). Here, we observe that $s_i s_{i+1} \cdots s_{n-1} \cdots s_{i+1} s_i$ (where the subscripts rise from i to $n-1$ and then decrease back to i) is a reduced expression for the transposition (i, n) . Thus,

$$T_{(i,n)} = T_i T_{i+1} \cdots T_{n-1} \cdots T_{i+1} T_i. \quad (\text{A.6})$$

Similarly,

$$T_{(i+1,n)} = T_{i+1} T_{i+2} \cdots T_{n-1} \cdots T_{i+2} T_{i+1}. \quad (\text{A.7})$$

Now, we can rewrite (A.6) as

$$T_{(i,n)} = T_i \underbrace{T_{i+1} T_{i+2} \cdots T_{n-1} \cdots T_{i+2} T_{i+1}}_{\substack{= T_{(i+1,n)} \\ \text{(by (A.7))}}} T_i = T_i T_{(i+1,n)} T_i.$$

Hence,

$$T_{(i,n)} + q T_{(i+1,n)} = T_i T_{(i+1,n)} T_i + q T_{(i+1,n)}. \quad (\text{A.8})$$

But Lemma A.1 (applied to $a = T_{(i+1,n)}$) shows that T_i commutes with $T_i T_{(i+1,n)} T_i + q T_{(i+1,n)}$. In other words, T_i commutes with $T_{(i,n)} + q T_{(i+1,n)}$ (by (A.8)). Thus, (A.5) is proved, and we are done proving Lemma 2.15. $\square$

APPENDIX B. ANOTHER PROOF OF COMMUTATIVITY

Another proof of Theorem 4.1 relies on the following lemma:

Lemma B.1. *Assume that $\mathbf{k} \subseteq \mathbb{R}$ and $q > 0$. Let $a \in \mathcal{B}_n^* \mathcal{H}_n$. If $\mathcal{B}_n a = 0$, then $a = 0$.*

Proof. Assume that $\mathcal{B}_n a = 0$. We know that $a \in \mathcal{B}_n^* \mathcal{H}_n$. In other words, $a = \mathcal{B}_n^* b$ for some $b \in \mathcal{H}_n$. Thus, $a^* a = (\mathcal{B}_n^* b)^* a = b^* \mathcal{B}_n a = 0$ (since $\mathcal{B}_n a = 0$), so that [1, Lemma 6.1 (1)] yields $a = 0$. This proves Lemma B.1. $\square$

Proof of Theorem 4.1. We induct on n (the case $n = 0$ being obvious).

Let $n > 0$ and $i, j \geq 0$. We must show that $\mathcal{R}_{n,i} \mathcal{R}_{n,j} = \mathcal{R}_{n,j} \mathcal{R}_{n,i}$. This is obvious if $i = 0$ (since $\mathcal{R}_{n,i} = 1$ in this case), obvious if $j = 0$ (for similar reasons), obvious if $i > n$ (since $\mathcal{R}_{n,i} = 0$ in this case, by (2.9)) and obvious if $j > n$ (for similar reasons). Hence, we WLOG assume that both i and j are positive integers $\leq n$. Furthermore, since our claim $(\mathcal{R}_{n,i} \mathcal{R}_{n,j} = \mathcal{R}_{n,j} \mathcal{R}_{n,i})$ is a polynomial identity in q , we assume without loss of generality that the ground field $\mathbf{k}$ is $\mathbb{R}$ and that $q \in \mathbb{R}_{>0}$.

We shall first show that $\mathcal{B}_n \mathcal{R}_{n,j} \mathcal{R}_{n,i} = \mathcal{B}_n \mathcal{R}_{n,i} \mathcal{R}_{n,j}$.

Indeed, Theorem 3.6 yields

$$\mathcal{B}_n \mathcal{R}_{n,i} = \left(q^i \mathcal{R}_{n-1,i} + \left([n+1-i]_q + q^{n+1-i} J_n \right) \mathcal{R}_{n-1,i-1} \right) \mathcal{B}_n.$$

Hence,

$$\begin{aligned} \mathcal{B}_n \mathcal{R}_{n,i} \mathcal{R}_{n,j} &= \left(q^i \mathcal{R}_{n-1,i} + \left([n+1-i]_q + q^{n+1-i} J_n \right) \mathcal{R}_{n-1,i-1} \right) \mathcal{B}_n \mathcal{R}_{n,j} \\ &= \left(q^i \mathcal{R}_{n-1,i} + \left([n+1-i]_q + q^{n+1-i} J_n \right) \mathcal{R}_{n-1,i-1} \right) \\ &\quad \left(q^j \mathcal{R}_{n-1,j} + \left([n+1-j]_q + q^{n+1-j} J_n \right) \mathcal{R}_{n-1,j-1} \right) \mathcal{B}_n \end{aligned}$$

(here, we have applied Theorem 3.6 again, this time to $\mathcal{B}_n \mathcal{R}_{n,j}$). Interchanging i and j , we obtain the equality

$$\begin{aligned} \mathcal{B}_n \mathcal{R}_{n,j} \mathcal{R}_{n,i} &= \left(q^j \mathcal{R}_{n-1,j} + \left([n+1-j]_q + q^{n+1-j} J_n \right) \mathcal{R}_{n-1,j-1} \right) \\ &\quad \left(q^i \mathcal{R}_{n-1,i} + \left([n+1-i]_q + q^{n+1-i} J_n \right) \mathcal{R}_{n-1,i-1} \right) \mathcal{B}_n. \end{aligned}$$

Our intermediate goal is to show that the left hand sides of these two equalities are the same. Clearly, it suffices to show that the right hand sides are the same. For this, it suffices to show that the two factors

$$\begin{aligned} q^i \mathcal{R}_{n-1,i} + \left([n+1-i]_q + q^{n+1-i} J_n \right) \mathcal{R}_{n-1,i-1} \quad \text{and} \\ q^j \mathcal{R}_{n-1,j} + \left([n+1-j]_q + q^{n+1-j} J_n \right) \mathcal{R}_{n-1,j-1} \end{aligned}$$

commute. But this is clear, since the elements $\mathcal{R}_{n-1,i}$, $\mathcal{R}_{n-1,i-1}$, $\mathcal{R}_{n-1,j}$ and $\mathcal{R}_{n-1,j-1}$ pairwise commute (by the induction hypothesis) and also commute with J_n (since they lie in $\mathcal{H}_{n-1}$, but any element of $\mathcal{H}_{n-1}$ commutes with J_n by Lemma 2.16). Thus, the left hand sides are the same. In other words, we have shown that

$$\mathcal{B}_n \mathcal{R}_{n,j} \mathcal{R}_{n,i} = \mathcal{B}_n \mathcal{R}_{n,i} \mathcal{R}_{n,j}. \quad (\text{B.1})$$

However, $\mathcal{R}_{n,i} = \frac{1}{[i]_q!} \mathcal{B}_{n,i}^* \mathcal{B}_{n,i} \in \mathcal{B}_n^* \mathcal{H}_n$ (since $i > 0$ entails $\mathcal{B}_{n,i}^* = \mathcal{B}_n^* \mathcal{B}_{n-1}^* \cdots \mathcal{B}_{n-i+1}^* \in \mathcal{B}_n^* \mathcal{H}_n$). Thus, $\mathcal{R}_{n,i} \mathcal{R}_{n,j} \in \mathcal{B}_n^* \mathcal{H}_n$. Similarly, $\mathcal{R}_{n,j} \mathcal{R}_{n,i} \in \mathcal{B}_n^* \mathcal{H}_n$. Hence, the difference $\mathcal{R}_{n,i} \mathcal{R}_{n,j} - \mathcal{R}_{n,j} \mathcal{R}_{n,i}$ belongs to $\mathcal{B}_n^* \mathcal{H}_n$ as well. But (B.1) shows that $\mathcal{B}_n (\mathcal{R}_{n,i} \mathcal{R}_{n,j} - \mathcal{R}_{n,j} \mathcal{R}_{n,i}) = 0$. Thus, Lemma B.1 (applied to $a = \mathcal{R}_{n,i} \mathcal{R}_{n,j} - \mathcal{R}_{n,j} \mathcal{R}_{n,i}$) yields $\mathcal{R}_{n,i} \mathcal{R}_{n,j} - \mathcal{R}_{n,j} \mathcal{R}_{n,i} = 0$. In other words, $\mathcal{R}_{n,i} \mathcal{R}_{n,j} = \mathcal{R}_{n,j} \mathcal{R}_{n,i}$. This is exactly what we wanted to show. Thus, the induction is complete. $\square$

Remark B.2. The above proof seems paradigmatic: It renders the commutativity of the $\mathcal{R}_{n,k}$ an almost predestined consequence of the recursion given in Theorem 3.6, provided that the general case can be reduced to the fair-weather case of $q > 0$. It appears likely that the same strategy can be used to prove commutativity of other families in the Hecke algebra, provided that similar recursions are found.

APPENDIX C. A THEORY OF SPLIT ELEMENTS

In this appendix, we shall give an alternative proof for the following part of Theorem 5.17:

Proposition C.1. *Let $n, k \geq 0$. All eigenvalues of $\mathcal{R}_{n,k}$ (acting by right multiplication on $\mathcal{H}_n$ over $\mathbf{k}$) are polynomials in q with integer coefficients. That is, the characteristic polynomial of $\mathcal{R}_{n,k}$ (acting by right multiplication on $\mathcal{H}_n$) factors as a product of linear terms $x - p_i(q)$, where each p_i is a univariate polynomial over $\mathbb{Z}$.*

Our proof will rely on a notion of *splitness* of an element over a commutative ring. This notion is somewhat similar to the classical notion of integrality over a commutative ring ([12, Section 15.3], [8, Section 4.1]), but also related to the notion of split algebras ([2], [7, Definition 2.8]). We believe that it might have uses beyond the proof of Proposition C.1; thus, we present it at a greater generality than directly necessary.

In this section, $\mathbf{k}$ shall be an arbitrary commutative ring (not necessarily a field). Its elements will be called *scalars*.

Definition C.2. *Let A be a $\mathbf{k}$ -algebra. An element $a \in A$ is said to be split (over $\mathbf{k}$) if there exist some scalars $u_1, u_2, \dots, u_n \in \mathbf{k}$ (not necessarily distinct) such that $\prod_{i=1}^n (a - u_i) = 0$.*

Note that above, the product here is well-defined, since all the $a - u_i$ commute. The elements of $\mathbf{k}$ itself are split, since each $a \in \mathbf{k}$ satisfies $\prod_{i=1}^1 (a - a) = a - a = 0$. Split elements are integral over $\mathbf{k}$, but not vice versa. The relation between the notion of splitness and Proposition C.1 is the following:

Proposition C.3. *Assume that $\mathbf{k}$ is a field. Let M be a square matrix over $\mathbf{k}$. Then, M is split (regarded as an element of the matrix ring) if and only if all eigenvalues of M belong to $\mathbf{k}$.*

Proof. $\implies$: Assume that M is split. Thus, there exist some scalars $u_1, u_2, \dots, u_n \in \mathbf{k}$ such that

$$\prod_{i=1}^n (M - u_i) = 0.$$

Consider these scalars. Hence, the minimal polynomial of M must divide the polynomial

$$\prod_{i=1}^n (x - u_i).$$

Therefore, all eigenvalues of M belong to $\{u_1, u_2, \dots, u_n\}$ (since all eigenvalues of a matrix are roots of its minimal polynomial), and thus belong to $\mathbf{k}$ as well.

$\Leftarrow$: Assume that all eigenvalues of M belong to $\mathbf{k}$. Let $u_1, u_2, \dots, u_n \in \mathbf{k}$ be these eigenvalues (with their algebraic multiplicities). Hence, by the Cayley–Hamilton theorem, we have $\prod_{i=1}^n (M - u_i) = 0$. This shows that M is split. $\square$

Proposition C.3 reduces the proof of Proposition C.1 to showing that $\mathcal{R}_{n,k}$ is split over $\mathbb{Z}[q]$. To show this, we will now show general properties of split elements.

Lemma C.4. *Let A be a $\mathbf{k}$ -algebra. Let a and b be two commuting elements of A . Let $u_1, u_2, \dots, u_n \in \mathbf{k}$ be scalars such that $\prod_{i=1}^n (a - u_i) = 0$. Let $v_1, v_2, \dots, v_m \in \mathbf{k}$ be scalars such that $\prod_{j=1}^m (b - v_j) = 0$. Then,*

$$\prod_{i=1}^n \prod_{j=1}^m (a + b - u_i - v_j) = 0 \quad (\text{C.1})$$

and

$$\prod_{i=1}^n \prod_{j=1}^m (ab - u_i v_j) = 0. \quad (\text{C.2})$$

Proof. Let B be the $\mathbf{k}$ -subalgebra of A generated by a and b . This subalgebra B is commutative, since its generators a and b commute.

For each $i \in [n]$, we set $s_i := \prod_{j=1}^m (a + b - u_i - v_j)$ and $p_i := \prod_{j=1}^m (ab - u_i v_j)$. Thus, our goal is to show that $\prod_{i=1}^n s_i = 0$ and $\prod_{i=1}^n p_i = 0$.

For each $i \in \{0, 1, \dots, n\}$, define the ideal

$$U_i := (a - u_1)(a - u_2) \cdots (a - u_i) B$$

of B . Clearly, $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n$. Furthermore, $U_0 = B$ (since the product $(a - u_1)(a - u_2) \cdots (a - u_0)$ is empty and thus equals 1) and $U_n = 0$ (since $(a - u_1)(a - u_2) \cdots (a - u_n) = \prod_{i=1}^n (a - u_i) = 0$).

Now, fix $i \in [n]$. Then, $U_{i-1} = (a - u_1)(a - u_2) \cdots (a - u_{i-1}) B$, so that

$$\begin{aligned} (a - u_i) U_{i-1} &= \underbrace{(a - u_i) \cdot (a - u_1)(a - u_2) \cdots (a - u_{i-1})}_{{}=(a-u_1)(a-u_2)\cdots(a-u_i)} B \\ &= (a - u_1)(a - u_2) \cdots (a - u_i) B = U_i. \end{aligned}$$

Hence, the element $a - u_i$ acts as 0 on the quotient B -module U_{i-1}/U_i . In other words, the element a acts as u_i on this B -module U_{i-1}/U_i . Therefore, for any polynomial $P(x) \in \mathbf{k}[x]$, the element $P(a) \in B$ acts as $P(u_i)$ on this B -module U_{i-1}/U_i . Applying this to $P(x) = \prod_{j=1}^m (x + b - u_i - v_j)$, we conclude that the element $\prod_{j=1}^m (a + b - u_i - v_j) \in B$ acts as $\prod_{j=1}^m \underbrace{(u_i + b - u_i - v_j)}_{{}=b-v_j} = \prod_{j=1}^m (b - v_j) = 0$ on this B -module U_{i-1}/U_i .

In other words, the element s_i acts as 0 on the B -module U_{i-1}/U_i (since $s_i = \prod_{j=1}^m (a + b - u_i - v_j)$). In other words, $s_i U_{i-1} \subseteq U_i$.

A similar argument shows that $p_i U_{i-1} \subseteq U_i$ (here, we need to observe that $p_i = \prod_{j=1}^m (ab - u_i v_j)$ acts as

$$\prod_{j=1}^m \underbrace{(u_i b - u_i v_j)}_{{}={u_i(b-v_j)}} = u_i^m \underbrace{\prod_{j=1}^m (b - v_j)}_{{}={0}} = 0 \text{ on } U_{i-1}/U_i.$$

Now, forget that we fixed i . We thus have shown that each $i \in [n]$ satisfies

$$s_i U_{i-1} \subseteq U_i \quad \text{and} \quad (\text{C.3})$$

$$p_i U_{i-1} \subseteq U_i. \quad (\text{C.4})$$

Now,

$$\begin{aligned} \prod_{i=1}^n s_i &= s_n s_{n-1} \cdots s_3 s_2 s_1 = s_n s_{n-1} \cdots s_3 s_2 s_1 \underbrace{1}_{\in B=U_0} \\ &\in s_n s_{n-1} \cdots s_3 s_2 \underbrace{s_1 U_0}_{\substack{\subseteq U_1 \\ \text{(by (C.3))}}} \subseteq s_n s_{n-1} \cdots s_3 \underbrace{s_2 U_1}_{\substack{\subseteq U_2 \\ \text{(by (C.3))}}} \subseteq s_n s_{n-1} \cdots \underbrace{s_3 U_2}_{\substack{\subseteq U_3 \\ \text{(by (C.3))}}} \subseteq \cdots \subseteq U_n = 0. \end{aligned}$$

In other words, $\prod_{i=1}^n s_i = 0$. Similarly, using (C.4), we obtain $\prod_{i=1}^n p_i = 0$. As explained above, this completes the proof of Lemma C.4. $\square$

Theorem C.5. *Let A be a $\mathbf{k}$ -algebra. Let a and b be two commuting split elements of A . Then, the elements $a + b$ and ab are split as well.*

Proof. This follows from Lemma C.4. $\square$

Proposition C.6. *Let A be a $\mathbf{k}$ -algebra. Let a be a split element of A . Let $\lambda \in \mathbf{k}$. Then, λa is also split.*

Proof. Since a is split, there exist scalars $u_1, u_2, \dots, u_n \in \mathbf{k}$ satisfying $\prod_{i=1}^n (a - u_i) = 0$. Consider these scalars. Then, $\prod_{i=1}^n (\underbrace{\lambda a - \lambda u_i}_{=\lambda(a-u_i)}) = \lambda^n \underbrace{\prod_{i=1}^n (a - u_i)}_{=0} = 0$, and this shows that λa is split. $\square$

Theorem C.7. *Let A be a commutative $\mathbf{k}$ -algebra that is generated by split elements. Then, any element of A is split.*

Proof. Let S be the set of all split elements of A . Then, S is closed under addition and multiplication (by Theorem C.5) and under scaling (by Proposition C.6), and contains $\mathbf{k}$ as a subset (since each element of $\mathbf{k}$ is split). Thus, S is a $\mathbf{k}$ -subalgebra of A . Since S contains a set of generators of A (because A is generated by split elements), we thus see that S must be A itself. Hence, any element of A lies in S , and thus is split. $\square$

Theorem C.7 provides a convenient way to infer the splitness of certain elements from the splitness of other elements. Another tool for such inferences (particularly useful in noncommutative $\mathbf{k}$ -algebras) is the following theorem:

Theorem C.8. *Let A be a $\mathbf{k}$ -algebra. Let b, c, d, f be elements of A such that f is split and such that $bc = fb$ and $c = db$. Then, c is split.*

Proof. Since f is split, there exist scalars $u_1, u_2, \dots, u_n \in \mathbf{k}$ such that $\prod_{i=1}^n (f - u_i) = 0$. Consider these scalars. For each $i \in [n]$, we have

$$b(c - u_i) = \underbrace{bc}_{=fb} - u_i b = fb - u_i b = (f - u_i) b. \quad (\text{C.5})$$

Now,

$$\begin{aligned} b \prod_{i=1}^n (c - u_i) &= \underbrace{b(c - u_1)}_{\substack{=(f-u_1)b \\ \text{(by (C.5))}}} (c - u_2) (c - u_3) \cdots (c - u_n) \\ &= (f - u_1) \underbrace{b(c - u_2)}_{\substack{=(f-u_2)b \\ \text{(by (C.5))}}} (c - u_3) \cdots (c - u_n) \\ &= (f - u_1) (f - u_2) \underbrace{b(c - u_3)}_{\substack{=(f-u_3)b \\ \text{(by (C.5))}}} \cdots (c - u_n) \\ &= \cdots \\ &= \underbrace{(f - u_1) (f - u_2) \cdots (f - u_n)}_{=\prod_{i=1}^n (f - u_i) = 0} b = 0. \end{aligned}$$

But $c = db$, and thus $c \prod_{i=1}^n (c - u_i) = db \underbrace{\prod_{i=1}^n (c - u_i)}_{=0} = 0$. In other words, $\prod_{i=0}^n (c - u_i) = 0$, where we set $u_0 := 0$. This shows that c is split. $\square$

We observe that Theorem C.8 has two remarkable corollaries, which we shall not use:

Corollary C.9. *Let A be a $\mathbf{k}$ -algebra. Let a and b be two elements of A such that the product ba is split. Then, ab is also split.*

Proof. We can apply Theorem C.8 to $c = ab$ and $f = ba$ and $d = a$ (since ba is split and since $b \cdot ab = ba \cdot b$ and $ab = ab$). Thus we conclude that ab is split. $\square$

Corollary C.10. *Let A be a $\mathbf{k}$ -algebra. Let a and b be two elements of A such that $b^2 = ab$. Assume that a is split. Then, b is also split.*

Proof. We can apply Theorem C.8 to $c = b$ and $f = a$ and $d = 1$ (since a is split, and since $bb = b^2 = ab$ and $b = 1b$). Thus we conclude that b is split. $\square$

To apply the above theory to the Hecke algebras, we need one input from representation theory:

Proposition C.11. *Assume that q is invertible in $\mathbf{k}$. Then, the Young–Jucys–Murphy elements $J_1, J_2, \dots, J_n$ in $\mathcal{H}_n$ are split (over $\mathbf{k}$).*

Proof. The formula [16, (5.2)] says that each $m \in [n]$ satisfies

$$\prod_{c \in \mathcal{R}(m)} (J_m - c) = 0,$$

where⁸

$$\mathcal{R}(m) := \begin{cases} \left\{ [k]_q \mid -m < k < m \right\}, & \text{if } m \geq 4; \\ \left\{ [k]_q \mid -m < k < m \right\} \setminus \{0\}, & \text{if } m < 4. \end{cases}$$

Since all the $c \in \mathcal{R}(m)$ are scalars in $\mathbf{k}$, this equality shows that each J_m is split. Proposition C.11 is proved. $\square$

We are now ready to prove the following:

Lemma C.12. *Assume that $\mathbf{k}$ contains the field of rational functions $\mathbb{Q}(q)$. Let $n, k \geq 0$. Then, the element $\mathcal{R}_{n,k}$ of $\mathcal{H}_n$ is split (over $\mathbf{k}$).*

The assumption on $\mathbf{k}$ will soon be disposed of, but we need it for the following proof:

Proof of Lemma C.12. We induct on n . The base case ($n = 0$) is obvious, so we pass to the induction step.

If $k = 0$, then our claim is obvious, since $\mathcal{R}_{n,0} = 1$. So is it for $k > n$, since $\mathcal{R}_{n,k} = 0$ in this case. Thus, without loss of generality, we assume that $1 \leq k \leq n$. Hence, Theorem 3.6 yields

$$\mathcal{B}_n \mathcal{R}_{n,k} = \mathcal{W}_{n,k} \mathcal{B}_n, \tag{C.6}$$

where

$$\mathcal{W}_{n,k} := q^k \mathcal{R}_{n-1,k} + \left([n+1-k]_q + q^{n+1-k} J_n \right) \mathcal{R}_{n-1,k-1}.$$

Now we shall show that $\mathcal{W}_{n,k}$ is split. Indeed, by our induction hypothesis, the elements $\mathcal{R}_{n-1,k}$ and $\mathcal{R}_{n-1,k-1}$ are split (in $\mathcal{H}_{n-1}$ and thus also in the larger algebra $\mathcal{H}_n$). So is the element J_n (by Proposition C.11). Moreover, these three elements $\mathcal{R}_{n-1,k}$ and $\mathcal{R}_{n-1,k-1}$ and J_n all commute (indeed, Theorem 4.1 shows that the elements $\mathcal{R}_{n-1,k}$ and $\mathcal{R}_{n-1,k-1}$ commute, whereas Lemma 2.16 yields that J_n commutes with both of them as well). Hence, the $\mathbf{k}$ -subalgebra A of $\mathcal{H}_n$ generated by these three elements $\mathcal{R}_{n-1,k}$ and $\mathcal{R}_{n-1,k-1}$ and J_n is commutative. Since its generators $\mathcal{R}_{n-1,k}$ and $\mathcal{R}_{n-1,k-1}$ and J_n are split, we thus conclude by Theorem C.7 that any element of this subalgebra A must be split. In particular, $\mathcal{W}_{n,k}$ must be split (since the definition of $\mathcal{W}_{n,k}$ shows that $\mathcal{W}_{n,k}$ belongs to A).

⁸If some of the q -integers $[k]_q$ are equal, then this $\mathcal{R}(m)$ should be understood as a multiset.

Moreover, (2.7) yields

$$\begin{aligned}\mathcal{R}_{n,k} &= \frac{1}{[k]!_q} \mathcal{B}_{n,k}^* \mathcal{B}_{n,k} \\ &= \frac{1}{[k]!_q} \mathcal{B}_{n,k}^* \mathcal{B}_{n-(k-1)} \cdots \mathcal{B}_{n-1} \mathcal{B}_n \quad (\text{by (2.5)}) \\ &= \mathcal{U}_{n,k} \mathcal{B}_n, \quad \text{where } \mathcal{U}_{n,k} := \frac{1}{[k]!_q} \mathcal{B}_{n,k}^* \mathcal{B}_{n-(k-1)} \cdots \mathcal{B}_{n-2} \mathcal{B}_{n-1}.\end{aligned}$$

Therefore, Theorem C.8 (applied to $\mathcal{H}_n$, $\mathcal{B}_n$, $\mathcal{R}_{n,k}$, $\mathcal{U}_{n,k}$ and $\mathcal{W}_{n,k}$ instead of A , b , c , d and f) shows that $\mathcal{R}_{n,k}$ is split (because of (C.6)). This completes the induction step, and thus proves Lemma C.12. $\square$

We can now prove Proposition C.1:

Proof of Proposition C.1. We must show that the characteristic polynomial of $\mathcal{R}_{n,k}$ (acting by right multiplication on $\mathcal{H}_n$ over $\mathbf{k}$) can be factored into monic linear factors of the form $x - p_i(q)$, where each p_i is a univariate polynomial over $\mathbb{Z}$.

Clearly, it suffices to show this in the case when our base ring $\mathbf{k}$ is $\mathbb{Z}[q]$. Thus, without loss of generality, we assume that we are in this case. Note that $\mathbb{Z}[q]$ is a unique factorization domain with quotient field $\mathbb{Q}(q)$; it is thus integrally closed in $\mathbb{Q}(q)$ (since any unique factorization domain is integrally closed in its quotient field — see [8, Theorem (4.1.9)]). In other words, any element of $\mathbb{Q}(q)$ that is integral over $\mathbb{Z}[q]$ must belong to $\mathbb{Z}[q]$.

The equality (2.20) contains no denominators or negative powers of q . Thus, the element $\mathcal{R}_{n,k} \in \mathcal{H}_n$ is defined over the ring $\mathbb{Z}[q]$. We shall view the Hecke algebra $\mathcal{H}_n$ defined over the ring $\mathbb{Z}[q]$ as a subring of the corresponding Hecke algebra $\mathcal{H}_n$ defined over the field $\mathbb{Q}(q)$. The element $\mathcal{R}_{n,k}$ belongs to both of these Hecke algebras, and thus acts on them both by right multiplication; the actions clearly have the same characteristic polynomial and the same eigenvalues. We must show that this characteristic polynomial can be factored into monic linear factors of the form $x - p_i(q)$ with $p_i \in \mathbb{Z}[q]$.

By the Cayley–Hamilton theorem, it suffices to show that all eigenvalues of $\mathcal{R}_{n,k}$ belong to $\mathbb{Z}[q]$.

Recall that $\mathcal{R}_{n,k}$ is defined over the ring $\mathbb{Z}[q]$. Hence, the characteristic polynomial of $\mathcal{R}_{n,k}$ is a monic polynomial of degree $n!$ with coefficients in $\mathbb{Z}[q]$. Therefore, all eigenvalues of $\mathcal{R}_{n,k}$ are integral over the ring $\mathbb{Z}[q]$ (since they are the roots of the characteristic polynomial of $\mathcal{R}_{n,k}$).

But Lemma C.12 shows that $\mathcal{R}_{n,k}$ is split over the field $\mathbb{Q}(q)$. Hence, Proposition C.3 (applied to the field $\mathbb{Q}(q)$) shows that all eigenvalues of $\mathcal{R}_{n,k}$ belong to $\mathbb{Q}(q)$. But at the same time, these eigenvalues are integral over the ring $\mathbb{Z}[q]$, as we just saw. Hence, they must belong to $\mathbb{Z}[q]$ (since any element of $\mathbb{Q}(q)$ that is integral over $\mathbb{Z}[q]$ must belong to $\mathbb{Z}[q]$). As explained above, this proves Proposition C.1. $\square$

APPENDIX D. THE EIGENVALUES $\mathcal{E}_{\lambda \setminus \mu}(k)$, EXPANDED

For $3 \leq n \leq 5$, we include the polynomial expansion of the nonzero eigenvalues of the $\mathcal{R}_{n,k}$; compare with the tables in Examples 5.18, 5.19, and 5.21.

$n = 3$.

$\lambda \setminus \mu$	k	$\mathcal{E}_{\lambda \setminus \mu}(k)$
$(3) \setminus \emptyset$	1	$q^4 + 2q^3 + 3q^2 + 2q + 1$
$(3) \setminus \emptyset$	2	$q^5 + 3q^4 + 5q^3 + 5q^2 + 3q + 1$
$(3) \setminus \emptyset$	3	$q^3 + 2q^2 + 2q + 1$
$(2, 1) \setminus (1, 1)$	1	$q^3 + q^2 + q + 1$
$(1, 1, 1) \setminus (1, 1)$	1	1

$n = 4$.

$\lambda \setminus \mu$	k	$\mathcal{E}_{\lambda \setminus \mu}(k)$
$(4) \setminus \emptyset$	1	$q^6 + 2q^5 + 3q^4 + 4q^3 + 3q^2 + 2q + 1$
$(4) \setminus \emptyset$	2	$q^9 + 3q^8 + 7q^7 + 11q^6 + 14q^5 + 14q^4 + 11q^3 + 7q^2 + 3q + 1$
$(4) \setminus \emptyset$	3	$q^9 + 4q^8 + 9q^7 + 15q^6 + 19q^5 + 19q^4 + 15q^3 + 9q^2 + 4q + 1$
$(4) \setminus \emptyset$	4	$q^6 + 3q^5 + 5q^4 + 6q^3 + 5q^2 + 3q + 1$
$(3, 1) \setminus (2, 1)$	1	$q^5 + q^4 + q^3 + q^2 + q + 1$
$(3, 1) \setminus (1, 1)$	1	$q^5 + 2q^4 + 2q^3 + 2q^2 + 2q + 1$
$(3, 1) \setminus (1, 1)$	2	$q^7 + 2q^6 + 3q^5 + 4q^4 + 4q^3 + 3q^2 + 2q + 1$
$(2, 2) \setminus (2, 1)$	1	$q^3 + q^2 + q + 1$
$(2, 1, 1) \setminus (2, 1)$	1	$q + 1$
$(2, 1, 1) \setminus (1, 1)$	1	$q^4 + q^3 + q^2 + 2q + 1$
$(2, 1, 1) \setminus (1, 1)$	2	$q^3 + q^2 + q + 1$

$n = 5$.

$\lambda \setminus \mu$	k	$\mathcal{E}_{\lambda \setminus \mu}(k)$
$(5) \setminus \emptyset$	1	$q^8 + 2q^7 + 3q^6 + 4q^5 + 5q^4 + 4q^3 + 3q^2 + 2q + 1$
$(5) \setminus \emptyset$	2	$q^{13} + 3q^{12} + 7q^{11} + 13q^{10} + 20q^9 + 26q^8 + 30q^7 + 30q^6 + 26q^5 + 20q^4 + 13q^3 + 7q^2 + 3q + 1$
$(5) \setminus \emptyset$	3	$q^{15} + 4q^{14} + 11q^{13} + 23q^{12} + 40q^{11} + 59q^{10} + 76q^9 + 86q^8 + 86q^7 + 76q^6 + 59q^5 + 40q^4 + 23q^3 + 11q^2 + 4q + 1$
$(5) \setminus \emptyset$	4	$q^{14} + 5q^{13} + 14q^{12} + 29q^{11} + 49q^{10} + 70q^9 + 86q^8 + 92q^7 + 86q^6 + 70q^5 + 49q^4 + 29q^3 + 14q^2 + 5q + 1$
$(5) \setminus \emptyset$	5	$q^{10} + 4q^9 + 9q^8 + 15q^7 + 20q^6 + 22q^5 + 20q^4 + 15q^3 + 9q^2 + 4q + 1$
$(4, 1) \setminus (3, 1)$	1	$q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$
$(4, 1) \setminus (2, 1)$	1	$q^7 + 2q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + 2q + 1$
$(4, 1) \setminus (2, 1)$	2	$q^{11} + 2q^{10} + 3q^9 + 4q^8 + 5q^7 + 6q^6 + 6q^5 + 5q^4 + 4q^3 + 3q^2 + 2q + 1$
$(4, 1) \setminus (1, 1)$	1	$q^7 + 2q^6 + 3q^5 + 3q^4 + 3q^3 + 3q^2 + 2q + 1$
$(4, 1) \setminus (1, 1)$	2	$q^{11} + 3q^{10} + 6q^9 + 9q^8 + 12q^7 + 14q^6 + 14q^5 + 12q^4 + 9q^3 + 6q^2 + 3q + 1$
$(4, 1) \setminus (1, 1)$	3	$q^{12} + 3q^{11} + 6q^{10} + 10q^9 + 14q^8 + 17q^7 + 18q^6 + 17q^5 + 14q^4 + 10q^3 + 6q^2 + 3q + 1$
$(3, 2) \setminus (2, 2)$	1	$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$
$(3, 2) \setminus (3, 1)$	1	$q^4 + q^3 + q^2 + q + 1$
$(3, 2) \setminus (2, 1)$	1	$q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + 2q + 1$
$(3, 2) \setminus (2, 1)$	2	$q^8 + 2q^7 + 3q^6 + 4q^5 + 4q^4 + 4q^3 + 3q^2 + 2q + 1$
$(3, 1, 1) \setminus (2, 1, 1)$	1	$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$
$(3, 1, 1) \setminus (3, 1)$	1	$q^2 + q + 1$
$(3, 1, 1) \setminus (2, 1)$	1	$q^6 + q^5 + q^4 + q^3 + 2q^2 + 2q + 1$
$(3, 1, 1) \setminus (2, 1)$	2	$q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + 2q + 1$
$(3, 1, 1) \setminus (1, 1)$	1	$q^6 + 2q^5 + 2q^4 + 2q^3 + 3q^2 + 2q + 1$
$(3, 1, 1) \setminus (1, 1)$	2	$q^9 + 2q^8 + 3q^7 + 5q^6 + 7q^5 + 7q^4 + 6q^3 + 5q^2 + 3q + 1$
$(3, 1, 1) \setminus (1, 1)$	3	$q^7 + 2q^6 + 3q^5 + 4q^4 + 4q^3 + 3q^2 + 2q + 1$
$(2, 2, 1) \setminus (2, 2)$	1	$q^2 + q + 1$
$(2, 2, 1) \setminus (2, 1, 1)$	1	$q^4 + q^3 + q^2 + q + 1$
$(2, 2, 1) \setminus (2, 1)$	1	$q^4 + q^3 + 2q^2 + 2q + 1$
$(2, 2, 1) \setminus (2, 1)$	2	$q^4 + 2q^3 + 2q^2 + 2q + 1$
$(2, 1, 1, 1) \setminus (1, 1, 1, 1)$	1	$q^5 + q^4 + q^3 + q^2 + q + 1$
$(2, 1, 1, 1) \setminus (2, 1, 1)$	1	$q + 1$
$(1, 1, 1, 1, 1) \setminus (1, 1, 1, 1)$	1	1