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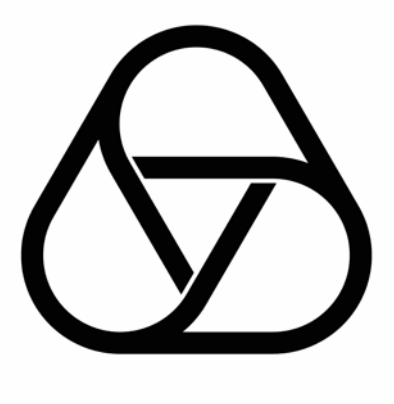

OWP 2014 - 12

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

RANDOM DYNAMICS OF TRANSCENDENTAL FUNCTIONS

VOLKER MAYER AND MARIUSZ URBAŃSKI

ABSTRACT. This work concerns random dynamics of hyperbolic entire and meromorphic functions of finite order and whose derivative satisfies some growth condition at infinity. This class contains most of the classical families of transcendental functions and goes much beyond. Based on uniform versions of Nevanlinna's value distribution theory we first build a thermodynamical formalism which, in particular, produces unique geometric and fiberwise invariant Gibbs states. Moreover, spectral gap property for the associated transfer operator along with exponential decay of correlations and a central limit theorem are shown. This part relies on our construction of new positive invariant cones that are adapted to the setting of unbounded phase spaces. This setting rules out the use of Hilbert's metric along with the usual contraction principle. However these cones allow us to apply a contraction argument stemming from Bowen's initial approach.

Random dynamics is actually a quite active field. An overview can be found in Arnold's book [1] and in Kifer and Liu's chapter in [14]. The first work on random rational functions is due to Fornaess and Sibony [8]. Related to this is Rugh's paper on random repellers [25] and Sumi's work on rational semi-groups (see for example [28, 29]). A complete picture including thermodynamics and spectral gap is contained in [16] which concerns a much wider class of distance expanding random maps, a class originally introduced by Ruelle [24]. Recently random dynamics of countable infinite Markov shifts [6, 27] and graph directed Markov systems [23] have been treated. Here we extend the picture to a situation where the maps are also countable infinite – to – one, where the phase space is not compact and where in addition there is no Markov structure.

Given a probability space $(X, \mathcal{F}, m)$ along with an invertible ergodic transformation $\theta : X \to X$, we consider the dynamics of

$$f_x^n = f_{\theta^{n-1}(x)} \circ ... \circ f_x \quad , \quad n \geq 1$$

where $f_x : \mathbb{C} \to \hat{\mathbb{C}}$, $x \in X$, is a family of transcendental functions depending measurably on $x \in X$. Like in the deterministic case, the normal family behaviour of $(f_x^n)_n$ splits the plane into two parts and one is interested in the chaotic part $\mathcal{J}_x$,

Date: July 12, 2014.

1991 *Mathematics Subject Classification.* 111.

This research was supported in part through the program "Research in Pairs" (resp. "Oberwolfach Leibniz Fellows") by the Mathematisches Forschungsinstitut Oberwolfach in 2013 and by the Labex CEMPI (ANR-11-LABX-0007-01).

called random Julia set. Quite general *transcendental random systems* $f_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$, $x \in X$, are considered in this paper and, as already has been mentioned among the major difficulties one encounters is that the phase space $\mathcal{J}_x$ is unbounded and the functions are of infinite degree.

In the deterministic case, this difficulty has been overcome in [19] for a very general class of functions, so called balanced meromorphic functions. They contain most classical families like all periodic functions (tangent, sine, exponential and elliptic), functions with polynomial Schwarzian derivative, the cosine-root family and many more (all these examples are discussed in detail in [19]). The key point there was to replace the Euclidean metric by a metric having an appropriate singularity at infinity. Once this is done, one can use Nevanlinna's value distribution theory to show that the corresponding transfer operator is well defined and bounded. The present paper treats random dynamics of the families of functions considered in [19]. Again we start with an appropriated choice of metric in order to be able to control the transfer operator. This time we make use of the uniform versions of Nevanlinna's theorems in Cherry-Ye's book [4] (all needed details of this theory are in the Appendix 7).

Then, since we are dealing with random dynamics, measurability of all involved operators, measures and functions has to be checked. This point has sometimes been neglected in the literature (see the discussion in [13]) or is the reason for additional assumptions. Here we take advantage of Crauel's framework [5] and treat measurability very carefully. Moreover, this allows us to have a global, in terms of skew product, approach which, for example, produces directly measurable families of conditional measures (see Section 3.2). This is in contrast to [16] where these objects are constructed fiberwise and then later proven to be measurable.

Having then good behaving transfer operators and measurability, we can proceed with building the thermodynamical formalism. As the result, we prove the existence and uniqueness of fiberwise conformal measures and the existence and uniqueness of invariant densities. This gives rise to the existence and uniqueness of fiberwise invariant measures absolutely continuous with respect to the conformal ones (see Theorem 3.1 and Theorem 5.1 (1)).

Contrary to Ionescu Tulcea-Marinescu's theorem [12] (or its generalization by Hennion's [11]), the method introduced by Birkhoff [2] and developed further by Liverani [15], based on positive cones and the Hilbert distance, can be employed in random dynamics. It especially permits us to obtain the spectral gap property. But this only does work if the phase spaces are compact. In the present paper this is not the case and so the Hilbert distance is of much less use. Indeed, cones of functions of finite distance are too small since all of its members must be comparable near infinity. Fortunately there is a very nice contraction lemma in Bowen's manuscript [3]. In order to be able to adapt it to the present setting, we first produce, via a delicate construction, non-standard appropriate invariant cones. Once this is done,

the Bowen-like argument is quite elementary. In this sense, the present work, incidentally, simplifies the deterministic work [19] which uses Marinescu-Ionescu-Tulcea Theorem.

In conclusion, we get the spectral gap property of Theorem 5.1 (2). It then almost immediately implies the version of exponential decay of correlations in Theorem 6.1 and the Central Limit Theorem 6.2.

1. Transcendental random systems

We consider random dynamics generated by a quite general class of transcendental entire or meromorphic functions of finite order. As in Arnold [1], the randomness is modeled by a measure preserving dynamical system $(X, \mathcal{F}, m, \theta)$, where $(X, \mathcal{F}, m)$ is a complete probability space and $\theta : X \to X$ an invertible measure preserving ergodic transformation. We do not assume the σ–algebra $\mathcal{F}$ to be countably generated. To every $x \in X$ associated is an entire or meromorphic function

$$f_x : \mathbb{C} \to \hat{\mathbb{C}}.$$

The order of this function is supposed to be finite and is denoted by $\rho(f_x)$. For every given $z \in \mathbb{C}$, the map $x \mapsto f_x(z)$ is assumed to be (at least) measurable as a map from $(X, \mathcal{F})$ to $(\mathbb{C}, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ–algebra of $\mathbb{C}$. We will often call

$$(f_x : \mathbb{C} \to \hat{\mathbb{C}})_{x \in X}$$

a system or, more fully, a *transcendental random system* or even a *transcendental random dynamical system* if it satisfies the following four natural conditions.

Condition 1 (Common growth of characteristic function)**.** *There are two constants* $\rho, C_\rho > 0$ *and an increasing function* $\omega : [0, \infty) \to [0, \infty)$ *satisfying* $\lim_{r\to\infty} \log r/\omega(r) = 0$ *such that*

$$\omega(r) \leq \mathring{T}_x(r) \leq C_\rho r^\rho \quad \textit{for all} \quad r > 0 \textit{ and all } x \in X\,.$$

Here, following the standard notation in Nevanlinna theory we denoted by $\mathring{T}_x(r) = \mathring{T}(f_x, r)$ the spherical characteristic function of f_x. All necessary details on Nevanlinna theory and his fundamental main theorems, in the form most convenient for us, are collected in 7. Appendix. Notice that the right hand side inequality of this condition implies that the orders $\rho(f_x) \leq \rho$ whereas the left hand side is simply a quantitative way of saying that the functions f_x are transcendental ($\mathring{T}_f(r) = \mathcal{O}(\log r)$ means that f is a rational function).

In order to study the behavior of the orbits $z \mapsto f_x(z) \mapsto f_{\theta(x)}(f_x(z)) \mapsto ...$ it is natural to use the notation

$$f_x^n := f_{\theta^{n-1}(x)} \circ ... \circ f_{\theta(x)} \circ f_x \ , \ \ n \geq 1\,.$$

The random Fatou set $\mathcal{F}_x$, $x \in X$, is the set of all points $z \in \mathbb{C}$ for which there exists a neighborhood U of z on which all the iterates f_x^n are well defined and form a normal family. The complements in the plane,

$$\mathcal{J}_x := \mathbb{C} \setminus \mathcal{F}_x,$$

are called the random Julia sets. Sometimes, they will be also be denoted by $\mathcal{J}(f_x)$, $x \in X$. We impose the following normalization which mainly signifies that the Julia set does not accumulate at infinity. Here and throughout the whole paper we will use the notation

$$\mathbb{D}_T = \mathbb{D}(0, T)$$

for the open Euclidean disk centered at the origin with radius equal to T.

Condition 2. *There exists $T > 0$ such that*

$$\left(\mathcal{J}_x \cap \mathbb{D}_T\right) \cap f_x^{-1}\left(\mathcal{J}_{\theta(x)} \cap \mathbb{D}_T\right) \neq \emptyset \ , \ \ x \in X\,.$$

Let $z_x \in \mathcal{J}_x \cap \mathbb{D}_T \cap f_x^{-1}\left(\mathcal{J}_{\theta(x)} \cap \mathbb{D}_T\right)$. We will see in Lemma 3.3) that these points can be chosen in a measurable way. Consider then the translations $T_x(z) = z + z_x$, $x \in X$. They conjugate $(f_x)_{x\in X}$ to a new system, say $(g_x)_{x\in X}$ which again does depend measurably on x and such that

$$0 \in \mathcal{J}(g_x) \quad \text{and} \quad |g_x(0)| \leq 2T\,, \ \ x \in X.$$

Notice that the family of translations $(T_x)_{x\in X}$ and the family of its inverses are equicontinuous since $|z_x| \leq T$, $x \in X$. In [17] such families of conjugations are called bi-equicontinuous and they are important since such conjugations preserve the dynamics. In particular they preserve corresponding Julia sets whereas general conjugations do not, as can be seen from Example 2.3 in [17]). In conclusion, up to such a conjugation and by replacing the constant T by $2T$ if necessary, we can use the following normalizing requirement instead of Condition 2:

$$0 = z_x \in \mathcal{J}_x \quad \text{and} \quad |f_x(0)| \leq T\,, \ \ x \in X. \tag{1.1}$$

A straightforward generalization of the notion of hyperbolicity used in [18, 19] to the random setting is the following.

Definition 1.1. *A transcendental random system $(f_x)_{x\in X}$ is called*

(1) topologically hyperbolic *if there exists $0 < \delta_0 \leq \frac{1}{4}$ such that for every $x \in X$, $n \geq 1$ and $w \in \mathcal{J}_{\theta^n(x)}$ all holomorphic inverse branches of f_x^n are well defined on $\mathbb{D}(w, 2\delta_0)$.*

(2) expanding *if there exists $c > 0$ and $\gamma > 1$ such that*

$$|(f_x^n)'(z)| \geq c\gamma^n$$

for every $z \in \mathcal{J}_x \setminus f_x^{-n}(\infty)$ and every $x \in X$. a

(3) hyperbolic *if it is both topologically hyperbolic and expanding.*

As in the papers [18, 19], dealing with deterministic systems, we will consider hyperbolic systems for which one has some more information about derivatives.

Condition 3 (Balanced growth condition)**.** *There are $\alpha_2 > \max\{0, -\alpha_1\}$ and $\kappa \geq 1$ such that for every $x \in X$ and every $z \in \mathcal{J}_x \setminus f_x^{-1}(\infty)$,*

$$\kappa^{-1}(1+|z|)^{\alpha_1}(1+|f_x(z)|)^{\alpha_2} \leq |f_x'(z)| \leq \kappa(1+|z|)^{\alpha_1}(1+|f_x(z)|)^{\alpha_2}\,. \tag{1.2}$$

Condition 4. *For every $R > 0$ and $N \geq 1$ there exists $C_{R,N}$ such that*

$$|\left(f_x^N\right)'(z)| \leq C_{R,N} \quad \textit{for all} \quad z \in \mathbb{D}_R \cap f_x^{-N}\left(\mathbb{D}_R\right) \ \textit{and } x \in X\,.$$

Remark 1.2. *As it is explained in* [18, 19]*, many families naturally satisfy the balanced growth condition. All other conditions, i.e. Conditions 1, 2 and 4, are automatically satisfied in the deterministic case.* [1] *Therefore, the present setting is a straightforward generalisation of the deterministic situation, the only difference being that α_2 in Condition 3 is constant whereas it is allowed to be a bounded function in* [19].

Throughout this section and also in the rest of this paper we use some standard notations. For example, $a \preceq b$ means that $a \leq cb$ for some constant c which does not depend on the involved variables. We also use $\mathcal{V}_\delta(K)$ for the δ–neighborhood of K in Hausdorff distance generated by the standard Euclidean metric.

1.1. **Mixing.** We shall prove the following mixing property.

Lemma 1.3. *Let $(f_x)_{x\in X}$ be a hyperbolic transcendental random system. Then, for all $r > 0$ and $R > 0$ there exists $N = N(r, R)$ such that*

$$f_x^n(\mathbb{D}(z,r)) \supset \overline{\mathbb{D}}_R \cap \mathcal{J}_{\theta^n(x)} \quad \textit{for every } n \geq N,\ \ z \in \mathcal{J}_x \cap \overline{\mathbb{D}}_R \textit{ and } x \in X\,.$$

Proof. Suppose to the contrary that there exist $r, R > 0$ and arbitrarily large integers $n \geq N$ such that for some $x_n \in X$ and $z_n \in \mathcal{J}_{x_n} \cap \overline{\mathbb{D}}_R$ there exists a point

$$w_n \in \left(\overline{\mathbb{D}}_R \cap \mathcal{J}_{\theta^n(x_n)}\right) \setminus f_{x_n}^n(\mathbb{D}(z_n, r)).$$

Define then $\varphi_n : \mathbb{D} \to \hat{\mathbb{C}}$ by $\varphi_n(\xi) = f_{x_n}^n(z_n + r\xi) - w_n$. Note that the family $(\varphi_n)_n$ is not normal at the origin. Consequently, there exist arbitrarily large integers n such that

$$\varphi_n\left(\mathbb{D}(0, 1/2)\right) \cap \mathbb{D}(0, \delta) \neq \emptyset\,.$$

But then, it follows from hyperbolicity and, in particular, from the expanding property that

$$f_{x_n}^{-n}(\mathbb{D}(w_n, \delta)) \subset \mathbb{D}(z_n, r)$$

[1] For some very special examples, the lower bound in Condition 1 can fail. Notice however that, if f is not a rational function, then $\mathring{T}$ growths faster than $\log r$ and this is exactly the property we really need.

provided that $n \geq q$ is sufficiently large, where $f_{x_n}^{-n}$ is a appropriated holomorphic inverse branch of $f_{x_n}^n$ defined on $\mathbb{D}(w_n, \delta)$. But this contradicts the fact that $w_n \notin f_{x_n}^n(\mathbb{D}(z_n, r))$. □

2. Transfer operators

Let $\mathcal{C}_b(\mathcal{J}_x)$ be the space of continuous bounded real–valued functions on $\mathcal{J}_x$ and $\mathcal{C}_0(\mathcal{J}_x)$ its subspace consisting of all functions converging to 0 at ∞. Let $(f_x)_{x\in X}$ be a hyperbolic system and define:

$$\mathcal{L}_x g(w) = \sum_{f_x(z)=w} e^{\varphi_x(z)} g(z)\,, \quad w \in \mathcal{J}_{\theta(x)} \text{ and } g \in \mathcal{C}_b(\mathcal{J}_x).$$

This is the associated family of transfer operators with potential $\varphi_x : \mathcal{J}_x \to \mathbb{R}$. A natural choice for the potentials is $e^{\varphi_x} = |f_x'|^{-t}$ since usually one can choose the parameter t such that these potentials encode the geometric properties of the dynamical system. However, since f_x is of infinite degree, $\mathcal{L}_x$ is in general not well–defined for such a potential. One might replace it by its spherical version. Then, at least for $t = 2$, $\mathcal{L}_x$ would be well-defined but the new obstacle would then arise, that, excepted for some special cases, $\mathcal{L}_x$ would not be a bounded operator; to see it the reader is invited just to try and write it down for the exponential family. However, using Nevanlinna theory, we showed in [18, 19], still for the deterministic case, that there is a Riemannian metric, somehow in between the Euclidean and spherical one, conformally equivalent to any of them, such that the transfer operator has all the properties needed for developing the thermodynamic formalism and that this holds for the values of parameter t in a sufficiently large domain, containing in particular the hyperbolic dimension. This method does work as soon as the derivative growth condition is satisfied.

So, suppose that $(f_x)_{x\in X}$ satisfies the Condition 3. Then, $\alpha = \alpha_1 + \alpha_2 > 0$. Given any $t > \frac{\rho}{\alpha}$ there is $\tau \in (0, \alpha_2)$ such that

$$t > \frac{\rho}{\hat{\tau}} > \frac{\rho}{\alpha} \quad \text{where} \quad \hat{\tau} = \alpha_1 + \tau\,. \tag{2.1}$$

Remark 2.1. *Notice that here τ can be chosen individually for each $t > \frac{\rho}{\alpha}$. In particular, we may suppose that $\alpha_2 - \tau > 0$ is arbitrarily small.*

Consider then the Riemannian metric

$$d\sigma_\tau(z) = \frac{|dz|}{(1+|z|)^\tau}\,.$$

We denote by $|f_x'|_\tau$ the derivative of f_x with respect to this metric, and, using Condition 3, we have,

$$|f_x'(z)|_\tau = |f_x'(z)|\frac{(1+|z|)^\tau}{(1+|f_x(z)|)^\tau} \asymp (1+|f_x(z)|)^{\alpha_2-\tau}(1+|z|)^{\hat{\tau}}$$

for every $z \in \mathcal{J}_x \setminus f_x^{-1}(\infty)$. Consequently, for all $w \in \mathcal{J}_{\theta(x)}$ we have,

$$\mathcal{L}_x \mathbb{1}(w) = \mathcal{L}_{x,t}\mathbb{1}(w) = \sum_{f_x(z)=w} |f_x'(z)|_\tau^{-t} \leq \frac{\kappa^t}{(1+|w|)^{(\alpha_2-\tau)t}} \sum_{f_x(z)=w} \left(1+|z|\right)^{-t\hat{\tau}} . \tag{2.2}$$

Remark 2.2. *Hyperbolicity of the functions f_x implies that, increasing κ if necessary,* (2.2) *does hold for all w in the δ_0–neighborhoods $\mathcal{V}_{\delta_0}(\mathcal{J}_{\theta(x)})$ of the Julia sets provided $\delta_0 > 0$ has been chosen sufficiently small.*

Since the factor with w, appearing in the right hand side of (2.2), converges to zero as $|w| \to \infty$, and applying also Nevanlinna theory (similarly as in [19], details can be found in 7 of Appendix)), we see that the series in (2.2) can be uniformly bounded from above. We therefore obtain the following good behavior of these operators $\mathcal{L}_x$.

Proposition 2.3. *For every $t > \frac{\rho}{\hat{\tau}} > \frac{\rho}{\alpha}$ there exists $M_0 = M_0(t,\tau) > 0$ such that for every $x \in X$, we have*

(1) $\sum_{f_x(z)=w} \left(1+|z|\right)^{-t\hat{\tau}} \leq M_0$ *for every* $w \in \mathcal{J}_{\theta(x)}$,
(2) $\|\mathcal{L}_x\|_\infty \leq M_0$ *and*
(3) $\mathcal{L}_x\mathbb{1}(w) \leq M_0 \left(1+|w|\right)^{-(\alpha_2-\tau)t} \longrightarrow 0$ *as* $|w| \to \infty$.

2.1. **Distortion and Hölder functions.** Koebe's Distortion Theorem (see Theorem 1.3 in [22]) and elementary calculus give:

Lemma 2.4. *Given $t, \tau > 0$ as in* (2.1)*, there exists $K = K_{t,\tau} > 0$ such that, for every $x \in X$, every integer $n \geq 1$ and every ψ, an inverse branch of f_x^n defined on some disk $\mathbb{D}(w, 2\delta_0)$, $w \in \mathcal{J}_{\theta^n(x)}$, we have that*

$$\frac{|\psi'(w_1)|_\tau^t}{|\psi'(w_2)|_\tau^t} \leq 1 + K|w_1 - w_2|, \quad w_1, w_2 \in \mathbb{D}(w, \delta_0) .$$

Remark 2.5. *It is reasonable to require that $1/2 \leq \tau \leq 2$. This would then imply that the constant K does depend only on the parameter t.*

A Straightforward application of Lemma 2.4 gives (remember that $\delta_0 \leq 1/4$):

Lemma 2.6. *There exists $K = K_{t,\tau}$ such that, for every $x \in X$, every integer $n \geq 1$, and every $w \in \mathcal{J}_{\theta^n(x)}$, we have that*

$$\frac{\mathcal{L}_x^n \mathbb{1}(w_1)}{\mathcal{L}_x^n \mathbb{1}(w_2)} \leq 1 + K|w_1 - w_2| \quad \textit{for all} \quad w_1, w_2 \in \mathbb{D}(w, \delta_0) .$$

In particular,

$$\mathcal{L}_x^n \mathbb{1}(w_1) \leq K \mathcal{L}_x^n \mathbb{1}(w_2) \quad \textit{for all} \quad w_1, w_2 \in \mathbb{D}(w, \delta_0) .$$

Let $\mathcal{H}_\beta(\mathcal{J}_x)$ be the set of real-valued bounded β–Hölder functions defined on $\mathcal{J}_x$. The β–variation of a function $g \in \mathcal{H}_\beta(\mathcal{J}_x)$ is defined to be

$$v_\beta(g) := \sup_{0<|w_1-w_2|\le\delta_0} \left\{ \frac{|g(w_1)-g(w_2)|}{|w_1-w_2|^\beta} \right\} \tag{2.3}$$

and

$$\|g\|_\beta := v_\beta(g) + \|g\|_\infty$$

is the corresponding β–Hölder norm on $\mathcal{H}_\beta(\mathcal{J}_x)$. The good distortion behavior established in Lemma 2.4 implies the following two-norm inequality which, in particular, yields invariance of Hölder spaces $\mathcal{H}_\beta(\mathcal{J}_x)$.

Proposition 2.7. *Let $c, \gamma > 0$ be the expanding constants from Definition 1.1. Then, for every $x \in X$, every integer $n \ge 1$, and every $g \in \mathcal{H}_\beta(\mathcal{J}_x)$, we have, with some $K = K_{t,\tau} > 0$ that*

$$v_\beta(\mathcal{L}_x^n g) \le \|\mathcal{L}_x^n\|_\infty \Big(\|g\|_\infty + K(c\gamma^n)^{-\beta} v_\beta(g)) \Big).$$

Proof. Let $x \in X$, $n \ge 1$, $g \in \mathcal{H}_\beta(\mathcal{J}_x)$ and let $w_1, w_2 \in \mathcal{J}_{\theta^n(x)}$ with $|w_1 - w_2| \le \delta_0$. The points z_1, z_2 are said to form a pairing if they are respectively preimages of w_1 and w_2 by the same holomorphic inverse branch of f_x^n. With this convention,

$$|\mathcal{L}_x^n g(w_1) - \mathcal{L}_x^n g(w_2)| \le I + II,$$

where

$$II =: \sum_{z_1,z_2 \ pairing} |(f_x^n)'(z_2)|_\tau^{-t} |g(z_1) - g(z_2)| \le \|\mathcal{L}_x^n\|_\infty v_\beta(g)(c\gamma^n)^{-\beta} K^\beta |w_1 - w_2|^\beta$$

and

$$\begin{aligned} I &= \sum_{z_1,z_2 \ pairing} \Big| |(f_x^n)'(z_1)|_\tau^{-t} - |(f_x^n)'(z_2)|_\tau^{-t} \Big| g(z_1) \\ &\le \|g\|_\infty \sum_{z_1,z_2 \ pairing} |(f_x^n)'(z_1)|_\tau^{-t} \left| 1 - \frac{|(f_x^n)'(z_2)|_\tau^{-t}}{|(f_x^n)'(z_1)|_\tau^{-t}} \right| \\ &\le \|g\|_\infty \|\mathcal{L}_x^n\|_\infty K |w_1 - w_2|, \end{aligned}$$

where the last inequality results from Lemma 2.4. It suffices now to combine the above estimates of both terms I and II. □

3. Random Gibbs states

In this section we establish the following key result which the rest of the paper relies on.

Theorem 3.1. *Let $(f_x)_{x\in X}$ be a hyperbolic transcendental random dynamical system satisfying Conditions 1-4. Fix $t > \rho/\alpha$. Then there exists a random Gibbs measure ν with disintegrations $\nu_x \in \mathcal{PM}(\mathcal{J}_x)$, $x \in X$, and a measurable function $\lambda : X \to (0,\infty)$ such that*

$$\mathcal{L}_x^* \nu_{\theta(x)} = \lambda_x \nu_x \quad \text{for } m \text{ a.e. } x \in X\,. \tag{3.1}$$

Moreover, there exists a constant $C \geq 1$ such that $C^{-1} \leq \lambda_x \leq C$ for m–a.e. $x \in X$.

Yes, we have not defined random measures yet. Roughly speaking, this concept means that the family of probability measures $(\nu_x)_{x\in X}$ is measurable. In order to get measurability of λ and ν, unlike to the previous sections, it is much better now to consider the global skew product map

$$(x,z) \stackrel{f}{\longmapsto} (\theta(x), f_x(z)),$$

the global transfer operator, and the associated global Julia set

$$\mathcal{J} = \bigcup_{x\in X} \{x\} \times \mathcal{J}_x \subset X \times \mathbb{C}, \tag{3.2}$$

along with the measurable structure of $\mathcal{J}$ induced by the σ–algebra $\mathcal{F} \otimes \mathcal{B}$ of $X \times \mathbb{C}$ where $\mathcal{B}$ is the Borel σ–algebra of $\mathbb{C}$. The advantage is that then one can use the framework described by Crauel in [5]. We now first present this framework along with some applications, and then prove Theorem 3.1.

3.1. **Random observables and measures.** Let us recall first that $\mathcal{F}$ is an arbitrary complete σ–algebra on X. Proposition 2.4 of [5] implies then that any measurable set $C \subset \mathcal{F} \otimes \mathcal{B}$, whose fibers $C_x = C \cap (\{x\} \times \mathbb{C})$ are closed, is a *closed random set*. We will take this characterization as a definition.

A particularly important feature of closed random sets is that they allow us to use a Measurable Selection Theorem, namely Theorem 2.6 in [5]. This theorem asserts that for any closed random set $C \subset \mathcal{F} \otimes \mathcal{B}$ there exists a countable family of measurable functions $(c_n : X \to \mathbb{C})_{n\geq 0}$ such that for m–a. e. $x \in X$,

$$C_x = \overline{\{c_n(x) : n \geq 0\}}. \tag{3.3}$$

We shall prove the following.

Lemma 3.2. *The global Julia set $\mathcal{J}$ is a closed random set.*

Proof. For every $x \in X$ denote by $\mathcal{C}_{f_x}$ the set of all critical points of f_x, i.e.

$$\mathcal{C}_{f_x} = \{z \in \mathbb{C} : f_x'(z) = 0\}.$$

Set

$$\mathcal{O}^+_{n,x} = f_x^n(\mathcal{C}_{f_x}) \cup f_{\theta(x)}^{n-1}(\mathcal{C}_{f_{\theta(x)}}) \cup \ldots \cup f_{\theta^{n-1}(x)}(\mathcal{C}_{f_{\theta^{n-1}(x)}}).$$

and then

$$\mathcal{P}_x = \overline{\bigcup_{n\geq 1} \mathcal{O}^+_{n,\theta^{-n}(x)}} \cap \mathbb{C}\,.$$

Since each set $\mathcal{C}_{f_x}$ is countable and its elements vary measurably with $x \in X$, Proposition 2.9 in [5] assures us that $\mathcal{P} = \bigcup_{x\in X}\{x\} \times \mathcal{P}_x$ is a closed random set. Hence, still by Proposition 2.4 in [5], $\mathcal{P}^\delta = \bigcup_{x\in X}\{x\} \times \overline{\mathcal{V}_\delta(\mathcal{P}_x)}$ is also a closed random set, where $\delta = \delta_0/2$. It now follows from Proposition 2.9 in [5] that $C = \bigcup_{x\in X}\{x\} \times C_x$, with $C_x = \mathbb{C} \setminus \mathcal{V}_\delta(\mathcal{P}_x)$, is a closed random set. The Measurable Selection Theorem, Theorem 2.6 in [5], thus applies, and, as $\mathcal{V}_\delta(\mathcal{J}_x) \subset C_x$ for all $x \in X$ by hyperbolicity of $(f_x)_x$, this theorem yields measurable maps $(c_k : X \to \mathbb{C})_{k\geq 0}$ such that $c_k(x) \in \mathcal{V}_\delta(\mathcal{J}_x)$ for every $k \geq 0$, and moreover, for m–a.e. $x \in X$, $\overline{\{c_k(x) : k \geq 0\}} \supset \mathcal{V}_\delta(\mathcal{J}_x)$. By definition of C, all holomorphic inverse branches of f_x^n are well–defined in the δ_0 neighborhoods of all points $c_{\theta^n(x)}$. Fix $1 < \eta < \gamma$ where γ is the expanding constant coming from Definition 1.1. We call a holomorphic inverse branch $f_{x,*}^{-n}$ of f_x^n, defined on $\mathbb{D}(c_{\theta^n(x)}, \delta)$, *shrinking*, if $|(f_{x,*}^{-n})'(c_{\theta^n(x)})| \leq \eta^{-n}$. It is now easy to check that

$$\mathcal{J}_x = \bigcap_N \overline{\bigcup_{n\geq N} \bigcup_{* \text{ shrinking}} f_{x,*}^{-n}(\{c_k(\theta^n(x) : k \geq 0\})}.$$

This shows that $\mathcal{J}$ is a closed random set. □

Lemma 3.3. *If Condition 2 holds, then there is a measurable choice*

$$x \mapsto z_x \in \mathcal{J}_x \cap \overline{\mathbb{D}}_T \cap f_x^{-1}\left(\mathcal{J}_{\theta(x)} \cap \overline{\mathbb{D}}_T\right).$$

Proof. Since, by Lemma 3.2, $\mathcal{J}$ is a closed random set, the sets with fibers $\mathcal{J}_x \cap \overline{\mathbb{D}}_T$ and $f_x^{-1}(\mathcal{J}_{\theta(x)} \cap \overline{\mathbb{D}}_T)$ are both closed random sets. The intersection of these closed random sets is again a closed random set, and Condition 2 implies that every fiber of this intersection is not empty. Therefore, again by the Measurable Selection Theorem, there exist a measurable map z such that $z_x \in \mathcal{J}_x \cap \overline{\mathbb{D}}_T \cap f_x^{-1}\left(\mathcal{J}_{\theta(x)} \cap \overline{\mathbb{D}}_T\right)$ for e.e. $x \in X$. □

We now introduce random observables. We recall from [5] that a function $g : \mathcal{J} \to \mathbb{R}$, $(x, z) \mapsto g_x(z)$, is called random continuous if $g_x \in \mathcal{C}_b(\mathcal{J}_x)$ for all $x \in X$, the function $x \mapsto \|g_x\|_\infty$ is measurable and, moreover, m–integrable. The vector space of all such functions is denoted by $\mathcal{C}_b(\mathcal{J})$. It becomes a Banach space when equipped with the norm

$$|g| = \int_X \|g_x\|_\infty \, dm(x).$$

We need more special functions.

Definition 3.4. *A random continuous function $g : \mathcal{J} \to \mathbb{R}$, $(x, z) \mapsto g_x(z)$, is said to vanish at infinity if*

$$\lim_{z\to\infty} g_x(z) = 0$$

for m-a.e. $x \in X$. The vector space of all such functions is denoted by $\mathcal{C}_0(\mathcal{J})$. It is a closed subspace of $\mathcal{C}_b(\mathcal{J})$ and inherits the norm $|\cdot|$ from $\mathcal{C}_b(\mathcal{J})$. Thus, it becomes a Banach space on its own.

Definition 3.5. *Random β–Hölder observables are defined to be all the functions $g \in \mathcal{C}_b(\mathcal{J})$ such that $g_x \in \mathcal{H}_\beta(\mathcal{J}_x)$ and such that $x \mapsto \|g_x\|_\beta$ is integrable. This space is denoted by $\mathcal{H}_\beta(\mathcal{J})$ and equipped with the norm*

$$|g|_\beta = \int_X \|g_x\|_\beta \, dm(x) \, .$$

Consider now the global transfer operator $\mathcal{L}$ defined by

$$(\mathcal{L}g)_x(w) = \mathcal{L}_{\theta^{-1}(x)} g_{\theta^{-1}(x)}(w) \, , \quad (x, w) \in \mathcal{J} \, .$$

Lemma 3.6. *If $g \in \mathcal{C}_b(\mathcal{J})$, then $\mathcal{L}g$ is measurable.*

Proof. First of all, it suffices to establish measurability of $\mathcal{L}g$ restricted to measurable sets of the form

$$E_w = \mathcal{J} \cap (X \times \mathbb{D}(w, \delta/2)) \, , \quad w \in \mathcal{J}_x \, , \ \ x \in X \, .$$

So, let $(x, w) \in \mathcal{J}$. Since $\mathcal{J}$ is a closed random set, it follows from Proposition 2.4 in [5]) that the set

$$Y = \{y \in X : \mathcal{J}_y \cap \mathbb{D}(w, \delta/2) \neq \emptyset\}$$

is measurable. Notice that, by definition of Y and by hyperbolicity of $(f_x)_x$, the function $\mathcal{L}g$ is in fact well defined on $Y \times \mathbb{D}(w, \delta/2)$. Therefore, we can consider the map $h : X \times \mathbb{D}(w, \delta/2) \to \mathbb{R}$ defined by

$$h_y(z) = \begin{cases} (\mathcal{L}g)_y(z) & \text{if } \ y \in Y \\ 0 & \text{if } \ y \notin Y. \end{cases}$$

Obviously, to show that $\mathcal{L}g_{|E_w}$ is measurable, it suffices to establish measurability of h. Also, since Y is measurable and $h \equiv 0$ on $Y^c \times \mathbb{D}(w, \delta/2)$ it suffices to show that h restricted to $Y \times \mathbb{D}(w, \delta/2)$ is measurable and, by virtue of Lemma 1.1 in [5], in order to prove this, it suffices to show that for every $y \in Y$ the map $\mathbb{D}(w, \delta/2) z \mapsto h_y(z)$ is continuous and that the map $Yy \mapsto h_y(z)$ is measurable for every $z \in \mathbb{D}(w, \delta/2)$. The continuity for fixed $y \in Y$ is obvious. So we are left to show measurability of

$$Yy \mapsto h_y(z) = (\mathcal{L}g)_y(z) = \mathcal{L}_{\theta^{-1}(y)} g_{\theta^{-1}(y)}(z)$$

for every fixed $z \in \mathbb{D}(w, \delta/2)$. Let $z \in \mathbb{D}(w, \delta/2)$. Then the set $C = \bigcup_{y \in Y} \{y\} \times f^{-1}_{\theta^{-1}(y)}(z)$ is a closed random set with discrete fibers. Therefore, the Selection Theorem yields the existence of countably measurable functions $(c_n :)_{n=1}^\infty$ such that $\{c_n(y) : n \geq 1\} = f^{-1}_{\theta^{-1}(y)}(z)$ for m–a.e. $y \in Y$. Consequently,

$$h_y(z) = \mathcal{L}_{\theta^{-1}(y)} g_{\theta^{-1}(y)}(z) = \sum_{n \geq 1} |f'_{\theta^{-1}(y)}(c_n(y))|_\tau^{-t} g_{\theta^{-1}(y)}(c_n(y)) \quad \text{for } m - \text{ a.e. } y \in Y.$$

This proves the desired measurability. □

Combining Lemma 3.6 with Proposition 2.3 leads to the following.

Proposition 3.7. *$\mathcal{L}$ acts continuously on both $\mathcal{C}_b(\mathcal{J})$ and $\mathcal{C}_0(\mathcal{J})$.*

Let $\mathcal{M}(\mathcal{J})$ be the space of all measures ν on $\mathcal{J}$ such that if $(\nu_x)_{x\in X}$ is the disintegration of ν, then $x \mapsto |\nu_x|$ is in $L^\infty(X)$, i.e. $ess\,sup_x|\nu_x| < +\infty$, where $|\nu_x|$ is the total variation of ν_x. These measures can be identified with continuous functionals on $\mathcal{C}_b(\mathcal{J})$:

$$g \mapsto \nu(g) = \int_X \int_{\mathcal{J}_x} g_x \, d\nu_x dm(x) \quad \text{and} \quad |\nu(g)| \le ess\,sup_x|\nu_x| \; |g| \,, \quad g \in \mathcal{C}_b(\mathcal{J}) \,.$$

We want to single out one particular subspace of $\mathcal{M}(\mathcal{J})$. This subspace will be essential in the sequel. In its definition, stated below, $\pi_X : \mathcal{J} \to X$ is the usual projection $\pi_x(x,z) = x$.

Definition 3.8. *A measure $\nu \in \mathcal{M}(\mathcal{J})$ with marginal m, i.e. $\nu \circ \pi_X^{-1}(\nu) = m$, or, equivalently, whose disintegrations ν_x belong to the space $\mathcal{P}(\mathcal{J}_x)$ of probability measures on $\mathcal{J}_x$ for all $x \in X$ is called random measure. The subspace of $\mathcal{M}(\mathcal{J})$ consisting of all random measures will be denoted by $\mathcal{P}(\mathcal{J})$.*

Random measures, as defined in Crauel's book [5], are measures on the set $X \times \mathbb{C}$. But here we are only interested in the subclasses $\mathcal{P}(\mathcal{J})$ and $\mathcal{M}(\mathcal{J})$ and they are the measures of $\mathcal{M}(X \times \mathbb{C})$ with support in $\mathcal{J}$.

Clearly $\mathcal{P}(\mathcal{J})$ is a convex subspace and, most importantly, we have the following compactness result (Theorem 4.4 in [5]).

Theorem 3.9 (Crauel's Prohorov Compactness Theorem)**.** *Suppose that $\mathcal{M} \subset \mathcal{P}(\mathcal{J})$. Then the set $\mathcal{M}$ is tight if and only if $\mathcal{M}$ is relatively compact with respect to the narrow topology. In this case, $\mathcal{M}$ is also relatively sequentially compact.*

Tightness of $\mathcal{M}$ means here that for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\nu(X \times \mathbb{D}_R) \ge 1 - \varepsilon \quad \text{for every} \;\; \nu \in \mathcal{M} \,.$$

Finally, a sequence of measures $(\nu^n)_n$ converges to ν in the narrow topology of $\mathcal{M}(\mathcal{J})$ if

$$\lim_{n\to\infty} \nu^n(g) = \nu(g)$$

for every $g \in \mathcal{C}_b(\mathcal{J})$.

3.2. **Random Gibbs states.** The aim of this section is to prove Theorem 3.1 and to provide some useful estimates. Concerning the measures $(\nu_x)_{x\in X}$ from Theorem 3.1 we will refer to them as both *Gibbs states* or *conformal measures*.

From the invariance relation (3.1) follows that $\lambda_x = \int \mathcal{L}_x \mathbb{1} \, d\nu_{\theta(x)}$ and so we look for measures $(\nu_x)_{x\in X}$ that are invariant under the map $\Phi : \mathcal{P}(\mathcal{J}) \to \mathcal{P}(\mathcal{J})$ whose fiber maps $\Phi_x : \mathcal{PM}(\mathcal{J}_{\theta(x)}) \to \mathcal{PM}(\mathcal{J}_x)$ are defined by

$$\Phi_x(\nu_{\theta(x)}) = \frac{\mathcal{L}_x^* \nu_{\theta(x)}}{\mathcal{L}_x^* \nu_{\theta(x)}(\mathbb{1})} = \frac{\mathcal{L}_x^* \nu_{\theta(x)}}{\nu_{\theta(x)}(\mathcal{L}_x \mathbb{1})} \,. \tag{3.4}$$

We want to obtain these measures in the usual way by employing Schauder–Tychonoff's fixed point theorem. But, since the sets J_x are unbounded, this can

be done only if a convex compact and Φ–invariant space of probability measures is found, and if in addition Φ acts continuously on this space. Towards this end, consider

$$\mathcal{M} = \mathcal{M}(R_0, \varepsilon) := \Big\{ \nu = (\nu_x)_{x\in X} \in \mathcal{P}(\mathcal{J}) : \text{(a) and (b) hold} \Big\}, \quad \text{where} \tag{3.5}$$

(a) $\nu_x(\overline{\mathbb{D}}_{R_0}) \geq \frac{1}{2}$ and
(b) $\nu_x(\overline{\mathbb{D}}_R^c) \leq \frac{1}{R^\varepsilon}$ for every $R \geq R_0$

are required to hold for m–a.e. $x \in X$.

For any $x \in X$, define further

$$\mathcal{M}_x = \{\nu_x : \text{ (a) and (b) hold}\}.$$

Clearly, property (b) implies (a) with some R_0 sufficiently large. The significance of (a) is to specify some radius R_0. Obviously, we have the following.

Lemma 3.10. *The set $\mathcal{M}$ is convex and tight, hence compact.*

We shall prove the following.

Proposition 3.11. *There are $R_0, \varepsilon > 0$ such that $\mathcal{M} = \mathcal{M}(R_0, \varepsilon)$ is invariant under the map $\Phi = (\Phi_x)_{x\in X}$ defined in* (3.4)*, i.e. $\Phi_x(\mathcal{M}_{\theta(x)}) \subset \mathcal{M}_x$ for all $x \in X$.*

In order to establish this result we first need two lemmas.

Lemma 3.12. *For every $0 < a < \hat{\tau}t - \rho$ there exists M_a such that for all $x \in X$ and all $R \geq 1$,*

$$\mathcal{L}_x \mathbb{1}_{\overline{\mathbb{D}}_R^c}(w) \leq \frac{M_a}{R^a} \quad , \quad w \in \mathcal{J}_{\theta(x)}.$$

Proof. Given $a \in (0, \hat{\tau}t - \rho)$, let $b = b(a) > 0$ such that $\hat{\tau}t = a + \rho + b$. Then,

$$\begin{aligned} \mathcal{L}_x \mathbb{1}_{\overline{\mathbb{D}}_R^c}(w) &\leq \kappa^t (1 + |w|)^{-(\alpha_2 - \tau)t} \sum_{z \in f_x^{-1}(w) \cap \overline{\mathbb{D}}_R^c} (1 + |z|)^{-t\hat{\tau}} \\ &\leq \kappa^t \sum_{z \in f_x^{-1}(w) \cap \overline{\mathbb{D}}_R^c} |z|^{-a} |z|^{-(\rho + b)} \leq \frac{M_a}{R^a}, \end{aligned}$$

where, the first inequality was written by (2.2), while the last one, with some constant $M_a < \infty$, was written due to Proposition 2.3 with $\hat{\tau}t$ replaced by $\rho + b$. □

Lemma 3.13. *There exists $\tilde{R}_0 > 0$ and, for every $R \geq \tilde{R}_0$, such that, for some constant $c > 0$,*

$$\mathcal{L}_x \mathbb{1}(w) \geq c R^{-(\alpha_2 - \tau)t} 8 \log R \, r_R^{-\hat{\tau}t} \quad \textit{for every} \quad w \in \mathcal{J}_{\theta(x)}\,, \;\; |w| \leq R\,, x \in X$$

where $r_R = \omega^{-1}\big(8 \log R\big)$.

Proof. This proof relies heavily on Nevanlinna's theory and especially on Theorem 7.2. The notation used in it is explained in 7′ Appendix. But first we need some preliminary observations.

Since Condition 2 holds we may assume that $0 \in \mathcal{J}_x$ and $|f_x(0)| \le T$ for all $x \in X$. Let Ω_x be the connected component of $f_x^{-1}\Big(\mathbb{D}(f_x(0), \delta_0)\Big)$ that contains 0. Since $0 \in \mathcal{J}_x$ we can use the expanding property of Definition 1.1 along with Condition 4 in order to get

$$c\gamma \le |f_x'(0)| \le C_T \quad \text{for every} \quad x \in X\,. \tag{3.6}$$

Koebe's 1/4–Theorem applies and, together with (3.6), implies that

$$\Omega_x \supset \mathbb{D}\left(0, \frac{1}{4}|f_x'(0)|^{-1}\delta_0\right) \supset \mathbb{D}\,(0,s) \quad \text{where} \quad s = \frac{\delta_0}{4C_T}\,.$$

Let from now on $x \in X$ and $w \in \mathcal{J}_{\theta(x)}$ with $|w| \le R$.

Case 1: Suppose that $f_x^{-1}(\mathbb{D}(w,\delta_0)) \cap \mathbb{D}_s \ne \emptyset$, i.e. that there exists $z' \in \mathbb{D}(0,s)$ with $w' = f_x(z') \in \mathbb{D}(w,\delta_0)$. Then

$$\mathcal{L}_x\mathbb{1}(w) \ge \frac{1}{K}\mathcal{L}_x\mathbb{1}(w') \ge \frac{1}{K}|f_x'(z')|_\tau^{-t} = \frac{|f_x'(z')|^{-t}}{K}\frac{(1+|z'|)^{-\tau t}}{(1+|w'|)^{-\tau t}} \ge \frac{(KC_T)^{-t}}{K}(1+s)^{-\tau t}$$

by Lemma 2.6, Lemma 2.4, and (3.6). Hence, in this case there is a uniform lower bound for $\mathcal{L}_x\mathbb{1}(w)$.

Case 2: Suppose that $f_x^{-1}(\mathbb{D}(w,\delta_0)) \cap \mathbb{D}_s = \emptyset$. Then we have to use the uniform SMT (Theorem 7.2) and, in order to do so, first to verify its assumptions. It follows from (3.6) that

$$\frac{c\gamma}{1+T^2} \le f_x^\#(0) = \frac{|f_x'(0)}{1+|f_x(0)|^2} \le C_T \quad , \quad x \in X\,.$$

In other words, the assumption (1) of Theorem 7.2 holds with $L = \max\left\{C_T,\ \frac{1+T^2}{c\gamma}\right\}$. Assumption (3) is exactly the uniform growth condition of the characteristic functions in Condition 1. It remains to choose appropriate points a_j. Let $a_1, a_2, a_3 \in \mathbb{D}(w,\delta_0)$ be any points such that $|a_i - a_j| \ge \frac{\delta_0}{3}$ for $i \ne j$. Notice that

$$f_x(0) \notin \{a_1,a_2,a_3\} \quad \text{and} \quad f_x^{-1}(a_j) \cap \mathbb{D}_s = \emptyset\,.$$

We need the following simple estimate:

$$\begin{aligned}\mathring{D}(a_1,a_2,a_3) = -\log\prod_{i\ne j}[a_i,a_j] + 2\log 2 &\le \log\left(1+|w|^2\right) + \log\frac{12}{\delta_0} + 2\log 2\\ &\le 2\log R + \log\frac{12}{\delta_0} + 3\log 2\,.\end{aligned}$$

Theorem 7.2 gives now the following inequality:

$$\begin{aligned}\sum_{j=1}^{3} N(a_j,r) &\geq \mathring{T}_x(r) - b_6 - 6\rho\log r - 2\log R - \log\frac{12}{\delta_0} - 3\log 2\\ &= \mathring{T}_x(r) - \tilde{b}_6 - 6\rho\log r - 2\log R\\ &= \mathring{T}_x(r)\left(1 - \frac{\tilde{b}_6 + 6\rho\log r + 2\log R}{\mathring{T}_x(r)}\right).\end{aligned}$$

Remember that $\mathring{T}_x(r) \geq \omega(r)$, that $\lim_{r\to\infty}\log(r)/\omega(r) = 0$ (Condition 1) and that $R \geq \tilde{R}_0$. If we define $r_R := \omega^{-1}\big(8\log R\big)$, then

$$\frac{2\log R}{\omega(r)} \leq \frac{1}{4} \quad \text{for every} \quad r \geq r_R\,.$$

Therefore, for every $R \geq \tilde{R}_0$ and provided that $\tilde{R}_0$ is sufficiently large, we have:

$$\frac{\tilde{b}_6 + 6\rho\log r + 2\log R}{\omega(r)} \leq \frac{1}{2} \quad \text{for every} \quad r \geq r_R.$$

This implies that

$$\sum_{j=1}^{3} N(a_j,r) \geq \frac{1}{2}\mathring{T}_x(r) \geq \frac{1}{2}\omega(r) \geq \frac{1}{2}\omega(r_R) \quad \text{for every} \quad r \geq r_R\,. \tag{3.7}$$

We can now conclude the proof of our lemma. Indeed, Lemma 2.6, the lower bound in Condition 3 and the fact that $f_x^{-1}(a_j)\cap\mathbb{D}_s = \emptyset$ imply, for every $j = 1,2,3$,

$$\begin{aligned}K\mathcal{L}_x\mathbb{1}(w) \geq \mathcal{L}_x\mathbb{1}(a_j) &\succeq (1+|a_j|)^{-(\alpha_2-\tau)t}\sum_{f_x(z)=a_j}(1+|z|)^{-\hat{\tau}t}\\ &\succeq R^{-(\alpha_2-\tau)t}\sum_{f_x(z)=a_j}|z|^{-\hat{\tau}t}\,.\end{aligned}$$

A standard argument (see 7. Appendix, [20] or Chapter 3 of [19]) and (3.7) shows that

$$\sum_{j=1}^{3}\sum_{f_x(z)=a_j}|z|^{-\hat{\tau}t} = (\hat{\tau}t)^2\int_s^\infty\frac{\sum_{j=1}^3 N(a_j,r)}{r^{\hat{\tau}t+1}}dr \succeq \omega(r_R)\int_{r_R}^\infty\frac{dr}{r_R^{\hat{\tau}t+1}} \succeq \omega(r_R)r_R^{-\hat{\tau}t}\,.$$

Finally, there exists $\tilde{R}_0 > 0$ and $c > 0$ such that

$$3K\mathcal{L}_x\mathbb{1}(w) \geq \sum_{j=1}^{3}\mathcal{L}_x\mathbb{1}(a_j) \geq cR^{-(\alpha_2-\tau)t}\omega(r_R)r_R^{-\hat{\tau}t}$$

for every $R \geq \tilde{R}_0$ and $w \in \mathcal{J}_{\theta(x)}$, $|w| \leq R$, $x \in X$. □

Proof of Proposition 3.11. Let $\nu_{\theta(x)} \in \mathcal{M}_{\theta(x)}$. We have to show that there are constants R_0, ε that do not depend on $x \in X$ such that $\Phi_x(\nu_{\theta(x)}) \in \mathcal{M}_x$, i.e. that the properties (a), (b) of (3.5) are satisfied. Let $\tilde{R}_0$ be the number given in Lemma 3.13, suppose that $R_0 \geq \tilde{R}_0$ and let $R \geq R_0$.

We have to choose the constant $a \in (0, \hat{\tau}t - \rho)$ from Lemma 3.12 and $\tau \in (0, \alpha_2)$. Let $a = \frac{1}{2}(\hat{\alpha}t - \rho)$ and, according to Remark 2.1, we may choose τ sufficiently close to α_2 such that $a < \hat{\tau}t - \rho$ and $b = \frac{a}{2} - (\alpha_2 - \tau)t > 0$. Lemma 3.12 implies

$$\mathcal{L}_x^* \nu_{\theta(x)}(\overline{\mathbb{D}}_R^c) = \int \mathcal{L}_x \mathbb{1}_{\overline{\mathbb{D}}_R^c} d\nu_{\theta(x)} \leq \frac{M_a}{R^a}\,.$$

On the other hand, Lemma 3.13 applied with $R = R_0$ yields

$$\mathcal{L}_x^* \nu_{\theta(x)} \mathbb{1} \geq \int_{\overline{\mathbb{D}}_{R_0}} \mathcal{L}_x \mathbb{1} d\nu_{\theta(x)} \geq c R_0{}^{-(\alpha_2-\tau)t} \left[\omega^{-1}\big(8 \log R_0\big)\right]^{-\hat{\tau}t} \nu_{\theta(x)}(\overline{\mathbb{D}}_{R_0})\,.$$

Notice that $\nu_{\theta(x)}(\overline{\mathbb{D}}_{R_0}) \geq \frac{1}{2}$ since $\nu_{\theta(x)} \in \mathcal{M}_{\theta(x)}$. Therefore,

$$\begin{aligned}\Phi_x(\nu_{\theta(x)})(\overline{\mathbb{D}}_R^c) = \frac{\mathcal{L}_x^* \nu_{\theta(x)}(\overline{\mathbb{D}}_R^c)}{\mathcal{L}_x^* \nu_{\theta(x)} \mathbb{1}} &\leq \frac{2M_a}{c} \frac{R_0{}^{(\alpha_2-\tau)t} \left[\omega^{-1}\big(8 \log R_0\big)\right]^{\hat{\tau}t}}{R_0^{a/2}} \frac{R_0^{a/2}}{R^a}\,. \\ &\leq \frac{2M_a}{c} \frac{\left[\omega^{-1}\big(8 \log R_0\big)\right]^{\hat{\tau}t}}{R_0^b} \frac{1}{R^{a/2}}\,.\end{aligned}$$

In order to conclude that $\Phi_x(\nu_{\theta(x)}) \in \mathcal{M}_x$ it suffices to set $\varepsilon = \frac{a}{2}$ and to show that there exists R_0 such that $\frac{2M_a}{c} \frac{\left[\omega^{-1}\left(8 \log R_0\right)\right]^{\hat{\tau}t}}{R_0^b} \leq 1$. But this results from an elementary calculation based on the properties of ω: $\lim_{r\to\infty} \log(r)/\log \omega(r) = 0$ and ω is increasing. □

Proposition 3.14. *Let $\mathcal{M}$ be the invariant set of random measures from Proposition 3.11. The map $\Phi : \mathcal{M} \to \mathcal{M}$ is continuous with respect to the narrow topology.*

Proof. Suppose that Λ is a directed set and $(\nu^\alpha)_{\alpha\in\Lambda}$ is a net in $\mathcal{P}(\mathcal{J})$ converging to a measure $\nu \in \mathcal{P}(\mathcal{J})$ in the narrow topology. If $h_{\theta(x),\alpha} = 1/\nu^\alpha_{\theta(x)}(\mathcal{L}_x \mathbb{1})$ then, by (3.4),

$$\Phi_x(\nu^\alpha_{\theta(x)}) = \mathcal{L}_x^* \left(\frac{1}{\nu^\alpha_{\theta(x)}(\mathcal{L}_x \mathbb{1})} \nu^n_{\theta(x)}\right) = \mathcal{L}_x^* \left(h_{\theta(x),\alpha}\, \nu^\alpha_{\theta(x)}\right)\,.$$

Proposition 3.7 implies that $\mathcal{L}^*$ is continuous with respect to the narrow topology of $\mathcal{M}(\mathcal{J})$. Thus, we have to investigate $h_{\theta(x),\alpha}\, \nu^\alpha_{\theta(x)}$. Item (a) of the definition of $\mathcal{M}$ in (3.5) and Lemma 3.13 imply that there are constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 \leq h_{\theta(x),\alpha} \leq c_2 \quad \text{for all} \quad n \geq 0 \text{ and } x \in X\,.$$

This implies that $(h_{x,\alpha}\ \nu_x^\alpha)_{x\in X}$ is a tight, hence relatively compact, family of $\mathcal{M}(\mathcal{J})$. Let $\mu \in \mathcal{M}(\mathcal{J})$ be an accumulation point. It is shown (as a matter of fact for sequences but the same argument works for all nets) in Lemma 2.9 of [23] that then

$$\mu = h\nu$$

for some measurable function $h : X \to (0,\infty)$. Notice that $\mathcal{L}^*\mu$ is a random (probability) measure. Hence, the disintegrations of this measure

$$\mathcal{L}_x^*\mu_{\theta(x)} = h_{\theta(x)}\mathcal{L}_x^*(\nu_{\theta(x)}) \in \mathcal{P}(\mathcal{J}_x)$$

are probability measures. Therefore, $1 = \mathcal{L}_x^*\mu_{\theta(x)}(\mathbb{1}) = h_{\theta(x)}\mathcal{L}_x^*(\nu_{\theta(x)})(\mathbb{1})$ which implies that the accumulation point μ is uniquely defined by

$$\mu_{\theta(x)} = \frac{1}{\mathcal{L}_x^*(\nu_{\theta(x)})(\mathbb{1})}\nu_{\theta(x)} \quad , \quad x \in X\,.$$

This shows that the net $(\Phi(\nu^\alpha)_{\alpha\in\Lambda}$ converges to $\Phi(\nu)$ in the narrow topology. The proof of continuity of Φ is complete. □

We are now ready to show the main result of this section.

Proof of Theorem 3.1. Proposition 3.11 yields a Φ–invariant convex and compact set $\mathcal{M} \subset \mathcal{P}(\mathcal{J})$ of random measures. By Proposition 3.14) the map Φ is continuous on $\mathcal{M}$ for the narrow topology . Therefore, one can apply Schauder-Tychonoff fixed point theorem in order to get a Φ–invariant random measure ν. This measure is the required Gibbs state. Finally, the bounds on

$$\lambda_x = \nu_{\theta(x)}(\mathcal{L}_x\mathbb{1})$$

follow again from item (a) of the definition of $\mathcal{M}$ in (3.5) and Proposition 2.3 together with Lemma 3.13. □

We have to study these conformal measures more in detail. Here and in the rest of the paper it is very useful to introduce normalized operators

$$\hat{\mathcal{L}}_x := \lambda_x^{-1}\mathcal{L}_x, \tag{3.8}$$

and to employ the notation

$$\lambda_x^n = \prod_{j=0}^{n-1}\lambda_{\theta^j(x)}^n \quad \text{and} \quad \hat{\mathcal{L}}_x^n = \lambda_x^{-n}\mathcal{L}_x^n\,.$$

We continue to use the radius R_0 of the definition of the invariant measure space $\mathcal{M}$, given in Proposition 3.11. Clearly we may suppose that $R_0 \geq T > 0$, T being the constant of Condition 2. Condition 4 is applied to get the following lower estimate.

Lemma 3.15. *For every $R \geq R_0 + 1$ and every $0 < \delta \leq \min\{\delta_0, 1\}$ there exists $A = A(\delta, R, t) \geq 1$ such that*

$$\nu_x\big(\mathbb{D}(z,\delta)\big) \geq A^{-1} \quad \textit{for all } x \in X \textit{ and all } z \in \mathcal{J}_x \textit{ with } |z| \leq R.$$

Proof. Covering $\overline{\mathbb{D}}_{R_0} \cap \mathcal{J}_x$ with δ–disks whose no more than three elements intersect, and using the fact that $\nu_x(\overline{\mathbb{D}}_{R_0}) \geq \frac{1}{2}$, we see that there exists $\tilde{a} = a(\delta, R_0) > 0$ such that

$$\nu_x\big(\mathbb{D}(w_x, \delta)\big) \geq \tilde{a} \quad \text{for } m - \text{a.e. } x \in X \text{ and some } w_x \in \overline{\mathbb{D}}_{R_0} \ . \tag{3.9}$$

Let now $N = N(\delta, R)$ be the number coming from the mixing property Lemma 1.3, applied with $r = \delta/2$ and $R \geq R_0 + 1$. Then, for every $z \in \mathcal{J}_x$, $|z| \leq R$,

$$f_x^N\left(\mathbb{D}(z, \delta/2)\right) \supset \overline{\mathbb{D}}_R \cap \mathcal{J}_{\theta^N(x)} \supset \mathbb{D}(w_{\theta^N(x)}, \delta) \cap \mathcal{J}_{\theta^N(x)} \, .$$

Consequently, there exists a holomorphic inverse branch $f_{x,*}^{-N}$ of f_x^N defined on the disk $\mathbb{D}(w_{\theta^N(x)}, \delta)$ such that $f_{x,*}^{-N}\left(\mathbb{D}(w_{\theta^N(x)}, \delta)\right) \cap \mathbb{D}(z, \delta/2) \neq \emptyset$. We may assume without loss of generality that N is so large that Koebe's distortion theorem together with the expanding property imply that $\operatorname{diam}\left(f_{x,*}^{-N}(\mathbb{D}(w_{\theta^N(x)}, \delta))\right) \leq \delta/2$. Then $f_{x,*}^{-N}\left(\mathbb{D}(w_{\theta^N(x)}, \delta)\right) \subset \mathbb{D}(z, \delta)$. Hence,

$$\begin{aligned}
\nu_x(\mathbb{D}(z, \delta)) &\geq \nu_x\left(f_{x,*}^{-N}\left(\mathbb{D}(w_{\theta^N(x)}, \delta)\right)\right) \\
&\geq \frac{1}{K}\lambda_x^{-N}\left|(f_x^N)'(f_x^{-N}(w_{\theta^N(x)})\right|_\tau^{-t} \nu_{\theta^N(x)}(\mathbb{D}(w_{\theta^N(x)}, \delta)) \\
&\geq a
\end{aligned}$$

for some $a = a(t, R, \delta) > 0$ by (3.9), Condition 4, and since $\lambda_x \leq C < \infty$ for all $x \in X$. □

4. Uniform bounds and invariant densities

We are now able to prove the following uniform bound for the normalized operators $\hat{\mathcal{L}}_x^n$.

Proposition 4.1. *There exists $M = M_t < \infty$ such that*

$$\|\hat{\mathcal{L}}_x^n\| \leq M \quad \textit{for every } n \geq 1 \textit{ and } m - \textit{a.e. } x \in X \, .$$

By combining this result with Proposition 2.7, we obtain the following:

Corollary 4.2. *For every $x \in X$, every $g \in \mathcal{H}_\beta(\mathcal{J}_x)$, and all $n \geq 1$, we have*

$$v_\beta(\hat{\mathcal{L}}_x^n g) \leq M\left(\|g\|_\infty + K(c\gamma^n)^{-\beta} v_\beta(g)\right) . \tag{4.1}$$

Proposition 4.1 and Corollary 4.2 together imply that the above uniform bound is also valid with respect to the Hölder norm $\|\cdot\|_\beta$. For simplicity we will use the same bound M in the sequel.

Corollary 4.3. *There exists $M = M_t < \infty$ such that*

$$\|\hat{\mathcal{L}}_x^n\|_\beta \leq M \quad \textit{for every} \quad n \geq 1 \ \textit{and} \ \ m - \textit{a.e. } x \in X \, .$$

We first need an auxiliary result. Let $0 < \delta \leq \delta_0$ and $R \geq R_0 + 1$.

Lemma 4.4. *For every $n \geq 1$ we have*

$$\hat{\mathcal{L}}_x^n \mathbb{1}(w) \leq KA \quad , \quad w \in \mathcal{J}_{\theta^n(x)} \ , \ |w| \leq R \, , \ x \in X \, ,$$

where K is the distortion constant from Lemma 2.6 and A is defined in Lemma 3.15.

Proof. Let $w \in \mathcal{J}_{\theta^n(x)}$, $|w| \leq R$. Then, using Lemma 2.6 and Lemma 3.15, we get

$$1 = \int \hat{\mathcal{L}}_x^n \mathbb{1} d\nu_{\theta^n(x)} \geq \int_{\mathbb{D}(w,\delta)} \hat{\mathcal{L}}_x^n \mathbb{1} d\nu_{\theta^n(x)} \geq \frac{1}{K} \hat{\mathcal{L}}_x^n \mathbb{1}(w) \nu_{\theta^n(x)}\big(\mathbb{D}(w,\delta)\big) \geq \frac{\hat{\mathcal{L}}_x^n \mathbb{1}(w)}{AK} \, .$$

We are done. □

Proof of Proposition 4.1. From the lower bound on λ_x given in Theorem 3.1 and from the fact that $\mathcal{L}_x \mathbb{1}(w) \to 0$ as $|w| \to \infty$ uniformly in $x \in X$ (see Proposition 2.3), it follows that there exists $R \geq R_0 + 1$ such that

$$\hat{\mathcal{L}}_x \mathbb{1}(w) \leq 1 \quad \text{for} \quad w \in \mathcal{J}_{\theta(x)} \cap \mathbb{D}_R^c \tag{4.2}$$

for m-a.e. $x \in X$.

Claim 4.5. *Set $M = KA(\delta_0, R, t)$, again with constants as in Lemma 4.4. Then $\hat{\mathcal{L}}_x^n \mathbb{1} \leq M$ for every $n \geq 1$ and m-a.e. $x \in X$.*

It suffices to prove this claim. It will be done by induction. The case $n = 1$ results directly from (4.2) and Lemma 4.4. So, fix $n \geq 1$ and suppose that Claim holds for this n. We have to show that

$$\hat{\mathcal{L}}_{\theta^{-(n+1)}(x)}^{n+1} \mathbb{1}(w) \leq M \quad \text{for every} \quad w \in \mathcal{J}_x \ \text{ and } a.e.\ x \in X \, .$$

If $w \in \mathcal{J}_x \cap \mathbb{D}_R$, then it suffices to apply Lemma 4.4. Otherwise, i.e. if $|w| \geq R$, then

$$\hat{\mathcal{L}}_{\theta^{-(n+1)}(x)}^{n+1} \mathbb{1}(w) = \hat{\mathcal{L}}_{\theta^{-1}(x)} \left(\hat{\mathcal{L}}_{\theta^{-(n+1)}(x)}^{n} \mathbb{1} \right)(w) \leq M \hat{\mathcal{L}}_{\theta^{-1}(x)} \mathbb{1}(w) \leq M \, .$$

□

5. Invariant positive cones and Bowen's contraction

G. Birkhoff in [2] reinterpreted Hilbert's pseudo-distance on positive cones in a way which allowed him to show that every linear map preserving cones is a weak contraction. This enabled him to give an elegant proof of the Perron-Frobenius theorem based on Banach's contraction principle. Various versions of Ruelle's Perron-Frobenius theorem have been obtained since then using Birkhoff's strategy (see, for example, Liverani [15] and Rugh [25, 26] who, at the same time, considered random dynamics and introduced a complexification scheme leading to real analyticity of the dimension).

In our setting, with unbounded phase spaces $\mathcal{J}_x$, we encounter several problems. First of all, because of the behavior of the functions at infinity, every reasonable

invariant cone contains many functions that all are at the infinite Hilbert distance from each other. These cones have many, in fact uncountably many, connected components that are at finite distances from each other. The second problem is that it is hard to get a strict-contraction property since the mixing property which is at our disposal (Lemma 1.3) is too weak; one only has mixing on bounded regions.

Our way to overcome these difficulties is to define appropriate invariant cones and then to avoid Birkhoff's strategy, but instead, to employ an argument inspired by Bowen's lemma [3, Lemma 1.9]. For compact phase spaces this lemma is indeed equivalent to a strict contraction in the Hilbert metric. In our situation this is not the case but it turns out that Bowen's lemma is sufficiently tricky so that we can use some appropriate version if it that leads to the following exponential convergence result.

From now on the number $\delta_0 > 0$ in the definition of the variation of Hölder functions will be replaced by a smaller number $0 < \delta \le \delta_0$ as explained in (5.5).

Theorem 5.1. *Let $(f_x)_x$ be a hyperbolic transcendental random system. We then have the following.*

(1) *There exists a unique $\rho \in \mathcal{H}_\beta(\mathcal{J})$, an invariant density, i.e. $\hat{\mathcal{L}}\rho = \rho$.*
(2) *There are $B > 0$ and $\vartheta \in]0,1[$ such that*

$$\left\| \hat{\mathcal{L}}_x^n g_x - \int g_x \, d\nu_x \, \rho_{\theta^n(x)} \right\|_\beta \le B\vartheta^n \|g_x\|_\beta$$

for every $g_x \in H_\beta(\mathcal{J}_x)$ and a.e. $x \in X$.

Remark 5.2. *Multiplying, as usually, the Gibbs state ν by the invariant function ρ of Theorem 5.1, (1) gives, again unique by Theorem 5.1, (2), invariant Gibbs state $\mu \in \mathcal{P}(\mathcal{J})$ whose disintegrations are*

$$\mu_x = \rho_x \nu_x \quad , \quad x \in X. \tag{5.1}$$

Moreover, μ is ergodic. Indeed, if there existed an invariant set $E \subset \mathcal{J}$ with $0 < \mu(E) < 1$, then $\mu_1 = \mathbb{1}_E \mu$ and $\mu_2 = \mathbb{1}_{E^c} \mu$ would be two invariant Gibbs states. But this would contradict the above uniqueness property.

Remark 5.3. *Notice that, as a straightforward consequence of point (2) of this theorem, we also get exponential convergence for random Hölder observables, i.e in $\mathcal{H}_\beta(\mathcal{J})$, with respect to the canonical norm of this space:*

$$|\hat{\mathcal{L}}^n g - \pi_\rho(\hat{\mathcal{L}}^n g)|_\beta \le B\vartheta^n |g|_\beta \quad , \quad g \in \mathcal{H}_\beta(\mathcal{J}) \, ,$$

where $\pi_\rho : \mathcal{H}_\beta(\mathcal{J}) \to <\rho>$ is the canonical projection defined by $\pi_\rho(g)_x = \int g_x \, d\nu_x \rho_x$.

In particular, $\hat{\mathcal{L}}^n \mathbb{1} \longrightarrow \rho$ exponentially fast.

5.1. **Invariant cones.** Consider the following cones:

$$\mathcal{C}_x := \left\{ g \geq 0 \ : \ \|g\|_\infty \leq \mathcal{A} \int g d\nu_x < \infty \ \text{ and } \ v_\beta(g) \leq H \int g d\nu_x \right\} . \tag{5.2}$$

$$\mathcal{C}_{x,0} := \left\{ g \in \mathcal{C}_x \ : \ g \leq 2M_t \mathcal{A} \left(\int g d\nu_x \right) \ \hat{\mathcal{L}}_{\theta^{-1}(x)} \mathbb{1} \right\} . \tag{5.3}$$

Since we are primarily interested in the projective features of these cones, it is convenient for us to use the following slices

$$\Lambda_x = \{ g \in \mathcal{C}_x \ , \ \ \nu_x(g) = 1 \} \quad \text{and} \quad \Lambda_{x,0} = \Lambda_x \cap \mathcal{C}_{x,0} \ , \ \ x \in X . \tag{5.4}$$

Hence both type of cones do depend on constants $\beta \geq 1$, $\mathcal{A} > 0$, $H > 0$, and even on δ. Whenever this is important we will indicate this and write $\mathcal{C}_x(\mathcal{A}, H)$ or even $\mathcal{C}_x(\mathcal{A}, H, \beta, \delta)$, and similarly for the second type of cones. In order to produce cones with good properties, for example invariance, we have to choose carefully these constants.

We continue to write $M = M_t$ for the uniform bound given in Proposition 4.1 and in Corollary 4.3 and $K = K_t$ for the distortion constant appearing in Lemma 2.4 and Lemma 2.6. First of all, let $0 < \delta \leq \delta_0$ be such that

$$\frac{1}{2} + (2MK + 4)\, \delta^\beta \leq 1 . \tag{5.5}$$

When we deal in the sequel with Hölder functions g, then we assume that the variation $v_\beta(g)$ is evaluated on disks of radius δ, i.e. δ_0 is replaced by δ in (2.3).

The radius R_0 has been defined in Lemma 3.13. Increasing it if necessary we may suppose that Lemma 3.15 is valid with $R = R_0$ and, for the same reason as in (4.2), that

$$2M\hat{\mathcal{L}}_x \mathbb{1} \leq 1 \quad \text{in} \quad \mathbb{D}^c_{R_0} \cap \mathcal{J}_x \ , \ \ x \in X . \tag{5.6}$$

Define now, with $A(\delta, R, t)$ from Lemma 3.15,

$$\mathcal{A} := 2 \max\{1, A(\delta, R_0, t), M\} \quad \text{and} \quad H = 2MK\mathcal{A} + 4 . \tag{5.7}$$

Notice that $\mathcal{A} \geq 1$. This ensures that the constant function $\mathbb{1} \in \mathcal{C}_x$, $x \in X$. Finally, let $N_0 \geq 1$ be such that

$$MK(c\gamma^{N_0})^{-\beta} H \leq 1 . \tag{5.8}$$

Proposition 5.4. *With the above choice of constants and for every $n \geq N_0$,*

$$\hat{\mathcal{L}}^n_x (\mathcal{C}_x) \subset \mathcal{C}_{\theta^n(x),0} \subset \mathcal{C}_{\theta^n(x)} \ , \ \ x \in X .$$

Proof. Let $g \in \mathcal{C}_x$. We may assume that $\int g d\nu_x = 1$. We will show that $\hat{\mathcal{L}}^n_x g \in \mathcal{C}_{\theta^n(x),0}$ for every $n \geq N_0$. Let in the following $n \geq N_0$. From the two-norm type inequality (4.1) and from the definition of the cone, we get that

$$v_\beta(\hat{\mathcal{L}}^n_x g) \leq M \left(\mathcal{A} + K(c\gamma^n)^{-\beta} H \right) \leq M\mathcal{A} + 1 \leq H, \tag{5.9}$$

where the last two inequalities result from the choice of N_0 and from the definition of H. Then,

$$\hat{\mathcal{L}}_{x\mathcal{G}}^n g = \hat{\mathcal{L}}_{\theta^{n-1}(x)}\left(\hat{\mathcal{L}}_x^{n-1} g\right) \le M\|g\|_\infty \hat{\mathcal{L}}_{\theta^{n-1}(x)}\mathbb{1} \le M\mathcal{A}\hat{\mathcal{L}}_{\theta^{n-1}(x)}\mathbb{1}\,. \tag{5.10}$$

In order to see that $\mathcal{L}_{x\mathcal{G}}^n g \in \mathcal{C}_{\theta^n(x),0}$ it remains to estimate $\|\hat{\mathcal{L}}_{x\mathcal{G}}^n g\|_\infty$. Since we already have (5.9), we obtain, for every $|z| \le R_0$, the following.

$$\begin{aligned} 1 &= \int \hat{\mathcal{L}}_{x\mathcal{G}}^n g \, d\nu_{\theta^n(x)} \ge \int_{\mathbb{D}(z,\delta)} \hat{\mathcal{L}}_{x\mathcal{G}}^n g \, d\nu_{\theta^n(x)} \\ &\ge \left(\hat{\mathcal{L}}_{x\mathcal{G}}^n g(z) - H\delta^\beta\right) \nu_{\theta^n(x)}(\mathbb{D}(z,\delta)) \ge \left(\hat{\mathcal{L}}_{x\mathcal{G}}^n g(z) - H\delta^\beta\right) A(\delta, R_0, t)^{-1}\,, \end{aligned}$$

where the last inequality was written due to Lemma 3.15. Therefore,

$$\hat{\mathcal{L}}_{x\mathcal{G}}^n g(z) \le \mathcal{A}/2 + H\delta^\beta \le \mathcal{A}\left(\frac{1}{2} + (2MK+4)\delta^\beta\right) \le \mathcal{A},$$

by (5.5). If $|z| \ge R_0$, then it suffices to combine (5.10) and (5.6) in order to conclude this proof with the inequality,

$$\hat{\mathcal{L}}_{x\mathcal{G}}^n g(z) \le M\mathcal{A}\hat{\mathcal{L}}_{\theta^{n-1}(x)}\mathbb{1}(z) \le \mathcal{A}\,.$$

The proof is complete. □

5.2. **Cone contraction via Bowen's lemma.** Let $R_1 \ge R_0$ be such that

$$2AM\hat{\mathcal{L}}_x\mathbb{1} \le 1 \quad \text{in} \quad \mathbb{D}_{R_1}^c\,. \tag{5.11}$$

Lemma 5.5. *For every $R \ge R_1$ there are $N = N_R \ge N_0$ and $a = a_R > 0$ such that*

$$\hat{\mathcal{L}}_x^N g_{\big|\mathbb{D}_{2R}} \ge a \quad \textit{for every} \quad g \in \Lambda_{x,0}\,,\ \ x \in X\,.$$

Proof. Let $g \in \Lambda_{x,0}$. Since $\int g d\nu_x = 1$, we have that $\|g\|_\infty \ge 1$. Hence, by the choice of R_1,

$$g \le 2M\mathcal{A}\,\hat{\mathcal{L}}_x\mathbb{1} \le 1 \quad \text{in} \quad \mathbb{D}_{R_1}^c.$$

Thus, there exists $z_{max} \in \overline{\mathbb{D}}_{R_1}$ with $g(z_{max}) = \|g\|_\infty \ge 1$.

Let $0 < r \le \delta$ be such that $Hr^\beta \le \frac{1}{4}$. The mixing property Lemma 1.3 implies the existence of $N = N(r,R) \ge 0$ such that every $w \in \mathcal{J}_{\theta^N(x)} \cap \mathbb{D}_{2R}$ has a preimage $z_0 \in f_x^{-N}(w) \cap \mathbb{D}(z_{max}, r)$. Therefore, using Condition 4, for every such w, we get that

$$\begin{aligned} \hat{\mathcal{L}}_x^N g(w) &\ge \lambda_x^{-N}\left|(f_x^N)'(z_0)\right|_\tau^{-t} g(z_0) \ge C^{-N}\left|(f_x^N)'(z_0)\right|_\tau^{-t}\left(g(z_{max}) - Hr^\beta\right) \\ &\ge C^{-N} \inf_{|z|\le R_1\ |f_x^N(z)|\le 2R}\left|(f_x^N)'(z)\right|_\tau^{-t}\left(1 - \frac{1}{4}\right) \\ &=: a > 0. \end{aligned}$$

The proof is complete. □

Notice that there is no way to get a global, valid on the whole Julia set, version of Lemma 5.5. This is why we have to work with the following truncated functions. We remark that our cones are chosen in such a way that such truncations can be made to lie inside them. This is not the case for the standard Bowen cones.

Let $\varphi_1 : \mathbb{C} \to [0,1]$ be a Lipschitz function such that $\varphi_1 \equiv 1$ on $\mathbb{D}_1$ and $\varphi_1 \equiv 0$ on $\mathbb{D}_2^c$. For $R \geq 1$ define $\varphi_R(z) = \varphi_1(z/R)$. Then φ_R is also Lipschitz with variation $v_1(\varphi_R) \to 0$ as $R \to \infty$. Define

$$\varphi_{x,R} := \varphi_R \hat{\mathcal{L}}_{\theta^{-1}(x)} \mathbb{1} \,. \tag{5.12}$$

Then, $0 \leq \varphi_{x,R} \leq \hat{\mathcal{L}}_{\theta^{-1}(x)}\mathbb{1}$, $\varphi_{x,R} \equiv \hat{\mathcal{L}}_{\theta^{-1}(x)}\mathbb{1}$ on $\mathbb{D}_R$ and $\varphi_{x,R} \equiv 0$ in D_{2R}^c. The functions $\varphi_{x,R}$ are Lipschitz with $v_1(\varphi_{x,R}) \to v_1(\hat{\mathcal{L}}_{\theta^{-1}(x)}\mathbb{1})$ uniformly as $R \to \infty$. Therefore, given the definition of the cones, especially the definitions of the constants $\mathcal{A}, H$ in (5.7), and the formulas established in the course of the proof of Proposition 5.4, it follows that

$$\varphi_{x,R} \in \mathcal{C}_{x,0} \quad , x \in X,$$

provided that R is sufficiently large. We will assume that R_1 is chosen so that these truncated functions belong to the cones for all $R \geq R_1$. Suppose also, in what follows, that $\eta > 0$ is chosen such that

$$0 < \eta \leq \min\left\{\frac{1}{3},\ \frac{1}{H},\ \frac{1}{2}\frac{a}{M}\right\} . \tag{5.13}$$

With these choices we will now obtain the following version of Bowen's [3, Lemma 1.9].

Lemma 5.6. *For every $R \geq R_1$ and with $N = N_R \geq N_0$ given by Lemma 5.5,*

$$\hat{\mathcal{L}}_x^N g - \eta \varphi_{\theta^N(x),R} \in \mathcal{C}_{\theta^N(x),0} \quad \textit{for every} \quad g \in \Lambda_{x,0} \,.$$

Proof. Let $x \in X$, let $g \in \Lambda_{x,0}$, and let $R \geq R_1$. Lemma 5.5 shows that for $0 < \eta < \frac{1}{2}\frac{a}{M}$,

$$\hat{\mathcal{L}}_x^N g - \eta \varphi_{\theta^N(x)} > \frac{a}{2} > 0 \quad \text{on} \quad \mathbb{D}_{2R} \cap \mathcal{J}_{\theta^N(x)} \,.$$

Set

$$g' = \frac{\hat{\mathcal{L}}_x^N g - \eta \varphi_{\theta^N(x),R}}{1 - \eta_{\theta^N(x),R}} \quad \text{where} \quad \eta_{\theta^N(x),R} := \eta \int \varphi_{\theta^N(x),R} d\nu_{\theta^N(x)} \,. \tag{5.14}$$

Then $\int g' d\nu_{\theta^N(x)} = 1$ and $g' > 0$. We have, by

$$(1 - \eta_{\theta^N(x),R})\, g' \leq M\|g\|_\infty \hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1} + \eta \hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1} \leq (M\mathcal{A} + \eta)\hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1} \,.$$

But $0 < \eta_{\theta^N(x),R} \leq \eta \leq \frac{1}{3}$ and thus $g' \leq 2M\mathcal{A}\hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1}$. This means that function $g' \in \Lambda_{\theta^N(x),0}$ provided that we can show that $g' \in \Lambda_{\theta^N(x)}$.

In order to estimate the variation of g' we use again the two-norm type inequality (4.1):

$$v_b(g') \le \frac{1}{1-\eta_{\theta^N(x),R}} \left(M\|g\|_\infty + MK(c\gamma^N)^{-\beta} v_\beta(g) + \eta v_\beta(\varphi_{\theta^N(x),R}) \right) .$$

Remember that $g, \varphi_{\theta^N(x),R} \in \mathcal{C}_x$, that $\eta \le \min\left\{\frac{1}{3}, \frac{1}{H}\right\}$, and that we have (5.8). Therefore,

$$v_b(g') \le 2\,(M\mathcal{A}+1+1) = 2M\mathcal{A}+4 \le H\,.$$

It remains to estimate $\|g'\|_\infty$. If $z \in \mathcal{J}_{\theta^N(x)} \cap \mathbb{D}_{R_0}$, then

$$1 = \int g' d\nu_{\theta^N(x)} \ge \int_{\mathbb{D}(z,\delta)} g' d\nu_{\theta^N(x)} \ge (g'(z) - H\delta^\beta)\nu_{\theta^N(x)}(\mathbb{D}(z,\delta))\,.$$

Using once more Lemma 3.15 and the choice of δ in (5.5), we obtain

$$g'(z) \le A(\delta, R_0, t) + H\delta^\beta \le \frac{\mathcal{A}}{2} + (2MK\mathcal{A}+4)\delta^\beta \le \mathcal{A}\left(\frac{1}{2} + (2MK+4)\delta^\beta\right) \le \mathcal{A}\,.$$

If $z \in \mathcal{J}_{\theta^N(x)} \cap \mathbb{D}^c_{R_0}$, then $g'(z) \le 2M\mathcal{A}\hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1}(z) \le \mathcal{A}$ by the choice of R_0 (see (5.6)). The proof is complete. □

Applying repeatedly Lemma 5.6 gives the desired contraction.

Proposition 5.7. *For every $\varepsilon > 0$ there exists $n_\varepsilon \ge 1$ such that for every $n \ge n_\varepsilon$ and a.e. $x \in X$,*

$$\left\|\hat{\mathcal{L}}^n_x g_x - \hat{\mathcal{L}}^n_x h_x\right\|_\beta \le \varepsilon \quad \textit{for all} \quad g_x, h_x \in \Lambda_{x,0}\,. \tag{5.15}$$

Proof. Let $R \ge R_1$ and $N = N_R \ge N_0$ be like in Lemma 5.6. and let $g = g^{(0)}_x \in \Lambda_{x,0}$. With the notation of the previous proof, and in particular with the numbers $\eta_{\theta^N(x),R}$ defined in (5.14), we get from Lemma 5.6 that

$$\hat{\mathcal{L}}^N_x g = \eta\varphi_{\theta^N(x),R} + (1-\eta_{\theta^N(x),R})g^{(1)}_{\theta^N(x)}$$

for some $g^{(1)}_{\theta^N(x)} \in \Lambda_{\theta^N(x),0}$. Applying $\hat{\mathcal{L}}^N_{\theta^N(x)}$ to this equation and using once more Lemma 5.6 gives

$$\hat{\mathcal{L}}^{2N}_x g = \eta\hat{\mathcal{L}}^N_{\theta^N(x)}\varphi_{\theta^N(x),R} + (1-\eta_{\theta^N(x),R})\eta\varphi_{\theta^{2N}(x),R} + (1-\eta_{\theta^N(x),R})(1-\eta_{\theta^{2N}(x),R})g^{(2)}_{\theta^{2N}(x)}$$

for some $g^{(2)}_{\theta^{2N}(x)} \in \Lambda_{\theta^{2N}(x),0}$. Inductively it follows that for every $k \ge 1$ there is a function $g^{(k)}_{\theta^{kN}(x)} \in \Lambda_{\theta^{kN}(x),0}$ such that

$$\hat{\mathcal{L}}^{kN}_x g = \eta\sum_{j=1}^{k}\left(\prod_{i=1}^{j-1}(1-\eta_{\theta^{iN}(x),R})\right)\hat{\mathcal{L}}^{(k-j)N}_{\theta^{jN}(x)}\varphi_{\theta^{jN}(x),R} + \prod_{i=1}^{k}(1-\eta_{\theta^{iN}(x),R})\,g^{(k)}_{\theta^{kN}(x)}\,.$$

Observe that the first of these two terms does not depend on g. Therefore, for every $g, h \in \Lambda_{x,0}$ there are $g^{(k)}_{\theta^{kN}(x)}, h^{(k)}_{\theta^{kN}(x)} \in \Lambda_{\theta^{kN}(x),0}$ such that

$$\hat{\mathcal{L}}^{kN}_x g - \hat{\mathcal{L}}^{kN}_x h = \prod_{i=1}^{k}(1-\eta_{\theta^{iN}(x),R}) \left(g^{(k)}_{\theta^{kN}(x)} - h^{(k)}_{\theta^{kN}(x)}\right). \tag{5.16}$$

Remember that $\eta_{y,R} = \eta \int \varphi_{y,R} d\nu_y \geq \eta \int_{\overline{\mathbb{D}}_{R_0}} \hat{\mathcal{L}}_{\theta^{-1}(y)} \mathbb{1} d\nu_y$ for all $R \geq R_0$, and that, by Lemma 3.13, there exists a constant $c = c(R_0) > 0$ such that $\hat{\mathcal{L}}_{\theta^{-1}(y)} \mathbb{1} \geq c$ on $\overline{\mathbb{D}}_{R_0}$. Therefore,

$$1 > \eta \geq \eta_{y,R} \geq \eta c \nu_y(\overline{\mathbb{D}}_{R_0}) \geq \eta \frac{c}{2} = \tilde{\eta} > 0.$$

Thus,

$$1 - \eta_{y,R} \leq 1 - \tilde{\eta}.$$

Along with (5.16), this allows us to deduce the the uniform bound of Proposition 4.1, with some $n_{1,\varepsilon} \geq 1$ sufficiently large, for the supremum norm rather than the Hölder one. In order to get the appropriate estimate for the β–variation we need once more (4.1). Write $n = m + n_{2,\varepsilon} + n_{1,\varepsilon}$ with some $n_{2,\varepsilon}$ to be determined in a moment and some $m \geq 0$. Then for all $g, h \in \Lambda_{x,0}$, we have

$$\begin{aligned} v_\beta\left(\hat{\mathcal{L}}^n_{x\mathcal{S}} g - \hat{\mathcal{L}}^n_x h\right) &= v_\beta\left(\hat{\mathcal{L}}^{m+n_{2,\varepsilon}}_{\theta^{n_{1,\varepsilon}}(x)}\left(\hat{\mathcal{L}}^{n_{1,\varepsilon}}_x (g-h)\right)\right) \\ &\leq M\left(\left\|\hat{\mathcal{L}}^{n_{1,\varepsilon}}_x (g-h)\right\|_\infty + K(c\gamma^{m+n_{2,\varepsilon}})^{-\beta} v_\beta\left(\hat{\mathcal{L}}^{n_{1,\varepsilon}}_x (g-h)\right)\right) \\ &\leq M\varepsilon + MK(c\gamma^{n_{2,\varepsilon}})^{-\beta} 2H, \end{aligned}$$

since $\hat{\mathcal{L}}^{n_{1,\varepsilon}}_x g, \hat{\mathcal{L}}^{n_{1,\varepsilon}}_x h \in \mathcal{C}_{\theta^{n_{1,\varepsilon}}(x),0}$. It suffices now to choose $n_{2,\varepsilon} \geq 0$ sufficiently large in order to conclude this proof. □

Proof of Theorem 5.1 (1). Consider $\rho^k = \hat{\mathcal{L}}^k \mathbb{1}$. First of all, Proposition 5.4 implies that $\rho^k_x \in \Lambda_{x,0}$ for every $k \geq N_0$. Hence Proposition 5.7 applies and gives

$$\|\rho^k_x - \rho^l_x\|_\beta \leq \varepsilon \quad \text{for every} \quad l \geq k \geq n_\varepsilon \ , \ \ x \in X.$$

This shows that $(\rho^k_x)_k$ is a, uniformly in $x \in X$, Cauchy sequence of $(\mathcal{H}_\beta(\mathcal{J}_x), \|.\|_\beta)$ and hence there is a limit $\rho \in \mathcal{H}_\beta(\mathcal{J})$. Clearly, $\hat{\mathcal{L}}\rho = \rho$ and $\rho_x \in \Lambda_{x,0}$, $x \in X$. Uniqueness of this function follows from the contraction given in (5.15). □

Proof of Theorem 5.1 (2). Since $\mathcal{A}, H \geq 2$, we have that

$$\left\{\mathbb{1} + h_x : \ \|h_x\|_\beta < 1/4\right\} \subset \mathcal{C}_x,$$

for all $x \in X$. Let $g \in H_\beta(\mathcal{J}_x)$, $g \not\equiv 0$ be arbitrary. Then

$$h := \frac{g}{8\|g\|_\beta} = (h + \mathbb{1}) - \mathbb{1}$$

is a difference of functions from $\mathcal{C}_x$. If $\varepsilon > 0$ and $n = n_\varepsilon$ is given by Proposition 5.7, then

$$\begin{aligned}\left\|\hat{\mathcal{L}}_x^n\left(h-\left(\int h d\nu_x\right)\rho_{\theta^n(x)}\right)\right\|_\beta &\le \left\|\hat{\mathcal{L}}_x^n h-\left(\int h d\nu_x\right)\rho_{\theta^n(x)}\right\|_\beta \le \\ &\le \left\|\hat{\mathcal{L}}_x^n(\mathbb{1}+h)-\int(\mathbb{1}+h)\,d\nu_x\,\rho_{\theta^n(x)}\right\|_\beta + \left\|\hat{\mathcal{L}}_x^n\mathbb{1}-\rho_{\theta^n(x)}\right\|_\beta \\ &\le \varepsilon\int(\mathbb{1}+h)\,d\nu_x+\varepsilon \\ &\le \frac{17}{8}\varepsilon.\end{aligned}$$

This shows that for every $\varepsilon > 0$ there exists $N = N_\varepsilon$ such that

$$\left\|\hat{\mathcal{L}}_x^N\left(g-\left(\int g d\nu_x\right)\rho_{\theta^N(x)}\right)\right\|_\beta \le \varepsilon\|g\|_\beta \quad \text{for every} \quad g\in H_\beta(\mathcal{J}_x)\,.$$

Fix $\varepsilon := 1/2$ and let $N = N_{1/2}$. Write any integer $n \ge 0$ in a unique form as $n = kN + m$, where $k \ge 0$ and $m \in \{0, ..., N-1\}$. Then, for every $g \in H_\beta(\mathcal{J}_x)$ we have,

$$\begin{aligned}\left\|\hat{\mathcal{L}}_x^n g-\int g d\nu_x\rho_{\theta^n(x)}\right\|_\beta &= \left\|\hat{\mathcal{L}}_{\theta^{kN}(x)}^m\left(\hat{\mathcal{L}}_x^{kN}(g-\int g d\nu_x\rho_x)\right)\right\|_\beta \\ &\le M\left(\frac{1}{2}\right)^k\left\|g-\int g d\nu_x\rho_x\right\|_\beta \\ &\le 2M\left(\frac{1}{2^{1/N}}\right)^n\left(1+\|\rho_x\|_\beta\right)\|g\|_\beta\,.\end{aligned}$$

This completes the proof of Theorem 5.1. □

6. Exponential decay of correlations and CLT

Exponential decay of correlations is now a fairly straightforward consequence of Theorem 5.1 (2). It will be valid for functions of the following spaces.

Let $\mathcal{H}_\beta^p(\mathcal{J})$ be the space of functions $g : \mathcal{J} \to \mathbb{R}$ with Hölder fibers $g_x \in \mathcal{H}_\beta(\mathcal{J}_x)$ and such that $\|g_x\|_\beta \in L^p(m)$. The canonical norm is

$$|g|_{\beta,p} = \left(\int_X \|g_x\|_\beta^p dm(x)\right)^{\frac{1}{p}}.$$

Replacing in this definition the β–Hölder condition on the fiber $\mathcal{J}_x$ by a $L^1(\nu_x)$ condition leads to a space of functions that will be denoted by $L_\nu^{1,p}(\mathcal{J})$. The natural norm is in this case

$$|g|_\nu^{1,p} = \left(\int_X \|g_x\|_{L^1(\nu_x)}^p dm(x)\right)^{\frac{1}{p}}.$$

Clearly, if $p = 1$ then $L^{1,1}_\nu(\mathcal{J}) = L^1(\nu)$. In both cases we also consider $p = \infty$ and then the L^p norms are replaced by the sup–norm.

Theorem 6.1. *Let $(f_x)_x$ be a hyperbolic transcendental random system and let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for every $g \in L^{1,p}_\nu(\mathcal{J})$, $h \in \mathcal{H}^q_\beta(\mathcal{J})$ with $\int_{\mathcal{J}_x} h_x \, d\mu_x = 0$ and for every $n \geq 1$, we have*

$$\left| \int_{\mathcal{J}} (g \circ f^n)\, h \, d\mu \right| = \left| \int_X \int_{\mathcal{J}_x} (g_{\theta^n(x)} \circ f^n_x)\, h_x d\mu_x dm(x) \right| \leq b\vartheta^n \, |g|^{1,p}_\nu \, |h|_{\beta,q}$$

for some positive constant b and some $\vartheta \in (0, 1)$.

Proof. A standard calculation and application of Theorem 5.1 (2) gives

$$\begin{aligned} \left| \int_{\mathcal{J}_x} (g_{\theta^n(x)} \circ f^n_x)\, h_x d\mu_x \right| &= \left| \int_{\mathcal{J}_{\theta^n(x)}} g_{\theta^n(x)} \, \hat{\mathcal{L}}^n_x \, (h_x \rho_x) \, d\nu_{\theta^n(x)} \right| \\ &\leq \left\| \hat{\mathcal{L}}^n_x \, (h_x \rho_x) \right\|_\beta \, \|g_{\theta^n(x)}\|_{L^1(\nu_{\theta^n(x)})} \\ &\leq B\vartheta^n \, \|h_x \rho_x\|_\beta \, \|g_{\theta^n(x)}\|_{L^1(\nu_{\theta^n(x)})} \leq b\vartheta^n \, \|h_x\|_\beta \, \|g_{\theta^n(x)}\|_{L^1(\nu_{\theta^n(x)})} \end{aligned}$$

for some constant $b > 0$ since $\|\rho_x\|_\beta \leq M$ for all $x \in X$ by Corollary 4.3 and Theorem 5.1 (2). Therefore,

$$\begin{aligned} \left| \int_{\mathcal{J}} (g \circ f^n)\, h \, d\mu \right| &= \left| \int_X \int_{\mathcal{J}_x} (g_{\theta^n(x)} \circ f^n_x)\, h_x d\mu_x dm(x) \right| \\ &\leq b\vartheta^n \int_X \|h_x\|_\beta \, \|g_{\theta^n(x)}\|_{L^1(\nu_{\theta^n(x)})} dm(x) \\ &\leq b\vartheta^n \left(\int_X \|h_x\|^q_\beta \, dm(x) \right)^{\frac{1}{q}} \left(\int_X \|g_x\|^p_{L^1(\nu_x)} dm(x) \right)^{\frac{1}{p}} . \end{aligned}$$

□

Finally, following Gordin and Liverani's method, one can obtain various versions of the central limit theorem (CLT). Here is the simplest one.

Theorem 6.2. *Let $\psi \in \mathcal{H}_\beta(\mathcal{J}) \cap L^\infty(\mathcal{J})$ such that $\int_{\mathcal{J}_x} \psi_x d\mu_x = 0$, $x \in X$. If ψ is not cohomologous to 0, then there exists $\sigma > 0$ such that, for every $t \in \mathbb{R}$,*

$$\mu \left(\{ z \in \mathcal{J} \, ; \, \frac{1}{\sqrt{n}} S_n \psi(z) \leq t \} \right) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^t exp(-u^2/2\sigma^2) \, du$$

Proof. The dual operator $U^*_x : L^2(\mathcal{J}_x, \mu_x) \to L^2(\mathcal{J}_{\theta(x)}, \mu_{\theta(x)})$ of the Koopman operator $U_x \psi_x = \psi_x \circ f_x$ is given by

$$U^*_x \psi_x = \frac{1}{\rho_\theta(x)} \hat{\mathcal{L}}_x (\rho_x \psi_x) \, .$$

By Gordin's result [10] it suffices to check that $\sum_k \|U^k U^{*k} \psi\|_{L^2(\mu)} < \infty$. We have

$$\|U^k U^{*k} \psi\|^2_{L^2(\mu)} = \int_{\mathcal{J}} (U^{*k} \psi)^2 \circ f^k \, d\mu = \int_{\mathcal{J}} (U^{*k} \psi)^2 \, d\mu$$

by invariance of the measure μ. Therefore,

$$\|U^k U^{*k}\psi\|^2_{L^2(\mu)} = \int_{\mathcal{J}} \psi\, U^k U^{*k}\psi\, d\mu \le \|\psi\|_\infty \int_{\mathcal{J}} |U^{*k}\psi| \circ f^k\, d\mu = \|\psi\|_\infty \int_{\mathcal{J}} |U^{*k}\psi|\, d\mu$$

by the same argument. Now,

$$\begin{aligned}\int_{\mathcal{J}} |U^{*k}\psi|\, d\mu &= \int_X \int_{\mathcal{J}_x} \frac{1}{\rho_x} \left|\hat{\mathcal{L}}_{\theta^{-k}(x)}(\rho_{\theta^{-k}(x)}\psi_{\theta^{-k}(x)})\right| d\mu_x dm \\ &= \int_X \int_{\mathcal{J}_x} \left|\hat{\mathcal{L}}_{\theta^{-k}(x)}(\rho_{\theta^{-k}(x)}\psi_{\theta^{-k}(x)})\right| d\nu_x dm\,.\end{aligned}$$

The fibers of ψ having μ_x–integral zero, it follows from Theorem 5.1 (2) as in the preceding proof that, for some constant $b > 0$,

$$\int_{\mathcal{J}} |U^{*k}\psi|\, d\mu \le b\vartheta^k \int_X \|\psi_{\theta^{-k}(x)}\|_\beta dm = b\vartheta^k |\psi|_\beta\,.$$

In conclusion,

$$\|U^k U^{*k}\psi\|^2_{L^2(\mu)} \le b\vartheta^k |\psi|_\beta \|\psi\|_\infty$$

which directly implies Gordin's L^2–summability condition. □

7. Appendix: Facts from Nevanlinna Theory and uniform bounds of transfer operators

7.1. **FMT and proof of Proposition 2.3.** The goal here is to establish the uniform bounds of the transfer operators claimed in Proposition 2.3. These bounds can be established by employing Nevanlinna's theory of value distribution similar to what we did in [18, 19]. The main tool we use is Nevalinna's first main theorem (FMT) which we now describe briefly. There are several complete accounts of it in the literature, for example in [20, 21, 4, 9].

The theory of value distribution of a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ relies on some naturally to f associated functions for which we use standard notations. For example, $n(r, w)$ or $n_f(r, w)$ is used for the counting function which designs the number of w–points (counted with multiplicity) of modulus at most r. The average or integrated counting number $N(r, w)$ is related to $n(r, w)$ by $dN(r, w)/dr = n(r, w)/r$.

Concerning the characteristic function $\mathring{T}(r) = \mathring{T}_f(r)$ of f, we use the Ahlfors-Shimizu spherical version of it which measures the average covering number of the Riemann sphere of the restriction of f to the disk of radius r:

$$\mathring{T}(r) = \int_0^r \left(\frac{1}{\pi}\int\int_{|x+iy|\le t} \frac{|f'|^2}{(1+|f|^2)^2} dxdy\right)\frac{dt}{t} = \int_0^r A_f(t)\frac{dt}{t}\,. \tag{7.1}$$

The exponential growth of this function determines the order $\rho(f)$ of f since we have

$$\rho(f) = \limsup_{r\to\infty} \frac{\mathring{T}(r)}{r}\,.$$

Nevanlinna's first main theorem (FMT) as stated in [7] (see also [4, 9]) yields:

Theorem 7.1. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic of finite order. Then, with the notations above,*

$$N(r,w) \le \mathring{T}(r) + \log \frac{1}{[f(0),w]} \quad \textit{for every } \ w \in \hat{\mathbb{C}} \textit{ and } r > 0$$

where $[a,b]$ denotes the chordal distance on the Riemann sphere (with, in particular, $[a,b] \le 1$, $a, b \in \hat{\mathbb{C}}$).

Proof of Proposition 2.3. Remember first that we have the normalization Condition 2 and thus (1.1): $0 \in \mathcal{J}_x$ and $|f_x(0)| \le T$, $x \in X$.

Secondly, by (2.2) along with the Remark 2.2,

$$\mathcal{L}_x \mathbb{1}(w) \le \frac{\kappa^t}{(1+|w|)^{(\alpha_2-\tau)t}} \sum_{f_x(z)=w} \big(1+|z|\big)^{-t\hat{\tau}} \quad \text{for every} \quad w \in \mathcal{V}_{\delta_0}(\mathcal{J}_{\theta(x)}) \, .$$

Combined with the distortion Lemma 2.6 it follows that the required estimations follow if there exists $C > 0$ such that

$$\sum_{f_x(z)=w} \big(1+|z|\big)^{-t\hat{\tau}} \le C \quad \text{for every} \quad w \in \mathcal{V}_{\delta_0}(\mathcal{J}_{\theta(x)}) \setminus \mathbb{D}(f_x(0), \delta_0/2) \, , \ \ x \in X \, .$$

Observe that

$$\sum_{f_x(z)=w} \big(1+|z|\big)^{-t\hat{\tau}} \le \sum_{f_x(z)=w} \max\{1,|z|\}^{-t\hat{\tau}} = n_{f_x}(1,w) + \sum_{f_x(z)=w,\, |z|>1} |z|^{-t\hat{\tau}} \, .$$

The second term can be treated by means of two integrations by part and an application of Theorem 7.1 (this is completely standard, compare also [19, p.16]):

$$\begin{aligned} \sum_{\substack{f_x(z)=w \\ |z|>1}} |z|^{-t\hat{\tau}} &= \int_1^\infty \frac{d\, n_{f_x}(r,w)}{r^{t\hat{\tau}}} = -n_{f_x}(1,w) + t\hat{\tau} \int_1^\infty \frac{n_{f_x}(r,w)}{r^{t\hat{\tau}+1}} dr \\ &\le -n_{f_x}(1,w) - t\hat{\tau} N_{f_x}(1,w) + (t\hat{\tau})^2 \int_1^\infty \frac{N_{f_x}(r,w)}{r^{t\hat{\tau}+1}} dr \\ &\le -n_{f_x}(1,w) + (t\hat{\tau})^2 \int_1^\infty \frac{\mathring{T}_{f_x}(r)}{r^{t\hat{\tau}+1}} dr + (t\hat{\tau})^2 \log \frac{1}{[f(0),w]} \int_1^\infty \frac{dr}{r^{t\hat{\tau}+1}} \, . \end{aligned}$$

Since $\mathring{T}_{f_x}(r) \le C_\rho r^\rho$ (Condition 1),

$$\sum_{f_x(z)=w} \big(1+|z|\big)^{-t\hat{\tau}} \le (t\hat{\tau})^2 \frac{C_\rho}{\hat{\tau}t - \rho} + \hat{\tau} t \log \frac{1}{[f(0),w]} \, .$$

The second term is uniformly bounded since we assumed $|w - f_x(0)| \ge \delta_0/2$ and since we know that $|f_x(0)| \le T$. The proof is complete.

□

7.2. **Uniform second main theorem (SMT).** Our construction of conformal measures relies on the SMT of Nevanlinna along with good estimates of the error term appearing in it. The later has been extensively studied in the 80's and 90's and the book [4] by Cherry and Ye is an excellent reference for this topic. In particular, Chapter 2 of this book fits perfectly well to what we are doing. The following result is a straightforward adaption of a particular case of Theorem 2.8.5 in [4]. We use here and throughout the whole section the notations of this book.

Theorem 7.2. *Let $L \geq 1$ and set $b_1 = b_1(L) = e(1 + (Le^e)^2)$ and $r_0 = r_0(L) = Le^e$. Let $\rho > 0$ and $C_\rho > 0$. Then, for every non–constant meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ and every three distinct points $a_1, a_2, a_3 \in \hat{\mathbb{C}}$ verifying*

(1) $\dfrac{1}{L} \leq f^\#(0) = \dfrac{|f'(0)|}{1 + |f(0)|^2} \leq L$,
(2) $f(0) \notin \{a_1, a_2, a_3\}$ *and*
(3) $\mathring{T}_f(r) \leq C_\rho r^\rho$, $r > 0$, *the following holds:*

$$\sum_{j=1}^{3} N_f(a_j, r) \geq \mathring{T}_f(r) - S(r, a_1, a_2, a_3) \quad \text{for every} \quad r \geq r_0$$

where

$$\begin{aligned} S(r, a_1, a_2, a_3) &= 2\log(108 + 18\log 2) + \frac{1}{2}\log b_1 + 1 + 4\log \mathring{T}_f(r) \\ &\quad + \left(\frac{3}{2}(\rho - 1) + \frac{1}{2}\right)\log r + \log L + \mathring{D}(a_1, a_2, a_3) \\ &\leq b_6 + 6\rho \log r + \mathring{D}(a_1, a_2, a_3)\,, \end{aligned}$$

$\mathring{D}(a_1, a_2, a_3) = -\log \prod_{i \neq j} [a_i, a_j] + 2\log 2$, $[a_i, a_j]$ *being the chordal distance, and where the constant b_6 does depend on L, C_ρ only.*

This, in fact uniform, version of the SMT deserves some comments.

First of all, the radius r_0 normally depends on the function f since it is chosen in order to have $\mathring{T}_f(r) \geq e$. However, as it is explained in Proposition 2.8.1 of [4], if f is any meromorphic function with

$$f^\#(0) \geq \frac{1}{L} \tag{7.2}$$

then $\mathring{T}_f(r) \geq \log r - \log L$. Consequently, given $L \geq 1$, there exists $r_0 = r_0(L)$ such that the above SMT does hold for every f that satisfies (7.2). Inspecting the proof of Proposition 2.8.1 of [4] gives the precise number r_0 indicated in the above theorem.

Various formulations of the SMT and especially the ones in Chapter 2 of [4] involve two functions, a *Khinchin function* ψ and an auxiliary function ϕ. Their role is to optimize the error term $S(r, a_1, a_2, a_3)$ often by the cost of a larger *exceptional set*

E, i.e. set of radii $r \geq r_0$ such that SMT does only hold if $r \notin E$ and this set satisfies

$$\int_E \frac{dr}{\phi(r)} \leq 2k_0(\psi) = 2\int_e^\infty \frac{dx}{x\psi(x)}\,. \tag{7.3}$$

For our application we do not care about a minimal error term and thus we did a more or less arbitrary chose $\psi(x) = x$. We equally well could have made Nevanlinna's choice $\psi(x) = (\log x)^{1+\varepsilon}$. But our choice leads to a nicer expression of the error term.

The choice of ϕ is more subtle since we need the SMT estimation for every $r \geq r_0$. A precise argument how to remove the exceptional set is in Nevanlinna's book [20, p. 257] and it is only possible since we deal with functions that have finite order. Indeed, the assumption (3) implies that the order $\rho(f) \leq \rho$ and that the variation of the characteristic function is bounded in the following way. From the definition of $\mathring{T}_f$ in (7.1) follows that $A_f(r) \leq \int_r^{er} A_f(t)\frac{dt}{t} \leq \mathring{T}_f(er) \leq C_\rho(er)^\rho$. Therefore, if $r_0 \leq r_1 < r_2$ then

$$\mathring{T}_f(r_2) - \mathring{T}_f(r_1) = \int_{r_1}^{r_2} A_f(t)\frac{dt}{t} \leq C_\rho e^\rho \rho^{-1}\left(r_2^\rho - r_1^\rho\right)\,.$$

Choose now, anf that's what we did in the above SMT, the function $\phi(r) = r^{-(\rho-1)}$. If the interval $(r_1, r_2) \subset E$ then it results from (7.3) that this variation is bounded

$$\mathring{T}_f(r_2) - \mathring{T}_f(r_1) \leq C_\rho e^\rho 2k_0(\psi)$$

and from this it is not hard to see how to remove the exceptional set.

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