Mathematisches Forschungsinstitut Oberwolfach

Report No. 48/2025

DOI: 10.4171/OWR/2025/48

Mini-Workshop: Resurgence, Difference Equations and Quantum Modularity

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26 October – 31 October 2025

Abstract. The resurgent analysis of asymptotic series has found diverse applications and led to profound insights into our understanding of enumerative geometry, difference equations, and quantum modular forms. Consequently, interactions among these three independent yet intertwined avenues of contemporary research have recently soared. This mini-workshop brought together both senior and junior researchers from each area to clarify the state of the art and establish common ground for collectively addressing the most relevant open questions in the field.

Mathematics Subject Classification (2020): 14N35, 14J15, 14D21, 11F67.

Introduction by the Organizers

The workshop *Resurgence, Difference Equations and Quantum Modularity*, organised by Murad Alim (Garching / Edinburgh), Veronica Fantini (Orsay), Lotte Hollands (Edinburgh), and Claudia Rella (Bures-sur-Yvette), was attended by 15 in-person participants and one remote participant, each of whom contributed a 45-minute talk followed by a short Q&A session. There were a maximum of four talks a day, leaving plenty of time for both informal and structured discussions. On the first afternoon, a supervised brainstorming session was held to identify a set of concrete problems for participants to investigate in working groups during the workshop. On the last afternoon, a representative from each group presented the core ideas from their meetings to update the rest of the participants.

The topic of the workshop

Factorially divergent formal power series have appeared ubiquitously in the perturbative regimes of quantum-mechanical problems. As observed by Dyson (1951) and later by Bender–Wu (1971), their divergence hints at hidden non-perturbative terms that dictate the large-order behaviour of the perturbative coefficients. By applying the theory of resurgence, developed by Écalle in the 1980s, it is often possible to construct a collection of analytic functions whose asymptotic expansion at the origin reproduces the initial series and that are related to one another via Stokes jumps. In a quantum-mechanical setting, Stokes jumps encode information about tunnelling effects. More generally, resurgence has been successfully applied to study non-perturbative effects associated with perturbative partition functions in various physical theories, including Chern–Simons theory and topological string theory. At the same time, it has found fruitful applications in enumerative geometry and mirror symmetry.

In the work of Kontsevich and Soibelman, a conjectural relation between resurgent asymptotic series and analytic wall-crossing structures has been outlined. Simultaneously, within the research program initiated by Bridgeland, a notion of BPS structure from the stability data of a three-dimensional Calabi–Yau (CY) category is associated with a Riemann–Hilbert problem. In fact, variations of BPS structures, which reproduce the behaviour of Donaldson–Thomas (DT) invariants under changes of stability, follow the Kontsevich–Soibelman wall-crossing formula. Moreover, solving the Riemann–Hilbert problem for the resolved conifold leads to an asymptotic expansion containing the Gromov–Witten (GW) invariants of the same geometry. The opposite direction, which starts with the asymptotic series given by the generating function of GW invariants of the resolved conifold and produces piecewise analytic functions related by Stokes jumps, was recently addressed by Alim–Saha–Teschner–Tulli. As expected, the Stokes factors contain information on the DT invariants. This link between resurgence, GW theory, and DT theory has been extended to other geometries by Alim and Bridgeland–Tulli.

An intriguing appearance of a quantum-mechanical problem in the context of mirror symmetry occurs in the case of local (hence, non-compact) CY threefolds. For these geometries, the mirror data is encoded in an algebraic curve in $\mathbb{C} \times \mathbb{C}$ or $\mathbb{C}^* \times \mathbb{C}^*$ describing the theory of variations of complex structures. As initially discussed in the work of Aganagic–Dijkgraaf–Klemm–Mariño–Vafa from the B-model perspective, one can consider the ambient space of the curve as phase space and the defining equation of the curve as quantum-mechanical Hamiltonian. It was later understood by Aganagic–Cheng–Dijkgraaf–Krefl–Vafa that this approach is connected to a quantisation problem considered by Nekrasov and Shatashvili, which moreover relates to quantum Hitchin integrable systems. Depending on the ambient space, these quantisation problems lead to differential or difference operators, whose exact WKB analysis is, in turn, intimately related to topological strings and GW theory. The Stokes phenomena are again captured by DT invariants and connected to the spectral networks of Gaiotto–Moore–Neitzke, as recently observed by Alim–Hollands–Tulli.

Very recently, in a series of works by Mariño and collaborators, the resurgence of topological string theory on arbitrary CY threefolds has been studied via the BCOV equations satisfied by the conventional and Nekrasov–Shatashvili free energies, producing formal solutions for their multi-instanton trans-series extensions. Although the values of the Stokes constants in these resurgent structures are undetermined, these are conjecturally identified with enumerative invariants counting BPS states of the topological string. A different perspective is offered by the so-called topological string/spectral theory (TS/ST) correspondence of Grassi–Hatsuda–Mariño. A family of quantum operators is constructed from the quantisation of the mirror curve of a toric CY threefold. Their fermionic spectral traces are conjectured to give access to the non-perturbative effects associated with the factorial divergence of the topological string partition functions in the spirit of large-N gauge/string dualities. In the works of Gu–Mariño and Rella, resurgence is applied to the spectral theory side of the correspondence, leading to a non-parametric version of the resurgent structures of the topological string with calculable Stokes constants.

Finally, in recent works by Fantini and Rella, a connection between resurgence and the theory of quantum modular forms of Zagier appears naturally in the context of quantum mirror curves. The resurgent structures of the spectral traces of local weighted projective planes satisfy an exact strong-weak resurgent symmetry and unique arithmetic, modularity, and summability properties. Abstracting these observations, the same authors introduced the framework of modular resurgence. Its implications, as well as the extent of its applicability and potential generalisations, remain to be fully understood, particularly in relation to the enumerative interpretation of the Stokes constants and to the S-duality of topological strings.

Working groups

After a collective discussion on overlapping topics of interest moderated by the organisers, the participants divided into two groups. The main questions addressed by each group are summarised below.

Group one. The first working group discussed the application of the Frobenius method for q-difference equations to the study of quantum curves arising within mirror symmetry of CY threefolds. Solutions to such q-difference equations encode the quantum periods of the CT, which in turn possess an interpretation as generating functions of (relative) GW invariants. The Frobenius method for q-difference equations was presented at this workshop in the talk by Wheeler.

The group investigated the application of this method to the quantum mirror curves of specific local CY geometries, that is, local $\mathbb{P}^1 \times \mathbb{P}^1$ and local $\mathbb{P}^2$. Throughout the discussions, it became progressively clear how the Frobenius method for q-difference equations is a natural counterpart to the exact WKB treatment of differential equations, providing an orthogonal perspective to earlier works by participants in the workshop. Furthermore, the role of q-Borel resummation as the appropriate procedure for studying the analytic structure of solutions to q-difference equations was discussed.

Finally, the group observed that an integrality property of the coefficients of the q-expansions of the formal solutions mentioned above, which had previously appeared in the mathematical physics literature, can be rigorously proven using the Frobenius method.

Group two. The second working group approached q-difference equations from a geometric perspective. In the standard approach to q-difference equations, these are defined on $\mathbb{C}^*$. The group considered possible ways to define q-difference equations on an arbitrary Riemann surface C. A promising idea, suggested by Nikolaev, is to describe them in terms of a time-$\log(q)$-flow of a meromorphic vector field on C. In physics, such a construction would be relevant in the study of 5d lifts of 4d $\mathcal{N} = 2$ theories.

The group also discussed the exact WKB analysis of known q-difference equations, with the aim of defining a cluster structure on the associated moduli space of q-connections. Longhi explained his joint paper with Del Monte, in which quantum periods and their Stokes jumps are analysed for the $\mathbb{P}^1 \times \mathbb{P}^1$-geometry. A possible definition of q-Fenchel–Nielsen coordinates was identified, with the aim of extracting 5d partition functions from the exact WKB analysis.

Finally, Teschner explained the relevance of apparent singularities of ODEs to the physics of 4d $\mathcal{N} = 2$ theories and associated integrable systems. A generalisation of this discussion to the q-difference context was initiated.

Mini-Workshop: Resurgence, Difference Equations and Quantum Modularity

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Abstracts

Analytic structure of generating functions of Gromov–Witten invariants of Calabi–Yau threefolds

Murad Alim

1. Introduction

Gromov-Witten (GW) invariants of Calabi-Yau (CY) threefolds are organized in the GW potential, which is an asymptotic series in a formal parameter λ and which can be obtained from topological string theory. For a CY threefold X it takes the form:

$$F_{\mathrm{GW}}(\lambda,t)=\sum_{g=0}^{\infty}\sum_{\beta}\lambda^{2g-2}[\mathrm{GW}]_{\beta,g}Q^{\beta}\,, \tag{1}$$

where $\beta\in H_2(X,\mathbb{Z})$ is a curve class, $Q^{\beta}:=\exp(2\pi \mathrm{i} t^{\beta})$ and t^{β} are formal parameters and $[\mathrm{GW}]_{\beta,g}$ denote the GW invariants. An example is the Gromov-Witten potential for the resolved conifold geometry which is a CY threefold given by the total space of the rank two bundle:

$$\mathbf{X}:=\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1\,, \tag{2}$$

over the projective line and corresponds to the resolution of the conifold singularity in $\mathbb{C}^4$. The GW potential for this geometry was determined, in physics [5, 6] as well as in mathematics [3], with the outcome

$$\tilde{F}^{\mathrm{con}}(\lambda,t)=\sum_{g=0}^{\infty}\lambda^{2g-2}F_g(t)=\frac{1}{\lambda^2}\mathrm{Li}_3(Q)+\sum_{g=1}^{\infty}\lambda^{2g-2}\,\frac{(-1)^{g-1}B_{2g}}{2g\,(2g-2)!}\,\mathrm{Li}_{3-2g}(Q)\,, \tag{3}$$

for the non-constant maps, where $Q=\exp(2\pi i t)$.

The Gopakumar-Vafa (GV) resummation of the GW potential [5, 6] reformulates the non-constant part of the GW potential in terms of the GV invariants $[\mathrm{GV}]_{\beta,g}\in\mathbb{Z}$. The GW potential can thus be written as:

$$F_{\mathrm{GV}}(\lambda,t)=\sum_{\beta>0}\sum_{g\geq 0}[\mathrm{GV}]_{\beta,g}\sum_{k\geq 1}\frac{1}{k}\left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2}Q^{k\beta}\,. \tag{4}$$

In [7], Maulik and Toda introduced sheaf invariants which are related to the GV invariants in the following way:

$$\sum_{n\in\mathbb{Z}}\Omega_{\beta,n}y^n=\sum_{g\geq 0}[\mathrm{GV}]_{\beta,g}\,(-1)^g(y^{1/2}-y^{-1/2})^{2g}\,. \tag{5}$$

2. Resurgence of the Gromov-Witten potential

In [2], the techniques of [4] were used to study the Borel resummation of the GW potential $\tilde{F}^{\mathrm{con}}(\lambda, t)$ for the resolved conifold. It was found that the exponential of the Borel sum of the GW potential along the real axis gives an analytic function which has the following infinite product expansion, spelled out in [1]:

$$Z_{\mathrm{np}}^{\mathrm{con}}(\lambda, t) = \prod_{m=0}^{\infty} (1 - Q\, q^{m+1})^{m+1} \cdot \exp\left(-\frac{\mathrm{Li}_2(Q' q'^m)}{2\pi i}\right) \cdot (1 - Q' q'^m)^{-\frac{1}{\check{\lambda}}(t+m)}, \tag{6}$$

where $Q = \exp(2\pi i t)$, $q = e^{\mathrm{i}\lambda}$, $Q' = \exp(2\pi i t/\check{\lambda})$, $q' = \exp(2\pi i/\check{\lambda})$, $\check{\lambda} = \frac{\lambda}{2\pi}$. It was furthermore proven in [1] that the GW potential for any CY threefold can be written in terms of the Maulik-Toda invariants in the following way:

$$\tilde{F}_{\mathrm{GW}}(\lambda, t) = \sum_{n\in\mathbb{Z}} \sum_{\beta>0} \Omega_{\beta,n} \tilde{F}^{\mathrm{con}}(\lambda, t^\beta + n\check{\lambda}). \tag{7}$$

Theorem (MA, work in progress)**.** *The Borel transform of the asymptotic series of the higher genus $g \geq 2$ GW invariants is given by:*

$$G(\xi, t) = \sum_{\beta>0} \sum_{k,n\in\mathbb{Z}} \frac{\Omega_{\beta,n}}{2} \left(\frac{2(k-t^\beta)^3}{\xi^3} + \left(n^2 - \frac{1}{6}\right) \frac{k-t^\beta}{\xi} - \frac{e^{\frac{\xi(n+1)}{k-t^\beta}} + e^{\frac{\xi(-n+1)}{k-t^\beta}}}{\xi\left(e^{\frac{\xi}{k-t^\beta}} - 1\right)^2} (k-t^\beta) \right) \tag{8}$$

The sum over Stokes jumps can be expressed as follows

$$F_{\mathrm{stokes}}(\lambda, t) = -\frac{1}{2\pi} \partial_\lambda \left(\lambda\, \tilde{F}^0_{\mathrm{def}} \left(\frac{2\pi}{\check{\lambda}}, \frac{t - 1/2}{\check{\lambda}} \right) \right). \tag{9}$$

in terms of the ε-deformation of the prepotential introduced in [1]

$$\tilde{F}^0_{\mathrm{def}}(\varepsilon, t) := \frac{1}{2} \sum_{k,\beta>0} \frac{[\mathrm{GV}]_{\beta,0}}{k^2} \frac{Q^{k\beta}}{\sin(k\varepsilon/2)}. \tag{10}$$

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Chiralization of quiver varieties

Ioana Coman
(joint work with Myungbo Shim, Masahito Yamazaki, and Yehao Zhou)

From a physical perspective, recent years have seen much progress in the construction of vertex operator algebras (VOAs) as distinguished algebras of operators supported on a surface within higher-dimensional supersymmetric quantum field theories (SQFTs). In particular, such VOAs can be defined on the two-dimensional (2d) boundary of a topologically-twisted three-dimensional (3d) $\mathcal{N} = 4$ SQFT "$\mathcal{T}$" [1], and thus provide a novel and intriguing analog in a supersymmetric setting of archetypal 3d/2d correspondences between 3d (Chern-Simons) gauge theories and a boundary (WZW) conformal field theory [2]. New examples of such boundary VOAs [3] are logarithmic VOAs constructed as extensions of $\mathcal{W}$-algebras, with interesting features such as non semi-simple categories of modules and characters which are quantum modular forms [4]. One particularly exciting outcome is that, by now, there exist numerous instances where the interplay between a 3d theory and its boundary VOA has proven to be a rich source of insights into both areas.

Within this context, this talk presents a systematic construction of VOAs, which will be denoted hereafter by $\mathcal{V}_{X[Q]}$, via a chiralization procedure [5] from a corresponding "classical" geometry $X[Q]$. The defining data is that of a framed doubled quiver Q, which specifies physically a 3d $\mathcal{N} = 4$ quiver gauge theory $\mathcal{T}[Q]$, and for which $X[\mathcal{T}[Q]]$ represents the Higgs branch of the moduli space of vacua.

Mathematically, the geometry $X[Q]$ is a symplectic variety (or, more generally, a symplectic scheme) defined from the data of the framed doubled quiver Q as the Nakajima quiver variety. On the one hand, the Nakajima quiver variety $X[Q]$ is a GIT quotient of a subspace of the representation space $\mathcal{R}[Q]$ of the framed doubled quiver Q, and is formed by tuples of linear maps between complex vector spaces associated to the nodes of the quiver, with a set of stability conditions. The *global* chiral counterpart of this construction defines the VOA $\mathcal{V}_{X[Q]}$ as the degree-0 relative semi-infinite BRST cohomology of a corresponding VOA $\mathcal{V}_Q \otimes \mathcal{V}_{f[Q]}$, where $\mathcal{V}_Q$ promotes the linear maps in the representation space $\mathcal{R}[Q]$ to VOA fields and $\mathcal{V}_{f[Q]}$ is a corresponding fermionic VOA [5]

$$\mathcal{V}_{X[Q]} = H_{rel}^{\frac{\infty}{2}+0}(\mathcal{V}_Q \otimes \mathcal{V}_{f[Q]}, d_Q) \ , \tag{1}$$

with respect to a nilpotent operator d_Q. The global chiralization procedure therefore replaces classical functions $f \in \mathbb{C}[X[Q]]$ with generating currents $f(z)$ of the VOA $\mathcal{V}_{X[Q]}$, quantizing the jet scheme of the given scheme $X[Q]$. The "classical" geometry $X[Q]$ is then recovered as the associated variety of $\mathcal{V}_{X[Q]}$.

A remarkable outcome of this systematic analysis is the explicit construction of homomorphisms from affine $\mathcal{W}$-algebras into the quiver VOAs $\mathcal{V}_{X[Q]}$

$$\mathcal{W}(\mathfrak{g}_{[Q]}, f_{[Q]}) \to \mathcal{V}_{X[Q]} \ ,$$

where $\mathfrak{g}_{[Q]}$ is a semi-simple Lie algebra and $f_{[Q]} \in \mathfrak{g}_{[Q]}$ is a nilpotent element, both of which are determined by the quiver data. This analysis therefore provides the tools for the construction and study of generalisations of the logarithmic VOAs which have appeared in [3], and which are in turn expected to be a rich source of functions with intriguing quantum modular properties.

Following [6], the *local* version of the chiralisation procedure developed in [5] produces a sheaf of $\hbar$-adic vertex algebras on an extension of Nakajima quiver varieties, which has proven to be a powerful tool for analysing relations among quiver VOAs. In particular, this has led to the definition of a *quiver reduction algorithm*, whose chiral counterpart constructs free-field realisations. It furthermore turns out that there exists a natural map between the VOAs defined by the global and respectively local chiralization procedures, whose injectivity or surjectivity is not clear in general, but which can nevertheless be made precise for a class of quivers [5, 7].

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WKB approximation for qDEs and BPS Riemann–Hilbert problems

Fabrizio Del Monte

(joint work with Tom Bridgeland, Luca Giovenzana, and Pietro Longhi)

Monodromies of ODEs have been a fruitful research area for many decades, linked to many fields of mathematics and mathematical physics. In topological string theory, q-difference equations (qDEs) arise as quantum curves and their monodromies are expected to solve infinite-dimensional Riemann-Hilbert Problems (RHPs) associated to BPS spectra in M-theory and corresponding TBA equations [3, 4, 2]. Yet even the appropriate notion of monodromy in the discrete setting is a rapidly evolving subject. In this talk I presented the systematic WKB approximation for second-order qDEs and explained how it can be used to canonically characterise their monodromies. We consider second-order qDEs in *involutive* form[1]

$$\psi(qx)+\psi(q^{-1}x)=2T(x)\psi(x), \tag{1}$$

[1] See [3] for the WKB recursion of the most general second-order qDE.

and construct asymptotic expansions of solutions in the limit $\hbar := \log q \to 0$:

$$(2)\quad \psi_{\pm,n}(x) = f(x)\, e^{\frac{2\pi i n}{\hbar}} \exp\left\{\frac{1}{\hbar}\int^x \log \mathcal{R}_\pm(x)\frac{dx}{x}\right\}, \quad \mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} \mathcal{R}_{\pm,k}(x)\hbar^n.$$

Here $\mathcal{R}_\pm$ are the two formal power series solution of the q-Riccati equation

$$(3)\qquad \mathcal{R}(x)\mathcal{R}(q^{-1}x) - 2T(x)\mathcal{R}(q^{-1}x) + 1 = 0, \qquad \mathcal{R}(x) := \frac{\psi(qx)}{\psi(x)},$$

and the coefficients $\mathcal{R}_{\pm,k}$ can be obtained recursively [5]. A basic difference with the differential case is that "constants" of qDE are q-periodic factors $c(qx) = c(x)$ naturally encoded by the monomials $e^{\frac{2\pi i n}{\hbar}}$. One may view a second-order qDE as having a two-dimensional basis of solutions $\psi_\pm$, defined up to such factors, or a $2\mathbb{Z}$-dimensional basis of solutions $\psi_{\pm,n}$. Monodromies are realised either as 2×2 q-periodic matrices, or as constant $2\mathbb{Z} \times 2\mathbb{Z}$ matrices, suggesting a loop group structure. Traditionally, "monodromies" of qDEs are defined very differently from the differential case, since discrete equations do not give direct ways to analytically continue their solutions. Following Birkhoff's classical work [1] (see also the contribution by Wheeler), one introduces the connection matrix

$$(4)\qquad C(x) = Y_\infty(x)^{-1} Y_0(x),$$

where Y_∞, Y_0 are fundamental matrices of solutions at $x = \infty$ and $x = 0$ respectively. Spaces of monodromies are identified with spaces of connection matrices [7, 6] modulo gauge equivalence. This approach obscures the intuitive picture of monodromy as analytic continuation: WKB analysis provides an alternative description that recovers this intuition, under the assumption that there exist analytic solutions with an asymptotic series in $\hbar$. The WKB expression (2) shows that the Stokes geometry is encoded in the semiclassical curve and differential

$$(5)\qquad \Sigma : y + y^{-1} = 2T(x) \subset (\mathbb{C}^*)^2, \qquad \lambda_\pm = \log y_\pm \frac{dx}{x} = \log \mathcal{R}_{\pm,0}(x)\frac{dx}{x}.$$

The Stokes lines, defined as the loci where a solution is maximally dominant with respect to the other, i.e. where the ratio $\left|\frac{\psi_{s_1,n_1}}{\psi_{s_2,n_2}}\right|$, $s_i = \pm$, $n_i \in \mathbb{Z}$ is maximised, determine a graph on the x-plane $\mathbb{C}^*$ known in the physical literature as *exponential network*. I described explicit Stokes matrices attached to such Stokes lines, which generalise the familiar Airy Stokes matrices from WKB theory of ODEs, and explained how the exponential networks provide natural cluster-type coordinates for monodromies, yielding an explicit *nonabelianization* map (see also the contribution by Murugesan). I further showed how sequences of Stokes transformations and analytic continuations compute monodromies. As a main example, I discussed the monodromy around the origin of the q-Mathieu equation, with $2T(x) = \Lambda(x + x^{-1}) - \kappa$. In the WKB coordinates, the trace of this monodromy coincides with a q-deformation of the classical curve, and of the Hamiltonian of a corresponding cluster integrable system related to the q-Painlevé III_3 equation.

Open problems. The WKB method for qDEs is still in its infancy, and several directions remain open. A major problem is to characterise connection matrices in the WKB framework, thereby relating WKB and RH approaches. Important technical ingredients that are currently missing include a detailed analysis of Voros symbols for open paths, and monodromies for equations with singularities at finite x, as well as the summability of WKB solutions (see contributions by Kashaev, Nikolaev, Wheeler). Perhaps the most intriguing open aspect is the expected presence of q-Painlevé dynamics both on spaces of qDEs and on the associated spaces of monodromies, motivated by the link to BPS RHPs [3, 4, 2]. These would form dual q-Painlevé systems, related by a modular transformation $q = e^{\hbar} \longleftrightarrow q^{\vee} = e^{\frac{4\pi^2}{\hbar}}$, suggesting a yet undexplored role of modularity, and possible connections with the TS/ST correspondence (see contributions by Fantini and Rella).

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Summability and resurgence of Andersen–Kashaev state integrals

Veronica Fantini

(joint work with Campbell Wheeler)

Given a hyperbolic knot, the Andersen–Kashaev state integrals are convergent integrals of products of Faddeev's quantum dilogarithm associated with certain triangulations of the knot complement [1]. Their asymptotic expansions are divergent power series, which conjecturally agree with the formal invariants of Dimofte–Garoufalidis [2, 7]. In addition, it has been conjectured by Garoufalidis, Gu and Mariño that these formal invariants are resurgent and Borel summable [5].

In this talk, I will discuss the proof of this conjecture for the knots 4_1 and 5_2, which is based on a joint project with C. Wheeler [4]. These are the simplest examples of hyperbolic knots for which the Andersen–Kashaev state integral are one-dimensional.

The theory of resurgence, introduced by J. Écalle [3], allows one to recontruct the so-called sub-leading order contributions to a formal divergent series. The simplest class of divergent series are the Gevrey-1 $\mathbb{C}[\![\hbar]\!]_1 \subset \mathbb{C}[\![\hbar]\!]$. A formal power series $\tilde{\Phi} = \sum_{n=0}^{\infty} a_n \hbar^{n+1} \in \mathbb{C}[\![\hbar]\!]_1$ if its coefficients grow at most factorially:

$$|a_n| \leq C A^n n!$$

for some constants $C, A > 0$. Then, a Gevrey-1 series is resurgent if its Borel transform admits *endless analytic continuation.* The Borel transform is a formal map, defined as

$$\mathcal{B}_\zeta[\hbar^{n+1}] := \frac{\zeta^n}{n!}$$

and extended by countable linearity on $\mathbb{C}[\![\hbar]\!]$. In addition, the Borel transform maps Gevrey-1 series to convergent formal series in $\mathbb{C}\{\zeta\}$. The convergent power series typically have singularities on the boundary of the disk of convergence, and if the series is resurgent we can get access to the full set of singularities by means of the analytic continuation.

When the singularities are simple poles or logarithmic branch points, the series is called simple resurgent. Given a Gevrey-1 simple resurgent series, we can compute its resurgent structure given by the following data:

- singularities $\omega \in \Omega$;
- the so-called Stokes constants S_ω associated to each singularity $\omega \in \Omega$;
- the so-called germ of second resurgent series $\tilde{\phi}_\omega \in \mathbb{C}\{\zeta\}$.

The theory of resurgence is also closely related to the theory of summability, aimed at building an analytic function from a divergent power series. More precisely, Borel summation in the direction ϑ associates to a divergent power series $\tilde{\Phi} \in \mathbb{C}[\![\hbar]\!]_1$ an analytic function $\mathcal{S}^\vartheta \tilde{\Phi}$ that is asymptotic to the original divergent series:

$$\tilde{\Phi} \in \mathbb{C}[\![\hbar]\!]_1 \xrightarrow{\text{Borel sum}} \mathcal{S}^\vartheta \tilde{\Phi}\,, \quad \text{and} \quad \tilde{\Phi} \in \mathbb{C}[\![\hbar]\!]_1 \xleftarrow{\text{asymptotics}} \mathcal{S}^\vartheta \tilde{\Phi}\,.$$

The divergent series associated with the hyperbolic knot $K = 4_1$ is the formal invariant of Dimofte–Garoufalidis Φ_{K,p_0}, which is defined as follows.[1] Let p_0 be the point

$$p_0 = (z_0, m_0) = \Big(\frac{\log(x_0)}{2\pi i}, \frac{1}{\pi i}\log(1-x_0) - \frac{\log(x_0)}{2\pi i} - \frac{1}{2}\Big)\,,$$

where x_0 solves $(-x_0) = (1-x_0)^2$. Then the formal power series Φ_{4_1,p_0} is defined by formal gaussian integration

$$\text{(1)} \quad \Phi_{p_0}(\hbar) = \int_{\mathcal{C}_{p_0}} \Psi(z;\hbar)^2 \, \exp\Big(-\frac{1}{2\hbar}(2\pi i)^2 z\Big(z + 1 - \frac{\hbar}{2\pi i} + (2\pi i)^2 \frac{m_0 z}{\hbar}\Big)\Big) dz\,,$$

where the contour $\mathcal{C}_{p_0}$ is close to z_0 and the formal series Ψ is defined as

$$\text{(2)} \qquad \Psi(z;\hbar) = e^{\frac{2\pi i}{8}} \, \exp\Big(-\frac{(2\pi i)^2}{24\hbar} - \frac{\hbar}{24} - \sum_{k=0}^{\infty} \frac{B_k}{k!} \mathsf{Li}_{2-k}\big(e^{2\pi i z}\big)\hbar^{k-1}\Big)\,,$$

and B_k denotes the k-th Bernoulli number. Notice that $\Psi(z;\hbar)$ is the asymptotic expansion of Faddeev's quantum dilogairthm $\Phi(z;\hbar)$, which is also known to be

[1]The divergent series associated to the knot 5_2 is similarly defined; we refer to [4] for details.

resurgent and Borel summable with sum along the reals given by the Faddeev's dilogarithm itself [6], i.e.

$$\mathcal{S}^0\Psi(z;\hbar) = \Phi(z;\hbar)\,.$$

Therefore, naively, one would be tempted to commute the resummation of Φ_{p_0} with the integration in the parameter space z to obtain

$$\mathcal{S}^0\Phi_{K,p_0} = \int_{\mathcal{C}_{p_0}} \Phi(z;\hbar)^2 \exp\Big(-\frac{(2\pi i)^2}{2\hbar}z\Big(z+1-\frac{\hbar}{2\pi i}+(2\pi i)^2\frac{m_0 z}{\hbar}\Big)\Big)dz\,. \tag{3}$$

The right-hand side of equation (3) is one of the descendants of the Andersen–Kashaev state integral for the knot 4_1, which are defined as

$$\mathcal{I}_{\ell,m}(\hbar) = \int_{\mathcal{J}_{\ell,\hbar}} \Phi((z-\ell)/\hbar;\hbar)^2 \exp\Big(-\frac{(2\pi i)^2}{2\hbar}z\Big(z+1-\frac{\hbar}{2\pi i}+(2\pi i)^2\frac{mz}{\hbar}\Big)\Big)dz\,, \tag{4}$$

where $m=0,1$, $\ell\in\mathbb{Z}$, and the contour $\mathcal{J}_{\ell,\theta}$ is illustrated in figure 1. The integral $\mathcal{I}_{0,0}$ corresponds to the Andersen–Kashaev state integral. Notice that thanks to the properties of Faddeev's dilogarithm the descendant state integrals have relations, and in the example of the knot 4_1 there are only two $\mathbb{Q}(q)$-independent integrals.

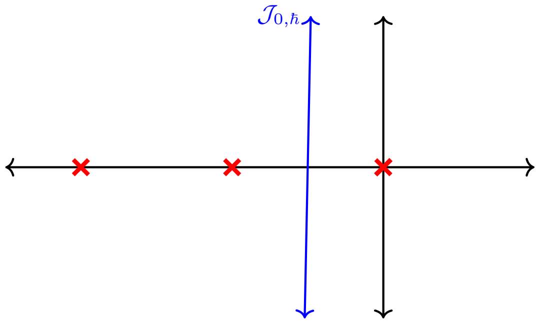

FIGURE 1. The contour of integration for the Andersen–Kashaev state integrals is defined as $\mathcal{J}_{\ell,\hbar} := (i\sqrt{\hbar}e^{-i\epsilon}\mathbb{R}_{\geq 0} - \frac{1}{2} + \ell) \cup (i\sqrt{\hbar}e^{i\epsilon}\mathbb{R}_{\leq 0} - \frac{1}{2} + \ell)$.

We can now state our main result:

The Borel sum in the direction ϑ of the divergent series Φ_{p_0} is given by a linear combination of state integrals $\mathcal{I}_{\ell,m}$.

The key idea of the proof is to define the resummation of Φ_{p_0} as a thimble integral, namely an integral over a steepest descent contour with respect to the potential

$$V(z,m,n) = 2\frac{\mathrm{Li}_2(e^{2\pi i z})}{(2\pi i)^2} + \frac{1}{12} + \frac{1}{2}z(z+1) + mz + n\,, \tag{5}$$

with $m,n\in\mathbb{Z}$ and where we fixed a branch of the dilogarithm $\mathrm{Li}_2(e^{2\pi i z})$. In other words, we can define V to be analytic on a suitable Riemann surface Σ. Then, by defining a relative homology theory for the function V, we showed that the steepest descent contours belong to this homology and that the contours $\mathcal{J}_{\ell,\hbar}$ define a basis.

As a corollary, we also computed the first Stokes constants by comparing the decomposition of the steepest descent contours for different directions. Our computations match the numerical evidence of Garoufalidis, Gu and Mariño [5], whose approach was based on studying the resurgent structure of the solutions of a q-difference equation satisfied by the Andersen–Kashaev state integral.

In a work in progress with J. Andersen and M. Kontsevich, we are generalizing these ideas to study the resummation of some higher-dimensional integrals, including the Andersen–Kashaev state integrals for other hyperbolic knots.

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WRT invariants as finite-dimensional integrals

Sam Hindson

(joint work with Jørgen Ellegaard Andersen and William Elbæk Mistegård)

Let M be a closed, oriented 3-manifold and let $L \subset M$ be a framed link inside of M. Let $k \in \mathbb{Z}_{\geq 0}$, let $\gamma = \pi/(k+2)$, and let $q = \mathrm{e}^{2\mathrm{i}\gamma}$. Suppose too that we have a coloring $\lambda \in \mathrm{Col}_k(L)$ of the components of L by the highest weight integrable representations of SU(2) whose levels are not more than k: that is, $\lambda : \pi_0(L) \to \Lambda_k \cong \{1, ..., k+1\}$. Moreover, let $\mathrm{CS} : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$ denote the Chern–Simons functional on the space of (gauge equivalence classes of) SU(2)-connections on the trivial SU(2)-bundle over M:

$$\mathrm{CS}([A]) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A) \mod \mathbb{Z}.$$

Witten in [1] proposed for each such k a topological invariant of the triple (M, L, λ) in the form of the following path integral:

$$Z_k^{\mathrm{W}}(M, L, \lambda) = \int_{\mathcal{A}/\mathcal{G}} \mathrm{e}^{2\pi\mathrm{i}k\mathrm{CS}(A)} \prod_{K_i \in \pi_0(L)} \mathrm{tr}(\lambda(K_i) \circ \mathrm{Hol}_{K_i}(A)) \mathcal{D}A. \tag{1}$$

The $M = \mathbb{S}^3$ case, Witten argued, provides a geometric interpretation of the colored Jones polynomial of the colored link $J^L_\lambda(q)$, itself a generalization of the original combinatorial construction by Jones in [2]. Witten's framework extends this notion to the case of colored links inside general 3-manifolds.

Soon afterwards, Reshetekhin and Turaev in [3] developed an approach to the same goal using techniques from the study of quantum groups: they defined the topological invariant

$$Z_k(M, L, \lambda) = \alpha_k(L_1) \sum_{\mu \in \mathrm{Col}_k(L_1)} J^{L_1 \cup L_2}_{\mu \cup \lambda}(q) \prod_{K_i \in \pi_0(L_1)} \frac{\sin(\pi\lambda(K_i)/(k+2))}{\sin(\pi/(k+2))}, \tag{2}$$

where $L_1, L_2 \subset \mathbb{S}^3$ are disjoint framed oriented links satisfying $(\mathbb{S}^3_{L_1}, L_2) \cong (M, L)$ via an orientation-preserving diffeomorphism, and $\alpha_k(L_1)$ is a factor depending on k and some basic properties of L_1.

In principle, applying standard perturbation theory arguments to $Z^{\mathrm{W}}_k(M) = Z^{\mathrm{W}}_k(M, \emptyset, \emptyset)$ leads to the following large-k asymptotic expansion for (1) in terms of the stationary points of the Chern–Simons action (corresponding to flat connections on M):

$$Z^{\mathrm{W}}_k(M) \overset{k\to\infty}{\sim} \sum_{\text{flat connections } A} \mathrm{e}^{2\pi \mathrm{i} k \mathrm{CS}(A)}(a^A_0 + a^A_1 k^{-1} + \cdots). \tag{3}$$

The asymptotic expansion conjecture (AEC) for closed, oriented 3-manifolds is the statement that the surgery formula (2) has this large-k asymptotic expansion as well [4]. The conjecture remains open for all but a few families of 3-manifolds, among these being torus bundles and lens spaces [5] and Seifert fibered homology spheres [6].

More generally, if $L \neq \emptyset$ and the colorings of L are chosen such that they grow linearly with the level k, the exponential weights in the asymptotic expansion of Z^{W}_k can be seen to come from connections having meridional holonomy fixed by the precise growth rate of the coloring. The equivalent statement about this behavior manifesting in Z_k is the AEC for 3-manifolds with links; see [7] for a precise statement of the conjecture and a proof in the case of links in finite-order mapping tori.

Towards understanding the large-k asymptotics of Z_k, we (in upcoming joint work with Andersen and Mistegård, [8]) propose a finite-dimensional integral representation thereof. Our approach is completely general, i.e. not limited to special classes of 3-manifolds, and elucidates how classical Chern–Simons actions appear from the combinatorial formula (2). Fix a planar diagram D of a link $L' = L_1 \cup L_2$, let E denote its set of edges, and let $\lambda \in \mathrm{Col}_k(L_2)$. We construct compact cycles $\Upsilon_k \subset \mathbb{C}^{|\pi_0(L_1)|}$, $\Gamma_{(y,\lambda)} \subset \mathbb{C}^{|E|}$ and a meromorphic $(|\pi_0(L_1)| + |E|)$-form $\Omega^{D,\lambda}_k$ on $\mathbb{C}^{|\pi_0(L_1)|} \times \mathbb{C}^{|E|}$ such that

$$Z_k(M, L, \lambda) = \oint_{y_1 \in (\Upsilon_k)^{|\pi_0(L_1)|}} \oint_{x \in \Gamma_{(y_1,\lambda)}} \Omega^{D,\lambda}_k. \tag{4}$$

The integrand is defined in such a way that a simple application of the residue theorem immediately yields (2); the resulting sums over simple poles correspond to the sums over colorings of L_1 as seen in (2), as well as the sums required in the universal R-matrix construction of the colored Jones polynomial. The latter of these relies heavily on a meromorphic extension of the quantum factorial, which is provided to us by a suitable normalisation of Faddeev's quantum dilogarithm [9], which we denote by $\tilde{S}_\gamma(z)$. We show that as $k \to \infty$, Ω_k can be written in terms of the exponential of semiclassical phase function $2\pi\mathrm{i}kW(y,x)$, whose critical values agree (modulo a discrepancy that is quadratic in y and x) with the classical Chern–Simons actions of flat connections on M with meridional holonomy around L prescribed by its coloring.

In other upcoming work (also joint with Andersen and Mistegård, [10]) we study the semiclassical asymptotics of a 1-dimensional contour integral expression for the colored Jones polynomial of $\mathbf{4_1}$, this time rooted in the cyclotomic expansion of Habiro [11]. The approach is inspired by work from Andersen–Hansen [12] and employs the residue theorem similarly to write

$$J_\lambda^{\mathbf{4_1}}(q) = \beta_k(\lambda) \oint_{\Gamma(\lambda)} \frac{\tilde{S}_\gamma(\lambda+t)}{\tilde{S}_\gamma(\lambda-t-1)} \cot(\pi t)\mathrm{d}t, \tag{5}$$

where $\Gamma(\lambda)$ is a closed compact contour encircling the points $\{0,...,\lambda-1\} \subset \mathbb{C}$, and $\beta_k(\lambda)$ is an overall factor. Towards understanding the AEC for $Z_k(\mathbb{S}^3, \mathbf{4_1}, \lambda)$, we set $\lambda = \eta k$ for a fixed $\eta \in \mathbb{Q}, 0 < \eta < 1$, and employ the known semiclassical behavior of $\tilde{S}_\gamma$ to write the integrand in terms of a semiclassical phase function which dominates the integrand as $k \to \infty$. We again find (up to a single discrepancy term) an agreement between the critical values of this semiclassical phase function and the Chern–Simons invariants of holonomy representations of $\pi_1(\mathbb{S}^3 \setminus \mathbf{4_1})$ with meridional holonomy controlled by η as computed by the Kirk–Klassen theorem [13].

Both of the above-mentioned approaches to asymptotic expansions of WRT invariants rely on Faddeev's quantum dilogarithm in a very deep way. Once it is clear how to treat the decomposition of compact contours in (4) and (5) into steepest descent contours, the abundance of resurgent structure associated with Faddeev's quantum dilogarithm (see e.g. [14]) will inform a full resurgent analysis of Z_k.

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2d N=(2,2) theories through spectral networks

Lotte Hollands

(joint work with Loïc Bramley and Subrabalan Murugesan)

In this talk I have introduced the main ideas and new results of [1]. The central object of my talk is a spectral network $\mathcal{W}^\vartheta$, which is a collection of real 1-dimensional trajectories on a (punctured) Riemann surface C described in terms of the data of a collection of k-differentials $(\varphi_2, \ldots, \varphi_N)$ and a phase $\vartheta \in [0, 2\pi)$ [2]. Spectral networks capture the BPS content of QFTs with 8 supercharges, and are closely related to Stokes graphs in the exact WKB analysis. In [1] we consider spectral networks in the context of 2d $\mathcal{N} = (2,2)$ theories.

The prime examples of 2d $\mathcal{N} = (2,2)$ theories are Landau–Ginzburg (LG) models.[1] These theories are defined in terms of a set of n chiral fields ϕ^i, which take values in a Kähler manifold X, and a superpotential $W(\phi^i)$, which is often considered a holomorphic function $W : X \to \mathbb{C}$, but can more generally be relaxed to a holomorphic 1-form dW on X. The perturbative space of ground states of the LG model is the Jacobian algebra $E = \mathbb{C}[\phi^i]/\langle \partial_i W \rangle$.

If we consider a family T_z of LG models, labeled by parameters $z \in C$, then the chiral rings E_z form a holomorphic bundle $\mathcal{E}$ over C with fiber E_z. (Note that C is allowed to have a larger complex dimension than 1.) The spectrum Σ_z of the algebra E_z defines a branched covering $\Sigma \in T^*C$ over C.

Not all perturbative vacua of the LG model are true vacua, since there may be BPS solitons that tunnel between two perturbative vacua and thereby lift their energy. The main reason for introducing LG models in this talk was to show how these BPS solitons of central charge Z, with $\arg Z = \vartheta$, are encoded in a family of

[1] The relation between 2d LG models and spectral networks was also discussed in [4].

spectral networks $\mathcal{W}^\vartheta$ on C. The point is that BPS solitons may be characterised in terms of a Morse flow in X, with respect to the Morse function $\mathrm{Re}(e^{-i\vartheta}W)$, and that an ij-trajectory in the spectral network $\mathcal{W}_\vartheta$, running through a point $z \in C$, indicates the presence of a BPS soliton tunneling from vacuum i to j in the 2d LG model T_z. Alternatively, they correspond to open special Lagrangian discs in the spectral geometry $\Sigma \in T^*C$ with boundaries on Σ as well as on the fiber T^*_z.

Some more intricate examples of 2d $\mathcal{N} = (2,2)$ theories are gauged linear sigma models (GLSMs). These models are defined in terms of gauge fields as well as matter fields (such as chiral fields). Their partition functions Z_i^{vor} are known to count solutions to the vortex equations in two dimensions.

At low energies, the moduli space of the GLSM is a combination of Higgs and Coulomb branches. On the Higgs branch the gauge group is spontaneously broken and the theory is described in terms of a nonlinear sigma model (NLSM) into a Kähler target manifold X. In this regime the vortex partition functions Z_i^{vor} count quasi-maps from $\mathbb{P}^1$ into X, with suitable boundary conditions (characterised by i) at infinity of $\mathbb{P}^1$. On the Coulomb branch the gauge group is broken into a product of abelian factors. In this regime the GLSM is characterised by an effective twisted superpotential $\widetilde{W}_{\mathrm{eff}}$, which generically has logarithmic singularities.

The vacuum structure of a GLSM can again be characterised in terms of a spectral geometry $\Sigma \in T^*C$, where C is parametrised by the exponentiated and complexified FI couplings z of the GLSM. Since the superpotential $\widetilde{W}$ should be described as a 1-form rather than a function, the family of spectral networks $\mathcal{W}^\vartheta$ is richer: the BPS spectrum changes at walls of marginal stability, and includes self-solitons that tunnel from the vacuum i to itself.

As an example we considered the so-called $\mathbb{P}^1$-model $T_z[\mathbb{P}^1]$, which is a certain GLSM that reduces on the Higgs branch to an NLSM with target space $\mathbb{P}^1$. Its spectral geometry is defined by the spectral curve

$$\Sigma: \quad \sigma^2 - m^2 = z \quad \subset \quad \mathbb{C}_\sigma \times \mathbb{C}^*_z,$$

as a ramified covering over the curve $C = \mathbb{C}^*_z$, with complex parameter m and tautological 1-form $\lambda = \sigma dz/z$. An approximately circular wall of marginal stability divides the $\mathbb{C}^*_z$-plane into two regions: the GLSM is in the Higgs phase in the region around the origin at $z = 0$, while it is in the Coulomb phase in the region near infinity. The BPS spectrum changes from a finite to an infinite spectrum after crossing the wall of marginal stability in the direction toward $z = 0$. The infinite spectrum is encoded in a special type of spectral network, that is known as a Fenchel–Nielsen network [5], at a particular phase ϑ_{FN}.

Considering the GLSM in the so-called Ω-background quantises the spectral geometry $\Sigma \subset T^*C$ into a holomorphic differential operator

$$D_\hbar = \hbar^2 \, (z\partial_z)^2 - z + m^2$$

on $\mathbb{C}^*_z$. The Stokes graph for this oper at $\arg(\hbar) = \vartheta$ is equivalent to the spectral network $\mathcal{W}^\vartheta$, and thus encodes the BPS solitons in the GLSM. In the exact WKB analysis one considers Borel sums, in the direction ϑ, of the asymptotic solutions ψ_i^{for} of the ODE $D_\hbar \psi(z) = 0$. This defines exact solutions that jump across the Stokes graph, and thereby encode the BPS spectrum of the GLSM. In [1] we analyse these Borel summed solutions in the $\mathbb{P}^1$-model, and we show that the vortex partition functions Z_i^{vor} appear as Borel sums in the Higgs region of $\mathbb{C}_z^*$, when we resum with respect to the phase ϑ_{FN}.

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Exact solutions to matrix models and string theories

Jasper Kager

(joint work with João Rodrigues, Ricardo Schiappa, Maximilian Schwick, and Noam Tamarin)

1. Context

String theories are generally defined through their genus expansions in a parameter g_{s}:

$$F = \sum_{g=0}^{\infty} g_{\mathrm{s}}^{2g-2} F_g \text{ where } F_g \sim (2g)! \tag{1}$$

Expansions of this type tend to suffer from two problems. On the one hand, the factorial growth of the expansion coefficients F_g renders it asymptotic with zero radius of convergence. On the other hand, non-analytic (i.e. non-perturbative) contributions of the type $\mathrm{e}^{-1/g_{\mathrm{s}}}$, which we know should be present based on physical principles, are absent and difficult to construct systematically. Resurgent analysis [1] teaches us that these two problems are two sides of the same coin, giving a systematic route to turn the above expansion into a well-defined function while constructing the non-perturbative pieces in the process. Still, the problem of non-perturbative completion remains challenging in general. Matrix models and their double-scaling limits to string theories are relatively tractable in this regard, and have seen a surge of research interest due to their connection to mathematical problems in algebraic geometry and combinatorics, and physical theories such as black hole physics and supersymmetric gauge theory.

2. Matrix Models And Their Exact Solutions

We will present advances in the study of the matrix model partition function, given by

$$Z = \int \mathrm{d}M \mathrm{e}^{-\frac{1}{g_{\mathrm{s}}}\mathrm{Tr}V(M)} \tag{2}$$

where $V(z)$ is a polynomial in z, and we integrate over all $N \times N$ Hermitian matrices M. It is known that a large N description of this object is given by a resonant transseries [2]

$$Z \sim \sum_{(n_1,m_1,\ldots,n_d,m_d)\in\mathbb{N}^{2d}} \left(\prod_{i=1}^{d} \sigma_i^{n_i}\tilde{\sigma}_i^{m_i}\right) \mathrm{e}^{-\frac{1}{g_{\mathrm{s}}}\sum_{i=1}^{d}(n_i-m_i)A_i} Z^{(n_1,m_1,\ldots,n_d,m_d)}(g_{\mathrm{s}}) \tag{3}$$

where the $\sigma_i, \tilde{\sigma}_i$ are parameters and the $Z^{(n_1,m_1,\cdots,n_d,m_d)}(g_{\mathrm{s}})$ are asymptotic expansions in g_{s}. In forthcoming work [3, 4, 5, 6] we present a conjecture of an alternative large N description of these objects, based on non-perturbative topological recursion [7, 8] and resurgent analysis, that accounts for all non-perturbative phenomena at play and is fully background independent. Under the assumption that the partition function is a simple resurgent function, analytic continuation to all couplings requires Stokes automorphisms, which we show to be given by the topological string connection formulae [9]. Under the same assumption, we use Borel-Padé-Laplace resummation to show numerically that our large N construction correctly reproduces finite N quantities.

3. Open problems

All of our results point to the partition functions being simple resurgent, but a formal proof remains lacking. We would like to see our results extended to resolvent correlation functions and more general models such as multi-matrix models, as well as models involving different matrix ensembles, such as unitary matrix models.

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Analytic interpolation of the coloured Jones polynomial of the figure-eight knot

Rinat Kashaev
(joint work with Vladimir Mangazeev)

The problem of interpolating coloured Jones polynomials to non-integer colours was first initiated by K. Habiro in [3] and studied subsequently in [4].

The Habiro expression for the coloured Jones polynomial

$$J_{K,n}(q) = \sum_{m=0}^{n-1} (-1)^m q^{-m(m+1)/2} (q^{1+n}, q^{1-n}; q)_m H_{K,m}(q),$$

where $H_{K,m}(q)$ are the Habiro polynomials[1], suggests the interpolation formula

$$J_{K,x}(q) = \sum_{m=0}^{\infty} (-1)^m q^{-m(m+1)/2} (q^{1+x}, q^{1-x}; q)_m H_{K,m}(q). \tag{1}$$

Here and below, we use the q-hypergeometric notation from [2].

For the figure-eight knot, $H_{4_1,m}(q) = 1$ for all $m \geq 0$, see [3, (9)], so that the series (1) is divergent if $0 < q < 1$ and $x \notin \mathbb{Z}_{\neq 0}$. In this case, (1) can be treated in the spirit of the Borel resummation technique [5], using the q-deformed version thereof.

q-Borel resummation method

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a series whose *q-Borel transform*

$$\tilde{f}(t) = (\mathcal{B}_q f)(t) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(q^{-1}; q^{-1})_n} a_n t^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} a_n t^n$$

converges and can be analytically continued along the ray from the origin through z. We define the *q-Borel resummation* of $f(z)$:

$$\hat{f}(z) = \frac{-1}{\log q} \int_0^{\infty} \frac{\tilde{f}(zt)}{(-t;q)_\infty}\, dt.$$

The method is based on Ramanujan's q-beta integral

$$\int_0^{\infty} t^{s-1} \frac{(-at;q)_\infty}{(-t;q)_\infty}\, dt = \frac{(a, q^{1-s}; q)_\infty}{(q, aq^{-s}; q)_\infty} \frac{\pi}{\sin(\pi s)}, \quad s > 0,\ |a| < q^s,$$

see [1, (2.8), (2.9)].

[1] In [3], Habiro uses the notation $a_m(K)$ for $H_{K,m}(q)$.

The figure-eight knot

We have $J_{4_1,x}(q) = f(1)$, where

$$f(z) = \sum_{m=0}^{\infty} (-1)^m q^{-m(m+1)/2} (q^{1+x}, q^{1-x}; q)_m z^m. \tag{2}$$

The q-Borel resummation of (2) can be written as follows:

$$\hat{f}(z) = \frac{\pi}{2\log q} \frac{(q^{1+x}, q^{1-x}; q)_\infty}{(q;q)_\infty^2} \int_{\mathbb{R}-i\epsilon} \frac{(q^{is}, q^{1-is}; q)_\infty}{(q^{is+x}, q^{is-x}; q)_\infty} \frac{z^{is-1}}{(\sinh(\pi s))^2}\, ds, \tag{3}$$

where $0 < \epsilon < 1$, $-\epsilon < x < \epsilon$, and $|\arg z| < \pi$.

For arbitrary x, one can treat the integral in (3) as a complex contour integral and make sense of it by a continuous deformation of the contour to avoid the moving poles of the integrand.

Theorem. The function $g_x = (1 - q^x)\hat{f}(1)$ satisfies the difference equation

$$\frac{q^x g_{x-1}}{1 - q^{2x-1}} + \frac{q^x g_{x+1}}{1 - q^{2x+1}} + \left(\frac{1 - q^{4x}}{(1 - q^{2x-1})(1 - q^{2x+1})} + q^x - q^{-x} \right) g_x = 1 + q^x.$$

Under the condition $-1 < x < 1$, we have

$$\hat{f}(1) = \frac{(q^{1+x}, q^{1-x}; q)_\infty}{(q;q)_\infty^3} \left(A_x(q) - \frac{1}{\log q} B_x(q) \right),$$

where

$$A_x(q) = \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{m(-1)^m q^{m(m+1)/2} (q^{1-m-x}; q)_\infty}{1 - q^m} {}_1\phi_1\big(0; q^{1-m-x}; q, q^{1-m+x}\big)$$

and

$$B_x(q) = (q^{1-x}; q)_\infty {}_1\phi_1\big(0; q^{1-x}; q, q^{1+x}\big).$$

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q-Difference equations and holomorphic curve counting

Pietro Longhi
(joint work with Sachin Chauhan and Tobias Ekholm)

The open-string topological A-model is a theory of maps to holomorphic curves in a Kähler target space X with Lagrangian boundary conditions [1]. Although a complete formulation of what would tentatively be termed "open Gromov–Witten theory" is yet to be achieved, amplitudes in the open-string A-model have been computed in physics by indirect means, such as gauge/string duality [2, 3], 3d-3d correspondence [4], and mirror symmetry [5, 6]. A consequence of this is that exact results remain limited to specific cases, such as toric Lagrangians in toric Calabi–Yau threefolds [7] or conormals of knots in T^*S^3 [8]. In mathematics, an alternative approach to holomorphic curve counting is provided by Legendrian Contact Homology [9, 10] which provides a rigorous—if somewhat implicit—definition of holomorphic curve counts, by constructing the genus-zero string-corrected A-brane moduli space, known as the augmentation variety.

We consider the open A-model in the resolved conifold with A-branes supported on Lagrangians which asymptote to cones over a knot Legendrian torus $\Lambda_K \approx S^1 \times S^1$ in the contact boundary $U^*S^3 \approx S^2 \times S^3$. For every K there exist (at least) two Lagrangian fillings, corresponding to the conormal $L_K \approx S^1 \times D^2$ and the knot complement $M_K \approx S^3 \setminus K$. The open string partition function on L_K is computed by colored HOMFLYPT polynomials of K [8, 11, 12]. On the other hand, no direct computations of open string partition functions for knot complements M_K have been obtained so far.

We show that when M_K is a fibered knot complement, admitting a fibration by a Seifert surface over S^1, the open string partition function can be evaluated directly by a localization argument and takes the general form

$$\psi_{M_K}(a,q,\mu) = \sum_{d_1,\ldots,d_m \geq 0} H_{(d_1)^\vee,\ldots,(d_m)^\vee}(a,q)\, q^{\sum_{k,l}\Gamma_{kl}d_k d_l}\, (\epsilon_j q^{\beta_j}\mu^{m_j})^{d_j}$$

where $H_{(d_1)^\vee,\ldots,(d_m)^\vee}(a,q)$ are colored HOMFLYPT polynomials of a certain m-component auxiliary link $\mathcal{L}$ in S^3 arising in the localization procedure. Here, $(d_j)^\vee$ corresponds to either symmetric or antisymmetric partitions with d_j boxes, Γ is the linking matrix of $\mathcal{L}$, $\epsilon_j = \pm 1$, $\beta_j, m_j \in \mathbb{Z}$, and μ is the meridian holonomy generator of $\pi_1(M_K) \simeq \mathbb{Z}$. We determine $\mathcal{L}$ explicitly for all torus knots. In the case of $T_{2,2p+1}$ torus knots, $\mathcal{L}$ is a Hopf link, and we provide a complete formula for $\psi_{M_K}(a,q,\mu)$ by determining $\Gamma_{ij}, \epsilon_j, \beta_j, m_j$. Our argument also extends directly to higher-rank knot invariants, and encodes all-genus open-string curve counts with boundaries on M_K valued in the HOMFLYPT skein of M_K.

Our results provide the first direct computation of open string partition functions on knot complements, and the techniques developed in our work can be applied to all fibered knots. The computation of open string partition functions on knot complements opens the way to several applications. On the one hand, the relation to HOMFLYPT polynomials of links establishes a novel relation between the augmentation curve of K and the augmentation variety of $\mathcal{L}$ which allows to compute the former from the latter. Second, ψ_{M_K} defines a quantization of the augmentation curve of K as a q-difference operator. An analogous quantization arises from the open string partition function on L_K, and comparing these enables the study of compatibility of the two quantization schemes, thus addressing the global consistency of the D-module conjectured in the context of higher-genus open topological string theory [10, 13]. Third, the localization technique adopted in our work provides a direct derivation of the quiver structures underlying open topological string theory [14, 15]. Last, but not least, direct computation of ψ_{M_K} can be used as a benchmark to address the enumerative interpretation of conjectural results obtained by other means, such as $\widehat{Z}$ invariants for knot complements [16].

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q-Nonabelianisation in conformal field theory

Subrabalan Murugesan
(joint work with Lotte Hollands)

The moduli space of flat G-connections on a Riemann surface, denoted by $\mathcal{M}_C(G)$, is ubiquitous in mathematical physics and mathematics. It is thus imperative to get a good handle on this moduli space. For $G = SL(N)$, and when the surface C

is decorated with some additional structure known as a **WKB spectral network** $\mathcal{W}$, we can define a diffeomorphism

$$\mathcal{M}_C^{\mathcal{W}}(SL(N)) \rightleftharpoons \mathcal{M}_\Sigma^{\mathrm{al}}(GL(1))$$

where $\Sigma \subset T^*C$ is a branched, K-sheeted cover of C, $\mathcal{M}_\Sigma^{\mathrm{al}}(GL(1))$ is the moduli space of abelian connections on Σ with a prescribed singularity at the branch points of Σ, and $\mathcal{M}_C^{\mathcal{W}}(SL(N)) \subset \mathcal{M}_C(SL(N))$ is an open, dense subset of those connections that are *compatible* with $\mathcal{W}$ in an appropriate sense. This construction is called **$\mathcal{W}$-abelianisation/$\mathcal{W}$-nonabelianisation** [1, 2, 3].

A canonical deformation quantisation of the moduli space of flat G-connections is given by the corresponding skein algebra $\mathrm{SkAlg}(C, G)$. It turns out that $\mathcal{W}$-nonabelianisation extends to a homomorphism between the skein algebras

$$\mathrm{SkAlg}(C, SL(2)) \to \mathrm{SkAlg}(\Sigma, GL(1))$$

for $G = SL(2)$. This construction is called ***q*-nonabelianisation** [4]. The skein algebra $\mathrm{SkAlg}(C, SL(2))$ is also known to be the algebra of loop operators in Liouville conformal field theory. In particular, these loop operators act as difference operators on the space of conformal blocks of Liouville theory [5, 6]. It is further expected that these difference operators should correspond to a special, highly singular spectral network called the **Fenchel–Nielsen network**. In this talk, I discuss how this statement can be made precise through q-nonabelianisation.

The rough idea behind it is that the image of q-nonabelianisation defined subordinate to a Fenchel–Nielsen network is a quantum toroidal algebra whose product structure mimics that of the difference operators mentioned above. However, q-nonabelianisation is not well-defined for singular networks. Nonetheless, we still try to make progress by simplifying the setup slightly. Namely, instead of applying q-nonabelianisation to skeins (i.e. closed loops), we restrict our attention to tangles (i.e. open paths). In this case, q-nonabelianisation indeed seems to reproduce the field-theoretic results.

Nevertheless, the ultimate goal is to define q-nonabelianisation for Fenchel–Nielsen networks and the full skein algebra. We believe that understanding the exact correspondence between q-nonabelianisation and Liouville theory will help advance our knowledge of both areas. Little is known about q-nonabelianisation for $SL(N)$ for $N > 2$, and the higher-rank counterparts of Liouville theory, known as Toda theories. The hope is that the exact correspondence mentioned above might help remedy that. In addition, as mentioned previously, conformal blocks are typically associated with Fenchel–Nielsen networks. However, we could contemplate defining conformal blocks associated with more generic networks, and a good place to start is with q-nonabelianisation.

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Geometry and resurgence of WKB solutions of Schrödinger equations

Nikita Nikolaev

Consider the Schrödinger equation

$$\hbar^2 \partial_x^2 \Psi(x, \hbar) = Q(x, \hbar)\Psi(x, \hbar)$$

where x is a local complex variable on a compact Riemann surface X, $\hbar \in \mathbb{C}$ is a small complex perturbation parameter, and Q is a polynomial in $\hbar$ with meromorphic coefficients. Away from turning points (which are the zeros and simple poles of the quadratic differential $\phi_0 = Q_0(x)dx^2$), this equation has a basis of *formal WKB solutions* of the form

$$\hat{\Psi}(x, \hbar) = e^{-S(x)/\hbar}\hat{A}(x, \hbar) \quad \text{where} \quad \hat{A}(x, \hbar) = \sum_{k=0}^{\infty} A_k(x)\hbar^k,$$

enumerated by the two sheets of the *spectral curve* Σ which is the Riemann surface of the square root of ϕ_0. The function S is the integral of the Liouville one-form λ along local paths on the two sheets of Σ. The power series $\hat{A}$ is explicitly computable using little more than simple algebraic manipulations, but it is divergent in all but the most trivial examples. Lifting $\hat{A}$ to an analytic function — and therefore lifting a formal WKB solution to an analytic one — is a major challenge in mathematical physics.

In [1], we prove that formal WKB solutions of Schrödinger equations on arbitrary compact Riemann surfaces are *resurgent*. Specifically, we prove that formal WKB solutions $\hat{\Psi}$ are *Borel summable* in almost every direction $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. Concretely, this means that $\hat{A}$ lifts to a holomorphic function A_α given as a Laplace integral and defined in a halfplane sector of the $\hbar$-plane bisected by a ray α, yielding the so-called *exact WKB solutions*:

$$\Psi_\alpha(x, \hbar) = e^{-S(x)/\hbar}A_\alpha(x, \hbar) = e^{-S(x)/\hbar}\int_0^{\infty e^{i\alpha}} \hat{\Phi}(x, t)e^{-t/\hbar}dt.$$

This is achieved by proving the *Borel transform* $\hat{\Phi}$ admits endless analytic continuation away from a discrete subset of singularities. As a corollary of our main result, we deduce an old conjecture of Voros and Écalle about the resurgence of formal WKB solutions of Schrödinger equations with polynomial potential.

Our approach is purely geometric, relying on understanding the global geometry of complex flows of meromorphic vector fields using techniques from holomorphic Lie groupoids and the geometry of spectral curves. This framework provides a fully geometric description of the Borel plane, Borel singularities, and the Stokes rays. In doing so, we introduce a geometric perspective on resurgence theory.

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Quantum modularity and arithmetic of q-series: Lessons from resurgence

Claudia Rella
(joint work with Veronica Fantini)

Several fundamental quantities in topological quantum field theories and topological string theories exhibit a decomposition as a finite sum of products of a q-series and a $\tilde{q}$-series, where we take

$$q = \mathrm{e}^{2\pi\mathrm{i}z}\,, \quad \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}/z}\,, \tag{1}$$

and z is usually a coupling constant. Asymptotically expanding a q-series $f(q)$ in the small-coupling limit $z \to 0$ often yields a formal power series

$$\varphi(z) = \sum_{n=0}^{\infty} a_n\, z^n \in \mathbb{C}[\![z]\!]\,, \quad a_n \sim n! \quad n \gg 1\,, \tag{2}$$

which has zero radius of convergence and thus does not determine the original function $f(q)$ uniquely. Typically, the resurgent analysis of a divergent asymptotic series allows us to reconstruct (some of) its hidden non-perturbative corrections from the large-order growth of its coefficients. These take the form of new asymptotic series $\varphi_\omega(z) \in \mathbb{C}[\![z]\!]$ along with exponentially small factors $\mathrm{e}^{-\zeta_\omega/z}$ and complex numbers $S_\omega \in \mathbb{C}$ known as Stokes constants. Here, $\zeta_\omega \in \mathbb{C}$ are the locations of the singularities of the Borel transform of $\varphi(z)$. Special arrangements of singularities called peacock patterns have been observed in theories controlled by quantum curves, particularly in examples from the spectral theory of quantum operators associated with local Calabi–Yau (CY) threefolds [2, 3]. In these remarkable cases, the Stokes constants turn out to be rational or integer numbers.

Under suitable assumptions, the data extracted from the asymptotic expansion of a q-series through resurgent methods possess intrinsic number-theoretic and quantum modular properties, reflecting the symmetries of the geometry underlying the given quantum theory. In this talk, I will describe the emerging bridge between the resurgent and arithmetic structures encoded in certain q-series that are quantum modular and illustrate it with examples from the spectral theory of local weighted projective planes.

Originally prompted by the resurgent analysis of the spectral trace of local $\mathbb{P}^2$ [3, 4], the framework of modular resurgence has been introduced in [5].

Definition. *A modular resurgent series (MRS) is a Gevrey-1, simple resurgent asymptotic series $\varphi(z) \in \mathbb{C}[\![z]\!]$ whose Borel transform displays an infinite tower of singularities at $\zeta_k = kA\mathrm{i}$, $k \in \mathbb{Z}_{\neq 0}$, for some $A \in \mathbb{R}$, the secondary resurgent series are trivial, and the Stokes constants $S_k \in \mathbb{C}$ are the coefficients of two L-functions*

$$L_{\pm}(s) = \pm \sum_{k=1}^{\infty} \frac{S_{\pm k}}{k^s} \,. \tag{3}$$

MRSs possess a rich arithmetic structure linking the resurgence of q-series with the analytic number-theoretic properties of L-functions and quantum modular forms.

When obtained from q-series, MRSs naturally occur in pairs satisfying the so-called modular resurgence paradigm, where one equals the asymptotic expansion of the discontinuity of the other in the appropriate limit, and the two L-functions are connected through a functional equation. More precisely, consider a q-series $f(q)$ with coefficients denoted R_k, $k \in \mathbb{Z}_{\neq 0}$, such that its asymptotic expansion $\varphi(z)$ for $z \to 0$ is modular resurgent. Then, its paired MRS $\psi(\tau)$ is the asymptotic expansion for $\tau = -1/z \to 0$ of the $\tilde{q}$-series $g(\tilde{q})$ whose coefficients are the Stokes constants S_k, $k \in \mathbb{Z}_{\neq 0}$, of $\varphi(z)$. In physical theories, we interpret the formal power series $\varphi(z), \psi(\tau)$ as perturbative expansions of the same analytic function at weak and strong coupling, respectively. Then, the exchange of perturbative and non-perturbative data between paired MRSs realises an exact strong-weak symmetry. We can represent it schematically as follows.

$$\begin{array}{ccc}
\mathrm{disc}[\psi](\tau) = f(q) & \xrightarrow{z \propto \tau^{-1} \to 0} & \varphi(z) \\
\updownarrow & & \updownarrow \\
\psi(\tau) & \xleftarrow{\tau \propto z^{-1} \to 0} & g(\tilde{q}) = \mathrm{disc}[\varphi](z)
\end{array}$$

Moreover, MRSs from q-series possess distinctive summability and quantum modularity properties. We adopt the definition of holomorphic quantum modular form introduced by Zagier.

Conjecture. *Let $f(q)$ be a q-series where $q = \mathrm{e}^{2\pi \mathrm{i} z}$. If its asymptotic expansion $\varphi(z)$ for $z \to 0$ is modular resurgent, then $f(q)$ is a holomorphic quantum modular form for some subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and is uniquely reconstructed from $\varphi(z)$ via median resummation.*

Examples of modular resurgent series arise in various contexts, including the theory of Maass cusp forms, combinatorics, and the quantum invariants of knots and 3-manifolds. An infinite class of examples, closely tied to the q-Pochhammer symbol, comes from the spectral theory of local CY threefolds [4, 6]. Recall that there is a canonical way of constructing a family of quantum operators acting on $L^2(\mathbb{R})$ with the identification $\hbar = 4\pi^2/g_s$, where g_s is the topological string coupling constant, via Weyl quantisation of the mirror curve. By the TS/ST correspondence, the spectral theory of these operators gives access to the non-perturbative sectors of the topological string perturbative series in a precise way. We consider the case of local $\mathbb{P}^{m,n}$, where $m, n \in \mathbb{Z}_{>0}$, and denote by $\rho_{m,n}$ the corresponding quantum operator, which is known to be positive definite and trace-class. Its spectral trace $\mathrm{Tr}(\rho_{m,n})$ explicitly factorises into the product of a q-series and a $\tilde{q}$-series expressed in terms of q-Pochhammer symbols [1]. Let $N = 1+m+n$ and identify $q = \mathrm{e}^{N\mathrm{i}\hbar}$ and $\tilde{q} = \mathrm{e}^{-\frac{4\pi^2\mathrm{i}}{N\hbar}}$.

For all choices of m, n, the resurgent structures of $\log \mathrm{Tr}(\rho_{m,n})$ at both weak ($\hbar \to 0$) and strong ($\tau \propto \hbar^{-1} \to 0$) coupling have been determined exactly. For $N = 3, 4$, the full modular resurgence paradigm applies. For higher values of N, the Dirichlet series of the Stokes constants are not L-functions but linear combinations thereof. A slightly weaker formulation of modular resurgence is at play. Remarkably, however, an exact strong-weak symmetry persists and can be thought of as a realisation of underlying physical mechanisms, which can be intuitively traced back to the S-duality between the standard and Nekrasov–Shatashvili free energies of the topological string.

Finally, albeit stripped of some arithmetic properties, weak MRSs preserve a non-trivial structure and, in particular, their connection with quantum modular forms. This observation paves the way for a forthcoming generalisation of modular resurgence to the vector-valued setting.

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Partition functions, modularity, and resurgence: A proposal

Jörg Teschner

In my presentation, I have discussed a proposal for geometrically defined generating functions encoding various enumerative invariants related to topological string theory. The proposal concerns local Calabi-Yau manifolds X defined by equations of the form $uv = P_X(e^x, e^y)$, with P_X being a polynomial. It takes inspiration from a research program referred to as TS/ST-correspondence, reviewed e.g. in [1]. Central objects of this program are finite difference operators $\mathsf{P}_X(e^{\mathsf{x}}, e^{\mathsf{y}})$ constructed from P_X using some ordering prescription, with x, y being self-adjoint operators on $L^2(\mathbb{R})$ satisfying $[\mathsf{x}, \mathsf{y}] = i\hbar$. Viewing P_X as a quantisation of the curve Σ defined by the equation $P_X(e^x, e^y) = 0$ motivates the terminology quantum curve for P_X. One then considers certain Fredholm determinants Ξ_X of operators defined from the quantum curves, defining entire functions Ξ_X of the parameters $\mathbf{t}$ of P_X.

A lot of evidence has been accumulated that the determinants Ξ_X admit asymptotic expansions in $g_s = \frac{4\pi^2}{\hbar}$ related to the generating series of Gromov-Witten (GW) invariants by resurgence. One expects, more precisely, that the determinants Ξ_X admit expansions of the form

$$\Xi_X = \sum_{\mu\in H^2} \mathcal{Z}_{\rm top}\big(\mathbf{t}+\mu\tfrac{\hbar}{2\pi\mathrm{i}},\hbar\big),$$

with $H^2 = H^2(X,\mathbb{Z})$, and $\mathcal{F}_{\rm top} = \log\mathcal{Z}_{\rm top}$ being a summation of the formal GW generating series, $\mathcal{Z}_{\rm top}$ having a factorisation of the form

$$\mathcal{Z}_{\rm top}(\mathbf{t},\hbar) := \mathcal{Z}_{\rm DT}(\mathbf{t},\hbar)e^{\mathcal{G}_{\rm np}(\mathbf{t},\hbar)}, \tag{1}$$

where $\mathcal{Z}_{\rm DT}(\mathbf{t},\hbar)$ is a generating function of framed DT-invariants of X related to the GW generating series by the MNOP-correspondence [2], and $e^{\mathcal{G}_{\rm np}(\mathbf{t},\hbar)}$ represents non-perturbative contributions to the trans-series defined by the GW generating series.

The proposal discussed here deviates from the TS/ST-correspondence in two ways. We first consider more general functions $\mathcal{T}_X$ defined by a Zak transform

$$\mathcal{T}_X(\eta,\mathbf{t};\hbar) = \sum_{\mu\in H^2} e^{2\pi\mathrm{i}\langle\mu,\eta\rangle}\mathcal{Z}_{\rm top}\big(\mathbf{t}+\mu\tfrac{\hbar}{2\pi\mathrm{i}},\hbar\big), \qquad \eta\in H_2(X,\mathbb{C}). \tag{2}$$

The second deviation is to interpret[1] these functions as local sections of a canonical holomorphic line bundle $\mathcal{L}$ on a moduli space $\mathcal{Z}$ containing the moduli space $\mathcal{B}$ of complex structures on X. The periods Z_γ of Σ, with $\gamma\in H_1(\Sigma,\mathbb{Z})$, define coordinates for $\mathcal{B}$. One may then consider $\mathcal{Z} = \mathcal{M}\times\mathbb{C}^*_\hbar$, with $\mathcal{M} = T\mathcal{B}$, together with the problem of defining a family of $\hbar$-deformed complex symplectic structures on $\mathcal{M}$ by charts $\mathcal{U}_\imath$, $\imath\in\mathcal{I}$, associated to canonical bases $(\gamma_\imath^r,\check{\gamma}_r^\imath)$, $r=1,\dots,d$, for $\Gamma = H_1(\Sigma,\mathbb{Z})$, and systems of local holomorphic Darboux coordinates $(t_\imath^1,\dots,t_\imath^d,\ \eta_1^\imath,\dots,\eta_d^\imath)$ on $\mathcal{U}_\imath$ such that $\Omega = \sum_{r=1}^d dt_\imath^r\wedge d\eta_r^\imath$,

- the changes of coordinates across $\{\hbar\in\mathbb{C}^\times; Z_\gamma/\hbar\in\mathrm{i}\mathbb{R}_-\}$ are of the form

$$X_{\gamma'}^\jmath = X_{\gamma'}^\imath\big(1-\sigma(\gamma)(X_\gamma^\imath)^{\mathrm{sgn}(\langle\gamma',\gamma\rangle\Omega(\gamma))}\big)^{\langle\gamma',\gamma\rangle\Omega(\gamma)}, \quad \gamma,\gamma'\in\Gamma, \tag{3}$$

  where $X_\gamma^\imath = e^{2\pi\mathrm{i}\sum_{r=1}^d(m_r^\imath t_\imath^r - \check{m}_\imath^r\eta_r^\imath)}$ if $\gamma = \sum_{r=1}^d(m_r^\imath\gamma_\imath^r - \check{m}_\imath^r\check{\gamma}_r^\imath)$, determined by BPS-indices $\Omega(\gamma)$ satisfying the wall-crossing formula of Kontsevich and Soibelman, and a sign $\sigma(\gamma)$ called quadratic refinement [3],
- and one may represent the asymptotic behaviour for $\hbar\to 0$ in the form $t_\imath^r \sim -\frac{1}{\hbar}Z_{\gamma_\imath^r} + \mathcal{O}(\hbar)$, $\eta_r^\imath\sim -\frac{1}{\hbar}Z_{\check{\gamma}_r^\imath} + \check{\theta}_r^\imath + \mathcal{O}(\hbar)$, with $\check{\theta}_r^\imath$ being coordinates on the tangent fibres of $T\mathcal{B}$.

We expect that Voros symbols of certain q-difference equations defined by the quantum curves, as defined and studied in [5], provide solutions to this problem.

There is a canonical holomorphic line bundle $\mathcal{L}$ on $\mathcal{Z}$, having the generating functions of the symplectomorphisms (3) as transition functions. It has been observed in [6] that the Zak transform structure exhibited by the right side of (2)

[1] This line of thought is strongly inspired by the programs reviewed in [3] and [4], but the precise relation to these programs is not clear at the moment.

is preserved by the transition functions of the canonical line bundle $\mathcal{L}$. A global holomorphic section s of $\mathcal{L}$ can be defined by requiring that it coincides with the Zak transform $\mathcal{T}_X$ defined in (2) in a chart around the large volume limit.

An explicit description of the Stokes jumps of the summation of the GW generating series has been proposed in [7] and references therein. By comparison with [6], one may identify these Stokes jumps as the generating functions defining $\mathcal{L}$. Taken together, these observations suggest that the section s might represent the summations of the GW generating series globally on $\mathcal{Z}$. Explicit computation of a simple example confirms this suggestion [8].

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Difference equations, relative Gromov–Witten theory and the geometry of the Nekrasov–Shatashvili limit

Noah Tischler
(joint work with Murad Alim)

Following the ideas of [1, 2], the physics community [3, 4] was able to compute quantum corrected periods coming from the Nekrasov–Shatashvili (NS) limit of the refined topological string on Calabi–Yau threefolds about ten to fifteen years ago. Up until recently [5, 7, 8], it was not entirely clear what was happening enumerative geometrically. Everything presented is work in progress with Prof. Murad Alim.

In this talk, we will discuss the enumerative geometric meaning of the refined topological string free energy

$$F^{\mathcal{X}}_{\mathsf{ref}}(Q_i,\epsilon_1,\epsilon_2)=\sum_{g_1,g_2\geq 0}F^{\mathcal{X}}_{g_1,g_2}(Q_i)\,(\epsilon_1+\epsilon_2)^{2g_1}\,(-\epsilon_1\epsilon_2)^{g_2-1}\,, \tag{1}$$

of the Calabi–Yau threefold

$$\mathcal{X}=\mathsf{Tot}_{\mathbb{P}^1}\left(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\right), \tag{2}$$

the resolved Conifold, in the NS limit $\epsilon_1 = 0, \epsilon_2 = \hbar$. Let $H_{\mathcal{X}} = \{\Sigma(X, Y, z_i) = 0\}$ for $X, Y \in \mathbb{C}^\times, z_i \in \mathbb{R}_{\geq 0}$ be the mirror curve of $\mathcal{X}$. We then promote X, Y to operators X, Y obeying the q-commutation rule

$$\mathsf{YX} = q\mathsf{XY}, \; q = \exp(-i\hbar) =: \exp(\varepsilon). \tag{3}$$

Identify $\mathsf{X} = \exp(x)$ as a multiplicative operator and $\mathsf{Y} = \exp(\varepsilon\partial_x)$ as a shift operator for $x \in \mathbb{C}$ where we understand the action of Y on a function $\Psi(x)$ of $\log \mathsf{X} = x$ as

$$\mathsf{Y}(\Psi)(x) = \Psi(x + \varepsilon).$$

Then the associated difference equation to our geometry $\mathcal{X}$ is

$$\left(1 + \mathsf{X} + \mathsf{Y} + zq^{-1/2}\mathsf{XY}^{-1}\right)\Psi(x, z, \varepsilon) = 0. \tag{4}$$

We can solve equation (4) order by order using the WKB Ansatz [3] in small ε.

However, one can also solve the above difference equation order by order in small z corresponding to the large radius limit, being non-perturbative in ε. Running this procedure, we generate q-expansions of our quantum A- and B-periods for $\mathcal{X}$. We then find an analogous result for $\mathcal{X}$ to what holds for toric local surface Calabi–Yau threefolds. It turns out that we can express the free energy of $\mathcal{X}$ via the relative Gromov–Witten (GW) invariants [6] $N_{g,d}^{\mathbb{P}^1\backslash\infty}$ of the pair $(\mathbb{P}^1, \mathsf{pt} = \infty)$, relative to the point at infinity, something already established in [7] for local surfaces using very different methods.

Not only can we express $\overline{F}_{\mathsf{NS}}^{\mathcal{X}}(Q(z), q)$ in terms of relative GW invariants of $(\mathbb{P}^1, \mathsf{pt})$, after a change of variables we can bring it into the form found in [5] after the (geometrically non-trivial) change of variable $q = \exp(\varepsilon)$. This leads us to the following theorem to appear in [9].

From the B-side computation, we get that the holomorphic contribution $\overline{F}_{\mathsf{NS}}^{\mathcal{X}}(Q(z), q)$ can be written as

$$\overline{F}_{\mathsf{NS}}^{\mathcal{X}}(Q(z), q) = \log q \sum_{k=0}^{\infty} \frac{1}{k^2} \frac{Q^k}{q^{k/2} - q^{-k/2}} \in \mathbb{Q}(q^{1/2})[\![Q]\!]. \tag{5}$$

The change of variables $q = \exp(\varepsilon)$ then leads us to be able to express $\overline{F}_{\mathsf{NS}}^{\mathcal{X}}(Q(z), q = \exp(\varepsilon))$ in terms of relative GW invariants of $\mathbb{P}^1$ [9]

$$Q\frac{\partial}{\partial Q} \sum_{k=0}^{\infty} \frac{1}{k^2} \frac{Q^k}{q^{k/2} - q^{-k/2}} = \sum_{g \in \mathbb{Z}_{\geq 0}} \sum_{d \in \mathbb{Z}_{\geq 1}} (-\varepsilon)^{2g-1} N_{g,d}^{\mathbb{P}^1\backslash\infty} Q^d. \tag{6}$$

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Modular monodromy

Campbell Wheeler
(joint work with Stavros Garoufalidis)

One of the basic invariants of a linear q-difference equation is its monodromy. This is a matrix of elliptic functions (functions $f(x;q)$ satisfying the equation $f(qx;q) = f(x;q)$) that relates a filtration preserving basis at $x = 0$ to that at $x = \infty$. This talk describes an effective way that this monodromy can be computed when the linear q-difference equation satisfies an additional modularity property. This modularity ends up implying that the monodromy should be a Jacobi function (which is a very strict requirement for an elliptic function). This allows for effective computations of this monodromy, which in turn allows for explicit descriptions of q-Borel resummations. This talk introduced the Frobenius method of computing q-difference equations and quantum modular q-difference equations.

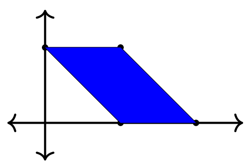
The Frobenius method for solving a q-difference equation is a simple algorithm based on the Newton polygon. The Newton polygon of a q-difference equation

$$\sum_{i,j} a_{ij}(q) x^j f(q^i x; q) = 0 \tag{1}$$

is defined as the convex hull of the points (i, j) such that $a_{ij} \neq 0$. For example, the q-difference equation

$$f(q^2 x, q) + (qx - 1) f(qx, q) - x f(x, q) = 0 \tag{2}$$

has Newton polygon given as follows:



Solutions to the equation can then be constructed for each edge. This is done first by flattening the edge, which is achieved by multiplication by a power of the θ-function. The θ-function is given by

$$\theta(x;q) = \sum_{k\in\mathbb{Z}} (-1)^k q^{k(k+1)/2} x^k = \prod_{j=0}^{\infty} (1 - q^{j+1} x)(1 - q^j x^{-1})(1 - q^{j+1}), \tag{3}$$

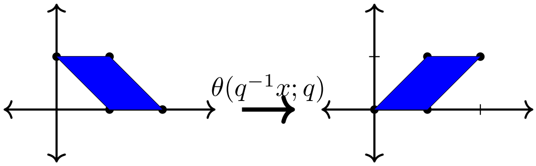
and satisfies the first order equation

$$\theta(qx;q) = -q^{-1}x^{-1}\theta(x;q)\,. \tag{4}$$

Multiplication by a θ-function acts by shearing the Newton polygon. In our example, if $g(x;q) = \theta(q^{-1}x;q)f(x;q)$, then $g(x;q)$ satisfies the equation

$$qxg(q^2x,q) + (1-qx)g(qx,q) - g(x,q) = 0\,, \tag{5}$$

which has Newton polygon:



After the edge is flat, we take an ansatz of the form

$$\frac{\theta(\rho^{-1}x;q)}{\theta(x;q)}\sum_{k=0}^{\infty} a_k(q)x^k\,, \tag{6}$$

where implicitly $a_k = 0$ for $k < 0$. This induces a linear recursion in the coefficients a_k. We can use the recursion to find a polynomial $P(\rho;q)$ so that $P(\rho;q)a_0 = 0$ where the RHS is some linear combination of a_k for $k < 0$. Therefore, to each ρ solving $P(\rho;q) = 0$ (counted with multiplicity) we find solutions to the q-difference equation setting $a_0(q) = 1$ and solving inductively for $a_k(q)$. (If ρ_0 is a root of $P(\rho;q)$ with multiplicity d, then take $\rho = \rho_0 + \epsilon + \mathrm{O}(\epsilon^d)$ when solving for $a_k(q)$. Roots ρ_0 and ρ_1 with $\rho_1 = q^d\rho_0$ are counted towards the same series solution and we take ρ_0 the root with all other roots ρ_1 with $d > 0$.)

In our example, we find two solutions

$$\theta(q^{-1}x;q)^{-1}\,, \qquad \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}x^k\,. \tag{7}$$

Notice that the second solution does not converge when $|q| < 1$. Therefore, to construct a meromorphic solution for $x, q \in \mathbb{C}^\times$ and $|q| < 1$ we must re-sum the second solution. We find divergent solutions whenever the Newton polygon has negative slopes.

To re-sum a divergent series, we use the q-Borel transform

$$\mathcal{B}\Big(\sum_{k=0}^{\infty} a_k(q)x^k\Big) = \sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}a_k(q)\xi^k\,, \tag{8}$$

and the q-Laplace transform for a function $f(\xi;q)$ we define

$$\mathcal{L}(f)(x,\lambda;q) = \frac{1}{\theta(\lambda;q)}\sum_{k\in\mathbb{Z}}(-1)^k q^{k(k+1)/2}\lambda^k f(q^k\lambda x;q) = \sum_{k\in\mathbb{Z}}\frac{f(q^k\lambda x;q)}{\theta(q^k\lambda;q)}\,. \tag{9}$$

We can always re-sum solutions to q-difference equations by a sequence of q-Borel and q-Laplace transforms.

In our example, we find that

$$\mathcal{B}\Big(\sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}x^k\Big) = \sum_{k=0}^{\infty}\xi^k = \frac{1}{1-\xi}\,, \tag{10}$$

and this has Laplace transform

$$\mathcal{L}\Big(\frac{1}{1-\xi}\Big)(x,\lambda;q) = \frac{1}{\theta(\lambda;q)}\sum_{k\in\mathbb{Z}}(-1)^k\frac{q^{k(k+1)/2}\lambda^k}{1-q^k\lambda x}\,. \tag{11}$$

This is the Appell-Lerch sum. We can apply the analogous method for the top of the Newton polygon with power series in x^{-1}. In our example, a theorem of Zwegers computes the monodromy as

$$\begin{pmatrix} 1 & 0 \\ -\frac{(q;q)_\infty^3\theta(q^{-1}t;q)\theta(\lambda^{-1}\mu;q)\theta(\lambda^{-1}\mu^{-1}t^{-1};q)}{\theta(\lambda^{-1};q)\theta(\mu;q)\theta(\lambda^{-1}t^{-1};q)\theta(\mu^{-1}t^{-1};q)} & 1\end{pmatrix}. \tag{12}$$

This is a Jacobi form.

One reason that this turned out to be a Jacobi form is that this difference equation is *quantum modular*. This states that the basis at $x=0$ and $x=\infty$ is a matrix-valued quantum modular form.

A quantum modular q-difference equation is a linear q-difference equation with a filtration preserving basis of solutions at $x=0$ and ∞ with fundamental matrices $F(x,\lambda;q)$ and $G(x,\mu;q)$ so that

$$\begin{aligned} &F(z/\tau,y/\tau;-1/\tau)\mathrm{diag}(\tau^{k_i})F(z,y;\tau)^{-1} \\ &= \Omega(z;\tau) = G(z/\tau,w/\tau;-1/\tau)\mathrm{diag}(\tau^{\ell_i})G(z,w;\tau)^{-1}\,, \end{aligned} \tag{13}$$

and $\Omega(z;\tau)$ has a meromorphic extension to $z\in\mathbb{C}$ and $\tau\in\mathbb{C}\setminus\mathbb{R}_{\le 0}$.

If this is true, then the monodromy is a Jacobi form as

$$F(\tfrac{z}{\tau},\tfrac{y}{\tau};\tfrac{-1}{\tau})^{-1}G(\tfrac{z}{\tau},\tfrac{w}{\tau};\tfrac{-1}{\tau}) = \mathrm{diag}(\tau^{k_i})F(z,y;\tau)^{-1}G(z,w;\tau)\mathrm{diag}(\tau^{-\ell_i}) \tag{14}$$

and hence is a matrix of Jacobi forms.

Our running example is quantum modular with $\Omega(z;\tau)$

(15)

$$\begin{pmatrix}0 & 1\\ 1 & -\tilde{t}\end{pmatrix}\begin{pmatrix} \tau^{-1} & 0 \\ \frac{\exp\left(\frac{\pi i}{4}\right)}{\sqrt{\tau}}\tilde{t}^{-\frac{1}{2}}\tilde{q}^{\frac{1}{8}}\exp\left(\pi i\frac{z^2}{\tau}\right)h(z;\tau) & \frac{\exp\left(\frac{\pi i}{4}\right)}{\sqrt{\tau}}\exp\left(-2\pi i\frac{(z+1/2-\tau/2)^2}{2\tau}\right)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & -t\end{pmatrix}^{-1}$$

where

$$h(z;\tau) = \int_{\mathbb{R}}\frac{e^{\pi i(\tau x^2+2izx)}}{2\cosh(\pi x)}dx\,, \tag{16}$$

is the Mordell integral.

This modularity is conjectured to be true for specialisations of hypergeometric equations. This can be effectively used to compute monodromy. This was done for examples coming from knots.

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