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OWP 2014 - 01

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

Varieties of Invariant Subspaces Given by Littlewood-Richardson Tableaux

Justyna Kosakowska and Markus Schmidmeier

Abstract: Given partitions α, β, γ, the short exact sequences

$$0 \longrightarrow N_\alpha \longrightarrow N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

of nilpotent linear operators of Jordan types α, β, γ, respectively, define a constructible subset $\mathbb{V}^\beta_{\alpha,\gamma}$ of an affine variety.
Geometrically, the varieties $\mathbb{V}^\beta_{\alpha,\gamma}$ are of particular interest as they occur naturally and since they typically consist of several irreducible components. In fact, each Littlewood-Richardson tableaux Γ of shape (α, β, γ) contributes one irreducible component $\overline{\mathbb{V}}_\Gamma$.
We consider the partial order $\Gamma \leq^*_{\mathsf{closure}} \tilde{\Gamma}$ on LR-tableaux which is the transitive closure of the relation given by $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma \neq \emptyset$. In this paper we compare the closure-relation with partial orders given by algebraic, combinatorial and geometric conditions. In the case where the parts of α are at most two, all those partial orders are equivalent. We discuss how the orders differ in general.

MSC 2010: Primary: 14L30, 16G20, Secondary: 16G70, 05C85, 47A15

Key words: degenerations, partial orders, Hall polynomials, nilpotent operators, invariant subspaces, Littlewood-Richardson tableaux

1. Introduction

We consider varieties given by short exact sequences of nilpotent k-linear operators where k is an algebraically closed field:
Each such operator is given uniquely, up to isomorphy, as a $k[T]$-module $N_\alpha = \bigoplus_{i=1}^s k[T]/(T^{\alpha_i})$ for some partition $\alpha = (\alpha_1, \ldots, \alpha_s)$ which represents

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The first named author is partially supported by Research Grant No. DEC-2011/02/A/ ST1/00216 of the Polish National Science Center.

This research is partially supported by a Travel and Collaboration Grant from the Simons Foundation (Grant number 245848 to the second named author).

the sizes of its Jordan blocks. The Theorem of Green and Klein [3] states that for given partitions α, β, γ, there exists a short exact sequence

$$0 \longrightarrow N_\alpha \longrightarrow N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

of nilpotent linear operators if and only if there is a Littlewood-Richardson (LR-) tableau of shape (α, β, γ).

In fact, the collection of all such short exact sequences forms a variety $\mathbb{V}^\beta_{\alpha\gamma}(k)$ which can be partitioned using LR-tableaux, as follows. Consider the affine variety $\mathrm{Hom}_k(N_\alpha, N_\beta)$ endowed with the Zariski topology, and assume that all subsets carry the induced topology. Define

$$\mathbb{V}^\beta_{\alpha\gamma}(k) \;=\; \big\{ f : N_\alpha \to N_\beta \;\;\big|\;\; f \text{ monomorphism of } k[T]\text{-modules} \\ \text{with cokernel isomorphic to } N_\gamma \big\}.$$

The irreducible components of $\mathbb{V}^\beta_{\alpha\gamma}(k)$ are counted by the Littlewood-Richardson coefficient. Namely, to each short exact sequence in $\mathbb{V}^\beta_{\alpha,\gamma}$ one can associate an LR-tableau Γ of shape (α, β, γ), as we will see in Section 2. The subset $\mathbb{V}_\Gamma$ of $\mathrm{Hom}_k(N_\alpha, N_\beta)$ of all such short exact sequences is constructible and irreducible. All $\mathbb{V}_\Gamma$ have the same dimension. We denote by $\overline{\mathbb{V}}_\Gamma$ the closure of $\mathbb{V}_\Gamma$ in $\mathbb{V}^\beta_{\alpha,\gamma}$; the sets $\overline{\mathbb{V}}_\Gamma$ define the irreducible components of $\mathbb{V}^\beta_{\alpha,\gamma}$, they are indexed by the set $\mathcal{T}^\beta_{\alpha,\gamma}$ of all LR-tableaux of shape (α, β, γ) (see [7, Theorem 4.3] and [8]). Our aim in this paper is to shed light on the geometry in the variety

$$\mathbb{V}^\beta_{\alpha,\gamma} \;=\; \dot{\bigcup_{\Gamma \in \mathcal{T}^\beta_{\alpha,\gamma}}} \mathbb{V}_\Gamma;$$

by studying the closure-relation given as follows.

$$\Gamma \leq_{\mathsf{closure}} \tilde{\Gamma} \quad \Leftrightarrow \quad \mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma \neq \emptyset \qquad \text{where} \qquad \Gamma, \tilde{\Gamma} \in \mathcal{T}^\beta_{\alpha,\gamma}. \tag{1.1}$$

First we show in Theorem 3.1 that the closure-relation is a preorder (that is, it is reflexive and antisymmetric) since the relation $\Gamma \leq_{\mathsf{closure}} \tilde{\Gamma}$ implies that the tableaux are in the dominance order $\Gamma \leq_{\mathsf{dom}} \tilde{\Gamma}$ given by the natural partial orders of the partitions defining the tableaux. Since in general, the closure-relation is not transitive, see Example 3, we consider the transitive closure $\leq^*_{\mathsf{closure}}$ which is a partial order.

We consider a second combinatorially given preorder on LR-tableaux, the box-relation. We say $\Gamma <_{\mathsf{box}} \tilde{\Gamma}$ if $\tilde{\Gamma}$ is obtained from Γ by exchanging two entries which are the only entries in their respective column in such a way that the lower entry is the higher position in $\tilde{\Gamma}$. Here is an example:

$$\begin{ytableau} \ & \ & 1 \\ \ & 2 \\ 1 \end{ytableau} \qquad <_{\mathsf{box}} \qquad \begin{ytableau} \ & \ & 1 \\ \ & 1 \\ 2 \end{ytableau}$$

(The box-order introduced in Section 5.1 is more general.) The box-order together with the dominance order provide two tests of combinatorial nature for the validity and for the failure of the closure-relation:

$$\Gamma \leq_{\mathsf{box}} \tilde{\Gamma} \quad \Rightarrow \quad \Gamma \leq_{\mathsf{closure}} \tilde{\Gamma} \quad \Rightarrow \quad \Gamma \leq_{\mathsf{dom}} \tilde{\Gamma}$$

The reductive algebraic group $G = \mathrm{Gl}_\alpha \times \mathrm{Gl}_\beta$ is acting on $\mathbb{V}_{\alpha,\gamma}^\beta$ via $(a,b)\cdot f = bfa^{-1}$. The orbits of this group action are in one-to-one correspondence with the isomorphism classes of embeddings. This gives rise to a preorder for LR-tableaux: We say $\Gamma \leq_{\mathsf{deg}} \tilde{\Gamma}$ if there are embeddings $f \in \mathbb{V}_\Gamma$, $\tilde{f} \in \mathbb{V}_{\tilde{\Gamma}}$ such that $f \leq_{\mathsf{deg}} \tilde{f}$, that is, $\mathcal{O}_{\tilde{f}} \subset \overline{\mathcal{O}}_f$. The degeneration relation is under control algebraically as the ext-relation implies the deg-relation, which in turn implies the hom-relation. As the deg-relation, also the hom- and ext-relations give rise to preorders for LR-tableaux. In the diagram below, the relations introduced so far on the set $\mathcal{T}_{\alpha,\gamma}^\beta$ are ordered vertically by containment, with the dominance order the weakest of the relations pictured.

$$\begin{tikzcd}
 & \leq_{\mathsf{box}} \arrow[d] & \\
 & \leq_{\mathsf{ext}} \arrow[d] & \\
 & \leq_{\mathsf{deg}} \arrow[dl] \arrow[dr] & \\
\leq_{\mathsf{hom}} \arrow[dr] & & \leq_{\mathsf{closure}} \arrow[dl] \\
 & \leq_{\mathsf{dom}} &
\end{tikzcd}$$

We show that the dominance order is in fact equivalent to the covariant hom-order restricted to certain objects called *pickets*. Thus we have also algebraic tests both for the validity and for the failure of the closure-relation:

$$\Gamma \leq_{\mathsf{ext}} \tilde{\Gamma} \quad \Rightarrow \quad \Gamma \leq_{\mathsf{closure}} \tilde{\Gamma} \quad \Rightarrow \quad \Gamma \leq_{\mathsf{hom-picket}} \tilde{\Gamma}$$

The case where all parts of α are at most two has been studied by the authors in [5] and [6]:

Theorem 1.2. *Suppose α, β, γ are partitions such that all parts of α are at most two.*

1. *The relations $\leq_{\mathsf{dom}}$, $\leq_{\mathsf{hom}}$, $\leq_{\mathsf{closure}}$, $\leq_{\mathsf{deg}}$, $\leq_{\mathsf{ext}}$, $\leq_{\mathsf{box}}$ are all partial orders.*

2. *The above partial orders are all equivalent.*

Several parts of this result have been shown in [6, Proposition 5.4]; in this paper we give in particular a formula for the dimension of any intersection of type $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma}$. More precisely, in [5, 6] we stratify the variety $\mathbb{V}^{\beta}_{\alpha,\gamma} = \bigcup^{\bullet} \mathbb{V}_{\Delta}$ using arc diagrams Δ. Each $\mathbb{V}_{\Delta}$ is contained in a unique $\mathbb{V}_{\Gamma}$, and by resolving intersections in the arc diagram Δ, one can decide if $\mathbb{V}_{\Delta} \subset \mathbb{V}_{\tilde{\Gamma}}$. The dimension of the intersection $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma}$ is the dimension of $\mathbb{V}_{\Gamma}$ minus the minimum number of crossings in an arc diagram Δ with $\mathbb{V}_{\Delta} \subset \mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma}$.

We conclude this introduction with two conjectures.

Conjecture 1.3. *Suppose α, β, γ are partitions such that $\beta\setminus\gamma$ is a horizontal and vertical strip. Then the partial orders $\leq_{\mathsf{dom}}$ and $\leq_{\mathsf{box}}$ are equivalent.*

Clearly, it suffices to show that any two LR-tableaux which are in the dominance order can be transformed into each other by using box moves as described above. It is a consequence of this conjecture that all partial orders $\leq_{\mathsf{dom}}$, $\leq^*_{\mathsf{hom}}$, $\leq^*_{\mathsf{closure}}$, $\leq^*_{\mathsf{deg}}$, $\leq^*_{\mathsf{ext}}$, $\leq_{\mathsf{box}}$ on $\mathcal{T}^{\beta}_{\alpha,\gamma}$ are equivalent in this case.

In our second conjecture we consider LR-tableaux of arbitrary shape. It links the geometric closure-relation and the combinatorial dominance order.

Conjecture 1.4. *For two LR-tableaux Γ, $\tilde{\Gamma}$ of the same shape, and for k an algebraically closed field, the following two statements are equivalent.*

1. $\Gamma \leq_{\mathsf{dom}} \tilde{\Gamma}$.

2. $\Gamma \leq^*_{\mathsf{closure}} \tilde{\Gamma}$.

Acknowledgement: This research project was started when the authors visited the Mathematische Forschungsinstitut Oberwolfach in a Research in Pairs project. They would like to thank the members of the Forschungsinstitut for creating an outstanding environment for their mathematical research.

2. The LR-tableau of an embedding

Definition: An *LR-tableau* of shape (α, β, γ) is a Young diagram of shape β in which the region $\beta \setminus \gamma$ contains α'_1 entries $\boxed{1}$, ..., α'_s entries $\boxed{s}$, where $s = \alpha_1$ is the largest entry and α' is the transpose of α, such that

- in each row, the entries are weakly increasing,
- in each column, the entries are strictly increasing,
- for each $\ell > 1$ and for each column c: on the right hand side of c, the number of entries $\ell - 1$ is at least the number of entries ℓ.

The skew diagram $\beta \setminus \gamma$ is said to be *a horizontal* (resp. *a vertical*) *strip*, if $\beta_i \leq \gamma_i + 1$ (resp. $\beta'_i \leq \gamma'_i + 1$), for all i.

Example: Let $\alpha = (4)$, $\beta = (741)$, $\gamma = (62)$. Then the transpose of α is $\alpha' = (1111)$, so we have to fill the skew diagram $\beta \setminus \gamma$ with a $\boxed{1}$, a $\boxed{2}$, a $\boxed{3}$ and a $\boxed{4}$. Due to the conditions on an LR-tableau, this can only be done in one way.

$$\Gamma : \quad \begin{array}{|c|c|c|}
\hline
 & & 1 \\
\hline
 & \\
\cline{1-2}
 & 2 \\
\cline{1-2}
 & 3 \\
\cline{1-2}
 \\
\cline{1-1}
 \\
\cline{1-1}
4 \\
\cline{1-1}
\end{array}$$

In this example, $\beta \setminus \gamma$ is a vertical but not a horizontal strip.

Notation: An LR-tableau Γ is given as a sequence of partitions

$$\Gamma = [\gamma^{(0)}, \ldots, \gamma^{(s)}]$$

where $\gamma^{(i)}$ denotes the region in the Young diagram β which contains the entries $\boxed{}$, $\boxed{1}$, ..., $\boxed{i}$. If Γ has shape (α, β, γ), then $\gamma = \gamma^{(0)}$, $\beta = \gamma^{(s)}$, and $\alpha'_i = |\gamma^{(i)} \setminus \gamma^{(i-1)}|$ for $i = 1, \ldots, s$.

In the example above, $\Gamma = [(62), (621), (631), (641), (741)]$.

Definition: Given an embedding $A \subset B$ of nilpotent operators of type α and β, respectively, the *LR-tableau of the embedding* $A \subset B$ is given by

$$\Gamma = [\gamma^{(0)}, \ldots, \gamma^{(s)}] \quad \text{where} \quad B/T^i A \cong N_{\gamma^{(i)}}.$$

Similarly, the LR-tableau of a short exact sequence $0 \to N_\alpha \xrightarrow{f} N_\beta \to N_\gamma \to 0$ is given as the LR-tableau of the embedding $\mathrm{Im}(f) \subset N_\beta$.

Two classes of examples will be important in this paper:

Definition: 1. An embedding $(A \subset B)$ is a *picket* if B is an indecomposable $k[T]$-module.

2. An indecomposable embedding $(A \subset B)$ is a *pole* if A is a cyclic $k[T]$-module.

Clearly, every picket with nonzero subspace is a pole. Poles have been classified, up to isomorphy, by Kaplansky [2, Theorem 24].

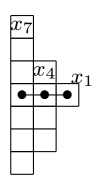
THEOREM 2.1. *A pole with submodule generator a is determined uniquely, up to isomorphy, by the radical layers of the elements $T^i a$.* □

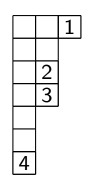
Notation: We denote by

1. P_ℓ^m the picket that represents the embedding $((T^{m-\ell}) \subset k[T]/(T^m))$ of an ℓ-dimensional subspace in an indecomposable nilpotent $k[T]$-module of dimension m,

2. $P(x_1, \ldots, x_k)$ the pole such that $T^i a \in T^{x_i} B \setminus T^{x_{i+1}} B$. For example for $\ell \geq 1$, the picket P_ℓ^m is the pole $P(m-\ell, m-\ell+1, \ldots, m-1)$.

Example: Let $\alpha = (4)$, $\beta = (741)$, $\gamma = (62)$, as before. We picture an embedding $(A \subset B)$ where $A \cong N_\alpha$, $B \cong N_\beta$ using the conventions from [10, (2.3)]. If the generators of B as a $k[T]$-module are x_7, x_4, x_1, as indicated, then the submodule generator $a = T^3 x_7 + T x_4 + x_1$ is given by the row of connected bullets.

X : 

Γ : 

The sequence of radical layers given by the T-powers $T^i a$ of the submodule generator a, and the corresponding quotients $B/(T^i a)$ are as follows.

$$\begin{array}{rccrcl}
a = & T^3x_7 + Tx_4 + x_1 & \in B \setminus TB & B/(a) & \cong & N_{62} \\
Ta = & T^4x_7 + T^2x_4 & \in T^2B \setminus T^3B & B/(Ta) & \cong & N_{621} \\
T^2a = & T^5x_7 + T^3x_4 & \in T^3B \setminus T^4B & B/(T^2a) & \cong & N_{631} \\
T^3a = & T^6x_7 & \in T^6B \setminus 0 & B/(T^3a) & \cong & N_{641}
\end{array}$$

We denote the pole $(A \subset B)$ by $P(0, 2, 3, 6)$ and observe that the entries $1, 2, 3, 4$ occur in rows $0 + 1, 2 + 1, 3 + 1, 6 + 1$ in the LR-tableau $\Gamma = [62, 621, 631, 641, 741]$.

In the remainder of this section, we present a formula for the number $\mu_{\ell,r}$ of boxes $\boxed{\ell}$ in the r-th row in the LR-tableau of an embedding $(A \subset B)$.

We denote the partition which consists of the first r rows of γ by $\gamma_{\leq r} = (\gamma'_1, \ldots, \gamma'_r)'$. Thus, if a $k[T]$-module C has type γ, then $C/T^r C$ has type $\gamma_{\leq r}$. The first rows of the above partitions $\gamma^{(\ell)}$ are given as follows.

$$\gamma^{(\ell)}_{\leq r} = \operatorname{type} \frac{B}{T^\ell A + T^r B}$$

In particular, the number of boxes $\boxed{\ell}$ in the first r rows of Γ is given by

$$|\gamma^{(\ell)}_{\leq r} \setminus \gamma^{(\ell-1)}_{\leq r}| = \dim \frac{T^{\ell-1} A + T^r B}{T^\ell A + T^r B}.$$

If $A = (a)$ is cyclically generated, then this number is 0 if $T^{\ell-1} a \in T^r B$ and 1 otherwise. Thus, in this case, Γ has a box $\boxed{\ell}$ in the r-th row if and only if

$$T^{\ell-1} a \in T^r B \setminus T^{r+1} B.$$

In general, denote by $\mu_{\ell,r}$ the number of boxes $\boxed{\ell}$ in the r-th row of the LR-tableau for an embedding $(A \subset B)$.

$$\begin{aligned} \mu_{\ell,r}(A \subset B) &= |\gamma^{(\ell)}_{\leq r} \setminus \gamma^{(\ell-1)}_{\leq r}| - |\gamma^{(\ell)}_{\leq r-1} \setminus \gamma^{(\ell-1)}_{\leq r-1}| \\ &= \dim \frac{T^{\ell-1} A + T^r B}{T^\ell A + T^r B} - \dim \frac{T^{\ell-1} A + T^{r-1} B}{T^\ell A + T^{r-1} B} \end{aligned}$$

As a consequence we obtain the following result.

LEMMA 2.2. *The isomorphism type of a pole $(A \subset B)$ where $A = (a)$ is determined uniquely by each of the following:*

1. *the type of B and the radical layers of the elements $T^i a$,*
2. *the LR-tableau for $(A \subset B)$, or*
3. *the type of B and the type of B/A.* $\square$

3. The closure-relation and its properties

In this section we present properties of the closure-relation defined by formula 1.1.

3.1. The closure-relation is a preorder

We show that the closure-relation for LR-tableaux implies the dominance order. As a consequence, the closure-relation is antisymmetric.

Definition:
- Two partitions $\gamma, \tilde{\gamma}$ are in the *natural partial order,* in symbols $\gamma \leq_{\mathsf{dom}} \tilde{\gamma}$, if the inequality

$$\gamma'_1 + \cdots + \gamma'_j \leq \tilde{\gamma}'_1 + \cdots + \tilde{\gamma}'_j$$

holds for each j.

- Two LR-tableaux $\Gamma = [\gamma^{(0)}, \ldots, \gamma^{(s)}]$, $\tilde{\Gamma} = [\tilde{\gamma}^{(0)}, \ldots, \tilde{\gamma}^{(s)}]$ of the same shape are in the *dominance order,* in symbols $\Gamma \leq_{\mathsf{dom}} \tilde{\Gamma}$, if for each i, $\gamma^{(i)} \leq_{\mathsf{dom}} \tilde{\gamma}^{(i)}$ holds.

Theorem 3.1. *Suppose that k is an algebraically closed field, and that the LR-tableaux Γ, $\tilde{\Gamma}$ have the same shape. If $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma} \neq \emptyset$ holds then $\Gamma \leq_{\mathsf{dom}} \tilde{\Gamma}$.*

Corollary 3.2. *The closure-relation is a preorder (i.e. reflexive and antisymmetric).*

We begin with a lemma.

Lemma 3.3. *Suppose A, B are vector spaces and $\mathcal{M} \subseteq \mathrm{Hom}_k(A, B)$ is a set of monomorphisms. For subspaces $U \subseteq A$, $V \subseteq B$ and a natural number n, the condition*

$$\dim(f(U) \cap V) \geq n$$

defines a closed subset in $\mathcal{M}$.

Proof. Recall that for a natural number m, the condition $\mathrm{rank}(f) > m$ defines an open subset in $\mathrm{Hom}_k(A, B)$ since it is given by the non-vanishing of a minor in the matrix representing f. By restricting that matrix to a basis for U and a basis for the complement of V, we see that the condition $\dim \frac{f(U)+V}{V} > m$ also defines an open subset in $\mathrm{Hom}_k(A, B)$. Let now $m = \dim U - n$. From the isomorphism $\frac{f(U)+V}{V} \cong \frac{f(U)}{f(U) \cap V}$ we obtain that the subset defined by

$\dim \frac{f(U)}{f(U)\cap V} > m$ is open, in particular it is open when restricted to $\mathcal{M}$. Since on $\mathcal{M}$, all spaces $f(U)$ have the same dimension (f is a monomorphism), the condition is equivalent to

$$\dim f(U) \cap V < \dim f(U) - m = n.$$

The complementary condition $\dim f(U) \cap V \geq n$ defines a closed subset of $\mathcal{M}$. ◻

PROPOSITION 3.4. *For all natural numbers i, ℓ, n, the subset*

$$\bigcup \big\{ \mathbb{V}_\Gamma : \Gamma \textit{ satisfies } (\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell \geq n \big\}$$

in $\mathbb{V}^\beta_{\alpha,\gamma}(k)$ is closed.

Proof. Denote by P^ℓ the $k[T]$-module $k[T]/T^\ell$ with only one Jordan block, so

$$B/f(T^iA) = \bigoplus_j P^{\gamma^{(i)}_j},$$

where $B = N_\beta$, $A = N_\alpha$ and $f \in \mathbb{V}^\beta_{\alpha\gamma}$. Recall that $\dim \mathrm{Hom}_{k[T]}(P^\ell, P^m) = \min\{\ell, m\} = \dim \frac{P^\ell}{T^m P^\ell}$. Thus:

$$\begin{aligned}
(\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell &= \sum_j \min\{\gamma^{(i)}_j, \ell\} \\
&= \dim \mathrm{Hom}_{k[T]}(B/f(T^iA), P^\ell) \\
&= \dim \frac{B/f(T^iA)}{T^\ell(B/f(T^iA))} \\
&= \dim \frac{B/f(T^iA)}{(T^\ell B + f(T^iA))/f(T^iA)}
\end{aligned}$$

Using the isomorphism $\frac{T^\ell B + f(T^iA)}{f(T^iA)} \cong \frac{T^\ell B}{T^\ell B \cap f(T^iA)}$ we obtain

$$(\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell = \dim B - \dim f(T^iA) - \dim T^\ell B + \dim T^\ell B \cap f(T^iA).$$

Since $\dim B - \dim f(T^iA) - \dim T^\ell B = c$ is constant on $\mathbb{V}^\beta_{\alpha\gamma}$, Lemma 3.3 implies that the set

$$\bigcup \big\{ \mathbb{V}_\Gamma : (\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell \geq n \big\} = \big\{ f \in \mathbb{V}^\beta_{\alpha\gamma} : \dim T^\ell B \cap f(T^iA) \geq n - c \big\}$$

is a closed subset of $\mathbb{V}^\beta_{\alpha\gamma}$. ◻

Proof of Theorem 3.1. We assume that $\Gamma \not\leq_{\mathsf{dom}} \tilde{\Gamma}$ and show that $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma} = \emptyset$. By assumption, there exist i, ℓ such that

$$n = (\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell > (\tilde{\gamma}^{(i)})'_1 + \cdots + (\tilde{\gamma}^{(i)})'_\ell$$

holds. By the proposition, $\mathbb{U} = \bigcup \left\{ \mathbb{V}_{\hat{\Gamma}} : (\hat{\gamma}^{(i)})'_1 + \cdots + (\hat{\gamma}^{(i)})'_\ell \geq n \right\}$ is a closed subset of $\mathbb{V}^\beta_{\alpha\gamma}$ such that

$$\mathbb{V}_\Gamma \subseteq \mathbb{U} \quad \text{and} \quad \mathbb{U} \cap \mathbb{V}_{\tilde{\Gamma}} = \emptyset.$$

Thus, $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma = \emptyset$. $\square$

We conclude this section with a result for later use.

Lemma 3.5. *Suppose $f, g : N_\alpha \to N_\beta$ are objects in $\mathbb{V}^\beta_{\alpha,\gamma}$. Let W be a subspace of N_β which is invariant under all automorphisms of N_β as a $k[T]$-module. If $\mathcal{O}_f \subset \overline{\mathcal{O}}_g$ then*

$$\dim \mathrm{Im} f \cap W \geq \dim \mathrm{Im} g \cap W.$$

Examples of possible invariant submodules of N_β are the powers of the radical $T^r N_\beta$, powers of the socle $T^{-s}0$, and their intersections $T^r N_\beta \cap T^{-s}0$.

Proof. Let $h_\lambda : N_\alpha \to N_\beta$ be a one-parameter family of objects in $\mathbb{V}^\beta_{\alpha,\gamma}$ such that $h_\lambda \cong g$ for $\lambda \neq 0$ and $h_0 \cong f$. Put $n = \dim \mathrm{Im} g \cap W$.
Any isomorphism $h_\lambda \cong g$ $(\lambda \neq 0)$ induces an isomorphism $\mathrm{Im} h_\lambda \cap W \cong \mathrm{Im} g \cap W$ since W is invariant under automorphisms of N_β. By Lemma 3.3, the set

$$\left\{ h \in \mathbb{V}^\beta_{\alpha,\gamma} : \dim \mathrm{Im} h \cap W \geq n \right\}$$

is closed in $\mathbb{V}^\beta_{\alpha,\gamma}$, so with h_λ, $\lambda \neq 0$, also h_0 is in the set. This shows $\dim \mathrm{Im} f \cap W = \dim \mathrm{Im} h_0 \cap W \geq n$. $\square$

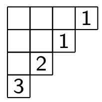
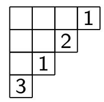
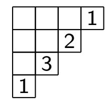
3.2. The closure-relation may not be transitive

In general, the relation for LR-tableaux given by

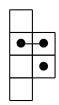
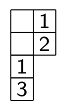
$$\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma \neq \emptyset$$

is not transitive. In this section, we provide an example.

Example: Let $\alpha = (3,1)$, $\beta = (4,3,2,1)$, $\gamma = (3,2,1)$. There are three LR-tableaux:

Γ_1 :  Γ_2 :  Γ_3 : 

Distributed over those three tableaux are five pairwise nonisomorphic embeddings $N_\alpha \to N_\beta$. Four of those are direct sums of poles, the fifth has an indecomposable summand which has a subspace that is not cyclically generated. Namely in the category $\mathcal{S}_4$, there is exactly one indecomposable object X with this property [10, (6.4)]:

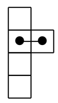
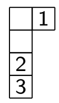
X :  Γ_X : 

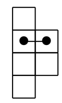
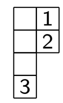
Thus, the object $M_2 = X \oplus P_0^3 \oplus P_0^1$ has LR-tableau Γ_2. (The LR-tableau of a direct sum is obtained by merging the rows of the LR-tableaux of the summands, starting at the top, and by sorting the entries in each row.)

The remaining four embeddings are direct sums of poles. Note that the height sequence of each pole is determined by the Klein tableau (since there is no row in any of the Klein tableaux which has the same entry twice, each with a different subscript). Moreover by [2, Theorem 24], each pole is determined uniquely, up to isomorphism, by its height sequence. Also note that Γ_1 can be refined in two different ways to a Klein tableau, and hence gives rise to two different pole decompositions. We are dealing with the following five isomorphism types of embeddings.

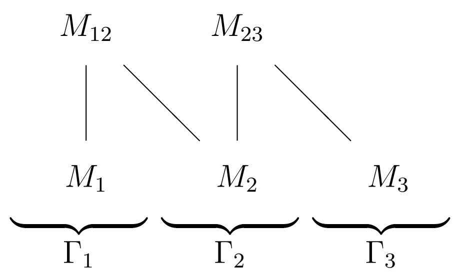
$$\begin{aligned} M_1 &= P_3^4 \oplus P_0^3 \oplus P_0^2 \oplus P_1^1, \\ M_{12} &= P(0,2,3) \oplus P_0^3 \oplus P_1^2, \\ M_2 &= X \oplus P_0^3 \oplus P_0^1, \\ M_{23} &= P(0,1,3) \oplus P_1^3 \oplus P_0^1, \\ M_3 &= P_1^4 \oplus P_3^3 \oplus P_0^2 \oplus P_0^1 \end{aligned}$$

The notation is such that M_i or M_{ix} has LR-tableau Γ_i. For the convenience of the reader, we picture the poles $P(0,2,3)$ and $P(0,1,3)$ and their LR-tableaux.

$P(0,2,3)$:  Γ_P : 

$P(0,1,3)$:  Γ_P : 

We show the containment relation of orbit closures is as follows.



The short exact sequence

$$0 \longrightarrow P_1^2 \longrightarrow M_2 \longrightarrow P(0,2,3) \oplus P_0^3 \longrightarrow 0$$

shows that $\mathcal{O}(M_{12}) \subset \overline{\mathcal{O}}(M_2)$, hence $\mathbb{V}_{\Gamma_1} \cap \overline{\mathbb{V}}_{\Gamma_2} \neq \emptyset$ and $\Gamma_1 >^*_{\mathsf{closure}} \Gamma_2$.

Similarly, the short exact sequence

$$0 \longrightarrow P_1^3 \longrightarrow M_3 \longrightarrow P(0,1,3) \oplus P_0^1 \longrightarrow 0$$

shows that $\mathcal{O}(M_{23}) \subset \overline{\mathcal{O}}(M_3)$, hence $\mathbb{V}_{\Gamma_2} \cap \overline{\mathbb{V}}_{\Gamma_3} \neq \emptyset$ and $\Gamma_2 >^*_{\mathsf{closure}} \Gamma_3$.

However, $\mathbb{V}_{\Gamma_1} \cap \overline{\mathbb{V}}_{\Gamma_3} = \emptyset$. The only possible orbit in the intersection is $\mathcal{O}(M_{12})$, since there are only two orbits in $\mathbb{V}_{\Gamma_1}$, and since the other orbit $\mathcal{O}(M_1)$ has the same dimension as $\mathbb{V}_{\Gamma_3} = \mathcal{O}(M_3)$.

Note that the module $M_{12} = (U \subset V)$ has the property that $\dim U \cap T^2 V \cap \operatorname{soc} V = 1$, while for the module M_3, the corresponding dimension is 2. It follows from Lemma 3.5 with $W = T^2 V \cap \operatorname{soc} V$ that $\mathcal{O}(M_{12}) \not\subseteq \overline{\mathcal{O}}(M_3)$.

This finishes the example which illustrates that in general, the condition for LR-tableaux that $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma} \neq \emptyset$ may not define a partial order. $\square$

4. Partial orders on the set of LR-tableaux

For modules of a fixed dimension over a finite dimensional algebra the three partial orders

$$\leq_{\mathsf{ext}}, \quad \leq_{\mathsf{deg}}, \quad \leq_{\mathsf{hom}}$$

have been studied extensively, see for example [1], [9]. In particular, the partial orders are available for invariant subspaces in $\mathbb{V}_{\alpha\gamma}^{\beta}$, see [5, Section 3.2]. For the convenience of the reader we recall these definitions. Let $f, g \in \mathbb{V}_{\alpha\gamma}^{\beta}$.

- The relation $f \leq_{\mathsf{ext}} g$ holds if there exist embeddings h_i, u_i, v_i in $\mathbb{V}_{\alpha\gamma}^{\beta}$ and short exact sequences $0 \to u_i \to h_i \to v_i \to 0$ of embeddings such that $f \cong h_1$, $u_i \oplus v_i \cong h_{i+1}$ for $1 \leq i \leq s$, and $g \cong h_{s+1}$, for some natural number s.
- The relation $f \leq_{\mathsf{deg}} g$ holds if $\mathcal{O}_g \subseteq \overline{\mathcal{O}_f}$ in $V_{\alpha,\gamma}^{\beta}(k)$.
- The relation $f \leq_{\mathsf{hom}} g$ holds if
$$[h, f] \leq [h, g]$$
for any embedding h in $\mathbb{V}_{\alpha\gamma}^{\beta}$, where $[h, f]$ denotes the dimension of the linear space $\mathrm{Hom}(h, f)$ of all homomorphisms of embeddings.

They induce three preorders on the set $\mathcal{T}_{\alpha\gamma}^{\beta}$: $\leq_{\mathsf{ext}}^{*}, \quad \leq_{\mathsf{deg}}^{*}, \quad \leq_{\mathsf{hom}}^{*}$ which, as we will see, are in fact partial orders.

Definition: Suppose Γ, $\tilde{\Gamma}$ are two LR-tableaux of shape (α, β, γ). We write $\Gamma \leq_{\mathsf{ext}}^{*} \tilde{\Gamma}$ ($\Gamma \leq_{\mathsf{deg}}^{*} \tilde{\Gamma}$; $\Gamma \leq_{\mathsf{hom}}^{*} \tilde{\Gamma}$) if there is a sequence

$$\Gamma = \Gamma^{(0)}, \Gamma^{(1)}, \ldots, \Gamma^{(s)} = \tilde{\Gamma}$$

such that for each $1 \leq i \leq s$ there are $f \in \mathbb{V}_{\Gamma^{(i-1)}}$, $g \in \mathbb{V}_{\Gamma^{(i)}}$ with $f \leq_{\mathsf{ext}} g$ ($f \leq_{\mathsf{deg}} g$; $f \leq_{\mathsf{hom}} g$).

It follows from the corresponding properties for modules that:

- $\Gamma \leq_{\mathsf{ext}}^{*} \tilde{\Gamma}$ implies $\Gamma \leq_{\mathsf{deg}}^{*} \tilde{\Gamma}$ and
- $\Gamma \leq_{\mathsf{deg}}^{*} \tilde{\Gamma}$ implies $\Gamma \leq_{\mathsf{hom}}^{*} \tilde{\Gamma}$.

Also, it is easy to see that

- $\Gamma \leq_{\mathsf{deg}}^{*} \tilde{\Gamma}$ implies $\Gamma \leq_{\mathsf{closure}}^{*} \tilde{\Gamma}$.

We have seen in Section 3.1 that the closure-relation implies the dominance order $\leq_{\mathsf{dom}}$. In the following section we show that also the hom-relation implies the dominance order.

4.1. Hom-relation implies dominance order

We start with an abstract result.

Denote by $\mathcal{N}$ the category $\mathrm{mod} k[T]_{(T)}$ of all nilpotent linear operators, and by $\mathcal{S} = \mathcal{S}(k[T]_{(T)})$ the category of all invariant subspaces. For each $i \in \mathbb{N}$, there is a pair of functors

$$\begin{aligned} R_i: \ \ & \mathcal{S} \to \mathcal{N}, \quad (A \subset B) \mapsto \frac{B}{T^i A} \\ L_i: \ \ & \mathcal{N} \to \mathcal{S}, \quad X \mapsto (\mathrm{soc}^{\,i} X \subset X). \end{aligned}$$

LEMMA 4.1. *For each $i \in \mathbb{N}$, the functors R_i, L_i form an adjoint pair.*

Proof. Given an operator $X \in \mathcal{N}$ and an invariant subspace $(A \subset B) \in \mathcal{S}$, we need to show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{S}}((A \subset B), L_i(X)) \ \cong \ \mathrm{Hom}_{\mathcal{N}}(R_i(A \subset B), X).$$

A morphism in $\mathcal{S}$ is given by a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f|_A"] \arrow[d] & \mathrm{soc}^{\,i} X \arrow[d] \\ B \arrow[r, "f"'] & X \end{tikzcd}$$

It gives rise to the commutative diagram:

$$\begin{tikzcd} \mathrm{rad}^i A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r, "f"'] & X \end{tikzcd}$$

Hence we obtain a morphism in $\mathcal{N}$:

$$\bar{f} : \frac{B}{\mathrm{rad}^i A} \longrightarrow X.$$

Conversely, the morphism in $\mathcal{N}$ gives rise to a commutative diagram and hence to a morphism in $\mathcal{S}$. Clearly, the two constructions are inverse to each other. $\square$

We recognize that the objects of the form $P_i^\ell = L_i(P^\ell)$ are pickets.

Proposition 4.2. *Suppose the objects* $(A \subset B)$ *and* $(\tilde{A} \subset \tilde{B})$ *have LR-tableaux* Γ *and* $\tilde{\Gamma}$*, respectively. The following assertions are equivalent:*

1. $\Gamma \leq_{\mathsf{dom}} \tilde{\Gamma}$
2. *For each picket* P_i^ℓ *the inequality holds:*

$$\dim \operatorname{Hom}_{\mathcal{S}}((A \subset B), P_i^\ell) \leq \dim \operatorname{Hom}_{\mathcal{S}}((\tilde{A} \subset \tilde{B}), P_i^\ell)$$

Proof. By the definition, given above the proof of Theorem 3.1, the condition $\Gamma \leq_{\mathsf{dom}} \tilde{\Gamma}$ is equivalent to

$$(\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell \leq (\tilde{\gamma}^{(i)})'_1 + \cdots + (\tilde{\gamma}^{(i)})'_\ell \qquad \text{for each } i \text{ and } \ell.$$

Let i and ℓ be natural numbers. We obtain from Lemma 4.1 and from the equality in the proof of Proposition 3.4 that

$$(\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell = \dim \operatorname{Hom}_{\mathcal{N}}(B/T^iA, P^\ell) = \dim \operatorname{Hom}_{\mathcal{S}}((A \subset B), P_i^\ell)$$

The claim follows from this and from the corresponding equality for $(\tilde{A} \subset \tilde{B})$. $\square$

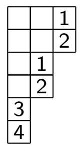
It follows that the covariant hom-relation implies the dominance order. Here is what we have. So far, we have not imposed any conditions on the triple (α, β, γ).

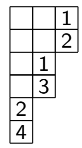
$$\begin{tikzcd} & \text{(ext*)} \arrow[d] & \\ & \text{(deg*)} \arrow[dl] \arrow[dr] & \\ \text{(covariant hom*)} \arrow[dr] & & \text{(closure*)} \arrow[dl] \\ & \text{(dominance)} & \end{tikzcd}$$

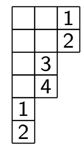
4.2. The ext- and deg-relations are not equivalent

It is well-known that for modules, the ext-relation $\leq_{\mathsf{ext}}$ implies the deg-relation $\leq_{\mathsf{deg}}$. In general for modules, the converse is not the case. Here we give an example for embeddings of linear operators.

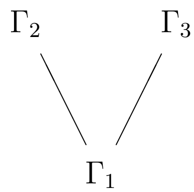
Example: For $\alpha = (4,2)$, $\beta = (6,4,2)$, $\gamma = (4,2)$, there are three LR-tableaux:

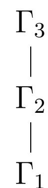
Γ_1 : 

Γ_2 : 

Γ_3 : 

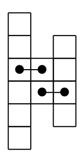
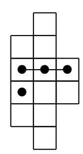
We show that the partial orders given by $\leq_{\mathsf{ext}}$ and $\leq_{\mathsf{deg}}$ are as follows:

ext: 

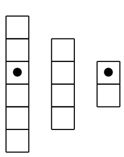
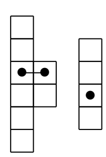
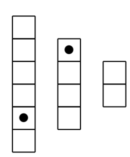
deg: 

(In each case, Γ_1 is the largest element in the poset.)

First we describe the embeddings which realize the tableaux. From [10] we know that there is a one-parameter family of indecomposable embeddings $M_2(\lambda)$ occurring on the mouths of the homogeneous tubes with tubular index 0; they all have type Γ_2. There are two additional indecomposables, they occur in the tube of circumference 2 at index 0; the modules are dual to each other and have type Γ_1 and Γ_2, respectively. We sketch the modules, using the conventions as in [10].

M_{12} : 
M_{23} : 

In addition, there are three decomposable configurations; note that M_1 is the dual of M_3 while M_{123} is self dual.

M_1 : 
M_{123} : 
M_3 : 

The modules $M_1 = P_4^6 \oplus P_0^4 \oplus P_2^2$ and $M_{123} = P(0,1,4,5) \oplus P_2^4$ have type Γ_1, and $M_3 = P_2^6 \oplus P_4^4 \oplus P_0^2$ has type Γ_3.

Consider the short exact sequences

$$0 \longrightarrow P_2^4 \longrightarrow M_{23} \longrightarrow P(0,1,4,5) \longrightarrow 0$$

and

$$0 \longrightarrow P_2^4 \longrightarrow M_3 \longrightarrow P(0,1,4,5) \longrightarrow 0.$$

In each, the sum of the end terms is M_{123}. It follows that $\Gamma_1 \geq_{\mathsf{ext}} \Gamma_2$ and $\Gamma_1 \geq_{\mathsf{ext}} \Gamma_3$, respectively. Note that $\Gamma_2 \not\geq_{\mathsf{ext}} \Gamma_3$ since there is no decomposable module of type Γ_2.

Since the ext-relation implies the deg-relation, it remains to show that $\Gamma_2 \geq_{\mathsf{deg}} \Gamma_3$. As mentioned, the modules M_1 and M_3 are dual to each other, so their orbits have the same dimension. As $\mathcal{O}_{M_3} = \mathbb{V}_{\Gamma_3}$, and since all varieties given by LR-tableaux are irreducible of the same dimension, it follows that $\mathcal{O}_{M_1}$ is dense in $\mathbb{V}_{\Gamma_1}$. In particular, $\mathcal{O}_{M_1}$ contains $\mathcal{O}_{M_{12}}$ in its closure. Applying duality again, we obtain that $\mathcal{O}_{M_3}$ contains $\mathcal{O}_{M_{23}}$ in its closure. Thus, $\mathcal{O}_{M_{23}}$ is in the closure of $\mathbb{V}_{\Gamma_3}$. □

5. Box-relation and ext-relation

We introduce box moves and the box-relation for LR-tableaux. We will show in the remainder of this paper that the box-relation is stronger than the ext-relation.

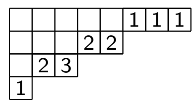
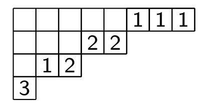
5.1. Box moves

Definition: We say an LR-tableau $\tilde{\Gamma}$ is obtained from the LR-tableau Γ via a *box move* if $\Gamma = \Gamma' \cup \Gamma'' \cup \Gamma'''$ and $\tilde{\Gamma} = \tilde{\Gamma}' \cup \tilde{\Gamma}'' \cup \tilde{\Gamma}'''$ are (rowwise) unions of LR-tableaux with the following properties:

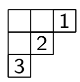
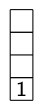
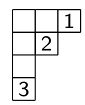
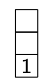
- Γ', Γ'', $\tilde{\Gamma}'$, $\tilde{\Gamma}''$ have no multiple entries.
- $\Gamma' \cup \Gamma''$ and $\tilde{\Gamma}' \cup \tilde{\Gamma}''$ have the same shape, but two entries are exchanged.
- $\Gamma''' = \tilde{\Gamma}'''$.

We write $\Gamma >_{\mathsf{box}} \tilde{\Gamma}$ if Γ has the smaller entry in the higher row. The *box-relation* is the reflexive and transitive closure of the relation given by box moves.

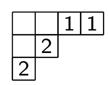
Example: Consider the following LR-tableaux

Γ :  and $\tilde{\Gamma}$: 

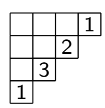
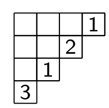
We set

Γ' :  Γ'' :  $\tilde{\Gamma}'$:  $\tilde{\Gamma}''$: 

and

$\Gamma''' = \tilde{\Gamma}'''$: 

Note that

$\Gamma' \cup \Gamma''$:  $\tilde{\Gamma}' \cup \tilde{\Gamma}''$: 

$\Gamma = \Gamma' \cup \Gamma'' \cup \Gamma'''$ and $\tilde{\Gamma} = \tilde{\Gamma}' \cup \tilde{\Gamma}'' \cup \tilde{\Gamma}'''$. Therefore $\tilde{\Gamma} >_{\mathsf{box}} \Gamma$.

This definition generalizes the example of a box move given in the introduction:

LEMMA 5.1. *Let Γ, $\tilde{\Gamma}$ be LR-tableaux of the same shape (α, β, γ) where $\beta \setminus \gamma$ is a horizontal strip. If $\tilde{\Gamma}$ is obtained from Γ by exchanging two entries (and by resorting the rows if necessary), then Γ and $\tilde{\Gamma}$ are in box-relation.*

Proof. First note that the (unordered) lists of columns in Γ and $\tilde{\Gamma}$ agree with the exception of two columns, which we call c', c'' and $\tilde{c}'$, $\tilde{c}''$, respectively. We assume that c' and $\tilde{c}'$ have the same length, say s, and that c'' and $\tilde{c}''$ have the same length, say r, where $r < s$. If $\Gamma <_{\mathsf{box}} \tilde{\Gamma}$, then the entries v of c' and u of c'' satisfy $u < v$. The corresponding entries in $\tilde{\Gamma}$ are u in $\tilde{c}'$ and v in $\tilde{c}''$.

To define Γ', take the column c', and successively the first column on the left starting from c' which contains an entry $v+1$, $v+2$, etc., up to the largest possible entry. In addition, take successively the first column on the right of c'' which contains an entry $v-1$, $v-2$, etc. For $\tilde{\Gamma}''$, take Γ' with column c' replaced by $\tilde{c}'$. Similarly, for Γ'', take the column c'' and successively from among the remaining columns the first column on the left of c' which contains an entry $u+1$, $u+2$, etc. In addition, take successively from among the remaining columns, starting at c'', the first column on the right which contains an entry $u-1$, $u-2$, etc. Let $\tilde{\Gamma}''$ be Γ'' with c'' replaced by $\tilde{c}''$. The remaining columns in Γ and $\tilde{\Gamma}$ form Γ''' and $\tilde{\Gamma}'''$, respectively.

Since both Γ, $\tilde{\Gamma}$ are LR-tableaux, so is Γ'''. $\square$

LEMMA 5.2. *Suppose embeddings X', X'' have LR-tableaux Γ', Γ'', respectively. Then the direct sum $X = X' \oplus X''$ has LR-tableau $\Gamma = \Gamma' \cup \Gamma''$.*

Proof. This is a consequence of the additivity of Formula 2.2. $\square$

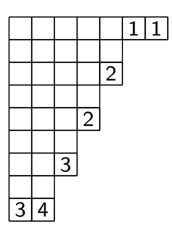
Returning to the set-up in the definition, since Γ', Γ'', $\tilde{\Gamma}'$, $\tilde{\Gamma}''$ have no multiple entries, each is the LR-type of a sum of a pole and a possibly empty sum of empty pickets. Moreover, each of these four tableaux determines an embedding uniquely, up to isomorphy.

5.2. Examples

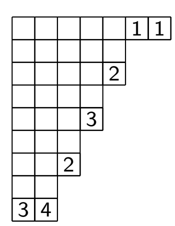
We present two examples where the box-relation implies the ext-relation.

In the first example, the LR-tableau $\tilde{\Gamma}$ is obtained from Γ by exchanging a box $\boxed{\mathsf{u}}$ where $u = 2$ occurs in row $r = 5$ with a $\boxed{\mathsf{v}}$ with $v = 3$ in row $s = 7$. Since in Γ, the box with the smaller entry is in the higher position, we have $\Gamma >_{\mathsf{box}} \tilde{\Gamma}$.

Γ :



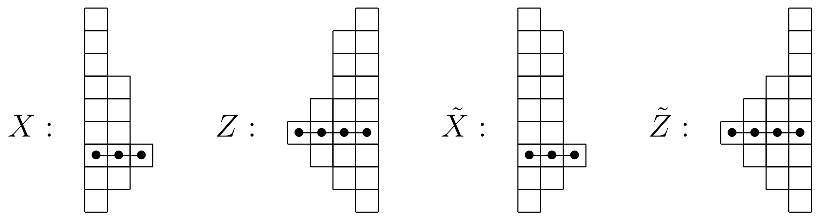
$\tilde{\Gamma}$:



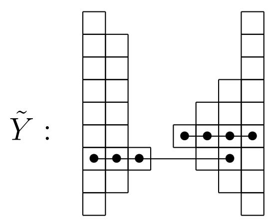
Our goal is to show that $\Gamma >_{\mathsf{ext}} \tilde{\Gamma}$.

Note that Γ is the LR-tableau for $X \oplus Z$ where $X = P(0,4,8)$ and $Z = P(0,2,6,8)$, while $\tilde{\Gamma}$ is the LR-tableau for $\tilde{X} \oplus \tilde{Z}$ where $\tilde{X} = P(0,6,8)$ and $\tilde{Z} = P(0,2,4,8)$.

We picture the four poles.



Let $\tilde{Y}$ be given by the following diagram. (Clearly, it is an extension of $\tilde{Z}$ by $\tilde{X}$.)

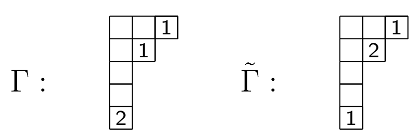


The statement $\Gamma >_{\mathsf{ext}} \tilde{\Gamma}$ is a consequence of the following two facts which we will show in a more general set-up in Section 5.4:

1. There is a short exact sequence $0 \to X \to \tilde{Y} \to Z \to 0$.
2. The LR-tableau for $\tilde{Y}$ is $\tilde{\Gamma}$.

Note that in the above example, the modules X, Z, $\tilde{X}$, $\tilde{Z}$ are all indecomposable. We present a second example in which X, Z, and $\tilde{X}$ are poles, hence indecomposable, but $\tilde{Z}$ is the direct sum of a pole and an empty picket.

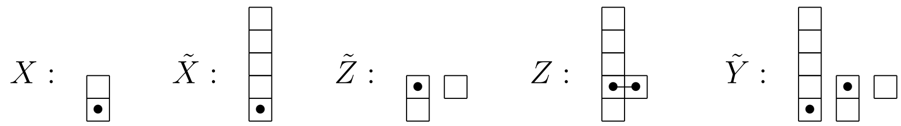
Example:



Here, $\tilde{\Gamma}$ is obtained from Γ by exchanging $\boxed{u}$ where $u = 1$ in row $r = 2$ by $\boxed{v}$ where $v = 2$ in row $s = 5$.

$$X = P(1); \quad \tilde{X} = P(4); \quad Z = P(0,4); \quad \tilde{Z} = P(0,1) \oplus P_1^5.$$

In this example we can put $\tilde{Y} = \tilde{X} \oplus \tilde{Z}$, then $\tilde{Y}$ has the same LR-tableau $\tilde{\Gamma}$ as $\tilde{X} \oplus \tilde{Z}$. It is easy to see that $\tilde{Y}$ occurs as the middle term of a short exact sequence $0 \to X \to \tilde{Y} \to Z \to 0$.



5.3. Gradings

To prove the main result of this section we need to work in the category of graded embeddings. Therefore we recall some facts and definitions.

Let $\mathrm{mod}_0^{\mathbb{Z}} k[T]$ denote the category of graded nilpotent $k[T]$-modules where T has degree 1. Each indecomposable object $P^m[d]$ has support given by an interval $[d, d+m-1]$; the object is generated as a $k[T]$-module by a homogeneous element (often denoted by g^m) in degree d. For $B \in \mathrm{mod}_0^{\mathbb{Z}} k[T]$, we denote the vector space in degree ℓ by B_ℓ.

By $\mathcal{S}^{\mathbb{Z}}$ we denote the category of graded embeddings between objects in $\mathrm{mod}_0^{\mathbb{Z}} k[T]$. Let H be a *height sequence*, i.e. a finite strictly increasing sequence of nonnegative integers. We show that the pole $P(H)$ is given by a graded embedding, also denoted by $P(H)$, by forgetting the grading. Here we choose the grading such that the subspace generator is in degree zero.

Definition: We say that a height sequence $H = (H(i))_{i=1,\dots,s}$ has a *jump* at i if $H(i+1) > H(i) + 1$, for $1 \le i \le s-1$, or if $i = s$.

The pole $P(H)$ is constructed as follows. Let $j_1 < \cdots < j_t$ be the list of all jumps. Put

$$\begin{aligned} B(H) &= \bigoplus_{i=1}^{t} P^{H(j_i)+1}[j_i - H(j_i) - 1] \\ a(H) &= (T^{H(j_i)+1-j_i})_{i=1,\dots,t} = \sum_{i=1}^{t} g^{H(j_i)+1} T^{H(j_i)+1-j_i} \\ A(H) &= a(H)k[T] \end{aligned}$$

Lemma 5.3. *For each height sequence H, there is a pole $P(H)$, unique up to isomorphy, which is gradable and which has height sequence H. It is isomorphic to the embedding $(A(H) \to B(H))$.*

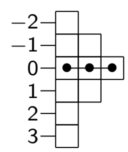
Proof. The embedding $(A(H) \to B(H))$ as above is degree preserving; clearly, the submodule generator $a(H)$ has height sequence H.

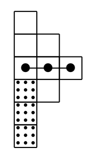
We show that the embedding is indecomposable. Note that the number of summands in $B(H)$ is the number of jumps in H (including the last jump at s). It suffices to show that each jump gives rise to an additional summand of B. Suppose $H(\ell+1) > H(\ell)+1$. Then there is $y \in B$ and $u \in \mathbb{N}$ such that $T^{\ell+1}a = T^u y$ but $T^\ell a \neq T^{u-1}y$. Thus, the map $T : B_\ell \to B_{\ell+1}$ is a not a monomorphism. Hence there is a summand in B which has support in degree ℓ but not in degree $\ell+1$.

The uniqueness statement follows from Kaplansky's Theorem. $\square$

Here is an example of a graded pole; the second picture is explained below.

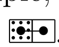
Example: The grading for the pole $P(0,2,4,5)$ is as follows.

$(A \subset B)$: 

$\left(\frac{A+T^3B}{T^3B} \subset \frac{B}{T^3B}\right)$: 

The height sequence $H = (0,2,4,5)$ gives rise to the jump sequence $j_1 = 1$, $j_2 = 2$, $j_3 = 4$, hence to the embedding given by $A(H) = P^4$, $B(H) = P^1 \oplus P^3[-1] \oplus P^6[-2]$, $a(H) = (1, T, T^2)$.

Graded embeddings facilitate the computation of the numbers $\dim \frac{B}{T^\ell A + T^r B}$ in Formula 2.2 which determine the LR-tableau of an embedding $(A \subset B)$: The numbers $\dim B/(T^\ell A + T^r B)$ can be computed componentwise for each degree i as the dimension of the subspace $(T^\ell A + T^r B)_i/(T^r B)_i$ within $B_i/(T^r B)_i$. Note that in the diagrams, $B_i/(T^r B)_i$ is given by the unshaded boxes in degree i, while the space $(T^\ell A + T^r B)_i/(T^r B)_i$ is given by the graded shifts of the dotted lines which mark the images of the generators of A in B.

In the above example, to verify that $\dim \frac{(A+T^3B)_1}{(T^3B)_1} = 1$ we read off from the picture in degree 1, , that A_1, being diagonally embedded, is not contained in the shaded region corresponding to $(T^3B)_1$.

5.4. Box-relation and ext-relation

We assume the set-up from the beginning of this chapter. To fix notation, suppose that the LR-tableau Γ has two boxes $\boxed{u}$ and $\boxed{v}$ in rows r and s

where $u < v$ and $r < s$. We assume that $\Gamma = \Gamma' \cup \Gamma'' \cup \Gamma'''$ is obtained by merging the rows of three LR-tableaux such that

(a) Γ' and Γ'' each have no multiple entries.

(b) Γ' contains $\boxed{u}$ in row r but no entries in rows $r+1, \ldots s$, and no other entry in the column of $\boxed{u}$.

(c) Γ'' contains $\boxed{v}$ in row s but no entries in rows $r, \ldots, s-1$ and no other entry in the column of $\boxed{v}$.

Then the tableau $\tilde{\Gamma}$ obtained by replacing the boxes $\boxed{u}$ and $\boxed{v}$ in Γ has the LR property.

Our goal is to construct a short exact sequence

$$0 \longrightarrow X \longrightarrow \tilde{Y} \longrightarrow Z \longrightarrow 0$$

such that the LR-type of $X \oplus Z$ is Γ and the LR-type of $\tilde{Y}$ is $\tilde{\Gamma}$.

Each tableaux Γ', Γ'' is the LR-tableau of a direct sum of a pole and one or several empty pickets. As we have seen in the second example in Section 5.2, some empty pickets are needed to perform a box move. We move all redundant empty pickets from Γ' and Γ'' to Γ'''. Moreover, we may assume that Γ''' is empty since an embedding which has LR-tableau Γ''' can be added later to both $\tilde{Y}$ and Z. Thus we may assume:

(d) Γ' is the LR-tableau of an embedding X where either $X = P(H_X)$ is a pole if the box with entry $u-1$ occurs in a row less than $r-1$; or $X = P(H_X) \oplus P^{r-1}$ is the direct sum of a pole and an empty picket P^{r-1} of height $r-1$.

(e) Γ'' is the LR-tableau of an embedding Z where either $Z = P(H_Z)$ is a pole if the box with entry $v+1$ occurs in a row larger than $s+1$; or $Z = P(H_Z) \oplus P^s$ is the direct sum of a pole and an empty picket P^s of height s.

(f) Γ''' is empty.

Let $\tilde{\Gamma}'$ be the tableau obtained from Γ' by replacing the column containing $\boxed{u}$ in row r by a column containing $\boxed{u}$ in row s. Because of (b), $\tilde{\Gamma}'$ is an LR-tableau. Similarly, let $\tilde{\Gamma}''$ be the tableau obtained from Γ'' by replacing the column containing $\boxed{v}$ in row s by a column containing $\boxed{v}$ in row r. Because of (c), $\tilde{\Gamma}''$ is an LR-tableau.

Let $H_{\tilde{X}}$ be the height sequence obtained from H_X by replacing $r-1$ by $s-1$. Then $\tilde{\Gamma}'$ is the LR-type of $\tilde{X}$, which is the pole $P(H_{\tilde{X}})$ or, in the case where $s+1$ occurs in $H_{\tilde{X}}$, the direct sum of $P(H_{\tilde{X}})$ and an empty picket of height s. Similarly, let $H_{\tilde{Z}}$ be the height sequence obtained from H_Z by replacing $s-1$ by $r-1$. Then $\tilde{\Gamma}''$ is the LR-type of $\tilde{Z}$ where $\tilde{Z}$ is the pole $P(H_{\tilde{Z}})$ or, in the case where $r-1$ occurs in $H_{\tilde{Z}}$, the direct sum of $P(H_{\tilde{Z}})$ and an empty picket of height $r-1$.

We summarize the definitions in a table.

module	tableau	entry in row?	module if NO:	if YES, add:
X	Γ'	$r-1$	$P(H_X)[v-u]$	$P^{r-1}[v-r]$
Z	Γ''	$s+1$	$P(H_Z)$	$P^s[v-s]$
$\tilde{X}$	$\tilde{\Gamma}'$	$s+1$	$P(H_{\tilde{X}})[v-u]$	$P^s[v-s]$
$\tilde{Z}$	$\tilde{\Gamma}''$	$r-1$	$P(H_{\tilde{Z}})$	$P^{r-1}[v-r]$

Observation: The embedding X embeds into $\tilde{X}$ as follows. Namely, H_X and $H_{\tilde{X}}$ differ by only one entry which gives $\tilde{X}$ a column that is by $s-r$ longer than the corresponding column in X. Note that if there is an entry in the $r-1$-st row in Γ', then $H_{\tilde{X}}$ has an additional jump, hence the ambient space for $\tilde{X}$ has an additional summand $P^{r-1}[v-r]$. Similarly, if there is an entry in the $s+1$-st row in Γ', then H_X has an additional jump, resulting in the ambient space for X having an additional summand $P^s[v-s]$. On the ambient spaces, the inclusion map is as follows.

$$\iota_{X,\tilde{X}}: \quad X_{\mathsf{amb}} \to \tilde{X}_{\mathsf{amb}}, \quad g_X^\ell \mapsto \begin{cases} g_{\tilde{X}}^s\, T^{s-r} & \text{if } \ell = r \\ g_{\tilde{X}}^\ell & \text{otherwise} \end{cases}$$

Here, $g_X^{r_i}$ denotes a homogeneous generator of the i-th summand in $X_{\mathsf{amb}} = \bigoplus_i P^{r_i}$. Note that $\operatorname{cok}(\iota_{X,\tilde{X}})$ is an empty picket of height $s-r$.

Dually, $\tilde{Z}$ embeds into Z with cokernel also an empty picket of height $s-r$.

$$\iota_{\tilde{Z},Z}: \quad \tilde{Z}_{\mathsf{amb}} \to Z_{\mathsf{amb}}, \quad g_{\tilde{Z}}^\ell \mapsto \begin{cases} g_Z^s\, T^{s-r} & \text{if } \ell = r \\ g_Z^\ell & \text{otherwise} \end{cases}$$

It remains to choose homogeneous submodule generators h_X, h_Z, $h_{\tilde{X}}$ and $h_{\tilde{Z}}$ for X, Z, $\tilde{X}$ and $\tilde{Z}$, respectively, such that the following compatibility conditions are satisfied.

- $\iota_{X,\tilde{X}}(h_X) = h_{\tilde{X}}$
- $\iota_{\tilde{Z},Z}(h_{\tilde{Z}}) = h_Z$
- If $h_{\tilde{X}} = \sum_\ell g^\ell_{\tilde{X}} T^{d_\ell}\mu_\ell$ and $h_{\tilde{Z}} = \sum_\ell g^\ell_{\tilde{Z}} T^{e_\ell}\nu_\ell$, then $\mu_s = \nu_r$ holds.

(The conditions are satisfied if X, Z, $\tilde{Z}$, $\tilde{Z}$ are poles as defined in Section 5.3. The compatibility conditions may give rise to nonzero coordinates in the empty pickets; they will not affect the height sequence of the submodule generator.)

We can now introduce the module $\tilde{Y}$. Define the ambient space for $\tilde{Y}$ as the sum $\tilde{Y}_{\mathsf{amb}} = \tilde{X}_{\mathsf{amb}} \oplus \tilde{Z}_{\mathsf{amb}}$ of graded modules, and let $\tilde{Y}_{\mathsf{sub}} = (h_{\tilde{X}}, g^r_{\tilde{Z}} T^{v-u}\nu_r) + (0, h_{\tilde{Z}})$. The maps in the short exact sequence

$$\mathcal{E}: \quad 0 \longrightarrow X \xrightarrow{\iota_{X,\tilde{Y}}} \tilde{Y} \xrightarrow{\pi_{\tilde{Y},Z}} Z \longrightarrow 0$$

are the following.

$$\iota_{X,\tilde{Y}}: \quad X_{\mathsf{amb}} \to \tilde{Y}_{\mathsf{amb}}, \quad g^\ell_X \mapsto \begin{cases} g^s_{\tilde{X}}\, T^{s-r} + g^r_{\tilde{Z}} & \text{if } \ell = r \\ g^\ell_{\tilde{X}} & \text{otherwise} \end{cases}$$

This map preserves the embedding since $\iota_{X,\tilde{Y}}(h_X) = (h_{\tilde{X}}, g^r_{\tilde{Z}} T^{v-u}\nu_r)$.

$$\pi_{\tilde{Y},Z}: \ \tilde{Y}_{\mathsf{amb}} \to Z_{\mathsf{amb}}, \ g^\ell_{\tilde{X}} \mapsto \begin{cases} 0 & \text{if } \ell \neq s \\ -g^s_Z & \text{if } \ell = s \end{cases} \ \text{and } g^\ell_{\tilde{Z}} \mapsto \begin{cases} g^\ell_Z & \text{if } \ell \neq r \\ g^s_Z\, T^{s-r} & \text{if } \ell = r \end{cases}$$

This map preserves the embedding since $\pi_{\tilde{Y},Z}(h_{\tilde{X}}, g^r_{\tilde{Z}} T^v \nu_r) = 0$ and $\pi_{\tilde{Y},Z}(0, h_{\tilde{Z}}) = \iota_{\tilde{Z},Z}(h_{\tilde{Z}}) = h_Z$.

It is straightforward to verify that the sequence $\mathcal{E}$ is exact. Namely, $\iota_{X,\tilde{Y}}$ is a monomorphism, $\pi_{\tilde{Y},Z}$ is an epimorphism, and the composition is zero.

It remains to show that the LR-tableau for $\tilde{Y}$ is $\tilde{\Gamma}$. We have already seen that $\tilde{X} \oplus \tilde{Z}$ has LR-tableau $\tilde{\Gamma}$. So we show that $\tilde{Y}$ and $\tilde{X} \oplus \tilde{Z}$ have the same LR-tableau. We write $\tilde{Y} = (A \subset B)$ and $\tilde{X} \oplus \tilde{Z} = (C \subset D)$. Recall that by construction, $B = D$. We have seen above in Section 5.3 that we need to verify that for each degree d, and for all exponents ℓ, q the subspaces

$$\left(\frac{T^\ell A + T^q B}{T^q B}\right)_d \text{ and } \left(\frac{T^\ell C + T^q D}{T^q D}\right)_d$$

have the same dimension. Note that the subspace generators are very similar: $A = (h_{\tilde{X}}, g^r_{\tilde{Z}} T^{v-u}\nu_r) + (0, h_{\tilde{Z}})$ versus $C = (h_{\tilde{X}}, 0) + (0, h_{\tilde{Z}})$; obviously, $\dim A_d, \dim C_d \leq 2$. Note that the subspaces differ only in degrees $v - u \leq d < v$. More precisely, the addition of $(0, g^r_{\tilde{Z}} T^{v-u}\nu_r)_d$ to the first generator $(h_{\tilde{X}}, 0)_d$ changes the dimension only if $(h_{\tilde{X}}, 0)_d \in (T^q B)_d$.

Note that $(h_{\tilde{X}}, 0)_d \in (T^q B)_d$ and $(0, g^r_{\tilde{Z}} T^{v-u}\nu_r)_d \notin (T^q B)_d$ is only possible if $d = v - 1$ (here we use that $\boxed{u}$ is the only entry in its column in Γ'). One can check that in this case, the ambient space $(B/T^q B)_d$ has dimension at most one. Note that whenever ℓ is such that $T^\ell(0, g^r_{\tilde{Z}} T^{v-u}\nu_r)_d$ is nonzero in $(B/T^q B)_d$, then so is $T^\ell(0, h_{\tilde{Z}})_d$. Hence, also in the case $d = v - 1$, the two subspaces have the same dimension.

This finishes the proof that $\tilde{Y}$ has LR-tableau $\tilde{\Gamma}$, as desired. As a corollary we obtain:

THEOREM 5.4. *Let Γ, $\tilde{\Gamma}$ be LR-tableaux of shape (α, β, γ). If $\Gamma \leq_{\mathsf{box}} \tilde{\Gamma}$, then $\Gamma \leq^*_{\mathsf{ext}} \tilde{\Gamma}$.*

Proof. The theorem follows from the arguments given above. □

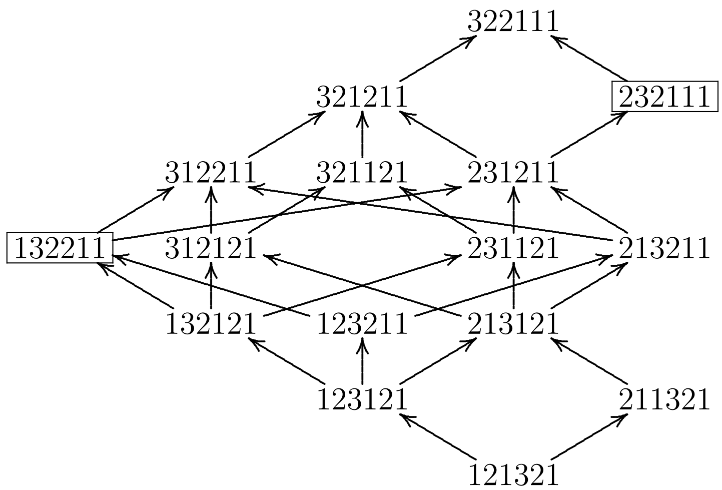
6. Combinatorial properties of the order $\leq_{\mathsf{box}}$

In this section we study combinatorial properties of the posets $(\mathcal{T}^\beta_{\alpha,\gamma}, \leq_{\mathsf{box}})$ and $(\mathcal{T}^\beta_{\alpha,\gamma}, \leq_{\mathsf{dom}})$, mainly in the case where $\beta \setminus \gamma$ is a horizontal and vertical strip. Recall that according to Conjecture 1.3, the two posets are equivalent in this case, and hence equivalent to all the algebraic and geometric poset structures considered in this paper.

Notation: Let Γ be an LR-tableau of shape (α, β, γ). Consider the chain ω_Γ consisting of the entries of Γ as they occur in Γ when read row-wise from the bottom up, and in each row from left to right. Note that given the shape (α, β, γ), this chain ω_Γ uniquely determines the tableau Γ.

6.1. An example

Consider the poset $(\mathcal{T}^\beta_{\alpha,\gamma}, \leq_{\mathsf{box}})$, where $\beta = (6, 5, 4, 3, 2, 1)$, $\gamma = (5, 4, 3, 2, 1)$ and $\alpha = (3, 2, 1)$. All LR-tableaux of this shape have entries: $1, 1, 1, 2, 2, 3$. The Hasse diagram of $(\mathcal{T}^\beta_{\alpha,\gamma}, \leq_{\mathsf{box}})$ is the following (instead of Γ we write ω_Γ):



Consider the LR-tableaux in frames. One is obtained from the other by a single box-move, but there is no chain of neighboring moves, i.e. moves that exchange neighbors. Moreover note that:

- $\beta \setminus \gamma$ is a horizontal and a vertical strip;
- in this poset there exists exactly one maximal and exactly one minimal element;
- all saturated chains have the same length;
- this poset is not a lattice;

6.2. Maximal and minimal elements

We have seen in [6, Proposition 5.5] that the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{dom}})$ has a unique maximal and a unique minimal element in the case where all parts of α are at most 2. We show that this statement also holds true if $\beta \setminus \gamma$ is a horizontal and vertical strip.

LEMMA 6.1. *Assume that $\beta \setminus \gamma$ is a horizontal and vertical strip. In the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$ there exists:*

1. *exactly one maximal element: the LR-tableau Γ such that the coefficients in ω_Γ are in non-increasing order,*

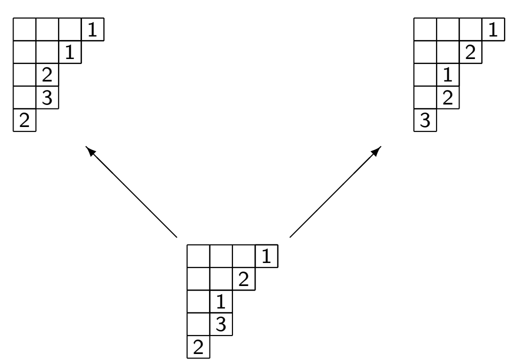
2. *exactly one minimal element: the LR-tableau* Γ *such that* ω_Γ *has the form* $(p_1^{k_1}, p_2^{k_2}, \ldots, p_n^{k_n})$*, for some* $k_1, \ldots, k_n$*, where* $p_i = (i, i-1, \ldots, 1)$ *and* $p^k = (p, p, \ldots, p)$ $(k$ *times* $)$.

Proof. 1. Let Γ be such that the coefficients in ω_Γ are not in non-increasing order. Then there exist j such that $\omega_{j-1} < \omega_j$, where $\omega_\Gamma = (\omega_1, \ldots, \omega_n)$. Then $\Gamma <_{\mathsf{box}} \tilde{\Gamma}$, where

$$\omega_{\tilde{\Gamma}} = (\omega_1, \ldots, \omega_{j-2}, \omega_j, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_n)$$

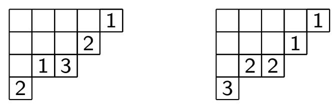
and Γ is not maximal.
2. Note that the element of the required form $(p_1^{k_1}, p_2^{k_2}, \ldots, p_n^{k_n})$ is created as follows. Starting from the right hand side we always take the largest possible entry. Let Γ be such that ω_Γ has not the required form. Choose i maximal with the property that ω_i is not the largest possible entry. Choose $j < i$ maximal with the property that ω_j is the largest possible entry that can be on the place i. It follows that $j < i$ and $\omega_j > \omega_k$ for all $k = j-1, \ldots, i$. We can replace ω_j and ω_{j-1} and get an LR-tableau $\tilde{\Gamma}$ such that $\tilde{\Gamma} <_{\mathsf{box}} \Gamma$. This finishes the proof. □

Example: 1. The first example shows that the condition that $\beta \setminus \gamma$ be a horizontal strip is needed for the uniqueness of the maximal element in $\mathcal{T}_{\alpha,\gamma}^{\beta}$. Consider the partition triple $\beta = (5,4,2,1)$, $\gamma = (4,2,1)$ and $\alpha = (3,2)$. The Hasse diagram of the poset $(\mathcal{T}_{\alpha,\gamma}^{\beta}, \leq_{\mathsf{dom}})$ has the following shape:

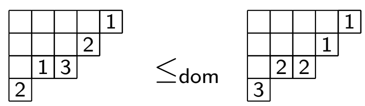


2. The second example shows that in Conjecture 1.3, the condition that $\beta \setminus \gamma$ be a horizontal and vertical strip is necessary. We also see that for horizontal strips, the poset $(\mathcal{T}_{\alpha,\gamma}^{\beta}, \leq_{\mathsf{box}})$ may have several minimal and several maximal elements.

Let $\beta = (4,3,3,2,1)$, $\gamma = (3,2,2,1)$ and $\alpha = (3,2)$. There are two LR-tableaux of type (α,β,γ):



They are incomparable in $\leq_{\mathsf{box}}$ relation, but



6.3. Saturated chains

We prove that, if $\beta \setminus \gamma$ is a horizontal and vertical strip, then all saturated chains in the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$ have the same length.
Throughout this subsection (α,β,γ) is a triple of partitions such that $\beta \setminus \gamma$ is a horizontal and vertical strip.

Lemma 6.2. *Let $y < x$.*

1. *If the words*

$$\omega = (\omega_1,\dots,\omega_{i-1},y,\omega_{i+1},\dots,\omega_{j-1},x,\omega_{j+1},\dots,\omega_{k-1},x,\omega_{k+1},\dots,\omega_n),$$

$$\omega' = (\omega_1,\dots,\omega_{i-1},x,\omega_{i+1},\dots,\omega_{j-1},x,\omega_{j+1},\dots,\omega_{k-1},y,\omega_{k+1},\dots,\omega_n)$$

have the lattice permutation property, then the word

$$\omega'' = (\omega_1,\dots,\omega_{i-1},x,\omega_{i+1},\dots,\omega_{j-1},y,\omega_{j+1},\dots,\omega_{k-1},x,\omega_{k+1},\dots,\omega_n)$$

has the lattice permutation property.

2. *If the words*

$$(\omega_1,\dots,\omega_{i-1},y,\omega_{i+1},\dots,\omega_{j-1},y,\omega_{j+1},\dots,\omega_{k-1},x,\omega_{k+1},\dots,\omega_n),$$

$$(\omega_1,\dots,\omega_{i-1},x,\omega_{i+1},\dots,\omega_{j-1},y,\omega_{j+1},\dots,\omega_{k-1},y,\omega_{k+1},\dots,\omega_n)$$

have the lattice permutation property, then the word

$$(\omega_1,\dots,\omega_{i-1},y,\omega_{i+1},\dots,\omega_{j-1},x,\omega_{j+1},\dots,\omega_{k-1},y,\omega_{k+1},\dots,\omega_n)$$

has the lattice permutation property.

Proof. We prove only the statement 1. The proof of the statement 2 is similar. It is clear that the word $\omega''_{(j+1)} = (\omega_{j+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n)$ has the lattice permutation property, because ω has this property. Note that

$$\#\{a \in \omega''_{(j+1)} \,;\, a = y-1\} = \#\{a \in \omega'_{(j+1)} \,;\, a = y-1\} \geq$$

$$\geq \#\{a \in \omega'_{(j)} \,;\, a = y\} > \#\{a \in \omega''_{(j)} \,;\, a = y\}.$$

Therefore $\omega''_{(j)} = (y, \omega_{j+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n)$ has the lattice permutation property. Moreover, for any $s = i+1, \dots, j-1$, we have

$$\#\{a \in \omega''_{(s)} \,;\, a = x+1\} = \#\{a \in \omega'_{(s)} \,;\, a = x+1\} \leq$$

$$\leq \#\{a \in \omega'_{(s)} \,;\, a = x\} = \#\{a \in \omega''_{(s)} \,;\, a = x\}.$$

Now, we easily conclude that ω'' has the lattice permutation property. $\square$

LEMMA 6.3. *Let $y < z < x$ and let the words*

$$\omega = (\omega_1, \dots, \omega_{i-1}, y, \omega_{i+1}, \dots, \omega_{j-1}, z, \omega_{j+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n),$$

$$\omega' = (\omega_1, \dots, \omega_{i-1}, x, \omega_{i+1}, \dots, \omega_{j-1}, z, \omega_{j+1}, \dots, \omega_{k-1}, y, \omega_{k+1}, \dots, \omega_n)$$

have the lattice permutation property. If $z+1 \notin \{\omega_{i+1}, \dots, \omega_{j-1}\}$, then

$$\omega'' = (\omega_1, \dots, \omega_{i-1}, z, \omega_{i+1}, \dots, \omega_{j-1}, y, \omega_{j+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n),$$

$$\omega''' = (\omega_1, \dots, \omega_{i-1}, z, \omega_{i+1}, \dots, \omega_{j-1}, x, \omega_{j+1}, \dots, \omega_{k-1}, y, \omega_{k+1}, \dots, \omega_n)$$

have the lattice permutation property.

Proof. Note that the words $(\omega_{j+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n)$ and $(\omega_{j+1}, \dots, \omega_{k-1}, y, \omega_{k+1}, \dots, \omega_n)$ have the lattice permutation property, because ω and ω' have this property. Similarly as in the proof of Lemma 6.2 we prove that $(y, \omega_{j+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n)$ and $(x, \omega_{j+1}, \dots, \omega_{k-1}, y, \omega_{k+1}, \dots, \omega_n)$ have the lattice permutation property. Since $z+1 \notin \{\omega_{i+1}, \dots, \omega_{j-1}\}$, it is easy to conclude that ω'' and ω''' have lattice permutation property. $\square$

For a LR-tableau Γ denote by x_Γ the numbers of pairs $i < j$ such that $\omega_i < \omega_j$.

LEMMA 6.4. *If $\Gamma <_{\mathsf{box}} \tilde{\Gamma}$, then $x_\Gamma > x_{\tilde{\Gamma}}$.*

Proof. Let $\omega_\Gamma = (\omega_1, \dots, \omega_{i-1}, y, \omega_{i+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n)$ and $\omega_{\tilde\Gamma} = (\omega_1, \dots, \omega_{i-1}, x, \omega_{i+1}, \dots, \omega_{k-1}, y, \omega_{k+1}, \dots, \omega_n)$ for $y < x$. Denote by

$$a = \#\{t \in \{i+1, \dots, k-1\}\ ;\ y < \omega_t < x\}$$
$$b = \#\{t \in \{i+1, \dots, k-1\}\ ;\ y < \omega_t > x\}$$
$$c = \#\{t \in \{i+1, \dots, k-1\}\ ;\ y > \omega_t < x\}$$
$$d = \#\{t \in \{i+1, \dots, k-1\}\ ;\ \omega_t = x\}$$
$$e = \#\{t \in \{i+1, \dots, k-1\}\ ;\ \omega_t = y\}$$

and note that $x_\Gamma = x_{\tilde\Gamma} - 1 - 2a - d - e$. We are done. □

PROPOSITION 6.5. *If $\Gamma <_{\mathsf{box}} \tilde\Gamma$, then there exist LR-tableaux $\Gamma_1, \dots, \Gamma_m$ such that*

$$\Gamma = \Gamma_1 <_{\mathsf{box}} \Gamma_2 <_{\mathsf{box}} \dots <_{\mathsf{box}} \Gamma_{m-1} <_{\mathsf{box}} \Gamma_m = \tilde\Gamma$$

and $x_{\Gamma_i} = x_{\Gamma_{i+1}} - 1$, for all $i = 1, \dots, m-1$.

Proof. Assume that $\omega_\Gamma = (\omega_1, \dots, \omega_{i-1}, y, \omega_{i+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n)$ and $\omega_{\tilde\Gamma} = (\omega_1, \dots, \omega_{i-1}, x, \omega_{i+1}, \dots, \omega_{k-1}, y, \omega_{k+1}, \dots, \omega_n)$ for $y < x$. If there exists $j = i+1, \dots, k-1$, such that $\omega_j = x$, then by Lemma 6.2 there exists the LR-tableau Γ' with

$$\omega_{\Gamma'} = (\omega_1, \dots, \omega_{i-1}, x, \omega_{i+1}, \dots, \omega_{j-1}, y, \omega_{j+1}, \dots, \omega_{k-1}, x, \omega_{k+1}, \dots, \omega_n),$$

$\Gamma <_{\mathsf{box}} \Gamma' <_{\mathsf{box}} \tilde\Gamma$ and $x_\Gamma > x_{\Gamma'} > x_{\tilde\Gamma}$. Therefore we can assume that $x \notin \{\omega_{i+1}, \dots, \omega_{k-1}\}$. Similarly we can assume that $y \notin \{\omega_{i+1}, \dots, \omega_{k-1}\}$. Moreover, we may assume that there exists $z \in \{\omega_{i+1}, \dots, \omega_{k-1}\}$ such that $y < z < x$, because otherwise $x_\Gamma = x_{\tilde\Gamma} - 1$ (compare with the formula in the proof of Lemma 6.4) and we are done. Therefore there exists $j = i+1, \dots, k-1$ such that $y < \omega_j < x$. Choose j minimal with this property and denote $z = \omega_j$. From our assumptions it follows that $z+1 \notin \{\omega_{i+1}, \dots, \omega_{j-1}\}$. Indeed, if $z+1 \in \{\omega_{i+1}, \dots, \omega_{j-1}\}$, then $z+1 = x$ or $z+1 < x$. If $z+1 = x$, we get a contradiction, because $x \notin \{\omega_{i+1}, \dots, \omega_{k-1}\}$. If $z+1 < x$, then we get a contradiction with the choice of j.
Finally, by Lemma 6.3, $\Gamma <_{\mathsf{box}} \Gamma'' <_{\mathsf{box}} \Gamma''' <_{\mathsf{box}} \tilde\Gamma$, where $\omega_{\Gamma''} = \omega''$ and $\omega_{\Gamma'''} = \omega'''$ (see Lemma 6.3). We finish by using induction. □

As a corollary we get the following fact.

THEOREM 6.6. *If $\beta \setminus \gamma$ is a horizontal and vertical strip, then all saturated chains in the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \le_{\mathsf{box}})$ have the same length.*

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