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INVERSION OF THE UNBOUNDED FINITE HILBERT TRANSFORM ON L^1

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ABSTRACT. The finite Hilbert transform T is a classical singular integral operator with its roots in aerodynamics, elasticity theory and image reconstruction. The setting has always been to consider T as acting in those rearrangement invariant spaces X over $(-1,1)$ which T maps boundedly into itself (e.g., L^p for $1<p<\infty$), a setting which excludes L^1. Our aim is to go beyond boundedness and to address the case $X=L^1$. For this, we need to consider T as an *unbounded* operator on L^1. Is there a "suitable" domain for T? Yes. Remarkably, for T acting on this domain, we prove a full inversion theorem, together with refined versions of both the Parseval and Poincaré-Bertrand formulae, which are crucial results needed for the proof. This domain, a somewhat unusual space, turns out to be a rather extensive subspace of L^1, fails to be an ideal and properly contains the Zygmund space $L\log L$ (which is the largest ideal of functions that T maps boundedly into L^1).

1. INTRODUCTION

Given $f\in L^1:=L^1(-1,1)$ its *finite Hilbert transform* $T(f)$ is the principal value integral

$$T(f)(x)=\lim_{\varepsilon\to0^+}\frac{1}{\pi}\left(\int_{-1}^{x-\varepsilon}+\int_{x+\varepsilon}^{1}\right)\frac{f(y)}{y-x}\,dy,$$

which exists for a.e. $x\in(-1,1)$ and is a measurable function. It is denoted briefly by FHT. Its initial study, in the 1920–30's, was intimately related to problems arising from mathematical physics, in particular, in aerodynamics and elasticity theory, via the solution of the *airfoil equation*. That is, for g a suitable given function, find all functions f which satisfy

$$g(y)=\text{p.v.}\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}\,dx,\quad \text{a.e. } y\in(-1,1). \tag{1.1}$$

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In the early 1990's a celebrated paper of Gel'fand and Graev, [8], triggered applications of the FHT to image reconstruction in tomography, thereby renewing interest in the investigation of this operator; see [2], for example.

The study of the *operator* $T\colon f \mapsto T(f)$ began in spaces of Hölder continuous functions. A consequence of the theorem of M. Riesz for the Hilbert transform operator on the real line is that the FHT maps $L^p := L^p(-1,1)$ continuously *into itself*, for $1 < p < \infty$. This allowed the development of the L^p-theory for the FHT, which was completed by the 1970's. In particular, an *inversion formula* for solving the airfoil equation (1.1) was established.

A consequence of a theorem of Boyd for the Hilbert transform on the real line is that the FHT maps a rearrangement invariant (briefly, r.i.) space X on $(-1,1)$ continuously *into itself* if and only if X has non-trivial Boyd indices. The family of all r.i. spaces includes many classical spaces appearing in analysis, such as the Lorentz $L^{p,q}$ spaces, Orlicz L^φ spaces, Marcinkiewicz M_φ spaces, Lorentz Λ_φ spaces, and the Zygmund $L^p(\log \mathrm{L})^\alpha$ spaces, In particular, this family includes $L^p = L^{p,p}$, for $1 \le p \le \infty$. The Boyd indices of X are two parameters which measure the effect of dilation on the space. From Boyd's theorem the theory of the FHT in r.i. spaces has been further developed in recent years; see [3], [4], [5]. In this development, the results from the L^p setting, for $1 < p < \infty$, carry over to the class of r.i. spaces with non-trivial Boyd indices. In particular, the validity of the inversion formula was extended. An overview of the above discussion can be found in the survey paper [7], for example.

In the recent paper [6] a different aspect was treated, namely an investigation was undertaken of the FHT when acting on the Zygmund space $L\log L$. This r.i. space, whose *Boyd indices are trivial*, strictly contains all spaces L^p with $1 < p < \infty$ and all r.i. spaces with non-trivial Boyd indices. A desirable feature is that the r.i. space $L\log L$ is rather *extensive* and, as it turned out, *continuity* of the FHT from $L\log L$ into L^1 is maintained. As a consequence of extrapolation theory, an *inversion formula* is also available. On the down side, one loses the property that the FHT maps the space $L\log L$ into itself. However, this is adequately compensated for by the following property of $L\log L$. Given a function $f \in L^1$ satisfying $T(f) \in L^1$, it need *not* follow in general that $T(f\chi_A) \in L^1$ for a measurable set $A \subseteq (-1,1)$. However, a result of Stein implies, if the previous property *does hold* for all such A, then necessarily $f \in L\log L$; see Remark 3.3(e) below. This special feature of $L\log L$ makes it a natural candidate to be considered as the domain space for T when acting in L^1.

In view of the previous paragraph, the novelty of the current paper is to place the emphasis on the FHT mapping $L\log L$, but now interpreted as a *subspace of* L^1 (i.e., equip $L\log L$ with the L^1*-norm* rather than its usual norm $\|\cdot\|_{L\log L}$), into the *'same space'* L^1. This is possible because T is actually defined on *all* of the ambient space L^1, although it does not map L^1 into itself. This approach allows one to interpret the FHT as the *unbounded* linear operator

$$T_1 \colon (L\log L, \|\cdot\|_1) \to L^1,$$

given by $T_1(f) := T(f)$, with dense domain $\mathrm{D}(T_1) := L\log L$ in L^1. In this new context we seek to establish a comprehensive inversion formula. To achieve this aim it is necessary to consider the *closure* of the operator T_1, rather than T_1 itself, thereby *extending* its domain beyond $L\log L$. This requires identifying explicitly both the closure of T_1 and its domain, which turns out to be the linear subspace

$$\begin{aligned} \mathbb{V} := \big\{ f \in L^1 : T(f) \in L^1, \exists \{f_n\}_{n=1}^{\infty} \subseteq L\log L, \\ \|f_n - f\|_1 \to 0, \|T(f_n) - T(f)\|_1 \to 0 \big\}. \end{aligned} \tag{1.2}$$

In contrast to the family of all r.i. spaces, $\mathbb{V}$ *fails* to be an ideal of functions in L^1 (cf. Remark 3.3(e)). So, the operator to be studied (i.e., the closure of T_1) is

$$T_{\mathbb{V}} \colon \mathbb{V} \to L^1,$$

defined by $T_{\mathbb{V}}(f) := T(f)$, for $f \in \mathbb{V}$. For the details we refer to Section 3.

There are two crucial tools required for investigating the FHT (hence, also $T_{\mathbb{V}}$), namely the Parseval and the Poincaré-Bertrand formulae. In this regard, recall that the Parseval formula is a classical result stating that

$$\int_{-1}^{1} fT(g) = -\int_{-1}^{1} gT(f), \quad f \in L^p,\ g \in L^{p'}, \tag{1.3}$$

whenever $1 < p < \infty$ and $1/p + 1/p' = 1$. The formula (1.3) was later extended to functions $f \in X$ and $g \in X^*$ (the dual Banach space of X), where X is any separable r.i. space on $(-1, 1)$ with non-trivial Boyd indices. The validity of (1.3) was recently further extended to the case when $f \in L\log L$ and $g \in L^\infty$. The Parseval formula is actually a version of *operator duality*. Indeed, if we denote $T_p := T \colon L^p \to L^p$, for $1 < p < \infty$, then the Parseval formula identifies the adjoint operator $T_p^* \colon L^{p'} \to L^{p'}$ of T_p as $T_p^* = -T_{p'}$. Hence, (1.3) corresponds to

$$\langle Tf, g\rangle = \langle f, T^* g\rangle, \quad f \in L^p,\ g \in L^{p'}.$$

Adopting this viewpoint, to establish the Parseval formula in the L^1-setting requires identifying the adjoint operator $T_{\mathbb{V}}^*$ of $T_{\mathbb{V}}$ and its domain. Somewhat unexpectedly, for the linear subspace

$$\mathbb{W} := L^\infty \cap T^{-1}(L^\infty) = \big\{ g \in L^\infty : T(g) \in L^\infty \big\},$$

it turns out that $T_{\mathbb{V}}^*$ is precisely the closed, unbounded operator

$$-T_{\mathbb{W}} \colon \mathbb{W} \to L^\infty,$$

where $T_{\mathbb{W}}(f) := T(f)$, for $f \in \mathbb{W}$. This allows us to establish the Parseval formula for the operator duality between $T_{\mathbb{V}}$ and $T_{\mathbb{V}}^*$ and between the pair of "dual spaces" $\mathbb{V}$ and $\mathbb{W}$ (cf. Theorem 4.1). The apparent simplicity of $\mathbb{W}$ is somewhat misleading; see Section 4.

The other crucial tool for studying the FHT, in addition to Parseval's identity, is the Poincaré-Bertrand formula (which Tricomi calls the *convolution theorem* for the FHT), namely

$$T\big(fT(g)+gT(f)\big)=T(f)T(g)-fg, \quad \text{a.e.} \tag{1.4}$$

This identity was first shown to be valid whenever $f\in L^{p_1}$ and $g\in L^{p_2}$ with $1/p_1+1/p_2<1$. In 1977, E. R. Love extended (1.4) in an essential way by allowing $1/p_1+1/p_2=1$. This result for L^p-spaces was extended to pairs of functions $f\in X$ and $g\in X'$, where X is any r.i. space on $(-1,1)$ with non-trivial Boyd indices and X' is the associate space of X; see [3]. Its validity was recently extended to the case when $f\in L\log L$ and $g\in L^\infty$, [6]. Surprisingly, the a priori unlikely aim of being able to establish the validity of the Poincaré-Bertrand formula (1.4) for pairs of functions $f\in\mathbb{V}$ and $g\in\mathbb{W}$ is actually achieved (by the use of Plemelj's formula) in Theorem 5.3.

Proving the Inversion Theorem for the FHT operator $T_{\mathbb{V}}\colon\mathbb{V}\to L^1$ requires an auxiliary operator $\widehat{T}\colon L^1\to L^0$ given by

$$\widehat{T}(f):=\frac{-1}{w}T\big(wf\big), \quad f\in L^1, \tag{1.5}$$

where

$$w(x):=\sqrt{1-x^2}, \quad x\in(-1,1). \tag{1.6}$$

The operator $\widehat{T}$ has the property that $\widehat{T}\colon L\log L\to L^1$ is bounded. As explained previously for T, we also need to consider the operator $\widehat{T}_1\colon(L\log L,\|\cdot\|_1)\to L^1$ and its *closure*, namely the closed, unbounded operator $\widehat{T}_{\widehat{\mathbb{V}}}\colon\widehat{\mathbb{V}}\to L^1$ defined by $\widehat{T}_{\widehat{\mathbb{V}}}(g):=\widehat{T}(g)$, for $g\in\widehat{\mathbb{V}}$; see (3.4) below. The proof of the Inversion Theorem is based on fundamental properties of $T_{\mathbb{V}}$ and $\widehat{T}_{\widehat{\mathbb{V}}}$ on their respective domains $\mathbb{V}$ and $\widehat{\mathbb{V}}$. These are studied in detail in Theorem 6.1.

With the above mentioned results in hand, we can now state (and prove in Section 6) the inversion formula for the operator $T_{\mathbb{V}}\colon\mathbb{V}\to L^1$, which is the highlight of the paper.

Inversion Theorem. *Let $g\in\widehat{\mathbb{V}}$. All solutions $f\in\mathbb{V}$ of the airfoil equation* (1.1) *are of the form $\widehat{T}(g)+c/w$, that is,*

$$f=\frac{-1}{w}\,T\,(wg)+\frac{c}{w}, \quad \text{a.e. in}\,(-1,1),$$

with $c\in\mathbb{C}$ arbitrary.

2. Preliminaries

The setting of this paper is the measure space $(-1,1)$ equipped with its Borel σ-algebra $\mathcal{B}$ and Lebesgue measure m (restricted to $\mathcal{B}$). Denote by $L^0:=L^0(-1,1)$ the space (of equivalence classes) of all $\mathbb{C}$-valued measurable functions, endowed with the topology of convergence in measure. The space weak-L^1, denoted by $L^{1,\infty} := L^{1,\infty}(-1,1)$, is a quasi-Banach space satisfying $L^1 \subseteq L^{1,\infty} \subseteq L^0$, with both inclusions continuous.

A *rearrangement invariant* space X on $(-1,1)$ is a Banach space $X \subseteq L^0$ satisfying the property that if $g^* \leq f^*$ with $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$. Here, given a function $h\colon (-1,1) \to \mathbb{C}$, its *decreasing rearrangement* $h^*\colon [0,2] \to [0,\infty]$ is the right continuous inverse of its distribution function $\lambda \mapsto m(\{t \in (-1,1) : |h(t)| > \lambda\})$ for $\lambda \geq 0$. The Boyd indices of a r.i. space consist of two parameters in $[0,1]$, [1, Definition III.5.12]. They are said to be non-trivial when both lie in $(0,1)$.

The Zygmund space $L\mathrm{log}L := L\mathrm{log}L(-1,1)$ consists of all functions $f \in L^0$ for which either one of the following two equivalent conditions hold:

$$\int_{-1}^{1} |f(x)| \log^+ |f(x)|\, dx < \infty, \quad \int_0^2 f^*(t) \log\left(\frac{2e}{t}\right) dt < \infty.$$

According to [1, §IV.6], it is a r.i. space for the norm given by

$$\|f\|_{L\mathrm{log}L} := \int_0^2 f^*(t) \log\left(\frac{2e}{t}\right) dt, \quad f \in L\mathrm{log}L.$$

It is appropriate to indicate some relevant references for results cited in the Introduction. For the Hilbert transform operator on the real line and Kolmogorov's theorem, [1, Theorem III.4.9(b)]; M. Riesz' theorem, [1, Theorem III.4.9(a)], Boyd's theorem, [1, Theorem III.5.18] and [16, pp.170–171]. For the theory of the FHT on Hölder spaces, see [19]. Concerning the L^p-theory for the FHT, see [9], [13], [14], [18], [20], [22]. Regarding the theory of the FHT in r.i. spaces, see the survey paper [7]. For facts about r.i. spaces we refer to [1], [16], for example.

3. Unbounded extension of the FHT

For the densely defined, unbounded operator

$$T_1\colon (L\mathrm{log}L, \|\cdot\|_1) \to L^1,$$

whose domain $L\mathrm{log}L$ is now endowed with the L^1-*norm*, we are in the desirable situation of being able to invoke the classical spectral theory for closed/closable operators.

Let us recall some terminology. The setting is a linear operator S acting in a Banach space Y, whose domain $\mathrm{D}(S)$ is equipped with the norm from Y, and taking values in Y. If its graph $\mathrm{G}(S) := \{(y, S(y)) \in Y \times Y : y \in \mathrm{D}(S)\}$ happens to be a closed linear subspace of $Y \times Y$, then S is said to be *closed*. In the event that S itself is not closed but, there exists a closed linear extension of S, still Y-valued, then S is said to be *closable*. In this case, there exists a *smallest closed extension* $\overline{S}$ of S with $\mathrm{G}(\overline{S}) = \overline{\mathrm{G}(S)}$, where the closure $\overline{\mathrm{G}(S)}$ of $\mathrm{G}(S)$ is taken in $Y \times Y$, having the property that any other closed extension of S is also an extension of $\overline{S}$. The operator $\overline{S}$ is called the *closure* of S. In particular, $\mathrm{D}(S) \subseteq \mathrm{D}(\overline{S})$ and $\overline{S}y = Sy$ for all $y \in \mathrm{D}(S)$. For those aspects of this theory used in the present section we refer to [10, Ch.II, §2] and [12, Ch.3, §5], for example.

When dealing with T_1 and its closed extension we rely on a particular feature of the FHT, namely that it is defined on *all* of the ambient space L^1. It turns out that T_1 is *not* a closed operator. To establish this fact one has to search for points in the closure of the graph of T_1 which are not in the graph itself. How to find such points? We take advantage of the fact that the auxiliary operator $\widehat{T}$ does *not* map $L\log L$ into itself, [6]. This assures the existence of functions $g \in L\log L$ satisfying $\widehat{T}(g) \notin L\log L$, which then generate the required point.

Proposition 3.1. *The unbounded, densely defined operator $T_1 \colon L\log L \to L^1$ is not closed.*

Proof. Recall that $\widehat{T}$ is defined in (1.5). Fix $g \in L\log L$ such that $\widehat{T}(g) \notin L\log L$, [6, Proposition 4.12(iv)], and define $f := \widehat{T}(g) \in L^1 \setminus L\log L$, [6, Theorem 4.10(ii)]. Then $(f, T(f)) \notin \mathrm{G}(T_1)$.

Choose any sequence $\{g_n\}_{n=1}^{\infty}$ from L^∞ which converges to g in the norm of $L\log L$ (possible as $L\log L$ has absolutely continuous norm, [1, pp.247–248]). For each $n \in \mathbb{N}$ observe that $\widehat{T}(g_n) \in L\log L$ because $\widehat{T}(g_n) \in \widehat{T}(L^p) \subseteq L^p \subseteq L\log L$ for any $1 < p < 2$, [9, Theorem I.4.2]. Define $f_n := \widehat{T}(g_n)$, for each $n \in \mathbb{N}$. Then $\{f_n\}_{n=1}^{\infty} \subseteq L\log L$ converges to $\widehat{T}(g) = f$ in L^1 because $g_n \to g$ in $L\log L$ and $\widehat{T} \colon L\log L \to L^1$ is bounded, [6, Proposition 4.5]. Moreover, $T(\widehat{T}(g_n)) = g_n$, for each $n \in \mathbb{N}$, [6, Theorem 4.10(ii)]. Since $T(f_n) = T(\widehat{T}(g_n)) = g_n$ converges to g in the norm of $L\log L$, the sequence $T(f_n) \to T(\widehat{T}(g)) = g$ in L^1 as $n \to \infty$. Since the sequence $(f_n, T(f_n)) \in \mathrm{G}(T_1)$, for $n \in \mathbb{N}$, and converges to $(f, T(f))$ in $L^1 \times L^1$, it follows that $(f, T(f)) \in \overline{\mathrm{G}(T_1)}$.

Thus, $(f, T(f)) \in \overline{\mathrm{G}(T_1)} \setminus \mathrm{G}(T_1)$. This proves that $T_1 \colon L\log L \to L^1$ is not a closed operator. □

Note that the proof of Proposition 3.1 shows, in fact, that *all* functions $g \in L\log L$ satisfying $\widehat{T}(g) \notin L\log L$ generate a point $(\widehat{T}(g), g)$ in $\overline{\mathrm{G}(T_1)}$ which is not in $\mathrm{G}(T_1)$.

The fact that $T_1 \colon L\log L \to L^1$ fails to be closed is not actually a disadvantage but, is a part of the inspiration for this paper. Consider the linear subspace $\mathbb{V}$ of L^1 given in (1.2) and define the linear operator

$$T_{\mathbb{V}} \colon \mathbb{V} \to L^1,$$

by $T_{\mathbb{V}}(f) := T(f)$, for $f \in \mathbb{V}$. The distinct advantage that $T_{\mathbb{V}}$ has over T_1 is made clear in the following result together with Remark 3.3(a).

Proposition 3.2. *The unbounded, densely defined operator $T_{\mathbb{V}} \colon \mathbb{V} \to L^1$ is a closed linear extension of $T_1 \colon L\log L \to L^1$.*

Proof. Since $L\log L \subseteq \mathbb{V}$ it is clear that $\mathbb{V}$ is dense in L^1. Moreover, $T_{\mathbb{V}} \colon \mathbb{V} \to L^1$ is an extension of $T_1 \colon L\log L \to L^1$ since $L\log L \subseteq \mathbb{V}$ and $T_{\mathbb{V}}(f) = T_1(f) = T(f)$, for $f \in L\log L$.

In order to show that $\mathrm{G}(T_{\mathbb{V}})$ is closed in $L^1 \times L^1$, fix any sequence $\{(f_n, T(f_n))\}_{n=1}^{\infty} \in \mathrm{G}(T_{\mathbb{V}})$ which converges to $(f, g) \in L^1 \times L^1$ in the norm of $L^1 \times L^1$. For each $n \in \mathbb{N}$, since $f_n \in \mathbb{V}$, there exists a sequence $\{f_{n,j}\}_{j=1}^{\infty} \subseteq L\log L$ such that $f_{n,j} \to f_n$ in L^1 as $j \to \infty$ and $T(f_{n,j}) \to T(f_n)$ in L^1 as $j \to \infty$. Via a diagonal-type procedure we can choose a sequence $\{h_n\}_{n=1}^{\infty} \subseteq L\log L$ with the property that $h_n \to f$ in L^1 and $T(h_n) \to g$ in L^1. The continuity of the operator $T \colon L^1 \to L^0$ (due to Kolmogorov's theorem) implies that $T(h_n) \to T(f)$ in L^0. On the other hand, since $T(h_n) \to g$ in L^1, the continuous inclusion $L^1 \subseteq L^0$ implies that also $T(h_n) \to g$ in L^0. Hence, $g = T(f)$ in L^0. But, $g \in L^1$ and so $T(f) \in L^1$. Consequently, the sequence $\{h_n\}_{n=1}^{\infty} \subseteq L\log L$ satisfies $h_n \to f$ in L^1 and $T(h_n) \to T(f)$ in L^1. Thus, $f \in \mathbb{V}$ and so $(f, g) = (f, T(f)) \in \mathrm{G}(T_{\mathbb{V}})$. We have thereby proved that $\mathrm{G}(T_{\mathbb{V}})$ is closed. □

Remark 3.3. (a) The definition of the subspace $\mathbb{V}$ in (1.2) is somewhat involved. However, $\mathbb{V}$ corresponds exactly, in the abstract setting, to the domain space $\mathrm{D}(\overline{S})$ of the closure $\overline{S}$ of a closable Banach space-valued operator $S \colon \mathrm{D}(S) \subseteq Y \to Y$; see [12, p.166].

(b) Using the facts that L^∞ is dense in $L\log L$, that the inclusion $L\log L \subseteq L^1$ is continuous and that $T \colon L\log L \to L^1$ is bounded, we may assume in (1.2) that the sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^\infty$.

(c) Recall that $L\log L \subseteq \mathbb{V}$. However, $L\log L \neq \mathbb{V}$, which follows from fact that the operator T_1 (with domain $L\log L$) is not closed, whereas its closure $T_{\mathbb{V}}$ (with domain $\mathbb{V}$) is a closed operator. Moreover, $\mathbb{V}$ is a *proper* subspace of L^1, since the L^1-function $h(t) := 1/(t(\log t)^2)\chi_{(0,1/2)}(t)$ does not belong to $\mathbb{V}$ as $T(h) \notin L^1$, [15, Theorem 1(b)]).

(d) It follows from $L\log L \subseteq \mathbb{V}$ that $\mathbb{V}$ contains all spaces L^p for $p > 1$, as well as all r.i. spaces over $(-1, 1)$ with non-trivial Boyd indices.

(e) It is relevant to observe that $\mathbb{V}$ is not an ideal of functions. That is, $f \in \mathbb{V}$ does not necessarily imply that $f\chi_A \in \mathbb{V}$, for all $A \in \mathcal{B}$. Indeed, as a consequence of a result of Stein for the space $L\log L$, [21, Theorem 3(b)], a function $f \in L^1$ satisfies $f\chi_A \in \mathbb{V}$ for every $A \in \mathcal{B}$ precisely when $f \in L\log L$; see [6, Remark 5.7]. Accordingly, $L\log L$ is the *largest* r.i. space (in fact, the largest Banach function space) contained in $\mathbb{V}$.

(f) A consequence of (e) is that r.i. spaces such as, for example, $L(\log L)^\alpha$ for $0 < \alpha < 1$ (cf. [1, Definition IV.2.12]), which satisfy $L\log L \subsetneq L(\log L)^\alpha \subsetneq L^1$, are not contained in $\mathbb{V}$.

Despite the fact that, given $f \in \mathbb{V}$, we cannot assure that $f\chi_A \in \mathbb{V}$ for $A \in \mathcal{B}$ (cf. Remark 3.3(e)), the following result shows, nevertheless, that $\mathbb{V}$ is stable under multiplication by Lipschitz functions. This important stability property is crucial for deducing the Poincaré-Bertrand formula (cf. Theorem 5.3) from the more general Poincaré-Bertrand formula for functions on $\mathbb{R}$ (cf. Theorem 5.1). This requires, via the classical Plemelj formula, the use of Lipschitz functions of the form $s \mapsto 1/(s-z)$ on $(-1, 1)$ for $z \in \mathbb{C}$ with $\Im(z) > 0$.

Proposition 3.4. *Let $\phi\colon (-1,1) \to \mathbb{C}$ be a Lipschitz function. For every $f \in \mathbb{V}$ the function $f\phi \in \mathbb{V}$.*

Proof. Since ϕ is bounded and $f \in L^1$, their product $f\phi \in L^1$. To show that $T(f\phi) \in L^1$, fix any $x \in (-1,1)$ such that $T(f)(x)$ exists. Then

$$\begin{aligned} \pi T(f\phi)(x) &= \text{ p.v.} \int_{-1}^{1} \frac{f(y)\phi(y) \pm f(y)\phi(x)}{y-x}\, dy \\ &= \text{ p.v.} \int_{-1}^{1} f(y)\frac{\phi(y)-\phi(x)}{y-x}\, dy + \phi(x)\,\text{p.v.} \int_{-1}^{1} \frac{f(y)}{y-x}\, dy. \end{aligned} \tag{3.1}$$

For the integrand in the first integral, with $K > 0$ being a Lipschitz constant for ϕ, we have

$$\Big| f(y)\frac{\phi(y)-\phi(x)}{y-x} \Big| \le |f(y)|K \quad \text{a.e. } y \in (-1,1).$$

So, the first integral in (3.1) is non-singular and bounded in absolute value by $K\|f\|_1$. Consequently,

$$\big|T(f\phi)(x)\big| \le (K/\pi)\|f\|_1 + ||\phi||_\infty \big|T(f)(x)\big|. \tag{3.2}$$

The previous inequality implies (as $T(f) \in L^1$) that $T(f\phi) \in L^1$.

Since $f \in \mathbb{V}$, by Remark 3.3(b) there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq L^\infty$ such that $f_n \to f$ in L^1 and $T(f_n) \to T(f)$ in L^1. Then $f_n\phi \to f\phi$ in L^1, as $\phi \in L^\infty$. Apply (3.2) to $(f - f_n)$ in place of f, to obtain

$$\big|T((f-f_n)\phi)(x)\big| \le (K/\pi)\|f-f_n\|_1 + ||\phi||_\infty \big|T(f-f_n)(x)\big|,$$

for a.e. $x \in (-1,1)$. Integrating over $(-1,1)$ yields

$$\|T(f\phi) - T(f_n\phi)\|_1 \le (2K/\pi)\|f-f_n\|_1 + ||\phi||_\infty \|T(f) - T(f_n)\|_1,$$

which implies that $T(f_n\phi) \to T(f\phi)$ in L^1. So, $f\phi \in \mathbb{V}$. □

In the sequel we will require the projection $P\colon L^1 \to L^1$ given by

$$P(f) := \Big(\frac{1}{\pi}\int_{-1}^{1} f(t)\, dt\Big)\frac{1}{w}, \quad f \in L^1, \tag{3.3}$$

which satisfies $\|P\|_{L^1\to L^1} = \frac{1}{\pi}\|\frac{1}{w}\|_1 = 1$.

The linear operator $\widehat{T}$ defined in (1.5) has the property that $\widehat{T}\colon L\log L \to L^1$ is bounded, [6, Proposition 4.5]. As done previously for T, we consider the operator

$$\widehat{T}_1\colon (L\log L, \|\cdot\|_1) \to L^1,$$

whose domain $\mathrm{D}(\widehat{T}_1) := L\log L$ is equipped with the L^1-norm. In this way, $\widehat{T}_1$ is a *non*-closed, densely defined operator from L^1 into itself. However, this time, showing that $\mathrm{G}(\widehat{T}_1)$ is not closed in $L^1 \times L^1$ is achieved by exhibiting a function $g \in L\log L$ satisfying $T(g) \notin L\log L$, [6].

Proposition 3.5. *The unbounded, densely defined operator $\widehat{T}_1\colon L\log L \to L^1$ is not closed.*

Proof. Select $g \in L\log L$ such that $T(g) \notin L\log L$, [6, Proposition 4.12(iii)], and define $f := T(g) \in L^1 \setminus L\log L$. Then $(f, \widehat{T}(f)) \notin \mathrm{G}(\widehat{T}_1)$.

Choose any sequence $\{g_n\}_{n=1}^\infty$ from L^∞ which converges to g in the norm of $L\log L$. For each $n \in \mathbb{N}$ observe that $T(g_n) \in L\log L$ because $T(g_n) \in T(L^p) \subseteq L^p \subseteq L\log L$ for any $1 < p < \infty$. Define $f_n := T(g_n)$, for each $n \in \mathbb{N}$. Then $\{f_n\}_{n=1}^\infty \subseteq L\log L$ converges to $f = T(g)$ in L^1 because $g_n \to g$ in $L\log L$ and $T\colon L\log L \to L^1$ is bounded, [6, Theorem 2.1]. Then $\widehat{T}(f_n) = \widehat{T}(T(g_n)) = g_n - P(g_n)$ for each $n \in \mathbb{N}$, [6, Theorem 4.10(iv)]. But, $g_n - P(g_n)$ converges to $g - P(g)$ in the norm of $L\log L$ and hence, $\widehat{T}(f_n) \to g - P(g)$ in L^1 as $n \to \infty$, [6, Proposition 4.5]. Since, $g - P(g) = \widehat{T}(T(g)) = \widehat{T}(f)$, it follows that $\widehat{T}(f_n) \to \widehat{T}(f)$ in L^1 as $n \to \infty$. So, the sequence $(f_n, \widehat{T}(f_n)) \in \mathrm{G}(\widehat{T}_1)$, for $n \in \mathbb{N}$, converges in $L^1 \times L^1$ to $(f, \widehat{T}(f))$, that is, $(f, \widehat{T}(f)) \in \overline{\mathrm{G}(\widehat{T}_1)}$.

Thus, $(f, \widehat{T}(f)) \in \overline{\mathrm{G}(\widehat{T}_1)} \setminus \mathrm{G}(\widehat{T}_1)$. This proves that $\widehat{T}_1\colon L\log L \to L^1$ is not closed. □

As done before with T_1, we take advantage of the fact that $\widehat{T}_1\colon L\log L \to L^1$ is not closed. Consider the linear subspace of L^1 given by

$$\begin{aligned}\widehat{\mathbb{V}} := \big\{ f \in L^1 : \widehat{T}(f) \in L^1, \exists \{f_n\}_{n=1}^\infty \subseteq L\log L,\\ \|f_n - f\|_1 \to 0, \|\widehat{T}(f_n) - \widehat{T}(f)\|_1 \to 0 \big\}\end{aligned} \tag{3.4}$$

and the unbounded operator

$$\widehat{T}_{\widehat{\mathbb{V}}}\colon \widehat{\mathbb{V}} \to L^1$$

defined by $\widehat{T}_{\widehat{\mathbb{V}}}(g) := \widehat{T}(g)$, for $g \in \widehat{\mathbb{V}}$. Since $L\log L \subseteq \widehat{\mathbb{V}}$ it is clear that $\widehat{\mathbb{V}}$ is dense in L^1. The proof of the next result concerning $\widehat{T}$ follows by suitably adapting the proof of Proposition 3.2 for T; the details are omitted.

Proposition 3.6. *The unbounded, densely defined operator $\widehat{T}_{\widehat{\mathbb{V}}}\colon \widehat{\mathbb{V}} \to L^1$ is a closed extension of $\widehat{T}_1\colon L\log L \to L^1$.*

Remark 3.7. (a) The same argument as in Remark 3.3(a) shows that the operator $\widehat{T}_{\widehat{\mathbb{V}}}$ is precisely the *closure* of $\widehat{T}_1$.

(b) By arguing as in Remark 3.3(b), we may assume in (3.4) that $\{f_n\}_{n=1}^\infty \subseteq L^\infty$.

(c) Since $\widehat{T}_1$ is not closed, we can conclude (argue as in Remark 3.3(c)) that $L\log L \subsetneq \widehat{\mathbb{V}}$. Moreover, $\widehat{\mathbb{V}}$ is a *proper* subspace of L^1. To see this, let $g := h/w$, where h is the function $h(t) := (t\log t)^{-2}\chi_{(0,1/2)}(t)$ appearing in Remark 3.3(c). Then $g \in L^1$ (as $\chi_{(0,1/2)}/w$ is bounded) and $\widehat{T}(g) = (-1/w)T(h)$ does not belong to $\widehat{\mathbb{V}}$ as $T(h) \notin L^1$.

4. The Parseval formula

The aim of this section is to establish a Parseval formula for the FHT operator $T_{\mathbb{V}}\colon \mathbb{V} \to L^1$. For this we need to identify the domain $\mathrm{D}(T_{\mathbb{V}}^*) \subseteq L^\infty$ of the unbounded adjoint operator $T_{\mathbb{V}}^*\colon \mathrm{D}(T_{\mathbb{V}}^*) \to L^\infty$.

Given a Banach space Y and a linear operator $S\colon \mathrm{D}(S) \to Y$ with *dense domain* $\mathrm{D}(S) \subseteq Y$, the *adjoint* of S is the operator $S^*\colon \mathrm{D}(S^*) \subseteq Y^* \to Y^*$, whose domain is the subspace of Y^* given by

$$\mathrm{D}(S^*) := \big\{ y^* \in Y^* : y \in \mathrm{D}(S) \mapsto \langle S(y), y^* \rangle \in \mathbb{C} \text{ is continuous} \big\}.$$

The adjoint operator S^* is then defined by $y^* \in \mathrm{D}(S^*) \longmapsto \overline{y^* S}$, where $\overline{y^* S}$ denotes the *extension* to all of Y of the continuous linear functional $y \in \mathrm{D}(S) \mapsto \langle S(y), y^* \rangle \in \mathbb{C}$. This extension exists and is unique because $\mathrm{D}(S)$ is dense in Y. Equivalently, $\mathrm{D}(S^*)$ consists of all $y^* \in Y^*$ for which there exists $u^* \in Y^*$, necessarily unique, satisfying

$$\langle Sy, y^* \rangle = \langle y, u^* \rangle, \quad y \in \mathrm{D}(S), \tag{4.1}$$

in which case $S^*(y^*) = \overline{y^* S} = u^*$. For those aspects of this theory used in the current section we refer to [10, Ch.II, §2], [11, §2.11] and [12, Ch.3, §5], for instance.

The previous abstract scheme applied to $T_{\mathbb{V}}\colon \mathbb{V} \to L^1$ yields that $\mathrm{D}(T_{\mathbb{V}}^*) = \big\{ g \in L^\infty : \overline{gT_{\mathbb{V}}} \in L^\infty \big\} \subseteq L^\infty$, that the adjoint operator $T_{\mathbb{V}}^*\colon \mathrm{D}(T_{\mathbb{V}}^*) \to L^\infty$ is defined by $T_{\mathbb{V}}^*(g) := \overline{gT_{\mathbb{V}}}$ for $g \in \mathrm{D}(T_{\mathbb{V}}^*)$, and that it satisfies

$$\int_{-1}^{1} gT(f) = \int_{-1}^{1} gT_{\mathbb{V}}(f) = \int_{-1}^{1} T_{\mathbb{V}}^*(g) f, \quad f \in \mathbb{V},\ g \in \mathrm{D}(T_{\mathbb{V}}^*).$$

To identify $\mathrm{D}(T_{\mathbb{V}}^*)$ more concretely we consider the subspace of L^∞ given by

$$\mathbb{W} := L^\infty \cap T^{-1}(L^\infty) = \big\{ g \in L^\infty : T(g) \in L^\infty \big\}$$

and equip it with the L^∞-*norm*. Note, since T does not map L^∞ into itself (actually, $T\colon L^\infty \to L_{\exp}$ boundedly, [5, (2.4)]), the requirement on g that $T(g) \in L^\infty$ is a restrictive one in the definition of $\mathbb{W}$. The restriction of T to the normed space $\mathbb{W}$ generates the *unbounded* linear operator

$$T_{\mathbb{W}}\colon \mathbb{W} \to L^\infty$$

defined by $T_{\mathbb{W}}(g) := T(g)$, for $g \in \mathbb{W}$.

The following result establishes the Parseval formula alluded to above which, in turn, leads to an unexpected identification of $\mathrm{D}(T_{\mathbb{V}}^*)$.

Theorem 4.1. *The following Parseval formula is valid, namely*

$$\int_{-1}^{1} fT(g) = -\int_{-1}^{1} gT(f), \quad f \in \mathbb{V},\ g \in \mathbb{W}. \tag{4.2}$$

Moreover, $\mathrm{D}(T_{\mathbb{V}}^*) = \mathbb{W}$ *and the adjoint of* $T_{\mathbb{V}}$ *is the closed, unbounded operator*

$$T_{\mathbb{V}}^* = (-T_{\mathbb{W}})\colon \mathbb{W} \to L^\infty.$$

Proof. Fix $g \in \mathbb{W}$. According to (1.2), for each $f \in \mathbb{V}$ the function $T(f) \in L^1$ and there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq L\log L$ such that $f_n \to f$ in L^1 and $T(f_n) \to T(f)$ in L^1. Since $g \in \mathbb{W}$, both functions $g, T(g) \in L^\infty$. The Parseval formula for $L\log L$, [6, Corollary 3.3], implies that

$$\begin{aligned}\int_{-1}^1 fT(g) &= \int_{-1}^1 \lim_n f_n T(g) = \lim_n \int_{-1}^1 f_n T(g) \\ &= -\lim_n \int_{-1}^1 gT(f_n) = -\int_{-1}^1 g \lim_n T(f_n) = -\int_{-1}^1 gT(f),\end{aligned}$$

where we have used the fact that both $f_n T(g) \to fT(g)$ in L^1 and $gT(f_n) \to gT(f)$ in L^1. This establishes (4.2).

Let $g \in \mathrm{D}(T_{\mathbb{V}}^*)$, that is, $g \in L^\infty$ and

$$\Big|\int_{-1}^1 gT(f)\Big| \le K\|f\|_1, \quad f \in \mathbb{V}, \tag{4.3}$$

for some constant $K > 0$. Then (4.2) and the density of $\mathbb{V}$ in L^1 imply that (4.3) is actually valid for all $f \in L^1$. Accordingly, $T(g) \in (L^1)^* = L^\infty$. So, $g \in \mathbb{W}$ and we have established that $\mathrm{D}(T_{\mathbb{V}}^*) \subseteq \mathbb{W}$.

For the reverse inclusion let $h \in \mathbb{W}$, that is, both $h, T(h) \in L^\infty$. The Parseval formula (4.2) shows that $u^* := -T(h) \in L^\infty$ satisfies

$$\langle T_{\mathbb{V}}(f), h\rangle = \langle T(f), h\rangle = \langle f, u^*\rangle, \quad f \in \mathbb{V} = \mathrm{D}(T_{\mathbb{V}}),$$

and hence, (4.1) yields that $h \in \mathrm{D}(T_{\mathbb{V}}^*)$.

The adjoint of a closed, densely defined operator is always closed, [11, Theorem 2.11.9]. □

Remark 4.2. (a) The classical Parseval formula for L^p with $1 < p < \infty$ occurs in [20, §2]. For r.i. spaces with non-trivial Boyd indices we refer to [3, Proposition 3.1] and for $L\log L$, see [6, Theorem 3.1 and Corollary 3.3].

(b) A closable, densely defined operator S and its closure $\overline{S}$ have the *same* adjoint operator. That is, $\mathrm{D}(S^*) = \mathrm{D}(\overline{S}^*)$ and $S^* = \overline{S}^*$, [10, Theorem II.2.11]. In our case, this implies that $T_1^* = T_{\mathbb{V}}^*$ with common domain $\mathbb{W}$.

The importance of $\mathbb{W}$, which is clear from its role in the Parseval formula (4.2), makes an examination of its properties necessary.

Proposition 4.3. *The following properties of* $\mathbb{W}$ *are valid.*

(i) $\mathbb{W} \neq \{0\}$ *and it is a proper subspace of* L^∞.
(ii) $\mathbb{W}$ *is a non-closed, weak*-dense subspace of* L^∞ *but, it is not norm-dense.*
(iii) *The following description of* $\mathbb{W}$ *is valid, namely*

$$\mathbb{W} = \Big\{g \in L^\infty : g = -w\,T\Big(\frac{h}{w}\Big),\ h \in L^\infty \cap T(L^\infty)\Big\}.$$

(iv) *The containment* $T(\mathbb{W}) \subseteq \{h \in L^\infty : \int_{-1}^1 \frac{h}{w} = 0\}$ *is proper.*

(v) *The operator T does not map $\mathbb{W}$ into $\mathbb{W}$.*
(vi) *The range space $T(\mathbb{W})$ is not norm-dense in L^∞.*

Proof. (i) The function $w \in L^\infty$ (cf. (1.6)) and $T(w)(x) = -x$, which also belongs to L^∞. Hence, $w \in \mathbb{W}$, [14, (12A.8) Appendix 1, p.519] (note that the FHT used in [14] is $-T$).

For $-1 \le a < b \le 1$, direct computation shows that $T(\chi_{(a,b)})(x) = \frac{1}{\pi}\log\left|\frac{b-x}{a-x}\right|$, which is unbounded on $(-1,1)$. So, $\chi_{(a,b)} \notin \mathbb{W}$.

(ii) Since T does not map L^1 boundedly into itself, the closed operator $T_{\mathbb{V}}\colon \mathbb{V} \to L^1$ is not continuous on all of L^1 and hence, the subspace $\mathbb{W} = \mathrm{D}(T_{\mathbb{V}}^*)$ is not closed in L^∞, [10, Corollary II.4.8].

That $\mathbb{W}$ is weak*-dense in L^∞ follows from Theorem 4.1 and [11, Theorem 2.11.9].

We prove that $\mathbb{W}$ is not norm-dense in L^∞ by showing, for $f_0 := \chi_{(-1,1)} \in L^\infty$, that $\operatorname{dist}\big(f_0, \mathbb{W}\big) := \inf\big\{\|f_0 - g\|_\infty : g \in \mathbb{W}\big\} > 0$. Fix any $g \in \mathbb{W}$. Then both $g, T(g) \in L^\infty$. Set $M := \|T(g)\|_\infty$ and define

$$A := \big\{x \in (-1,1) : |T(f_0)(x)| > M\big\}.$$

Since $T\colon L^\infty \to L_{\exp}$ is bounded and, for a.e. $x \in A$, we have

$$\Big|T(f_0)(x) - T(g)(x)\Big| \ge |T(f_0)(x)| - |T(g)(x)| \ge |T(f_0)(x)| - M > 0,$$

it follows (as $T\colon L^\infty \to L_{\exp}$ boundedly) that

$$\begin{aligned} \|f_0 - g\|_\infty \cdot \|T\|_{L^\infty \to L_{\exp}} &\ge \|T(f_0) - T(g)\|_{L_{\exp}} \\ &\ge \big\|\big(T(f_0) - T(g)\big)\chi_A\big\|_{L_{\exp}} \ge \big\|\big(|T(f_0)| - M\big)\chi_A\big\|_{L_{\exp}}. \end{aligned} \tag{4.4}$$

Recall, for $h \in L_{\exp}$ (cf. [1, Lemma IV.6.12]), that

$$\|h\|_{L_{\exp}} = \sup_{0<t<2} \frac{\int_0^t h^*(s)\,ds}{t\log(2e/t)} \ge \sup_{0<t<2} \frac{h^*(t)}{\log(2e/t)} \ge \lim_{t\to 0^+} \frac{h^*(t)}{\log(2e/t)}. \tag{4.5}$$

So, from (4.4) and (4.5), we proceed to compute

$$\lim_{t\to 0^+} \frac{\psi^*(t)}{\log(2e/t)} \tag{4.6}$$

for the function $\psi := \big(|T(f_0)| - M\big)\chi_A$.

Since $T(f_0)(x) = \frac{1}{\pi}\log\left(\frac{1-x}{1+x}\right)$, for $x \in (-1,1)$, for each $\lambda > 0$ we have that

$$m\Big(\big\{x \in (-1,1) : \big|T(f_0(x))\big| > \lambda\big\}\Big) = \frac{4}{e^{\lambda\pi} + 1}.$$

Hence, for the decreasing rearrangement of $T(f_0)$, direct calculation shows that

$$\big(T(f_0)\big)^*(t) = \frac{1}{\pi}\log\left(\frac{4}{t} - 1\right), \quad 0 < t < 2.$$

From the equimeasurability between a function and its rearrangement, it follows that

$$m(A) = m\Big(\Big\{t \in (0,2) : \frac{1}{\pi}\log\Big(\frac{4}{t} - 1\Big) > M\Big\}\Big) = \frac{4}{e^{M\pi}+1}.$$

Accordingly,

$$\psi^*(t) = \begin{cases} \frac{1}{\pi}\log\left(\frac{4}{t}-1\right) - M, & \text{if } 0 < t < \frac{4}{e^{M\pi}+1}, \\ 0, & \text{if } \frac{4}{e^{M\pi}+1} < t < 2. \end{cases} \tag{4.7}$$

From (4.5), (4.6) and (4.7) we can conclude, for each $g \in \mathbb{W}$, that

$$\|f_0 - g\|_\infty \cdot \|T\|_{L^\infty \to L_{\exp}} \geq \lim_{t\to 0} \frac{\frac{1}{\pi}\log\left(\frac{4}{t}-1\right) - \|T(g)\|_\infty}{\log(2e/t)} = \frac{1}{\pi}.$$

The proof is thereby complete.

(iii) Let $g \in \mathbb{W}$, that is, $g \in L^\infty$ and $h := T(g) \in L^\infty \cap T(L^\infty)$. Then, for any $2 < p < \infty$, we have $g \in L^p$ and $h = T(g) \in L^p$. Since $\int_{-1}^1 \frac{h}{w} = \int_{-1}^1 \frac{T(g)}{w} = 0$, [20, Corollary 2.8], the inversion theorem for T on L^p, [20, Corollary 2.8], implies that $g = -w\,T(h/w)$.

Conversely, let $h \in L^\infty \cap T(L^\infty)$. Then $h = T(f)$ for some unique $f \in L^\infty$ and $\int_{-1}^1 \frac{h}{w} = 0$. Arguing as above yields $f = -wT(h/w)$ and so $-wT(h/w) \in \mathbb{W}$.

(iv) For each $2 < p < \infty$ the range of $T\colon L^p \to L^p$ is the subspace $\{h \in L^p : \int_{-1}^1 \frac{h}{w} = 0\}$, [20, Corollary 2.8]. Thus, if $h \in T(L^\infty)$ then, for any choice of $2 < p < \infty$, the function $h \in T(L^p)$ and so the condition $\int_{-1}^1 \frac{h}{w} = 0$ is satisfied. Hence, $T(\mathbb{W}) \subseteq \big\{h \in L^\infty : \int_{-1}^1 \frac{h}{w} = 0\big\}$.

To show that the containment is strict, consider the function $h(x) := w(x)(\chi_{(0,1)} - \chi_{(-1,0)}) \in L^\infty$. First, it satisfies $\int_{-1}^1 \frac{h}{w} = 0$. Direct computation shows that $T(h/w)(x) = T(\chi_{(0,1)})(x) - T(\chi_{(-1,0)})(x) = \frac{1}{\pi}\log\left(\frac{1-x^2}{x^2}\right)$. The singularities of $T(h/w)(x)$ at $x = \pm 1$ get cancelled when multiplying by $w(x)$ but, not the singularity at $x = 0$. So, $g := -wT(h/w) \notin L^\infty$ and hence, from part (iii), $g \notin \mathbb{W}$ but, $T(g) = h$. Since T is injective on $\mathbb{W}$, the function $h \notin T(\mathbb{W})$.

(v) Note that $T(w)(x) = -x$ and $T(t)(x) = \frac{2}{\pi} + \frac{x}{\pi}\log\left(\frac{1-x}{1+x}\right)$, [14, (12A.8), (12A.2) Appendix 1, p.519]. Thus, $w \in \mathbb{W}$ whereas $T(T(w))$ is unbounded on $(-1,1)$ and so $T(w) \notin \mathbb{W}$.

(vi) Since $T_{\mathbb{V}}\colon \mathbb{V} \to L^1$ fails to be injective, it follows from [11, Theorem 2.11.13] that $T(\mathbb{W})$ is not weak*-dense in L^∞ and hence, it is not norm-dense in L^∞. □

In view of the Parseval formula and the previous result it is important to have criteria available for membership in $\mathbb{W}$. The next proposition exhibits some classes of functions within $\mathbb{W}$.

Proposition 4.4. (i) *For each $n \geq 0$, the n-th Chebyshev polynomial of the second kind U_n satisfies $wU_n \in \mathbb{W}$.*

(ii) *Let* $\phi\colon (-1,1) \to \mathbb{C}$ *be a Lipschitz function. For every* $g \in \mathbb{W}$ *the function* $g\phi \in \mathbb{W}$.

(iii) *Let* h *be a Lipschitz function satisfying* $\int_{-1}^{1}(h/w) = 0$. *Then the function* $-wT(h/w) \in \mathbb{W}$.

(iv) *Let* $\alpha \in (0,1]$ *and* g *be a Hölder function of class* α *such that* $g(x) = O\big(\log^{-1}\big(\frac{1-x}{1+x}\big)\big)$ *when* $|x| \to 1$. *Then* $g \in \mathbb{W}$.

Proof. (i) Clearly wU_n, for each $n \geq 0$, is bounded in $(-1,1)$. Moreover, for the n-th Chebyshev polynomial of the first kind T_n, it is known that $T(wU_n) = -T_{n+1} \in L^\infty$, [22, §4.3 (23)]. Hence, $wU_n \in \mathbb{W}$.

(ii) Since ϕ is bounded and $g \in L^\infty$, the function $g\phi$ is bounded. The proof of Proposition 3.4 can be adapted to show that also $T(g\phi) \in L^\infty$. Hence, $g\phi \in \mathbb{W}$.

(iii) Since $h \in L^\infty$ and $1/w \in L^1$, the function $h/w \in L^1$. Fix any point $x \in (-1,1)$ such that $T(h/w)(x)$ exists. Recalling that $T(1/w) = 0$ yields

$$\begin{aligned}\pi T(h/w)(x) &= \text{p.v.}\int_{-1}^{1} \frac{h(y)}{w(y)(y-x)}\,dy - h(x)\,\text{p.v.}\int_{-1}^{1} \frac{dy}{w(y)(y-x)}\,dy \\ &= \int_{-1}^{1} \frac{h(y)-h(x)}{w(y)(y-x)}\,dy.\end{aligned}$$

Taking absolute values and using $|h(y) - h(x)| \leq K|y-x|$, where $K > 0$ is a Lipschitz constant for h, we arrive at

$$\Big|T(h/w)(x)\Big| \leq \frac{1}{\pi}K\int_{-1}^{1}\frac{dy}{w(y)} = K.$$

Hence, $g := -wT(h/w)$ is bounded.

Fix $2 < p < \infty$. Noting that $L^\infty \subseteq L^p$, it follows from Theorem 3.3 of [3] and the assumption $\int_{-1}^{1}\frac{h}{w} = 0$ that $T(g) = h$. Since $g \in L^\infty$, it follows that $h \in T(L^\infty)$. So, we can conclude from part (iii) of Proposition 4.3 that $g \in \mathbb{W}$.

(iv) Since $g \in L^\infty$, it suffices to show that also $T(g) \in L^\infty$. Fix any point $x \in (-1,1)$ such that $T(g)(x)$ exists. Then

$$\begin{aligned}\pi T(g)(x) &= \text{p.v.}\int_{-1}^{1}\frac{g(y) \pm g(x)}{y-x}\,dy \\ &= \int_{-1}^{1}\frac{g(y)-g(x)}{y-x}\,dy + g(x)\,\text{p.v.}\int_{-1}^{1}\frac{dy}{y-x}. \qquad (4.8)\end{aligned}$$

In the first integral, since g is of Hölder class α, there exists $K > 0$ such that

$$\begin{aligned}\int_{-1}^{1}\Big|\frac{g(y)-g(x)}{y-x}\Big|\,dy &\leq K\int_{-1}^{1}\frac{|y-x|^\alpha}{|y-x|}\,dy \\ &= K\int_{-1}^{x}\frac{dy}{(x-y)^{1-\alpha}}\,dy + K\int_{x}^{1}\frac{dy}{(y-x)^{1-\alpha}}\,dy.\end{aligned}$$

For $\alpha = 1$ the right-side equals $2K$ and for $0 < \alpha < 1$ the right-side is equal to $\frac{K}{\alpha}((1-x)^\alpha + (1+x)^\alpha)$. In both cases the right-side is a bounded function.

Concerning the second integral in (4.8), the so defined function

$$x \mapsto g(x)\,\text{p.v.}\int_{-1}^{1} \frac{dy}{y-x} = g(x)\log\Big(\frac{1-x}{1+x}\Big),$$

is also bounded because both functions are continuous in $(-1,1)$, the function g is bounded and, by assumption, the product function $g(x)\log\big(\frac{1-x}{1-x}\big)$ is also bounded as $|x| \to 1$. So, both g and $T(g)$ are bounded, that is, $g \in \mathbb{W}$. $\square$

Remark 4.5. The Lipschitz condition on ϕ in Proposition 4.4(ii) cannot be replaced, in general, by a Hölder condition. Using the inequality $|\sqrt{a} - \sqrt{b}| \le |a-b|^{1/2}$, for $a, b > 0$, it follows that the continuous function $\phi := w$ is of Hölder class $\alpha = 1/2$. Moreover, recall that $h := w \in \mathbb{W}$. However, the bounded product function $\phi h = w^2 \notin \mathbb{W}$ as $T(w^2) \notin L^\infty$. This follows from the formulae

$$T(w^2)(x) = T(1)(x) - T(t^2)(x) = \frac{(1+x^2)}{\pi}\log\Big(\frac{1-x}{1+x}\Big) + \frac{2x}{\pi}, \quad x \in (-1,1);$$

see (12A.1) and (12A.3) in [14, Appendix 1, p.519].

Although it will not be needed in the sequel, we mention a further application of the previous abstract scheme to identify the adjoint $(\widehat{T}_{\widehat{\mathbb{V}}})^*$ of the auxiliary operator $\widehat{T}_{\widehat{\mathbb{V}}}$. To achieve this we require the linear, L^0-valued operator $\check{T}$ defined by

$$f \mapsto \check{T}(f) := -wT\Big(\frac{f}{w}\Big),$$

for all $f \in L^1$ satisfying $f/w \in L^1$. This operator appears in the solution of the airfoil equation when considering the FHT acting on L^p, for $2 < p < \infty$, [20, Proposition 2.6], and on r.i. spaces X with Boyd indices in the interval $(0, 1/2)$, [3, Theorem 3.3]. To determine the domain of $(\widehat{T}_{\widehat{\mathbb{V}}})^*$, consider the subspace of L^∞ given by

$$\widehat{\mathbb{W}} := \Big\{g \in L^\infty : wT\Big(\frac{g}{w}\Big) \in L^\infty\Big\} = \Big\{g \in L^\infty : \check{T}(g) \in L^\infty\Big\},$$

equipped with the L^∞-norm. We point out that $\widehat{\mathbb{W}}$ is a *proper* subspace of L^∞. Indeed, $f_1 = \chi_{(0,1)} \in L^\infty$ but, $\check{T}(f_1)(x) = \frac{-1}{\pi}\log\big|\frac{1+\sqrt{1-x^2}}{x}\big|$, for $x \in (-1,1)$, is an unbounded function; see the proof of Lemma 4.3 in [20].

The proof of Theorem 4.1 can be adapted to establish the next result.

Theorem 4.6. *The following Parseval formula is valid, namely*

$$\int_{-1}^{1} g\widehat{T}(f) = -\int_{-1}^{1} f\check{T}_{\widehat{\mathbb{W}}}(g) = \int_{-1}^{1} fwT\Big(\frac{g}{w}\Big), \quad f \in \widehat{\mathbb{V}},\ g \in \widehat{\mathbb{W}}.$$

Moreover, $\mathrm{D}((\widehat{T}_{\widehat{\mathbb{V}}})^*) = \widehat{\mathbb{W}}$ *and the adjoint of* $\widehat{T}_{\widehat{\mathbb{V}}}$ *is the closed, unbounded operator*

$$(\widehat{T}_{\widehat{\mathbb{V}}})^* = (-\check{T}_{\widehat{\mathbb{W}}})\colon \widehat{\mathbb{W}} \to L^\infty,$$

where $\breve{T}_{\widehat{\mathbb{W}}}(g) := \breve{T}(g)$, *for* $g \in \widehat{\mathbb{W}}$.

5. The Poincaré-Bertrand formula

The main result of this section is Theorem 5.3 below in which the Poincaré-Bertrand formula (5.8) for the pair of spaces $\mathbb{V}$ and $\mathbb{W}$ is established.

We begin with the following 'Poincaré-Bertrand type' result. Its deviation from the usual Poincaré-Bertrand theorem is that membership in the spaces to which the functions belong (e.g., $\phi_1 \in L^{p_1}, \phi_2 \in L^{p_2}$) is now replaced by conditions on the functions ϕ_1, ϕ_2 themselves and their Hilbert transforms. The proof is inspired by Love's proof of [17, Theorem]. Let $H\colon L^1(\mathbb{R}) \to L^0(\mathbb{R})$ denote the Hilbert transform on $\mathbb{R}$.

Theorem 5.1. *Let* $\phi_1, \phi_2 \in L^1(\mathbb{R})$ *satisfy the following two conditions.*

(i) *Both of the product functions* $H(\phi_1)\,\phi_2$ *and* $\phi_1\, H(\phi_2)$ *belong to* $L^1(\mathbb{R})$.

(ii) *For all* $z = x + iy$ *with* $x \in \mathbb{R}$ *and* $y > 0$, *the following Parseval-type formula*

$$\int_{-\infty}^{\infty} \phi_z(s) H(\phi_2)(s)\, ds = -\int_{-\infty}^{\infty} H(\phi_z)(s)\phi_2(s)\, ds,$$

holds, where $\phi_z(s) := \phi_1(s)/(s-z)$, *for* $s \in \mathbb{R}$, *and both* $\phi_z H(\phi_2), \phi_2 H(\phi_z)$ *belong to* $L^1(\mathbb{R})$.

Then the following Poincaré-Bertrand formula is valid, namely

$$H\big(\phi_1 H(\phi_2) + \phi_2 H(\phi_1)\big) = H(\phi_1)H(\phi_2) - \phi_1\phi_2, \quad \textit{a.e. in } \mathbb{R}.$$

Proof. It suffices to prove the result for ϕ_1, ϕ_2 real functions.

Fix $z = x + iy$ with $x \in \mathbb{R}$ and $y > 0$. Since $\phi_1 \in L^1(\mathbb{R})$ and $y > 0$ imply that $\rho_z(s) := 1/(s-z)$, for $s \in \mathbb{R}$, belongs to $L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, we have $\phi_z = \phi_1\rho_z \in L^1(\mathbb{R})$ and so, its Hilbert transform $H(\phi_z)(s)$ exists for a.e. $s \in \mathbb{R}$. Then, because both $H(\phi_z)(s)$ and $H(\phi_1)(s)$ exist simultaneously for a.e. $s \in \mathbb{R}$, we have

$$\begin{aligned} H(\phi_z)(s) &= \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{\phi_z(t)}{t-s}\, dt = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{\phi_1(t)}{(t-s)(t-z)}\, dt \\ &= \frac{1}{(s-z)}\frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\left(\frac{1}{t-s} - \frac{1}{t-z}\right)\phi_1(t)\, dt \qquad (5.1) \\ &= \frac{H(\phi_1)(s)}{(s-z)} - \frac{1}{(s-z)}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_1(t)}{t-z}\, dt, \end{aligned}$$

where the last integral is a Lebesgue integral as $\phi_1\rho_z \in L^1(\mathbb{R})$.

The identity (5.1) and the fact that $\phi_2 H(\phi_z) \in L^1(\mathbb{R})$ yield

$$\begin{aligned} \int_{-\infty}^{\infty} H(\phi_z)(s)\phi_2(s)\, ds = &\int_{-\infty}^{\infty} \frac{H(\phi_1)(s)}{s-z}\phi_2(s)\, ds \\ &- \left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_2(s)}{s-z}\, ds\right)\left(\int_{-\infty}^{\infty}\frac{\phi_1(t)}{t-z}\, dt\right), \qquad (5.2) \end{aligned}$$

where the separation into two terms in the right-side is possible because both $\phi_1\rho_z$ and $\phi_2\rho_z$ belong to $L^1(\mathbb{R})$.

Condition (ii) and (5.2) imply that

$$\begin{aligned}\int_{-\infty}^{\infty}\phi_z(s)H(\phi_2)(s)\,ds &= -\int_{-\infty}^{\infty}H(\phi_z)(s)\phi_2(s)\,ds\\ &= -\int_{-\infty}^{\infty}\frac{H(\phi_1)(s)}{s-z}\phi_2(s)\,ds\\ &\quad+\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_2(s)}{s-z}\,ds\right)\left(\int_{-\infty}^{\infty}\frac{\phi_1(t)}{t-z}\,dt\right).\end{aligned}\tag{5.3}$$

The identity (5.3) can be rewritten as

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_1(s)H(\phi_2)(s)}{s-z}\,ds+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(\phi_1)(s)\phi_2(s)}{s-z}\,ds\\ &\qquad=\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_2(s)}{s-z}\,ds\right)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_1(t)}{t-z}\,dt\right).\end{aligned}\tag{5.4}$$

We are now in the situation where we can apply Plemelj's formula, [17, Theorem 92'], to each of the four integrals in (5.4).

Accordingly, for a.e. $x\in\mathbb{R}$, taking the limit in (5.4) when $y\to 0^+$, we have

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_1(s)H(\phi_2)(s)}{s-z}\,ds &\longrightarrow \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_1(s)H(\phi_2)(s)}{s-x}\,ds+i\phi_1(x)H(\phi_2)(x),\\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(\phi_1)(s)\phi_2(s)}{s-z}\,ds &\longrightarrow \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(\phi_1)(s)\phi_2(s)}{s-x}\,ds+iH(\phi_1)(x)\phi_2(x),\\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_2(s)}{s-z}\,ds &\longrightarrow H(\phi_2)(x)+i\phi_2(x),\\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_1(t)}{t-z}\,dt &\longrightarrow H(\phi_1)(x)+i\phi_1(x).\end{aligned}\tag{5.5}$$

After taking the limit in (5.4), we add together the terms in the right-side of (5.5) and arrive at the identity

$$\begin{aligned}&\text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\phi_1(s)H(\phi_2)(s)}{s-x}\,ds+\text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(\phi_1)(s)\phi_2(s)}{s-x}\,ds\\ &\qquad=H(\phi_1)(x)H(\phi_2)(x)-\phi_1(x)\phi_2(x),\end{aligned}$$

which is what was to be proved. □

For $f\in L^0$ define $\tilde{f}\in L^0(\mathbb{R})$ by $\tilde{f}:=f\chi_{(-1,1)}$. Then $\tilde{f}\in L^1(\mathbb{R})$ whenever $f\in L^1$ and $H(\tilde{f})(x)=T(f)(x)$, for all $x\in(-1,1)$. With these identifications we have the following consequence of Theorem 5.1.

Corollary 5.2. *Let functions $f,g\in L^1$ satisfy the following two conditions.*

(i) *Both of the product functions $gT(f)$ and $fT(g)$ belong to L^1.*

(ii) *For all* $z = x + iy$ *with* $x \in (-1,1)$ *and* $y > 0$ *the following Parseval-type formula*

$$\int_{-1}^{1} f_z(s)T(g)(s)\,ds = -\int_{-1}^{1} g(s)T(f_z)(s)\,ds, \tag{5.6}$$

is valid, where $f_z(s) := f(s)/(s-z)$, *for* $s \in (-1,1)$, *and both functions* $f_zT(g), gT(f_z)$ *belong to* L^1.

Then the following Poincaré-Bertrand formula holds, namely

$$T\big(fT(g) + gT(f)\big) = T(f)T(g) - fg, \quad \textit{a.e. in } (-1,1). \tag{5.7}$$

From Corollary 5.2 we can deduce the Poincaré-Bertrand formula (1.4) for the pair of spaces $\mathbb{V}$ and $\mathbb{W}$.

Theorem 5.3. *Let* $f \in \mathbb{V}$ *and* $g \in \mathbb{W}$. *Then*

$$T\big(fT(g) + gT(f)\big) = T(f)T(g) - fg, \quad \textit{a.e. in } (-1,1). \tag{5.8}$$

Proof. Observe that $f \in \mathbb{V}$ (resp. $g \in \mathbb{W}$) implies that both $f, T(f)$ belong to L^1 (resp. both $g, T(g)$ belong to L^∞). So, condition (i) of Corollary 5.2 is satisfied.

Fix $z = x + iy$ with $x \in (-1,1)$ and $y > 0$. Then $\rho_z(s) = 1/(s-z)$, for $s \in (-1,1)$, is a Lipschitz function. Since $f \in \mathbb{V}$, it follows from Proposition 3.4 that $f_z := g\rho_z \in \mathbb{V}$ and hence, both $f_z, T(f_z) \in L^1$, which implies that both $f_zT(g)$ and $gT(f_z)$ belong to L^1. Moreover, Theorem 4.1 for $f_z \in \mathbb{V}$ and $g \in \mathbb{W}$ yields

$$\int_{-1}^{1} f_zT(g) = -\int_{-1}^{1} gT(f_z),$$

which is precisely (5.6). All the assumptions in Corollary 5.2 are satisfied and so the formula (5.7) in Corollary 5.2 yields (5.8). □

Remark 5.4. (a) The Poincaré-Bertrand formula for $\phi_1 \in L^{p_1}$ and $\phi_2 \in L^{p_2}$ with $1/p_1 + 1/p_2 < 1$ can be found in [22, §4.3 (4)]. Love's extension to the case when $1/p_1 + 1/p_2 = 1$ occurs in [17, Corollary]. A further extension to pairs of functions $f \in X$ and $g \in X'$, where X is any r.i. space on $(-1,1)$ with non-trivial Boyd indices and associate space X', is given in [3, Proposition 3.1]. For functions $f \in L\log L$ and $g \in L^\infty$ we refer to [6, Corollary 3.3].

(b) A proof of Theorem 5.3 is also possible by using Proposition 4.4(ii) for $\mathbb{W}$ instead of Proposition 3.4 for $\mathbb{V}$.

6. Inversion of the unbounded FHT

The aim of this final section is to establish the Inversion Theorem for $T_{\mathbb{V}} \colon \mathbb{V} \to L^1$; see Section 1. For this, the main tool is the following theorem which exposes, in detail, the properties of both $T_{\mathbb{V}}$ on $\mathbb{V}$ and $\widehat{T}$ on $\widehat{\mathbb{V}}$. The proof will also require various consequences derived from the Parseval and Poincaré-Bertrand formulae in Sections 4 and 5.

Recall that w is the function in (1.6), $\widehat{T}$ is the operator in (1.5) and P is the projection in (3.3).

Theorem 6.1. *The following assertions are valid.*

(i) *The identity*
$$\widehat{T}(T(f)) = f - P_{\mathbb{V}}(f), \quad f \in \mathbb{V},$$
holds, where $P_{\mathbb{V}}\colon \mathbb{V} \to \mathbb{V}$ is the restriction of P to $\mathbb{V}$.

(ii) *The operator $T_{\mathbb{V}}\colon \mathbb{V} \to \widehat{\mathbb{V}}$ is surjective.*

(iii) $\mathrm{Ker}(T_{\mathbb{V}})$ *is the 1-dimensional subspace $\left\langle \frac{1}{w} \right\rangle$ of $\mathbb{V}$ spanned by $1/w$.*

(iv) *The operator $\widehat{T}$ maps $\widehat{\mathbb{V}}$ into $\mathbb{V}$ and satisfies both*
$$T(\widehat{T}(g) = g, \quad g \in \widehat{\mathbb{V}},$$
and
$$\int_{-1}^{1} \widehat{T}(g)(x)\, dx = 0, \quad g \in \widehat{\mathbb{V}}.$$

(v) *The operator $\widehat{T}_{\widehat{\mathbb{V}}}$ is a bijection of $\widehat{\mathbb{V}}$ onto its range $\mathrm{R}(\widehat{T}_{\widehat{\mathbb{V}}})$. Moreover,*
$$\mathrm{R}(\widehat{T}_{\widehat{\mathbb{V}}}) = \left\{ f \in \mathbb{V} : \int_{-1}^{1} f = 0 \right\}.$$

(vi) *The following decomposition of $\mathbb{V}$ holds, namely,*
$$\mathbb{V} = \mathrm{R}(\widehat{T}_{\widehat{\mathbb{V}}}) \oplus \left\langle \frac{1}{w} \right\rangle.$$

Proof. (i) Note, since $1/w \in L\log L \subseteq \mathbb{V}$, that $P(\mathbb{V}) \subseteq \mathbb{V}$.

We follow the steps of the classical proof given by Tricomi in [22, §4.3 (12)], adapted by using the Poincaré-Bertrand formula (5.8) for $\mathbb{V}$ and $\mathbb{W}$. Fix $f \in \mathbb{V}$. Since $w \in \mathbb{W}$, we can apply Theorem 5.3 to obtain, for a.e. $x \in (-1,1)$, that
$$\begin{aligned} T\big(wT(f) + fT(w)\big)(x) &= T(w)(x)T(f)(x) - w(x)f(x) \\ &= -xT(f)(x) - w(x)f(x). \end{aligned} \tag{6.1}$$

Moreover, for a.e. $x \in (-1,1)$, we have
$$\begin{aligned} T(tf(t))(x) &= \frac{1}{\pi}\text{p.v.}\int_{-1}^{1} \frac{tf(t)}{t-x}\, dt = \frac{1}{\pi}\int_{-1}^{1} f(t)\, dt + \frac{x}{\pi}\text{p.v.}\int_{-1}^{1} \frac{f(t)}{t-x}\, dt \\ &= \frac{1}{\pi}\int_{-1}^{1} f(t)\, dt + xT(f)(x). \end{aligned} \tag{6.2}$$

It follows from (6.1) and (6.2) that
$$T\big(wT(f)\big)(x) - \frac{1}{\pi}\int_{-1}^{1} f(t)\, dt - xT(f)(x) = -xT(f)(x) - w(x)f(x),$$

that is,

$$T\big(wT(f)\big)(x) - \frac{1}{\pi}\int_{-1}^{1} f(t)\,dt = -w(x)f(x).$$

So, via (1.5), we can conclude that

$$f = -\frac{1}{w}T(wT(f)) + \left(\frac{1}{\pi}\int_{-1}^{1} f\right)\frac{1}{w} = \widehat{T}\big(T(f)\big) + P_{\mathbb{V}}(f).$$

(ii) Let $f \in \mathbb{V}$. Then both $f, T(f) \in L^1$ and there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq L^\infty$ such that $f_n \to f$ in L^1 and $T(f_n) \to T(f)$ in L^1; see Remark (3.3)(b). Let $g_n := T(f_n)$, for $n \in \mathbb{N}$, and $g := T(f) \in L^1$. Then $\{g_n\}_{n=1}^\infty \subseteq L\log L$ (cf. Proof of Proposition 3.5) and $g_n \to g \in L^1$. Moreover, $\widehat{T}(g_n) = \widehat{T}(T(f_n)) = f_n - P(f_n)$, [6, Theorem 4.10(iv)], which converges to $f - P(f)$ in L^1 because $P\colon L^1 \to L^1$ boundedly. But, by part (i), we have that $f - P(f) = \widehat{T}(T(f)) = \widehat{T}(g)$. Thus, $\widehat{T}(g) \in L^1$ and $\widehat{T}(g_n) \to \widehat{T}(g)$ in L^1. These facts guarantee, via (3.4), that $g \in \widehat{\mathbb{V}}$, that is, $T(f) \in \widehat{\mathbb{V}}$ and so $T_{\mathbb{V}}(\mathbb{V}) \subseteq \widehat{\mathbb{V}}$.

Let $h \in \widehat{\mathbb{V}}$. Then both $h, \widehat{T}(h) \in L^1$ and there exists a sequence $\{h_n\}_{n=1}^\infty \subseteq L^\infty \subset L\log L$ such that $h_n \to h$ in L^1 and $\widehat{T}(h_n) \to \widehat{T}(h)$ in L^1; see Remark 3.7(b). Let $u_n := \widehat{T}(h_n) \in L\log L$ and $u := \widehat{T}(h) \in L^1$. That is, $\{u_n\}_{n=1}^\infty \subseteq L\log L$ and $u_n \to u \in L^1$. Moreover, $T(u_n) = T(\widehat{T}(h_n)) = h_n$, [6, Theorem 4.10(ii)], which converges to h in L^1. Since $h_n \to h$ in L^1 we have (via Kolmogorov's theorem) that $T(u_n) \to T(u)$ in L^0. Thus, $T(u) = h$ and $T(u_n) \to T(u)$ in L^1. These facts guarantee, via (1.2), that $u \in \mathbb{V}$ and $h = T(u)$. Thus $h \in T_{\mathbb{V}}(\mathbb{V})$ and so, $\widehat{\mathbb{V}} \subseteq T_{\mathbb{V}}(\mathbb{V})$.

(iii) Let $f \in \mathrm{Ker}(T_{\mathbb{V}})$, that is, $T(f) = 0$. We can apply part (i) to deduce that $f = P(f) + \widehat{T}\big(T(f)\big) = P(f)$, that is, $f = \big(\frac{1}{\pi}\int_{-1}^{1} f\big)\frac{1}{w} \in \langle\frac{1}{w}\rangle$. Since $T(1/w) = 0$, we also have the reverse inclusion $\langle\frac{1}{w}\rangle \subseteq \mathrm{Ker}(T_{\mathbb{V}})$.

(iv) In the proof of part (ii) we showed, for $h \in \widehat{\mathbb{V}}$, that $\widehat{T}(h) \in \mathbb{V}$. Thus, $\widehat{T}_{\widehat{\mathbb{V}}}(\widehat{\mathbb{V}}) \subseteq \mathbb{V}$.

Let $g \in \widehat{\mathbb{V}}$. By (3.4) both $g, \widehat{T}(g) \in L^1$ and there exists a sequence $\{g_n\}_{n=1}^\infty \subseteq L\log L$ such that $g_n \to g$ in L^1 and $\widehat{T}(g_n) \to \widehat{T}(g)$ in L^1. Kolmogorov's theorem implies that $T(\widehat{T}(g_n)) \to T(\widehat{T}(g))$ in L^0. Since $T(\widehat{T}(g_n)) = g_n$ for all $n \in \mathbb{N}$, [6, Theorem 4.10(ii)], it follows that $g_n \to T(\widehat{T}(g))$ in L^0. Moreover, $g_n \to g$ in L^1 implies that a subsequence of $\{g_n\}_{n=1}^\infty$ converges to g a.e. Hence, $T(\widehat{T}(g)) = g$.

Since $\widehat{T}(g_n) \to \widehat{T}(g)$ in L^1, surely $\int_{-1}^{1}\widehat{T}(g_n) \to \int_{-1}^{1}\widehat{T}(g)$. But, $\int_{-1}^{1}\widehat{T}(g_n) = 0$ for all $n \in \mathbb{N}$, [6, Theorem 4.10(ii)], and so $\int_{-1}^{1}\widehat{T}(g) = 0$.

(v) Part (iv) implies that $\widehat{T}_{\widehat{\mathbb{V}}}(\widehat{\mathbb{V}}) \subseteq \big\{f \in \mathbb{V} : \int_{-1}^{1} f = 0\big\}$. Conversely, suppose that $f \in \mathbb{V}$ satisfies $\int_{-1}^{1} f(x)\,dx = 0$, that is, $P_{\mathbb{V}}(f) = 0$. Then part (i) implies that $\widehat{T}(T(f)) = f$ and so $f \in \mathrm{R}(\widehat{T}_{\widehat{\mathbb{V}}})$ since $T(f) \in \widehat{\mathbb{V}}$. The identity stated in (iv) is thereby established.

Part (iv) yields that $T_{\mathbb{V}}\widehat{T}_{\widehat{\mathbb{V}}} = I_{\widehat{\mathbb{V}}}$, where $I_{\widehat{\mathbb{V}}}$ is the identity operator on $\widehat{\mathbb{V}}$. This implies that $\widehat{T}_{\widehat{\mathbb{V}}}$ is injective. So, $\widehat{T}_{\widehat{\mathbb{V}}}\colon \widehat{\mathbb{V}} \to \widehat{T}_{\widehat{\mathbb{V}}}(\widehat{\mathbb{V}})$ is a linear bijection.

(vi) The identity $\widehat{T}(T(f)) = f - P(f)$ in part (i) shows that each $f \in \mathbb{V}$ has the decomposition $f = \widehat{T}(T(f)) + P(f)$ with $\widehat{T}(T(f)) \in \mathrm{R}(\widehat{T}_{\widehat{\mathbb{V}}})$ and $P(f) \in \langle 1/w\rangle$. Let $h \in \mathrm{R}(\widehat{T}_{\widehat{\mathbb{V}}}) \cap \langle 1/w\rangle$. Then $h = \widehat{T}(g)$ for some $g \in \widehat{\mathbb{V}}$ and $h = c/w$ for some $c \in \mathbb{C}$, that is, $\widehat{T}(g) = c/w$. Integrating both sides of this identity over $(-1,1)$ and recalling that $\int_{-1}^{1} \widehat{T}(g) = 0$ (cf. part (iv)) shows that $c = 0$. Hence, $h = 0$. □

Remark 6.2. (a) For each space L^p, with $1 < p < 2$, parts (i), (ii) and (iv) of Theorem 6.1 are classical, [20, Proposition 2.4 and (2.6)]. For r.i. spaces with Boyd indices in the interval $(1/2, 1)$ we refer to [3, Theorem 3.2] and for $L\log L$, see [6, Theorem 4.10].

(b) Part (iii) of Theorem 6.1 is a classical result for $f \in L^p$, with $1 < p < \infty$, [20, Lemma 2.2]. It was extended to functions f in r.i. spaces with non-trivial Boyd indices in [3, Theorem 3.2] and to functions $f \in L\log L$ in [6, Theorem 3.4].

We can now proceed to establish the inversion formula (1.1) for $T_{\mathbb{V}}\colon \mathbb{V} \to L^1$.

Proof of the Inversion Theorem. Given $c \in \mathbb{C}$ define

$$f := \frac{-1}{w}\, T(wg) + \frac{c}{w}, \quad \text{a.e. in } (-1,1). \tag{6.3}$$

Since $g \in \widehat{\mathbb{V}}$ and $\widehat{T}_{\widehat{\mathbb{V}}}(\widehat{\mathbb{V}}) \subseteq \mathbb{V}$ (by Theorem 6.1(iv)), we have that $\widehat{T}(g) \in \mathbb{V}$. Since $f = \widehat{T}(g) + c/w$ with $1/w \in \mathbb{V}$, it follows that $f \in \mathbb{V}$. But, $T(1/w) = 0$ and so $T(f) = T(\widehat{T}(g))$. Theorem 6.1(iv) gives $T(\widehat{T}(g)) = g$ and so $T(f) = g$, that is, f is a solution of the airfoil equation.

Conversely, let $f \in \mathbb{V}$ satisfy the airfoil equation $T(f) = g$. Theorem 6.1(i) implies that

$$\widehat{T}(g) = \widehat{T}(T(f)) = f - P(f),$$

where $P(f) = c/w$ with $c := \frac{1}{\pi}\int_{-1}^{1} f(t)\,dt$. Accordingly, f has the form (6.3). □

Remark 6.3. The Inversion Theorem for the FHT is long since known to hold in L^p, for $1 < p < 2$, [20, Proposition 2.4 and Corollary 2.5]. The result also holds in r.i. spaces whose Boyd indices lie in the interval $(1/2, 1)$, [3, Corollary 3.5]. Its validity was recently extended to the case when $g \in T(L\log L)$, [6, Theorem 4.10(iv)].

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