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A Relation Between N-Qubit and 2^{N-1}-Qubit Pauli Groups via Binary $LGr(N,2N)$

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Imprint:

A Relation Between N-Qubit and 2^{N-1}-Qubit Pauli Groups via Binary $LGr(N, 2N)$

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Abstract

Employing the fact that the geometry of the N-qubit ($N \geq 2$) Pauli group is embodied in the structure of the symplectic polar space $\mathcal{W}(2N - 1, 2)$ and using properties of the Lagrangian Grassmannian $LGr(N, 2N)$ defined over the smallest Galois field, it is demonstrated that there exists a bijection between the set of maximum sets of mutually commuting elements of the N-qubit Pauli group and a certain subset of elements of the 2^{N-1}-qubit Pauli group. In order to reveal finer traits of this correspondence, the cases $N = 3$ (also addressed recently by Lévay, Planat and Saniga (JHEP 09 (2013) 037)) and $N = 4$ are discussed in detail. As an apt application of our findings, we use the stratification of the ambient projective space $PG(2^N - 1, 2)$ of the 2^{N-1}-qubit Pauli group in terms of G-orbits, where $G \equiv SL(2, 2) \times SL(2, 2) \times \cdots \times SL(2, 2) \rtimes S_N$, to decompose $\underline{\pi}(LGr(N, 2N))$ into non-equivalent orbits. This leads to a partition of $LGr(N, 2N)$ into distinguished classes that can be labeled by elements of the above-mentioned Pauli groups.

Keywords: Multi-Qubit Pauli Groups – Symplectic Polar Spaces $\mathcal{W}(2N - 1, 2)$ – Lagrangian Grassmannians $LGr(N, 2N)$ over the smallest Galois field

1 Introduction

Generalized Pauli groups (also known as Weyl-Heisenberg groups) associated with finite-dimensional Hilbert spaces play an important role in quantum information theory, in particular in quantum tomography, dense coding, teleportation, error correction/cryptography, and black-hole–qubit correspondence. A special class of these groups are the so-called N-qubit Pauli groups, N being a positive integer, whose elements are simply N-fold tensor products of the famous Pauli matrices and the two-by-two unit matrix. A remarkable property of these particular groups is that their structure can be completely recast in the language of symplectic polar spaces of rank N and order two, $\mathcal{W}(2N - 1, 2)$ (see, for example, [1]–[7] and references therein). The elements of the group (discarding the identity) answer to the points of $\mathcal{W}(2N - 1, 2)$, a maximum set of pairwise commuting elements has its representative in a maximal

subspace (also called a generator) $PG(N-1,2)$ of $\mathcal{W}(2N-1,2)$ and, finally, commuting translates into collinear (or, perpendicular). In the case of the *real* N-qubit Pauli group, the structure of the corresponding $\mathcal{W}(2N-1,2)$ can be refined in terms of the orthogonal polar space $\mathcal{Q}^+(2N-1,2)$, that is, a hyperbolic quadric of the ambient projective space $PG(2N-1,2)$, which is the locus accommodating all symmetric elements of the group [4]. Given this finite-geometrical picture of (real) N-qubit Pauli groups, one can invoke properties of the Lagrangian Grassmannian $LGr(N,2N)$ defined over the Galois field of two elements, $GF(2)$, to establish a very interesting bijection between the generators of $\mathcal{W}(2N-1,2)$ and points lying on a sub-configuration of $\mathcal{W}(2^N-1,2)$ defined by a set of quadratic equations. This furnishes an intriguing mapping of maximum sets of mutually commuting N-qubit observables into observables of 2^{N-1}-qubits. For $N=3$, all essential technicalities of this relation have recently been worked out in detail in [8]. In this paper, we shall first give a short rigorous proof that this bijection holds for any N. Then, after a brief addressing of a rather trivial $N=2$ case, we shall again discuss in detail the $N=3$ case using, however, a more "projective-slanted" view to be compared with an "affine" approach of the latter reference, as well as the $N=4$ case to see some novelties and get a feeling of the kind of problems one can envisage/encounter when addressing higher rank cases.

The paper is organized as follow. In Section 2, we recall the definition of the symplectic polar space $\mathcal{W}(2N-1,2)$ and how this space encodes the geometry of the N-Pauli group. In Section 3, we prove our main result by establishing the existence of a projection which maps bijectively the aggregate of maximum sets of mutually commuting N-qubit observables into a distinguished subset of 2^{N-1}-qubit observables. Then, in Section 4, we illustrate our construction for $N=2,3$, and 4 by explicitly computing the equations defining the image of the projection in $PG(2^N-1,2)$. In Section 5, one shows how our findings can be used to partition the set of generators of $\mathcal{W}(2N-1,2)$. Finally, in Section 6 we point out a relation between our construction and similar ones done over the field of complex numbers.

Notations

In what follows, we will denote by $\mathbb{K}$ the Galois field $GF(2)$ and, if V is a $\mathbb{K}$-vector space, we will use the symbol $\mathbb{P}(V)$ to represent the corresponding projective space over $\mathbb{K}$; thus, $\mathbb{P}(\mathbb{K}^N)$ will be an alternative expression for $PG(N-1,2)$, the projective space of dimension $N-1$ over $GF(2)$. Given a nonzero vector $v \in V$, we will denote by $[v] \in \mathbb{P}(V)$ the corresponding point in the associated projective space. On the other hand, for any $X \subset \mathbb{P}(V)$, we define the cone over X, $\widehat{X} \subset V$, to be the pre-image of X in V, i.e. the set of all vectors $x \in V$ such that $[x] \in X$. A tensorial basis of $(\mathbb{K}^2)^{\otimes n} \equiv \mathbb{K}^2 \otimes \cdots \otimes \mathbb{K}^2$ (n factors) will be denoted by $x_1^{i_1} \otimes x_2^{i_2} \otimes \cdots \otimes x_n^{i_n}$, where $i_j \in \{0,1\}$; obviously, $\{x_i^0, x_i^1\}$ is a basis of $(\mathbb{K}^2)_i$.

Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in $\mathbb{K}$ and let $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$ be subsets of $\{1, \ldots, n\}$. The symbol $\Delta_{I,J}$ will stand for the corresponding $k \times k$ minor of A, i.e. $\Delta_{I,J}(A) = \det((a_{i,j})_{i \in I, j \in J})$; when $I = J$, $\Delta_{I,I}(A)$ will be called a principal minor of A and simply referred to as $\Delta_I(A)$.

2 The symplectic polar space $\mathcal{W}(2N-1,2)$ and the associated N-qubit Pauli group

Let $\mathbb{K}^{2N}$ be the $2N$-dimensional vector space over $GF(2)$ equipped with a non-degenerate symplectic form σ (which exists because the dimension is even). The symplectic polar space $\mathcal{W}(2N-1,2)$ is the space which describes the geometry of the vector space $\mathbb{K}^{2N}$ endowed with σ in the following sense: all non-zero vectors $x \in \mathbb{K}^{2N}$ are points of $\mathcal{W}(2N-1,2)$, because $\sigma(x,x) = 0$ by definition, and moreover two points $x, y \in \mathcal{W}(2N-1,2)$ will be said to be collinear if and only if $\sigma(x,y) = 0$. Recall that a subspace of $\mathbb{K}^{2N}$ is said to be totally isotropic if σ vanishes identically on it. $\mathcal{W}(2N-1,2)$ is, loosely speaking, the space of all totally isotropic subspaces of $PG(2N-1,2)$. The maximal totally isotropic subspaces of $PG(2N-1,2)$, also called *generators* of $\mathcal{W}(2N-1,2)$, have all the same dimension $N-1$. The number of points of the polar space is equal to $|PG(2N-1,2)| = 2^{2N}-1 = 4^N-1$ and the number of generators amounts to $(2+1)(2^2+1)\ldots(2^N+1)$.

The generalized real N-qubit Pauli group, denoted by $\mathcal{P}_N$, is generated by N-fold tensor products of the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Explicitly,

$$\mathcal{P}_N = \{\pm A_1 \otimes A_2 \otimes \cdots \otimes A_N : \; A_i \in \{I, X, Y, Z\}, \; i = 1, 2, \cdots, N\}.$$

The associated factor group $\overline{\mathcal{P}}_N \equiv \mathcal{P}_N/\mathcal{Z}(\mathcal{P}_N)$, where the center $\mathcal{Z}(\mathcal{P}_N)$ consists of $\pm I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$, features 4^N elements. The elements of $\overline{\mathcal{P}}_N \backslash \{I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}\}$ can be bijectively identified with the same number of points of $\mathcal{W}(2N-1, 2)$ in such a way that two commuting elements of the group will lie on the same totally isotropic line of this polar space. If one selects a basis of $\mathcal{W}(2N-1, 2)$ in which the symplectic form $\sigma(x, y)$ is given by

$$\sigma(x, y) = (x_1 y_{N+1} - x_{N+1} y_1) + (x_2 y_{N+2} - x_{N+2} y_2) + \cdots + (x_N y_{2N} - x_{2N} y_N), \tag{1}$$

then this bijection acquires the form:

$$A_i \leftrightarrow (x_i, x_{i+N}), \; i \in \{1, 2, 3, 4\}, \tag{2}$$

with the understanding that

$$I \leftrightarrow (0, 0), \; X \leftrightarrow (0, 1), \; Y \leftrightarrow (1, 1), \; Z \leftrightarrow (1, 0); \tag{3}$$

thus, for example, in $\mathcal{W}(7, 2)$ the point having coordinates $(0, 1, 1, 0, 0, 1, 0, 1)$ corresponds to the element $I \otimes Y \otimes Z \otimes X \equiv IYZX$.

The elements of the group $\overline{\mathcal{P}}_N$ whose square is $+I_{(1)} I_{(2)} \cdots I_{(N)}$ (i.e., symmetric elements) lie on a certain $\mathcal{Q}^+(2N-1, 2)$ of the ambient space $PG(2N-1, 2)$. It follows from the definition of the bijection that the equation of *the* $\mathcal{Q}^+(2N-1, 2)$ accommodating all symmetric elements must have the following standard form

$$\mathcal{Q}(x) = x_1 x_{N+1} + x_2 x_{N+2} + \cdots + x_N x_{2N} = 0. \tag{4}$$

This can readily be inspected using the fact that the matrix Y is the only skew-symmetric element in the set $\{I, X, Y, Z\}$ and, so, any symmetric element of the group must contain an even number of Ys.

It should also be added that generators, of both $\mathcal{W}(2N-1, 2)$ and $\mathcal{Q}^+(2N-1, 2)$, correspond to *maximal* sets of mutually commuting elements of the group.

3 Mapping $LGr(N, 2N)$ to $PG(2^N - 1, 2)$

Recently, Lévay, Planat and Saniga [8] found and analyzed in detail an explicit bijection between the set of 135 maximum sets of mutually commuting elements of the three-qubit Pauli group (that is, the set of generators of $\mathcal{W}(5, 2)$) and the set of 135 symmetric operators of the four-qubit Pauli group (that is, the set of points lying on a particular $\mathcal{Q}^+(7, 2)$ of $\mathcal{W}(7, 2)$). Following the spirit of this work, we will generalize this physically important result and prove the existence of a similar bijection between *any* N-qubit and 2^{N-1}-qubit Pauli groups by considering first the Grassmaniann variety $Gr(N, 2N)$, then its associated Lagrangian Grassmannian $LGr(N, 2N)$ and, finally, using a crucial fact that we work in characteristic 2.

To this end in view, let us first recall the definition of the variety of N-planes in $\mathbb{K}^{2N}$, i.e. the Grassmannian variety $Gr(N, 2N)$. An N-plane (respectively an $(N-1)$-projective-plane) P, spanned by N non-zero vectors $u_1, u_2, \ldots, u_N$ of $\mathbb{K}^{2N}$ (respectively by N points $[u_1], [u_2], \ldots, [u_N]$ of $PG(2N-1, 2)$) is a point of the Grassmannian variety $Gr(N, 2N) \subset \mathbb{P}(\bigwedge^N \mathbb{K}^{2N}) = PG(\binom{2N}{N} - 1, 2)$. The embedding of the Grassmannian variety is given by the so-called Plücker map:

$$P = \text{span}\langle u_1, u_2, \ldots, u_N \rangle \mapsto [u_1 \wedge u_2 \wedge \cdots \wedge u_N] \in Gr(N, 2N) \subset \mathbb{P}(\bigwedge^N \mathbb{K}^{2N}).$$

In other words, the Grassmanian variety is the set of all skew symmetric tensors that can be factorized (i.e., are separable). The algebraic equations defining $Gr(N, 2N)$ are known as the Plücker equations. Let $(e_i)_{1 \leq i \leq 2N}$ be a basis of the vector space $\mathbb{K}^{2N}$ and let $P \in \mathbb{P}(\bigwedge^N \mathbb{K}^{2N})$, i.e.

$$P = \sum_{1 \leq i_1 < i_2 < \cdots < i_N \leq 2N} p_{i_1, i_2, \ldots, i_N} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_N}.$$

If P belongs to $Gr(N, 2N)$, then for any two sequences $1 \leq i_1 < \cdots < i_{N-1} \leq 2N$ and $1 \leq j_1 < \cdots < j_{N+1} \leq 2N$, the coordinates of P satisfy the following relations (see [12] page 94)

$$\sum_{a=1}^{N+1} (-1)^a p_{i_1, i_2, \ldots, i_{N-1}, j_a} p_{j_1, j_2, \ldots, \hat{j_a}, \ldots, j_{N+1}} = 0, \tag{5}$$

where the symbol $\hat{j_a}$ means that the corresponding index is omitted. Equivalently, the coordinates $[p_{1,2,\ldots,N}, \ldots, p_{N,N+1\ldots,2N}]$ of $P \in Gr(N, 2N)$ can be expressed as follow. Let M be an $N \times 2N$ matrix whose rows are coordinates of N vectors that are spanning the N-plane P. Then, we have

$$p_{i_1, \ldots, i_N} = \Delta_{\{i_1, \ldots, i_N\}, \{1, \ldots, N\}}(M) \tag{6}$$

We are only interested in those $(N-1)$-planes of $PG(2N-1, 2)$ which are totally isotropic with respect to our symplectic form σ (i.e., in the generators of $\mathcal{W}(2N-1, 2)$). The extension of σ to $\mathbb{P}(\bigwedge^N \mathbb{K}^{2N})$ defines (see [9]) linear conditions on the coordinates $[p_{1,\ldots,N}, p_{1,\ldots,N-1,N+1}, \ldots, p_{N,\ldots 2N}]$ of P to insure that P is totally isotropic. These linear conditions define a projective space $\mathbb{P}(L)$ whose intersection with $Gr(N, 2N)$ is a sub-variety of $Gr(N, 2N)$ called the Lagrangian variety,

$$LGr(N, 2N) = Gr(N, 2N) \cap \mathbb{P}(L).$$

The Lagrangian variety is thus the variety of all the generators $PG(N-1, 2)$ of $\mathcal{W}(2N-1, 2)$. We will now show that over $\mathbb{K}$ the variety $LGr(N, 2N)$ can further be projected bijectively to a subset of points of $PG(2^N - 1, 2)$, where $PG(2^N - 1, 2)$ is the projective space obtained by eliminating the variables involved in the linear conditions which define $\mathbb{P}(L)$ (i.e., the linear conditions given by the extension of σ to $\mathbb{P}(\bigwedge^N \mathbb{K}^{2N})$).

Let $P = (e_1 + \sum_j a_{1,j} e_{N+j}) \wedge (e_2 + \sum_j a_{2,j} e_{N+j}) \wedge \cdots \wedge (e_N + \sum_j a_{N,j} e_{N+j}) \in Gr(N, 2N)$. Expanding this expression, we obtain the local parametrization of $Gr(N, 2N)$:

$$P = e_1 \wedge \cdots \wedge e_N + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{N+j} \wedge e_{i+1} \wedge e_N$$

$$+ \sum_{i,j} \sum_{s,t} (a_{is} a_{jt} - a_{it} a_{js}) e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{N+s} \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge e_{N+t} \wedge e_{j+1} \wedge \cdots \wedge e_N + \cdots.$$

This shows that locally the coordinates of P can be written as

$$[1, a_{11}, \ldots, a_{NN}, a_{11} a_{22} - a_{21} a_{12}, \ldots ...] = [1, \Delta_1(A), \ldots, \Delta_{I,J}(A), \ldots],$$

where $A = (a_{i,j})$. Requiring P to be totally isotropic means that the vectors spanning P must be totally isotropic. Denoting $u_i = e_i + \sum_j a_{i,j} e_{N+j}$, we get $\sigma(u_s, u_t) = a_{st} - a_{ts}$, which is zero if and only if $A = (a_{ij})$ is a symmetric matrix. Thus P will be totally isotropic if its coordinates locally correspond to minors of a symmetric matrix A over $\mathbb{K}$.

The linear conditions defining $\mathbb{P}(L)$ correspond locally to the fact that the minors $\Delta_{I,J}(A)$ and $\Delta_{J,I}(A)$ are equal for $I \neq J$. Moreover, these conditions do not involve the coordinates corresponding to principal minors. Thus, we obtain a splitting $\bigwedge^N \mathbb{K}^{2N} = V + W$ such that the coordinates defining V are locally given by minors of type $\Delta_{I,J}$, whereas the coordinates defining W are principal minors

$\Delta_I(A)$. But for symmetric matrices over $\mathbb{K} = GF(2)$ all off-diagonal entries are completely determined by the principal minors $\Delta_{\{i\}}(A)$ and $\Delta_{\{i,j\}}(A)$. This fact readily stems from the following equation $a_{ii}a_{jj} - a_{ij}^2 = \Delta_{\{i,j\}}(A)$, i.e., $a_{i,j}^2 = \Delta_i(A)\Delta_j(A) - \Delta_{i,j}(A)$. Thus, all minors $\Delta_{I,J}(A)$, with $I \neq J$, are (over $\mathbb{K}$) *uniquely* determined by the principal minors $\Delta_K(A)$ of A. In other words, once the coordinates in W of a point of $LGr(N, 2N)$ are chosen, the coordinates in V are automatically fixed. If we consider the cone $\widehat{LG}(N, 2N) \subset \bigwedge^N \mathbb{K}^{2N} = V + W$, this can be regarded as a graph over mere W and mapped bijectively to a subset of W. The dimension of W is given by the number of principal minors: $\sum_{i=0}^N \binom{N}{i} = 2^N$. Since all principal minors cannot vanish simultaneously, we obtain a well-defined projective map $\underline{\pi} : LGr(N, 2N) \to \mathbb{P}(W) = PG(2^N - 1, 2)$. The map $\underline{\pi}$ sends P to $p \in \mathbb{P}(W)$, where p is defined by the coordinates of P not occurring in the equations defining V. All in all, we obtain a bijective mapping by projecting $LGr(N, 2N)$ to $PG(2^N - 1, 2)$ after eliminating all the variables involved/occurring in the linear conditions.

4 An explicit construction of the bijection: a few examples

The above-given proof of the existence of the mapping

$$\underline{\pi} : LGr(N, 2N) \to PG(2^N - 1, 2)$$

provides us with a recipe of how to obtain the equations of the image

$$\underline{\pi}(LGr(N, 2N) \subset PG(2^N - 1, 2).$$

Indeed, following our reasoning one first needs to find the ideal $I(Gr(N, 2N))$ (i.e., a set of equations) defining $Gr(N, 2N)$, as well as the linear conditions $J = (l_1, \dots, l_m)$ induced by the associated symplectic form. These two sets of equations will then define the ideal of $LGr(N, 2N)$, i.e.,

$$I(LGr(N, 2N)) = I(Gr(N, 2N)) \cup J.$$

Then we calculate the ideal of the projection $\underline{\pi}(LGr(N, 2N))$ by eliminating in $I(LGr(N, 2N))$ all the variables appearing in J. The last step can be done by hand when cases are rather simple, or be handled with the formalism of the Elimination Theory [10] when calculations become more tedious. We shall now illustrate this approach on the first three cases in the sequence.

4.1 The smallest non-trivial $\left(Gr(2, 4) \mapsto LGr(2, 4) \simeq PG(3, 2)\right)$ case

The set of lines in $PG(3, 2)$ (or 2-planes in $\mathbb{K}^4$) is the first non-trivial Grassmannian variety. From eqs. (5) it follows that $Gr(2, 4)$ is defined by a single equation,

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0, \tag{7}$$

representing a quadric surface in $PG(\binom{4}{2} - 1, 2) = PG(5, 2)$. Hence, $Gr(2, 4)$ is a variety of dimension 4. The canonical symplectic form (see eq. (1)) $\sigma(x, y) = x_1y_3 - x_3y_1 + x_2y_4 - x_4y_2$ imposes that P (of projective coordinates $[p_{12} : p_{13} : p_{14} : p_{34} : p_{24} : p_{23}]$) is isotropic if and only if $p_{13} = p_{24}$. The linear conditions stemming from σ tell us that

$$LGr(2, 4) = Gr(2, 4) \cap PG(4, 2),$$

where $PG(4, 2) = \{[x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \in PG(5, 2), x_2 = x_5\}$. The variety $LGr(2, 4)$ is of dimension 3, being mapped down to $PG(3, 2)$ when we take into account the projection $\underline{\pi} : LGr(2, 4) \to PG(3, 2)$ given by $\underline{\pi}([x_1 : x_2 : x_3 : x_4 : x_5 : x_6]) = [x_1 : x_3 : x_4 : x_6]$. It is a one-to-one mapping (because both p_{13} and p_{24} are fixed by the other minors), so we have

$$\underline{\pi}(LGr(2, 4)) = PG(3, 2).$$

4.2 The 'Lévay-Planat-Saniga' $\big(Gr(3,6) \mapsto LGr(3,6) \mapsto \mathcal{Q}^+(7,2)\big)$ case revisited

For $N = 3$, eqs. (5) reduce into the following form

$$\sum_{a=1}^{4} (-1)^a p_{i_1 i_2 j_a} p_{j_1 \dots \widehat{j_a} \dots j_4} = 0. \tag{8}$$

For each choice of the pair of indices of $\{i_1, i_2\}$ we find eight equations with three terms each and one equation featuring all four terms. There will be, of course, an overlap and what we get are only 30 independent three-term equations and five four-term ones. We start with $\{i_1, i_2\} = \{1, 2\}$ and for each subsequent choice of these two indices we list only those equations that have not appeared in the preceding steps. Each equation of the former set is, apart from the pair i_1, i_2, characterized by the string $\{j_1, j_2, j_3, j_4\}$ shown after the equation. The 30 equations read:

$\{i_1, i_2\} = \{1, 2\}:$
$p_{123}p_{145} + p_{124}p_{135} + p_{125}p_{134} = 0,$ $\{1, 3, 4, 5\}$,
$p_{123}p_{146} + p_{124}p_{136} + p_{126}p_{134} = 0,$ $\{1, 3, 4, 6\}$,
$p_{123}p_{156} + p_{125}p_{136} + p_{126}p_{135} = 0,$ $\{1, 3, 5, 6\}$,
$p_{124}p_{156} + p_{125}p_{146} + p_{126}p_{145} = 0,$ $\{1, 4, 5, 6\}$,
$p_{123}p_{245} + p_{124}p_{235} + p_{125}p_{234} = 0,$ $\{2, 3, 4, 5\}$,
$p_{123}p_{246} + p_{124}p_{236} + p_{126}p_{234} = 0,$ $\{2, 3, 4, 6\}$,
$p_{123}p_{256} + p_{125}p_{236} + p_{126}p_{235} = 0,$ $\{2, 3, 5, 6\}$,
$p_{124}p_{256} + p_{125}p_{246} + p_{126}p_{245} = 0,$ $\{2, 4, 5, 6\}$,

$\{i_1, i_2\} = \{1, 3\}:$
$p_{134}p_{156} + p_{135}p_{146} + p_{136}p_{145} = 0,$ $\{1, 4, 5, 6\}$,
$p_{123}p_{345} + p_{134}p_{235} + p_{135}p_{234} = 0,$ $\{2, 3, 4, 5\}$,
$p_{123}p_{346} + p_{134}p_{236} + p_{136}p_{234} = 0,$ $\{2, 3, 4, 6\}$,
$p_{123}p_{356} + p_{135}p_{236} + p_{136}p_{235} = 0,$ $\{2, 3, 5, 6\}$,
$p_{134}p_{356} + p_{135}p_{346} + p_{136}p_{345} = 0,$ $\{3, 4, 5, 6\}$,

$\{i_1, i_2\} = \{1, 4\}:$
$p_{124}p_{345} + p_{134}p_{245} + p_{145}p_{234} = 0,$ $\{2, 3, 4, 5\}$,
$p_{124}p_{346} + p_{134}p_{246} + p_{146}p_{234} = 0,$ $\{2, 3, 4, 6\}$,
$p_{124}p_{456} + p_{145}p_{246} + p_{146}p_{245} = 0,$ $\{2, 4, 5, 6\}$,
$p_{134}p_{456} + p_{145}p_{346} + p_{146}p_{345} = 0,$ $\{3, 4, 5, 6\}$,

$\{i_1, i_2\} = \{1, 5\}:$
$p_{125}p_{345} + p_{135}p_{245} + p_{145}p_{235} = 0,$ $\{2, 3, 4, 5\}$,
$p_{125}p_{356} + p_{135}p_{256} + p_{156}p_{235} = 0,$ $\{2, 3, 5, 6\}$,
$p_{125}p_{456} + p_{145}p_{256} + p_{156}p_{245} = 0,$ $\{2, 4, 5, 6\}$,
$p_{135}p_{456} + p_{145}p_{356} + p_{156}p_{345} = 0,$ $\{3, 4, 5, 6\}$,

$\{i_1, i_2\} = \{1, 6\}:$
$p_{126}p_{346} + p_{136}p_{246} + p_{146}p_{236} = 0,$ $\{2, 3, 4, 6\}$,
$p_{126}p_{356} + p_{136}p_{256} + p_{156}p_{236} = 0,$ $\{2, 3, 5, 6\}$,
$p_{126}p_{456} + p_{146}p_{256} + p_{156}p_{246} = 0,$ $\{2, 4, 5, 6\}$,
$p_{136}p_{456} + p_{146}p_{356} + p_{156}p_{346} = 0,$ $\{3, 4, 5, 6\}$,

$\{i_1, i_2\} = \{2, 3\}:$
$p_{234}p_{256} + p_{235}p_{246} + p_{236}p_{245} = 0,$ $\{2, 4, 5, 6\}$,
$p_{234}p_{356} + p_{235}p_{346} + p_{236}p_{345} = 0,$ $\{2, 4, 5, 6\}$,
$\{i_1, i_2\} = \{2, 4\}:$
$p_{234}p_{456} + p_{245}p_{346} + p_{246}p_{345} = 0,$ $\{3, 4, 5, 6\}$,

$\{i_1, i_2\} = \{2, 5\}:$
$p_{235}p_{456} + p_{245}p_{356} + p_{256}p_{345} = 0,$ $\{3, 4, 5, 6\}$,

$\{i_1, i_2\} = \{2, 6\}:$
$p_{236}p_{456} + p_{246}p_{356} + p_{256}p_{346} = 0$ $\{3, 4, 5, 6\}$.

The five independent four-term equations (followed by the corresponding pair $\{i_1, i_2\}$) are:

$$p_{123}p_{456} + p_{124}p_{356} + p_{125}p_{346} + p_{126}p_{345} = 0, \quad \{1, 2\},$$
$$p_{123}p_{456} + p_{134}p_{256} + p_{135}p_{246} + p_{136}p_{245} = 0, \quad \{1, 3\},$$
$$p_{124}p_{356} + p_{134}p_{256} + p_{145}p_{236} + p_{146}p_{235} = 0, \quad \{1, 4\},$$
$$p_{125}p_{346} + p_{135}p_{246} + p_{145}p_{236} + p_{156}p_{234} = 0, \quad \{1, 5\},$$
$$p_{126}p_{345} + p_{136}p_{245} + p_{146}p_{235} + p_{156}p_{234} = 0, \quad \{1, 6\}.$$

We are only interested in the Lagrangian grassmannian $LGr(3, 6)$, that is in those planes of $PG(5, 2)$ that are totally isotropic with respect to a given symplectic polarity. Choosing the latter to have again the 'canonical' form (eq. (1)),

$$(x_1 y_4 - x_4 y_1) + (x_2 y_5 - x_5 y_2) + (x_3 y_6 - x_6 y_3) = 0,$$

the coordinates of such planes have to meet the following constraints

$$p_{125} = p_{136}, \ p_{235} = p_{134}, \ p_{124} = p_{236}, \ p_{245} = p_{346}, \ p_{256} = p_{146}, \ p_{145} = p_{356}, \tag{9}$$

which reduce the set of 30 three-term equations into

$$p_{123}p_{356} + p_{236}p_{135} + p_{136}p_{235} = 0, \qquad p_{236}p_{456} + p_{356}p_{246} + p_{256}p_{346} = 0,$$
$$p_{123}p_{256} + p_{236}p_{136} + p_{126}p_{235} = 0, \qquad p_{235}p_{456} + p_{356}p_{346} + p_{256}p_{345} = 0,$$
$$p_{236}p_{156} + p_{136}p_{256} + p_{126}p_{356} = 0, \qquad p_{136}p_{456} + p_{356}p_{256} + p_{156}p_{346} = 0,$$
$$p_{123}p_{346} + p_{236}p_{235} + p_{136}p_{234} = 0, \qquad p_{123}p_{156} + p_{136}^2 + p_{126}p_{135} = 0,$$
$$p_{236}p_{256} + p_{136}p_{246} + p_{126}p_{346} = 0, \qquad p_{123}p_{246} + p_{236}^2 + p_{126}p_{234} = 0,$$
$$p_{235}p_{156} + p_{135}p_{256} + p_{136}p_{356} = 0, \qquad p_{123}p_{345} + p_{235}^2 + p_{135}p_{234} = 0,$$
$$p_{235}p_{356} + p_{135}p_{346} + p_{136}p_{345} = 0, \qquad p_{135}p_{456} + p_{356}^2 + p_{156}p_{345} = 0,$$
$$p_{236}p_{345} + p_{235}p_{346} + p_{356}p_{234} = 0, \qquad p_{126}p_{456} + p_{256}^2 + p_{156}p_{246} = 0,$$
$$p_{236}p_{346} + p_{235}p_{246} + p_{256}p_{234} = 0, \qquad p_{234}p_{456} + p_{346}^2 + p_{246}p_{345} = 0,$$

and the set of five four-term ones into

$$p_{123}p_{456} + p_{236}p_{356} + p_{136}p_{346} + p_{126}p_{345} = 0,$$
$$p_{123}p_{456} + p_{235}p_{256} + p_{135}p_{246} + p_{136}p_{346} = 0,$$
$$p_{136}p_{346} + p_{135}p_{246} + p_{356}p_{236} + p_{156}p_{234} = 0,$$
$$p_{126}p_{345} + p_{136}p_{346} + p_{256}p_{235} + p_{156}p_{234} = 0.$$

These last four equations are, however, not independent, as each of them is equal to the sum of the remaining three. Moreover, summing the first of them with the third one, or the second with the fourth, yields

$$p_{123}p_{456} + p_{126}p_{345} + p_{135}p_{246} + p_{156}p_{234} = 0, \tag{10}$$

which after relabeling the variables as $x_1 = p_{123}$, $x_2 = p_{126}$, $x_3 = p_{135}$, $x_4 = p_{156}$, $x_5 = p_{456}$, $x_6 = p_{345}$, $x_7 = p_{246}$ and $x_8 = p_{234}$ reads

$$x_1 x_5 + x_2 x_6 + x_3 x_7 + x_4 x_8 = 0 \tag{11}$$

and is readily recognized to represent a hyperbolic quadric $\mathcal{Q}^+(7,2)$ in a particular subspace $PG(7,2)$ of the ambient projective space $PG(19,2)$ of $Gr(3,6)$. More precisely, the quadric defined by eq. (10) lives in the ideal $I_2(LGr(3,6))$ and it is the *only* quadric that does not depend on the coordinates p_{136}, p_{236}, p_{235}, p_{356}, p_{256}, and p_{346}. Thus, if we consider the splitting of the linear space $\mathbb{K}^{14} = \mathbb{K}^8 \oplus \mathbb{K}^6$, where $\mathbb{K}^8$ represents the vector space defined by the set of coordinates $\{p_{123}, p_{126}, p_{135}, p_{156}, p_{456}, p_{345}, p_{246}, p_{234}\}$ and $\mathbb{K}^6$ that defined by $\{p_{136}, p_{236}, p_{235}, p_{356}, p_{256}, p_{346}\}$, and employ the fact that each coordinate from the latter set can be expressed as a linear combination of the coordinates from the former set, then we can represent $\widehat{LGr}(3,6) \subset \mathbb{K}^{14}$ as a graph over the quadric $\mathcal{Q}^+(7,2)$ defined by eq. (10) in $\mathbb{K}^8$, i. e.,

$$\widehat{LGr}(3,6) = \{(\mathbf{x}, g(\mathbf{x})) \in \mathbb{K}^{14}, \mathbf{x} \in \widehat{\mathcal{Q}}^+(7,2) \subset \mathbb{K}^8\}.$$

One thus automatically gets a bijection between $LGr(3,6)$ and the $\mathcal{Q}^+(7,2)$ by taking the projection to the base of the graph $\mathbb{K}^8 \oplus \mathbb{K}^6 \to \mathbb{K}^8$.

This procedure can be rephrased in algebraic terms in the framework of Elimination Theory [10]. Given an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$, the l-th elimination ideal, with $1 \leq l \leq n$, is the ideal of $\mathbb{K}[x_{l+1}, \ldots, x_n]$ defined by

$$I_l = I \cap \mathbb{K}[x_{l+1}, \ldots, x_n].$$

Let π_l be the projection $\mathbb{K}^n \to \mathbb{K}^{n-l}$ defined by $\pi_l(a_1, \ldots, a_n) = (a_{l+1}, \ldots, a_n)$. If $V(I) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n, f(a_1, \ldots, a_n) = 0, \forall f \in I\}$ is an affine variety corresponding to the ideal I, then $\pi(V(I)) \subset V(I_l)$, i. e., the projection of $V(I)$ is contained in the algebraic variety defined by the elimination ideal (which is, in fact, the smallest affine variety containing $\pi(V)$). Using the notion of Groebner basis, one can compute the elimination ideal I from the fact that

$$G_l = G \cap \mathbb{K}[x_{l+1}, \ldots, x_n],$$

where G is the Groebner basis of I and G_l that of I_l. To perform this calculation, it suffices to choose a monomial order adapted to eliminate the first l variables.

Going back to the (cone over the) Lagrangian Grassmanian, $\widehat{LG}(3,6) \subset \mathbb{K}^{14}$, we know that the ideal of this variety is defined by the above-given 21 equations of degree 2. This ideal is obtained from the Plücker relations defining $Gr(3,6)$ by imposing the six constraints (9), i.e.

$$I(LGr(3,6)) = I(Gr(3,6) \cup J,$$

where $J = (p_{125}+p_{136}, p_{235}+p_{134}, p_{124}+p_{236}, p_{245}+p_{346}, p_{256}+p_{146}, p_{145}+p_{356})$. The Plücker coordinates appearing in J are exactly those we want to eliminate to project down to $\mathbb{K}^8$, because they depend linearly on the remaining ones. Starting form the ideal $I(\widehat{LG}(3,6))$, we compute the desired elimination ideal using Macaulay2:

$$I(\widehat{LG}(3,6)) \cap \mathbb{K}[p_{123}, p_{126}, p_{135}, p_{156}, p_{456}, p_{345}, p_{246}, p_{234}] =$$

$$= (p_{123}p_{456} + p_{126}p_{345} + p_{135}p_{246} + p_{156}p_{234}).$$

Thus, $\pi(\widehat{LG}(3,6)) \subset \widehat{\mathcal{Q}}^+(7,2)$. Everything holds projectively, as we worked only with homogeneous polynomials; then $\underline{\pi}(LGr(3,6)) \subset \mathcal{Q}^+(7,2)$, where $\underline{\pi} : PG(13,2) \backslash PG(5,2) \to PG(7,2)$.[1] Moreover, since both $\sharp\underline{\pi}(LGr(3,6)) = 135$ and $\sharp\mathcal{Q}^+(7,2) = 135$ and because $\underline{\pi}$ is a bijection (the values of $p_{125}, \cdots, p_{356}$ are completely determined by those of $p_{123}, \cdots, p_{234}$), we finally get

$$\underline{\pi}(LGr(3,6)) = \mathcal{Q}^+(7,2).$$

4.3   A more intricate $\left(Gr(4,8) \mapsto LGr(4,8) \mapsto \mathcal{Q}^+(15,2)\right)$ case

We shall follow the same strategy to show that the 2295 maximal subspaces of $\mathcal{W}(7,2)$ are mapped to a subset of points of a hyperbolic quadric $\mathcal{Q}^+(15,2)$. $Gr(4,8)$ is a variety of the $\binom{8}{4} - 1 = 69$-dimensional projective space defined by

$$\sum_{a=1}^{5}(-1)^a p_{i_1 i_2 i_3 j_a} p_{j_1 \ldots \widehat{j_a} \ldots j_5} = 0. \tag{12}$$

The Lagrangian Grassmannian associated with the following symplectic polarity

$$(x_1 y_5 - x_5 y_1) + (x_2 y_6 - x_6 y_2) + (x_3 y_7 - x_7 y_3) + (x_4 y_8 - x_8 y_4) = 0,$$

has to meet 24 constraints of the type

$$p_{1345} = p_{2346}, \; p_{1245} = p_{2347}, \; p_{1235} = p_{2348}, \; p_{1246} = p_{1347}, \; p_{1236} = p_{1348}, \; p_{1237} = p_{1248},$$
$$p_{1358} = p_{2368}, \; p_{1258} = p_{2378}, \; p_{1257} = p_{2478}, \; p_{1268} = p_{1378}, \; p_{1267} = p_{1478}, \; p_{1367} = p_{1468},$$
$$p_{1457} = p_{2467}, \; p_{1456} = p_{3467}, \; p_{1356} = p_{3468}, \; p_{2456} = p_{3457}, \; p_{2356} = p_{3458}, \; p_{2357} = p_{2458},$$
$$p_{1578} = p_{2678}, \; p_{1568} = p_{3678}, \; p_{1567} = p_{4678}, \; p_{2568} = p_{3578}, \; p_{2567} = p_{4578}, \; p_{3567} = p_{4568},$$

and three conditions of the type

$$p_{1256} + p_{1357} + p_{1458} = 0,$$
$$p_{1256} + p_{2367} + p_{2468} = 0,$$
$$p_{1357} + p_{2367} + p_{3478} = 0.$$

We thus find 27 independent linear relations; hence, our $LGr(4,8)$ lives in a subspace of the $PG(69,2)$ that is isomorphic to $PG(42,2)$.

To find the projection $\pi : \mathbb{K}^{70} \to \mathbb{K}^{16}$ we compute the elimination ideal

$$I = (I(Gr(4,8)) \cup J) \cap \mathbb{K}[x_1, \ldots, x_{16}],$$

---

[1] It is worth mentioning here that the projection map is well defined only outside $PG(5,2) = \{p \in PG(13,2), x = [a_1, \cdots, a_6, 0, \ldots, 0]\}$; however, $\underline{\pi}_{|LGr(3,6)}$ is well defined because the variables $p_{123}, \ldots, p_{234}$ do not vanish simultaneously on $LGr(3,6)$.

where J is a homogeneous ideal of degree one generated by the above-given 27 equations and x_i's, $i \in \{1, 2, \cdots, 16\}$, stand for the following 16 Plücker coordinates that do not appear in the expression defining J:

$$x_1 = p_{1234}, \quad x_2 = p_{1238}, \quad x_3 = p_{1247}, \quad x_4 = p_{1278},$$
$$x_5 = p_{1346}, \quad x_6 = p_{1368}, \quad x_7 = p_{1467}, \quad x_8 = p_{1678},$$
$$x_9 = p_{5678}, \quad x_{10} = p_{4567}, \quad x_{11} = p_{3568}, \quad x_{12} = p_{3456},$$
$$x_{13} = p_{2578}, \quad x_{14} = p_{2457}, \quad x_{15} = p_{2358}, \quad x_{16} = p_{2345}.$$

Using Macaulay2, we get the ideal

$$I = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5, \mathcal{Q}_6, \mathcal{Q}_7, \mathcal{Q}_8, \mathcal{Q}_9, \mathcal{Q}_{10}), \tag{13}$$

where $\mathcal{Q}_i$ are the following quadratic forms

$$\mathcal{Q}_1 = x_{12}x_{13} + x_{11}x_{14} + x_{10}x_{15} + x_9x_{16}, \quad \mathcal{Q}_2 = x_1x_{13} + x_2x_{14} + x_3x_{15} + x_4x_{16},$$
$$\mathcal{Q}_3 = x_1x_{11} + x_2x_{12} + x_5x_{15} + x_6x_{16}, \quad \mathcal{Q}_4 = x_4x_5 + x_3x_6 + x_2x_7 + x_1x_8,$$
$$\mathcal{Q}_5 = x_1x_{10} + x_3x_{12} + x_5x_{14} + x_7x_{16}, \quad \mathcal{Q}_6 = x_5x_9 + x_6x_{10} + x_7x_{11} + x_8x_{12},$$
$$\mathcal{Q}_7 = x_3x_9 + x_4x_{10} + x_7x_{13} + x_8x_{14}, \quad \mathcal{Q}_8 = x_2x_9 + x_4x_{11} + x_6x_{13} + x_8x_{15},$$
$$\mathcal{Q}_9 = x_1x_9 + x_4x_{12} + x_6x_{14} + x_7x_{15}, \quad \mathcal{Q}_{10} = x_2x_{10} + x_3x_{11} + x_5x_{13} + x_8x_{16}.$$

Moreover, the quadric

$$\mathcal{Q}_0 \equiv \mathcal{Q}_9 + \mathcal{Q}_{10} = 0$$

is a hyperbolic quadric in $PG(15, 2)$ defined by eq. (4). The ideal I defines a subvariety $V(I)$ of $PG(15, 2)$ that $LGr(4, 8)$ is mapped to. Moreover, $V(I) \subset \mathcal{Q}^+(15, 2)$ because $\mathcal{Q}_0 \in I$. Again, by comparing the number of points of $V(I)$ and $LGr(4, 8)$, we conclude that

$$\underline{\pi}(LGr(4, 8)) = V(I) \subset \mathcal{Q}^+(15, 2).$$

5 Stratification of $PG(2^N - 1, 2)$

Let us consider the natural action of the group $G = SL(2, 2) \times SL(2, 2) \times \ldots, \times SL(2, 2) \rtimes S_N$ on $PG(2^N - 1, 2)$. This action partitions the set of points of the projective space in terms of G-orbits and allows us, thanks to the bijection described in Section 3, to partition $LGr(N, 2N)$ in terms of G-equivalent classes. Knowing a representative of each orbit, we can use the equations obtained in the previous section to check whether a particular orbit does or does not belong to $\underline{\pi}(LGr(N, 2N))$. In the following examples, we use the classification of G-orbits of points of $PG(7, 2)$ and $PG(15, 2)$ obtained by Bremner and Stavrou [11].

5.1 Two distinguished classes of mutually commuting two-qubit operators

It is well known (see, e. g., [13]) that the projective space $PG(3, 2)$ is the union of two G orbits,

$$PG(3, 2) = \mathcal{O}_1 \cup \mathcal{O}_2,$$

with $\sharp\mathcal{O}_1 = 9$ and $\sharp\mathcal{O}_2 = 6$. The orbit $\mathcal{O}_1$, comprising the points lying on a hyperbolic quadric $\mathcal{Q}^+(3, 2)$, corresponds to the G-orbit of any separable vector in the tensorial basis (for example, the orbit of $x_1^1 \otimes x_2^1$). On the other hand, $\mathcal{O}_2$, consisting of six off-quadric points, is the orbit corresponding to non-separable tensors (for example, the orbit of $x_1^0 \otimes x_2^0 + x_1^1 \otimes x_2^1$). Our bijection associates the two orbits of $PG(3, 2)$ with two distinguished classes of maximal sets of mutually commuting two-qubit operators, as described in Table 1.

The projective line $PG(1, 2)_a$, spanned by $\langle XI, IX \rangle$, is obviously mapped to $[0 : 0 : 1 : 0]$ by our construction. Indeed, according to eqs. (2) and (3), the observables XI and IX correspond to the points

Orbit	Size	Representative	Observable	Set of mutually commuting two-qubit observables
$\mathcal{O}_1$	9	$[0:0:1:0]$	XI	$PG(1,2)_a$
$\mathcal{O}_2$	6	$[1:0:1:0]$	YI	$PG(1,2)_b$

Table 1: Classes of mutually commuting 2-qubits operators; here, $PG(1,2)_a = \langle XI, IX \rangle$ and $PG(1,2)_b = \langle ZX, XZ \rangle$.

of $\mathcal{W}(3,2)$ having the coordinates $(0,0,1,0)$ and $(0,0,0,1)$ and these two points define the line represented by the matrix

$$P_a = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whose Plücker coordinates are $[0:0:1:0]$, i.e., $p_{12} = p_{14} = p_{23} = 0$ and $p_{34} = 1$. Similarly, the line defined by

$$P_b = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

satisfies $p_{12} = p_{34} = 1$ and $p_{14} = p_{23} = 0$, i.e., it is mapped to $[1:0:1:0]$. This line is spanned by the points $(1,0,0,1)$ and $(0,1,1,0)$, being thus generated by ZX and XZ.

The partition of $PG(3,2)$ into two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ tells us that we can similarly partition $LGr(2,4)$ into two classes of lines; a class of cardinality 9, which is the G-orbit of $PG(1,2)_a$, and a class of cardinality 6, which is the G-orbit of $PG(1,2)_b$.

5.2 Three distinguished classes of mutually commuting three-qubit operators

The projective space $PG(7,2)$ is the union of five G-orbits (see [11, 13]),

$$PG(7,2) = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5,$$

with $\sharp\mathcal{O}_1 = 27$, $\sharp\mathcal{O}_2 = 54$, $\sharp\mathcal{O}_3 = 108$, $\sharp\mathcal{O}_4 = 54$ and $\sharp\mathcal{O}_5 = 12$. It is also well known (see, e.g., [13]) that

$$\mathcal{Q}^+(7,2) = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_4.$$

Hence, in light of our main result of Sec. 4.2, the variety $\underline{\pi}(LGr(3,6))$ is partitioned into three different G-orbits whose properties are summarized in Table 2; here, we used the explicit expression of $\underline{\pi}$ given in [8] and the representatives of the orbits $\mathcal{O}_i$ were taken from [11] (transformed, of course, into our adopted system of coordinates).

Orbit	Size	Representative	Symmetric four-qubit observable	Set of mutually commuting three-qubit observables
$\mathcal{O}_1$	27	$[0:0:0:0:1:0:0:0]$	$XIII$	$PG(2,2)_a$
$\mathcal{O}_2$	54	$[0:0:0:1:0:0:1:0]$	$IIXZ$	$PG(2,2)_b$
$\mathcal{O}_4$	54	$[0:0:0:1:0:1:1:0]$	$IXXZ$	$PG(2,2)_c$

Table 2: Classes of mutually commuting 3-qubits operators; here, $PG(2,2)_a = \langle XII, IXI, IIX \rangle$, $PG(2,2)_b = \langle ZZI, XXI, IIX \rangle$, and $PG(2,2)_c = \langle XIX, IXX, ZZZ \rangle$.

To illustrate how we assign a projective plane of order two to a representative of $\mathcal{O}_i$, let us detail the calculation for the orbit $\mathcal{O}_2$. A representative of the second non-trivial orbit in the classification of [11] is, in the tensorial basis, $x_1^0 \otimes x_2^1 \otimes x_3^1 + x_1^1 \otimes x_2^0 \otimes x_3^1$, which in our notation corresponds to $x_4 = x_7 = 1$.

Using the labeling of the Plücker coordinates given in Section 4.2 this means that $p_{156} = p_{246} = 1$, the remaining coordinates being zero. The 3×6 matrix satisfying these conditions is of the form

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

The 3 vectors represented by the rows of the matrix are the coordinates of the three points that span the $PG(2,2)_b$, which is obviously mapped to $[0:0:0:1:0:0:1:0]$. A similar partition of $LGr(3,6)$ into three different classes is also obvious.

5.3 Six distinguished classes of mutually commuting four-qubit operators

The stratification of $PG(15,2)$ in terms of 29 G-orbits was also established in [11]. In order to identify the orbits which partition the variety $\underline{\pi}(LGr(4,8))$, we checked the representative of each orbit, taken from Table 5 of [11], and found out that six of them annihilate the polynomials of the ideal I (see eq. (13)). The results of our calculations are portrayed in Table 3 (here the first non-trivial orbit is denoted $\mathcal{O}_2$ to be compatible with the numbering of [11], which also takes into account the trivial orbit).

Orbit	Size	Representative	Symmetric 8-qubit obs'le	Set of mut. comm. 4-qubit obs'les
$\mathcal{O}_2$	81	$[0:0:0:0:0:0:0:0:1:0:0:0:0:0:0:0]$	$XIIIIIII$	$PG(3,2)_a$
$\mathcal{O}_3$	324	$[0:0:0:0:0:0:0:0:0:1:1:0:0:0:0:0]$	$IXXIIIII$	$PG(3,2)_b$
$\mathcal{O}_6$	648	$[0:0:0:0:0:0:0:0:0:1:1:0:1:0:0:0]$	$IXXIXIII$	$PG(3,2)_c$
$\mathcal{O}_{14}$	162	$[0:0:0:0:0:0:0:1:0:1:1:0:1:0:0:0]$	$IXXIXIIZ$	$PG(3,2)_d$
$\mathcal{O}_{17}$	108	$[0:0:0:0:0:1:1:0:0:0:0:0:0:1:1:0]$	$IIIIIYYI$	$PG(3,2)_e$
$\mathcal{O}_{18}$	972	$[0:0:0:0:0:1:1:0:1:0:0:0:0:1:1:0]$	$XIIIIYYI$	$PG(3,2)_f$

Table 3: Classes of mutually commuting 4-qubits operators; here, $PG(3,2)_a = \langle XIII, IXII, IIXI, IIIX \rangle$, $PG(3,2)_b = \langle XIII, IXII, IIZZ, IIYY \rangle$, $PG(3,2)_c = \langle XIII, IZZZ, IYYZ, IYZY \rangle$, $PG(3,2)_d = \langle ZYYY, YZYY, YYZY, YYYZ \rangle$, $PG(3,2)_e = \langle XXII, ZZII, IIZZ, IIYY \rangle$ and $PG(3,2)_f = \langle XIZZ, IXZZ, ZZXI, ZZIX \rangle$.

To identify the $PG(3,2)$'s that correspond to the representatives of the orbits we proceed similarly as in the previous two cases, that is, we create the 4×8 matrix whose minors satisfy the conditions implied by the corresponding representative. $LGr(4,8)$ is likewise partitioned into six non-equivalent classes.

6 The Lagrangian Grassmannian and the variety of principal minors

At this point is is worth mentioning several papers [14, 15, 16] that deal with similar problems over the field of complex numbers and which are deeply related to the construction over $GF(2)$ considered in this paper.

Let $\mathbb{K} = \mathbb{C}$ and let $\mathcal{Z}_N \subset \mathbb{P}(\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N \text{ times}})$ be the image of the following rational map [15]:

$$\begin{array}{rccc} \phi: & \mathbb{P}(S^2 \mathbb{C}^n \oplus \mathbb{C}) & \dashrightarrow & \mathbb{P}((\mathbb{C}^2)^{\otimes N}) \\ & [A,t] & \mapsto & [t^{n-|I|} \Delta_I(A) X^I], \end{array}$$

with A being a symmetric complex matrix and $X^I = x_1^{i_1} \otimes \cdots \otimes x_N^{i_N}$, $i_j = \left\{ \begin{array}{ll} 0 & \text{if } j \notin I \\ 1 & \text{if } j \in I \end{array} \right.$, a tensorial basis of $(\mathbb{C}^2)^{\otimes N}$. $\mathcal{Z}_N$ is an algebraic variety, called the variety of principal minors of symmetric matrices,

corresponding to the linear projection of the Lagrangian Grassmannian (over the complex numbers). The linear projection means that from the set of minors of cardinality $\binom{2N}{N}$ we only keep the set of principal minors of cardinality 2^N, i.e. it is the type of the projection $\underline{\pi}$ defined in Section 3 over $GF(2)$. However, in the complex case this projection is no longer a bijection, that is, the principal minors do not contain all the information on the Lagrangian Grassmannian.

The motivation for studying $\mathcal{Z}_N$ in the complex case comes from the Principal Minors Assignment Problem [17, 16]. This problem asks for necessary and sufficient conditions for a collection of 2^N numbers to arise as the principal minors of an $N \times N$ matrix. In the case of a symmetric matrix, a collection of 2^N numbers corresponds to its principal minors if and only of the corresponding point in $\mathbb{P}((\mathbb{C}^2)^{\otimes N})$ belongs to the variety $\mathcal{Z}_N$. This problem for symmetric matrices was solved by Oeding [14, 15], who successfully described a set of degree-four polynomials which cut out the variety $\mathcal{Z}_N$. In particular, Oeding proved, using representation theory techniques, that a set of equations defining $\mathcal{Z}_N$ are obtained by taking the $G = SL_2(\mathbb{C}) \times \cdots \times SL_2(\mathbb{C}) \ltimes S_N$ orbit of a certain peculiar quartic polynomial (the so-called $2 \times 2 \times 2$ Cayley hyperdeterminant).

It is, naturally, tempting to rephrase Oeding's result into the $GF(2)$-regime and check if one can recover the equations obtained in Sections 4.2 and 4.3. Over $GF(2)$, the equations provided by Corrolary 1.4 of [15] lead to the $SL(2,2) \times \cdots \times SL(2,2) \ltimes S_N$ orbit of

$$\mathcal{Q} = p_{\mathbf{123}4\ldots N}p_{\overline{\mathbf{123}}4\ldots N} + p_{\mathbf{12}\overline{\mathbf{3}}4\ldots N}p_{\overline{\mathbf{312}}4\ldots N} + p_{\mathbf{13}\overline{\mathbf{2}}4\ldots N}p_{\mathbf{2}\overline{\mathbf{13}}4\ldots N} + p_{\mathbf{1}\overline{\mathbf{23}}4\ldots N}p_{\mathbf{23}\overline{\mathbf{1}}4\ldots N},$$

where $\overline{k} = N + k$. For $N = 3$, $\mathcal{Q} = 0$ is readily recognized to be identical to eq. (10) defining $\underline{\pi}(LGr(3,6))$. For $N = 4$, the distinguished polynomial $\mathcal{Q}$ coincides with our $\mathcal{Q}_8$ appearing in the ideal defined by eq. (13). It can be shown that the G-orbit of $\mathcal{Q} = \mathcal{Q}_8$ consists of the polynomials $\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_8$ and $\mathcal{Q}_0$. However, via the repeated action of the generators of G we did not manage to get the remaining four-term polynomials $\mathcal{Q}_9$ and $\mathcal{Q}_{10}$, merely their sum $\mathcal{Q}_0 = \mathcal{Q}_9 + \mathcal{Q}_{10}$. This case thus features over $GF(2)$ some subtle properties that have no counterpart over the field of complex numbers. Nevertheless, we are convinced that the approach developed by Oeding is a very promising one, which can be appropriately adjusted/modified to be meaningful also over the smallest Galois field.

7   Conclusion

In this paper, we gave, for any $N \geq 2$, a rigorous proof of the existence of a bijection between the set of generators of the symplectic polar space $\mathcal{W}(2N - 1, 2)$ and a distinguished subset of points of $\mathcal{W}(2^N - 1, 2)$. Physically speaking, we established a one-to-one mapping between the maximal sets of pairwise commuting operators of the N-qubit Pauli group and a subset of the 2^{N-1} qubit observables. Proving this correspondence, we also found a method how to get the defining equations of the image of the mapping within $PG(2^N - 1, 2)$ and explicitly illustrated this method for the cases $N = 2$, $N = 3$ and $N = 4$.

The image of our mapping is over the complex numbers known as the variety of principal minors of symmetric matrices [14, 15]. We have also pointed out that the calculations in the $GF(2)$-regime deserve a special treatment and are, in general, not the direct translation of the results obtained over the field of the complex numbers. Since $GF(2)$-settings have already acquired a firm footing in the context of Quantum Information Theory, we aim at getting deeper insights into the variety of principal minors of $GF(2)$-symmetric matrices employing, if possible, a more coordinate-free approach.

Acknowledgments

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