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MARIA CHARINA, COSTANZA CONTI, NICOLA GUGLIELMI, AND VLADIMIR PROTASOV

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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Non-Stationary Multivariate Subdivision: Joint Spectral Radius and Asymptotic Similarity

Maria Charina, Dortmund University of Technology, Germany
Costanza Conti, University of Firenze, Italy
Nicola Guglielmi, University of L'Aquila, Italy
Vladimir Protasov, Moscow State University, Russia

Abstract

In this paper we study scalar multivariate non-stationary subdivision schemes with a general integer dilation matrix. We present a new numerically efficient method for checking convergence and Hölder regularity of such schemes. This method relies on the concepts of approximate sum rules, asymptotic similarity and the so-called joint spectral radius of a finite set of square matrices. The combination of these concepts allows us to employ recent advances in linear algebra for exact computation of the joint spectral radius that have had already a great impact on studies of stationary subdivision schemes. We also expose the limitations of non-stationary schemes in their capability to reproduce and generate certain function spaces. We illustrate our results with several examples.

Keywords: multivariate non-stationary subdivision schemes, approximate sum rules, asymptotic similarity, joint spectral radius.

Classification (MSCS): 65D17, 15A60, 39A99

1 Introduction

In this paper we study multivariate non-stationary subdivision schemes with a general dilation matrix $M \in \mathbb{Z}^{s\times s}$ all of whose eigenvalues are in absolute value larger than 1, i.e. $\rho(M^{-1}) < 1$.

Subdivision schemes are efficient iterative methods for generating smooth surfaces or functions from a given initial set of data $\boldsymbol{c}^{(1)} := \{c^{(1)}(\alpha),\ \alpha \in \mathbb{Z}^s\}$ by means of local refinement with rules stored in $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$. The subdivision operators $S_{\mathbf{a}^{(k)}} : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ are linear and, at each step of the subdivision recursion, the refinement rules define a map from the coarser sequence $\boldsymbol{c}^{(k)} \in \ell(\mathbb{Z}^s)$ to the denser finer sequence $\boldsymbol{c}^{(k+1)} \in \ell(\mathbb{Z}^s)$ via

$$\boldsymbol{c}^{(k+1)} := S_{\mathbf{a}^{(k)}}\boldsymbol{c}^{(k)}, \quad S_{\mathbf{a}^{(k)}}\boldsymbol{c}^{(k)}(\alpha) := \sum_{\beta\in\mathbb{Z}^s} \mathrm{a}^{(k)}(\alpha - M\beta)c^{(k)}(\beta), \quad k \geq 1, \quad \alpha \in \mathbb{Z}^s. \tag{1}$$

A subdivision scheme whose refinement rules are level dependent is said to be *non-stationary* (see, [14], for example), or, sometimes, *non-homogeneous* (see, [3], for example). In this paper we opt for using the term non-stationary and, therefore, denote by *stationary* a scheme whose refinement rules are level independent (the scheme is therefore specified by a unique set of refinement coefficients $\mathbf{a}^{(k)} = \mathbf{a} = \{\mathrm{a}(\alpha),\ \alpha \in \mathbb{Z}^s\}$ for all $k \geq 1$). Moreover, we consider non-stationary subdivision schemes $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ whose *subdivision masks* - the sets of refinement coefficients $\mathbf{a}^{(k)} = \{\mathrm{a}^{(k)}(\alpha),\ \alpha \in \mathbb{Z}^s\}$, i.e. the sequence of real numbers indexed by $\mathbb{Z}^s$, - are each supported in $\{0, \ldots, N\}^s$, $N \in \mathbb{N}$.

A notion of convergence for the non-stationary scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ is established using the sequence $\{F^{(k)},\ k \geq 1\}$ of continuous functions $F^{(k)}$ that interpolate the data $\boldsymbol{c}^{(k)}$ at the parameter values $M^{-k}\alpha,\ \alpha \in \mathbb{Z}^s$, namely

$$F^{(k)}(M^{-k}\alpha) = c^{(k)}(\alpha), \quad \alpha \in \mathbb{Z}^s, \quad k \geq 1. \tag{2}$$

Definition 1. *The scheme* $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ *applied to the initial data* $\mathbf{c}^{(1)} \in \ell(\mathbb{Z}^s)$ *is called* convergent*, if there exists the* limit function $g_{\mathbf{c}^{(1)}} \in C(\mathbb{R}^s)$ *(which is nonzero for at least one initial nonzero sequence* $\mathbf{c}^{(1)}$*) such that the sequence* $\{F^{(k)},\ k \geq 1\}$ *in* (2) *converges to* $g_{\mathbf{c}^{(1)}}$*, i.e.*

$$g_{\mathbf{c}^{(1)}} := \lim_{k\to\infty} S_{\mathbf{a}^{(k)}}S_{\mathbf{a}^{(k-1)}} \cdots S_{\mathbf{a}^{(1)}}\boldsymbol{c}^{(1)} = \lim_{k\to\infty} F^{(k)}. \tag{3}$$

We call the scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ $C^\ell-$convergent, $\ell \geq 0$, *if* $g_{\mathbf{c}^{(1)}} \in C^\ell(\mathbb{R}^s)$.

Let $\delta_{\mathbf{0},\alpha}$, $\alpha \in \mathbb{Z}^s$, be the Kroneker delta symbol, i.e. $\delta_{\mathbf{0},\mathbf{0}} = 1$ and zero otherwise. An interesting fact about convergent schemes is that the compactly supported *basic limit functions* ϕ_k obtained from the initial sequence $\boldsymbol{\delta} := \{\delta(\alpha) = \delta_{0,\alpha},\ \alpha \in \mathbb{Z}^s\}$, when starting from the mask at level k,

$$\phi_k := \lim_{\ell\to\infty} S_{\mathbf{a}^{(k+\ell)}}S_{\mathbf{a}^{(k+1)}} \cdots S_{\mathbf{a}^{(k)}}\boldsymbol{\delta}, \quad k \in \mathbb{N},$$

are mutually refinable, i.e., they satisfy the functional equations

$$\phi_k = \sum_{\alpha \in \mathbb{Z}^s} \mathrm{a}^{(k)}(\alpha)\phi_{k+1}(M \cdot -\alpha), \quad k \in \mathbb{N}. \tag{4}$$

The popularity of stationary and non-stationary schemes is due to their applications in geometric modeling [3, 14], non-stationary multiresolution analysis [2, 9, 17, 23], isogeometric analysis [25]. From 1995 and up to recently, convergence and regularity of non-stationary subdivision schemes have been usually determined through a comparison to the so-called asymptotically equivalent stationary scheme, see [15] and its univariate generalization in [16].

Let $\ell \geq 0$. We recall that $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ and a stationary scheme $S_{\mathbf{a}}$ are called *asymptotically equivalent (of order ℓ)*, if they satisfy

$$\sum_{k=1}^{\infty} m^{k\ell/s} \|S_{\mathbf{a}^{(k)}} - S_{\mathbf{a}}\|_\infty < \infty, \quad \text{where} \quad \|S_{\mathbf{a}^{(k)}}\|_\infty := \max_{\varepsilon \in E} \left\{ \sum_{\alpha \in \mathbb{Z}^s} |\mathrm{a}^{(k)}(M\alpha + \varepsilon)| \right\}, \tag{5}$$

where the set $E := \{\varepsilon_0, \dots, \varepsilon_{m-1}\}$ is a set of representatives of $\mathbb{Z}^s/M\mathbb{Z}^s$, $m := |\det(M)|$ and $\varepsilon_0 = \mathbf{0} := (0, \dots, 0)$.

The main contribution of this paper is a new numerically efficient method for checking convergence and Hölder regularity of non-stationary schemes. The importance of this method is that it helps to enrich the family of existing non-stationary schemes and, thus, the class of functions generated by such subdivision schemes. The method we propose, see section 3, relies on the concepts of approximate sum rules, asymptotic similarity and the joint spectral radius of a bounded set of square matrices [12, 29].

Definition 2. *The joint spectral radius (JSR) of a finite collection of square matrices $\mathcal{A} := \{A_j\ :\ j = 1, \dots, J\}$, $J \in \mathbb{N}$, is defined by*

$$\rho(\mathcal{A}) := \lim_{n \to \infty} \max_{j_\ell = 1, \dots, J} \left\| \prod_{\ell=1}^{n} A_{j_\ell} \right\|^{1/n}.$$

Note that this definition of $\rho(\mathcal{A})$ is independent of the choice of the matrix norm $\|\cdot\|$. Note also that for a bounded - but not finite - set of matrices this definition is also applicable after replacing lim max by lim sup. In the stationary case, it is well-known that the Hölder regularity of the subdivision limits, as well as the rate of convergence of the corresponding subdivision scheme, are given explicitly in terms of JSR of special matrices constituting $\mathcal{A}$ (so-called *transition matrices*) restricted to their common invariant subspace, see [28] and references therein. In the non-stationary setting, the concept of the joint spectral radius is not directly applicable and has no straightforward generalization as it heavily relies on certain polynomial reproduction property of subdivision, which determines this invariant subspace. Indeed, in the non-stationary case, the set $\mathcal{A}$ is replaced by a sequence of sets whose matrix elements may have no common invariant subspaces at all. The latter phenomenon creates the main obstacle for the generalization of the matrix techniques to the non-stationary case.

To make our method work, we established a link between stationary and non-stationary settings by means of approximate sum rules and asymptotic similarity. These notions of approximate sum rules and asymptotic similarity further relax the sufficient criteria for regularity given in [8, 15, 16]. The new weaker sufficient conditions, we propose, are close to being necessary, see Section 4.1. Thus, already in the univariate binary case, our sufficient conditions for convergence and regularity of non-stationary schemes cannot be relaxed much further. An important intrinsic difference between stationary and non-stationary settings as well as the estimates of Hölder regularity are discussed in Section 3.3. We also expose the limitations of non-stationary schemes in their capability to reproduce and generate certain function spaces, see Section 4.2. In Sections 3.2 and 4.2, we illustrate our results with several examples. Note that, due to our assumption on the supports of the masks $\mathbf{a}^{(k)}$, one cannot expect that the non-stationary schemes whose properties can be analyzed using our method generate or reproduce functions of smaller support or higher smoothness than their asymptotic similar stationary counterparts.

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2 Background and preliminary definitions

In this section we recall basic important facts about subdivision and provide background and preliminary definitions and results.

We start by recalling that a stationary subdivision scheme $S_{\mathbf{a}}$, which is convergent to a C^ℓ limit, has a symbol $a_*(z) := \sum_{\alpha\in\mathbb{Z}^s} \mathrm{a}(\alpha) z^\alpha$, $z \in (\mathbb{C} \setminus 0)^s$, $z^\alpha := z^{\alpha_1} z^{\alpha_2} \cdots z^{\alpha_s}$, that satisfies the so-called *sum rules of order* $\ell+1$, see e.g. [3, 23], or reference therein. Define the set

$$\Xi = \{e^{2\pi i M^{-T}\xi} \ : \ \xi \text{ is a coset representative of } \mathbb{Z}^s/M^T\mathbb{Z}^s\}\,, \tag{6}$$

which contains $\mathbf{1} := (1,1,\ldots,1)$.

Definition 3. *The symbol* $a_*(z)$ *satisfies the* sum rules *of order* $\ell+1$ *if*

$$a_*(\mathbf{1}) = m, \quad \textit{and} \quad \max_{|\eta|\le \ell} \ \max_{\epsilon\in\Xi\setminus\{\mathbf{1}\}} |D^\eta a_*(\epsilon)| = 0\,. \tag{7}$$

In the above definition, D^η, $\eta\in\mathbb{N}_0^s$, denotes the $\eta-$th directional derivative.

In the stationary setting, it is well-known that the JSR of a suitably defined set of square matrices characterizes the subdivision convergence. To state the corresponding result in the non-stationary setting, we need to introduce some additional notation. By $\ell_0(\mathbb{Z}^s)$ we denote a space of finitely supported scalar sequences. For $\varepsilon\in E$, we denote by $\mathcal{A}_\varepsilon$ the linear operator on $\ell_0(\mathbb{Z}^s)$

$$\mathcal{A}_\varepsilon \boldsymbol{v} := \sum_{\beta\in\mathbb{Z}^s} v(\beta)\mathrm{a}(\varepsilon + M\cdot - \beta), \quad \boldsymbol{v}\in\ell_0(\mathbb{Z}^s), \tag{8}$$

derived from the subdivision mask $\mathbf{a}$. The linear operators $\mathcal{A}_\varepsilon$ constitute the finite collection $\mathcal{A} := \{\mathcal{A}_\varepsilon : \ \varepsilon\in E\}$. For a given finite set $K\subset\mathbb{Z}^s$, we denote by $\ell(K)\subset\ell_0(\mathbb{Z}^s)$ the linear space of all sequence supported in K.

Definition 4. *A set* $V\subset\ell(K)$ *is called admissible for* $\mathcal{A}$ *or* $\mathcal{A}$*-admissible, if* V *is invariant under all* $\mathcal{A}_\varepsilon$*,* $\varepsilon\in E$*, i.e., if* $\boldsymbol{v}\in V$*, then* $\mathcal{A}_\varepsilon\boldsymbol{v}\in V$ *for any* $\varepsilon\in E$*.*

If V is a finite dimensional subset of $\ell(K)$ which is $\mathcal{A}$-admissible, then the finite collection

$$\mathcal{A}|_V := \{\mathcal{A}_\varepsilon|_V \ : \ \varepsilon\in E\} \quad \text{of linear operators} \quad \mathcal{A}_\varepsilon|_V : V\to V,$$

has a corresponding finite collection of matrix representations (in a basis of V) denoted by

$$\mathcal{T}|_V := \{T_{\varepsilon,\mathbf{a}|_V} = [\mathrm{a}(\varepsilon + M\alpha - \beta)]_{\alpha,\beta\in K}\,|_V, \quad \varepsilon\in E\}. \tag{9}$$

By [6], for a given $\mathbf{a}\in\ell_0(\mathbb{Z}^s)$, one can choose $K = \sum_{r=1}^{\infty} M^{-r}G$, where G is given by

$$G := (\mathrm{supp}(\mathbf{a})\cup\{0\}) - E + \{-1,1\}^s = \{0,\ldots,N\}^s - E + \{-1,1\}^s.$$

Then, by e.g. [2, 6, 21, 22],

$$V := \{\boldsymbol{v} \in \ell(K) : \sum_{\alpha \in K} v(\alpha) = 0\}, \tag{10}$$

and the following characterization establishes the connection between the *convergence of stationary subdivision schemes* and the value of the *JSR* of $\mathcal{T}|_V$.

Proposition 1. *Let* $\mathbf{a} \in \ell_0(\mathbb{Z}^s)$ *be a subdivision mask. The subdivision scheme* $S_{\mathbf{a}}$ *is convergent if and only if* $\rho(\mathcal{T}|_V) < 1$, *for* V *defined in (10).*

The result of Proposition 1 has been extended for the C^ℓ-regularity analysis of stationary schemes, see e.g. [6, 4, 22]. In this case, one considers the restriction of $T_{\varepsilon,\mathbf{a}}$ to the finite dimensional invariant subspaces V_ℓ, $V_0 = V$, that "mimic" taking the derivatives of associated basic limit function, see e.g. [2, 22] for the structure of V_ℓ. Once the subspaces V_ℓ are properly defined the corresponding regularity result, see e.g. [6], states the following.

Proposition 2. *Let* $\mathbf{a} \in \ell_0(\mathbb{Z}^s)$ *be a subdivision mask. The subdivision scheme* $S_{\mathbf{a}}$ *is* $C^\ell-$*convergent if and only if* $\rho(\mathcal{T}|_{V_\ell}) < m^{-\ell/s}$, *for* V_ℓ *suitably defined.*

Although there is a multitude of results that deal with Hölder and Sobolev regularity of subdivision limits in both stationary and non-stationary settings, in this paper, we are interested in "extensions" of Propositions 1 and 2 to the multivariate non-stationary setting under milder sufficient conditions than in [8, 15, 16]. For this purpose we note that, due to non-stationarity, instead of one subdivision symbol we have a sequence of subdivision symbols $\{a_*^{(k)}(z),\ k \geq 1\}$ defined by

$$a_*^{(k)}(z) := \sum_{\alpha \in \mathbb{Z}^s} \mathrm{a}^{(k)}(\alpha) z^\alpha, \quad z \in (\mathbb{C} \setminus 0)^s, \quad k \geq 1,$$

and a sequence of associated trigonometric polynomials

$$a_*^{(k)}(\boldsymbol{e}^{-i2\pi\omega}) = \sum_{\alpha \in \mathbb{Z}^s} \mathrm{a}^{(k)}(\alpha) \boldsymbol{e}^{-i2\pi\omega\cdot\alpha}, \quad \boldsymbol{e}^{-i\omega} := (e^{-i\omega_1}, \ldots, e^{-i\omega_s}), \quad \omega \in \mathbb{R}^s, \quad k \geq 1.$$

We continue by introducing what we call the *approximated sum rules*, which are the main ingredient of our analysis.

Definition 5. *Let* $\ell \geq 0$. *The sequence of symbols* $\{a_*^{(k)}(z),\ \ k \geq 1\}$ *satisfies* approximate sum rules of order $\ell + 1$, *if*

$$\mu_k := a_*^{(k)}(\mathbf{1}) - m \quad \textit{and} \quad \delta_k := \max_{|\eta| \leq \ell} \max_{\epsilon \in \Xi \setminus \{\mathbf{1}\}} m^{-k|\eta|/s} |D^\eta a_*^{(k)}(\epsilon)| \tag{11}$$

satisfy

$$\sum_{k=1}^{\infty} \mu_k < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} m^{k\ell/s} \delta_k < \infty\,. \tag{12}$$

Note that one can always rescale each of the symbols $a_*^{(k)}(z)$ in such a way that all $\mu_k = 0$ and, thus, trivially summable. Note also that the approximate sum rules of order 1 are weaker than the asymptotic equivalence in (5) as (11)-(12) require

$$\sum_{k=1}^{\infty} \max_{\varepsilon \in E} \left\{ \big| \sum_{\alpha \in \mathbb{Z}^s} \mathrm{a}^{(k)}(M\alpha + \varepsilon) - \mathrm{a}(M\alpha + \varepsilon) \big| \right\} < \infty.$$

To comply with the notation in [8], we recall the notion of *asymptotic similarity.*

Definition 6. *A stationary* $S_{\mathbf{a}}$ *and a non-stationary* $\{S_{\mathbf{a}^{(k)}},\ \ k \geq 1\}$ *subdivision schemes are called* asymptotically similar*, if* $\lim_{k \to \infty} \mathbf{a}^{(k)} = \mathbf{a}\,.$

In Definition 7, we also generalize the notion of asymptotic similarity to the cases, when the sequence $\{\mathbf{a}^{(k)},\ k \geq 1\}$ has infinitely many accumulation points that define stationary schemes. This allows us, for example, to analyze the smoothness of non-stationary schemes that at different levels of subdivision recursion have unrelated symbols.

3 Convergence and regularity of non-stationary schemes

In this section we first derive sufficient conditions for convergence of a certain big class of non-stationary subdivision schemes. In Subsection 3.1, we show that a non-stationary subdivision scheme satisfying approximated sum rules of order 1 and asymptotically similar to a convergent stationary scheme is convergent as well. We also present a generalization of this convergence result for schemes that are asymptotically similar in terms of Definition 7, see Theorem 2. In Subsection 3.2, we illustrate the differences between the concepts of asymptotic equivalence and the generalized asymptotic similarity. In the last Subsection 3.3, we point out one of the crucial differences between stationary and non-stationary settings which makes it necessary to separate convergence and regularity analysis in the latter case. We also state the regularity results that are in preparation.

3.1 Convergence

In this section we always assume that the non-stationary schemes satisfy Definition (7) for $\ell = 0$, i.e. they satisfy the following.

a) "Approximate sum rules of order 1.**"** The sequence of masks $\{\mathbf{a}^{(k)},\ k \geq 1\}$ satisfies "approximate sum rules of order 1".

Theorem 1. *If the non-stationary scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ satisfies a), i.e. the approximate sum rules of order 1, and is asymptotically similar to a convergent stationary scheme $S_{\mathbf{a}}$, then it is $C^0(\mathbb{R}^s)-$convergent.*

`Proof.` As recalled in Section 2, in the stationary case, we denote by V the subspace of $\ell(K)$ invariant under all $T_{\varepsilon,\mathbf{a}}$, which we denote by T_ε in the sequel. After a suitable change of basis, the canonical row unit vector $(1\ 0 \ldots 0)$ becomes a common left-eigenvector of the operators T_ε, $\varepsilon \in E$, which thus can be assumed to have block diagonal structure,

$$T_\varepsilon = \begin{pmatrix} 1 & b_\varepsilon \\ 0 & \\ \vdots & Q_\varepsilon \\ 0 & \end{pmatrix}^T .$$

It is well-known that convergence of a non-stationary scheme is equivalent to the convergence of the associated cascade algorithm [17], i.e. for every $\boldsymbol{v} \in V^\perp$ with V in (10), the sequence with elements $T_\varepsilon^{(1)} \ldots T_\varepsilon^{(k)} \boldsymbol{v}$ converges as k goes to infinity for every choice of $\varepsilon \in E$ (we remark that ε varies depending on k). After an appropriate change of basis, the canonical row unit vector $(1\ 0 \ldots 0)$ becomes a *quasi*-common left-eigenvector of the operators $T_\varepsilon^{(k)}$, $\varepsilon \in E$ (this can be derived similarly to what is done in [2, 21] in the stationary case). In other words, we get the decomposition

$$T_\varepsilon^{(k)} = \widetilde{T}_\varepsilon^{(k)} + \Delta_\varepsilon^{(k)}, \quad \varepsilon \in E, \quad k \geq 1, \tag{13}$$

with

$$\widetilde{T}_\varepsilon^{(k)} = \begin{pmatrix} 1 & b_\varepsilon^{(k)} \\ 0 & \\ \vdots & Q_\varepsilon^{(k)} \\ 0 & \end{pmatrix}^T, \qquad \Delta_\varepsilon^{(k)} = \begin{pmatrix} 0 & 0 \cdots 0 \\ & \\ c_\varepsilon^{(k)} & \mathrm{O} \\ & \end{pmatrix}^T,$$

where, due to to the asymptotic similarity assumption, $b_\varepsilon^{(k)}$ and $Q_\varepsilon^{(k)}$ converge as $k \to \infty$ to b_ε and Q_ε, respectively, and $c_\varepsilon^{(k)}$ converges to a zero vector (we denote by O the zero matrix of suitable dimension).

Moreover a) implies that the sequence $\|\Delta_\varepsilon^{(k)}\|$ is summable with respect to k.

By assumption, $S_{\mathbf{a}}$ is convergent, thus, we have $\rho(\{Q_\varepsilon,\ \varepsilon \in E\}) < 1$. The existence of the extremal operator norm of $\{Q_\varepsilon,\ \varepsilon \in E\}$ and the continuity of the joint spectral radius imply that there exists $\bar{k}$ such that $\rho\left(\{Q_\varepsilon^{(k)}, \varepsilon \in E, k \geq \bar{k}\}\right) < 1$. By well-known results (see e.g. [1]) on the joint spectral radius of block triangular families of matrices we obtain that $\rho\left(\{\widetilde{T}_\varepsilon^{(k)},\ \varepsilon \in E, k \geq \bar{k}\}\right) = 1$. Moreover, the family of matrices $\{\widetilde{T}_\varepsilon^{(k)},\ \varepsilon \in E, k \geq \bar{k}\}$ is non-defective (see e.g. [18]), thus by [1, 29], there exists an extremal operator norm $\|\cdot\|$ such that

$$\|\widetilde{T}_\varepsilon^{(k)}\| \leq 1 \quad \text{for all} \quad \varepsilon \in E, \quad k \geq \bar{k}. \tag{14}$$

This implies that for all vectors $\boldsymbol{v} \in V^\perp$, the product $\widetilde{T}_\varepsilon^{[1]} \ldots \widetilde{T}_\varepsilon^{[k]} \boldsymbol{v}$ converges as k goes to infinity for every choice of $\varepsilon \in E$. By assumption a), we also have

$$\|\Delta_\varepsilon^{(k)}\| \leq C(\delta_k + \mu_k) := \tilde{\delta}_k \quad \text{where} \quad \sum_{k=1}^{\infty} \tilde{\delta}_k < \infty, \tag{15}$$

where C is a constant which does not depend on k. For $n, \ell \in \mathbb{N}$, consider any specific sequence

$$T_\varepsilon^{(n)} \ldots T_\varepsilon^{(n+\ell)} = \left(\widetilde{T}_\varepsilon^{(n)} + \Delta_\varepsilon^{(n)}\right) \ldots \left(\widetilde{T}_\varepsilon^{(n+\ell)} + \Delta_\varepsilon^{(n+\ell)}\right) = \widetilde{T}_\varepsilon^{(n)} \ldots \widetilde{T}_\varepsilon^{(n+\ell)} + R_{n,\ell}$$

where $R_{n,\ell}$ is obtained by expanding all the products. From (14)-(15) we get $\lim_{n\to\infty} R_{n,\infty} = \mathrm{O}$ implying convergence of $\prod_{j=1}^{k} T_\varepsilon^{[j]} \boldsymbol{v}$ as $k \to \infty$. The reasoning for $\lim_{n\to\infty} R_{n,\infty} = \mathrm{O}$ is as follows

$$\|R_{n,\infty}\| \leq \sum_{j=1}^{\infty} \left(\sum_{k=n}^{\infty} \tilde{\delta}_k\right)^j = \sum_{j=0}^{\infty} \left(\sum_{k=n}^{\infty} \tilde{\delta}_k\right)^j - 1 = \frac{\sum_{k=n}^{\infty} \tilde{\delta}_k}{1 - \sum_{k=n}^{\infty} \tilde{\delta}_k}.$$

□

It is worthwhile to remark that Theorem 1 generalizes [8, Theorem 10] dealing with the univariate case under the assumption that the non-stationary scheme reproduces constants. The result of Theorem 1 is also a generalization of the corresponding results in [15, 16] that require that stationary and non-stationary schemes are asymptotically equivalent.

The previous result can be extended in several directions. One possible extension, whose proof is similar to the proof of Proposition 1, makes use of the following notions.

Definition 7. *For the mask sequence* $\{\mathbf{a}^{(k)},\ \ k \ge 1\}$ *define* $\partial\mathcal{A}$ *to be the set of its limit points, i.e.*

$$\mathbf{a} \in \partial\mathcal{A}, \textit{if} \quad \exists\{k_n,\ n \in \mathbb{N}\} \quad \textit{such that} \quad \lim_{n\to\infty} \mathbf{a}^{(k_n)} = \mathbf{a}\,.$$

Remark 1. (*i*) *In general, for an arbitrary compact set* B *of masks, there exists a non-stationary subdivision scheme* $\{S_{\mathbf{a}^{(k)}},\ k \ge 1\}$ *for which* $\partial A = B$. (*ii*) *For asymptotically similar schemes the sets of limit points* ∂A *coincide. However, the converse is not true in general. Indeed, if the sets of limit points coincide, then the schemes may not be asymptotically similar.*

Definition 8. *A set of masks* $\partial\mathcal{A}$ *is said to satisfy* simultaneous contractibility (of order 0) *if for all* $\mathbf{a} \in \partial\mathcal{A}$ *the joint spectral radius of the finite sequence* $\mathcal{T}|_V = \{T_{\varepsilon,\mathbf{a}}|_V,\ \varepsilon \in E\}$ *satisfies* $\rho(\mathcal{T}|_V) < 1$.

We conclude this section with an useful generalization of Theorem 1.

Theorem 2. *Assume that the non-stationary scheme* $\{S_{\mathbf{a}^{(k)}},\ k \ge 1\}$ *satisfies the approximate sum rules of order 1. Assume further that the corresponding set* $\partial\mathcal{A}$ *satisfies* simultaneous contractibility of order 0. *Then the non-stationary scheme* $\{S_{\mathbf{a}^{(k)}},\ k \ge 1\}$ *is* C^0-*convergent.*

The proof is analogous to that of Theorem 1 and is, therefore, omitted.

In the next subsection, we illustrate our convergence results with several examples.

3.2 Examples

Already in the binary univariate case we give several simple examples showing the advantage of relaxing the condition of asymptotic equivalence.

Example 1. *We start by considering a non-stationary scheme based on masks which are level dependent convex combination of two masks* $\mathbf{a}, \mathbf{b} \in \ell_0(\mathbb{Z})$ *that define two stationary convergent subdivision schemes. See also [7]. The new non-stationary subdivision scheme has a mask given by*

$$\mathbf{a}^{(k)} = \left(1 - \frac{1}{k}\right)\mathbf{a} + \frac{1}{k}\mathbf{b}, \quad k \ge 1. \tag{16}$$

This non-stationary scheme does not satisfy the conditions in (5) *since*

$$\sum_{k\in\mathbb{N}} \max_{\varepsilon\in\{0,1\}} \left\{ \sum_{\alpha\in\mathbb{Z}} |\mathrm{a}^{(k)}(\varepsilon+2\alpha) - \mathrm{a}(\varepsilon+2\alpha)| \right\} = \max_{\varepsilon\in\{0,1\}} \left\{ \sum_{\alpha\in\mathbb{Z}} |\mathrm{b}(\varepsilon+2\alpha) - \mathrm{a}(\varepsilon+2\alpha)| \right\} \cdot \sum_{k\in\mathbb{N}} \frac{1}{k}.$$

Nevertheless, it satisfies the assumptions of Theorem 1, since all symbols satisfy

$$a_*^{(k)}(1) - 2 = 0, \quad a_*^{(k)}(-1) = 0, \quad k \geq 1\,,$$

i.e. $\mu_k = 0$, $\delta_k = 0$ *and, by construction,* $\lim\limits_{k\to\infty} \mathbf{a}^{(k)} = \mathbf{a}$.

Figure 1 shows the result of 12 *iterations of the non-stationary subdivision scheme based on the cubic and linear splines that is on* $\mathbf{a} = \{\frac{1}{8},\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{2},\ \frac{1}{8}\}$ *and* $\mathbf{b} = \{\frac{1}{2},\ 1,\ \frac{1}{2}\}$ *when starting with the initial delta sequence* $\boldsymbol{\delta} := \{0,\ 0,\ 1,\ 0,\ 0\}$.

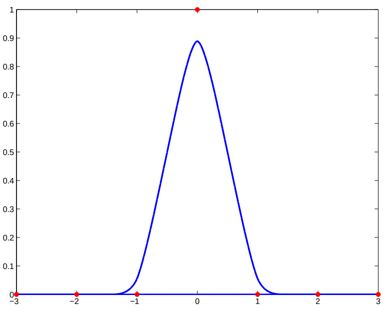

Fig. 1. *Result of* 12 *iterations of the non-stationary subdivision scheme in* (16)

One could also use the results of [8] for checking the convergence of this non-stationary scheme.

In the next example neither the results of [8] nor the ones in [15, 16] are applicable.

Example 2. *Generalizing the previous example we construct the non-stationary scheme based on the sequence of masks*

$$\mathbf{a}^{(k)} = \begin{cases} \left(1-\frac{1}{k}\right)\mathbf{a} + \frac{1}{k}\mathbf{b}, & k \ \ \textit{even;} \\ \left(1-\frac{1}{k}\right)\mathbf{c} + \frac{1}{k}\mathbf{d}, & k \ \ \textit{odd;} \end{cases} \qquad k \geq 1\,, \tag{17}$$

with $\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d} \in \ell_0(\mathbb{Z})$ *defining four stationary convergent subdivision schemes and with* $\mathbf{a},\ \mathbf{c}$, *having the same support. In this case, even though the notion of asymptotically equivalence is not applicable, but Theorem 2 allows us to establish the convergence of the scheme in* (17).

Figure 2 shows the result of 12 *iterations of the non-stationary subdivision scheme based on the stationary masks* $\mathbf{a} = \{\frac{1}{32},\ \frac{6}{32},\ \frac{15}{32},\ \frac{20}{32},\ \frac{15}{32},\ \frac{6}{32},\ \frac{1}{32}\}$, $\mathbf{b} = \{\frac{1}{8},\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{2},\ \frac{1}{8}\}$, $\mathbf{c} = \{-\frac{1}{16},\ 0,\ \frac{9}{16},\ 1,\ \frac{9}{16},\ 0,\ -\frac{1}{16}\}$ *and* $\mathbf{d} = \{\frac{1}{2},\ 1,\ \frac{1}{2}\}$ *when starting with the initial delta sequence* $\boldsymbol{\delta} := \{0,\ 0,\ 1,\ 0,\ 0\}$.

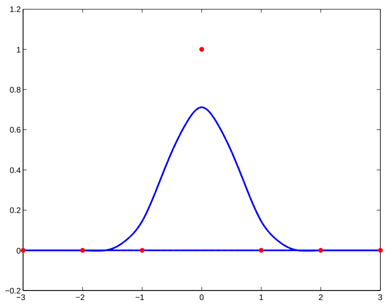

Fig. 2. *Result of* 12 *iterations of the non-stationary subdivision scheme* (17)

Example 3. *We continue by constructing the sequence of masks* $\{\mathbf{a}^{(k)},\ k \geq 1\}$ *where*

$$\mathbf{a}^{(k)} = \left\{ \left(\frac{1}{4} - \frac{1}{k}\right),\ \left(\frac{3}{4} - \frac{1}{k^2}\right),\ \left(\frac{3}{4} + \frac{1}{k}\right),\ \left(\frac{1}{4} - \frac{1}{k^2}\right) \right\}, \quad k \geq 1\,. \tag{18}$$

Obviously, $\lim_{k\to\infty} \mathbf{a}^{(k)} = \mathbf{a}$ *with* $\mathbf{a} = \{\frac{1}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{1}{4}\}$ *the mask of the Chaikin subdivision scheme. This non-stationary scheme does not satisfy the conditions in* (5) *since*

$$\sum_{k\in\mathbb{N}} \max_{\varepsilon\in\{0,1\}} \left\{ \sum_{\alpha\in\mathbb{Z}} |\mathrm{a}^{(k)}(\varepsilon + 2\alpha) - \mathrm{a}(\varepsilon + 2\alpha)| \right\} = \max \left\{ \sum_{k\in\mathbb{N}} \frac{1}{k^2},\ \sum_{k\in\mathbb{N}} \frac{1}{k} \right\} = \infty.$$

Nevertheless, the non-stationary scheme (18) *satisfies the assumptions of Theorem 1 since all symbol are such that*

$$a_*^{(k)}(1) - 2 = -\frac{2}{k^2} \quad a_*^{(k)}(-1) = \frac{2}{k^2}, \quad k \geq 1\,,$$

i.e. $\mu_k = -\delta_k = \frac{2}{k^2}$ *which are summable sequences.*

Figure 3 shows the result of 12 *iterations of the non-stationary subdivision scheme based on* (18) *when starting with the initial delta sequence* $\boldsymbol{\delta} := \{0,\ 0,\ 1,\ 0,\ 0\}$.

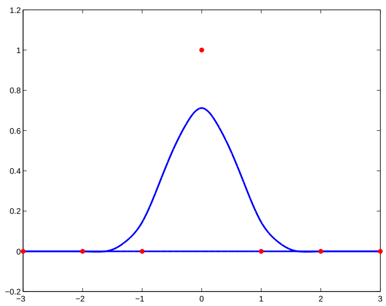

Fig. 3. *Result of* 12 *iterations of the non-stationary subdivision scheme* (17)

3.3 Outlook: Regularity

One of the striking facts about the non-stationary setting is that the well-known fact that $C^\ell-$convergence of a stationary scheme implies its C^0-convergence is no longer true.

Remark 2. Consider again the univariate binary case with $M = 2$. It is a well-known fact that the convergence of $S_{\mathbf{a}}$ in the stationary case is equivalent to the fact that the difference scheme $S_{\mathbf{b}}$ with

$$a_*(z) = (1+z)\, b_*(z), \quad z \in \mathbb{C} \setminus 0,$$

is zero convergent, i.e for every $\boldsymbol{u} \in V$ the product $\|T_{\varepsilon_1} \dots T_{\varepsilon_n} \boldsymbol{u}\|$ goes to zero as n goes to ∞. In the non-stationary case, this characterization is not valid. Consider a non-stationary scheme with the masks

$$\mathbf{a}^{(k)} = \left(1 + \frac{1}{k}\right) \mathbf{a}, \quad k \geq 1. \tag{19}$$

Note that $\mu_k = \frac{2}{k}$ and $\delta_k = 0$, thus, the non-stationary scheme is asymptotically similar to $S_{\mathbf{a}}$ and the associated trigonometric satisfies

$$a_*{}^{(k)}(k) = (1+z) \left(1 + \frac{1}{k}\right) b_*(z), \quad z \in \mathbb{C} \setminus 0.$$

We show next that the zero convergence of the associated difference schemes with the symbols $(1 + 1/k)b_*(z)$ does not imply the convergence of the corresponding non-stationary scheme. Indeed, for $\boldsymbol{u} \in V$ and $\varepsilon_j \in \{0, 1\}$, we get

$$\|T^{(1)}_{\varepsilon_1} \dots T^{(n)}_{\varepsilon_n} \boldsymbol{u}\| = \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) \|T_{\varepsilon_1} \dots T_{\varepsilon_n} \boldsymbol{u}\| = (n+1) \|T_{\varepsilon_1} \dots T_{\varepsilon_n} \boldsymbol{u}\|.$$

The convergence of $S_{\mathbf{a}}$ implies the existence of an extremal operator norm such that

$$\|T_{\varepsilon_1} \dots T_{\varepsilon_n} \boldsymbol{u}\| \leq C\gamma^n, \quad \gamma < 1.$$

Therefore, the norm $\|T^{(1)}_{\varepsilon_1} \dots T^{(n)}_{\varepsilon_n} \boldsymbol{u}\|$ goes to zero as n goes to ∞, but the corresponding non-stationary scheme is not convergent. Otherwise, its basic limit function ϕ_1 would satisfy

$$\hat{\phi}_1(\omega) = \hat{\phi}_1(0) \prod_{k=1}^{\infty} a_*^{(k)}(e^{-i2\pi 2^{-k}\omega}), \quad \omega \in \mathbb{R},$$

where

$$\hat{\phi}_1(0) = \lim_{n\to\infty} 2\prod_{k=1}^{n}\left(1+\frac{1}{k}\right) = \lim_{n\to\infty} 2(n+1) = \infty.$$

Not to overwhelm the reader with the technical details of the proofs of our regularity results, we postpone those to a forthcoming publication [5]. For completeness, we only state them here to show that the method we propose can also be used to check the regularity of limits of non-stationary schemes. The constant α_{ϕ_1} in the following two results determines the Hölder exponent of the refinable limit function ϕ_1 in (4).

Theorem 3. *Let $\ell \geq 0$. If the masks $\{\mathbf{a}^{(k)}, k \geq 1\}$, satisfy sum rules of order $\ell+1$ and*

$$\rho_\ell \;:=\; \rho\left\{T_{\varepsilon,\mathbf{a}}|_{V_\ell} \;\middle|\; \varepsilon \in E,\ \mathbf{a} \in \partial\mathcal{A}\right\} \;<\; m^{-\ell/s},$$

then the scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ converges, its basic limit function $\phi_1 \in C^\ell(\mathbb{R}^s)$ and

$$\alpha_{\phi_1} \;\geq\; -s\log_m \rho_\ell.$$

Theorem 4. *Let $\ell \geq 0$. Assume that the non-stationary scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ satisfies approximate sum rules of order $\ell+1$ and is asymptotically similar to a convergent stationary scheme $S_\mathbf{a}$ with a stable refinable function $\varphi \in C^\ell(\mathbb{R}^s)$. Then $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ converges to a refinable function $\phi_1 \in C^\ell(\mathbb{R}^s)$ with*

$$\alpha_{\phi_1} \;\geq\; \max\left\{\alpha_\varphi,\ -s\limsup_{k\to\infty}\frac{\log_m \delta_k}{k}\right\}. \tag{20}$$

4 Further properties of non-stationary schemes

In this section we consider *binary univariate case* only with $M=2$ and the symbols $a_*^{(k)}$ scaled so that all $\mu_k = 0$. For such schemes, in Subsection 4.1, we show that the approximate sum rules in Definition 5 are very close to being necessary for the Hölder regularity of a non-stationary schemes. This resembles the stationary setting, motivates our multivariate convergence results and, more important, their generalization to the smoothness analysis in [5]. Indeed, in the binary univariate case, we show that under stability assumptions the $C^\ell-$regularity of a non-stationary scheme implies that the sum rules defects $\delta_k \;:=\; \max_{j\leq\ell} 2^{-kj}|D^j a_*^{(k)}(-1)|$ and μ_k must decay faster than $2^{-\ell k}$. Although there is still a gap, even in the case $\ell=0$, between this necessary condition $\lim_{k\to\infty}\delta_k = 0$ and the sufficient condition $\sum_{k\in\mathbb{N}}\delta_k < \infty$. Nevertheless, even in the simplest binary univariate case, our weaker conditions for convergence and regularity of non-stationary scheme cannot be relaxed by much to improve even further the results in [8, 15, 16].

In the last Subsection 4.2, we show that already in the binary univariate case the generation and reproduction properties of non-stationary schemes are limited to some special subspaces of functions.

4.1 Necessary conditions for regularity of non-stationary schemes

Before proving the announced result we consider a preliminary Proposition that studies the infinite products of certain trigonometric polynomials. To this purpose we need to discuss some preliminary facts:

Let $\{a_k,\ k \in \mathbb{N}\}$ be a sequence of algebraic polynomials of degree N. We write $p_k(x) = a_k(e^{-2\pi i x})$ for the corresponding trigonometric polynomials and assume that $a_k(1) = p_k(0) = 1$ for all k. A pair of complex numbers $\{z, -z\}$ is *a pair of symmetric roots* for a polynomial a, if $a(z) = a(-z) = 0$.

It is known that if a refinable function is stable, then the symbol a of the corresponding refinement equation has no symmetric roots on the unit circle $\{z \in \mathbb{C}\ :\ |z| = 1\}$. We consider the following function

$$f(x) \;=\; \prod_{k=1}^{\infty} p_k(2^{-k}x)\,, \quad x \in \mathbb{R}\,. \tag{21}$$

By [11], if the sequence $\{a_k,\ k \in \mathbb{N}\}$ is bounded, then this product converges uniformly on each compact subset of $\mathbb{R}$, and hence, f is an analytic function.

We continue with the following proposition.

Proposition 3. *Assume a sequence of polynomials $\{a_k,\ k \in \mathbb{N}\}$ tends to a polynomial a that has no symmetric roots on the unit circle. Then, for every $\ell \geq 0$, the assertion $f(x) = o(x^{-\ell})$ as $x \to +\infty$ implies that $\delta_k = o(2^{-\ell k})$ as $k \to \infty$, where δ_k is defined in (12).*

`Proof.` We consider the assumption $f(x) = o(x^{-\ell})$ for points of the form $x = 2^{k-1}d + t$, where d is a fixed natural number, t is an arbitrary number from a fixed segment $[0, \sigma]$, and $k \to \infty$. The parameters $d \in \mathbb{N}$ and $\sigma > 0$ will be chosen in a special way.

First, we define σ. Since p_k tends to the polynomial $p(\cdot) = a(e^{-2\pi i \cdot})$ as $k \to \infty$, it follows that the sequence $\{p_k,\ k \in \mathbb{N}\}$ is bounded. Moreover, $p_k(0) = 1$ for all k. Hence, product (21) converges uniformly on each compact set to an analytic function f, and $f(0) = 1$. This implies that there are $\sigma \in (0,1)$ and $C_0 > 0$ such that for every $r \geq 0, R \in \mathbb{N} \cup \{\infty\}$ we have

$$\Bigl| \prod_{j=1}^{R} p_{j+r}\bigl(2^{-j}x\bigr) \Bigr| \ \geq \ C_0 \, , \quad x \in [0, \sigma] \, . \tag{22}$$

Now let us choose the number d. To this end we consider the binary tree defined as follows: the number $1/2$ is at the root, the numbers $1/4$ and $3/4$ are its children, and so on. Every vertex α has two children $\alpha/2$ and $(\alpha+1)/2$. For convenience we shall identify a vertex and the corresponding number. Thus, all vertices of the tree are dyadic points of the interval $(0,1)$: the nth level of the tree (i.e., the set of vertices with the distance to the root equal to n) consists of points $2^{-n-1}j$, where j is an odd number from 1 to $2^{n+1}-1$.

The trigonometric polynomial p has at most N zeros on the period $[0,1)$, and hence, on the tree. Therefore, there is a number q such that all roots of p on the tree are contained on levels $j \leq q$. Since the polynomial a has no symmetric roots, it follows that for every vertex of the tree, at least one of two its children is not a root of p. Whence, there is a path of length q along the tree starting at the root (all paths are without backtracking) that does not contain any root of p. Let $2^{-q-1}d$ be the final vertex of that path, d is an odd number, $1 \leq d \leq 2^{q+1}-1$. Denote as usual by $\{x\}$ the fractional part of a number x. We see that the sequence $\{2^{-1}d\}, \ldots, \{2^{-q-1}d\}$ does not contain roots of p. The sequence $\{2^{-q-2}d\}, \{2^{-q-3}d\}, \ldots$ does not contain them either, because there are no roots of p on levels bigger than q. Let n be the smallest natural number such that $2^{-q-n}d < \sigma/2$. We have $p(2^{-1}d) \cdots p(2^{-q-n}d) \neq 0$. Since $p_k \to p$ as $k \to \infty$, and all p_k are equi-continuous on $\mathbb{R}$, it follows that there is a constant $C_1 > 0$ such that

$$\Bigl| \prod_{j=1}^{q+n} p_{k+j}\bigl(2^{-j}d + 2^{-k-j}x\bigr) \Bigr| \ \geq \ C_1 \, , \quad x \in [0, \sigma] \tag{23}$$

for all sufficiently large k. Now we are ready to estimate the value $f(2^{k-1}d + t)$. We have

$$\Bigl| f\bigl(2^{k-1}d + t\bigr) \Bigr| \ = \ \Bigl| \prod_{j=1}^{k-1} p_j\bigl(2^{k-1-j}d + 2^{-j}t\bigr) \Bigr| \times \Bigl| p_k\bigl(2^{-1}d + 2^{-k}t\bigr) \Bigr| \times$$

$$\times \Bigl| \prod_{j=1}^{q+n} p_{k+j}\bigl(2^{-j}d + 2^{-k-j}t\bigr) \Bigr| \times \Bigl| \prod_{j=1}^{\infty} p_{k+q+n+j}\bigl(2^{-j}(2^{-q-n}d + 2^{-k-q-n}t)\bigr) \Bigr| .$$

To estimate the first product, we note that $2^{k-1-j}d \in \mathbb{Z}$, whenever $j \leq k-1$, and hence $p_j\bigl(2^{k-1-j}d + 2^{-j}t\bigr) = p_j(2^{-j}t)$. So, the first product is equal to $\bigl| \prod_{j=1}^{k-1} p_j(2^{-j}t) \bigr|$, which is, by (22), bigger than or equal to C_0, for every $t \in [0, \sigma]$.

The third product $\left|\prod_{j=1}^{q+n} p_{k+j}(2^{-j}d+2^{-k-j}t)\right|$, by (23), is at least C_1. Finally, the latter product is bigger than or equal to C_0. To see this it suffices to use (22) for $R = \infty, r = k+q+n, x = 2^{-q-n}d + 2^{-k-q-n}t$ and note that $x < \sigma$ by the choice of n. Thus,

$$|f(2^{k-1}d+t)| \ \geq \ C_0^2 C_1 \bigl|p_k\bigl(2^{-1}d+2^{-k}t\bigr)\bigr| .$$

On the other hand, $f(2^{k-1}d+t) = o(2^{-\ell k})$ as $k \to \infty$, consequently $p_k(2^{-1}d+2^{-k}t) = o(2^{-\ell k})$. The number d is odd, hence, by periodicity, $p_k(2^{-1}d+2^{-k}t) = p_k(1/2+2^{-k}t)$. Thus, we arrive at the following asymptotic relation: for every $t \in [0, \sigma]$ we have

$$p_k\bigl(1/2+2^{-k}t\bigr) \ = \ o\left(2^{-\ell k}\right) \quad \text{as } k \to \infty . \tag{24}$$

This already implies that $D^j p_k(1/2) = o(2^{(j-\ell)k})$ as $k \to \infty$, for every $j = 0, \ldots, \ell$. Indeed, consider the Tailor expansion of the function $h(t) = p_k\bigl(1/2 + 2^{-k}t\bigr)$ at the point 0 with the remainder in Lagrange form:

$$h(t) \ = \ \sum_{j=0}^{\ell} \frac{D^j h(0)}{j!}\, t^j \ + \ \frac{D^{\ell+1} h(\theta)}{(\ell+1)!}\, t^{\ell+1} , \ t \in [0, \sigma] ,$$

where $\theta = \theta(t) \in [0, t]$. Substituting $D^j h(0) = 2^{-jk} D^j p_k(1/2)$, we get

$$p_k(1/2+2^{-k}t) \ = \ \sum_{j=0}^{\ell} \frac{D^j p_k(1/2)}{j!}\, 2^{-jk}\, t^j \ + \ \frac{D^{\ell+1} p_k(1/2+2^{-k}\theta)}{(\ell+1)!}\, 2^{-(\ell+1)k}\, t^{\ell+1} , \ t \in [0, \sigma] .$$

First, we estimate the remainder. Since the sequence of trigonometric polynomials $\{p_k, \ k \in \mathbb{N}\}$ is bounded, the norms $\|D^{\ell+1} p_k\|_{C[0,\sigma]}$ do not exceed some constant C_2. Therefore,

$$\left|\frac{D^{\ell+1} p_k(1/2+2^{-k}\theta)}{(\ell+1)!}\, 2^{-(\ell+1)k}\, t^{\ell+1}\right| \ \leq \ \frac{C_2}{(\ell+1)!} 2^{-(\ell+1)k}\, \sigma^{\ell+1} \ = \ o(2^{-\ell k}) \quad \text{as } k \to \infty .$$

Combining this with (24), we get

$$\left\| \sum_{j=0}^{\ell} \frac{D^j p_k(1/2)}{j!}\, 2^{-jk}\, t^j \right\|_{C\,[0,\sigma]} \ = \ o(2^{-\ell k}) \quad \text{as } k \to \infty . \tag{25}$$

Since, in a finite-dimensional space, all norms are equivalent, the norm of an algebraic polynomial of degree ℓ in the space $C[0, \sigma]$ is equivalent to its largest coefficient. Whence, (25) implies that $\max\limits_{j=0,\ldots,\ell} \frac{|D^j p_k(1/2)|}{j!}\, 2^{-jk} = o(2^{-\ell k})$ as $k \to \infty$. Expressing now the derivatives of a_k by derivatives of p_k, we obtain $\max_{j=0,\dots,\ell} 2^{-jk}|D^j a_k(-1)| = o(2^{-\ell k})$ as $k \to \infty$, which completes the proof.

$\square$

We are finally ready to prove the announced necessary conditions for convergence of non-stationary subdivisions schemes.

Theorem 5. *Let a binary subdivision scheme with the mask* $\mathbf{a}$ *be convergent to a continuous stable function; let* $\{\mathbf{a}^{(k)},\ k \ge 1\}$ *be a sequence of masks supported on* $[-N, N]$ *such that*

$$\lim_{k\to\infty} \mathbf{a}^{(k)} = \mathbf{a}\,.$$

If the non-stationary subdivision scheme $\{S_{\mathbf{a}^{(k)}},\ k \ge 1\}$ *converges to a* $C^\ell(\mathbb{R})$, $\ell \ge 0$, *function, then* $\lim_{k\to\infty} 2^{\ell k}\delta_k = 0$.

`Proof of Theorem 5.` Let $p_k(\omega) = a_*^{(k)}(e^{-2\pi i\omega})$ be the symbol of the $k-$th mask in the trigonometric form. If the non-stationary scheme converges to a continuous compactly supported refinable function ϕ, then its Fourier transform $\widehat{\phi}(\omega) = \int_{\mathbb{R}} \phi(x)e^{-2\pi i x\omega}dx$ is given by

$$\widehat{\phi}(\omega) = \prod_{k=1}^{\infty} p_k(2^{-k}\omega), \quad \omega \in \mathbb{R}\,. \tag{26}$$

If $\phi \in C^\ell(\mathbb{R})$, then $\widehat{\phi}(\omega) = o(\omega^{-\ell})$ as $\omega \to \infty$. Since the refinable function of the limit mask $\mathbf{a}$ is stable, it follows that its symbol $a_*(z)$ has no symmetric roots on the unit circle. The claim follows by Proposition 3.

$\square$

4.2 Reproduction and generation properties of non-stationary schemes

To expose certain limitations of non-stationary schemes, it suffices again to consider only the univariate case. In the univariate case, M is simply an integer larger than 2. In this subsection, we show that the generation and reproduction properties of non-stationary schemes are limited to some special subspaces of functions. More precisely, we will show that the zero sets of the Fourier transforms of the limit functions ϕ_k of such schemes are certain unions of the sets

$$\Gamma_r = \{\omega \in \mathbb{C} \ :\ a_*^{(r)}(e^{-i2\pi M^{-r}\omega}) = 0\}, \quad r \ge k,$$

and that the sets Γ_r are such that $\Gamma_r + M^r\mathbb{Z} = \Gamma_r$. Thus, some elementary functions cannot be generated by non-stationary schemes, see example 6. Another requirement that

$$\hat{\phi}_k(\omega) = \int_{\mathbb{R}} \phi_k(x) e^{-i2\pi x\omega} dx, \quad \omega \in \mathbb{C}, \quad k \in \mathbb{N},$$

is an entire function, also limits the reproduction and generation properties of non-stationary subdivision schemes.

Proposition 4. *Let* $\{\phi_k,\ k \in \mathbb{N}\}$ *be continuous functions of compact support satisfying*

$$\phi_k(x) = \sum_{\alpha \in \mathbb{Z}} \mathrm{a}^{(k)}(\alpha)\phi_{k+1}(Mx - \alpha), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}.$$

Then

$$\{\omega \in \mathbb{C}\ : \hat{\phi}_k(\omega) = 0\} = \bigcup_{r \geq k} \Gamma_r,$$

such that the sets Γ_r *satisfy*

$$\Gamma_r + M^r\mathbb{Z} = \Gamma_r.$$

`Proof.` Let $k \in \mathbb{N}$. By Paley-Wiener theorem, the Fourier transform $\hat{\phi}_k$ defined on $\mathbb{R}$ has an analytic extension

$$\hat{\phi}_k(\omega) = \int_{\mathbb{R}} \phi_k(x) e^{-i2\pi x\omega} dx, \quad \omega \in \mathbb{C},$$

to the whole complex plane $\mathbb{C}$ and $\hat{\phi}_k$ is an entire function. By Weierstraß theorem [10], every entire function can be represented by a product involving its zeroes. Define the sets

$$\Gamma_r = \{\omega \in \mathbb{C}\ :\ a_*^{(r)}(e^{-i2\pi M^{-r}\omega}) = 0\}, \quad r \in \mathbb{N}.$$

Let $z_{r,1}, \ldots, z_{r,N}$ be the zeros of the polynomials $a_*^{(r)}(z)$, $z = e^{-i2\pi M^{-r}\omega}$, counting their multiplicities. Then

$$\Gamma_r = iM^r \bigcup_{\ell=1}^{N} \mathrm{Ln}(z_{r,\ell}),$$

where, by the properties of the complex logarithm, each of the sets $iM^r\mathrm{Ln}(z_{r,\ell})$ consists of sequences of complex numbers and is M^r-periodic. Thus, each of the sets Γ_r satisfy

$$\Gamma_r + M^r\mathbb{Z} = \Gamma_r, \quad r \in \mathbb{N}.$$

The definition of $\hat{\phi}_k$ as an infinite product of the trigonometric polynomials $a_*^{(r)}(e^{-i2\pi M^{-r}\omega})$, $r \geq k$, yields the claim.

$\square$

The first two examples we present here illustrate the capability of non-stationary schemes to reproduce and generate exponential polynomials.

Example 4. *The basic limit function of the simplest stationary scheme is given by* $\phi_1 = \chi_{[0,1)}$. *Its Fourier transform is*

$$\hat{\phi}_1(\omega) = \frac{1 - e^{-i2\pi\omega}}{i2\pi\omega}, \quad \textit{and} \quad \{\omega \in \mathbb{C} \ : \ \hat{\phi}_1(\omega) = 0\} = \mathbb{Z} \setminus \{0\}.$$

The mask symbol $a_*(z) = 1 + z$ *has a single zero at* $z = -1$, *i.e.* $e^{-i2\pi 2^{-r}\omega} = -1$ *for* $\omega = 2^r\{\frac{1}{2} + k \ \ k \in \mathbb{Z}\}$, $r \in \mathbb{N}_0$. *In other words,* $\Gamma_1 = \{1 + 2k \ : \ k \in \mathbb{Z}\}$ *and* $\Gamma_r = 2\Gamma_{r-1}$ *for* $r \geq 2$. *Therefore,*

$$\{\omega \in \mathbb{C} \ : \ \hat{\phi}_1(\omega) = 0\} = \bigcup_{r \in \mathbb{N}} \Gamma_r.$$

Example 5. *One of the basic limit function of the simplest non-stationary scheme is given by* $\phi_1(x) = \chi_{[0,1)}(x)e^{\lambda x}$, $\lambda \in \mathbb{C}$. *Its Fourier transform is*

$$\hat{\phi}_1(\omega) = \frac{e^{-i2\pi\omega+\lambda} - 1}{-i2\pi\omega + \lambda}, \quad \omega \in \mathbb{C}, \quad \textit{and} \quad \{\omega \in \mathbb{C} \ : \ \hat{\phi}_1(\omega) = 0\} = -\frac{i\lambda}{2\pi} + \mathbb{Z} \setminus \{0\}.$$

The mask symbol $a_*^{(k)}(z) = 1 + e^{\lambda 2^{-r}}z$ *has a single zero at* $z = -e^{-\lambda 2^{-r}}$, *i.e.* $e^{-i2\pi 2^{-r}\omega} = -e^{-\lambda 2^{-r}}$ *for* $\omega = -\frac{i\lambda}{2\pi} + 2^r\{\frac{1}{2} + k \ : \ k \in \mathbb{Z}\}$, $r \in \mathbb{N}$. *Note that* $\Gamma_1 = -\frac{i\lambda}{2\pi} + \{1 + 2k \ : \ k \in \mathbb{Z}\}$ *and*

$$\bigcup_{r \in \mathbb{N}} 2^r\{\frac{1}{2} + k \ : \ k \in \mathbb{Z}\} = \mathbb{Z} \setminus \{0\}.$$

Therefore,

$$\{\omega \in \mathbb{C} \ : \ \hat{\phi}_1(\omega) = 0\} = \bigcup_{r \in \mathbb{N}} \Gamma_r.$$

The next example shows that not all compactly supported functions can be reproduced by any non-stationary subdivision scheme.

Example 6. *For example the compactly supported function*

$$f(x) = \chi_{[-1,1]}(x)\frac{2}{\sqrt{1 - x^2}}, \quad x \in \mathbb{R},$$

cannot be a limit of any non-stationary subdivision scheme. Indeed, its Fourier transform

$$J_0(\omega) = \int_{\mathbb{R}} f(x)e^{-ix\omega}dx, \quad \omega \in \mathbb{C},$$

is the Bessel function J_0 of the first kind, which is entire, but has only positive zeros. The lower bound for its zeros $j_{0,s}$, $s \in \mathbb{N}$, is given by $j_{0,s} > \sqrt{(s-\frac{1}{4})^2\pi^2}$, see [26]. Thus, proposition 4 implies the claim. We used a different definition of the Fourier transform to be consistent with the literature.

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