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VASILY VASYUNIN AND ALEXANDER VOLBERG

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SHARP CONSTANTS IN THE CLASSICAL WEAK FORM OF THE JOHN–NIRENBERG INEQUALITY

VASILY VASYUNIN AND ALEXANDER VOLBERG

Abstract. The sharp constants in the classical John–Nirenberg inequality are found by using Bellman function approach.

1. Introduction

Bellman function method in Harmonic Analysis was introduced by Burkholder for finding the norm in L^p of the martingale transform. Later it became clear that the scope of the method is quite wide. After Burkholder a lot of papers followed this method (see, e.g., [11], [9], [10], [22], [17], [14] and the references section of the present article). It became clear that magic Burkholder function from [1] is a natural dweller of the area called stochastic optimal control. Many harmonic analysis problems have their analog of the stochastic optimal control Bellman function, which is a solution of a certain partial differential equation (Bellman equation).

In the present paper we solve an extremal problem related to the famous John–Nirenberg inequality. This is the situation when the Bellman equation turns out to be the degenerated Monge–Ampère equation. The Bellman function of the corresponding extremal problem (the definition see below) is found explicitly. This function carries all the information about the problem: not only the sharp constants, but, for example, a construction of extremal test functions (*extremizers*).

Now we start the formal description of the problem.

For an interval I and a real-valued function $\varphi \in L^1(I)$, let $\langle\varphi\rangle_I$ be the average of φ over I, i.e.,

$$\langle\varphi\rangle_I = \frac{1}{|I|}\int_I \varphi,$$

where $|I|$ stands for Lebesgue measure of I. For $1 \le p < \infty$, let

$$\mathrm{BMO}(J) = \left\{\varphi \in L^1(J)\colon \langle|\varphi - \langle\varphi\rangle_I|^p\rangle_I \le C^p < \infty,\ \forall I \subset J\right\} \tag{1.1}$$

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with the best (smallest) such C being the corresponding "norm" of φ. For $\varepsilon \geq 0$, let

$$\mathrm{BMO}_\varepsilon(J) = \{\varphi \in \mathrm{BMO}(J) \colon \|\varphi\| \leq \varepsilon\}.$$

The classical definition of John and Nirenberg uses $p = 1$; it is known that the norms are equivalent for different p. A crucial property of elements of BMO-space, the exponential decay of their distribution function, was established in the classical paper [4]; it is known as the John–Nirenberg inequality. For every $\varphi \in \mathrm{BMO}(J)$ and every $\lambda \in \mathbb{R}$ the classical John–Nirenberg inequality consists in the following assertion.

Theorem (John, Nirenberg; weak form)

$$\frac{1}{|J|}|\{s \in J \colon |\varphi(s) - \langle\varphi\rangle_J| \geq \lambda\}| \leq c_1 e^{-c_2\lambda/\|\varphi\|_{\mathrm{BMO}(J)}}. \tag{1.2}$$

We refer to this statement as to the weak form of the John–Nirenberg inequality to distinguish it from the following equivalent assertion.

Theorem (John, Nirenberg; integral form) *There exists $\varepsilon_0 > 0$ such that for every ε, $0 \leq \varepsilon < \varepsilon_0$, there is $C(\varepsilon) > 0$ such that for any function φ,* $\varphi \in \mathrm{BMO}_\varepsilon(J)$, the following inequality holds

$$\langle e^\varphi\rangle_J \leq C(\varepsilon) e^{\langle\varphi\rangle_J}\,.$$

The sharp constants in the integral form were found in [17] and [14]. In the second paper the dyadic analog BMO^d is considered as well, for which every subinterval I of J in definition (1.1) is an element of the dyadic lattice rooted in J. It appears that the constants in the dyadic case and the usual one are different.

The Bellman function corresponding to the integral John–Nirenberg inequality was found by solving the boundary value problem for the Bellman equation. In that case the Bellman equation was a second order partial differential equation with two variables, and due to a natural homogeneity of the problem, the Bellman partial differential equation was reduced to an ordinary differential equation, which was successfully solved. The corresponding Bellman equation for the week John–Nirenberg inequality has an additional parameter λ preventing a similar reduction of the Bellman partial differential equation to an ordinary differential equation.

The Bellman equations for all these problems are in fact special cases of the Monge–Ampère equation. After finding possibility to solve this type of equation explicitly (see [13], [19]) we are able to find the Bellman function (and therefore, the sharp constants) for the weak John–Nirenberg inequality as well ([20]).

We work with L^2-based BMO-norm, i.e., $p = 2$ is chosen in (1.1). For the classical case $p = 1$, Korenovskii [5] established the exact value $c_2 = 2/e$ using the equimeasurable rearrangements of the test function and the "sunrise lemma". But to apply the Bellman function method the L^2-based BMO-norm is more appropriate. Some Bellman-type function (so-called supersolution) for the weak John–Nirenberg inequality was proposed by Tao in [16], where there was no attempt to find true Bellman function and sharp constants. In the present paper it will be proved that for $p = 2$ the sharp constant are $c_1 = \frac{4}{e^2}$ and $c_2 = 1$.

2. Definitions and statements of the main results

2.1. Bellman functions. Now the main subject of the paper will be introduced, the Bellman function corresponding to the John–Nirenberg inequality. First of all we define the following set of test functions

$$S_\varepsilon(x) = S(x_1, x_2; \varepsilon) =$$
$$\{\varphi \in \mathrm{BMO}(J)\colon \langle\varphi\rangle_J = x_1, \langle\varphi^2\rangle_J = x_2, \langle|\varphi - \langle\varphi\rangle_I|^2\rangle_I \le \varepsilon^2 \ \forall I \subset J\}\,. \quad (2.1)$$

For any test function φ the point $x = (x_1, x_2) = (\langle\varphi\rangle_J, \langle\varphi^2\rangle_J)$ belongs to the parabolic strip

$$\Omega_\varepsilon = \{x = (x_1, x_2)\colon x_1^2 \le x_2 \le x_1^2 + \varepsilon^2\}\,. \quad (2.2)$$

Indeed, the left inequality $x_1^2 \le x_2$ is simply the Cauchy inequality, but the right one $x_2 \le x_1^2 + \varepsilon^2$ follows from the fact that $\varphi \in \mathrm{BMO}_\varepsilon(J)$:

$$x_2 - x_1^2 = \langle\varphi^2\rangle_J - \langle\varphi\rangle_J^2 = \langle|\varphi - \langle\varphi\rangle_J|^2\rangle_J \le \varepsilon^2\,.$$

Now we define the Bellman $\mathbf{B}$ function corresponding to the weak John–Nirenberg inequality:

$$\mathbf{B}(x;\lambda) \overset{\text{def}}{=} \mathbf{B}(x;\lambda,\varepsilon) \overset{\text{def}}{=} \frac{1}{|I|}\sup\big\{|\{s \in I\colon |\varphi(s)| \ge \lambda\}|\colon \varphi \in S_\varepsilon(x)\big\}\,. \quad (2.3)$$

This function is defined on Ω and it supplies us with the sharp estimate of the distribution function

$$\frac{1}{|J|}|\{s \in J\colon |\varphi(s) - \langle\varphi\rangle_J| \ge \lambda\}| \ \ \le \sup_{\xi\in[0,\varepsilon^2]} \mathbf{B}(0,\xi;\lambda) \quad \forall\varphi \in \mathrm{BMO}_\varepsilon\,. \quad (2.4)$$

To check this, we consider a new function $\tilde\varphi \overset{\text{def}}{=} \varphi + c$. If $\varphi \in S_\varepsilon(x)$, then $\tilde\varphi \in S_\varepsilon(\tilde x)$, where $\tilde x_1 = x_1 + c$ and $\tilde x_2 = x_2 + 2cx_1 + c^2$. Therefore, by definition (2.3), we have

$$\frac{1}{|J|}|\{s \in J\colon |\tilde\varphi(s)| \ge \lambda\}| \le \mathbf{B}(\tilde x;\lambda)\,.$$

If we take now $c = -\langle\varphi\rangle_J = -x_1$, we get $\tilde x_1 = 0$, $\tilde x_2 = x_2 - x_1^2$, and the latter inequality turns into

$$\frac{1}{|J|}|\{s \in J\colon |\varphi(s) - \langle\varphi\rangle_J| \ge \lambda\}| \le \mathbf{B}(0,\tilde x_2;\lambda) \le \sup_{\xi\in[0,\varepsilon^2]} \mathbf{B}(0,\xi;\lambda)\,.$$

So, to find the sharp constants in the weak John–Nirenberg inequality we prove the following theorem.

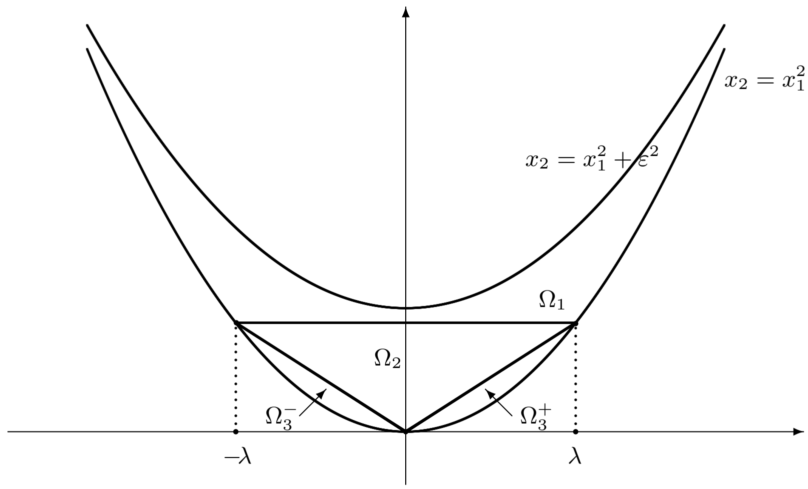

FIGURE 1

Theorem 1. *For* $0 \le \lambda \le \varepsilon$ *split* Ω *in three subdomains* (see Fig. 1):

$$\begin{aligned} \Omega_1 &= \{x \in \Omega\colon x_2 \ge \lambda^2\}\,, \\ \Omega_2 &= \{x \in \Omega\colon \lambda|x_1| \le x_2 \le \lambda^2\}\,, \\ \Omega_3 &= \{x \in \Omega\colon x_2 < \lambda|x_1|\}\,, \end{aligned}$$

then

$$\mathbf{B}(x;\lambda,\varepsilon) = \begin{cases} 1\,, & x \in \Omega_1\,, \\ \dfrac{x_2}{\lambda^2}\,, & x \in \Omega_2\,, \\ \dfrac{x_2 - x_1^2}{x_2 + \lambda^2 - 2\lambda|x_1|}\,, & x \in \Omega_3\,. \end{cases} \tag{2.5}$$

For $\varepsilon < \lambda \le 2\varepsilon$ *split* Ω *in four subdomains* (see Fig. 2):

$$\begin{aligned} \Omega_1 &= \{x \in \Omega\colon |x_1| \ge \lambda \text{ and } x_2 \le 2(\lambda+\varepsilon)|x_1| - \lambda^2 - 2\varepsilon\lambda \text{ for } |x_1| < \lambda+\varepsilon,\}, \\ \Omega_2 &= \{x \in \Omega\colon \lambda-\varepsilon \le |x_1| \le \lambda+\varepsilon,\, x_2 \ge \max\{2\lambda|x_1| - \lambda^2 \pm 2\varepsilon(|x_1|-\lambda)\}\}, \\ \Omega_3 &= \{x \in \Omega\colon x_2 < \lambda|x_1|\}, \\ \Omega_4 &= \{x \in \Omega\colon x_2 \ge \lambda|x_1| \text{ and } x_2 \le 2(\lambda-\varepsilon)|x_1| - \lambda^2 + 2\varepsilon\lambda \text{ for } |x_1| > \lambda-\varepsilon\}, \end{aligned}$$

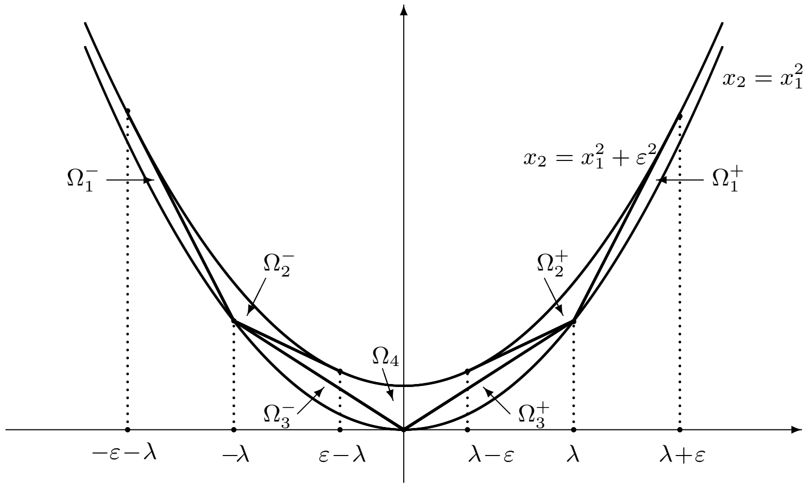

FIGURE 2

then

$$\mathbf{B}(x;\lambda,\varepsilon) = \begin{cases} 1\,, & x \in \Omega_1\,, \\ \dfrac{2(\lambda^2-\varepsilon^2)|x_1| - (\lambda-\varepsilon)x_2 + \lambda(2\varepsilon^2+\varepsilon\lambda-\lambda^2)}{2\varepsilon\lambda^2}\,, & x \in \Omega_2\,, \\ \dfrac{x_2 - x_1^2}{x_2+\lambda^2-2\lambda|x_1|}\,, & x \in \Omega_3\,, \\ \dfrac{x_2}{\lambda^2}\,, & x \in \Omega_4\,. \end{cases} \tag{2.6}$$

For $\lambda > 2\varepsilon$ split Ω in five subdomains (see Fig. 3):

$$\Omega_1 = \{x \in \Omega\colon |x_1| \geq \lambda \textit{ and } x_2 \leq 2(\lambda+\varepsilon)|x_1| - \lambda^2 - 2\varepsilon\lambda \textit{ for } |x_1| < \lambda+\varepsilon,\},$$
$$\Omega_2 = \{x \in \Omega\colon \lambda-\varepsilon \leq |x_1| \leq \lambda+\varepsilon,\, x_2 \geq \max\{2\lambda|x_1| - \lambda^2 \pm 2\varepsilon(|x_1|-\lambda)\}\},$$
$$\Omega_3 = \{x \in \Omega\colon x_2 < 2(\lambda-\varepsilon)|x_1| - \lambda^2 + 2\varepsilon\lambda\},$$
$$\Omega_4 = \{x \in \Omega\colon x_2 \geq 2(\lambda-\varepsilon)|x_1| - \lambda^2 + 2\varepsilon\lambda \textit{ and } x_2 \leq 2\varepsilon|x_1| \textit{ for } |x_1| < \varepsilon\},$$
$$\Omega_5 = \{x \in \Omega\colon x_2 \geq 2\varepsilon|x_1|\},$$

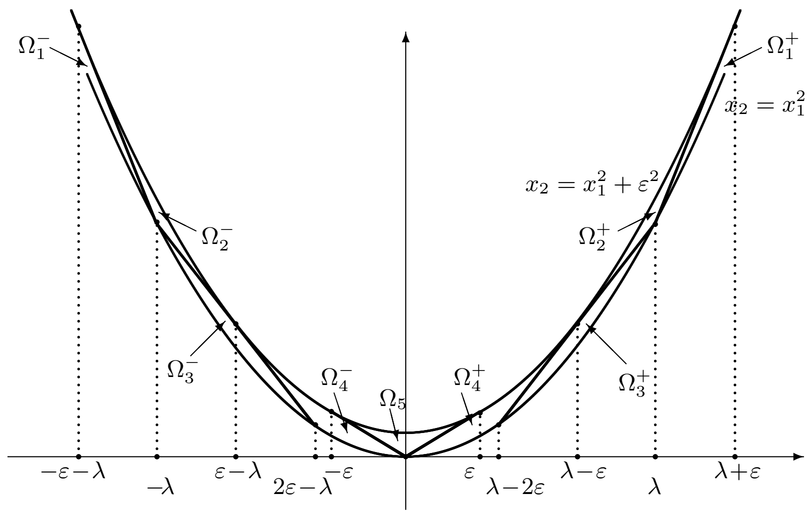

FIGURE 3

then

$$\mathbf{B}(x;\lambda,\varepsilon) = \begin{cases} 1\,, & x\in\Omega_1\,,\\[2mm] 1-\dfrac{x_2-2(\lambda+\varepsilon)|x_1|+\lambda^2+2\varepsilon\lambda}{8\varepsilon^2}\,, & x\in\Omega_2\,,\\[2mm] \dfrac{x_2-x_1^2}{x_2+\lambda^2-2\lambda|x_1|}\,, & x\in\Omega_3\,,\\[2mm] \dfrac{e}{2}\left(1-\sqrt{1-\dfrac{x_2-x_1^2}{\varepsilon^2}}\right)\exp\left\{\dfrac{|x_1|-\lambda}{\varepsilon}+\sqrt{1-\dfrac{x_2-x_1^2}{\varepsilon^2}}\right\}, & x\in\Omega_4\,,\\[2mm] \dfrac{x_2}{4\varepsilon^2}\exp\left\{2-\dfrac{\lambda}{\varepsilon}\right\}, & x\in\Omega_5\,. \end{cases} \tag{2.7}$$

Corollary. *If* $\varphi\in\mathrm{BMO}_\varepsilon(I)$*, then*

$$\frac{1}{|I|}\left|\{s\in I\colon |\varphi(s)-\langle\varphi\rangle_I|\ge\lambda\}\right| \le \begin{cases} 1, & \textit{if}\quad 0\le\lambda\le\varepsilon,\\[2mm] \dfrac{\varepsilon^2}{\lambda^2} & \textit{if}\quad \varepsilon\le\lambda\le 2\varepsilon,\\[2mm] \dfrac{e^2}{4}e^{-\lambda/\varepsilon} & \textit{if}\quad 2\varepsilon\le\lambda, \end{cases}$$

and this bound is sharp.

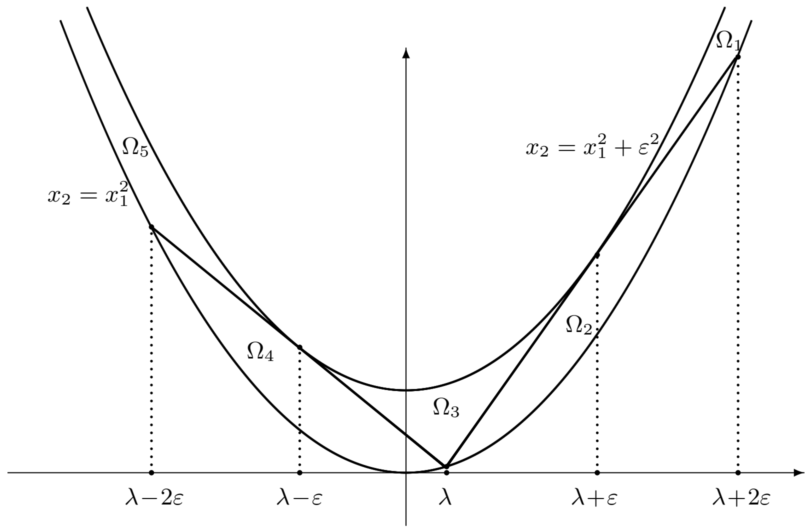

FIGURE 4

Proof. According to formula (2.4) it is sufficient to calculate

$$\sup_{\xi\in[0,\varepsilon^2]} \mathbf{B}(0,\xi;\lambda,\varepsilon)\,.$$

Since $\mathbf{B}(0,x_2;\lambda,\varepsilon)$ is an increasing function in x_2, this supremum is just the value $\mathbf{B}(0,\varepsilon^2;\lambda,\varepsilon)$, what yields the stated formula. □

Before we start to prove Theorem 1, where the Bellman function has two singularities on the boundary at the points $x=(\pm\lambda,\lambda^2)$, let us consider the simplest possible extremal problem with one singularity. We shall consider two extremal problems simultaneously: one estimate from above and the second estimate from below. So, we define two Bellman functions: $\mathbf{B}_{\max}$ and $\mathbf{B}_{\min}$.

$$\mathbf{B}_{\max}(x;\lambda,\varepsilon) \stackrel{\text{def}}{=} \frac{1}{|I|}\sup\big\{|\{s\in I\colon \varphi(s)\ge\lambda\}|\colon \varphi\in S_\varepsilon(x)\big\}\,,$$

$$\mathbf{B}_{\min}(x;\lambda,\varepsilon) \stackrel{\text{def}}{=} \frac{1}{|I|}\inf\big\{|\{s\in I\colon \varphi(s)\ge\lambda\}|\colon \varphi\in S_\varepsilon(x)\big\}\,,$$

For these function the following formula will be proved:

Theorem 2. *Split Ω in the following five subdomains* (see Fig. 4):

$$\Omega_1=\{x\in\Omega\colon x_1\ge\lambda+\varepsilon,\ x_2\ge 2(\lambda+\varepsilon)x_1-\lambda^2-2\varepsilon\lambda\}\,,$$
$$\Omega_2=\{x\in\Omega\colon x_2\le 2(\lambda+\varepsilon)x_1-\lambda^2-2\varepsilon\lambda\}\,,$$
$$\Omega_3=\{x\in\Omega\colon \lambda-\varepsilon\le x_1\le\lambda+\varepsilon,\ x_2\ge 2\lambda x_1-\lambda^2+2\varepsilon|x_1-\lambda|\}\,,$$
$$\Omega_4=\{x\in\Omega\colon x_2\le 2(\lambda-\varepsilon)x_1-\lambda^2+2\varepsilon\lambda\}\,,$$
$$\Omega_5=\{x\in\Omega\colon x_1\le\lambda-\varepsilon,\ x_2\ge 2(\lambda-\varepsilon)x_1-\lambda^2+2\varepsilon\lambda\}\,.$$

Then

$$\mathbf{B}_{\max}(x;\lambda,\varepsilon)=\begin{cases} 1\,, & x\in\Omega_1\cup\Omega_2\,,\\ 1-\dfrac{x_2-2(\lambda+\varepsilon)x_1+\lambda^2+2\varepsilon\lambda}{8\varepsilon^2}\,, & x\in\Omega_3\,,\\ \dfrac{x_2-x_1^2}{x_2+\lambda^2-2\lambda x_1}\,, & x\in\Omega_4\,,\\ \dfrac{e}{2}\left(1-\sqrt{1-\dfrac{x_2-x_1^2}{\varepsilon^2}}\right)\exp\left\{\dfrac{x_1-\lambda}{\varepsilon}+\sqrt{1-\dfrac{x_2-x_1^2}{\varepsilon^2}}\right\}, & x\in\Omega_5\,, \end{cases} \tag{2.8}$$

and

$$\mathbf{B}_{\min}(x;\lambda,\varepsilon)=\begin{cases} 0\,, & x\in\Omega_5\cup\Omega_4\,,\\ \dfrac{x_2-2(\lambda-\varepsilon)x_1+\lambda^2-2\varepsilon\lambda}{8\varepsilon^2}\,, & x\in\Omega_3\,,\\ 1-\dfrac{x_2-x_1^2}{x_2+\lambda^2-2\lambda x_1}\,, & x\in\Omega_2\,,\\ 1-\dfrac{e}{2}\left(1-\sqrt{1-\dfrac{x_2-x_1^2}{\varepsilon^2}}\right)\exp\left\{\dfrac{\lambda-x_1}{\varepsilon}+\sqrt{1-\dfrac{x_2-x_1^2}{\varepsilon^2}}\right\}, & x\in\Omega_1\,. \end{cases} \tag{2.9}$$

3. How do we proceed?

The consideration of the theorems above can be split to four parts.

- I. In the first part one observes that just by definition Bellman functions **B** satisfy a certain concavity condition in their domain of definition and boundary conditions on (part of) the boundary of this domain.
- II. In the second part one considers **all** function satisfying these concavity and boundary conditions. We denote this class by $\mathcal{V}$. And one makes the following supposition: as b belongs to $\mathcal{V}$ and is the "best" such function, it has to satisfy not only the concavity condition but also this concavity should become degenerate, i.e., the inequality has to turns into equality along some vector field on our domain Ω. This brings to the picture the Monge–Ampère equation. One solves it using the boundary conditions mentioned above. The result is a function $B\in\mathcal{V}$. Function B carries an interesting geometric information to be used later.

These two steps are in fact not necessary for the proof of the results, they are needed only to finding a function B, a candidate for a role of the Bellman function **B**. For example, in the series of excellent papers [6, 7, 8] very complicated Bellman functions appear as *deus ex machina*. As we shall see the analysis of Monge–Ampère equation not only supplies us with a candidate, but it helps in the next two steps as well, namely, in the prove that the found candidate really is the desired Bellman function.

- III. The third part consists of proving that $B \ge \mathbf{B}$. In convex domains of definition this is usually not difficult. Otherwise it may require a non-trivial proof, see [14] and Section 6 below.
- IV. The fourth part consists of proving $B \le \mathbf{B}$. This is achieved by presenting the extremal functions or extremal sequences of functions. In its turn, such functions are found from the geometric structure of B (mentioned above in II).

4. Heuristic for the Monge–Ampère equation. Boundary condition

If in (2.3) we would take the supremum over functions lying only in dyadic BMO^d, then we would get quite easily that the corresponding dyadic Bellman function $\mathbf{B}^d$ would satisfy the following inequality

Notice first, the local concavity of the function $\mathbf{B}$:

$$\mathbf{B}^d\Big(\frac{x^+ + x^-}{2}\Big) \ge \frac{1}{2}\big(\mathbf{B}^d(x^+) + \mathbf{B}^d(x^-)\big)\,, \quad \alpha_\pm \ge 0,\ \alpha_+ + \alpha_- = 1, \tag{4.1}$$

for any pair $x^\pm \in \Omega_\varepsilon$ such that the middle point $x = \frac{1}{2}(x^- + x^+)$ is in Ω_ε.

In fact, the definition of $\mathbf{B}^d$ does not depend on the dyadic interval I. This is just the scale invariance of the problem. Therefore, let us choose an almost best (up to an error η) function $\varphi^+ \in \mathrm{BMO}^d(I_+)$ for data $x^+ \in \Omega$, (and for fixed λ, ε), and let us do the same for x^-, we call a corresponding almost best (again up to an error η) function $\varphi^- \in \mathrm{BMO}^d(I_-)$. Consider function φ equal to $\varphi^\pm$ on $I_\pm$ correspondingly. We can check that $\varphi \in \mathrm{BMO}^d(I)$. In fact one should only check that $\langle\varphi^2\rangle_I - \langle\varphi\rangle_I^2 \le \varepsilon^2$. This easily follows from the requirement that $x = \frac{1}{2}(x^- + x^+)$ is in Ω_ε. The function φ competes to be the extremal function for the data $x \in \Omega$. The supremum given by $\mathbf{B}^d(x)$ is not less than $\frac{1}{|I|}|\{t \in I\colon \varphi(t) \ge \lambda\}|$. But

$$\begin{aligned}\frac{1}{|I|}|\{t\in I\colon \varphi(t)\ge\lambda\}| &= \frac{1}{2}\Big(\frac{1}{|I_+|}|\{t\in I_+\colon \varphi(t)\ge\lambda\}| + \frac{1}{|I_-|}|\{t\in I_-\colon \varphi(t)\ge\lambda\}|\Big)\\ &= \frac{1}{2}\Big(\frac{1}{|I_+|}|\{t\in I_+\colon \varphi^+(t)\ge\lambda\}| + \frac{1}{|I_-|}|\{t\in I_-\colon \varphi^-(t)\ge\lambda\}|\Big)\,,\end{aligned}$$

which is at least $\frac{1}{2}\big(\mathbf{B}^d(x_+) + \mathbf{B}^d(x_+)\big) - \eta$. We get (4.1) by choosing η as small as we wish.

The same consideration for $\mathbf{B}$ instead of $\mathbf{B}^d$ hits the following difficulty: we cannot readily say that φ built of two functions $\varphi^\pm$ is in $\mathrm{BMO}(I)$ and has norm at most ε. To do that we would need to check the too many intervals, namely, all intervals containing the center of interval I inside them.

Our function $\mathbf{B}$ does not satisfy (4.1) for all three points $x^{\pm}$ and $x = \frac{1}{2}(x^+ + x^-)$ lying in Ω_ε. However (4.1) is true for $\mathbf{B}$ if the entire straight line segment $[x^-, x^+]$ is in Ω_ε. This means that our function is locally concave, i.e., it is concave in every convex subset of Ω_ε. Even though there is no formal justification of this property, we will see that this is exactly what happens. For the smooth enough functions, the concavity condition can be rewritten in the differential form:

$$\frac{d^2\mathbf{B}}{dx^2} \overset{\text{def}}{=} \begin{pmatrix} \mathbf{B}_{x_1x_1} & \mathbf{B}_{x_1x_2} \\ \mathbf{B}_{x_2x_1} & \mathbf{B}_{x_2x_2} \end{pmatrix} \le 0\,, \qquad \forall x \in \Omega\,, \tag{4.2}$$

where by $\mathbf{B}_{x_ix_j}$ we denote the partial derivatives $\dfrac{\partial^2\mathbf{B}}{\partial x_i \partial x_j}$.

Proposition (Boundary condition)**.**

$$\mathbf{B}(x_1, x_1^2) = \begin{cases} 1 & \textit{if } x_1 \ge \lambda\,, \\ 0 & \textit{if } x_1 < \lambda\,. \end{cases} \tag{4.3}$$

This is obvious, because at every point of the boundary, where $x_2 = x_1^2$, we can have inly one test function, and this test function is a constant function $\varphi(t) = x_1$.

If we look for a sharp estimate we need to choose the minimal possible function from the class of concave functions satisfying this boundary condition, which must be the Bellman function $\mathbf{B}$. This function, "a candidate in the true Bellman function" will be denoted by the usual letter B. For every point $x \in \Omega$ there exists an extremal function φ (or "almost extremal" function φ_n, i.e., a sequence of functions) realizing the supremum in the definition (2.3). The usual procedure of using the Bellman function consists in the consecutive application of (6.1), when splitting the interval I, where a test function is defined. For the extremal test function there has to be no loss in such procedure, therefore, for the Bellman function the equality has to occur at least for one splitting the point x into a pair $\{x^+,\, x^-\}$. For a concave function this means that it is linear at some direction. If we have almost extremal functions, i.e., an extremal sequence, then we must have "almost equality" in (6.1), at least up to the terms of second order. In any case this means that the Hessian matrix (4.2) has to be degenerated. Thus, we are looking for the "best" B, on the top of this condition of negativity of Hessian we will impose the following degeneration condition:

$$\forall x \in \Omega\ \exists \Theta(x) \in \mathbb{R}^2 \setminus \{0\}\colon\ \Big(\frac{d^2B}{dx^2}(x)\Theta(x), \Theta(x)\Big) = 0\,. \tag{4.4}$$

Since the matrix $\frac{d^2B}{dx^2}$ is negatively defined, we conclude from (4.4) the following degeneration condition on the Hessian:

$$\det\begin{pmatrix} B_{x_1x_1} & B_{x_1x_2} \\ B_{x_2x_1} & B_{x_2x_2} \end{pmatrix} = 0\,, \quad \forall x \in \Omega\,. \tag{4.5}$$

5. Foliation

Our heuristic tells us that to find $\mathbf{B}$ we need to guess the right foliations of Ω. Actually this is rather difficult. Originaly it was made "by hand" using some heuristic arguments. But now it is clear how to build such a foliation for almost arbitrary boundary condition. Explanation how to do this and a general approach to this geometric problem can be found in [2], [3]. In these papers quite general ruling surfaces with given boundary curve are built.

6. Proofs of the theorems

Let us show that it is sufficient to prove Theorem 2 only for $\mathbf{B}_{\max}$, then we get the lower Bellman function automatically. Indeed, since $\mathbf{B}_{\max}$ is a continuous function in λ for any fixed x except one point on the lower boundary (i.e. $x_2 > x_1^2$), for any such x and any $\eta > 0$ we have:

$$|\{s \in I \colon \varphi(s) \geq \lambda + \eta\}| \leq |\{s \in I \colon \varphi(s) > \lambda\}| \leq |\{s \in I \colon \varphi(s) \geq \lambda\}|\,,$$

which yields

$$\mathbf{B}(x; \lambda + \eta) \leq \sup\big\{|\{s \in I \colon \varphi(s) > \lambda\}| \colon \varphi \in S_\varepsilon(x)\big\} \leq \mathbf{B}(x; \lambda)\,.$$

Therefore, the Bellman function for the strict inequality in the definition is the same as the Bellman function for the non strict inequality, except one point on the boundary $x = (\lambda, \lambda^2)$, where we know the Bellman function from the beginning, because for the points of the lower boundary the set $S_\varepsilon(x)$ consists of only the constant test function $\varphi = x_1 = \lambda$.

At the point $x = (\lambda, \lambda^2)$, where both Bellman function are equal to 1, $\mathbf{B}_{\max}(x) = \mathbf{B}_{\min}(x) = 1$. At all other points we have the following relation

$$\mathbf{B}_{\min}(x_1, x_2; \lambda) = 1 - \mathbf{B}_{\max}(-x_1, x_2; -\lambda).$$

Indeed,

$$\begin{aligned} \mathbf{B}_{\min}(x_1, x_2; \lambda) &= \frac{1}{|J|} \inf\big\{|\{s \in J \colon \varphi(s) \geq \lambda\}| \colon \varphi \in S_\varepsilon(x)\big\} \\ &= 1 - \frac{1}{|J|} \sup\big\{|\{s \in J \colon \varphi(s) < \lambda\}| \colon \varphi \in S_\varepsilon(x)\big\} \\ &= 1 - \frac{1}{|J|} \sup\big\{|\{s \in J \colon -\varphi(s) > -\lambda\}| \colon \varphi \in S_\varepsilon(x)\big\} \\ &= 1 - \mathbf{B}_{\max}(-x_1, x_2; -\lambda)\,. \end{aligned}$$

Using this relation we obtain (2.9) from (2.8).

When proving Theorem 1 we denote by B the function from the right-hand side of either (2.5), or (2.6), or (2.7), depending on the relation between λ and ε, and B will be the function from the right-hand side of (2.8) in the proof of Theorem 2. In any case B will be a candidate for the role of the Bellman function, and to prove the theorem we need in each case to check two inequalities for the corresponding pair $\mathbf{B}$ and B: $\mathbf{B}(x) \le B(x)$ and $\mathbf{B}(x) \ge B(x)$ for every point $x \in \Omega_\varepsilon$.

To prove the upper estimate, we need, first, the local concavity of the function B:

$$B(\alpha_+ x^+ + \alpha_- x^-) \ge \alpha_+ B(x^+) + \alpha_- B(x^-)\,, \quad \alpha_\pm > 0,\ \alpha_+ + \alpha_- = 1, \tag{6.1}$$

for any pair $x^\pm \in \Omega_\varepsilon$ such that the whole straight line segment $[x^-, x^+]$ is in Ω_ε, and, second, the following splitting lemma that can be found in [17] or [14]:

Lemma 3 (Splitting lemma)**.** *Fix two positive numbers ε, δ, with $\varepsilon < \delta$. For an arbitrary interval I and any function $\varphi \in \mathrm{BMO}_\varepsilon(I)$, there exists a splitting $I = I_+ \cup I_-$ such that the whole straight line segment $[x^{I_-}, x^{I_+}]$ is inside Ω_δ. Moreover, the parameters of splitting $\alpha_\pm \stackrel{\mathrm{def}}{=} |I_\pm|/|I|$ are separated form 0 and 1 by constants depending on ε and δ only, i.e. uniformly with respect to the choice of I and φ.*

Here the following notation was used: for a function $\varphi \in \mathrm{BMO}_\varepsilon(J)$ and a subinterval $I \subset J$ we define a *Bellman point* $x^I \stackrel{\mathrm{def}}{=} (\langle\varphi\rangle_I, \langle\varphi^2\rangle_I)$ in the domain Ω_ε.

Using this lemma we prove the following result.

Lemma 4. *Let G be a locally concave bounded function on Ω_δ, $\delta > \varepsilon$, and continuous at almost every point of the lower boundary of Ω. Let E be a measurable subset of $\mathbb{R}$. If the function G satisfies the following boundary condition*

$$G(x_1, x_1^2) = \begin{cases} 1, & \text{if } x_1 \in E; \\ 0, & \text{if } x_1 \notin E, \end{cases} \tag{6.2}$$

then

$$\frac{1}{|I|}|\{s\colon \varphi(s) \in E\}| \le G(x)$$

for all $\varphi \in S_\varepsilon(x)$.

We will use this lemma to prove the theorem putting $G(x) = B(x; \lambda, \delta)$ and then, using continuity of $B(x; \lambda, \delta)$ in δ, we pass to the limit $\delta \to \varepsilon$. In such a way we get the upper estimate

$$\mathbf{B}(x; \lambda, \varepsilon) \le B(x; \lambda, \varepsilon).$$

Proof of Lemma 4. Procedure of the proof is standard, as in [17] or [14]: we apply repeatedly main inequality (6.1) each time splitting the interval according to Lemma 3.

Fix a function $\varphi \in S_\varepsilon(x)$. By the splitting lemma we can split every subinterval $I \subset J$, in such a way that the segment $[x^{I_-}, x^{I_+}]$ is inside Ω_δ. Since G is locally concave, we have (we drop temporarily parameter δ)

$$|I| G(x^I) \ge |I_+| G(x^{I_+}) + |I_-| G(x^{I_-})$$

for any such splitting. Repeating this procedure n times we get 2^n subintervals of n-th generation (this set of intervals we denote by $\mathcal{D}_n$). So, we can write the following chain of inequalities:

$$|J|G(x^J) \geq |J_+|G(x^{J_+}) + |J_-|G(x^{J_-}) \geq \sum_{I\in\mathcal{D}_n} |I|G(x^I) = \int_J G(x^{(n)}(s))\,ds\,,$$

where $x^{(n)}(s) = x^I$, when $s \in I$, $I \in \mathcal{D}_n$. By the Lebesgue differentiation theorem we have $x^{(n)}(s) \to (\varphi(s), \varphi^2(s))$ almost everywhere. (We have used here the fact that we split the intervals so that all coefficients $\alpha_\pm$ are uniformly separated from 0 and 1, and, therefore, $\max\{|I|\colon I \in \mathcal{D}_n\} \to 0$ as $n \to \infty$.) Since G is bounded, we can pass to the limit in this inequality by the Lebesgue dominated convergence theorem. Using the boundary condition (6.2) we obtain:

$$|J|G(x^J) \geq \int_J G(\varphi(s), \varphi^2(s))\,ds = \int_{\{s\colon\,\varphi(s)\in E\}} ds = |\{s\colon \varphi(s) \in E\}|\,.$$

Dividing the obtained inequality by $|J|$, we come to the desired inequality. □

To complete proving the upper estimate $\mathbf{B} \leq B$ both in Theorems 1 and 2 we need to check local concavity of the functions B defined by (2.5), (2.6), (2.7), and (2.8).

Let us check the most difficult case (2.7). In all other cases the consideration is analogous.

$$\frac{\partial B}{\partial x_1} = \begin{cases} 0\,, & x \in \Omega_1\,, \\ \dfrac{\lambda+\varepsilon}{4\varepsilon^2}\operatorname{sign} x_1\,, & x \in \Omega_2\,, \\ \dfrac{2(x_2-\lambda|x_1|)(\lambda-|x_1|)}{(x_2+\lambda^2-2\lambda|x_1|)^2}\operatorname{sign} x_1\,, & x \in \Omega_3\,, \\ \dfrac{e}{2}\cdot\dfrac{\varepsilon-|x_1|-\sqrt{\varepsilon^2-x_2+x_1^2}}{\varepsilon^2}\exp\left\{\dfrac{|x_1|-\lambda}{\varepsilon}+\sqrt{1-\dfrac{x_2-x_1^2}{\varepsilon^2}}\right\}\operatorname{sign} x_1, & x \in \Omega_4\,, \\ 0, & x \in \Omega_5\,; \end{cases}$$

$$\frac{\partial B}{\partial x_2}=\begin{cases} 0\,, & x\in\Omega_1\,,\\ -\dfrac{1}{8\varepsilon^2}\,, & x\in\Omega_2\,,\\ \Big(\dfrac{|x_1|-\lambda}{x_2+\lambda^2-2\lambda|x_1|}\Big)^2\,, & x\in\Omega_3\,,\\ \dfrac{e}{4\varepsilon^2}\exp\left\{\dfrac{|x_1|-\lambda}{\varepsilon}+\sqrt{1-\dfrac{x_2-x_1^2}{\varepsilon^2}}\right\}, & x\in\Omega_4\,,\\ \dfrac{1}{4\varepsilon^2}\exp\left\{2-\dfrac{\lambda}{\varepsilon}\right\}, & x\in\Omega_5\,. \end{cases} \tag{6.3}$$

We see that the function B is C^1-smooth on the boundaries $\Omega_5\cap\Omega_4$, where

$$B_{x_1}=0\,,\qquad B_{x_2}=\frac{1}{4\varepsilon^2}\exp\big\{2-\frac{\lambda}{\varepsilon}\big\}\,,$$

and on $\Omega_4\cap\Omega_3$, where

$$B_{x_1}=-\frac{\lambda-2\varepsilon}{2\varepsilon^2}\,,\qquad B_{x_2}=\frac{1}{4\varepsilon^2}\,.$$

On the boundary of Ω_2 the first derivatives have jumps of the needed signs to keep concavity of B. First of all, we note that it is sufficient to consider a jump along any direction transversal to the boundary, because along the boundary our functions coincide and their derivatives coincide as well. (By the way, to check C^1-smoothness of B on the boundary of Ω_4, it was sufficient to verify the continuity of any partial derivatives, another one would be continuous automatically.) We check the value of jumps of B_{x_2}, because this direction is transversal to the boundary for any ε. According to (6.3), on Ω_2 the derivative B_{x_2} is strictly negative and on Ω_1 and Ω_3 it is nonnegative, therefore B_{x_2} monotonously decreases in x_2, as we need. To prove local concavity of B everywhere, it remains to check that the Hessian matrix

$$\frac{d^2B}{dx^2}=\begin{pmatrix}B_{x_1x_1} & B_{x_1x_2}\\ B_{x_2x_1} & B_{x_2x_2}\end{pmatrix}$$

is non-positive. On $\Omega_1\cup\Omega_2\cup\Omega_5$ the function is linear, and therefore there is nothing to check. On Ω_3 we have

$$\frac{d^2B}{dx^2}=\begin{pmatrix} -\dfrac{2(\lambda^2-x_2)^2}{(x_2+\lambda^2-2\lambda|x_1|)^3} & \dfrac{2(\lambda^2-x_2)(\lambda-|x_1|)}{(x_2+\lambda^2-2\lambda|x_1|)^3}\operatorname{sign}x_1\\ \dfrac{2(\lambda^2-x_2)(\lambda-|x_1|)}{(x_2+\lambda^2-2\lambda|x_1|)^3}\operatorname{sign}x_1 & -\dfrac{2(\lambda-|x_1|)^2}{(x_2+\lambda^2-2\lambda|x_1|)^3}\end{pmatrix}\le 0\,,$$

and on Ω_4

$$\frac{d^2B}{dx^2}=\frac{e^{1+r-\frac{\lambda}{\varepsilon}}}{8\varepsilon^3\sqrt{\varepsilon^2-x_2+x_1^2}}\begin{pmatrix}-4\varepsilon^2r^2 & 2\varepsilon r\\ 2\varepsilon r & -1\end{pmatrix}\le 0\,,$$

where $r = \frac{1}{\varepsilon}\big(|x_1| + \sqrt{\varepsilon^2 - x_2 + x_1^2}\,\big)$.

In a similar way it is possible to check local concavity of the functions B defined by (2.5), (2.6), and (2.8). Thus we completed the proof of the upper estimate $\mathbf{B} \le B$ both in Theorems 1 and 2.

To prove the converse inequality we construct extremal test functions (*extremizers*) realizing supremum in the definition of the Bellman function. Again, we restrict ourself by the consideration of the most difficult case (2.7) only. Moreover, it is sufficient to consider only the points with $x_1 \ge 0$, because if φ is an extremizer for a point (x_1, x_2), then the function $-\varphi$ is an extremizer for the point $(-x_1, x_2)$.

Any point of Ω_1 can be represented as a convex combination of the points of the boundary, where $|x_1| \ge \lambda$, i.e. $B(x) \ge 1$. Therefore, the corresponding extremizer can be constructed as a step function consisting of two constants. Namely, for an arbitrary $x \in \Omega_1$ we draw the tangent line to the upper boundary so that the tangent point is to the right from x. First coordinates of two points of intersection of that tangent line with the lower boundary are $u^\pm = x_1 \pm \varepsilon + \sqrt{\varepsilon^2 - x_2 + x_1^2}$, and the corresponding extremizer is

$$\varphi(t) = \begin{cases} u^-, & \text{if } 0 < t < \frac{u^+ - x_1}{2\varepsilon}, \\ u^+, & \text{if } \frac{u^+ - x_1}{2\varepsilon} < t < 1. \end{cases}$$

By direct calculation we check that $(\langle\varphi\rangle_{[0,1]}, \langle\varphi^2\rangle_{[0,1]}) = x$ and $\varphi \ge \lambda$. First of all we note that

$$\begin{aligned} u^+ - u^- &= 2\varepsilon, \\ u^+ + u^- &= 2\Big(x_1 + \sqrt{\varepsilon^2 - x_2 + x_1^2}\Big), \\ u^+u^- &= \Big(x_1 + \sqrt{\varepsilon^2 - x_2 + x_1^2}\Big)^2 - \varepsilon^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle\varphi\rangle_{[0,1]} &= u^- \frac{u^+ - x_1}{2\varepsilon} + u^+\Big(1 - \frac{u^+ - x_1}{2\varepsilon}\Big) \\ &= u^+ - \frac{(u^+ - x_1)(u^+ - u^-)}{2\varepsilon} = x_1, \\ \langle\varphi^2\rangle_{[0,1]} &= (u^-)^2 \frac{u^+ - x_1}{2\varepsilon} + (u^+)^2\Big(1 - \frac{u^+ - x_1}{2\varepsilon}\Big) \\ &= (u^+)^2 - \frac{(u^+ - x_1)(u^+ - u^-)(u^+ + u^-)}{2\varepsilon} = x_1(u^+ + u^-) - u^+u^- \\ &= 2x_1\Big(x_1 + \sqrt{\varepsilon^2 - x_2 + x_1^2}\Big) - \Big(x_1 + \sqrt{\varepsilon^2 - x_2 + x_1^2}\Big)^2 + \varepsilon^2 = x_2. \end{aligned}$$

To prove that $\varphi \ge \lambda$ we need to check that $u_- \ge \lambda$. If $x_1 \ge \lambda + \varepsilon$, then everything is trivial:

$$u_- \ge x_1 - \varepsilon \ge \lambda.$$

If $x_1 < \lambda + \varepsilon$, then the second coordinate of a point x from Ω_1^+ satisfies the following additional condition $x_2 \le 2(\lambda+\varepsilon)x_1 - \lambda^2 - 2\varepsilon\lambda$. Therefore,

$$\varepsilon^2 - x_2 + x_1^2 \ge \varepsilon^2 + x_1^2 - 2(\lambda+\varepsilon)x_1 + \lambda^2 + 2\varepsilon\lambda = (\lambda+\varepsilon-x_1)^2,$$

and hence,

$$u_- \ge x_1 - \varepsilon + |\lambda+\varepsilon-x_1| = \lambda.$$

What we need more to check is the fact that the BMO-norm of our extremizer does not exceed ε. In fact it is equal to ε, since the BMO-norm of any step function consisting of two steps is equal to the half of the jump and in our case $u^+ - u^- = 2\varepsilon$. So, we have proved that $\mathbf{B} \ge 1$ in Ω_1.

Now, we consider a point x from Ω_3^+. A similar step function consisting of two steps will be an extremizer here. We have to draw a straight line through the points x and (λ,λ^2). It intersects the lower boundary in one more point with the first coordinate $u = \frac{\lambda x_1 - x_2}{\lambda - x_1}$. We take a step function consisting of steps λ and u:

$$\varphi(t) = \begin{cases} \lambda, & \text{if } 0 < t < a, \\ u, & \text{if } a < t < 1, \end{cases}$$

where $a = \frac{x_2 - x_1^2}{x_2 + \lambda^2 - 2\lambda x_1}$. By direct calculation we can check that

$$\langle \varphi \rangle_{[0,1]} = \lambda a + u(1-a) = x_1,$$
$$\langle \varphi^2 \rangle_{[0,1]} = \lambda^2 a + u^2(1-a) = x_2.$$

The fact that $\varphi \in \mathrm{BMO}_\varepsilon$ is geometrically clear, because a Bellman point corresponding to φ and any subinterval of $[0,1]$ is in Ω_3. However this is easy to check formally as well. The jump is

$$\lambda - u = \lambda - x_1 + \frac{x_2 - x_1^2}{\lambda - x_1}\,.$$

Since $x_2 \le 2(\lambda-\varepsilon)x_1 - \lambda^2 + 2\lambda\varepsilon$ for $x \in \Omega_3^+$, we have

$$x_2 - x_1^2 \le (\lambda - x_1)(2\varepsilon - \lambda + x_1),$$

and hence $\lambda - u \le 2\varepsilon$. So, we conclude that

$$\mathbf{B} \ge a = \frac{x_2 - x_1^2}{x_2 + \lambda^2 - 2\lambda x_1}\,.$$

To consider a point $x \in \Omega_2^+$ we note that this point is a convex combination of three point Λ and $\Lambda^\pm$ on the lower boundary with the first coordinates λ and $\lambda \pm 2\varepsilon$ respectively. As a result we construct an extremizer as a step function consisting of these three steps:

$$\varphi(t) = \begin{cases} \lambda - 2\varepsilon, & \text{if } 0 < t < a, \\ \lambda, & \text{if } a < t < b, \\ \lambda + 2\varepsilon, & \text{if } b < t < 1. \end{cases}$$

For φ to be a test function corresponding the point x (i.e. for $\langle\varphi\rangle_{[0,1]} = x_1$ and $\langle\varphi^2\rangle_{[0,1]} = x_2$) we need to take

$$a = \frac{x_2 + \lambda^2 - 2\lambda x_1 - 2\varepsilon(x_1 - \lambda)}{8\varepsilon^2}$$

and

$$b = 1 - \frac{x_2 + \lambda^2 - 2\lambda x_1 + 2\varepsilon(x_1 - \lambda)}{8\varepsilon^2}\,.$$

The easiest way to prove that $\varphi \in \mathrm{BMO}$ is the following geometric consideration. Take any straight line, say L, passing through x and not intersecting the upper parabola. Note that we need to consider the oscillation of φ only over intervals $[\alpha,\beta]$ containing $[a,b]$, because in other case φ would have on $[\alpha,\beta]$ only one jump of size 2ε, but as we know the BMO-norm of such step function is just ε. Our point x is a convex combination of three Bellman points $x^{[0,\alpha]}$, $x^{[\alpha,\beta]}$, and $x^{[\beta,1]}$. But since the points $x^{[0,\alpha]} = \Lambda^-$ and $x^{[\beta,1]} = \Lambda^+$ are above the line L, the point $x^{[\alpha,\beta]}$ has to be below this line and therefore in Ω_ε. This means just what we need that the oscillation over $[\alpha,\beta]$ does not exceed ε.

It remains to note that the measure of the set where $\varphi \ge \lambda$ is $1-a$, i.e. in Ω_2 we have

$$\mathbf{B} \ge 1 - a = 1 - \frac{x_2 + \lambda^2 - 2\lambda x_1 - 2\varepsilon(x_1 - \lambda)}{8\varepsilon^2}\,.$$

To get an extremizer for a point x on the intersection of the upper parabola with Ω_4^+ we need to concatenate the logarithmic function with the step function corresponding to the upper right corner of Ω_4^+, i.e. with the step function consisting of two steps of equal size with the values λ and $\lambda - 2\varepsilon$. For an arbitrary point $x \in \Omega_4^+$ we have cut the latter function from below on the corresponding level. As a result we get the following

$$\varphi(t) = \begin{cases} \lambda, & \text{if } 0 < t < a,\\ \lambda - 2\varepsilon, & \text{if } a < t < 2a,\\ \lambda - 2\varepsilon + \varepsilon\log\frac{2a}{t}, & \text{if } 2a < t < b,\\ \lambda - 2\varepsilon + \varepsilon\log\frac{2a}{b}, & \text{if } b < t < 1. \end{cases}$$

As in the previous case, we could write down two equations $\langle\varphi\rangle_{[0,1]} = x_1$ and $\langle\varphi^2\rangle_{[0,1]} = x_2$ and solving them to find the appropriate value of the parameters a and b. However it is easier to find a and b using other arguments and after that simply to check that the averages have the desired values. For this aim we consider splitting of the interval $[0,1]$ at the point b. In result we get two Bellman points $V = x^{[0,b]}$ and $U = x^{[b,1]}$. The point $U = (u, u^2)$ is on the lower boundary, it corresponds to the constant function $u = \lambda - 2\varepsilon + \varepsilon\log\frac{2a}{b}$. The point V has to be on the on the upper boundary and the segment $[U,V]$ has to be a segment of the extremal line passing through x, i.e. a segment of the tangent line to the upper parabola. (We mean here the extremal lines of the solution of the corresponding Monge–Ampère equation, which is lurking behind all our considerations.) But it is easy to calculate the coordinates of the points of intersection the tangent line to the upper parabola passing through the point x:

$$u = x_1 - \varepsilon + \sqrt{\varepsilon^2 - x^2 + x_1^2}\,,$$

whence

$$\log\frac{2a}{b} = 1 + \frac{x_1 - \lambda}{\varepsilon} + \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}}\,.$$

Furthermore, the length of the horizontal projection of $[U, V]$ is just ε, i.e. the splitting ratio is

$$b = \frac{x_1 - u}{\varepsilon} = 1 - \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}}\,,$$

and finally

$$a = \frac{e}{2}\left(1 - \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}}\right)\exp\left\{\frac{x_1 - \lambda}{\varepsilon} + \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}}\right\}.$$

We omit verification that for this parameters a and b averages of φ and φ^2 have the prescribed values. To finish our proof of the desired estimate

$$\mathbf{B}(x) \ge a$$

for any $x \in \Omega_4$, it remains to verify that the norm of our test function φ does not exceed ε. Again this verification will be geometric. Consider the following curve in Ω_ε built by using φ mentioned above:

$$\psi(t) = x^{[0,t]}, \qquad t \in [0, 1].$$

For $t \in [0, a]$ the point $\psi(t)$ stands at $\Lambda = (\lambda, \lambda^2)$. At the moment $t = a$ it starts to move to the left along the tangent line to the upper boundary. At the moment $t = 2a$ it reaches the upper parabola and continue its movement along this upper boundary till the point V. It reaches V at the moment $t = b$ and then continues along $[U, V]$. The destination point is $\psi(1) = x$. Note that this curve is convex. Take now an arbitrary subinterval $[\alpha, \beta] \subset [0, 1]$ and draw a straight line L passing through $\psi(\beta)$ and tangent to our curve ψ (i.e. tangent to the upper parabola). Since ψ is concave, the point $\psi(\alpha) = x^{[0,\alpha]}$ is above L (more precisely, not below L). And we conclude that the point $x^{[\alpha,\beta]}$ has to be below L (more precisely, not above L), because the point $\psi(\beta)$ (on L) is a convex combination of the point $\psi(\alpha)$ (above L) and $x^{[\alpha,\beta]}$. Therefore, the latter point is in Ω_ε, i.e. the oscillation of φ over this interval does not exceed ε.

Finally, we have to consider the most difficult case $x \in \Omega_5$. We shall proceed as in the triangle domain Ω_2^+. Arbitrary point of Ω_5 is a convex combination of three points: the origin and $E^\pm = (\pm\varepsilon, 2\varepsilon^2)$. Since $E^\pm \in \Omega_4^\pm$, we already know the extremizers for these points, but for the origin there is the only test function, namely, the constant zero function. We concatenate these three function in the proper order (to get a monotonous function in result). This will be the desired extremizer:

$$\varphi(t)=\begin{cases} -\lambda, & \text{if } 0<t<a_-,\\ -\lambda+2\varepsilon, & \text{if } a_-<t<2a_-,\\ \varepsilon\log\frac{t}{2a_-}-\lambda+2\varepsilon, & \text{if } 2a_-<t<b_-,\\ 0, & \text{if } b_-<t<1-b_+,\\ \varepsilon\log\frac{2a_+}{1-t}+\lambda-2\varepsilon, & \text{if } 1-b_+<t<1-2a_+,\\ \lambda-2\varepsilon, & \text{if } 1-2a_+<t<1-a_+,\\ \lambda, & \text{if } 1-a_+<t<1. \end{cases}$$

The continuity of φ at the points $t=b_-$ and $t=1-b_+$ yields

$$\frac{b_-}{2a_-}=\frac{b_+}{2a_+}=\exp\big(\frac{\lambda}{\varepsilon}-2\big).$$

From the representation

$$x=b_-E^-+b_+E^++(1-b_--b_+)\mathbf{0}$$

we get two equations for $b_\pm$:

$$x_1=-\varepsilon b_-+\varepsilon b_+,$$
$$x_2=2\varepsilon^2b_-+2\varepsilon^2b_+,$$

whence

$$b_\pm=\frac{x_2\pm2\varepsilon x_1}{4\varepsilon^2}\,,$$

and therefore,

$$a_\pm=\frac{1}{2}b_\pm\exp\big(2-\frac{\lambda}{\varepsilon}\big)=\frac{x_2\pm2\varepsilon x_1}{8\varepsilon^2}\exp\big(2-\frac{\lambda}{\varepsilon}\big)\,.$$

Again we omit verification that $\langle\varphi\rangle_{[0,1]}=x_1$ and $\langle\varphi^2\rangle_{[0,1]}=x_2$, we only say few words how to check that the norm of φ does not exceed ε. We shall proceed as in the triangle domain Ω_2^+. Take any straight line L passing through x and not intersecting the upper parabola. Note that we need to consider the oscillation of φ only over intervals $[\alpha,\beta]$ containing $[b_-,1-b_+]$, because in other case φ on $[\alpha,\beta]$ is a part of test function considered for the domain Ω_4. Our point x is a convex combination of three Bellman points $x^{[0,\alpha]}$, $x^{[\alpha,\beta]}$, and $x^{[\beta,1]}$. It is clear that the points $x^{[0,\alpha]}$ and $x^{[\beta,1]}$ are above the line L (they are somewhere on the left and right curves considered for the points from $\Omega^4_\pm$). Therefore, the point $x^{[\alpha,\beta]}$ has to be below the line L, i.e. in Ω_ε. This means just what we need that the oscillation over $[\alpha,\beta]$ does not exceed ε.

It remains to note that the measure of the set where $\varphi\ge\lambda$ is a_+ and the measure of the set where $\varphi\le-\lambda$ is a_-, i.e. in Ω_5 we have

$$\mathbf{B}\ge a_-+a_+=\frac{x_2}{4\varepsilon^2}\exp\big(2-\frac{\lambda}{\varepsilon}\big)\,.$$

This completes the proof of formula (2.7). Extremizers for all other cases of Theorem 1 and Theorem 2 are absolutely similar to those just built. □

7. How to find the expression for the Bellman function and formulas for extremizers

The theorems presented in this paper were proved in 2006, when the problem of finding a Bellman function was a kind of art. Using some heuristic arguments the whole domain was splitting in several subdomains, thereafter the corresponding boundary value problem for the homogeneous Monge–Ampère equation was solved. The solutions were glued together continuously to get a locally convex function in the entire domain. After that, using known foliation of the domain by the extremal lines of the solution of the Monge–Ampr̀e equation, the extremizers were constructed for every point of the domain. We refer the reader to two papers [2] and [15] for explanation of methods of solving Monge–Ampère equation in the parabolic strip, and to [19] for more general cases.

The same can be said about finding extremal test functions and especially about the proof that the found function has the desired BMO-norm. The geometric method of proving that the BMO-norm of the extremizers does not exceed ε first appeared in [15] for some special cases and then was generalized in [2], where the notion of *delivery curves* appeared. Traces of this notion the reader can see in the presented proof. We have to say that this part of the proof is modernized, not the original one. The calculation of the BMO-norms of extremizers in 2006 was made by the straightforward calculation. These were awful calculations, enormous amount of calculations. There were impossible to place them in any paper. That was one of the reasons why this result is prepared for publication six years after it was proved.

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