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Homogeneous Structures: Model Theory meets Universal Algebra

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Abstract. Many fundamental mathematical structures, such as the rationals or the random graph, are homogeneous, meaning that local isomorphisms extend to global automorphisms. Such structures arise as limits of classes of finite structures and encode these classes in a single object. This viewpoint has proved fruitful in model theory, universal algebra, and computer science, with applications to constraint satisfaction, automata theory, and verification. Homogeneous structures have rich automorphism groups, which makes them interesting for topological dynamics. For many applications, however, automorphism groups do not store enough information about the homogeneous structure, and one must instead consider polymorphism clones. Universal algebra has recently achieved major results for polymorphism clones on finite structures, culminating in the 2017 resolution of the Feder–Vardi dichotomy conjecture. An analogous conjecture for homogeneous structures remains open despite growing structural insights.

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Introduction by the Organizers

This workshop brought together researchers at the intersection of model theory, universal algebra, combinatorics, and theoretical computer science to explore recent developments in the study of homogeneous structures and their applications. The workshop was attended in person by 46 participants, with additional 6 participants attending virtually.

Homogeneous structures—those where every isomorphism between finite substructures extends to an automorphism—have long been central objects in model theory, and exhibit rich symmetry properties. The workshop emphasized the increasingly fruitful dialog between model-theoretic methods and algebraic techniques, particularly in the context of constraint satisfaction problems (CSPs) and structural Ramsey theory.

The meeting featured talks spanning several interconnected themes: the classification of homogeneous structures, the complexity of constraint satisfaction problems through polymorphisms, topological reconstruction in spaces of symmetries, Ramsey theory and amalgamation properties, and the dynamics of automorphism groups. A recurring motif was the power of algebraic invariants—particularly polymorphisms and their generalizations—to capture both structural properties and computational complexity, bridging classical model theory with contemporary applications in computer science.

The program included 4 tutorial talks (50 min), 20 talks (40 min), and 6 short talks by junior participants (20 min). The tutorials made the diverse audience familiar with the basics of the most recent developments.

Concrete topics of the contributions were as follows.

- **CSPs.** Andrei Krokhin in his tutorial introduced a generalization of finite-domain CSPs that gained prominence in the last 7 years – Promise CSPs. In this context, a generalization and abstraction of polymorphism clones emerged: the concept of a minion. This new algebraic development also has a major influence to the original, non-promise setting. The categorical perspective on the theory was explained by Jakub Opršal. Sebastian Meyer presented results about the minion homomorphism order of clones, for example that finite simple groups are tightly related to the first non-trivial top layer of this order.

  Recent results on other prominent generalizations of finite-domain CSPs were given in the talks by Dmitriy Zhuk (algorithmic results on temporal CSPs, i.e., CSPs over reducts of the order of the rational numbers), Johanna Brunar (expressive power of temporal CSPs), Žaneta Semanišinová (valued temporal CSPs), and Andrei Bulatov (modular counting of the number of CSP solutions).

  Different problems related to CSPs were discussed by Samuel Braunfeld (χ-bounded graph classes) and in a virtual talk by Szymon Toruńczyk (model checking problem).
- **CSPs and set theory.** Surprising connections recently emerged between finite-domain CSPs and infinite combinatorics. In his tutorial, Zoltán Vidnyánszky explained how each finite-domain (Promise) CSP gives rise to a compactness statement, whose strength over ZF aligns with the computational complexity of that CSP. The tutorial of Jan Grebík presented exciting connections to Borel combinatorics, e.g., the descriptive complexity of the set of Borel instances of a CSP that admit a Borel solution is, again, related to the computational complexity of the CSP.

- **Reconstruction.** An extensively studied question in the context of ω-categorical structures is the following: to what extend does the algebraic structure of the automorphism group determine its topological structure? The tutorial by Paolo Marimon presented motivation and state of the art on the analogous question when automorphism groups are replaced by monoids of elementary embeddings. A fresh look at a specific reconstruction theorem (by Lascar) was presented by Mira Tartarotti. A related question is how complex is the topological isomorphism relation between the invariants. Recent results were presented by Gianluca Paolini (for oligomorphic groups with additional properties and for procountable groups) and Roman Feller (for the monoid and clone setting).
- **Limit objects.** The Urysohn space is the universal and homogeneous metric space with respect to isometries. Katrin Tent presented an appropriate analogue for hyperbolic spaces and discussed open problems about its isometry group. Adam Bartoš described an analogue of the Urysohn space for ultrametric spaces and properties of its automorphism group, in particular the existence of a generic distance-carrying automorphism. A general, categorical viewpoint on generic maps was provided by Wiesław Kubiś. A logic that is naturally suited for studying topological objects, the logic of co-valuation, was proposed by Maciej Malicki in his talk.

  A new amalgamation class coming from combinatorial topology (namely, the stellar moves on simplicial complexes) was presented by Sławomir Solecki. Notably, the core arguments are neither geometric nor topological, they are carried by developing a set theoretic calculus of finite sequences of finite sets.

  Interesting limit objects also come from algebra. Peter Mayr presented a recent result in which he showed ample generics of the automorphism group of the Fraïssé limit of finite powers of a finite simple non-abelian Mal'cev algebra (e.g., a finite simple non-abelian group). Dragan Mašulović showed in his talk how to prove strong properties (metrizable universal minimal flow) of the projective limit of finite members of a variety by categorification of model-theoretic methods.
- **Homogeneity.** Itay Kaplan discussed the question of when adding a new relation symbol for a parameter-definable set preserves homogeneity, and presented a positive answer for NIP structures.

  In connection to CSPs, particularly important classes of structures are finitely bounded ones. Jakub Rydval presented a procedure (conditioned on the existence of suitable Ramsey expansions) deciding whether such a class has the amalgamation property.

  A positive classification result, namely for homomorphism-homogeneous countable oriented loopless graphs, was presented by Maja Pech.
- **Automorphism groups.** The center of interest of some of the contributions was the structure of automorphism groups. David Evans discussed the relation between questions about permutation modules of structures and central problems for homogeneous structures (such as the Thomas conjecture)

and reported on recent progress concerning ω-categorical Ramsey structures. Todor Tsankov presented his classification of ergodic measures invariant under an action of a primitive oligomorphic group with no algebraicity, and discussed some applications. Friedrich Martin Schneider provided a new source of examples of groups with uncountable cofinality and the Bergman property, coming from von Neumann's continuous geometry.

- **Ramsey theory.** Jan Hubička presented an emerging systematic approach to constructing Ramsey expansions for amalgamation classes and discussed the connection to big Ramsey degrees. Maximilian Hadek provided a close connection between the Ramsey property and the categorical approach to CSPs, in particular providing a common generalization of several Ramsey transfer theorems.

The workshop organizers would like to thank all speakers for their genuine efforts to make their presentations accessible to participants from diverse research backgrounds. We also thank all participants for their active and constructive engagement in productive cross-area discussions. These interactions will strengthen the growing synergy between our research areas. Finally, the organizers wish to express their appreciation to the Mathematisches Forschungsinstitut Oberwolfach for their hospitality and comprehensive organizational support.

Workshop: Homogeneous Structures: Model Theory meets Universal Algebra

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Abstracts

Amalgamation of partial dc-automorphisms

Adam Bartoš

(joint work with Wiesław Kubiś, Aleksandra Kwiatkowska, and Maciej Malicki)

In this work in progress we study ultrametric spaces as two-sorted structures $\langle X, D_X\rangle$ where the distance set D_X is a linear order with the minimum 0. The ultrametric $d_X\colon X \times X \to D_X$ satisfies the standard axioms including the ultrametric triangle inequality $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$.

Given two ultrametric spaces X and Y, an *isometric embedding* is an embedding $f\colon X \to Y$ of sets such that (assuming $D_X \subseteq D_Y$)

$$d_Y(f(x), f(x')) = d_X(x, x') \quad \text{for every } x, x' \in X.$$

A *dc-embedding* ("distance-carrying") is an embedding $f\colon X \to Y$ of sets together with an embedding $D_f\colon D_X \to D_Y$ of linear orders fixing 0 satisfying

$$d_Y(f(x), f(x')) = D_f(d_X(x, x')) \quad \text{for every } x, x' \in X.$$

By $\mathfrak{U}$ we denote the category of ultrametric spaces and dc-embeddings, $\mathfrak{L}^0$ is the category of linear orders with minimum 0 and embeddings preserving 0, and for every $E \in \mathfrak{L}^0$, $\mathfrak{I}^E \subseteq \mathfrak{U}$ is the subcategory consisting of spaces X with $D_X \subseteq E$ and of isometric embeddings. By $\mathfrak{U}_{\mathit{fin}}$, $\mathfrak{L}^0_{\mathit{fin}}$, and $\mathfrak{I}^E_{\mathit{fin}}$ denote the corresponding full subcategories of finite structures.

It is well-known that finite rational metric spaces with isometric embeddings form a Fraïssé category whose limit is the rational Urysohn metric space $\mathbb{M}$. Its completion $\overline{\mathbb{M}}$ is the Urysohn space universal for all separable metric spaces. Similarly, every $\mathfrak{I}^E_{\mathit{fin}}$, for a countable set of distances $E \subseteq \mathbb{R}^{\geqslant 0}$, is a Fraïssé category with limit $\mathbb{U}_E$. But in the ultrametric case the completion does not add any new distances, and so $\overline{\mathbb{U}_E}$ is not universal for $\mathfrak{I}^{\mathbb{R}^{\geqslant 0}}_{\mathit{fin}}$. In fact, $\mathfrak{I}^{\mathbb{R}^{\geqslant 0}}_{\mathit{fin}}$ is not Fraïssé, and there is no isometrically universal separable ultrametric space. However, if we change perspective from isometric embeddings to dc-embeddings, we recover a single generic space.

Theorem 1. *$\mathfrak{U}_{\mathit{fin}}$ is a Fraïssé category, and its limit $\mathbb{U}$ is the countable rational Urysohn ultrametric space, which is homogeneous both with respect to dc-automorphisms and isometries. Moreover, its completion $\overline{\mathbb{U}}$ is dc-universal for all separable ultrametric spaces.*

For a Fraïssé limit $\mathbb{F}$, an automorphism $g \in \mathrm{Aut}(\mathbb{F})$ is called *generic* if its conjugacy class is comeager.

Theorem 2 (Kechris–Rosendal [3], cf. Ivanov [1]). *The generic automorphism of the limit $\mathbb{F}$ of a Fraïssé category $\mathcal{F}_{\mathit{fin}}$ is exactly the weak Fraïssé limit of the category of finite partial automorphisms $\mathcal{F}^*_{\mathit{fin}}$. Hence, it exists if and only if $\mathcal{F}^*_{\mathit{fin}}$ has the joint embedding property and the weak amalgamation property.*

We have proved that $\mathfrak{U}^*_{\mathit{fin}}$ has even the cofinal amalgamation property (CAP) and that the empty space is an amalgamation base. Hence, by Theorem 2 we have that $\mathbb{U}$ has a generic dc-automorphism. The proof of CAP employs a general strategy based on determined partial automorphisms (cf. Kuske–Truss [4]), applicable when total automorphisms can be amalgamated. A similar approach was used by Kaplan–Rzepecki–Siniora [2] to obtain a generic automorphism of the universal meet-tree. Now we shall explain the general strategy.

By a *nice category of structures* we mean a hereditary class $\mathcal{C}$ of locally finite structures that is closed under unions of countable chains and such that $\mathcal{C}_{\mathit{fin}}$ is Fraïssé. From now on let us fix such $\mathcal{C}$.

A partial automorphism $\langle A, p\rangle \in \mathcal{C}^*_{\mathit{fin}}$ is called *determined* if it has an extension $\langle A, p\rangle \to \langle \hat{A}, \hat{p}\rangle$ (called *unique totalization*) to a total automorphism in $\mathcal{C}^*$ such that every extension $\langle A, p\rangle \to \langle X, f\rangle$ to a total automorphism has a unique factorization $\langle \hat{A}, \hat{p}\rangle \to \langle X, f\rangle$, i.e. orbits of $\langle A, p\rangle$ can be completed to full orbits in a unique way. It turns out that for a nice category of structures $\mathcal{C}$, every amalgamation base in $\mathcal{C}^*_{\mathit{fin}}$ is determined.

Theorem 3. *Suppose that the full subcategory of $\mathcal{C}^*$ of total automorphisms has AP. Then $\mathcal{C}^*_{\mathit{fin}}$ has CAP iff determined partial automorphisms are cofinal in $\mathcal{C}^*_{\mathit{fin}}$. In that case, $\langle A, p\rangle \in \mathcal{C}^*_{\mathit{fin}}$ is an amalgamation base iff $\langle A, p\rangle$ is determined.*

A forward *one-point orbit extension* of $\langle A, p\rangle \in \mathcal{C}^*_{\mathit{fin}}$ at $x \in A \setminus \mathrm{dom}(p)$ is an extension $\langle B, q\rangle \geqslant \langle A, p\rangle$ such that B is generated by $A \cup \{q(x)\}$, $\mathrm{dom}(q)$ is generated by $\mathrm{dom}(p) \cup \{x\}$, and $\mathrm{rng}(q)$ is generated by $\mathrm{rng}(p) \cup \{q(x)\}$. A backward one-point orbit extension is defined analogously.

A determined partial automorphism has unique one-point orbit extensions, and these are themselves determined. On the other hand, the following theorem shows how we can detect determined partial automorphisms by considering one-point extensions.

Theorem 4. *Let $\mathcal{S} \subseteq \mathcal{C}^*_{\mathit{fin}}$ be a full subcategory such that*

(1) $\mathcal{S}$-objects have unique one-point orbit extensions,

(2) one-point orbit extensions of $\mathcal{S}$-objects are in $\mathcal{S}$.

Then any $\langle A, p\rangle \in \mathcal{S}$ is determined. Consequently, if we additionally have that

(3) $\mathcal{S}$ is cofinal in $\mathcal{C}^*_{\mathit{fin}}$,

then determined partial automorphisms are cofinal in $\mathcal{C}^*_{\mathit{fin}}$.

Strategy. Now we are ready to describe our strategy for proving CAP and simultaneously characterizing amalgamation bases in $\mathcal{C}^*_{\mathit{fin}}$ for a nice category of structures $\mathcal{C}$.

(1) Show that total automorphisms in $\mathcal{C}^*$ have AP so Theorem 3 applies.
(2) Pick a full subcategory $\mathcal{S} \subseteq \mathcal{C}^*_{\mathit{fin}}$ of so-called *sufficient* partial automorphisms satisfying the assumptions of Theorem 4.
(3) Then by Theorems 3 and 4, $\mathcal{C}^*_{\mathit{fin}}$ has CAP, the amalgamation bases are exactly determined partial automorphisms, and $\mathcal{S}$ forms a cofinal family of those.

(4) If we also show that every amalgamation base is in $\mathcal{S}$, then $\mathcal{S}$ is exactly the family of amalgamation bases.

We consider the following choice of $\mathcal{S}$. An orbit $p^{\mathbb{Z}}(a)$ of $\langle A, p\rangle \in \mathcal{C}^*$ is *stable* if it does not merge with another orbit in any extension and if it does not become closed in any extension (unless already closed). It turns out that every orbit of an amalgamation base $\langle A, p\rangle \in \mathcal{C}^*_{fin}$ is stable. As an application of the strategy, we obtain the following theorems.

Theorem 5 (simple example)**.** *$(\mathfrak{L}^0)^*_{fin}$ has CAP, and for $\langle A, p\rangle \in (\mathfrak{L}^0)^*_{fin}$ the following are equivalent:*

(a) $\langle A, p\rangle$ is an amalgamation base.
(b) $\langle A, p\rangle$ is determined.
(c) Every orbit of $\langle A, p\rangle$ is stable.

Theorem 6 (main application)**.** *$\mathfrak{U}^*_{fin}$ has CAP, and for $\langle A, p\rangle \in \mathfrak{U}^*_{fin}$ the following are equivalent:*

(a) $\langle A, p\rangle$ is an amalgamation base.
(b) $\langle A, p\rangle$ is determined.
(c) Every orbit of $\langle A, p\rangle$ is stable.
(d) Every orbit of the point sort of $\langle A, p\rangle$ is stable and $\langle D_A, D_p\rangle$ is an amalgamation base in $(\mathfrak{L}^0)^*_{fin}$.

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Coloring in graphs definable in a pure set

Samuel Braunfeld
(joint work with Sarosh Adenwalla, Tomáš Hons, John Sylvester, Viktor Zamaraev)

In analogy with how semi-algebraic graph classes are obtained from the real ordered field *set-defined classes* were introduced in [1] as hereditary graph classes obtained by taking a graph definable in the structure $(\mathbb{N}, =)$, passing to its class of finite induced subgraphs, and then possibly passing to a further hereditary subclass. A *complete* set-defined class is the full class of finite induced subgraphs of some graph definable in $(\mathbb{N}, =)$. A graph class $\mathcal{C}$ is *χ-bounded* if there is $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$, we have $\chi(G) \leqslant f(\omega(G))$, where $\chi(G)$ is the chromatic number and $\omega(G)$ is the size of the largest clique. Thus the topic of χ-boundedness is concerned with the question of to what extent cliques are the only reason for graphs in some class to have large chromatic number.

We study the question of when a (complete) set-defined class is χ-bounded. As noted in [1], the class of shift graphs (induced subgraphs of the infinite graph defined on $\mathbb{N}^2$ with edge-relation defined by $(x, y)R(x', y') \iff y' = x \vee x' = y$) is set-defined and not χ-bounded. Our results show that this is in some sense the only obstruction.

Theorem 1. *Let $\mathcal{C}$ be a complete set-defined graph class. Then one of the following holds.*

(1) $\mathcal{C}$ is (polynomially) χ-bounded.
(2) $\mathcal{C}$ contains a subclass of shift graphs that is not χ-bounded.

Furthermore, which outcome holds is decidable given the formula defining the edge-relation of the infinite graph whose finite induced subgraphs form $\mathcal{C}$.

Corollary 1. *Let $\mathcal{C}$ be a complete set-defined graph class. Then $\mathcal{C}$ satisfies the Gyárfás-Sumner conjecture, i.e. if $\mathcal{C}$ is not χ-bounded then it contains every tree.*

Theorem 2. *Let $\mathcal{C}$ be a set-defined graph class. If $\mathcal{C}$ is not χ-bounded, then its subgraph-closure contains a subclass of shift graphs that is not χ-bounded.*

The proof ultimately relies on a dichotomy theorem for tropical linear programs. After various reductions to simpler subclasses, we write a pair of tropical linear programs that, if they can both be satisfied, witness that the subclass contains shift graphs via a particular construction. If one cannot be satisfied, then the dichotomy theorem of [2] says that this is witnessed by a winning strategy for a player in a two-player (mean payoff) game on an auxiliary graph. This strategy is then exploited to show that the subclass has bounded chromatic number, and if all subclasses we consider have bounded chromatic number then the original class is χ-bounded.

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Ianus-faces of Temporal Constraint Languages

Johanna Brunar
(joint work with Michael Pinsker, Moritz Schöbi)

The Bodirsky-Kára classification [2] of temporal constraint languages stands as one of the earliest and most seminal complexity classifications within infinite-domain Constraint Satisfaction Problems (CSPs), yet it remains one of the most mysterious in terms of algorithms and algebraic invariants for the tractable cases.

The reason is that the general procedure from [3], which relies on canonical polymorphisms [4] to show polynomial-time solvability of an infinite-domain CSP by reducing the problem to the CSP of a finite template, is not applicable in the temporal case. While ongoing research aims to gain deeper insight into the cases that are solvable in polynomial time through new algorithmic approaches [5], our contribution is to shed light on the algebraic aspects.

We show that those temporal languages which do not pp-construct every finite structure (and thus by the classification are solvable in polynomial time) have, in fact, very limited expressive power as measured by the digraphs they can pp-interpret.

Theorem. *Let $\mathbb{A}$ be a temporal constraint language that does not pp-construct every finite structure. If $\mathbb{G}$ is any smooth digraph that is pp-interpretable in $\mathbb{A}$ and has pseudo-algebraic length* 1 *modulo the automorphism group of* $(\mathbb{Q};<)$*, then $\mathbb{G}$ contains a pseudo-loop modulo* $\mathrm{Aut}((\mathbb{Q};<))$.

By standard techniques, first observed in [6], this limitation yields an infinite family of algebraic consequences. While most of these were previously unknown, we also provide new, uniform proofs for known invariance properties. In particular, we show that any temporal constraint language that fails to pp-construct some finite structure admits 4-ary pseudo-Siggers polymorphisms:

Corollary. *Let $\mathbb{A}$ be a temporal constraint language that does not pp-construct every finite structure. Then $\mathbb{A}$ has polymorphisms u, v, s such that for all evaluations of the variables the following identity is satisfied:*

$$u \circ s(a, r, e, a) = v \circ s(r, a, r, e).$$

While in the finite, the existence of 4-ary Siggers polymorphisms is known to characterise those finite structures that do not pp-construct all finite structures [7], our result sustains the possibility that the existence of such polymorphisms extends to the much broader context of the Bodirsky-Pinsker conjecture [8].

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Modular counting: automorphism groups and finite fields

Andrei A. Bulatov
(joint work with Amirhossein Kazeminia)

In a counting Constraint Satisfaction Problem (CSP) the task is to find the number of homomorphisms between two relational structures. If the second (the target) structure is fixed, say, $\mathcal{H}$, this problem is denoted $\#CSP(\mathcal{H})$. We study how the complexity of this problem depends on the structure $\mathcal{H}$.

The complexity of just finding the number of homomorphisms is very well studied, so in this presentation we focus on the finding the number of homomorphisms modulo a prime number. The corresponding version of the counting CSP is denoted $\#_pCSP(\mathcal{H})$. The complexity of such problems is completely characterized when $\mathcal{H}$ is a graph [1], but the case of general relational structures remains widely open.

We review recent progress in this direction; the majority of the results can be found in [2] and [3]. The main results we report on include:

(1) Applications of multi-sorted relational structures and there automorphisms to strengthen sufficient conditions for polynomial time solvability.

(2) A generalized type of automorphisms, automorphisms polynomials, and there applications to simplify modular counting CSPs.

(3) Applications of the above methods to classify the complexity of $\#CSP(\mathcal{H})$ for conservative $\mathcal{H}$, that is, structures that define every possible unary relations.

(4) Applications of the above methods to classify the complexity of $\#CSP(\mathcal{H})$ for 3-element structures $\mathcal{H}$.

Classifications (3) and (4) are incomplete, as our understanding of modular partition functions is lacking.

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Permutation modules for Ramsey structures

David M. Evans

1. Background and Questions

Suppose G is a group acting on a set W and R is a commutative ring with 1. Let RW denote the free R-module with basis W. By extending the permutation action on the basis linearly, we obtain an action of G on RW. Formally, this makes RW into a module for the group ring RG, but we shall say simply that RW is a G-module. We refer to this as a *permutation module.* We will represent an element x of RG as a formal sum

$$x = \sum_{w \in W} \alpha_w w$$

where the α_w are elements of R and only finitely many of these are non-zero. The finite set $\{w : \alpha_w \neq 0\}$ is called the *support* of x.

We are interested in submodules of this module in the case where M is a countably infinite ω-categorical structure, $G = \mathrm{Aut}(M)$ is the automorphism group of M and W is a sort in M^{eq} (for example, W is M^n for some $n \in \mathbb{N}$). We will mainly be interested in the case where R is a field F. In this case, FW is the F-vector space with basis W and G is acting on this as F-linear maps. Our aim is develop tools in the case where M is a *Ramsey structure.* As will be clear from the proofs, there are strong similarities between what we are doing here and the use of canonical functions in work on infinite domain CSPs (see [4] for a comprehensive account) and the decidability results in [5].

Permutation modules first arose in model theory in the work of Ahlbrandt and Ziegler [1, 2] and in work on finite covers (see [8] for a survey). Abstracting some ideas from [1], the paper [7] gave a number of general results and questions. Independently, permutation modules arising from ω-categorical structures have also been studied more recently in the computer science literature: for example, the papers [6, 9]. The following question appears in [7] and more recently in [6]:

Question. Suppose M is a countable ω-categorical structure and $G = \mathrm{Aut}(M)$. If W is a sort in M^{eq}, does FW have the ascending chain condition (acc) on submodules (equivalently, is every submodule finitely generated)?

It would be enough to prove acc in the cases where W is M^n. A model-theoretic method for proving acc in Question 1 is to show that W has an *AZ-enumeration* (the idea is due to Ahlbrandt and Ziegler and is sometimes called a nice enumeration, or a good enumeration). Further details and references can be found in [7] or [8]. The paper [11] show that this method has limitations by giving an example of an ω-categorical structure *without* an AZ-enumeration (answering a question in [3, 7]).

It is known that the descending chain condition for FW can fail for general W (in fact, for the case where F is a finite field, M is the vector space of countable dimension over F and W is the corresponding projective space: see [2, 6]). However, for structures M which are homogeneous in a finite relational language we have the following stronger question from [6]:

Question. Suppose that M is a structure which is homogeneous in a finite relational language and $G = \mathrm{Aut}(M)$. Let $k \in \mathbb{N}$ and consider G acting on $W = M^k$. Is FW of finite length for every field F?

Here, we say that a module U has *finite length* if there is a bound on the length n of a chain of submodules $\{0\} < U_1 < U_2 \ldots < U_n = U$. The maximum length is then called the length of U.

Remarks. It is not hard to show that a negative answer to Question 1 with F finite would yield:

(i) a structure which is homogeneous for a finite relational language whose automorphism group has infinitely many closed normal subgroups;
(ii) a structure which is homogeneous in a finite relational language which has infinitely many first-order reducts.

So (i) would give a negative answer to a question of Macpherson (Question 2.2.7 (4) in [13]) and (ii) would give a counterexample to a well-known conjecture of Simon Thomas [14] on reducts of finitely homogeneous structures.

2. Ramsey structures and a decision procedure

If M is a (ordered) *Ramsey structure*, then by the theorem of Kechris, Pestov and Todorčević ([12]), the group $G = \mathrm{Aut}(M)$, considered as a topological group with the usual topology on a permutation group, is *extremely amenable*: meaning that every non-empty compact space on which it acts continuously has a G-fixed point. Under this hypothesis, we can develop a duality between the module FW and the topological module $F[W]$ of *definable functions* $W \to F$. These are functions which are constant on the parts of a partition of W into finitely many definable sets. This approach has several consequences which seem to us to be surprising, including the following 'decision procedure'.

Suppose that R is a commutative ring, $x, v_1, \ldots, v_r \in RW$ and we wish to decide whether or not x is in $Y = \langle v_1, \ldots, v_r \rangle_{RG}$, the RG-submodule of RW generated by $v_1, \ldots, v_r$. Let $S \subseteq W$ be the support x and let $G_{(S)}$ be the (pointwise) stabiliser of S in G. Let $W_1, \ldots, W_l$ be the $G_{(S)}$-orbits on W (there are only finitely many such orbits, by the ω-categoricity). Define a map

$$\Omega_S : RW \to R^l$$

by

$$\Omega_S(\sum_{w \in W} \gamma_w w) = (\sum_{w \in W_1} \gamma_w, \ldots, \sum_{w \in W_l} \gamma_w).$$

If $x \in Y$, then clearly $\Omega_S(x) \in \Omega_S(Y)$. In general, there is no reason to expect that the converse should hold. However, we show in Theorem 1 that if M is an ω-categorical Ramsey structure, then the converse does indeed hold. Using this we can reduce our original decision problem to a computation in the finite-rank, free R-module R^l. Let $v_1, \dots, v_t$ be representatives for the $G_{(S)}$-orbits on the union of the G-orbits on RW containing $v_1, \dots, v_r$ (the ω-categoricity of M and finiteness of S guarantee that there are finitely many such orbits). Then we have the following (of course, t and l here will increase with the size of the support of x).

Theorem 1. *Suppose R is a commutative ring, M is an ω-categorical Ramsey structure and W is a sort in M^{eq}. Let $x, v_1, \dots, v_t$, S, Ω_S be as defined above. Then*

$$x \in \langle v_1, \dots, v_r \rangle_{RG} \Leftrightarrow \Omega_S(x) \in \langle \Omega_S(v_1), \dots, \Omega_S(v_t) \rangle_R. \square$$

Remarks. In the case $M = (\mathbb{Q}; \leqslant)$, this result has been proved using different methods (relying on the results of [6]) in [10]. Decision processes for reducts of ω-categorical Ramsey structures are proved in [5]. The methods used here are clearly based on those in [5], but it is less clear whether our results formally follow from those in [5].

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Decidability of Interpretability

Roman Feller
(joint work with Michael Pinsker)

Recent work by Paolini and Nies [8] shows that, within the class of automorphism groups of ω-categorical structures without algebraicity, the equivalence relation of topological isomorphism is smooth. Loosely speaking, this means that it is as simple as possible from the perspective of descriptive set theory. In more formal terms, the set of closed oligomorphic subgroups of $\mathrm{Sym}(\omega)$ denoted by $\mathcal{O}$, corresponding to automorphism groups of ω-categorical structures on ω, can be viewed as a Borel space in a natural way [7]. Asserting smoothness of the equivalence relation of topological isomorphism (restricted to those groups that are without algebraicity) now means that there is a Borel reduction to equality on a standard Borel space, i.e., a map f from the space $\mathcal{O}$ into a standard Borel space, such that for any two oligomorphic groups without algebraicity $G, H \in \mathcal{O}$, it holds

$$G \cong_{\mathrm{top}} H \iff f(G) = f(H).$$

This result can be viewed as a corollary of a result originally due to Rubin [9], which, reformulated in the language of permutation groups, reads as follows.

Theorem (Rubin)**.** *Let G, H be closed oligomorphic subgroups of* $\mathrm{Sym}(\omega)$ *that are without algebraicity. Then the groups G and H are isomorphic as topological groups precisely when they are isomorphic as permutation groups.*

Namely, to show smoothness of topological isomorphism it suffices to show that the much simpler relation of permutation group isomorphism is smooth. Inspired by this result, we ask the same question in the setting of monoids and clones: is the equivalence relation of topological isomorphism between closed oligomorphic submonoids/subclones of $\mathrm{End}(\omega)$ $(= \omega^\omega)$/$\mathrm{Pol}(\omega)$ $(= \bigcup_{n<\omega} \omega^{\omega^n})$ that are without algebraicity smooth?

We make progress on this question by observing that, similarly to the group case, the equivalence relation of isomorphism between transformation monoids and function clones is smooth, so to extend this to topological isomorphism we study in what capacity the above-mentioned theorem translates to monoids and clones.

So, how does the title of this abstract come about? Inspired by the tractability results in the descriptive set theory sense, i.e., smoothness, we also investigate a similar equivalence relation from the computational point of view. This is motivated by research on constraint satisfaction problems, CSPs for short, which shall be sketched briefly now to provide context for the decidability result.

Given any relational structure $\mathbb{A}$, $\mathrm{CSP}(\mathbb{A})$ is the computational problem of deciding if a given finite relational structure homomorphically maps into $\mathbb{A}$. While in general CSPs are undecidable [6], there are natural classes of relational structures whose CSP is known to be in NP, and it has been a very active area of research to investigate their computational complexity. Arguably, one of the premier results is a P vs. NP-complete dichotomy for CSPs of finite relational structures, independently established by Bulatov [4] and Zhuk [10, 11] in 2017. A similar conjecture has been stated in the infinite-domain setting by Bodirsky and Pinsker, asserting a P vs. NP-complete dichotomy for first-order reducts of finitely bounded homogeneous relational structures, see e.g., [1] for details. While the conjecture is still wide open as of today, significant progress has been made to understand the infinite ω-categorical landscape of CSPs (a broader class than the scope of the aforementioned conjecture) better. One such result being that, for an ω-categorical structure $\mathbb{A}$ there is a homomorphically equivalent structure $\mathbb{A}^{\mathrm{core}}$, its *model-complete core*, which is unique up to isomorphism and has the same CSP as $\mathbb{A}$. This model-complete core can be seen as a minimal representative of the class of all relational structures that have the same CSP as $\mathbb{A}$. It was further found that the computational complexity (up to log-space reductions) of $\mathrm{CSP}(\mathbb{A})$ is captured by $\mathrm{Pol}(\mathbb{A})$ viewed as a topological clone with the topology of pointwise convergence. Combining the two results, we see that $\mathbb{A}$ and $\mathbb{B}$ have CSPs of the same complexity (up to log-space reductions) if $\mathrm{Pol}(\mathbb{A}^{\mathrm{core}}) \cong_{\mathrm{top}} \mathrm{Pol}(\mathbb{B}^{\mathrm{core}})$; the latter assertion being equivalent to $\mathbb{A}^{\mathrm{core}}$ and $\mathbb{B}^{\mathrm{core}}$ being bi-interpretable by means of primitive positive formulas [2].

We establish the following decidability result for various kinds of interpretability; in this statement $\mathrm{xy} \in \{\text{first-order}, \text{existential positive}, \text{primitive positive}\}$. For details on the setup, we refer the reader to [3], where instead of bi-interpretability the relation of interdefinability is investigated.

Theorem ([5])**.** *The following (promise-)problem is decidable:*

- ***Given:*** *Finitely bounded homogeneous transitive Ramsey structures $\mathbb{C}$, $\mathbb{D}$ and first-order reducts $\mathbb{A}$, $\mathbb{B}$ in a finite language whose model-complete cores are without algebraicity.*
- ***Question:*** *Are $\mathbb{A}^{\mathrm{core}}$, $\mathbb{B}^{\mathrm{core}}$ xy-bi-interpretable?*

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CSPs and Borel combinatorics

Jan Grebík

Recent years have brought remarkable progress in the study of combinatorial properties of measurable graphs and their applications to various areas of mathematics, including descriptive set theory, dynamics, ergodic theory, random processes, and distributed computing, see the surveys [19, 22, 2, 15].

Definition 1. A *Borel graph* $\mathcal{G}$ is a triplet $(V, \mathcal{B}, E)$, where $(V, \mathcal{B})$ is a standard Borel space[1] and E is a Borel subset of $\binom{V}{2}$, the standard Borel space of unorder pairs of V that inherits its standard Borel structure from $V \times V$.

Prominent examples of Borel graphs arise from dynamics. For instance, let $\alpha_1, \ldots, \alpha_k$ be Borel isomorphisms of a standard Borel space $(V, \mathcal{B})$. For concreteness, one may take k rotations of the unit circle $\mathbb{S}^1$. Define

$$E = \left\{ \{x, y\} \in \binom{V}{2} : \exists 1 \leqslant i \leqslant k \ \alpha_i(x) = y \right\}.$$

It can be readily checked that $\mathcal{G} = (V, \mathcal{B}, E)$ is a Borel graph.

A typical problem in Borel combinatorics is the following: *Given a Borel graph $\mathcal{G}$, what is the minimal $k \in \mathbb{N}$ such that there is a Borel coloring of $\mathcal{G}$ with k colors?* Formally, when can we find a map $\varphi : V \to \{1, \ldots, k\}$ that is a graph homomorphism from $\mathcal{G}$ to the complete graph on k vertices such that $\bigsqcup_{i=1}^{k} \varphi^{-1}(\{i\})$ is a partition of V into sets from $\mathcal{B}$? Compactly phrased, *what is the* Borel chromatic number $\chi_B(\mathcal{G})$ *of a Borel graph $\mathcal{G}$?*

General results on the existence of Borel colorings of Borel graphs have shown a striking similarity with the theory of *distributed computing* in theoretical computer science, which aims to understand which coloring problems in large networks can be produced locally [21, 17, 23]. It is now understood that this connection is not a coincidence; rather, it has developed into a formal theory over the past years [3, 4, 13, 7]. Let us state a foundational positive result on the side of Borel combinatorics, and refer the reader to recent surveys [22, 2, 15] for the analogous results in distributed computing, as well as for a more detailed discussion of the connections between the two areas.

Theorem 2 (Borel greedy coloring [20])**.** *Let $\mathcal{G}$ be a Borel graph of maximum degree bounded by $\Delta < \infty$. Then $\chi_B(\mathcal{G}) \leqslant \Delta + 1$.*

[1] Recall that $(V, \mathcal{B})$ is a *standard Borel space* if there is a separable complete metric on V such that its σ-algebra of Borel sets is equal to $\mathcal{B}$.

As the existence of a Borel coloring also implies the existence of a coloring without any definability requirements, general combinatorial results about Borel graphs can be viewed as strengthenings of their classical counterparts. What does it mean, however, if no such generalization exists? Can we find a reasonable structural characterization of Borel graphs for which these general results fail? These questions motivate another part of Borel combinatorics that has seen substantial development—*complexity problems in Borel combinatorics.*

A fundamental (positive) result in this direction is the so-called $\mathbb{G}_0$-*dichotomy* [20] that characterizes Borel graphs that do not admit Borel colorings with finitely many colors by providing a concrete structural obstacle – an existence of a Borel graph homomorphism from the Borel graph $\mathbb{G}_0$.

Is there a similar characterization for Borel graphs of Borel chromatic number at most some fixed $k \in \mathbb{N}$*?* A breakthrough came with the work of Todorčević and Vidnyánszky [25]. To parse their result, we first recall a general framework for studying complexity questions in descriptive set theory, see e.g. [18, Chapter V]. These considerations are in complete, though merely illustrative, analogy with the classes P and NP from computational complexity theory, which correspond below to the classes $\boldsymbol{\Pi}^1_1$ and $\boldsymbol{\Sigma}^1_2$, respectively.

Given a Polish space X, a set $A \subseteq X$ is $\boldsymbol{\Sigma}^1_2$ if it is the projection of a $\boldsymbol{\Pi}^1_1$ (co-analytic) set. Borel graphs (up to isomorphism) can be encoded as elements of the Baire space ω^ω in such a way that the collections of all *codes of Borel graphs* is $\boldsymbol{\Pi}^1_1$, see [25, 11]. It can be shown that (in a fixed coding) the set of (codes of) locally finite Borel graphs that have Borel chromatic number at most some $k \in \mathbb{N}$ is a $\boldsymbol{\Sigma}^1_2$ set. Additionally, we say that A is $\boldsymbol{\Sigma}^1_2$-*hard* if for every Polish space Y and every $\boldsymbol{\Sigma}^1_2$ set $B \subseteq Y$, there is a Borel map $f : Y \to X$ such that $f^{-1}(A) = B$, i.e., $f(y) \in A$ if and only if $y \in B$. Finally, a set $A \subseteq X$ is called $\boldsymbol{\Sigma}^1_2$-*complete* if it is simultaneously $\boldsymbol{\Sigma}^1_2$-hard and $\boldsymbol{\Sigma}^1_2$.

Theorem 3 ([25]). *The set of codes of locally finite Borel graphs of Borel chromatic number at most* 3 *is* $\boldsymbol{\Sigma}^1_2$*-complete.*

Theorem 3 is a hardness result that rules out any analogue of the $\mathbb{G}_0$-dichotomy for Borel graphs of Borel chromatic number strictly larger than 3. The machinery developed in [25] has been successfully used to prove hardness results for various other Borel structures [7, 24, 11, 14, 12]. In the remainder, we discuss a systematic study of the complexity landscape for Borel coloring problems that has been initiated by Thornton [24], who found connections with the theory of finite template CSPs.

Recall that a (fixed-template) *constraint satisfaction problem (CSP)* takes the following form: *Given a finite (relational) structure* H*, how complicated is it to decide whether a finite structure in the same signature admits a homomorphism to* H*?* The target structure H is called the *template*, and the class of finite structures admitting a homomorphism to H is denoted $\mathrm{CSP}(H)$, see e.g. [5] for a comprehensive treatment of the area. The well-known *CSP dichotomy conjecture*

[10], now a theorem [8, 26], states that every CSP is *tractable*, in which case $\mathrm{CSP}(H)$ admits a polynomial-time decision algorithm, or *NP-complete*.

In analogy with the definition of Borel chromatic numbers, one can consider a Borel version of finite template CSPs, denoted for fixed $\mathcal{D}$ by $\mathrm{CSP}_B(\mathcal{D})$. That is, the set $\mathrm{CSP}_B(\mathcal{D})$ consists of the codes of Borel instances of $\mathcal{D}$ that admit a Borel homomorphism to $\mathcal{D}$. In [24], Thornton utilized the algebraic approach from the finite setting to prove the following general result.

Theorem 4 ([24])**.** *Let $\mathcal{D}$ be a finite template CSP that is not tractable. Then $\mathrm{CSP}_B(\mathcal{D})$ is $\mathbf{\Sigma}^1_2$-complete.*

Theorem 4 shows that a large class of Borel coloring problems is hard. This raises the question of whether an analogue of the $\mathbb{G}_0$-dichotomy holds for tractable CSPs. Early results suggested a positive answer: the Hell–Nešetřil theorem [16] and, more generally, the dichotomy for smooth digraphs [1] have exact analogues in the Borel context [9, 24]. The following theorem, however, shows that the split in the Borel setting occurs at a different point, since solving finite systems of linear equations can be done in polynomial time in the finite case.

Theorem 5 (Borel systems of linear equations [14])**.** *The set of codes of Borel systems of linear equation over $\mathbb{F}_p$, $p \in \mathbb{N}$ prime, that admit a Borel solution is $\mathbf{\Sigma}^1_2$-complete.*

After Theorem 5, it remains an interesting problem to determine whether CSPs of so-called *bounded width*, that are all tractable in the finite context see [5, Section 3], are easy in the Borel context.

Problem 1 (Problem 1.14 in [24])**.** *Is every bounded width Borel CSP $\mathbf{\Pi}^1_1$?*

We conclude by mentioning the following general direction. Recall that a *measurable graph* $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ is a Borel graph $(V, \mathcal{B}, E)$ that is additionally endowed with a Borel probability measure μ on $(V, \mathcal{B})$. As in the Borel case, one can study the existence of measurable colorings of $\mathcal{G}$, that is, Borel colorings of $(V, \mathcal{B}, E)$ that may differ from a proper coloring on a μ-null set. The following is widely open due to the lack of any hardness result in the spirit of [25].

Problem 2. *Develop a complexity theory for measurable coloring problems on measurable graphs.*

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Kőnig = Ramsey

Maximilian Hadek

Given a some sets and some maps between them, is there a selection of elements, one from each set, which is compatible with all given maps? The simplest case, when there are no maps, is precisely the axiom of choice: given any family of nonempty sets, one can pick a single element out of each of them. Slightly less simple, Kőnigs infinity lemma deals with the case when the sets and maps are arranged in sequence.

Theorem 1 (Kőnig, [1]). *Consider finite nonempty sets $E_1, E_2, E_3, \ldots$ and maps $E_{n+1} \xrightarrow{R_n} E_n$. We can find in every set E_n an element a_n such that $R_n(a_{n+1}) = a_n$ for all $n \in \mathbb{N}$.*[1]

More complicated shapes in which sets and maps can be arranged are categories $\mathcal{C}$, which are collections of points and arrows together with a formal composition of arrows. A diagram in the shape of $\mathcal{C}$ assigns to every point a set and to every arrow a map, such that composition is respected, and *solutions* to a diagram are the selections of elements, one from each set, which are invariant under the given maps. Our main theorem characterizes those categories, for which a generalized Kőnigs lemma holds, i.e. those shapes where every diagram of finite nonempty sets has a solution.

Theorem 2. *If $\mathcal{C}$ is an essentially small and locally finite category, then the following are equivalent:*

(1) every diagram in the shape of $\mathcal{C}^{\mathrm{op}}$ which consists of finite nonempty sets has a solution
(2) $\mathcal{C}$ is confluent and has the Ramsey property.

While confluence is a minor connectivity assumption, the theorem also serves as a characterization of the Ramsey property of categories and, as a special case, classes of relational structures. As such, we study applications of Theorem 2 to structural Ramsey theory. Namely we provide the following refinement of a theorem which initially appeared in the context of topological dynamics; see [2] and [3].

Theorem 3. *If a locally finite and essentially small confluent category has a precompact confluent Ramsey expansion, then it has a precompact confluent Ramsey expansion with the expansion property. Moreover, this expansion is unique up to isomorphism of expansions.*

Moreover, we prove a new Ramsey transfer theorem about Grothendieck opfibrations, providing a common generalization of three known Ramsey transfers, namely products [4], discrete opfibrations [5] and blowups.

Theorem 4. *If a Grothendieck opfibration is locally finite and essentially small, has confluent Ramsey fibers and a confluent Ramsey base, then it is confluent and Ramsey.*

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[1]While this is almost the original wording of the lemma, some readers might be more familiar with a graph theoretic formulation in terms of trees. To see the similarity, draw all elements of all E_n as vertices and connect them by an edge whenever $R_n(a_{n+1}) = a_n$.

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Systematic approach to Ramsey expansions

Jan Hubička

1. Classification of Ramsey classes

When referring to a *structure* we encompass graphs, hypergraphs, and other model-theoretic L-structures. (Language L may contain both relation and function symbols.) Given structures $\mathbf{A}$ and $\mathbf{B}$, we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all embeddings from $\mathbf{A}$ to $\mathbf{B}$. We write $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,\ell}$ to denote the following statement:

> For every colouring χ of $\binom{\mathbf{C}}{\mathbf{A}}$ with k colours, there exists an embedding $f\colon \mathbf{B} \to \mathbf{C}$ such that χ does not take more than ℓ values on $\binom{f(\mathbf{B})}{\mathbf{A}}$.

A class $\mathcal{K}$ of finite structures is a *Ramsey class* if for every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2,1}$ (see e.g. [7]).

Recall that $\mathcal{K}$ is an *amalgamation class* if it is hereditary (closed for substructures), closed for isomorphisms, has only countably many mutually non-isomorphic structures and satisfies the amalgamation property (informally, it is possible to "glue" every pair of structures in the class over any common substructure and obtain another structure in the class).

It was observed by Nešetřil in the 1980s that under mild (and natural) assumptions, every Ramsey class is an amalgamation class [12]. The converse is, however, not necessarily true. It is a known fact that the automorphism group of every Ramsey structure fixes a linear order on the underlying set [11]. Consequently, every homogeneous structure without this property, such as the random graph, is a counterexample to the question. But yet this is a good question which points in the right direction if we allow modification of structures by an *expansion*, i.e. adding additional symbols to the language and interpreting them in the structure in a suitable way.

The *Nešetřil's classification programme of Ramsey classes* was initiated in 2005 [13] and seeks Ramsey expansions of homogeneous structures given by the classification programme of homogeneous structures [3]. This project became very specific thanks to connections to topological dynamics [11] and the notion of precompact expansions with the expansion property [15].

Ramsey classes were introduced in the 1970s and their identification was (and to a degree still is) a challenging problem [14]. Nowadays, the usual approach of determining the Ramsey expansion of a given class is to first guess the expansion and then verify the Ramseyness by application of the main result of [8]. To fix a linear order, the Ramsey expansion usually adds a binary symbol $<$ which determines the order of the vertices. However, the Ramsey expansions are more subtle. For example Ramsey expansion of the partial order fixes its linear extension, the Ramsey expansion of ultrametric space needs to be ordered convexly (such that every ball of a given diameter forms a linear order), while the Ramsey expansion of the generic local order adds a unary relation. See recent survey [7] for details.

While this approach has been successful in determining Ramsey expansions of all main catalogues from the classification programme [10, 1] there are several open problems where the question about the existence of Ramsey expansion is open, as well as known examples of classes where it is known that precompact Ramsey expansion does not exist [6]. In the open cases, it is often unclear what the Ramsey expansion should be and if there is a need for new proof techniques (see e.g. [4, 7]).

2. Big Ramsey degrees

The process of finding the Ramsey expansion is usually based on experience and understanding of other examples. However, recently, a more systematic approach has emerged from the study of the infinitary extension of the notion of Ramsey class.

For a countably infinite structure $\mathbf{B}$ and any of its finite substructures $\mathbf{A}$, the *big Ramsey degree* of $\mathbf{A}$ in $\mathbf{B}$ is the least number $\ell \in \mathbb{N} \cup \{\infty\}$ such that $\mathbf{B} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,\ell}$ for every $k \in \mathbb{N}$ [11]. We say that *the big Ramsey degrees of* $\mathbf{B}$ *are finite* if for every finite substructure $\mathbf{A}$ of $\mathbf{B}$ the big Ramsey degree of $\mathbf{A}$ in $\mathbf{B}$ is finite.

It is easy to check that, if a structure $\mathbf{H}$ has finite big Ramsey degrees, then the age of $\mathbf{H}$ (that is, the class of all finite substructures of $\mathbf{H}$) has a precompact Ramsey expansion. It thus follows that, as a natural extension of the classification programme of Ramsey classes, one may ask which homogeneous structures have finite big Ramsey degrees.

The proof techniques giving upper boundon big Ramsey degrees, however, differ significantly from techniques used to identify Ramsey classes. While the main technique of finite structural Ramsey theory is the Nešetřil–Rödl's partite construction, the proofs about infinite structures always start by understanding the tree of types which arises from an enumeration of a given countable structure.

For us, a *tree* is a (possibly empty) partially ordered set $(T, <_T)$ such that, for every $t \in T$, the set $\{s \in T : s <_T t\}$ is finite and linearly ordered by $<_T$.

An *enumerated structure* is simply a structure $\mathbf{A}$ with underlying set $A = |A| = \{0, 1, \dots |A|-1\}$. Fix a countably infinite enumerated structure $\mathbf{A}$. Given vertices u, v and an integer n satisfying $\min(u, v) \geqslant n \geqslant 0$, we write $u \sim^{\mathbf{A}}_n v$ and say that u *and* v *have the same (quantifier-free) type over* $\{0, 1, \dots, n-1\}$, if the structures induced by $\mathbf{A}$ on $\{0, 1, \dots, n-1, u\}$ and $\{0, 1, \dots, n-1, v\}$ are isomorphic via the map which is the identity on $\{0, \dots, n-1\}$ and sends u to v. We write $[u]^{\mathbf{A}}_n$ for the $\sim^{\mathbf{A}}_n$-equivalence class of vertex u.

Definition 1 (Tree of 1-types)**.** Let $\mathbf{A}$ be a countably infinite (relational) enumerated structure. Given $n < \omega$, write $\mathbb{T}_{\mathbf{A}}(n) = \omega / \sim_n^{\mathbf{A}}$. A (quantifier-free) *1-type* is any member of the disjoint union $\mathbb{T}_{\mathbf{A}} \coloneqq \bigsqcup_{n<\omega} \mathbb{T}_{\mathbf{A}}(n)$. We turn $\mathbb{T}_{\mathbf{A}}$ into a tree as follows. Given $x \in \mathbb{T}_{\mathbf{A}}(m)$ and $y \in \mathbb{T}_{\mathbf{A}}(n)$, we declare that $x \leqslant_{\mathbf{A}}^{\mathbb{T}} y$ if and only if $m \leqslant n$ and $x \supseteq y$.

While our current understanding of big Ramsey degrees and structures is less advanced then understanding of Ramsey classes (see surverys [9, 5]) and gets very complicated even with free amalgamation [2] we notice that the analysis of the tree of types can be used to determine Ramsey expansion. In particular, the tree of types fixes a lexicographic order, which then projects to an order presented in every Ramsey class. The lexicographic order of the tree of types of the Urysohn ultrametric spaces is convex, and the lexicographic order of the tree of types of the generic partial order forms a linear extension. This explains phenomena seen earlier. Recent joint work with Evans and Sullivan shows that this can be carried beyond simple examples and can be used to recover the main result of [6]. In the talk, we discuss these examples and suggest future directions.

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Expansions of finitely homogeneous structures preserving being finitely homogeneous

Itay Kaplan

In this talk we discuss the following question: given a finitely homogeneous structure, i.e., a structure admitting quantifier elimination in a finite relational language, does adding a new relation symbol for a definable set (with parameters) preserve finite homogeneity?

I do not know the full answer to this question, but there are some partial results. In the talk, I presented a proof that if the original structure is NIP then the answer is positive. This generalizes a result of Lachlan [1, Lemma 5.1] who proved it for stable structures.

The original question remains open. In the NIP case it is natural to try to generalize the result further by replacing "definable set" by "externally definable set", but I do not know the answer to this either. In this case, it is even unknown whether the resulting expansion is still ω-categorical.

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Introduction to Promise Constraint Satisfaction Problems

Andrei Krokhin

The Constraint Satisfaction Problem (CSP) framework attracts much attention in theoretical computer science due to the versatility of the framework and strong relationship with many branches of mathematics. The basic version of CSP, parameterised by a relational structure A, denoted $CSP(A)$, asks whether a given relational structure in the same signature as A admits a homomorphism to A. Examples include many versions of the satisfiability problem from propositional logic, graph colouring problems from combinatorics, and systems of equations over an algebra. The (universal-)algebraic approach, developed by many researchers in the last 30 years (see survey [2], written specifically for non-algebraists), dominated the study of the computational complexity of problems $CSP(A)$.

In the last 7 years, a new version of CSP gained prominence - the Promise CSP (PCSP), see survey [3]. In this version, problems are parameterised by pairs of relational structures (A, B) of the same signature such that there exists a homomorphism from A to B. In this talk, we always assume that A and B are finite. The problem, denoted by $PCSP(A, B)$, is to algorithmically distinguish the relational structures having a homomorphism to A from those not admitting a homomorphism even to B. The promise here is that (exactly) one of the two cases holds. Note that, in the special case when $A = B$, $PCSP(A, A)$ is exactly $CSP(A)$. The most prominent example of PCSP is the approximate graph colouring problem, where A and B are complete graphs, B larger than A. The overall goal of this direction is to understand how exactly the computational complexity $PCSP(A, B)$ depends on the mathematical properties of (A, B).

This talk is an introduction to the PCSP, where the basic set-up and prominent examples will be discussed. I will also explain the core of the algebraic theory of PCSP developed in [1]. This theory is based on polymorphisms from A to B, which are homomorphisms from a finite direct power of A to B (typically viewed as multivariable functions from A to B). The set of all polymorphisms from A to B is denoted by $Pol(A, B)$. For an n-ary function f from A to B and a map $\pi : [n] \to [m]$, the function $g(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ is called a (π-)minor of f. In this case we write $g = f^\pi$. A minion is a set of multivariate functions from one set to another that is closed under taking minors. It is easy to see that every set $Pol(A, B)$ is a minion. (As an aside, each minion can be seen a functor from the category of finite sets to the category of sets).

Systems of equations of the form $g = f^\pi$ (also known as minor conditions) play a special role in this theory. In such an equation, the functions are considered as unknowns. A minor condition is called trivial if it is satisfied in every minion, or, equivalently, in the minion of all projections (i.e. functions of the form $g(x_1, \dots, x_n) = x_i$) on a set with at least two elements. The first key result of the algebraic theory of [1] states that every problem $PCSP(A, B)$ is polynomial-time equivalent to the problem of distinguishing trivial minor conditions from those not satisfied even in $Pol(A, B)$. This means that the computational complexity of any problem $PCSP(A, B)$ depends only on the abstract properties of the minion $Pol(A, B)$ - specifically, on minor conditions satisfied in $Pol(A, B)$.

The second key result of the theory of [1] allows one to relate the complexity of different PCSPs on the basis of their polymorphism minions. Call a map $\xi : Pol(A, B) \to Pol(A', B')$ a minion homomorphism if it preserves the arity of functions and the operation of taking minors (i.e. for any n-ary f in $Pol(A, B)$ and any map $\pi : [n] \to [m]$, we have $\xi(f^\pi) = \xi(f)^\pi$). The result states that the following conditions are equivalent for all problems $PCSP(A, B)$ and $PCSP(A', B')$.

(1) There exists a minion homomorphism from $Pol(A, B)$ to $Pol(A', B')$.
(2) Each minor condition satisfiable in $Pol(A, B)$ is satisfiable in $Pol(A', B')$.
(3) The structures (A', B') can primitively positively constructed from (A, B).

If these conditions are satisfied then $PCSP(A', B')$ reduces to (i.e. is not harder than) $PCSP(A, B)$.

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Generic morphisms

Wiesław Kubiś

Let $\mathcal{F}$ be a fixed Fraïssé class and let $\sigma\mathcal{F}$ denote the class of all countable structures with age contained in $\mathcal{F}$. Given $\mathbb{V}, \mathbb{W} \in \sigma\mathcal{F}$, a homomorphism $u\colon \to V\mathbb{W}$ is called *generic* if its isomorphic copies form a residual set in $\hom(\mathbb{V}, \mathbb{W})$, the space of all homomorphisms from $\mathbb{V}$ to $\mathbb{W}$.

So far, this definition is a bit vague. One can easily guess that $\hom(\mathbb{V}, \mathbb{W})$ carries the product (pointwise convergence) topology. On the other hand, it is not clear what "isomorphic copies" is meant to be.

Two homomorphisms f_0, f_1 from $\mathbb{V}$ to $\mathbb{W}$ could be called *isomorphic* if there are automorphisms g, h such that $f_1 = g \circ f_0 \circ h$. This is perhaps the most general variant.

We could also define f_0, f_1 to be *left-isomorphic* if there is an automorphism h of $\mathbb{V}$ such that[1] $f_1 = f_0 \circ h$ (so g is the identity). Obviously, in the same manner one can define the concept of being *right-isomorphic*, although it will play no role here.

When $\mathbb{V} = \mathbb{W}$, it is natural to define the notion of being *symmetrically isomorphic*, requiring $g = h$ in the definition above. We say that $f\colon \to V\mathbb{V}$ is *symmetrically generic* if its symmetrical isomorphic copies form a residual set in $\hom(\mathbb{V}, \mathbb{V})$.

Our goal is to study when there are symmetrically generic endomorphisms of Fraïssé limits. Using the concept of domination, we are able to find a simple sufficient condition. Among the applications, let us mention the following result.

Let $\mathfrak{R}$ denote the random graph (the Fraïssé limit of all finite graphs). By [3, 2], there exists a universal homogeneous endomorphism $\Omega\colon \mathfrak{R} \to \mathfrak{R}$.

Theorem 1. *Ω is symmetrically generic.*

The same result holds for many Fraïssé classes with free amalgamations; also for linear orderings. It fails for the class of finite groups. In particular, we do not know whether Hall's group admits a (symmetrically) generic endomorphism.

Generic morphisms have also been studied and significantly applied in the new theory of ultrametric Fraïssé limits, see [1].

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A logic of co-valuations

Maciej Malicki

A co-valuation is, essentially, a minimal finite cover. We define a logic of co-valuations, and show that basic model theory, e.g., ultraproducts, compactness, omitting types, etc., can be developed in this framework. Moreover, using a duality between certain posets and second-countable compact T_1-spaces, recently discovered by A. Bartoš, T. Bice and A. Vignati, we show that these spaces play the role of countable sets in classical first-order logic. Thus, our logic - even though there is no topology involved at the start - seems to be naturally suited for studying topological objects. Indeed, standard topological notions, such as connectedeness, dimension, etc., can be easily expressed in it, and model-theoretic properties, such as atomicity, can be effectively investigated. It also turns out to get along with Fraïssé-type constructions of homogeneous topological structures.

On the topological reconstruction of monoids of elementary embeddings of ω-categorical structures

Paolo Marimon

(joint work with J. de la Nuez Gonzales, Zaniar Ghadernezhad, and Michael Pinsker)

Let $\mathbb{A}$ be a structure with countably infinite domain Ω. The automorphism group $\mathrm{Aut}(\mathbb{A})$ and the monoid of elementary embeddings $\mathrm{EEmb}(\mathbb{A})$ can be considered as a Polish topological group and semigroup (respectively) with the topology of pointwise convergence τ_{pw}. This is obtained by looking at the topology induced on these spaces from the product topology on Ω^Ω (where Ω is endowed with the discrete topology). Such topological structure on these spaces of symmetries raises two natural questions, which can be made more precise in several ways:

Question 1. What information can we recover about the original structure $\mathbb{A}$ from a given space of symmetries (as a topological group/monoid)?

Question 2. To what extent does the algebraic structure of a space of symmetries determine its topological structure?

Question 1 has a satisfying answer for ω-categorical structures. These are countable structures whose automorphism group has finitely many orbits on n-tuples of elements for each $n \in \mathbb{N}$. In this context, $\mathrm{Aut}(\mathbb{A})$ as a topological group allows us to recover $\mathbb{A}$ up to bi-intepretation [1]. This is also the case for $\mathrm{EEmb}(\mathbb{A})$ as a topological monoid [8]. These results make Question 2 more interesting, and whilst variants of the latter question have been studied extensively in the context of automorphism groups of ω-categorical structures, analogous questions for monoids have only received attention more recently. In this talk, I will survey the state of the art of work on topological reconstruction problems for monoids of elementary embeddings, following my recent survey with Pinsker [9]. In an ω-categorical context $\mathrm{EEmb}(\mathbb{A})$ is the topological closure of $\mathrm{Aut}(\mathbb{A})$ in Ω^Ω, so one may expect topological reconstruction problems to behave similarly for these two spaces of symmetries. Instead, we will see that the interactions between topological reconstruction properties of $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{EEmb}(\mathbb{A})$ are highly non-trivial. This is a consequence of the tension between the fact that $\mathrm{EEmb}(\mathbb{A})$ has a weaker algebraic structure than $\mathrm{Aut}(\mathbb{A})$ (not allowing inverses), but is a richer space, which often encodes additional information.

Sometimes we can transfer results from Aut($\mathbb{A}$) to EEmb($\mathbb{A}$) (automatic homeomorphicity)

A topological semigroup $\mathcal{S}$ has **automatic homeomorphicity** if every semigroup isomorphism between $\mathcal{S}$ and a closed submonoid of Ω^Ω is a homeomorphism. The same definition makes sense for topological groups with respect to closed subgroups of the full symmetric group S_Ω. For $\mathrm{Aut}(\mathbb{A})$, there are well-established techniques to prove automatic homeomorphicity, making it interesting to see whether we could lift it from $\mathrm{Aut}(\mathbb{A})$ to $\mathrm{EEmb}(\mathbb{A})$. Completing a line of work pursued in special cases in [3, 2, 10], we prove the following:[1]

Theorem 1 ([9])**.** *Let $\mathbb{A}$ be an ω-categorical structure. If* $\mathrm{Aut}(\mathbb{A})$ *has automatic homeomorphicity, then so does* $\mathrm{EEmb}(\mathbb{A})$.

Sometimes EEmb($\mathbb{A}$) behaves much more wildly than Aut($\mathbb{A}$) (on the number of Polish topologies)

For $\mathrm{Aut}(\mathbb{A})$ one can often prove there is a unique Polish group topology. Indeed, it is compatible with ZF that every Polish group has a unique Polish group topology [13, 14]. Instead, $\mathrm{EEmb}(\mathbb{A})$ always has at least two Polish semigroup topologies: τ_{pw}, and another topology in which $\mathrm{Aut}(\mathbb{A})$ is closed [6]. Moreover, under the assumption that algebraically closed sets in $\mathbb{A}$ form a pregeometry,[2] there are $\geqslant \aleph_0$ many Polish semigroup topologies on $\mathrm{EEmb}(\mathbb{A})$ [9]. Nevertheless, all of these topologies are finer than τ_{pw}. The situation for coarser topologies than τ_{pw} is much nicer:

Sometimes EEmb($\mathbb{A}$) seems to behave better than Aut($\mathbb{A}$) (minimality of $\boldsymbol{\tau_{\mathrm{pw}}}$)

It is often possible to prove τ_{pw} is the coarsest Hausdorff semigroup topology on $\mathrm{EEmb}(\mathbb{A})$ by showing it coincides with the Zariski topology:

Definition 2. Let $\mathcal{S}$ be a semigroup. The (semigroup) **Zariski** topology $\tau_{\mathbb{Z}}$ has a sub-basis of open sets given by *solutions to semigroup inequalities*:

$$\{s \in \mathcal{S} \mid t_k s t_{k-1} s \dots t_1 s t_0 \neq q_l s q_{k-1} s \dots q_1 s q_0\},$$

for $k, l \geqslant 1$ and $t_0, \dots, t_k, q_0, \dots q_l \in \mathcal{S}$.

Any Hausdorff semigroup topology on $\mathcal{S}$ contains $\tau_{\mathbb{Z}}$ [5]. We know that $\tau_{\mathbb{Z}} = \tau_{\mathrm{pw}}$ for $\mathrm{EEmb}(\mathbb{A})$ whenever $\mathbb{A}$ is ω-categorical and has no algebraicity:

[1] For brevity, in this extended abstract we give statements for ω-categorical structures even when they hold under more general assumptions.

[2] This is the case, for example, for ω-categorical structures with no algebraicity (e.g., the random graph, $(\mathbb{N}, =)$, $(\mathbb{Q}, <)$, the generic K_n-free graph) or for countably infinite vector spaces over finite fields.

Theorem 3 ([11]). *Let $\mathbb{A}$ be an ω-categorical structure with no algebraicity. Then, τ_{pw} is the minimal Hausdorff semigroup topology on* $\mathrm{EEmb}(\mathbb{A})$ *and equals $\tau_{\mathbb{Z}}$.*

There is no analogue of such result for $\mathrm{Aut}(\mathbb{A})$. Interestingly, we can also see that when $\mathbb{A}$ has algebraicity, $\tau_{\mathbb{Z}}$ often fails to be Hausdorff on $\mathrm{EEmb}(\mathbb{A})$. This is in particular the case whenever $\mathrm{Aut}(\mathbb{A})$ has non-trivial centre [4].

Sometimes hard problems for Aut($\mathbb{A}$) are "easy" for EEmb($\mathbb{A}$) (automatic action reconstruction)

What if we were interested in reconstructing more than just topology when given $\mathrm{Aut}(\mathbb{A})$ or $\mathrm{EEmb}(\mathbb{A})$ as an algebraic object? Can we also reconstruct the action of $\mathrm{Aut}(\mathbb{A})$ or $\mathrm{EEmb}(\mathbb{A})$ on Ω (up to conjugation by a bijection), thus recovering $\mathbb{A}$ up to bi-definability? This gives rise to the following notion:

Definition 4. Let $\mathbb{A}$ be an ω-categorical structure with no algebraicity. We say $\mathrm{Aut}(\mathbb{A})$ has **automatic action reconstruction** if whenever $\mathbb{B}$ is another ω-categorical structure with no algebraicity and $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ (as groups), then $\mathbb{A}$ and $\mathbb{B}$ are bi-definable. We define analogously automatic action reconstruction for $\mathrm{EEmb}(\mathbb{A})$.

Note that we need the "no algebraicity" requirement since otherwise adding fixed points to $\mathbb{A}$ would create structures with the same algebraic space of symmetries but different underlying actions. For $\mathrm{Aut}(\mathbb{A})$, Rubin developed a sophisticated method for proving automatic action reconstruction [12] which holds for several structures, such as the random graph and the random poset. However, it is currently an open problem whether ω-categorical structures with no algebraicity always have automatic action reconstruction. Surprisingly, the analogue of this problem for $\mathrm{EEmb}(\mathbb{A})$ has a fully general solution by combining existing results:

Theorem 5 ([9]). *Let $\mathbb{A}$ be an ω-categorical structure with no algebraicity. Then* $\mathrm{EEmb}(\mathbb{A})$ *has automatic action reconstruction.*

This result is a simple consequence of Theorem 3 and the fact (implicit in [12], explicit in [7]) that bi-interpretations between ω-categorical structures with no algebraicity yield bi-definitions.

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Projective KPT and automorphism groups of some profinite algebras

Dragan Mašulović

(joint work with Andy Zucker)

1. Introduction

Free algebras in a variety are well known and much loved. But, as in many noble families, there is also a rather curious cousin: the collection of all finite algebras in a nontrivial variety over a finite language forms a projective Fraïssé class whose projective limit is a profinite (topological) algebra. In this talk, I'll weave together several results to show that all of these curious cousins come equipped with metrizable universal minimal flows.

The proof strategy is based on an observation that Zucker's 2016 proof of a very natural generalization of the KPT-correspondence (see [5]) is, perhaps unexpectedly, deeply categorical in nature. I'll unpack the categorical heart of Zucker's argument, present its categorical dual, and then connect it with recent work by Solecki [4] (and independently by Mašulović [3]) on small dual Ramsey degrees for classes of finite algebras in nontrivial varieties. Putting these pieces together leads to a neat proof of the metrizability result.

2. The greatest ambit from Fraïssé sequences

Let us start with an overview of Zucker's construction [5] from the point of view of Fraïssé sequences as presented in [2]. We start with a Fraïssé limit K in a category $\mathbf{C}$:

$$\begin{tikzcd}
 & & & K \\
A_1 \arrow[r, "\sigma_1^2"'] \arrow[urrr, "\sigma_1"] & A_2 \arrow[r, "\sigma_2^3"'] \arrow[urr, "\sigma_2"] & A_3 \arrow[r, "\sigma_3^4"'] \arrow[ur, "\sigma_3"] & \ldots
\end{tikzcd}$$

where all the morphisms are mono. To this diagram we apply the *contravariant* hom-functor $\hom(-,K)$:

$$
\begin{tikzcd}
 & & & \hom(K,K) \arrow[dlll, "\widehat{\sigma_1}"'] \arrow[dll, "\widehat{\sigma_2}"'] \arrow[dl, "\widehat{\sigma_3}"] & \\
H_1 & H_2 \arrow[l, "\widehat{\sigma_1^2}"] & H_3 \arrow[l, "\widehat{\sigma_2^3}"] & & \cdots \arrow[ll, "\widehat{\sigma_3^4}"]
\end{tikzcd}
$$

where for the sake of simplicity we let $H_n = \hom(A_n, K)$. Let $G = \mathrm{Aut}(K) \subseteq \hom(K,K)$. The above diagram then becomes:

$$
\begin{tikzcd}
 & & & G \arrow[dlll, "\widehat{\sigma_1}"'] \arrow[dll, "\widehat{\sigma_2}"'] \arrow[dl, "\widehat{\sigma_3}"] \\
H_1 & H_2 \arrow[l, "\widehat{\sigma_1^2}"] & H_3 \arrow[l, "\widehat{\sigma_2^3}"] & \cdots \arrow[l, "\widehat{\sigma_3^4}"]
\end{tikzcd}
\tag{1}
$$

where $\widehat{\sigma_n}$'s now denote the appropriate restrictions. Applying the functor β : **Set** $\to$ **CHaus** to (1) produces the following diagram in **CHaus**:

$$
\begin{tikzcd}
 & & & \beta G \arrow[dlll, "\widetilde{\sigma_1}"'] \arrow[dll, "\widetilde{\sigma_2}"'] \arrow[dl, "\widetilde{\sigma_3}"] \\
\beta H_1 & \beta H_2 \arrow[l, "\widetilde{\sigma_1^2}"] & \beta H_3 \arrow[l, "\widetilde{\sigma_2^3}"] & \cdots \arrow[l, "\widetilde{\sigma_3^4}"]
\end{tikzcd}
$$

where $\widetilde{\sigma_m^n} = \beta\,\widehat{\sigma_m^n}$ and $\widetilde{\sigma_n} = \beta\,\widehat{\sigma_n}$.

Let $\varprojlim \beta H_n$ denote the limit of the diagram $(\beta H_n, \widetilde{\sigma_m^n})_{m \leqslant n \in \mathbb{N}}$ and let $\widetilde{\pi}_m$: $\varprojlim \beta H_n \to \beta H_m$ be the associated projection morphisms. Note that $\varprojlim \beta H_n$ is an object of **CHaus** because **CHaus** has limits of all small diagrams.

Since $(\beta G, \widetilde{\sigma}_m)_{m\in\mathbb{N}}$ is another compatible cone for the same diagram, there is a unique continuous map $\widetilde{\pi} : \beta G \to \varprojlim \beta H_n$ such that the following diagram commutes:

$$
\begin{tikzcd}
\varprojlim \beta H_n \arrow[d, "\widetilde{\pi_1}"'] \arrow[dr, "\widetilde{\pi_2}"'] \arrow[drr, "\widetilde{\pi_3}"] & & & \beta G \arrow[lll, dashed, "\widetilde{\pi}"'] \arrow[dlll, "\widetilde{\sigma_1}"'] \arrow[dll, "\widetilde{\sigma_2}"'] \arrow[dl, "\widetilde{\sigma_3}"] \\
\beta H_1 & \beta H_2 \arrow[l, "\widetilde{\sigma_1^2}"] & \beta H_3 \arrow[l, "\widetilde{\sigma_2^3}"] & \cdots \arrow[l, "\widetilde{\sigma_3^4}"]
\end{tikzcd}
$$

Then $(\varprojlim \beta H_n, 1_L)$ is the greatest G-ambit, where G is topologized by an appropriate topology, and Zucker's arguments apply to show the relationship between small Ramsey degrees in $\mathbf{C}$ and the metrizability of the UMF of G.

3. The greatest ambit from projective Fraïssé sequences

We start with a projective Fraïssé limit K in a category $\mathbf{C}$:

$$\begin{tikzcd}
 & & & K \arrow[dlll, "\sigma_1"'] \arrow[dll, "\sigma_2"'] \arrow[dl, "\sigma_3"] \\
A_1 & A_2 \arrow[l, "\sigma_1^2"] & A_3 \arrow[l, "\sigma_2^3"] & \ldots \arrow[l, "\sigma_3^4"]
\end{tikzcd}$$

where all the morphisms are epi. We then apply the *covariant* $\hom(K, -)$ functor to obtain:

$$\begin{tikzcd}
 & & & \hom(K,K) \arrow[dlll, "\widehat{\sigma_1}"'] \arrow[dll, "\widehat{\sigma_2}"'] \arrow[dl, "\widehat{\sigma_3}"] \\
H_1 & H_2 \arrow[l, "\widehat{\sigma_1^2}"] & H_3 \arrow[l, "\widehat{\sigma_2^3}"] & \ldots \arrow[l, "\widehat{\sigma_3^4}"]
\end{tikzcd}$$

The major insight now is that the rest of the construction is the same as above! So, as an immediate consequence of [5] and the Duality Principle for Category Theory reads as follows:

Theorem. *Let* $\mathbf{K}$ *be a projective Fraïssé age in the sense of* [1], $\mathcal{K}$ *its projective Fraïssé limit and* $G = \mathrm{Aut}(\mathcal{K})$ *with the co-pointwise-convergence topology. The following are equivalent:*

- *G has metrizable universal minimal flow;*
- *$\mathbf{K}$ has finite dual small Ramsey degrees.*

This result combines nicely with the well-known (and easy to prove) fact that for every equationally definable class $\mathbf{V}$ of algebras over a finite algebraic language, the subclass of all finite elements $\mathbf{V}^{fin}$ is a projective Fraïssé class and has the projective Fraïssé limit $\mathcal{K}_{\mathbf{V}}$. It was shown in [3] that $\mathbf{V}^{fin}$ has dual small Ramsey degrees, so as an immediate consequence of the above theorem we now have:

Corollary. *Let* $\mathbf{V}$ *be an equationally definable class of algebras over a finite algebraic language, and let* $\mathbf{V}^{fin}$ *the subclass of* $\mathbf{V}$ *containg all of its finite elements. Let* $\mathcal{K}_{\mathbf{V}}$ *be the projective Fraïssé limit of* $\mathbf{V}^{fin}$ *and let* G *be the automorphism group of* $\mathcal{K}_{\mathbf{V}}$ *topologized appropriately. Then* G *has metrizable universal minimal flow.*

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Boolean powers of finite simple Mal'cev algebras

Peter Mayr
(joint work with Nik Ruškuc)

An algebraic structure (*algebra* for short) is *Mal'cev* if it has a ternary term operation m satisfying $m(x,x,y) = y = m(y,x,x)$. Groups are Mal'cev via the term $m(x,y,z) = xy^{-1}z$, as are loops, rings and all their expansions. An algebra is *simple* if it has exactly two congruence, namely the equality and the total congruence. Hence simple Mal'cev algebras generalize the classical notions of simple groups and simple modules.

For the remainder let $\mathbf{A}$ be a finite simple Mal'cev algebra. In [6] we noted that the class K of finite direct powers of $\mathbf{A}$ satisfies the joint embedding property (JEP) and the amalgamation property (AP) by the Foster-Pixley Theorem. Further K is closed under substructures iff every proper subalgebra of $\mathbf{A}$ has size one. The (generalized) Fraïssé limit of K can be explicitly constructed as a filtered Boolean power of $\mathbf{A}$ by the countable atomless Boolean algebra $\mathbf{B}$ as defined by Arens and Kaplansky [1] (see also [2]). Here a *filtered Boolean power* $(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{a_1,\dots,a_n}$ is a subalgebra of $\mathbf{A}^{2^\omega}$ with universe

$$\{f\colon 2^\omega \to A \mid f \text{ is continuous and } f(x_1) = a_1, \dots, f(x_n) = a_n\},$$

where 2^ω denotes the Cantor space (the Stone space of the countable atomless Boolean algebra $\mathbf{B}$), $n \geqslant 0$, $x_1,\dots,x_n \in 2^\omega$ are distinct, and $a_1,\dots,a_n \in A$ generate one-element subalgebras of $\mathbf{A}$.

We continue our investigation of the automorphism group G of such a filtered Boolean power $(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{a_1,\dots,a_n}$ from [6]. Note that G is a Polish group under the topology of pointwise convergence and G has *ample generics* if for every $k \geqslant 1$ the diagonal conjugation action of G on G^k has a comeager orbit [4]. Our main result is the following.

Theorem 1. *Let $\mathbf{A}$ be a simple non-abelian Mal'cev algebra, i.e., not polynomially equivalent to a module (see* [3] *for a generalization of the commutator theory from groups to arbitrary algebras), and let $\mathbf{B}$ be the countable atomless Boolean algebra.*

Then the automorphism group G of every filtered Boolean power $(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{a_1,\dots,a_n}$ has ample generics.

This yields a new unified approach to our previous results from [6] that G as in the theorem has the small index property, uncountable cofinality and the Bergman property.

Our proof starts from the observation that G splits into a semidirect product of a normal subgroup with complement H isomorphic to the pointwise stabiliser of $x_1,\dots,x_n$ in the homeomorphism group of 2^ω. We then extend Kwiatkowska's result [5] that the homeomorphism group of 2^ω has ample generics to pointwise stabilisers. In the final step we show that the generic tuples over H actually have comeager orbits over G as well.

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A survey on polymorphism minions in classical settings

Sebastian Meyer

The polymorphism minion of a finite structure, a generalization of the endomorphism monoid, has established itself to be a very fruitful tool in the study of the computational complexity of constraint satisfaction problems. We have a look at classical structures such as groups, modules, simplicial sets and graphs and give an intuition and classification of the polymorphism minion in these cases.

More precisely, we define a preorder on finite structures by $\mathbb{A} \leqslant \mathbb{B}$ if and only if there is a homomorphism from the polymorphism minion of $\mathbb{A}$ to the polymorphism minion of $\mathbb{B}$ or, equivalently, if $\mathbb{A}$ primitively positively constructs $\mathbb{B}$. On undirected graphs, this poset is tame and consists of 4 equivalence classes: graphs with loops, graphs without edges, bipartite graphs with edges and all other graphs [1]. On digraphs, it is already truly complicated even though homomorphic equivalent structures are in the same equivalence class. Digraphs with at most 4 vertices give already a poset with 16 equivalence classes and no obvious pattern, see Figure 1 (based on [2]). On structures with exactly two elements, the resulting poset is a countable infinite quotient of posts lattice [3]. On permutation groups (or group actions), the poset is essentially given by the epimorphism poset of groups with trivial Frattini subgroup. This fact leads to finite simple groups appearing in the first non-trivial layer of this poset [4]. For simplicial complexes, one gets that contractible complexes are difficult to classify while non-contractible complexes are all equivalent [5]. These classifications have applications in the study of the computational complexity of constraint satisfaction problems as primitively positively constructions between structures induce logspace reductions among the associated constraint satisfaction problems.

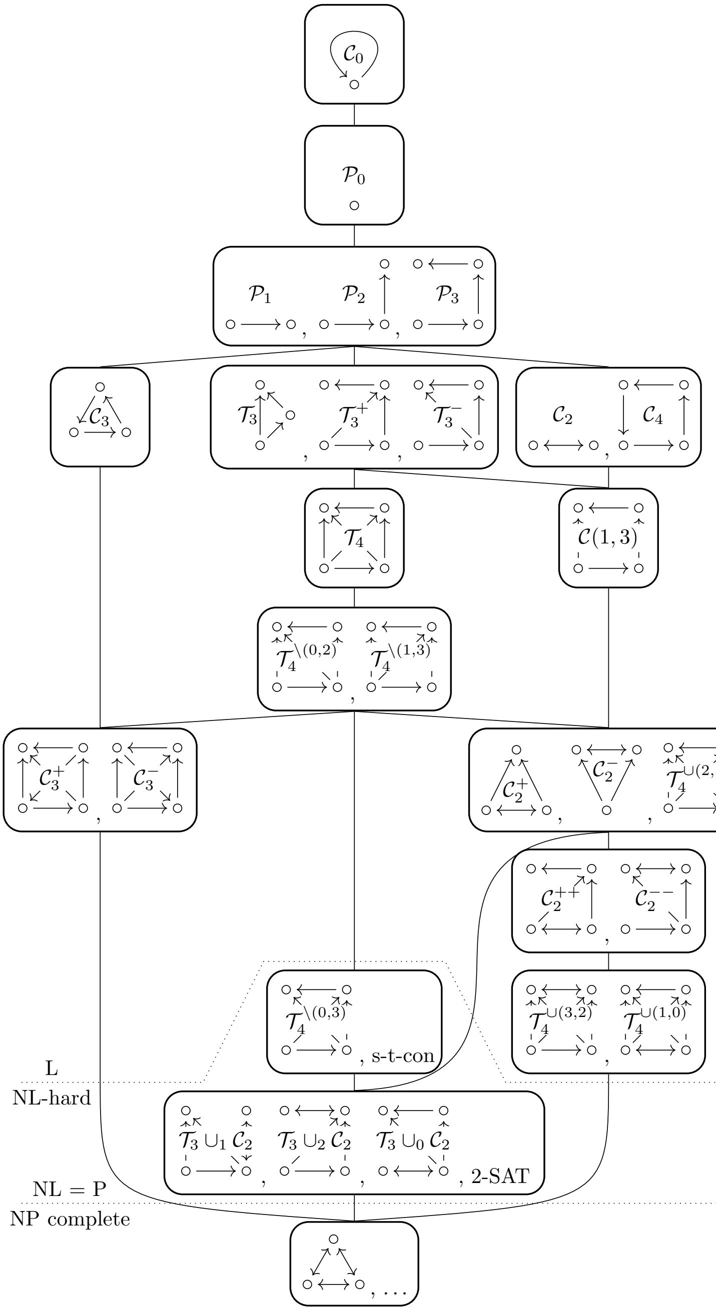

FIGURE 1. The primitive positive constructability poset on digraphs with at most 4 vertices.

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A gentle introduction to the categorical approach to CSP

Jakub Opršal

(joint work with Max Hadek, Tomáš Jakl)

The algebraic approach to the *constraint satisfaction problem (CSP)* is arguably the most influential framework for the study of the complexity of finite-template CSPs and promise CSPs. In the talk, I gave a categorical perspective on finite-template CSPs and the algebraic approach. I have introduced two versions of a CSP with a given template A which is now a functor from a finite category $\mathcal{B}$ to the category of finite sets Fin:

(1) A homomorphism problem $\mathrm{CSP}_{hom}(A)$ in the category of functors from $\mathcal{B}$ to Fin.

(2) The problem $\mathrm{CSP}^*(A)$ of deciding whether, for a given functor $D\colon \mathcal{E} \to \mathcal{B}$ from a finite category $\mathcal{E}$, there exists a *section* of $A \circ D\colon \mathcal{E} \to \mathsf{Fin}$, i.e., an assignment $s \in \prod_{u\in\mathcal{E}} A(D(u))$ such that for each morphism $\pi\colon u \to v$ in $\mathcal{E}$, we have $A(D(\pi))(s(u)) = s(v)$.

The problem $\mathrm{CSP}^*(1)$, where $1\colon \mathsf{Fin} \to \mathsf{Fin}$ denotes the identity functor, is also known as the *label cover problem*.

In short, the content of the talk can be described on the following diagram which shows reductions between the corresponding problems:

$$\begin{tikzcd}
\mathrm{CSP}_{hom}(A) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{CSP}^*(A) \arrow[l, shift right] \arrow[d, "A\circ-"] \\
\mathrm{CSP}_{hom}(1, \mathrm{pol}(A)) \arrow[u, "-\circ A"] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{CSP}^*(1, \mathrm{pol}(A)) \arrow[l, shift right]
\end{tikzcd}$$

The problems on the bottom line are promise problems defined as the natural generalisations of CSP_{hom} and CSP^*.

I explained why the two problems on the top row are equivalent. Namely, that the *Grothendieck construction* provides a reduction from $\mathrm{CSP}_{hom}(A)$ to $\mathrm{CSP}^*(A)$. The reduction in the opposite direction is given by the left-inverse to the Grothendieck construction (which is computed as the left adjoint).

The main strength of this perspective on CSPs is that now both reductions from $\mathrm{CSP}^*(1, \mathrm{pol}(A))$ to $\mathrm{CSP}^*(A)$ and from $\mathrm{CSP}_{hom}(A)$ to $\mathrm{CSP}_{hom}(1, \mathrm{pol}(A))$ become easy. The former is pre-composition with A and the latter post-composition with A. The proofs of these statements are relatively straightforward using the fact that $\mathrm{pol}(A, -)$ is right adjoint to $- \circ A$, i.e., that there is a bijection, for all $M\colon \mathsf{Fin} \to \mathsf{Fin}$ and $B\colon \mathcal{B} \to \mathsf{Fin}$,

$$\hom(M \circ A, B) \simeq \hom(M, \mathrm{pol}(A, B))$$

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On the complexity of topological isomorphism on non-archimedean Polish groups

Gianluca Paolini

(joint work with Su Gao, Feng Li, André Nies, Saharon Shelah)

In this talk we present results from a cluster of recent papers investigating the Borel complexity of topological isomorphism on various natural classes of non-archimedean Polish groups. Non-archimedean Polish groups are precisely the closed subgroups of the infinite symmetric group $\mathrm{Sym}(\mathbb{N})$, where the basic open subgroups are the pointwise stabilizers of finite subsets of $\mathbb{N}$. Following a programme initiated by Kechris, Nies, and Tent [7], we determine whether natural classes of such groups form Borel subsets of the space of closed subgroups, and study the Borel complexity of topological isomorphism on these classes.

Procountable groups. A topological group G is called *procountable* if it is topologically isomorphic to the inverse limit of an inverse system $(G_n, p_n)_{n\in\mathbb{N}}$ of countable discrete groups with surjective connecting homomorphisms $p_n : G_{n+1} \to G_n$. The procountable closed subgroups of $\mathrm{Sym}(\mathbb{N})$ form a Borel class. Since every countable discrete group is procountable, graph isomorphism (GI) provides a lower bound for the complexity of isomorphism on procountable groups. In joint work with Gao and Nies [1], we establish that topological isomorphism on procountable groups is strictly more complex than GI: we prove that the equivalence relation ℓ_∞ on $\mathbb{R}^\mathbb{N}$ (given by $(x_n)\ell_\infty(y_n)$ iff $\sup_n |x_n - y_n| < \infty$) is Borel reducible to topological isomorphism on procountable groups. By a theorem of Rosendal, ℓ_∞ is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group. Consequently, topological isomorphism on procountable groups is not classifiable by countable structures. The proof proceeds in two steps: first reducing ℓ_∞ to uniform homeomorphism between path spaces of pruned trees, then reducing this to topological isomorphism of inverse limits of free Coxeter groups constructed from these trees, using techniques from shape theory [8].

Oligomorphic groups. A permutation group $G \leqslant \mathrm{Sym}(\mathbb{N})$ is called *oligomorphic* if for every $n \in \omega$, the canonical action of G on $\mathbb{N}^n$ has only finitely many orbits. Oligomorphic groups are precisely the automorphism groups of ω-categorical countably infinite structures. The set of oligomorphic subgroups of $\mathrm{Sym}(\mathbb{N})$ forms a Borel subset. Work of Nies, Schlicht, and Tent [9] showed that topological isomorphism on oligomorphic groups is Borel reducible to a Borel equivalence relation with countable classes, but the question of whether this isomorphism relation is smooth (Borel reducible to equality on $\mathbb{R}$) remains open. In joint work [4, 3], we establish smoothness for several natural subclasses: oligomorphic groups with weak elimination of imaginaries, and oligomorphic groups with no algebraicity. These results rely on analyzing the structure of outer automorphism groups of oligomorphic groups. We prove that for any oligomorphic group $G \leqslant \mathrm{Sym}(\mathbb{N})$, the group $\mathrm{Aut}(G)$ of topological automorphisms has a unique Polish group topology making the action $\mathrm{Aut}(G) \times G \to G$ continuous, and the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is totally disconnected and locally compact [3]. For oligomorphic groups with weak elimination of imaginaries or no algebraicity, we show that $\mathrm{Out}(G)$ is actually profinite and $\mathrm{Aut}(G)$ is quasi-oligomorphic (isomorphic to an oligomorphic group). A key open question is whether there exists an oligomorphic group whose outer automorphism group is not profinite; resolving this appears essential to determining whether isomorphism on all oligomorphic groups is smooth.

Complete analytic quasi-orders. Moving beyond isomorphism to study quasi-orders on countable groups, we investigate the epimorphism relation: for countable groups A, B, we write $A \leqslant_{\mathrm{epi}} B$ if there exists a surjective homomorphism from B onto A (equivalently, A is isomorphic to a quotient of B). In joint work with Gao, Li, and Nies [6], we prove that the relation $\{(A, B) : A \leqslant_{\mathrm{epi}} B\}$ on the Borel space of countable groups is a complete analytic quasi-order, meaning that any analytic quasi-order is Borel reducible to it. This parallels Camerlo's result that epimorphism between countable graphs is complete analytic. The proof involves a novel construction associating to each countable graph Γ a Coxeter group $W(C_\Gamma)$ built from an expanded Coxeter graph C_Γ, such that there is a graph homomorphism $\Gamma \to \Delta$ if and only if $W(C_\Gamma) \leqslant_{\mathrm{epi}} W(C_\Delta)$. Related work with Shelah [5, 2] establishes that pure embeddability between countable torsion-free abelian groups is also a complete analytic quasi-order, extending work of Calderoni and Thomas on embeddability.

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The classification of homomorphism homogeneous oriented graphs

Maja Pech
(joint work with Bojana Pavlica, Christian Pech)

The notion of homomorphism homogeneity was introduced by Cameron and Nešetřil in [4]. A relational structure is called *homomorphism homogeneous* (shortly HH) if every homomorphism between finite substructures extends to an endomorphism of the structure in question. This notion is closely related to the notion of homogeneous relational structures where every isomorphism between finite substructures extends to an automorphism.

One of the central pillars in the research on (homomorphism-) homogeneous structures has been their classification. Of particular interest are relational structures with just one basic relation that is in addition supposed to be binary (also called binary relational structures). Prior to our research the following types of countable HH binary relational structures were classified:

- HH strict partial orders (Cameron and Lockett in [3])
- HH reflexive partial orders (Mašulović in [8], Cameron and Lockett in [3])
- HH tournaments (Ilić, Mašulović and Rajković in [7], Feller, Pech and Pech in [6])
- finite uniform HH oriented graphs (Mašulović in [10])

A big effort went into the classification of countable undirected HH graphs (with or without loops) (see [4, 13, 1, 2]), but this question remains open.

In this talk we present our recent results on the classification of countable HH oriented graphs without loops [11]. Our main result is:

Theorem. *Let Γ be a countable homomorphism homogeneous weakly connected oriented graph. Then Γ is one of the following oriented graphs:*

(1) I_1,
(2) $(\mathbb{Q},<)$,
(3) a tree with no minimal elements such that no finite subset of vertices has a maximal lower bound,
(4) a dual tree with no maximal elements such that no finite subset of vertices has a minimal upper bound,
(5) a poset such that:
  - *every finite subset of vertices is bounded from above and from below,*

- *no finite subset of vertices has a maximal lower bound or a minimal upper bound,*
- *no X_4-set has a midpoint,*

(6) an extension of the countable universal homogeneous strict poset,
(7) C_3,
(8) $S(2)[f]$ for a surjective function f with kernel classes of size at most 2,
(9) $T^\infty[f]$ for a surjective function f with a countable domain.

Here, for a countable oriented graph Γ, a countable set S, and a surjective $f\colon S \to \mathrm{V}(\Gamma)$, the oriented graph $\Gamma[f]$ is given by

$$\mathrm{V}(\Gamma[f]) = S, \text{and } \mathrm{E}(\Gamma[f]) = \{(s,t) \mid (f(s), f(t)) \in \mathrm{E}(\Gamma)\}.$$

The proof of this theorem bases on the following key observations:

(1) Every countable binary HH structure is weakly oligomorphic (see [9]).
(2) Every weakly oligomorphic HH structure has a unique (up to isomorphism) homogeneous, HH core (see [12]).
(3) The possible cores of countable HH oriented graphs are I_1, C_3, $(\mathbb{Q},<)$, $S(2)$, and T^∞.

Additionally, we compare the classification of HH oriented graphs with Cherlin's classification of homogeneous oriented graphs [5], and outline perspectives for future research.

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Finitely bounded homogeneity turned inside-out

Jakub Rydval

Deciding the amalgamation property for a given class of finite structures is an important subroutine in classifying countable finitely homogeneous structures. In this talk I discuss the computational complexity of the amalgamation decision problem for finitely bounded classes, i.e., classes specified by a finite set of forbidden finite substructures, or equivalently by a finite set of universal axioms.

I first present a link between the amalgamation decision problem and the problem of testing the containment between the reducts of two given finitely bounded amalgamation classes to a given common subset of their signatures. On the one hand, this link enables polynomial-time reductions from various decision problems that can be represented within the reduct containment problem for finitely bounded amalgamation classes, e.g., the 2-exponential square tiling problem, leading to a new lower bound for the complexity of the amalgamation decision problem: 2NEXPTIME-hardness. On the other hand, the link also allows me to show that the amalgamation decision problem is decidable under the assumption that every finitely bounded strong amalgamation class has a computable finitely bounded Ramsey expansion. The runtime of my conditional decision procedure depends 2-exponentially on the size of a minimal Ramsey expansion.

Subsequently, I briefly address the fact that the closely related problem of testing homogenizability is already undecidable, by a polynomial-time reduction from the regularity of context-free languages.

My results [1] indicate that the relationship between finitely bounded amalgamation classes and arbitrary finitely bounded classes shares similarities with the relationship between regular grammars and context-free grammars. A key difference is that the regularity of context-free grammars can be tested in linear time, while the problem of testing the amalgamation property for finitely bounded classes is 2NEXPTIME-hard.

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Bergman's property for unit groups of continuous rings

Friedrich Martin Schneider

In [1], Bergman initiated the study of a strong boundedness phenomenon for groups. Following Droste and Göbel [4], we say that a group G has *uncountable strong cofinality* if, for every countable chain $\mathcal{C}$ of subsets of G with $\bigcup\mathcal{C} = G$, there exist $m \in \mathbb{N}$ and $C \in \mathcal{C}$ such that $C^m = G$. Equivalently, a group possesses this property if and only if it has

— *uncountable cofinality*, i.e., it is not the union of a countable chain of proper subgroups, and

— the *Bergman property*, that is, it has finite width with respect to every generating set.

Bergman's work [1] established this property for the symmetric group over an infinite set, and his result was soon following by a stream of further natural examples (for instance, see [4, 8, 3, 5], to name but a few).

The talk revolved around a new source of examples of groups with uncountable strong cofinality, namely von Neumann's *continuous geometry* [6]. More precisely, the following result concerning the *unit group* $\mathrm{GL}(R)$, i.e., the group of invertible elements, of the coordinate rings of such continuous geometries was discussed.

Theorem ([7])**.** *Let R be a non-discrete irreducible, continuous ring. Then* $\mathrm{GL}(R)$ *has uncountable strong cofinality.*

By another result of [7], the *projective unit group* $\mathrm{PGL}(R) = \mathrm{GL}(R)/\mathrm{Z}(\mathrm{GL}(R))$, i.e., the quotient of the unit group modulo its center, of any non-discrete irreducible, continuous ring R is simple, and by force of the theorem above it thus follows that $\mathrm{PGL}(R)$ must have bounded normal generation. This answers a question of Carderi and Thom [2].

Among other things, an abstract method for establishing uncountable strong cofinality of groups of units in unital rings was discussed.

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Temporal Valued Constraint Satisfaction Problems

Žaneta Semanišinová

(joint work with Manuel Bodirsky, Édouard Bonnet)

We study the computational complexity of the valued constraint satisfaction problem (VCSP) for all valued structures over the domain $\mathbb{Q}$ preserved by all order-preserving bijections. Such VCSPs will be called temporal, in analogy to temporal constraint satisfaction problems whose complexity has been classified in [2]. Many optimization problems that have been studied intensively in the literature can be phrased as a temporal VCSP, for example, the min correlation clustering problem or the minimum feedback arc set problem. We present a P vs. NP-complete dichotomy for temporal VCSPs, based on the concepts of fractional polymorphisms and expressibility in valued structures. This is the first dichotomy result for VCSPs over infinite domains which is complete in the sense that it treats all valued structures that contain a given automorphism group.

The talk is based on [1].

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An amalgamation theorem for simplicial complexes

Sławomir Solecki

We use a set theoretic formalism to explore connections between stellar moves on simplicial complexes, amalgamation classes, and projective Fraïssé limits.

We identify a class of simplicial maps that naturally arise from the stellar moves. We call these maps weld-division maps. Our main theorem asserts that the category of weld-division maps fulfills the projective amalgamation property. Despite the geometric character of the theorem, the method of proof is neither geometric nor topological, but rather it consists of combinatorial calculations performed on finite sequences of finite sets crucially relying on a set theoretic formalism.

Aside from developing a new method of dealing with stellar moves and simplicial maps, the theorem gives (1) an example of an amalgamation class that substantially differs from known classes, (2) a combinatorial description of the geometric realization of a simplicial complex, and (3) an example of a combinatorially defined projective Fraïssé class whose canonical quotient space has topological dimension strictly bigger than 1.

The heart of the work is a proof of the amalgamation property for a natural class of simplicial maps acting among simplicial complexes. In the definition of the amalgamation class, we take as our departure point the two operations, known as stellar moves, on simplicial complexes that are fundamental to combinatorial topology—stellar subdivision and, its inverse, welding. We start with decoupling the two operations:

— we use stellar subdivision as a generating procedure for our amalgamation class; it produces new simplicial complexes and, together with composition, new simplicial maps;
— we use welding to define the base family of simplicial maps in our amalgamation class; simplicial maps in the class are produced by closing the base family under stellar subdivision and composition.

We prove the amalgamation theorem for the class of simplicial maps sketched out above. Our results reveal that the class is small enough to have strong combinatorial properties and large enough to remember the topology of simplicial complexes.

Now, we outline the content of the work in more detail.

Stellar moves, that is, stellar subdivision and welding, provide a combinatorial method of modifying simplicial complexes while retaining their geometric and topological structures. They go back to the papers of Alexander [2] and Newman [21], [22]. Stellar moves have been shown to be "geometrically complete" in various senses in [21], [2], [24], [1]; see [19, Theorem 4.5] for an exposition of the theorem from [21] and [2] and see also [20] for a related result. Stellar moves have formed the basis of combinatorial topology since the publication of the three papers by Alexander and Newman mentioned above; see [13] and [19].

Amalgamation classes, that is, classes of structures that fulfill the amalgamation property, are a well-researched area of combinatorics with firm connections to the study of homogeneous structure and Ramsey theory. A number of amalgamation classes have been described, both of the direct and projective kinds, and classifications of amalgamation classes in prescribed contexts have been achieved. One may consult [10], [14], [18], [21] and the papers cited in the paragraph below for more background information.

Projective Fraïssé theory is a method of producing "generic" compact topological spaces from classes of finite combinatorial objects by taking canonical projective limits and quotients. The method makes it possible to approach topological questions using combinatorial arguments. Projective Fraïssé theory was developed in [15]. It builds on and extends model theoretic ideas coming from Fraïssé [12]. This approach was applied in various topological situations; see [3], [4], [5], [6], [7], [8], [9], [11], [16], [17], [23].

Our work starts with an arbitrary (abstract) simplicial complex $\mathbf{A}$. We consider the class $\langle \mathbf{A} \rangle$ consisting of simplicial complexes produced from $\mathbf{A}$ by iterated application of stellar subdivisions. These are our objects. Next, we define a class of simplicial maps among complexes in $\langle \mathbf{A} \rangle$, which we name **weld-division maps**. These are our morphisms. Now, the complexes in $\langle \mathbf{A} \rangle$, as objects, and weld-division maps among them, as morphisms, form a natural category $\mathcal{D}(\mathbf{A})$ associated with the simplicial complex $\mathbf{A}$.

Since the class of weld-division maps is new, we comment briefly on the way it is defined. First, we introduce the operation of stellar subdivision of a simplicial map that is parallel to the notion of stellar subdivision of a simplicial complex. Weld-division maps are then defined as follows. The basic building blocks in this definition are what we call **weld maps** between simplicial complexes, which are inverses of stellar subdivisions of simplicial complexes and are a refinement of the operation of welding. Weld-division maps are obtained from weld maps by closing them under composition and stellar subdivision of simplicial maps.

We investigate properties of the category $\mathcal{D}(\mathbf{A})$, which amounts to a combinatorial study of weld-division maps. Our principal result asserts that the class of weld-division maps has an amalgamation property—the **projective amalgamation property**. That is, given two weld-division maps $f'\colon B \to A$ and $g'\colon C \to A$, where A, B, C are in $\langle \mathbf{A} \rangle$, there exist weld-division maps $f\colon D \to B$ and $g\colon D \to C$, for some D in $\langle \mathbf{A} \rangle$, such that

$$f' \circ f = g' \circ g.$$

We deduce from this result that the category $\mathcal{D}(\mathbf{A})$ forms a **transitive projective Fraïssé class**. In particular, it has a canonical limit, called the **projective Fraïssé limit**. From the limit, we extract, again following the general theory, a compact zero-dimensional metric space $\mathbb{A}$ with a compact equivalence relation $R^{\mathbb{A}}$ on it. Now, we are in a position to form the quotient space $\mathbb{A}/R^{\mathbb{A}}$—the **canonical quotient space of $\mathcal{D}(\mathbf{A})$**. A natural question arises of topologically identifying this space. We prove that the canonical quotient space is homeomorphic to the geometric realization of the simplicial complex $\mathbf{A}$.

A comment about the method of proof is in order. The interest in the class $\mathcal{D}(\mathbf{A})$ comes, to a large degree, from the geometric, multidimensional nature of its objects and morphisms. However, in proving the amalgamation property for $\mathcal{D}(\mathbf{A})$, we found it impossible to employ geometric or topological methods. Consequently, the proof of this property is rather unexpected. The high dimensional geometric problems are handled by forming a calculus of finite sequences of finite sets. To be a bit more specific, we note that finite sets are fundamental to our considerations. They are the building blocks of simplicial complexes. They are also operators on simplicial complexes, that is, a finite set applied to a complex subdivides it. Since we consider iterative stellar subdivisions, that is, subdivisions implemented by finite sequences of finite sets, and simplicial maps among so subdivided complexes, we are naturally led to a study of finite sequences of finite sets and appropriately defined functions among such sequences. Developing these ideas, we carry out the main arguments by performing computations and combinatorial manipulations on finite sequences of finite sets and functions among them. Curiously, crucial to these considerations is the set theoretic character of the entries of the sequences—in particular, the cumulative hierarchy of sets, the relation $\in$ of membership and its well foundedness, boolean operations, formation of singletons, etc.

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Notes on a reconstruction theorem of Lascar

Mira Tartarotti

By a result attributed to Coquand by Ahlbrandt and Ziegler [1] and proved independently by Makkai [5], any $\aleph_0$-categorical structure $\mathcal{M}$ is reconstructed up to bi-interpretability by its automorphism group $\mathrm{Aut}(\mathcal{M})$ endowed with the topology of pointwise convergence. A natural question to ask is under which circumstances one can disregard topology and already reconstruct $\mathcal{M}$ from the automorphism group $\mathrm{Aut}(\mathcal{M})$ or from the monoid of elementary embeddings $\mathrm{EEmb}(\mathcal{M})$ as purely algebraic objects. By a counterexample of Evans and Hewitt [3] and later work of Bodirsky, Evans, Kompatscher and Pinsker [2] this is not possible in general, even for $\mathrm{EEmb}(\mathcal{M})$. Lascar proved in 1982 [4] that when $\mathcal{M}$ satisfies a property called *G-finiteness* then in fact, $\mathcal{M}$ is reconstructed up to bi-interpretability from the monoid $\mathrm{EEmb}(\mathcal{M})$.

For a set $X \subseteq \mathcal{M}$, write $\mathrm{Aut}_X(\mathcal{M})$ for the group of pointwise stabilizers of X in $\mathrm{Aut}(\mathcal{M})$. Write $\mathrm{Autf}(\mathcal{M}/X)$ for the subgroup of $\mathrm{Aut}_X(\mathcal{M})$ generated by

$$\left\{f \in \mathrm{Aut}_X(\mathcal{M}) \;\middle|\; \exists \mathcal{N} \succ \mathcal{M}\; \exists \mathcal{M}' \prec \mathcal{N}\; \exists g \in \mathrm{Aut}_{\mathcal{M}'}(\mathcal{N}) :\; (X \subseteq \mathcal{M}' \;\wedge\; g|_{\mathcal{M}} = f)\right\},$$

i.e. by the set of automorphisms that can be extended to an automorphism fixing a submodel containing X. Then $\mathrm{Autf}(\mathcal{M}/X)$ is called the group of *Lascar-strong* automorphisms of $\mathcal{M}$ over X. When $\mathcal{M}$ is $\aleph_0$-categorical, $\mathcal{M}$ is called *G-finite* if for every finite set $X \subseteq \mathcal{M}$ there exists a finite set $Y \subseteq \mathcal{M}$ with $\mathrm{Aut}_Y(\mathcal{M}) \subseteq \mathrm{Autf}(\mathcal{M}/X)$, and additionally, the quotient group $\mathrm{Aut}_X(\mathcal{M})/\mathrm{Autf}(\mathcal{M}/X)$ is finite for every finite set X.

In my talk, I attempt to highlight some key tools used in Lascar's reconstruction theorem and to clarify in particular the role of G-finiteness in the result, following work from [6].

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Universal-homogeneous hyperbolic graphs and spaces and their isometry groups

Katrin Tent

The Urysohn space is the unique separable metric space that is universal and homogeneous for finite metric spaces, i.e., it embeds any finite metric space and any isometry between finite subspaces extends to an isometry of the whole space. The Urysohn space can easily be constructed by amalgamating all finite metric spaces with rational distances and taking the completion of the resulting metric space, see e.g. [2]. As is often the case with very homogeneous structures, the isometry group of the Urysohn space has a natural maximal normal subgroup, namely the group of isometries of bounded displacement and the quotient group is boundedly simple.

It is a natural question to ask whether an analog of this Urysohn space can exist for hyperbolic spaces. Here, we use the following definition of δ-hyperbolicity:

Definition. Let (X,d) be a metric space, $\delta \geqslant 0$. Then X is δ-hyperbolic if and only if for all $x_1,x_2,x_3,x_4 \in X$ the following holds: put

$$\begin{aligned} E =& d(x_1,x_2)+d(x_3,x_4),\\ F =& d(x_1,x_3)+d(x_2,x_4),\\ G =& d(x_1,x_4)+d(x_2,x_3). \end{aligned}$$

Then $G \leqslant \max\{E,F\}+2\delta$, or, equivalently, if $E \leqslant F \leqslant G$, then $G-F \leqslant 2\delta$.

In my talk I explained why for a class of structures the amalgamation property is a necessary condition for having an universal-homogeneous limit and I gave an explicit example showing that for $\delta > 0$ the class of finite δ-hyperbolic spaces fails to have this amalgamation property.

However, we can obtain a weaker limit structure when we restrict the amalgamation problem to certain strong embedddings.

To this end we define a property of subsets of metric spaces inspired by buildings and a notion of strong embeddings for metric spaces that will be appropriate for classes of hyperbolic metric spaces to allow amalgamation:

Definition. Fix $\delta \geqslant 0$ and let X be a metric space, $A \subset X$. Then we say that A is δ-*closed* (or: strongly embedded) in X and write $A \leqslant_\delta X$ if

(1) for every $b \in X \setminus A$ there exists a *gate* $g = g_A(b) \in A$ such that for all $a \in A$ we have $d(b,a) = d(b,g)+d(g,a)$.
(2) for $b,b' \in X \setminus A$ with $d(g_A(b),g_A(b')) > \delta$ we have
$$d(b,b') = d(b,g_A(b))+d(g_A(b),g_A(b'))+d(g_A(b'),b').$$

It turns out that amalgamation works over strongly embedded subspaces:

Theorem. Let A,B_1,B_2 be δ-hyperbolic spaces and assume that $A \leqslant_\delta B_1,B_2$ and let D be the canonical amalgam $D = B_1 \otimes_A B_2$ of B_1 and B_2 over A. I.e. D is defined as the disjoint union of A with $B_1 \setminus A$ and $B_2 \setminus A$ with the metric extending the metric on B_1,B_2 as follows: for $x \in B_1 \setminus A, y \in B_2 \setminus A$ we put

$$d(x,y) = d(x,g_A(x))+d(g_A(x),g_A(y))+d(g_A(y),y).$$

Then D is δ-hyperbolic and $B_1,B_2 \leqslant_\delta D$. Thus we obtain

Corollary. Fix $\delta \geqslant 0$ and let $(\mathcal{C}_\delta,\leqslant_\delta),({\mathcal{C}'}_\delta,\leqslant_\delta)$ be the class of finite δ-hyperbolic spaces with rational distances and the class of δ-hyperbolic graphs, respectively, partially ordered by strong embeddings $\leqslant_\delta$. Then there is a unique δ-hyperbolic space $\mathbb{H}_\delta$ and Γ_δ, and graph respectively, which is homogeneous and universal for finite δ-closed subspaces with rational distances (and subgraphs, respectively).

It can be shown that the isometry group of $\mathbb{H}_\delta$ does not contain elements of bounded displacement and it seems natural to conjecture that, as in the case of the

Urysohn space, the group $G = Isom(\mathbb{H}_\delta)$ is (essentially) a simple group. However, it is shown in [1] that G does not have a dense conjugacy class and does not act primitively on $\mathbb{H}_\delta$. Thus, proving simplicity will need different methods than the ones used in [2, 3]

I ended the talk with a number of open questions:

- Is there a natural class of δ-hyperbolic spaces $\mathcal{C}$, $\delta > 0$, such that the limit structure is stable?
- Is there a δ-hyperbolic analog of a bounded Urysohn space for $\delta > 0$?
- Is there a natural class of δ-hyperbolic spaces $\mathcal{C}$ for $\delta > 0$, such that the limit structure is ω-categorical?

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Monadically dependent classes and the model checking problem

Szymon Toruńczyk

A central problem in theoretical computer science concerns the complexity of the model-checking problem: to decide if a given formula holds in a given structure. Here, I will focus on formulas of first-order logic, and assume that the structure comes from a fixed hereditary class of graphs, allowing to leverage tools from structural graph theory. For some 'tame' graph classes – such as the class of planar graphs, or every class of graphs with bounded maximum degree – the model-checking problem is tractable in the sense of parameterized complexity, while for some others, it is hard. It is conjectured that the dividing line between the tractable and the hard hereditary classes corresponds precisely to the notion of monadic dependence (or NIP), originating in model theory. By now, it known that all monadically stable graph classes are tractable. I will survey the recent progress in this area, which connects various notions originating in structural graph theory – such as nowhere denseness, treewidth and twin-width – with notions originating in model theory.

Ergodic theory of automorphism groups of homogeneous structures

Todor Tsankov

Ergodic theory studies group actions on measure spaces, either by preserving a measure or a measure class. It turns out that for large groups (such as rich automorphism groups), it is sometimes possible to prove classification results which have applications to probability theory (exchangeability) and topological dynamics (classifying invariant measures on flows). The methods are usually inspired from model theory.

The most detailed results so far concern generalizations of de Finetti's theorem. The setting is as follows. Let $G \curvearrowright M$ be a permutation group and consider the action $G \curvearrowright [0,1]^M$ given by

$$(g \cdot \xi)(a) = \xi(g^{-1} \cdot a) \quad \text{for } g \in G, \xi \in [0,1]^M, a \in M.$$

The de Finetti theorem states that when $G = \mathrm{Sym}(M)$, the only ergodic invariant probability measures are the of the form $\nu^{\otimes M}$, where ν is a measure on $[0,1]$. We recall that a permutation group $G \curvearrowright M$ is *oligomorphic* if the diagonal action $G \curvearrowright M^n$ has only finitely many orbits for every n. It *has no algebraicity* if for every finite $A \subseteq M$, the stabilizer G_A has infinite orbits outside of A.

Generalizing a result of [2], we prove the following in [3]:

Theorem 1. *Let $G \curvearrowright M$ be an oligomorphic permutation group and suppose that the action is primitive and has no algebraicity. Then the only G-invariant ergodic measures on $[0,1]^M$ are of the form $\nu^{\otimes M}$, where ν is a measure on $[0,1]$.*

The assumption that the action is oligomorphic can be omitted at the expense of strengthening one of the other hypotheses.

Some applications of this and related theorems to dynamics can be found in [2] and [1].

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CSPs in the Choiceless Context

Zoltán Vidnyánszky
(joint work with Tamás Kátay, László Tóth)

For a finite relational structure $\mathcal{D}$, the $\mathcal{D}$-homomorphism problem—or CSP($\mathcal{D}$)—asks whether a given finite structure of the same signature admits a homomorphism to $\mathcal{D}$. The celebrated CSP Dichotomy Theorem of Bulatov [2] and Zhuk [8] asserts that CSP($\mathcal{D}$) is solvable in polynomial time or NP-complete.

In fact, a good understanding of easy and hard homomorphism problems has been reached, through the central notion of *polymorphisms*.

Definition. A map homomorphism $f : \mathcal{D}^n \to \mathcal{D}$ is called a *polymorphism.*

Here $\mathcal{D}^n$ stands for the categorical power of $\mathcal{D}$, i.e., tuples are in a relation of they are related in each coordinate.

It turns out that the polymorphism structure completely determines the hardness of a CSP. Intuitively, the lack of non-trivial polymorphisms of $\mathcal{D}$ is equivalent to the CSP being NP-complete. The following statement is not hard to show and illustrates well our general goal.

Proposition. Let K_3 be the complete graph on 3 vertices, with unary symbols for each vertex. If $f : K_3^n \to K_3$ is a polymorphism then the set

$$\mathcal{U} = \{A \subseteq n : \forall x \in K_3^n \ (x|_A \equiv i \implies f(x) = i)\},$$

is an ultrafilter. In particular, every such polymorphism is a projection.

A nice form of non-trivial polymorphisms plays a crucial role in our investigations.

Definition. A *cyclic polymorphism (of arity n)* is a polymorphism $f : \mathcal{D}^n \to \mathcal{D}$ such that for all $x_1, \dots, x_n$ we have

$$f(x_1, x_2 \dots, x_n) = f(x_2, \dots, x_n, x_1).$$

Barto and Kozik [1] has shown that a structure falls in the easy case of the Dichotomy, precisely if it admits a cyclic polymorphism of every large enough prime arity. Let us denote this property by $(*)_{\mathcal{D}}$.

Now we turn to the investigation of infinitary versions. Given a finite template $\mathcal{D}$, we consider the compactness statement $K_{\mathcal{D}}$:

> For every $\mathcal{G}$, if every finite substructure $\mathcal{F}$ of $\mathcal{G}$ admits a homomorphism to $\mathcal{D}$, then so does $\mathcal{G}$.

Note that these statements are consequences of the Axiom of Choice; for example, when $\mathcal{D}$ is K_n, this statement is known as the de Bruijn-Erdős theorem. However, working only over ZF (that is, assuming all the axioms except for Choice), it makes sense to compare them.

It was shown by Lévy [4] and Mycielski [5] that K_{K_n}, i.e., the compactness for n-coloring is equivalent to the Ultrafilter Lemma over ZF for $n \geqslant 3$, while K_{K_2} is strictly weaker (see also Rorabaugh-Tardif-Wehlau [6]). This means that there exists a model of ZF, in which K_{K_2} holds, but K_{K_n} fails for every $n \geqslant 3$.

The following is our main result.

Theorem. ([3]) Let $\mathcal{D}$ be a finite relational structure.

(1) If $\mathcal{D}$ admits a cyclic polymorphism (equivalently, satisfies the algebraic tractability condition $(*)_{\mathcal{D}}$), then $K_{\mathcal{D}}$ is strictly weaker than the Ultrafilter Lemma.

(2) If $\mathcal{D}$ does not admit a cyclic polymorphism, then $K_{\mathcal{D}}$ is equivalent to the Ultrafilter Lemma over ZF.

Moreover, there is a model of ZF in which $K_{\mathcal{D}}$ holds exactly for those $\mathcal{D}$ satisfying $(*)_{\mathcal{D}}$.

The proof does not use the CSP Dichotomy, instead, it relies on the characterization of Barto-Kozik. For us, what makes this really surprising, is the fact that the need for these polymorphisms naturally arose from the theory of rather abstract, choiceless, so called ZFA models and forcing constructions. In these constructions, one finds a *Boolean valued* solution, and uses the existence of cyclic polymorphisms to produce a classical one, resembling the linear relaxation method of solving CSPs.

Let us finish with mentioning a possible further direction. Tardif [7] has recently suggested to consider the promise version of compactness results. Assume that $\mathcal{D}, \mathcal{E}$ are finite structures with $\mathcal{D}$ admitting a homomorphism to $\mathcal{E}$. Denote by $K_{\mathcal{D},\mathcal{E}}$ the statement:

> For every $\mathcal{G}$, if every finite substructure $\mathcal{F}$ of $\mathcal{G}$ admits a homomorphism to $\mathcal{D}$, then $\mathcal{G}$ admits a homomorphism to $\mathcal{E}$.

The following problem is quite natural.

Problem. Characterize the strength of the compactness principles $K_{\mathcal{D},\mathcal{E}}$. For which pairs $\mathcal{D}, \mathcal{E}$ is $K_{\mathcal{D},\mathcal{E}}$ equivalent to the Ultrafilter Lemma?

In an ongoing work Bodor has shown that K_{K_3,K_4} and K_{K_3,K_5} are equivalent to UL, hence, yet again, the boundary between (known) hard and easy problems exactly matches the current finitary knowledge.

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Singleton algorithms for temporal CSPs

Dmitriy Zhuk

A natural strengthening of an algorithm for the (promise) constraint satisfaction problem is its singleton version: we first fix a constraint to some tuple from the constraint relation, then run the algorithm, and remove the tuple from the constraint if the answer is negative. We characterize the power of the singleton versions of standard universal algorithms for the (promise) CSP over a fixed template by polymorphisms with certain symmetries, called palette symmetric polymorphisms.

In the talk we demonstrate how palette polymorphisms work for CSPs over temporal relational structures, where a relational structure $\mathbb{A}$ is *temporal* if its domain is $\mathbb{Q}$ and its relations are definable by Boolean combinations of atomic formulas of the form $x < y$. The classification of the complexity of temporal CSPs has been known for a long time [1], but only recently Mottet proposed a new uniform algorithm based on universal algorithms [2]. Note that most of the universal algorithms cannot be applied directly to infinite domain CSPs. For instance, in linear relaxations one introduces a variable for every element of the domain, which would result in a system with infinitely many variables.

One of the ways to overcome this difficulty is "sampling": to solve an instance with N variables one first restricts the domain to any N-element subset of $\mathbb{Q}$, then runs a universal algorithm on the restricted instance, and returns its answer. Mottet proved in [2] that every tractable temporal CSP can be solved this way using local consistency and AIP algorithms. The soundness of the algorithm follows from the fact that $\mathrm{PCSP}(\mathbb{A}|_{[N]}, \mathbb{A})$ is solvable by the corresponding universal algorithms, where $\mathbb{A}|_{[N]}$ is the induced substructure obtained by restricting the domain to $\{1, 2, \ldots, N\}$.

Another singleton version of a universal algorithm is the algorithm in which, instead of fixing a tuple of a constraint, we fix an element of the domain of a variable and remove the element from the domain if the answer is negative. We denote the "tuple" version by CSingl and the "element" version by Singl. By ArcCons and AIP we denote the arc-consistency algorithm and the AIP algorithm, respectively (see [3] for more details).

In the talk, we present an alternative proof of the Mottet result by providing concrete palette symmetric functions from $\mathrm{Pol}(\mathbb{A}|_{[N]}, \mathbb{A})$, which can be used to obtain a (true) solution from the outputs of the algorithms SinglArcCons and CSinglAIP. Thus, we prove the following slight strengthening of Mottet's result, in which local consistency is replaced by CSinglArcCons:

Theorem. *Let $\mathbb{A}$ be a temporal structure, $N \in \mathbb{N}$. Then one of the following holds:*

- $\mathrm{PCSP}(\mathbb{A}|_{[N]}, \mathbb{A})$ *is solvable by* CSinglArcCons *or* SinglAIP*;*
- $\mathrm{CSP}(\mathbb{A})$ *is NP-hard.*

Meanwhile, we found a temporal relational structure $\mathbb{A}$ such that $\mathrm{PCSP}(\mathbb{A}|_{[N]}, \mathbb{A})$ can be solved by CSinglArcCons but not by SinglArcCons. This is a surprising result, since earlier we showed that for finite relational structures $\mathbb{A}$ and $\mathbb{B}$ the algorithm SinglArcCons solves PCSP$(\mathbb{A}, \mathbb{B})$ if and only if the algorithm CSinglArcCons does.

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