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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
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Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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Mehler–Heine Asymptotics of a Class of Generalized Hypergeometric Polynomials*

Cleonice F. Bracciali†
Departamento de Matemática Aplicada
UNESP - Univ. Estadual Paulista, SP, Brazil
E-mail address: cleonice@ibilce.unesp.br

Juan José Moreno–Balcázar‡
Departamento de Matemáticas
Universidad de Almería, Spain
E–mail address: balcazar@ual.es

Abstract

We obtain a Mehler–Heine type formula for a class of generalized hypergeometric polynomials. This type of formula describes the asymptotics of polynomials scale conveniently. As a consequence of this formula, we obtain the asymptotic behavior of the corresponding zeros. We illustrate these results with numerical experiments and some figures.

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Key words: Generalized Hypergeometric Polynomials; Asymptotics; Zeros; Bessel functions.

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1 Introduction

One of the main topics in the theory of orthogonal polynomials is the study of their asymptotics. Several types of asymptotics of polynomials can be studied, giving valuable information about the polynomials with degrees large enough. Mehler–Heine formulae provide us with the local asymptotics of polynomials which are scaled adequately, and they establish a limit relation between polynomials and Bessel functions of the first kind. As a consequence, we can deduce asymptotic relations between the zeros of the polynomials under study and the zeros of the corresponding Bessel function. These formulae were introduced for classical orthogonal polynomials by Mehler and Heine in the $19th$–century. For example, if we denote the Jacobi polynomials by $P_n^{(\alpha,\beta)}(x)$, the Laguerre ones by $L_n^{(\alpha)}(x)$, and by $J_\alpha(x)$ the Bessel function of the first kind and order α, then the Mehler–Heine formulae are (see [10]):

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta)}\left(1-\frac{z^2}{2n^2}\right)}{n^\alpha} = 2^\alpha z^{-\alpha} J_\alpha(z), \quad \lim_{n\to\infty} \frac{L_n^{(\alpha)}\left(\frac{z}{n}\right)}{n^\alpha} = z^{-\alpha/2} J_\alpha(2\sqrt{z}),$$

which hold uniformly on compact subsets of the complex plane, and where

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!\Gamma(\alpha+k+1)}. \tag{1}$$

Later this type of formulae has been studied in other frameworks such as: multiple orthogonal polynomials (see [5], [11]), orthogonal polynomials on the unit disk (see [4]), exceptional orthogonal polynomials (see [6]), Sobolev orthogonal polynomials (see, among others, the survey [8] though there has been a wide literature about this topic after that survey), etc. In those papers, the polynomials considered satisfy some type of orthogonality (standard or not).

It is well known that the orthogonal polynomials in the Askey scheme can be expressed in terms of terminating generalized hypergeometric functions (see [7]). Moreover, other families of polynomials, which are not necessarily orthogonal, such as Sister Celine polynomials, Cohen polynomials, Prabhakar and Jain polynomials, Laguerre–Sobolev type polynomials, etc can be also expressed in this way (see, for example, [9] and the references cited in that paper). Thus, all these families lie in the class of generalized hypergeometric polynomials. On the other hand, mathematical and physical applications of generalized hypergeometric functions can be found, for example, in [2, Sect. 16.23 and 16.24], and the references therein.

The main goal of this paper is to consider this wide class of polynomials, the generalized hypergeometric polynomials, and establish hypothesis under which Mehler–Heine type formulae can be obtained, so that we can describe asymptotically the scaled zeros of the corresponding polynomials in terms of the zeros of Bessel functions.

The structure of the paper is the following. In Section 2, we establish the Mehler–Heine asymptotics of these generalized hypergeometric polynomials and describe the asymptotic behavior of the corresponding zeros. We also illustrate the application of the main result by means of well-known families of polynomials. In Section 3, we provide some numerical results and plots.

2 Mehler–Heine type asymptotics

We consider generalized hypergeometric series (see, for example, [1] or [2])

$$ {}_pF_q\left(\begin{matrix} a_0, a_1, \ldots, a_{p-1} \\ b_0, b_1, \ldots, b_{q-1} \end{matrix}; z\right) := \sum_{i=0}^{\infty} \frac{(a_0)_i \cdots (a_{p-1})_i}{(b_0)_i \cdots (b_{q-1})_i} \frac{z^i}{i!}, $$

where b_j must not be nonpositive integers for $j = 0, \ldots, q-1$, and $(\cdot)_j$ denotes the Pochhammer symbol defined as

$$ (c)_j = \prod_{k=0}^{j-1} (c+k), \quad (c)_0 = 1. $$

The above series is convergent provided that either $p \leq q$, or $p = q+1$ and $|x| < 1$ (see [2]). Clearly, if we take $a_0 = -n$, then this series becomes a polynomial of degree at most n, i.e.,

$$ {}_pF_q\left(\begin{matrix} -n, a_1, \ldots, a_{p-1} \\ b_0, b_1, \ldots, b_{q-1} \end{matrix}; z\right) = \sum_{i=0}^{n} \frac{(-n)_i (a_1)_i \cdots (a_{p-1})_i}{(b_0)_i (b_1)_i \cdots (b_{q-1})_i} \frac{z^i}{i!}. \tag{2} $$

We denote by $\mathbb{Z}_- := \{0, -1, -2, -3, \ldots\}$. For our purposes, we take $b := b_0 \in \mathbb{R} \setminus \mathbb{Z}_-$, and use the following notation

$$ \begin{aligned} \alpha_{\mathbf{n}} &= (k_1 n + \ell_1, \ldots, k_{p-1} n + \ell_{p-1}), \quad p \geq 2, \\ \beta_{\mathbf{n}} &= (s_1 n + t_1, \ldots, s_{q-1} n + t_{q-1}), \quad q \geq 2, \end{aligned} \tag{3} $$

where $k_j > 0$ (resp. $s_j > 0$) and $k_j n + \ell_j$ (resp. $s_j n + t_j$) must not be nonpositive integers for $j = 1, \ldots, p-1$ (resp. $j = 1, \ldots, q-1$). With these assumptions we guarantee that (2) is a polynomial of degree exactly n. In fact, we can relax the assumption on $k_jn+\ell_j$, simply assuming that k_jn+l_j is not a negative integer greater than $-n$ for any j. However, this is irrelevant from the asymptotic point of view since $k_j>0$.

Thus, we are going to work with the generalized hypergeometric polynomials

$$ {}_pF_q(-n,\alpha_{\mathbf{n}};b,\beta_{\mathbf{n}};z) := {}_pF_q\left(\begin{matrix}-n, k_1n+\ell_1,\ldots,k_{p-1}n+\ell_{p-1}\\ b, s_1n+t_1,\ldots,s_{q-1}n+t_{q-1}\end{matrix};z\right), \tag{4} $$

where b must not be nonpositive integers. If $p=1$ (resp. $q=1$), then $\alpha_{\mathbf{n}}$ (resp. $\beta_{\mathbf{n}}$) does not appear in the expression of ${}_pF_q$, for example, ${}_1F_1(-n;b;z)$.

In this way, we can establish the main result.

Theorem 1. *Let $b\in\mathbb{R}\setminus\mathbb{Z}_-$. Using the notation given in (3)–(4), we have*

$$ \begin{aligned} \lim_{n\to\infty} {}_pF_q\left(-n,\alpha_{\mathbf{n}};b,\beta_{\mathbf{n}};\frac{z}{n^{p-q+1}}\right) &= 2^{b-1}\Gamma(b)\left(4\frac{k_1\cdots k_{p-1}}{s_1\cdots s_{q-1}}z\right)^{-(b-1)/2} \\ &\quad\times J_{b-1}\left(2\sqrt{\frac{k_1\cdots k_{p-1}}{s_1\cdots s_{q-1}}z}\right), \end{aligned} \tag{5} $$

uniformly on compact subsets of the complex plane.

Proof. We do the scaling $z\mapsto \frac{z}{n^{p-q+1}}$ in (4), and we get

$$ \begin{aligned} {}_pF_q\left(-n,\alpha_{\mathbf{n}};b,\beta_{\mathbf{n}};\frac{z}{n^{p-q+1}}\right) = \sum_{i=0}^{n} &\frac{(-n)_i(k_1n+\ell_1)_i\cdots(k_{p-1}n+\ell_{p-1})_i}{(b)_i(s_1n+t_1)_i\cdots(s_{q-1}n+t_{q-1})_i} \\ &\times\frac{1}{i!}\frac{z^i}{n^{i(p-q+1)}}. \end{aligned} $$

Taking into account the well–known relations (see, for example, [3])

$$ (c)_i=\frac{\Gamma(c+i)}{\Gamma(c)} \quad\text{and}\quad \lim_{n\to\infty}\frac{n^{b-a}\Gamma(n+a)}{\Gamma(n+b)}=1, $$

we can deduce for i and j fixed

$$ \lim_{n\to\infty} n^{-i}(k_jn+\ell_j)_i=k_j^i \quad\text{and}\quad \lim_{n\to\infty} n^{-i}(s_jn+t_j)_i=s_j^i\,, $$

where we have used $k_j > 0$ and $s_j > 0$. We also have $\lim_{n\to\infty} n^{-i}(-n)_i = (-1)^i$, for i fixed. Thus, we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(-n)_i(k_1 n+\ell_1)_i\cdots(k_{p-1}n+\ell_{p-1})_i}{(b)_i(s_1 n+t_1)_i\cdots(s_{q-1}n+t_{q-1})_i}\frac{1}{i!}\frac{z^i}{n^{i(p-q+1)}} \\
&= \left(\frac{-k_1\cdots k_{p-1}}{s_1\cdots s_{q-1}}z\right)^i \frac{1}{i!(b)_i}.
\end{aligned} \tag{6}$$

On the other hand, for each n positive integer we have,

$$\left|\frac{(-n)_i(k_1 n+\ell_1)_i\cdots(k_{p-1}n+\ell_{p-1})_i}{(b)_i(s_1 n+t_1)_i\cdots(s_{q-1}n+t_{q-1})_i}\frac{1}{i!}\frac{z^i}{n^{i(p-q+1)}}\right| \le C\frac{z^i}{i!(b)_i} := Cg_i(z),$$

and taking z into a compact subset K of the complex plane, we get

$$\sum_{i=0}^{\infty} Cg_i(z) = C\,{}_0F_1(-;b;z) < \infty.$$

The above expression together with (6) allow us to apply Lebesgue's dominated convergence theorem obtaining

$$\lim_{n\to\infty} {}_pF_q\left(-n,\alpha_{\mathbf{n}};b,\beta_{\mathbf{n}};\frac{z}{n^{p-q+1}}\right) = {}_0F_1\left(-;b;\frac{-k_1\cdots k_{p-1}}{s_1\cdots s_{q-1}}z\right).$$

To prove the result given in (5), it only remains to express the hypergeometric function ${}_0F_1$ in terms of a Bessel function of the first kind given in (1), that is,

$${}_0F_1(-;b;-z) = \Gamma(b)z^{-(b-1)/2}J_{b-1}(2\sqrt{z}).$$

□

Remark 1. When $p = 1$ or $q = 1$ we assume $\prod_{i=1}^{0} k_i = \prod_{i=1}^{0} s_i = 1$.

Remark 2. Notice that if we have a generalized hypergeometric polynomial like in (4), then this theorem points out how to scale the variable to obtain Mehler–Heine type asymptotics. In fact, the scaling depends only on the values p and q.

Remark 3. Notice that we have written (5) in that way to highlight that the limit function can be expressed as $\omega^{-d}J_d(\omega)$ with $d = b-1$ and $\omega = 2\sqrt{\frac{k_1\cdots k_{p-1}}{s_1\cdots s_{q-1}}z}$, and therefore it has only real zeros for $b > 0$. When b is negative and noninteger, complex zeros can occur. Since

$$\omega^{-d}J_d(\omega) = 2^{-d}\sum_{j=0}^{\infty}\frac{(-1)^j\omega^{2j}}{2^{2j}j!\Gamma(d+j+1)},$$

this function is even. Thus, if $z_0 \in \mathbb{C}$ is any zero of $\omega^{-d} J_d(\omega)$, then $-z_0$ and $\pm\overline{z_0}$ are zeros too. Indeed, the number of complex zeros of the function $z^{1-b} J_{b-1}(z)$ is $2\lfloor 1-b \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. Moreover, if $\lfloor 1-b \rfloor$ is odd, then two of these zeros are pure imaginary numbers (see [12, pp. 483-484]).

This remark will help us establish Corollary 1. In this way, we are going to introduce some notation about the zeros of these functions. We denote by $j_{b-1,i}$ the i–th positive zero of $z^{1-b} J_{b-1}(z)$, that is, $j_{b-1,1} < j_{b-1,2} < \dots$, and by $i_{b-1,k}$ with $k = 1, 2$ the two pure imaginary zeros of $z^{1-b} J_{b-1}(z)$, when they exist, i.e., when $b < 0$ and $\lfloor 1-b \rfloor$ is odd. When $b \in (-\infty, -1)$, the complex zeros of $z^{1-b} J_{b-1}(z)$ with real part different from 0 form a set

$$\mathfrak{A}_{b-1} := \{z_{b-1,1}, -z_{b-1,1}, \dots, z_{b-1,k}, -z_{b-1,k}\},$$

with $k = \lfloor 1-b \rfloor - 1$ when $\lfloor 1-b \rfloor$ is odd, and $\lfloor 1-b \rfloor$ when this value is even.

According to Remark 3, $i_{b-1,1} = -i_{b-1,2}$ and therefore we will write

$$i^2_{b-1} := i^2_{b-1,1} = (-i_{b-1,2})^2 \in (-\infty, 0).$$

On the other hand, we denote

$$c^2_{b-1,j} := z^2_{b-1,j} = (-z_{b-1,j})^2, \quad j = 1, \dots, k.$$

We denote by $x_{n,i}$, $i = 1, \dots, n$, the zeros of the generalized hypergeometric polynomial ${}_pF_q\left(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x\right)$. These zeros can be complex (see Remark 4), and therefore with this notation we only enumerate them. Now, Mehler–Heine type asymptotics given in Theorem 1 has a simple and nice consequence about the asymptotic behavior of the zeros of the generalized hypergeometric polynomials. In this way, applying Hurwitz's Theorem in the previous theorem and taking Remark 3 into account, we deduce the following result.

Corollary 1. *We have*

- *When* $b > 0$,
$$n^{p-q+1} x_{n,i} \to \frac{s_1 \cdots s_{q-1}}{4\, k_1 \cdots k_{p-1}}\, j^2_{b-1,i}\ .$$

- *When* $b \in (-1, 0)$,
$$n^{p-q+1} x_{n,1} \to \frac{s_1 \cdots s_{q-1}}{4\, k_1 \cdots k_{p-1}}\, i^2_{b-1},$$
$$n^{p-q+1} x_{n,i} \to \frac{s_1 \cdots s_{q-1}}{4\, k_1 \cdots k_{p-1}}\, j^2_{b-1,i-1}, \quad i \geq 2.$$

- *When $b \in (-\infty, -1)$, we get,*
  - *If $\lfloor 1-b \rfloor$ is odd,*

$$\begin{aligned} n^{p-q+1} x_{n,i} &\to \frac{s_1 \cdots s_{q-1}}{4\, k_1 \cdots k_{p-1}}\, c_{b-1,i}^2, \quad i = 1, \ldots \lfloor 1-b \rfloor - 1, \\ n^{p-q+1} x_{n,\lfloor 1-b \rfloor} &\to \frac{s_1 \cdots s_{q-1}}{4\, k_1 \cdots k_{p-1}}\, i_{b-1}^2, \\ n^{p-q+1} x_{n,i} &\to \frac{s_1 \cdots s_{q-1}}{4\, k_1 \cdots k_{p-1}}\, j_{b-1,i-\lfloor 1-b \rfloor}^2, \quad i \geq \lfloor 1-b \rfloor + 1. \end{aligned}$$

  - *If $\lfloor 1-b \rfloor$ is even,*

$$\begin{aligned} n^{p-q+1} x_{n,i} &\to \frac{s_1 \cdots s_{q-1}}{4\, k_1 \cdots k_{p-1}}\, c_{b-1,i}^2, \quad i = 1, \ldots \lfloor 1-b \rfloor, \\ n^{p-q+1} x_{n,i} &\to \frac{s_1 \cdots s_{q-1}}{4\, k_1 \cdots k_{p-1}}\, j_{b-1,i-\lfloor 1-b \rfloor}^2, \quad i \geq \lfloor 1-b \rfloor + 1. \end{aligned}$$

Remark 4. Observe that in Theorem 1 the values ℓ_j $(j = 0, \ldots, p-1)$ and t_k $(k = 0, \ldots, q-1)$ can be complex numbers. Then, the generalized hypergeometric polynomial (4) is a polynomial with complex coefficients, and all their zeros can be complex. But, for example, if $b > 0$, then according to Corollary 1 these zeros must go to real zeros when $n \to \infty$. We show an example in Table 1.

Table 1: Scaled zeros of ${}_pF_q(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x)$, $n^{p-q+1} x_{n,i}$ and limit values with $p = 4$, $q = 3$, $b = 3.2$, $\alpha_{\mathbf{n}} = (5n + 2 - i, 2n + 4.4 + 2i, 2n + 1.3)$ and $\beta_{\mathbf{n}} = (4n + 5.5 + 2i, n + 3, 2n + 1 - 2i)$.

$n^{p-q+1} x_{n,i}$	$i = 1$	$i = 2$	$i = 3$
$n = 50$	2.772974-0.07136404i	7.208271-0.18526461i	13.51223-0.3466352i
$n = 100$	2.837294-0.03669921i	7.378569-0.09540667i	13.83973-0.1788655i
$n = 200$	2.870103-0.01860900i	7.464701-0.04839507i	14.00345-0.0907761i
$n = 300$	2.881135-0.01246414i	7.493548-0.03241677i	14.05798-0.0608109i
$n = 400$	2.886670-0.00936998i	7.507997-0.02437004i	14.08523-0.0457175i
Limit values	2.903345890	7.551440462	14.16691692

Next, we are going to apply Theorem 1 to some families of polynomials. Obviously, we can recover the well-known Mehler–Heine formulae for classical Laguerre and Jacobi polynomials.

Laguerre orthogonal polynomials, $L_n^{(\alpha)}$, are defined (see [2, p. 443]) as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n,\\ \alpha+1\end{matrix};x\right),$$

then according to Theorem 1, we must scale as $x \mapsto x/n$, and we get

$$\lim_{n\to\infty}\frac{L_n^{(\alpha)}(x/n)}{n^\alpha} = \lim_{n\to\infty}\frac{(\alpha+1)_n}{n!n^\alpha}\,{}_1F_1\left(\begin{matrix}-n,\\ \alpha+1\end{matrix};\frac{x}{n}\right) = x^{-\alpha/2}J_\alpha(2\sqrt{x}).$$

Jacobi orthogonal polynomials are given by their explicit expression (see [2, p.442])

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right).$$

Then, doing the scaling $x \mapsto 1 - x^2/(2n^2)$ and applying Theorem 1 with $k_1 = 1$ and $\ell_1 = \alpha+\beta+1$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{P_n^{(\alpha,\beta)}(1-x^2/(2n^2))}{n^\alpha} &= \frac{1}{\Gamma(\alpha+1)}\lim_{n\to\infty}{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{x^2}{4n^2}\right)\\ &= \left(\frac{x}{2}\right)^{-\alpha}J_\alpha(x).\end{aligned}$$

We also obtain this type of formulae for some special cases of varying Jacobi polynomials. These polynomials have been studied broadly in the literature. Here, we consider the polynomials $P_n^{(\alpha,\beta_n)}(x)$ where $\beta_n = kn+\ell$. Then, for $k > -1$ and $kn+\ell$ are nonnegative integers, we can apply Theorem 1 with $k_1 = k+1$ and $\ell_1 = \alpha+\ell+1$ and we have

$$\begin{aligned}\lim_{n\to\infty}\frac{P_n^{(\alpha,kn+\ell)}(\cos(x/n))}{n^\alpha} &= \lim_{n\to\infty}\frac{P_n^{(\alpha,kn+\ell)}(1-x^2/(2n^2))}{n^\alpha}\\ &= 2^\alpha\left(\sqrt{k+1}\,x\right)^{-\alpha}J_\alpha\left(\sqrt{k+1}\,x\right).\end{aligned}\tag{7}$$

As far as we know, formula (7) is new and we can see the influence of the varying sequence β_n on the local asymptotics of the varying Jacobi polynomials. Taking into account the symmetry relation (see [10]) $P_n^{(\alpha,\beta)}(x) = (-1)^nP_n^{(\beta,\alpha)}(-x)$ we can also give a Mehler–Heine type formula for $P_n^{(kn+\ell,\beta)}(x)$.

Using Corollary 1 we get the asymptotic behavior of the scale zeros of these varying Jacobi polynomials, i.e.,

$$\lim_{n\to\infty} n\arccos(x_{n,i}) = \frac{j_{\alpha,i}}{\sqrt{k+1}}.$$

We illustrate this result in Table 2.

Table 2: $n \arccos(x_{n,i})$, for $i = 1, 2, 3, 4$, where $x_{n,i}$ are zeros of varying Jacobi polynomials $P_n^{(\alpha,kn+\ell)}(x)$, with $\alpha = 3$, $k = 2$, and $\ell = 3$.

$n \arccos(x_{n,i})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$n = 50$	3.553808938	5.437485980	7.251180615	9.040061345
$n = 100$	3.617406099	5.534411383	7.379743345	9.199244152
$n = 200$	3.650163324	5.584427086	7.446253216	9.281863874
$n = 300$	3.661229898	5.601337650	7.468768650	9.309873879
$j_{\alpha,i}/\sqrt{k+1}$	3.683588189	5.635529334	7.514329641	9.366622556

3 Numerical results and plots

We illustrate each case of Corollary 1 with numerical experiments. Tables 3, 4, 5, and 6 show examples of the scaled zeros of the polynomials ${}_pF_q(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x)$ for several values of the parameters covering the different cases in Corollary 1.

Table 3: Scaled zeros of ${}_pF_q(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x)$, $n^{p-q+1}x_{n,i}$, and limit values with $p = 4$, $q = 2$, $b = 2.7$, $\alpha_{\mathbf{n}} = (2n+5,\ 5n+4, 10n+2)$, and $\beta_{\mathbf{n}} = (7n+3)$.

$n^{p-q+1}x_{n,i}$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$n = 50$	0.3685097848	1.044570556	2.040260372	3.353764336
$n = 100$	0.3815670147	1.082163558	2.115364638	3.480873986
$n = 200$	0.3883188099	1.101464927	2.153534298	3.544641366
$n = 300$	0.3906034387	1.107974105	2.166343940	3.565906695
Limit values	0.3952246348	1.121105987	2.192087683	3.608429948

Table 4: Scaled zeros of ${}_pF_q(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x)$, $n^{p-q+1}x_{n,i}$ and limit values with $p = 3$, $q = 2$, $b = -0.51$, $\alpha_{\mathbf{n}} = (n+5,\ 2n+4)$, and $\beta_{\mathbf{n}} = (3n-2)$.

$n^{p-q+1}x_{n,i}$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$n = 100$	-0.5125961704	2.712821650	13.11913963	30.45738847
$n = 200$	-0.5285867673	2.797610057	13.53168903	31.42490016
$n = 300$	-0.5341169337	2.826909856	13.67388803	31.75698846
$n = 400$	-0.5369210238	2.841761889	13.74589783	31.92488576
Limit values	-0.5454935730	2.887148128	13.96565918	32.43614705

Table 5: Scaled zeros of ${}_pF_q(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x)$, $n^{p-q+1}x_{n,i}$ and limit values with $p = 5$, $q = 3$, $b = -4.7$, $\alpha_{\mathbf{n}} = (n + 5,\ 2n - 3, 4n - 2, n + 2.5)$, and $\beta_{\mathbf{n}} = (3n + 2, n + 3.2)$.

$n^{p-q+1}x_{n,i}$	$i = 1, 2$	$i = 3, 4$	$i = 5$
$n = 100$	-0.91580431 $\pm$ 1.1888420i	0.58363774 $\pm$ 1.7954946i	-1.4027762
$n = 200$	-0.92602579 $\pm$ 1.2019119i	0.58992949 $\pm$ 1.8156009i	-1.4182381
$n = 300$	-0.92972125 $\pm$ 1.2066715i	0.59224276 $\pm$ 1.8228585i	-1.4238619
$n = 400$	-0.93162543 $\pm$ 1.2091301i	0.59344144 $\pm$ 1.8265962i	-1.4267656
Limit values	-0.93757075 $\pm$ 1.2168298i	0.59721014 $\pm$ 1.8382584i	-1.4358546

$n^{p-q+1}x_{n,i}$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
$n = 100$	3.890351663	10.51334686	18.56598941	28.33916299
$n = 200$	3.934641689	10.63780780	18.79603886	28.70934750
$n = 300$	3.950503560	10.68157125	18.87525535	28.83385381
$n = 400$	3.958650613	10.70390684	18.91538509	28.89638312
Limit values	3.983985499	10.77280667	19.03799349	29.08527041

In Table 5, we have taken $b = -4.7$, then $\lfloor 1 - b \rfloor = 5$ is odd. Thus, according to Corollary 1, four complex numbers and one negative real number belong to the set of the limit values.

Table 6: Scaled zeros of ${}_pF_q(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x)$, $n^{p-q+1}x_{n,i}$ and limit values with $p = 3$, $q = 4$, $b = -1.3$, $\alpha_{\mathbf{n}} = (n + 5, 2n + 4)$, and $\beta_{\mathbf{n}} = (3n + 2, n + 3, 3n + 1)$.

$n^{p-q+1}x_{n,i}$	$i = 1, 2$	$i = 3$	$i = 4$	$i = 5$
$n = 100$	-1.5281402 $\pm$ 3.0432995i	22.07846924	68.17653987	136.0098247
$n = 200$	-1.5402918 $\pm$ 3.0675759i	22.25275073	68.70429887	137.0321472
$n = 300$	-1.5445394 $\pm$ 3.0760486i	22.31389119	68.89124261	137.3996579
$n = 400$	-1.5467015 $\pm$ 3.0803592i	22.34505034	68.98681754	137.5884435
Limit values	-1.5533451 $\pm$ 3.0935962i	22.44093282	69.28205198	138.1749534

In Table 6, we have taken $b = -1.3$, then $\lfloor 1 - b \rfloor = 2$ is even. Thus, according to Corollary 1, two complex numbers belong to the set of the limit values.

We have also included some plots with b negative, but $|b|$ larger than in the previous tables. Our choices have been $b = -10.4$ and $b = -11.5$, then we get $\lfloor 1 - b \rfloor = 11$ and $\lfloor 1 - b \rfloor = 12$, respectively. Therefore, ten complex numbers and one negative real number belong to the set of the limit values in the case $b = -10.4$, and twelve complex numbers are limit values of the scaled zeros in the case $b = -11.5$. See Figures 1 and 2.

Furthermore, we consider varying Jacobi polynomials $P_n^{(\alpha,\beta_n)}(x)$ with complex zeros converging to positive real zeros, and we obtain some nice plots in Figure 3.

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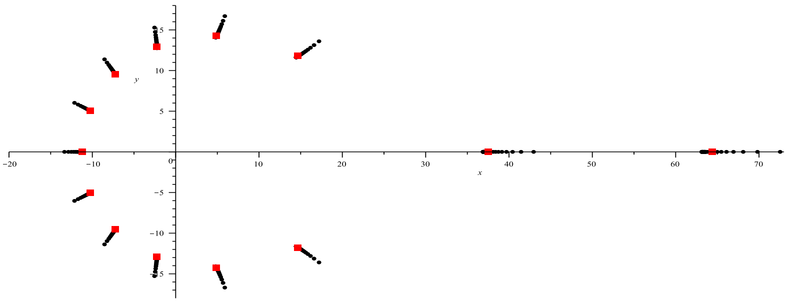
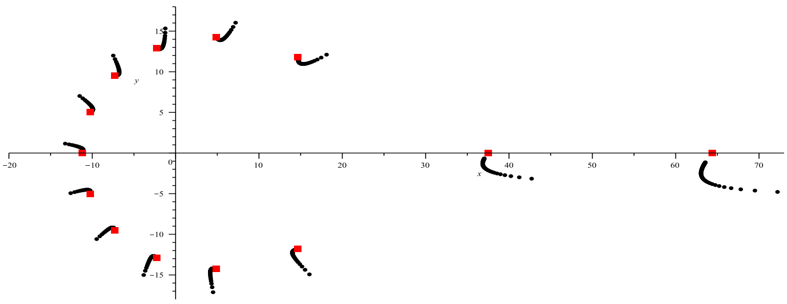
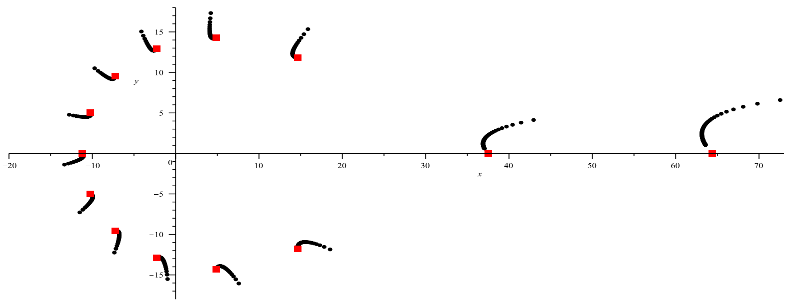


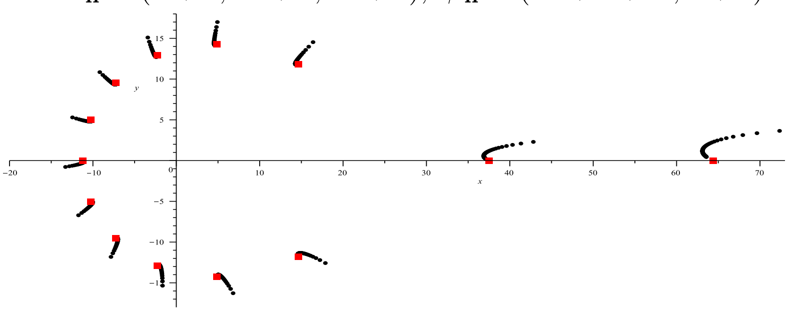
a. $\alpha_{\mathbf{n}} = (n+5, 2n+3, 2n+1)$, $\beta_{\mathbf{n}} = (3n+2, n+3)$



b. $\alpha_{\mathbf{n}} = (n+5+2i, 2n+3, 2n+1)$, $\beta_{\mathbf{n}} = (3n+2, n+3)$

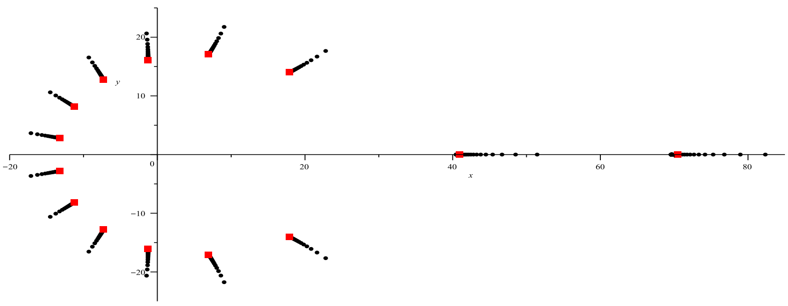


c. $\alpha_{\mathbf{n}} = (n+5, 2n+3, 2n+1)$, $\beta_{\mathbf{n}} = (3n+2+5i, n+3)$

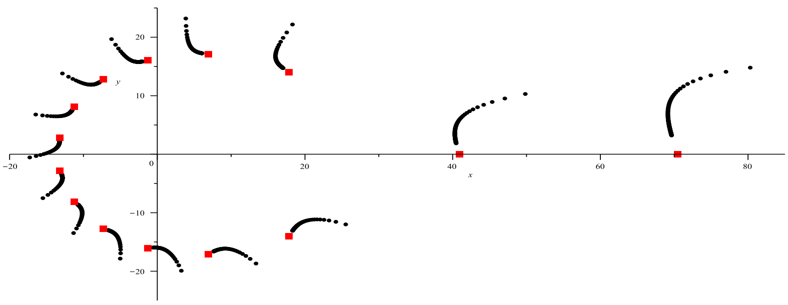


d. $\alpha_{\mathbf{n}} = (n+5+2i, 2n+3+4i, 2n+1-2i)$, $\beta_{\mathbf{n}} = (3n+2+5i, n+3+2i)$

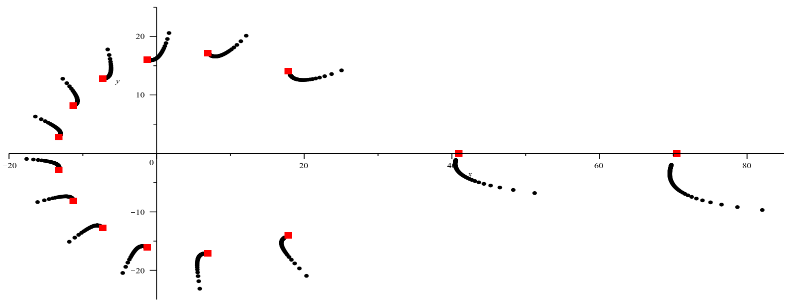
Figure 1: Scaled zeros of ${}_pF_q(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x)$, $n^{p-q+1}x_{n,i}$ (circles), and limit values (squares), with $p = 4$, $q = 3$, $b = -10.4$, for $n = 14, \ldots, 100$.



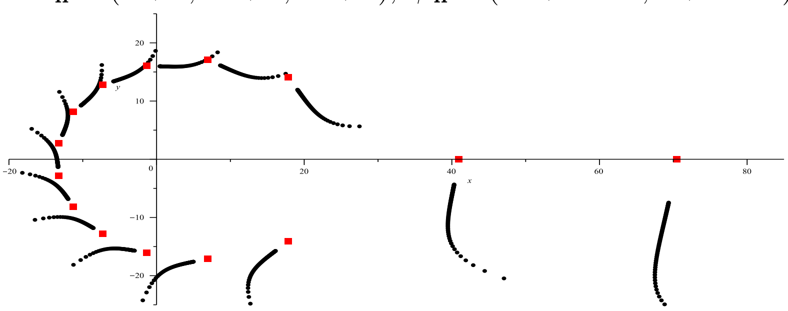
a. $\alpha_{\mathbf{n}} = (n+5, 2n+3, 2n+1)$, $\beta_{\mathbf{n}} = (3n+2, n+3)$



b. $\alpha_{\mathbf{n}} = (n+5-3i, 2n+3-4i, 2n+1)$, $\beta_{\mathbf{n}} = (3n+2, n+3)$

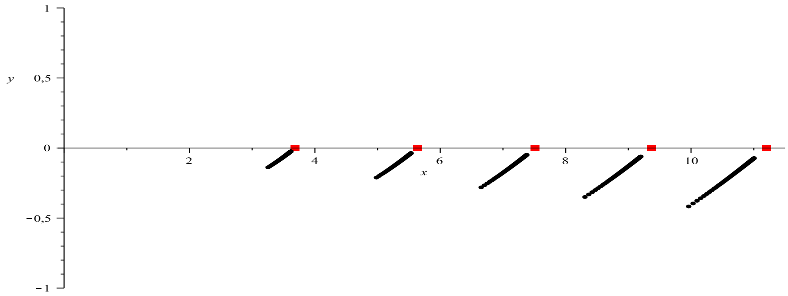


c. $\alpha_{\mathbf{n}} = (n+5, 2n+3, 2n+1)$, $\beta_{\mathbf{n}} = (3n+2-3i, n+3-2i)$

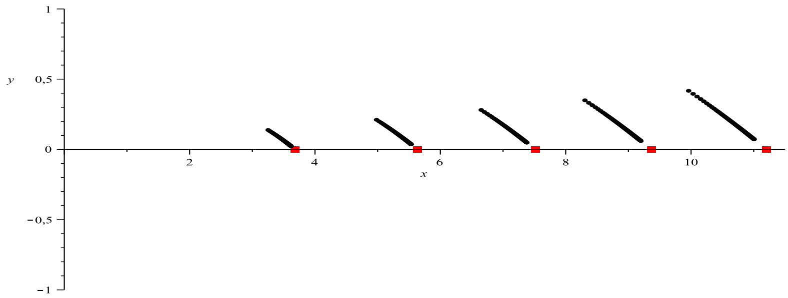


d. $\alpha_{\mathbf{n}} = (n+5+6i, 2n+3-5i, 2n+1+4i)$, $\beta_{\mathbf{n}} = (3n+2+2i, n+3-7i)$

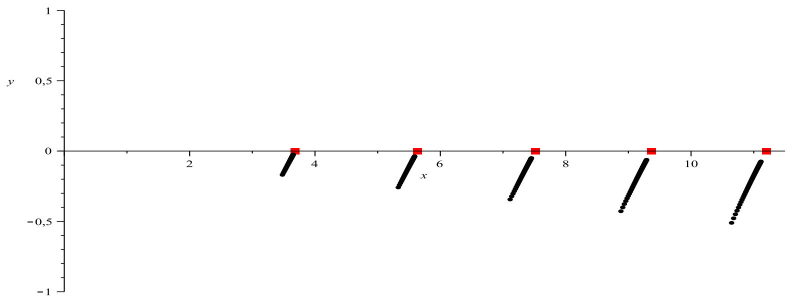
Figure 2: Scaled zeros of ${}_pF_q(-n, \alpha_{\mathbf{n}}; b, \beta_{\mathbf{n}}; x)$, $n^{p-q+1}x_{n,i}$ (circles) and limit values (squares), with $p=4$, $q=3$, $b=-11.5$ and $n=14,\dots,100$.



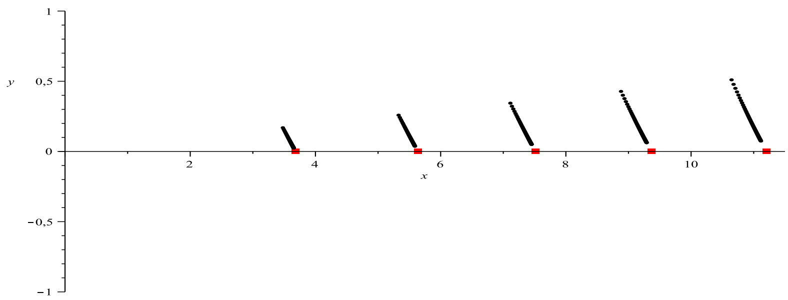
a. $\alpha = 3$, $\beta_n = 2n + 3 + 4i$



b. $\alpha = 3$, $\beta_n = 2n + 3 - 4i$



c. $\alpha = 3$, $\beta_n = 2n - 3 + 4i$



d. $\alpha = 3$, $\beta_n = 2n - 3 - 4i$

Figure 3: $n \arccos(x_{n,i})$ (circles), where $x_{n,i}$ are zeros of varying Jacobi polynomials $P_n^{(\alpha,\beta_n)}(x)$ and limit values (squares), for $n = 14, \ldots, 100$.