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Very general monomial valuations of $\mathbb{P}^2$ and a Nagata type conjecture

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1 Introduction

Ever since Zariski's pioneering work, valuations have been considered to be natural generalizations of points. However, in the context of linear systems defined by multiple base points on projective varieties, positivity, and Seshadri constants, this point of view seems to have been explored explicitly only recently.

In [6] and [5], S. Boucksom, M. Dumnicki, A. Küronya, C. Maclean, and T. Szemberg introduced the constant $a_{\max}$ of a valuation (here denoted μ), analogous to the s-invariant introduced by L. Ein, S. D. Cutkosky and R. Lazarsfeld in [8] for ideals (see also [16, 5.4]). For a valuation v centered at the origin of $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$, one has by definition

$$\hat{\mu}(v) = \lim_{d\to\infty} \frac{\max\{v(f) \mid f \in \mathbb{C}[x, y], \deg f \leq d\}}{d}.$$

All such invariants encode essentially the same information as the Seshadri constant does in the case of points and, as is the case for Seshadri constants, they turn out to be extremely hard to compute.

The last decade has also seen the blossoming of a geometric study of spaces of *real* valuations [11] or spaces of *seminorms*, usually called Berkovich spaces [2], which essentially coincide in dimension two (see [15, section 6] for a description in the plane case). Being compact and arcwise connected, the topology of such spaces has very interesting and useful properties. The work of S. Boucksom, C. Favre and M. Jonsson [3], [4] implicitly reveals connections between such valuation spaces, positivity, and birational geometry.

In this paper the invariant $\hat{\mu}$ is studied as a function on the space $\mathcal{V}$ of plane valuations of real rank 1, which is continuous along arcs in $\mathcal{V}$. Motivated by what is known in the case of points and by the conjectures of Nagata and Segre–Harbourne–Gimigliano–Hirschowitz, our focus will be on valuations along a *very general* half-line in $\mathcal{V}$.

We let $\hat{\mu}(t) = \hat{\mu}(v_t)$ with $t \in [1, \infty)$, where v_t is a very general quasimonomial valuation with characteristic exponent t (we refer the reader to Sections 2 and 3 for precise definitions). Divisorial valuations are dense in each arc of the valuation space; we will be primarily interested in such valuations; therefore we often work on the minimal proper birational model X_t where the center of v_t is a divisor.

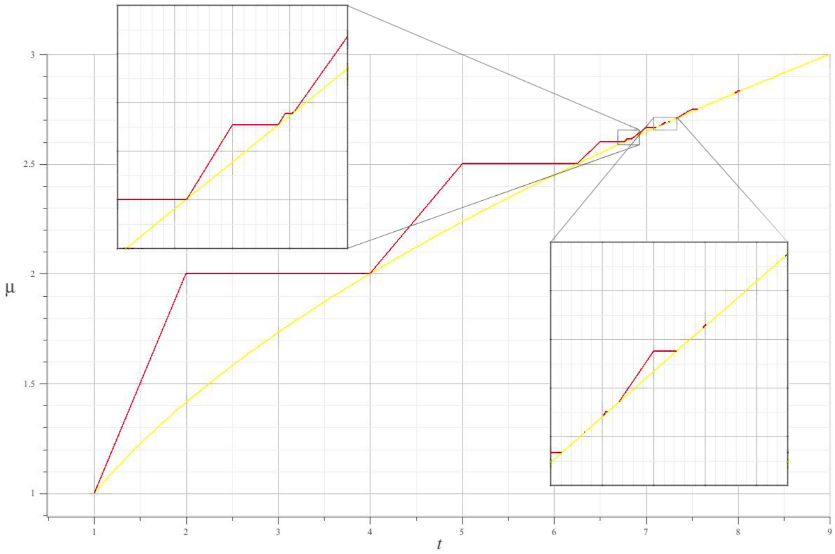

Figure 1: In red, the known behavior of $\hat{\mu}(t)$ for $t \leq 9$; in yellow, the lower bound $\sqrt{t}$.

It is not hard to see that $\hat{\mu}(t) \geq \sqrt{t}$, and the equality is expected to hold unless there is a good geometric reason, in the form of a (-1)-curve on X_t with value higher than expected. When X_t supports an effective anticanonical divisor, the cone of curves is generated by the $(-n)$-curves on the model, allowing us to compute $\hat{\mu}$. In this case there exists in fact a (-1)-curve computing $\hat{\mu}(t)$, which turns out to be piecewise linear near t; see Theorem 6.9.

Section 6 contains a description of a (countably infinite) family of (-1)-curves determining $\hat{\mu}(t)$ for $t \leq 7+1/9$ and other small values of t. We conjecture that this list is complete. If that is indeed so, then in particular $\hat{\mu}(t) = \sqrt{t}$ for $t \geq 8 + 1/36$, which implies Nagata's conjecture.

Indeed, integer values of t can be interpreted as the *number of points* that have been blown up, and we can look at $\hat{\mu}(t)$ as a continuous function with the geometric interpretation that it interpolates between the inverses of Seshadri constants at t very general points. In addition, it is not hard to show (Proposition 3.10) that for integer values of t that are squares, $\hat{\mu}(t) = \sqrt{t}$ holds.

As an unexpected connection, we want to mention that, except for 9 cases, the (-1)-curves of Section 6 are the same unicuspidal curves which give the asymptotically extremal ratio between degree and multiplicity, as explained in Orevkov's work [19] (see also the review [12]).

In what follows we work over the field of complex numbers.

2 Preliminaries

First we briefly recall a few facts from the general theory of valuations and complete ideals we shall need from [20, Chapter VI. and Appendix 5.] and [7, Chapter 8], applied to the field of functions F of a surface.

On every projective model of F (i.e. a smooth projective surface S with a fixed isomorphism $K(S) \cong F$) such a valuation v has a *center* $\mathfrak{p} \in S$; this means that for every affine chart $U \subset S$ containing $\mathfrak{p}$, v is nonnegative on the ring of regular functions $A[U] \subset K(S)$, and the ideal of functions with positive value is $\mathfrak{p}$.

For every effective divisor $D \subset S$, we will use the notation $v(D)$ for the value of any equation of $D \cap U$, which is independent of the choices made. If $\mathfrak{p}$ is the generic point of a curve C, then v is (up to a constant) simply the order of vanishing along C; thus, $v(D) = \max\{k \mid D - kC \geq 0\}$.

Valuations with a 0-dimensional center are much more varied, and are classified according to their *cluster of centers*, which we define next. To begin with, let $p_1 = \mathfrak{p}$ be the center of the valuation v. Consider the blowup $\pi_1 : S_1 \to S$ centered at p_1 and let E_1 be the correspinding exceptional divisor. The center of v on S_1 may be (the generic point of) E_1 or a point $p_2 \in E_1$.

Iteratively blowing up the centers $p_1, p_2, \ldots$ of v either ends with a model where the center of v is an exceptional divisor E_n, in which case

$$v(f) \;=\; c \cdot \operatorname{ord}_{E_n} f$$

for some constant c, and v is called a divisorial valuation, or this process goes on indefinitely. For each center p_i of v, general curves through p_i and smooth at p_i have the same value $v_i = v(E_i)$.

In this case $K = (p_1, p_2, \ldots)$ is a weighted possibly infinite cluster of points in the sense of [7, Chapter 4] with weights v_i which completely determines v, as for every effective divisor $D \subset S$,

$$v(D) = \sum_i v_i \cdot \operatorname{mult}_{p_i} \tilde{D}_i, \tag{$*$}$$

where $\tilde{D}_i$ denotes strict transform at S_i. The sum may be infinite, but for valuations with real rank 1, which are the ones we consider here, $\tilde{D}$ can have positive multiplicity only at a finite number of centers [7, 8.2].

Sometimes we shall say that a divisor goes through an infinitely near point to mean that its strict transform on the appropriate surface goes through it.

Definition 2.1. With notation as above, given indices $j < i$, the center p_i is called *proximate to* p_j ($p_i \succ p_j$) if p_i belongs to the strict transform $\tilde{E}_j$ of the exceptional divisor of p_j. Each p_i with $i > 0$ is proximate to p_{i-1} and to at most one other center p_j, $j < i - 1$; in this case $p_i = \tilde{E}_j \cap E_{i-1}$ and p_i is called a *satellite point*. A point which is not a satellite point is called *free*.

Remark 2.2. For every valuation v, and every center p_i such that v is not the divisorial valuation associated to p_i, Equation ($*$) applied to $D = E_j$ gives rise to the so-called *proximity equality*

$$v_j \;=\; \sum_{p_i \succ p_j} v_i \;.$$

For effective divisors D on S, the intersection number $\tilde{D} \cdot \tilde{E}_j \geq 0$ yields the proximity inequality

$$\operatorname{mult}_{p_j}(\tilde{D}_j) \geq \sum_{p_i \succ p_j} \operatorname{mult}_{p_i}(\tilde{D}_i) \;.$$

Assume now that $v = \mathrm{ord}_{E_n}$ is the divisorial valuation with cluster of centers $K = (p_1, \dots, p_n)$, while $\pi_K : S_K \to S$ denotes the composition of the blowups of all points of K. Then, for every $m > 0$, the valuation ideal sheaf, defined for any open affine $U \subseteq S$ by

$$\mathcal{I}_m(U) := \{f \in \mathcal{O}_S(U) \,|\, v(f) \geq m\},$$

can be described as

$$\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(-mE_n)) \ .$$

Remark 2.3. As soon as $n > 1$, the negative intersection number $-mE_n \cdot \tilde{E}_{n-1} = -m$ implies that all global sections of $\mathcal{O}_{S_K}(-mE_n)$ vanish along $\tilde{E}_{n-1}$, and therefore

$$\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(-mE_n - \tilde{E}_{n-1})) = (\pi_K)_*(\mathcal{O}_{S_K}(-E_{n-1} - (m-1)E_n)) \ .$$

This *unloads* a unit of multiplicity from p_n to p_{n-1}; iteratively subtracting all exceptional divisors that have negative intersection is a finite process [7, 4.6] which ends with a uniquely determined system of weights $\bar{m}_i$ such that

$$D_m = -\sum \bar{m}_i E_i \quad \text{is nef,}$$

and

$$\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(D_m)) \ .$$

In this case, general sections of $\mathcal{I}_m$ have multiplicity exactly $\bar{m}_i$ at p_i, and no other singularity. More precisely, for any ample divisor class A on S, the complete system $|kA + D_m|$ for $k \gg 0$ is base-point-free, and has smooth general element meeting E transversely.

It follows using ($*$) that the valuation of an effective divisor D on S can be computed as a local intersection multiplicity

$$v(D) = I_{p_1}(D, C)$$

where C is a general element of $|kA + D_m|$.

The unloading procedure just described also yields the following.

Lemma 2.4. *Let $v = \mathrm{ord}_{E_n}$ be the divisorial valuation whose cluster of centers is $K = (p_1, \dots, p_n)$ with weights v_i, and for every $m > 0$ denote $D_m = -\sum \bar{m}_i E_i$ the unique nef divisor on S_K with $\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(D_m))$. If $m = k\sum v_i^2$ for some integer k, then $\bar{m}_i = kv_i$ for all i.*

Remark 2.5. In the context of Zariski's theory of factorizations of complete ideals (where we write m_0 for $\sum v_i^2$) this translates into

$$\mathcal{I}_{km_0} = \mathcal{I}_{m_0}^k \ ,$$

and to the fact that $\mathcal{I}_{m_0}$ is a simple complete ideal.

For other values of m one has the equality

$$\mathcal{I}_{km_0+\delta} = \mathcal{I}_{m_0}^k \mathcal{I}_\delta$$

instead.

Non-divisorial valuations can be considered to be limits of divisorial valuations, and their valuation ideals turn out to be complete as well, determined by finitely many centers. The ideal $\mathcal{I}_{km}$ is then never a power of $\mathcal{I}_m$, rather there exists $\delta > 0$ such that

$$\mathcal{I}_m^k \subset \mathcal{I}_{km} \subset \mathcal{I}_{m-\delta}^k$$

for all m and k. Such bounds actually hold in greater generality, namely for Abhyankar valuations in arbitrary dimension; see [9] by L. Ein, R. Lazarsfeld and K. Smith.

The volume of a real valuation with zero-dimensional center on S, as defined in [9], is

$$\operatorname{vol}(v) := \lim_{m\to\infty} \frac{\dim_{\mathbb{C}}(\mathcal{O}_S/\mathcal{I}_m)}{m^2/2}$$

(note that $\mathcal{O}_S/\mathcal{I}_m$ is an artinian $\mathbb{C}$-algebra supported at the center of the valuation).

Lemma 2.6. *Let $v = \operatorname{ord}_{E_n}$ be the divisorial valuation with cluster of centers $K = (p_1, \ldots, p_n)$ and weights v_i. Then*

$$\operatorname{vol}(v) = \left(\sum v_i^2\right)^{-1} .$$

Proof. For $m = k\sum v_i^2$, $\dim_{\mathbb{C}}(\mathcal{O}_X/\mathcal{I}_m) = \sum kv_i(kv_i+1)/2$ by [7, 4.7]. □

Following [5], given a valuation v and an effective divisor D on S we denote

$$\mu_D(v) = \max\{v(D') \mid D' \in |D|\} ,$$

and

$$\hat{\mu}_D(v) = \lim_{k\to\infty} \mu_{kD}(v)/k .$$

Consider the group of linear equivalence classes of $\mathbb{R}$-divisors $N_1(S_K) = \operatorname{Pic}(S_K) \otimes_{\mathbb{Z}} \mathbb{R}$, where S_K is the blowup at the cluster of centers of v. Since numerical and linear equivalence coincide on rational surfaces, no confusion will arise from this abuse of notation.

A class $\eta \in N_1(S_K)$ is *integral* (respectively *rational*) if it belongs to $\operatorname{Pic}(S_K)$ (resp. to $\operatorname{Pic}(S_K)\otimes_{\mathbb{Z}}\mathbb{Q}$). A ray in $N_1(S_K)$ is called *rational* if it is generated by a rational class. One calls a rational ray in $N_1(S_K)$ *effective*, if it is generated by an effective class. The *Mori cone* $\overline{\operatorname{NE}}(S_K)$ is the closure in $N_1(S_K)$ of the set $\operatorname{NE}(S_K)$ of all effective rays, and it is the dual of the *nef cone* $\operatorname{Nef}(S_K)$ which is the closed cone described by all nef rays.

Remark 2.7. According to [5, proposition 2.9], one has

$$\hat{\mu}_D(v) \geq \sqrt{\operatorname{vol}(D)/\operatorname{vol}(v)} ,$$

which for D big and nef is the same as

$$\hat{\mu}_D(v) \geq \sqrt{D^2/\operatorname{vol}(v)} .$$

Remark 2.8. Using the language of $\mathbb{R}$-divisors, it is not hard to see that

$$D^2/\hat{\mu}_D(v) = \max\{s \in \mathbb{R} \mid \pi_K^* D - s\sum v_i E_i \text{ is nef}\} ,$$

and therefore, since nef divisors have nonnegative self-intersection numbers,

$$\hat{\mu}_D(v) \geq \sqrt{D^2/\operatorname{vol}(v)} .$$

By [5, Proposition 2.9] one also has the slightly stronger bound

$$\hat{\mu}_D(v) \geq \sqrt{\operatorname{vol}(D)/\operatorname{vol}(v)} .$$

Since we are mostly interested in cases where D is ample (more precisely, D will be a line in $\mathbb{P}^2$) the two bounds will be equivalent.

A (-1)-*ray* in $N_1(S_K)$ is a ray generated by a (-1)-*curve*, i.e., a smooth, irreducible, rational curve C with $C^2 = -1$ (hence $C \cdot K_n = -1$). Mori's Cone Theorem says that

$$\overline{\mathrm{NE}}(S_K) = \overline{\mathrm{NE}}(S_K)^{\succcurlyeq} + R_n ,$$

where $\overline{\mathrm{NE}}(S_K)^{\succcurlyeq}$ denotes the subset of $\overline{\mathrm{NE}}(S_K)$ described by rays generated by nonzero classes η such that $\xi \cdot \kappa \geq 0$ with κ being the canonical class, and

$$R_n \;=\; \sum_{\rho \text{ a } (-1)-\text{ray}} \rho \;\subseteq\; \overline{\mathrm{NE}}(S_K)^{\preccurlyeq} .$$

In the current work we are mostly interested in cases where the equality

$$\hat{\mu}_D(v) = \sqrt{D^2/\operatorname{vol}(v)}$$

holds; we call such valuations *minimal*.

Remark 2.9. In cases when $\overline{\mathrm{NE}}(S_K)$ is a polyhedral cone, Remark 2.8 yields that $\hat{\mu}_D(v)$ is a rational number, and therefore v can only be minimal if $\sqrt{\operatorname{vol}(v)}$ is rational. In fact, all examples of divisorial minimal valuations included here correspond to rational values of $\sqrt{\operatorname{vol}(v)}$, even for nonpolyhedral $\overline{\mathrm{NE}}(S_K)$.

3 Quasimonomial valuations

Our objects of study will be very general quasimonomial valuations on $\mathbb{P}^2$.

Remark 3.1. Quasimonomial valuations are exactly the valuations whose cluster of centers consists of a few free points followed by satellites, which may be finite or infinite in number, but not infinitely many proximate to the same center.

The genericity condition refers to the position of the free centers; it will be made precise below, after describing the continuity and semicontinuity properties of $\hat{\mu}$ on the space of quasimonomial valuations.

Regularity properties of $\hat{\mu}$ can be presented in various ways; for the sake of simplicity we specialize to the case when $S = \mathbb{P}^2$ and the center of v is the origin $O = (0,0) \in \mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x,y] \subset \mathbb{P}^2 = \operatorname{Proj}\mathbb{C}[X,Y,Z]$, with $x = X/Z, y = Y/Z$.

In this situation we write

$$\mu_d(v) = \max\{v(f) | f \in \mathbb{C}[x,y], \deg f \leq d\} ,$$

and

$$\hat{\mu}(v) = \lim_{d\to\infty} \frac{\mu_d(v)}{d} .$$

Definition 3.2. Given a series $\xi(x) \in \mathbb{C}[[x]]$ with $\xi(0) = 0$ and a real number $t \geq 1$, let

$$v(\xi, t; f) := \mathrm{ord}_x(f(x, \xi(x) + \theta x^t)) \ ,$$

where the symbol θ is transcendental over $\mathbb{C}$.

Equivalently, expand f as a Laurent series

$$f(x, y) = \sum a_{ij} x^i (y - \xi(x))^j \ ,$$

and put

$$v(\xi, t; f) := \min\{i + tj | a_{ij \neq 0}\} \ .$$

$f \mapsto v(\xi, t; f)$ is a valuation which we denote $v(\xi, t)$. Such valuations are called *monomial* if $\xi = 0$, and *quasimonomial* in general. Slightly abusing language, t will be called the *characteristic exponent* of $v(\xi, t)$ (even if it is an integer).

For simplicity we also write

$$\mu_d(\xi, t) = \mu_d(v(\xi, t)) \ ,$$

and

$$\hat{\mu}(\xi, t) = \hat{\mu}(v(\xi, t)) \ .$$

Remark 3.3. The valuation $v(\xi, t)$ only depends on the $\lfloor t \rfloor$-th jet of ξ, so for fixed t this series can be safely assumed to be a polynomial; however, later on we'll let t vary for a fixed ξ.

Remark 3.4. The cluster K of centers of $v(\xi, t)$ can be easily described from the continued fraction expansion

$$t = n_1 + \cfrac{1}{n_2 + \frac{1}{n_3 + \frac{1}{\ddots}}} \ .$$

The cluster K consists of $n = \sum n_i$ centers; if $t = n_1$ then they are all on the strict transform of the germ

$$\Gamma \colon y = \xi(x) \ ,$$

otherwise the first $n_1 + 1$ are on Γ and the rest are satellites: starting from $p_{n_1} + 1$ there are n_2 points proximate to p_{n_1}, followed by n_3 points proximate to $p_{n_1+n_2}$ and so on. The weights are $v_i = 1$ for $i = 1, \ldots, n_1$, then $v_i = t - n_1$ for $i = n_1 + 1, \ldots, n_1 + n_2$, and $v_i = v_{n_1+\cdots+n_{j-1}} - n_j v_{n_1+\cdots+n_j}$ for $i = n_1 + \cdots + n_j + 1, \ldots, n_1 + \cdots + n_{j+1}$. See Figure 2 for an example.

If t is rational, the continued fraction is finite, and so the valuation is divisorial. More precisely,

$$v(\xi, t; f) = v_n \cdot \mathrm{ord}_{E_n}(f) \ .$$

The factor v_n serves normalization purposes: in this way the function $t \mapsto v(\xi, t)$ becomes a continuous map from $[1, \infty)$ to the tree of valuations centered at O [11].

If t is irrational, then the sequence of centers is infinite and the group of values has rational rank 2.

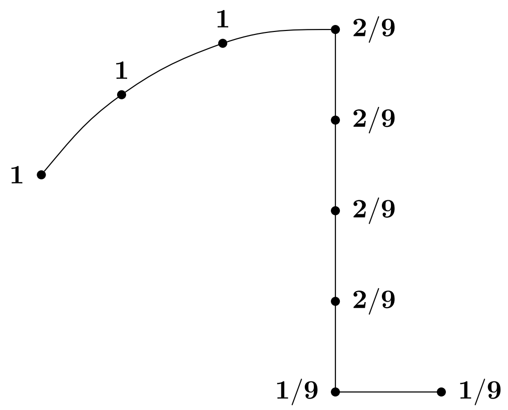

Figure 2: Enriques diagram of the centers of $v(\xi, 3 + \frac{1}{4+1/2})$.

Corollary 3.5. *With notation as above,*

$$\operatorname{vol}(v(\xi,t)) = t^{-1} \ , \ \mu_d(\xi,t) \geq d\sqrt{t} \ ,$$

and

$$\hat{\mu}(\xi,t) \geq \sqrt{t} \ ,$$

so the quasimonomial valuation $v(\xi,t)$ is minimal whenever $\hat{\mu}(\xi,t) = \sqrt{t}$.

Proposition 3.6. *Fix a real number $t > 1$ and a natural number d. Set $k = \lceil t \rceil$ and denote by $\mathcal{J}_k \subset \mathbb{C}[[x]]$ the space of $(k-1)$-jets of power series with $\xi(0) = 0$, endowed with the Zariski topology coming from the map $\mathcal{J}_k \cong \mathbb{A}^{k-1}$.*

Then the function $\xi \mapsto \mu_d(\xi,t)$ descends to an upper semicontinuous function

$$\mathcal{J}_k \to \langle 1, t\rangle_{\mathbb{Q}} \subset \mathbb{R}$$

which takes on only finitely many values.

It follows that for fixed t, $\hat{\mu}(\xi,t)$ takes its smallest value for ξ with very general jet ξ_{n-1}.

Proof. Because only the k free centers of $v(\xi,t)$ depend on ξ ($k = n_1$ in the continued fraction expansion if t is an integer and $k = n_1 + 1$ otherwise), it is clear that the valuation only depends on the $(k-1)$-th jet of ξ, and the existence of the function

$$\mathcal{J}_k \to \langle 1, t\rangle_{\mathbb{Q}} \subset \mathbb{R}$$

is clear. We will prove that it only takes a finite number of values and that for fixed m, the preimage of $[m, \infty)$ is Zariski-closed.

Given fixed t and d, there exists $m_{t,d} \in \langle 1, t\rangle_{\mathbb{Q}}$ such that $f \in \mathbb{C}[x,y]$, $v(\xi,t;f) \geq m_{t,d}$ implies $f \in (x,y)^{d+1}$ independently on ξ. Thus

$$\mu_d(\xi,t) < m_{t,d}$$

for all ξ.

Similarly, there exists $i_{t,d}$ such that no $f \in \mathbb{C}[x,y]_d$ has a strict transform going through any center p_i of $v(\xi,t)$ with $i > i_{t,d}$. Therefore for every $f \in \mathbb{C}[x,y]_d$, the value $v(\xi,t;f)$ belongs to the finite set

$$(\bigoplus_{i=1}^{i_{t,d}} \mathbb{N}v_i) \cap [1, m_{t,d}) \ ,$$

and the $\mu_d(\xi,t)$ belong to this set.

Now let V be the $\mathbb{C}$-subspace of $\mathbb{C}[\theta, x, x^t]$ consisting of polynomials P with $\deg_\theta(P) \leq d$ and $\deg_x(P) < m_{t,d}$. The space V is obviously finite-dimensional, $V \cong \mathbb{C}^N$ after taking the basis given by monomials.

Consider the composition of the substitution map

$$\mathcal{J}_k \times \mathbb{C}[x,y]_d \to \mathbb{C}[\theta][[x,x^t]] \ ,$$

given by $(s,f) \mapsto f(x, \xi(x) + \theta x^t)$, with truncation $\mathbb{C}[\theta][[x,x^t]] \to V$, seen as an algebraic morphism of $\mathbb{C}$-schemes.

For each value m, the 'incidence' subset

$$\{(\xi,f) \in \mathcal{J}_k \times \mathbb{C}[x,y]_d \,|\, v(\xi,t;f) \geq m\}$$

is by definition the preimage of the Zariski-closed set

$$\{\eta \in V \mid \operatorname{ord}_x(\eta(x)) \geq a\}$$

hence Zariski-closed. It is also closed under scalar multiplication on the second component, so it determines a closed subset $I_m \subset \mathcal{J}_k \times \mathbb{P}(\mathbb{C}[x,y]_d)$.

The locus in $\mathcal{J}_k$ where $\mu_d(\xi,t) \geq m$ is the projection of I_m to $\mathcal{J}_k$, therefore it is Zariski-closed. □

Proposition 3.7. *For every $\xi(x)$, the function $t \mapsto \hat{\mu}(\xi,t)$ is continuous.*

Proof. For every $f \in \mathbb{C}[x,y]$, the function $\mu_f : t \mapsto v(\xi,t;f)/\deg(f)$ is continuous and piecewise affine linear with nonnegative integer slopes (compare with [4, Corollary C]). In particular, it is nondecreasing and satisfies the property that

$$\frac{\mu_f(t_1)}{t_1} \geq \frac{\mu_f(t_2)}{t_2}$$

whenever $t_1 < t_2$.

Therefore

$$\mu_f(t) \leq \mu_f(t+\epsilon) \leq \mu_f(t) + (\mu_f(t)/t)\epsilon$$

for $\epsilon > 0$, and an analogous bound works for $\epsilon < 0$.

The function $t \mapsto \hat{\mu}(\xi,t)$ in the claim is $\sup_{f \in \mathbb{C}[x,y]}\{\mu_f\}$. The family of functions μ_f is equicontinuous by the remarks above and so this supremum is continuous; for instance,

$$a(t) \leq a(t+\epsilon) \leq a(t) + (a(t)/t)\epsilon$$

whenever $\epsilon > 0$. □

Remark 3.8. We proved in Proposition 3.6 that for a fixed t, very general series $\xi(x)$ give the same, minimal, value $\hat{\mu}(\xi,t)$ which we denote $\hat{\mu}(t)$.

By the countability of the rational number field, it follows that very general series $\xi(x)$ give the same (minimal) function $\hat{\mu}(\xi, t)$ of $t \in \mathbb{Q}$. Continuity of the functions $\hat{\mu}(\xi, t)$ then imply that very general series give the same function over all of $\mathbb{R}$, and also the following:

Corollary 3.9. *The function $t \mapsto \hat{\mu}(t)$ is continuous.*

The next claim will show the first analogy to Nagata's conjecture.

Proposition 3.10. *If t is the square of an integer, then a very general quasi-monomial valuation $v(\xi, t)$ is minimal.*

Proof. For integral values of t, the cluster of centers of $v(\xi, t)$ consists of the first t points infinitely near to the origin along the branch $y = \xi(x)$, and for each integer $m = qt + r$ (with $0 \leq r < t$) the corresponding valuation ideal is

$$\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(-q(E_1 + \cdots + E_t) - (E_1 + \cdots + E_r))) \ .$$

Now for every $d > 0$ and very general ξ, we will prove that $\mu_d(\xi, t) \leq d/\sqrt{t}$. To this end, we need to see that for every integer $m > d\sqrt{t}$ and very general ξ, the valuation ideal $\mathcal{I}_m$ has no sections of degree d, equivalently,

$$H^0(\mathcal{O}_{S_K}(dL - q(E_1 + \cdots + E_t) - (E_1 + \cdots + E_r))) = 0 \ ,$$

where L denotes the pullback of a line to S_K. By semicontinuity (Proposition 3.6) it will be enough to see this for a particular choice of ξ, e.g., an irreducible polynomial of degree $a = \sqrt{t}$. But the strict transform on S_K of the projectivized curve

$$D \colon YZ^{a-1} = Z^a \xi(X/Z)$$

defined by ξ is then an irreducible curve of self-intersection zero, therefore nef, and

$$D \cdot (dL - q(E_1 + \cdots + E_t) - (E_1 + \cdots + E_r))) = d\sqrt{t} - m < 0 \ ,$$

hence we are done. $\square$

4 Anticanonical surfaces

The rational surface S_K obtained by blowing up the cluster of centers of a valuation v will be called anticanonical if there is an effective anticanonical divisor on it. Under this hypothesis, adjunction becomes a very powerful tool to study the geometry of S_K. This section contains a complete description of the Mori cone of S_K for $v = v(\xi, t)$ with $t \leq 7$, and substantial information for $t = 7 + \frac{1}{n_2}$, $n_2 \in \mathbb{N}$.

Proposition 4.1. *Let $v(\xi, t)$ be a divisorial quasimonomial valuation, and S_K the blowup of its cluster of centers. Let $A = [1, 7] \cup \{7 + \frac{1}{n}\}_{n \in \mathbb{N}} \cup \{9\} \subset \mathbb{R}$.*

1. *If $t \in A$, then S_K is anticanonical.*

2. *If S_K is anticanonical and $v(\xi, t)$ is very general, then $t \in A$.*

Proof. The question is whether the anticanonical class $-\kappa = 3L - \sum E_i$ on S_K (where L denotes the pullback of a line) has nonzero global sections.

If t is an integer, then K consists of t free points; if $t \leq 9$, for a general choice of these points there is a smooth cubic going through them all so $-\kappa$ is effective, whereas for $t > 9$ there is no such plane cubic.

If $t = n_1 + \frac{1}{n_2}$ is a nonintegral rational, then $K = (p_1, \dots, p_{n_1+n_2})$ has $n_1 + 1$ free centers and $n_2 - 1 > 0$ satellites, all of them proximate to p_{n_1}; so $\tilde{E}_{n_1} = E_{n_1} - E_{n_1+1} - \cdots - E_{n_1+n_2}$. A quick unloading computation (see Section 2) then shows that

$$\begin{aligned} H^0(\mathcal{O}_{S_K}(-\kappa)) &= H^0(\mathcal{O}_{S_K}(3L - \sum E_i)) \\ &= H^0(\mathcal{O}_{S_K}(3L - 2E_1 - (E_2 + \cdots + E_{n_1})) \ . \end{aligned}$$

Consequently, S_K is anticanonical exactly when there exists a nodal cubic having the node at O and going through the free points $p_2, \dots, p_{n_1}$. For a general choice of the free points, this is the case exactly when $n_1 \leq 7$.

Finally, if the continued fraction of t has more than 2 coefficients, the corresponding unloading computation leads to $H^0(\mathcal{O}_{S_K}(3L - 2E_1 - (E_2 + \cdots + E_{n_1+1}))$, so S_K is anticanonical exactly when there exists a nodal cubic having the node at O and going through the free points $p_2, \dots, p_{n_1+1}$. This is the case exactly when $n_1 \leq 6$, and covers all rationals $t \in [1, 7]$. $\square$

Remark 4.2. Note that if $t \leq 7$, then K has at most 7 free centers, so there is always a divisor $\tilde{\Gamma}$ in $|3L - 2E_1 - \sum_{i>1, p_i \text{ free}} E_i|$. For general ξ, p_1, p_2, p_3 are not aligned and $p_1, \dots, p_6$ do not belong to a conic, so $\tilde{\Gamma}$ can be assumed to be the strict transform of an irreducible nodal cubic Γ, and $\Gamma_K = \tilde{\Gamma} + \sum \tilde{E}_i$ on S_K is a particular anticanonical divisor which contains all exceptional components (independently of t). For nongeneral ξ, $\tilde{\Gamma}$ may be reducible, but one can still determine an effective anticanonical divisor which contains all exceptional components, possibly with multiplicities; the details of each case are left to the reader.

Proposition 4.3. *Let $v(\xi, t)$ be a divisorial quasimonomial valuation with $t \leq 7$, and S_K the blowup of its cluster of centers. Then $\overline{\mathrm{NE}}(S_K)$ is a polyhedral cone, spanned by the classes of the $\tilde{E}_i$, $\tilde{\Gamma}$ and finitely many (-1)-curves, where Γ is a nodal cubic as above.*

Proof. Let Γ_K be an effective anticanonical divisor containing all exceptional components; for general ξ we can write $\Gamma_K = \tilde{\Gamma} + \sum \tilde{E}_i$, where Γ is a nodal cubic. Particular cases in which the cubic is irreducible are treated similarly and we leave the details to the reader. We claim that every irreducible curve $C \subset S_K$ which is not a component of Γ_K lies in $\overline{\mathrm{NE}}(S_K)^{\preceq}$. Indeed, C is the strict transform of a curve $\pi_K(C) \subset \mathbb{P}^2$; if $\pi_K(C)$ does not go through the origin, then C intersects $\tilde{\Gamma}$ and so

$$C \cdot \kappa = -(C \cdot (\Gamma_K)) \leq -(C \cdot \tilde{\Gamma}) < 0 \ ,$$

otherwise C intersects some $\tilde{E}_i$ and again

$$C \cdot \kappa = -(C \cdot (\Gamma_K)) \leq -(C \cdot \tilde{E}_i) < 0 \ .$$

By Mori's cone theorem, $\overline{\mathrm{NE}}(S_K)$ is generated by the rays spanned by the components of Γ_K and the (-1)-curves, so it only remains to see that there are finitely many (-1)-curves.

Now, a (-1)-curve C satisfies $C \cdot \kappa = -1$, so if it is not a component of Γ_K, it must intersect it in exactly one component. Write $C = dL - \sum m_i E_i$. If C meets $\tilde{E}_k$ only, it must satisfy $m_j = \sum_{p_i \succ p_j} m_i$ for all $j \neq k$, $m_k = \sum_{p_i \succ p_k} m_i + 1$ and $3d = \sum m_i$. These are $n+1$ linearly independent conditions which uniquely determine the class of C; so there is at most one (-1)-curve meeting $\tilde{E}_k$. On the other hand, C can not meet $\tilde{\Gamma}$ only, because it would satisfy $m_j = \sum_{p_i \succ p_j} m_i = 0$ for all j, and $3d = \sum m_i + 1 = 1$. In all, there are at most n (-1)-curves in addition to the components of Γ_K. □

For $7 < t < 8$, it is not clear which values of t give polyhedral Mori cones, but C. Galindo and F. Monserrat [13, corollary 5] give some positive results in this context. In particular, they show that for $t = 7 + 1/n_2$ with $n_2 = 1, 2, \ldots, 8$, $\overline{\mathrm{NE}}(S_K)$ is polyhedral. Proposition 4.3 can be seen as a strengthening of [13, Corollary 5.(3)].

Computations suggest that there might be only finitely many (-1)-curves, and $\overline{\mathrm{NE}}(S_K)$ might be polyhedral, for $t = 7 + 1/n_2$ without restrictions on n_2, provided that ξ is general, but this requires delicate arguments beyond the scope of this work; see however Remark 4.6. Nevertheless, irreducible curves C with $C^2 < -1$ can be completely determined.

Proposition 4.4. *Let $v(\xi, t)$ be a very general divisorial quasimonomial valuation with $t = 7 + 1/n_2$ for $n_2 \geq 1$, and let S_K be the blowup of its cluster of centers. The only curves C in S_K with $C^2 \leq -2$ are components of the exceptional divisors.*

Proof. As before, let Γ be a nodal cubic curve which has its node at the origin and goes through six additional free centers, $p_2, \ldots, p_7 \in K$. Then $\Gamma_K = \tilde{\Gamma} + \sum_{i=1}^{7} \tilde{E}_i$ on S_K is the unique effective anticanonical divisor.

By adjunction we have $C^2 + \kappa_{S_K} \cdot C = 2g - 2$, so $C^2 < -2$ implies $\Gamma_K \cdot C < 0$, hence C is a component of Γ_K. Computing the self-intersection of each of them shows that the only possibility is $C = \tilde{E}_7 = E_7 - E_8 - \cdots - E_n$ with $C^2 = -1 - n_2$.

By adjunction again, if $C^2 = -2$, then C is rational and $\kappa_{S_K} \cdot C = 0$, i.e., it is a (-2)-curve. Thus the question is what (-2)-curves can occur on S_K. The exceptional components $\tilde{E}_i$ for $i \neq 7, n$ are (-2)-curves. Now assume that C is not one of them. Then $\Gamma_K \cdot C = 0$ implies $C \cdot \tilde{E}_i = 0$ for $i = 0, \ldots, 7$, and $C \cdot \tilde{E}_i \geq 0$ for $i > 7$.

Write $C = dL - m_1 E_1 - \cdots - m_n E_n$. The constraint $C \cdot \tilde{E}_7 = 0$ gives $m_7 = m_8 + \cdots + m_n$. The constraints $C \cdot \tilde{E}_i = 0$ for $i = 1, \ldots, 6$ give $m_1 = \cdots = m_7$. Taking $m = m_1$, $C \cdot \tilde{\Gamma} = 0$ gives $3d = 7m + m_8 + \cdots + m_n = 8m$, so $d = 8m/3$. Note that d is an integer.

Consider the case that $n_2 = 1$. Then $-2 = C^2 = (8m/3)^2 - 8m^2 = -8m^2/9$. This has no integer solutions, so no C exists.

Next consider the case that $n_2 = 2$, so $n = 9$. The possible solutions C to $C^2 = -2$, $C \cdot \kappa_{S_K} = 0$ with $C \cdot L \geq 0$ are known (see the second half of the proof of [17, Proposition 25.5.3]); they are: $(E_i - E_j) - s\kappa_{S_K}$ with $1 \leq i, j \leq 9$, $i \neq j$, $s \geq 0$; $(L - E_i - E_j - E_k) - s\kappa_{S_K}$ with $1 \leq i, j, k \leq 9$, i, j, k distinct, $s \geq 0$; $(2L - E_{i_1} - \cdots - E_{i_6}) - s\kappa_{S_K}$ with $1 \leq i_j \leq 9$, i_j distinct for $1 \leq j \leq 6$, $s \geq 0$; and $(3L - 2E_{i_1} - E_{i_2} - \cdots - E_{i_8}) - s\kappa_{S_K}$ $1 \leq i_j \leq 9$, i_j distinct for $1 \leq j \leq 8$,

$s \geq 0$. An exhaustive check shows that each of these divisors intersects some exceptional component or Γ negatively, and thus is either itself a component of an exceptional curve, or is not reduced or irreducible.

Now consider the case that $n_2 \geq 3$, so $n \geq 10$, and we can write $C = dL - m(E_1 + \cdots + E_7) - m_8E_8 - \cdots - m_nE_n = (8m/3)L - m(E_1 + \cdots + E_7) - m_8E_8 - \cdots - m_nE_n$. Let $m = 3b$, so $C = 8bL - 3b(E_1 + \cdots + E_7) - m_8E_8 - \cdots - m_nE_n$. Then $\Gamma_K \cdot C = 0$ gives $3b - m_8 - \cdots - m_n = 0$ and $C^2 = -2$ gives $b^2 - m_8^2 - \cdots - m_n^2 = -2$. Numerical considerations no longer suffice; there are many solutions to $3b - m_8 - \cdots - m_n = 0$ and $b^2 - m_8^2 - \cdots - m_n^2 = -2$. For example, we have $C = 8L - 3(E_1 + \cdots + E_7) - E_8 - E_9 - E_{10}$ (i.e., $n = 10$, $n_2 = 3$, $b = 1$, and $m_8 = m_9 = m_{10} = 1$).

So consider any such $C = dL - m(E_1 + \cdots + E_7) - m_8E_8 - \cdots - m_nE_n = b(8L - 3(E_1 + \cdots + E_7)) - m_8E_8 - \cdots - m_nE_n$. Let $B = 8L - 3(E_1 + \cdots + E_7)$. We first check that B is nef. Note that $3B = (3L - E_1 - \cdots - E_7) + (3L - 2E_1 - \cdots - E_7) + (3L - E_1 - 2E_2 - E_3 - \cdots - E_7) + \cdots + (3L - E_1 - \cdots - E_6 - 2E_7)$. Each divisor in this sum is itself a sum of $\tilde{\Gamma}$ and exceptional components. But for each such divisor D we have $3B \cdot D \geq 0$, hence $3B$ (and therefore B) is nef. By [14, Proposition III.2], we may therefore assume that B is a reduced and irreducible divisor on S_K and since $B \cdot \tilde{\Gamma} = B \cdot \tilde{E}_i = 0$ for $i < 7$, we see $B|_{\Gamma_K}$ is a divisor which vanishes on each component $\tilde{E}_i$, $i < 7$ of Γ_K, and consists of a divisor B' of degree 3 on the interior of component $\tilde{E}_7$. Since $E_i|_{\Gamma_K} = E_n|_{\Gamma_K}$ for $i \geq 8$ and E_i is disjoint from $\tilde{E}_j$ for $i \geq 8$ and $j < 7$, we see $(-m_8E_8 - \cdots - m_nE_n)|_{\Gamma_K}$ is a divisor which is trivial on each component of Γ_K except $\tilde{E}_7$, and on $\tilde{E}_7$ it gives the divisor $(m_8 + \cdots + m_n)p_8 = mp_8 = 3bp_8$.

Consider the restriction exact sequence

$$0 \to \mathcal{O}_{S_K}(C - \Gamma_K) \to \mathcal{O}_{S_K}(C) \to \mathcal{O}_{\Gamma_K}(C) \to 0$$

and assume that C is a prime divisor. Then we have, since $C^2 < 0$ and so C is fixed, we have $h^0(S_K, \mathcal{O}_{S_K}(C - \Gamma_K)) = 0$ and we have $h^0(\mathcal{O}_{S_K}(C)) = 1$, which we will use in a moment. By taking cohomology of the short exact sequence and using what we have just seen about the cohomology of the terms, we conclude that $h^0(\mathcal{O}_{\Gamma_K}(bB' - 3bp_8)) > 0$. But $\deg(bB' - 3bp_8) = 0$ so $h^0(\mathcal{O}_{\Gamma_K}(bB' - 3bp_8)) > 0$ implies $bB' - 3bp_8 \sim 0$ (where $\sim$ denotes linear equivalence). Since B' is fixed of positive degree but p_8 is very general, this is impossible. Thus there is no such (-2)-curve C. □

Remark 4.5. When $8 \leq n \leq 15$, it is enough for p_8 to be a general, not very general, point of $\tilde{E}_7$ in order to conclude that S_K has no (-2)-curves other than those arising as components of the exceptional loci of the points blown up. To see this, consider a prime divisor $C \subset S_K$ such that $K_{S_K} \cdot C = 0$ and $C \cdot L > 0$. Write $C \sim dL - m_1E_1 - \cdots - m_nE_n$. Then, as above, $C = dL - m(E_1 + \cdots + E_7) - m_8E_8 - \cdots - m_nE_n = b(8L - 3(E_1 + \cdots + E_7)) - m_8E_8 - \cdots - m_nE_n$ and $m = m_8 + \cdots + m_n$, so

$$-2 = C^2 = b^2 - m_8^2 - \cdots - m_n^2 \leq b^2 - \frac{m^2}{(n-7)^2}(n-7) = b^2\frac{n-16}{n-7},$$

hence for $8 \leq n \leq 15$ we have

$$d^2 = 8b^2 \leq 8\frac{2n-14}{16-n}.$$

Thus for $8 \le n \le 15$ we have $d^2 \le 128$, so $d \le 11$.

I.e., for $8 \le n \le 15$ we see that d is bounded (i.e., $C \cdot L \le 11$) and hence that there are only finitely many possible (-2)-classes C. Since it is only for these classes that we must avoid $C|_{-K_X} = 0$ in order for C not to be effective, it is enough for p_8 to be general, in order to know that every (-2)-class is a component of the exceptional locus of a blow up.

Remark 4.6. It has proved more difficult so far to say as much about (-1)-curves. For a general divisorial quasimonomial valuation $v(\xi, t)$, consider the cases where either $t = 7$, or $t = 7 + 1/n_2$ and $1 \le n_2 \le 2$. Let $X = S_K$ be the blowup of its cluster of centers; thus X is a blow up $7 \le n \le 9$ points. Then there are only finitely many (-1)-curves and we can be fairly specific about finding them. We will content oursleves here with merely finding finite lists of divisor classes such that the class of every (-1)-curve is on the list. It would take more somewhat arduous work to confirm that each class on the list is the class of a reduced irreducible curve.

If $n < 9$, then X is a seven or eight point blow up of $\mathbb{P}^2$, so there are only finitely many divisor class solutions to $C^2 = C \cdot K_X = -1$, and they can be listed explicitly (see [17, Poposition 26.1]). In addition to E_7 (if $n = 7$) or E_8 (if $n = 8$), among all such solutions only those listed below meet each exceptional component nonnegatively.

- $L - E_1 - E_2$,
- $2L - E_1 - \cdots - E_5$,
- $3L - 2E_1 - E_2 - \cdots - E_7$,
- $4L - 2E_1 - 2E_2 - 2E_3 - E_4 - \cdots - E_8$,
- $5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8$ and
- $6L - 3E_1 - 2E_2 - \cdots - 2E_8$.

An essentially similar argument works for $n = 9$: write down all solutions to $C^2 = K_X \cdot C = -1$, and cull from these those that meet each exceptional component nonnegatively. However, there now are infinitely many solutions to $C^2 = K_X \cdot C = -1$. The set of all such C is the set of all C of the form $C = E_1 + N + N^2 K_X/2$, where N is an arbitrary divisor with $N \cdot E_1 = N \cdot K_X = 0$. (To see this, note on the one hand that for $C = E_1 + N + N^2 K_X/2$ we have $C^2 = K_X \cdot C = -1$, while on the other, if C is any class with $C^2 = K_X \cdot C = -1$, then for $N = (C - E_1 + (C \cdot E_1 + 1)K_X)$ we have $N \cdot K_X = 0$ and $N \cdot E_1 = 0$. Moreover, $N^2 = -2(1 + C \cdot E_1)$, so $E_1 + N + N^2 K_X/2 = E_1 + N - (C \cdot E_1 + 1)K_X = C$.)

The subgroup of classes consisting of all N satisfying $N \cdot E_1 = N \cdot K_X = 0$ is spanned over the integers by $L - E_2 - E_3 - E_4$, $E_2 - E_3$, $E_3 - E_4$, $\ldots$, $E_8 - E_9$. This subgroup is negative definite with respect to the intersection form. Because of the negative definiteness, for all N with $N^2 \gg 0$ we have $(E_1 + N + (N^2/2)K_X) \cdot \tilde{E}_7 = N \cdot \tilde{E}_7 + N^2/2 < 0$. Thus there are only finitely many (-1)-curves C on X. More specifically, for $C = E_1 + N + (N^2/2)K_X$, one can check that $C \cdot L \ge 78$ implies $C \cdot \tilde{E}_7 < 0$. (One can carry out this check using elements such as $8L - 3E_2 - \cdots - 3E_9$, $E_2 - E_3$, $E_2 + E_3 - 2E_4$, $E_2 + E_3 + E_4 - 3E_5$, $\ldots$, $E_2 + \cdots + E_8 - 7E_9$, which give an orthogonal basis over the rationals for the space defined by $N \cdot E_1 = N \cdot K_X = 0$.) One can, using quadratic transformations, generate the set of all classes C satisfying $C^2 = C \cdot K_X = -1$ with $C \cdot L \leq 78$. Choosing those which meet each $\tilde{E}_i$ nonnegatively gives the following set:

- E_9;
- $L - E_1 - E_2$;
- $2L - E_1 - \cdots - E_5$;
- $3L - 2E_1 - E_2 - \cdots - E_7$;
- $4L - 2E_1 - 2E_2 - 2E_3 - E_4 - \cdots - E_8$;
- $5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8$;
- $6L - 3E_1 - 2E_2 - \cdots - 2E_8$;
- $7L - 3E_1 - \cdots - 3E_4 - E_5 - 2E_6 - 2E_7 - E_8 - E_9$;
- $8L - 3E_1 - \cdots - 3E_7 - E_8 - E_9$;
- $9L - 4E_1 - 4E_2 - 3E_3 - \cdots - 3E_7 - 2E_8 - E_9$; and
- $11L - 4E_1 - \cdots - 4E_7 - 3E_8 - E_9$.

5 A variation on Nagata's conjecture

In this section we elaborate on the close analogy with Nagata's conjecture.

Let K be a finite union of finite weighted clusters on $\mathbb{P}^2$, and assume that the proximity inequalities

$$m_p \geq \sum_q m_q$$

are satisfied, with the sum taken over all points proximate to p.

Then

$$\mathcal{H}_{K,m} = \pi_*(\mathcal{O}_{S_K}(-\sum_{p \in K} m_p E_p))$$

is an ideal sheaf for which

$$h^0(\mathcal{H}_{K,m}(d)) = (d+1)(d+2)/2 - \sum m_p(m_p+1)/2$$

for $d \gg 0$, and its general member defines a degree d curve with multiplicity m_p at each $p \in K$.

It is expected that, if K is suitably general, then the dimension count is correct as soon as it gives a nonnegative value:

Conjecture 5.1 (Greuel-Lossen-Shustin, [10, Conjecture 6.3])**.** *Let K be a finite union of weighted clusters on the plane, satisfying the proximity inequalities, and $\mathcal{H}_{K,m}$ the corresponding ideal sheaf. Assume that K is general among all clusters with the same proximities, and let d be an integer which is larger than the sum of the three biggest multiplicities of m. Then*

$$h^0(\mathcal{H}_{K,m}(d)) = \max\left\{0, \frac{(d+1)(d+2)}{2} - \sum \frac{m_p(m_p+1)}{2}\right\}.$$

Proposition 5.2. *If the Greuel-Lossen-Shustin conjecture holds, then $\forall t \geq 9$ a very general quasimonomial valuation $v(\xi, t)$ is minimal.*

Proof. By continuity of $\hat{\mu}(t)$, it is enough to consider rational $t > 9$. Let $K = (p_1, \dots, p_n)$ be the sequence of centers, with weights $(v_1, \dots, v_n)$. For each integer $k > 0$, set $m_k = kt/v_n$. We shall prove that there is a sequence of integers d_k with $m_k > d_k\sqrt{t}$ and $\lim_{k\to\infty} m_k/d_k = \sqrt{t}$ such that if ξ is very general, then the valuation ideal $\mathcal{I}_{m_k}$ has no sections of degree d. It will follow that $\hat{\mu}(\xi, t) \leq \lim_{k\to\infty} m_k/d \leq \sqrt{t}$ and $v(\xi, t)$ is minimal.

By lemma 2.4, the ideal $\mathcal{I}_{m_k} = (\pi_K)_*(\mathcal{O}_{S_K}(-\sum \bar{m}_i E_i)$ is simple and the three largest multiplicities are $\bar{m}_1 = \bar{m}_2 = \bar{m}_3 = k/v_n$. Hence $\bar{m}_1 + \bar{m}_2 + \bar{m}_3 = 3k/v_n < \sqrt{t}k/v_n$ and for large k (which is not restrictive), there exist integers $d_k < m_k/\sqrt{t}$ which also satisfy $d_k > \bar{m}_1 + \bar{m}_2 + \bar{m}_3$. Thus we may assume that this inequality holds, so the hypothesis in conjecture 5.1 is satisfied and $h^0(\mathcal{H}_{K,m}(d)) = \max\{0, (d+1)(d+2)/2 - \sum \bar{m}_i(\bar{m}_i + 1)/2\}$. By way of contradiction, assume $\mathcal{I}_{m_k}$ has sections of degree d_k. Then $(d_k+1)(d_k+2)/2 \geq \sum \bar{m}_i(\bar{m}_i+1)/2$, which together with $d_k < m_k/\sqrt{t} = \sum \bar{m}_i^2$ implies $3d_k + 2 > \sum \bar{m}_i \geq 10\,\bar{m}_1 > kt/v_n = m_k$, a contradiction. □

With this in mind, we propose the following:

Conjecture 5.3 (Nagata for quasimonomial valuations)**.** $\forall t \geq 9$ $\hat{\mu}(t) = \sqrt{t}$.

Proposition 5.4. *Conjecture 5.3 implies Nagata's conjecture.*

Proof. Let $t > 9$ be a nonsquare integer. By a "collision de front" and semincontinuity, Nagata's conjecture for t points would follow by showing that, for a very general $\xi(x) \in \mathbb{C}[[x]]$, and for every couple of integers d, m with $0 < d < m\sqrt{t}$, the ideal $(x^t, y - \xi(t))^m \cap \mathbb{C}[x, y]$ has no nonzero element in degree d. But this is an immediate consequence of $\hat{\mu}(t) = \sqrt{t}$. □

In view of the computations in next section, we expect that in fact the range of t for which $\hat{\mu}(t) = \sqrt{t}$ is larger; see Conjecture 6.10.

6 Supraminimal curves

If some valuation v is not minimal, this is due to the existence of a curve C (which may be taken irreducible and reduced) with larger valuation than what one would expect from the degree. These curves will be called *supraminimal*, and are the subject of this section. For simplicity, we fix $O \in \mathbb{A}^2 \subset \mathbb{P}^2$ as before.

Lemma 6.1. *If there is an irreducible polynomial $f \in \mathbb{C}[x, y]$ with*

$$v(\xi, t; f) > \frac{1}{\sqrt{\mathrm{vol}(v(\xi, t))}} \deg(f) \ ,$$

then $v(\xi, t; f) = \hat{\mu}(\xi, t) \deg(f)$.

Moreover, if $\hat{\mu}(\xi, t) > \frac{1}{\sqrt{\mathrm{vol}(v(\xi,t))}}$, then there is such an irreducible polynomial f.

In the case above we say that f computes $\hat{\mu}(\xi, t)$.

Proof. By continuity of $\hat{\mu}(\xi, t)$ as a function of t, it is enough to consider the case $t \in \mathbb{Q}$. Let $v = v(\xi, t)$.

Let f be as in the claim, and $d = \deg f$. It will be enough to prove that, for every polynomial g with degree e and $v(g) = w > \frac{e}{\sqrt{\mathrm{vol}(v)}}$, f divides g. Choose an integer k such that $kw \in \mathbb{N}$ is an integer multiple of t, and consider the ideal

$$I_{kw} = \{h \in \mathbb{C}[x, y] \,|\, v(h) \geq kw\}.$$

A general $h \in I_{kw}$ has kw/t Puiseux series roots, each of them of the form $\xi(x) + ax^t + \dots$; therefore the local intersection multiplicity of $h = 0$ with $f = 0$ is

$$I_0(h, f) \geq \frac{kw}{t} v(f) > \frac{kwd}{t\sqrt{\mathrm{vol}(v)}} = \frac{kwd}{\sqrt{t}}. \tag{†}$$

Since obviously $g^k \in I$, the intersection multiplicity $I_0(g^k, f)$ is bounded below by (†), and therefore

$$I_0(g, f) > \frac{wd}{\sqrt{t}} = dw\sqrt{\mathrm{vol}(v)} > de,$$

so f is a component of g.

Now assume $\hat{\mu}(v) > \frac{1}{\sqrt{\mathrm{vol}(v)}}$. So there is a polynomial $g \in \mathbb{C}[x, y]$ of degree e with $v(g) > \frac{e}{\sqrt{\mathrm{vol}(v)}}$. Since $v(f_1 \cdot f_2) = v(f_1) + v(f_2)$, it follows that at least one irreducible component f of g, satisfies $v(f) > \frac{\deg f}{\sqrt{\mathrm{vol}(v)}}$. □

Proposition 6.2. *Assume that $d \in \mathbb{N}, m_1/n_1, \dots, m_r/n_r \in \mathbb{Q}$, with $\gcd\{m_i, n_i\} = 1$ are such that, for a very general $\xi(x)$, there exists an irreducible $f \in \mathbb{C}[x, y]$ with $\deg(f) = d$ which decomposes in $\mathbb{C}[[x, y]]$ as a product of r irreducible series $f = f_1 \dots f_r$ with $\mathrm{ord}_x f_i(x, \xi(x)) = m_i$, $\mathrm{ord}_0 f_i(x, y) = n_i$. Consider the tropical polynomial*

$$\mu_f(t) = \sum_{i=1}^{r} \min(n_i t, m_i).$$

Then $\hat{\mu}(t) \geq \mu_f(t)/d$, with equality at all values of t such that $\mu_f(t) > d\sqrt{t}$.

Proof. It is immediate that $v(\xi, t; f) = \mu_f(t)$, so the inequality $\hat{\mu}(t) \geq \mu_f(t)/d$ is clear. Now assume that $\mu_f(t) > d\sqrt{t}$. This implies that $v(\xi, t)$ is not minimal, and therefore by lemma 6.1, f computes $\hat{\mu}(v(\xi, t)) = \hat{\mu}(t)$. □

Example 6.3. The easiest examples of the situation described in Proposition 6.2 are given by (smooth) curves of degree 1 and 2.

Namely, for $d = 1$, $m_1/n_1 = 2$, it is trivial that for every $\xi(x)$, there exists a degree 1 polynomial f with $\mathrm{ord}_x f(x, \xi(x)) = 2$, $\mathrm{ord}_0 f_i(x, y) = 1$; one simply has to take the equation of the tangent line to $y - \xi(x) = 0$, or $f = y - \xi_1(x)$ (where ξ_1 denotes the 1-jet).

Along the same line, for $d = 2$, $m_1/n_1 = 5$, it is easy to show that for every $\xi(x)$, there exists a degree 2 polynomial f with $\mathrm{ord}_x f(x, \xi(x)) = 5$, $\mathrm{ord}_0 f_i(x, y) = 1$, which for general ξ is irreducible; one simply has to take the equation of the conic through the first five points infinitely near to $(0, 0)$ on the curve $y - s(x) = 0$ (more fancily, the curvilinear ideal $(y - \xi(x)) + (x, y)^5 \subset \mathbb{C}[x, y]$

has maximal Hilbert function and colength 5, and therefore a unique element in degree 2).

Proposition 6.2 then gives that

$$\hat{\mu}(t) = \begin{cases} t & \text{if } 1 \le t \le 2, \text{ computed by a line,} \\ 2 & \text{if } 2 \le t \le 4, \text{ computed by a line,} \\ t/2 & \text{if } 4 \le t \le 5, \text{ computed by a conic,} \\ 5/2 & \text{if } 5 \le t \le 25/4, \text{ computed by a conic.} \end{cases}$$

In order to construct the supraminimal curves in general position computing the function $\hat{\mu}$ for small values of t, we need certain Cremona maps, presumably well known, which have been used by Orevkov in [19] to show sharpness of his bound on the degree of cuspidal rational curves.

Proposition 6.4. *Let $K = (p_1, \ldots, p_7)$ be a general cluster with p_{i+1} infinitely near to p_i for $i = 1, \ldots, 6$. There exists a degree 8 plane Cremona map Φ_8 whose cluster of fundamental points is K, with all points weighted with multiplicity 3, and satifying the following properties:*

1. *The characteristic matrix of Φ_8 is*

$$\begin{pmatrix} 8 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 & -1 \\ -3 & -1 & -1 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -2 & -1 \\ -3 & -1 & -1 & -1 & -1 & -2 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix}.$$

2. *The inverse Cremona map is of the same type, i.e., it has the same characteristic matrix and its fundamental points are a sequence, each infinitely near to the preceding one.*

3. *The only curve contracted by Φ_8 is the nodal cubic which is singular at p_1 and goes through $(p_2, \ldots, p_6)$. The only expansive fundamental point is p_6, whose relative principal curve is the nodal cubic going through the fundamental points of the inverse map, and singular at the first of them.*

Recall that the characteristic matrix of a plane Cremona map is the matrix of base change in the Picard group of the blow up $\pi : S \to \mathbb{P}^2$ that resolves the map, from the natural base formed by the class of a line and the exceptional divisors, to the natural base in the *image* $\hat{\mathbb{P}}^2$, formed by the class of a line there (the homaloidal net in the original $\mathbb{P}^2$) and the divisors contracted by the map (which are the exceptional divisors of $\pi' : S \to \hat{\mathbb{P}}^2$), see [1]. We use it later on to compute images of curves under Φ_8.

Proof. This proof is taken from [19, p. 667]; the only modification lies in the remark that K can be taken general. Indeed, for K is general, there exists a unique irreducible nodal cubic Γ with multiplicity 2 at p_1 and going through $p_2, \dots, p_7$. Φ_8 is then defined as follows: let $\pi_K : S_K \to \mathbb{P}^2$ be the blowup of all points on K. The (strict) exceptional divisors $\tilde{E}_i$ are (-2)-curves, E_7 is a (-1) curve. The strict transform $\tilde{\Gamma} \subset S_K$ is another (-1)-curve that meets the (strict) exceptional divisors $\tilde{E}_1$ and E_7. Blow down $\tilde{\Gamma}$, $\tilde{E}_1, \dots \tilde{E}_6$ to obtain another map $\pi'_K : S_K \to \hat{\mathbb{P}}^2$. Then take $\Phi_8 = \pi'_K \circ \pi_K^{-1}$. All the stated properties are easy to check. □

Denote $F_{-1} = 1$, $F_0 = 0$ and $F_{i+1} = F_i + F_{i-1}$ the Fibonacci numbers, and $\phi = (1 + \sqrt{5})/2 = \lim F_{i+1}/F_i$ the "golden ratio".

Proposition 6.5. *For each odd $i \geq 1$, there exist rational curves C_i of degree F_i with a single cuspidal singularity of characteristic exponent F_{i+2}/F_{i-2} whose six singular free points are in general position. These curves become (-1)-curves in their embedded resolution, and are supraminimal for t in the interval* $\left(\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}^2}{F_i^2}\right)$.

Note that for $i = 1$ the line is actually not singular (the "characteristic exponent" is 2, an integer) but the statement in that case means that the line goes through the first two of six infinitely near points in general position, i.e., the exponent is interpreted as $m_i/n_i = 2$ in proposition 6.2.

Proof. The existence of such curves, without the generality statement, is [19, Theorem C, (a) and (b)]. Since the construction goes by recursively applying the rational map Φ_8, and the free singular points of C_i are exactly the seven fundamental points of Φ_8, it follows from 6.4 that these can be chosen to be general. They are (-1)-curves after resolution because the starting point of the construction are the two lines tangent to the two branches of the nodal cubic Γ (which becomes an exceptional divisor after Φ_8) i.e., (-1)-curves (each is a line through a point and an infinitely near point).

Now, with notation as in Proposition 6.2,

$$\mu_f(t) = \begin{cases} \frac{F_i - 2}{F_i} t & \text{if } t \leq \frac{F_{i+2}}{F_{i-2}}, \\ \frac{F_{i+2}}{F_i} & \text{if } t \geq \frac{F_{i+2}}{F_{i-2}}. \end{cases}$$

supraminimality in the claimed interval follows. □

Corollary 6.6.

$$\hat{\mu}(t) = \begin{cases} \frac{F_i - 2}{F_i} t & \text{if } t \in \left[\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}}{F_{i-2}}\right], \\ \frac{F_{i+2}}{F_i} & \text{if } t \in \left[\frac{F_{i+2}}{F_{i-2}}, \frac{F_{i+2}^2}{F_i^2}\right]. \end{cases}$$

Remark 6.7. In addition to the preceding family of curves, nine additional (-1)-curves compute $\hat{\mu}(t)$ for some range of t (see table 6.1). The existence of these curves is proved as follows. D_1 and D_2 are well known. The rest are obtained by applying the Cremona map Φ_8 to already constructed curves (the names chosen indicate that curve X^* is built from curve X).

Name	$(d;v_i)$	m_i/n_i	t
D_1	$(3;2,1^{\times 6})$	1,7	$\left[\phi^4, \left(\frac{8}{3}\right)^2\right]$
D_2^*	$(48;18^{\times 7},3,2^{\times 7})$	$7,\ \left(7+\frac{1}{8}\right)^{\times 2},\ 8$	$\left[\left(\frac{24+\sqrt{457}}{17}\right)^2, \left(24-\sqrt{455}\right)^2\right]$
C_1^{**}	$(64;24^{\times 7},3^{\times 7},1^{\times 2})$	$7^{\times 2},\ 7+\frac{1}{7+1/2},\ 7+\frac{1}{7}$	$\left[\left(\frac{32-\sqrt{177}}{7}\right)^2, \left(\frac{16+\sqrt{179}}{11}\right)^2\right]$
D_1^*	$(24;9^{\times 7},2,1^{\times 6})$	$7,\ 7+\frac{1}{7},\ 8$	$\left[\left(\frac{6+\sqrt{22}}{4}\right)^2, \left(12-\sqrt{87}\right)^2\right]$
C_5^*	$(40;15^{\times 7},2^{\times 6},1^{\times 2})$	$7^{\times 2},\ 7+\frac{1}{6+1/2}$	$\left[\left(\frac{20+\sqrt{218}}{13}\right)^2, \left(\frac{107}{40}\right)^2\right]$
C_3^*	$(16;6^{\times 7},1^{\times 5})$	$7,\ 7+\frac{1}{5}$	$\left[\left(\frac{8+\sqrt{29}}{5}\right)^2, \left(\frac{43}{16}\right)^2\right]$
D_3	$(35;13^{\times 7},4,3^{\times 3})$	$7+\frac{1}{4},\ 8$	$\left[\left(\frac{35}{13}\right)^2, \left(\frac{35-\sqrt{877}}{2}\right)^2\right]$
C_1^*	$(8;3^{\times 7},1^{\times 2})$	$7,\ 7+\frac{1}{2}$	$\left[\frac{4+\sqrt{2}}{2}, \left(\frac{22}{8}\right)^2\right]$
D_2	$(6;3,2^{\times 7})$	1, 8	$\left[\left(\frac{3+\sqrt{7}}{2}\right)^2, \left(\frac{17}{6}\right)^2\right]$

Table 6.1: Sporadic supraminimal curves. $(d;v_i)$ denote degree and multiplicities sequence, with ${}^{\times k}$ meaning k-tuple repetition. m_i/n_i follows the notation of proposition 6.2, with ${}^{\times 2}$ meaning repetition again.

Example 6.8. As an example, let us show the existence of D_1^*. Let $K = (p_1,\dots,p_8)$ be a general cluster with each point infinitely near to the preceding one; we want to show that there is an irreducible curve of degree 24 with three branches, two smooth, one of which goes through $(p_1,\dots,p_7)$ and the other through all of K, and one singular, with characteristic exponent $50/7$. Because K is general, there exist a cubic D_1 with multiplicities $[2,1^6,0]$ on K and another cubic Γ through K that has a node at some other point q_1. Choose one of the branches of Γ and let $q_2,\dots,q_7$ be the points infinitely near to q_1 on that branch. Apply the Cremona map Φ_8 based on $(q_1,\dots,q_7)$: then $D_1^* = \Phi_8(D_1)$.

All these computations together show that indeed, (-1)-curves compute $\hat\mu$ in the anticanonical range:

Theorem 6.9. *For $t \in A$, $\hat\mu(t)$ is computed by (-1)-curves; more precisely, the (infinitely many) curves C_i and 7 of the curves in table 6.1.*

Figure 1 shows $\hat\mu(t)$ in the ranges where it is known, together with the lower bound $\sqrt{t}$.

The two curves C_1^{**} and C_5^* compute $\hat\mu$ in ranges of t which do not intersect the anticanonical locus A. We expect that there are no more curves with such behavior, and so propose the following strengthening of conjecture 5.3:

Conjecture 6.10. *Let $t \in \mathbb{R}$ be such that $\hat\mu(t) > \sqrt{t}$. Then $\mu_C(t) > \sqrt{t}$ for a curve C which is either on the list of table 6.1 or one of the C_i. Equivalently, if $t > 7 + 1/9$ is not contained in any one of the intervals of table 6.1, then a very general valuation $v(\xi, t)$ is minimal.*

Remark 6.11. For $t > (17/6)^2$, it is possible to show (using Cremona maps) that no (-1)-curve is ever supraminimal. Thus conjecture 6.10 splits naturally into two conjectures: first, that all supraminimal curves are (-1)-curves, and second, that the only supraminimal (-1)-curves in the interval $[7, 8]$ are the ones above. Our evidence for the latter statement is experimental, obtained by a computer search.

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