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OWP 2015 - 09

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

Discrete Translates in $L^p(\mathbb{R})$

Alexander Olevskii and Alexander Ulanovskii

Abstract

A set Λ is called $p-$spectral if there is a function $g \in L^p(\mathbb{R})$ such that all $\Lambda-$translates $\{g(t-\lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$. We prove that exponentially small non-zero perturbations of the integers are $p-$spectral for all $p > 1$.

1 Introduction

1. Spectral sets. In what follows we will use the standard form of Fourier transform

$$f(x) = \hat{F}(x) := \int_{\mathbb{R}} e^{-2\pi i t x} F(t)\, dt, \ \ F \in L^2(\mathbb{R}).$$

Classical Wiener's Tauberian theorems provide necessary and sufficient condition on a function $g = \hat{G}$ whose translates $\{g(t-s), s \in \mathbb{R}\}$ span the space $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$:

(i) The translates of $g \in L^1(\mathbb{R})$ span $L^1(\mathbb{R})$ if and only if G does not vanish;

(ii) The translates of $g \in L^2(\mathbb{R})$ span $L^2(\mathbb{R})$ if and only if G is non-zero almost everywhere on $\mathbb{R}$.

There is no similar result for $1 < p < 2$, since the spanning property of the translates of $g \in L^p(\mathbb{R})$ cannot be expressed in terms of the zero set of G, see [LO11].

It is well–known that sometimes even discrete set of translates may span $L^p(\mathbb{R})$.

Definition. We say that a discrete set $\Lambda \subset \mathbb{R}$ is $p-$*spectral* if there is a function $g \in L^p(\mathbb{R})$ such that the family of translates

$$\{g(t-\lambda), \lambda \in \Lambda\}$$

spans $L^p(\mathbb{R})$. Such a function g is called *a $\Lambda-$generator*.

A natural question is *Which discrete sets Λ are $p-$spectral?*

We will present a brief account of known results in the area.

A simple result ([Bl06]) shows that if Λ is $p-$spectral, then it is $p'-$spectral, for every $p' > p$. As we will now demonstrate, the results are indeed very different for different values of p.

2. The case $\mathbf{p = 2}$. Recall that the Fourier transform is a unitary operator in $L^2(\mathbb{R})$. So, one may see that Λ is $2-$spectral if and only if there exists $G \in L^2(\mathbb{R})$ such that the system $\{G(t)e^{i\lambda t}, \lambda \in \Lambda\}$ spans the whole space $L^2(\mathbb{R})$. Using this, one may easily check that the set of integers $\Lambda = \mathbb{Z}$ is not $2-$spectral.

On the other hand, small perturbations of $\mathbb{Z}$ are $2-$spectral. More precisely, we call a set

$$\Lambda = \{\lambda_n := n + a_n, n \in \mathbb{Z}\} \tag{1}$$

an *almost integer set*, if the "perturbations" a_n satisfy

$$a_n \neq 0, \text{ for all } \ n \in \mathbb{Z}; \quad a_n \to 0, \quad |n| \to \infty.$$

Theorem 1 ([O97]) *Every almost integer set Λ is $2-$spectral.*

Observe that to obtain a completeness spectrum for $p = 2$, one does not need to perturb all integers. Even a sparse subset is suffice, see details in [NO07].

Let us say that Λ in (1) is an *exponentially small perturbation of the integers*, if a_n tend to zero exponentially fast:

$$0 < |a_n| < Cr^{|n|}, \ n \in \mathbb{Z}; \quad \text{for some } C > 0, 0 < r < 1. \tag{2}$$

Such sets appeared in [Ul01] in connection with completeness property of exponential systems on large sets. In [OU04] we show that each exponentially small perturbation of the integers admits a "nice" $\Lambda-$generator in $L^2(\mathbb{R})$, that is a Schwartz function, $g \in \mathcal{S}(\mathbb{R})$. We observe that not every almost integer set admits one, see [OU04] (This result may seem surprising, since when (2) holds, we are "closer" to the limiting case $\Lambda = \mathbb{Z}$ when no generator exists).

3. The case $\mathbf{p = 1}$. This is the only case where a complete description of spectral sets is known. The L^1 case is "easier" because every $\Lambda-$generator g is integrable, and G is a non-vanishing continuous function. This makes it possible to reduce the problem to the classical problem of completeness of the exponential system on intervals.

Given a discrete set Λ, let $R(\Lambda)$ denote the completeness radius of the exponential system $\{e^{i\lambda t}, \lambda \in \Lambda\}$, i.e. the supremum over all $R > 0$ such that the system is complete in $L^2(-R, R)$, where one sets $R(\Lambda) = 0$ if no such R exists. In particular, condition $R(\Lambda) = \infty$ means that the exponential system $\{e^{i\lambda t}, \lambda \in \Lambda\}$ is complete in $L^2(-R, R)$, for every $R > 0$.

Theorem 2 ([BOU06]) *Λ is $1-$spectral if and only if $R(\Lambda) = \infty$.*

We remark that the classical results of Beurling and Malliavin [BM67] states that the value $R(\Lambda)$ can be expressed in terms of a certain density of Λ.

Let us say that a set $\Lambda \subset \mathbb{R}$ is uniformly discrete (u.d.), if

$$\inf_{\lambda,\lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.$$

It is well-known that $R(\Lambda) < \infty$, for every u.d. set Λ. In particular, $R(\mathbb{Z}) = 1$. Theorem 2 shows that no u.d. set Λ is $p-$spectral for $p = 1$.

4. The case $\mathbf{p > 2}$. The Fourier transform of functions from $L^p(\mathbb{R})$ for $p > 2$ are in general temperate distributions. Almost integer sets Λ remain to be $p-$spectral for all $p > 2$. However, in a sharp difference with the case $p = 2$ we have

Theorem 3 ([AO96]) *The set of integers $\mathbb{Z}$ is $p-$spectral, for every $p > 2$.*

5. The case $\mathbf{1 < p < 2}$. This case is less investigated. In particular, it has been an open question if there exist u.d. spectral sets. The main result of this paper shows that exponentially small perturbations of the integers are $p-$spectral for $p > 1$:

Theorem 4. *Every set Λ satisfying (1) and (2) is $p-$spectral, for every $p > 1$.*

Our proof of Theorem 4 is based on a uniqueness theorem for tempered distributions.

2 Tempered Distributions with Deep Zeros

1. Notations. Set $e_a(x) := \exp(-2\pi i a x)$. Let $\|f\|_p$ denote the L^p-norm of a function f and

$$(g * f)(x) := \int_{\mathbb{R}} g(x-s) f(s)\, ds$$

the usual convolution. Further, $S_d, d > 0$, denotes the subspace of the Schwartz space $\mathcal{S}(\mathbb{R})$ of functions vanishing for $|x| \geq d$. Given a tempered distribution $F \in \mathcal{S}'(\mathbb{R})$ we denote by $\langle F, G \rangle$ the action of F on $G \in \mathcal{S}(\mathbb{R})$. Finally, throughout the rest of this note we denote by C different positive absolute constants.

2. Class K. Let us say that a continuous function H defined on $\mathbb{R}$ has an exponentially deep zero at a point a if

$$|H(t)| \leq Ce^{-\frac{C}{|t-a|}},\ t \in \mathbb{R},$$

and it has an exponentially deep zero at ∞ if

$$|H(t)| \leq Ce^{-C|t|},\ \ t \in \mathbb{R},$$

We will consider functions $H \in \mathcal{S}(\mathbb{R})$ whose every derivative has an exponentially deep zero at each integer point and at infinity:

$$|H^{(k)}(t)| \leq Ce^{-C|t| - \frac{C}{\rho(t,\mathbb{Z})}},\ t \in \mathbb{R},\ k = 0, 1, 2, ..., \tag{3}$$

where $\rho(t, \mathbb{Z}) = \min_{n\in\mathbb{Z}} |t-n|$ is the distance from t to $\mathbb{Z}$.

Denote by K the class of all distributions HF, where H satisfies (3) and $F \in \mathcal{S}'(\mathbb{R})$ is a tempered distribution. We also assume that H satisfies $H(-t) = \bar{H}(t), t \geq 0$, so that the function $h(x) = \hat{H}(x)$ is real for $x \in \mathbb{R}$. Let $\hat{K}$ denote the class of Fourier transforms $g = \hat{G}$, $G \in K$.

Recall that every tempered distribution $F \in \mathcal{S}'(\mathbb{R})$ is the distributional derivative of finite order, $F = D^{(k)}H$, of a continuous function $H(t)$ having at most polynomial growth on the real axis (see [FJ98], Theorem 8.3.1). Using this fact and (3), one may easily get the following

Lemma 1. *(i) For every $G \in K$ there exists $k \geq 0$ such that the inequality*

$$|\langle G, \Phi(t-n) \rangle| \leq C\|\Phi^{(k)}\|_\infty e^{-C|n| - \frac{C}{d}}, n \in \mathbb{Z}. \tag{4}$$

holds for every $\Phi \in S_d$.

(ii) Every $g \in \hat{K}$ *admits analytic continuation into some strip* $\{x + iy : |y| < \mathbb{C}\}$, *and there exits* $k = k(g) \geq 0$ *such that*

$$|g(x)| \leq C(1 + |x|^k), \ x \in \mathbb{R}.$$

(iii) If $g(x) \in \hat{K}$, *then* $\Re g(x), \Im g(x) \in \hat{K}$ *and* $g' \in \hat{K}$.

3 A Uniqueness Theorem for Distributions

We say that Λ is a uniqueness set for a space M of continuous functions, if

$$\varphi, \psi \in M, \varphi(\lambda) = \psi(\lambda), \lambda \in \Lambda \Rightarrow \varphi = \psi.$$

Theorem 5. *Every set* Λ *satisfying (1) and (2) is a uniqueness set for* $\hat{K}$.

1. The proof is based on

Main Lemma. *Assume* $G \in K$ *and* Λ *satisfies (1) and (2). If* $g = 0$ *on* Λ, *then* $g(n) = 0, n \in \mathbb{Z}$.

Proof. Take any function $g \in \hat{K}$ and consider the function

$$R(t) := \sum_{n \in \mathbb{Z}} g(n) e^{2\pi i n t}, \ t \in \mathbb{R}.$$

To prove the lemma, it suffices to show that $R(t)$ has the properties:

(A) $R(t)$ admits analytic continuation to the strip

$$\{|\Im z| < C \log(1/r)\}, \ z \in \mathbb{C},$$

where $0 < r < 1$ is the number in (2);

(B) $R(t)$ has a zero of infinite order at the origin.

By Lemma 1, g' has at most polynomial growth. So, since g vanishes on Λ, it follows from (1) and (2) that

$$|g(n)| < C|n|^C r^{|n|}, n \in \mathbb{Z}.$$

This proves (A).

Fix any function $\Phi \in S_d, d < 1$. Choose a large even integer N so that the function $g_\epsilon := gh_\epsilon$ is integrable on the real axis, where

$$h_\epsilon(x) := \left(\frac{\sin(2\pi\epsilon x)}{2\pi\epsilon x}\right)^N.$$

It is easy to see that its inverse Fourier transform is

$$H_\epsilon(t) = \left(\frac{1}{2\epsilon}1_\epsilon(t)\right)^{N*},$$

where 1_ϵ is the indicator function of $[-\epsilon, \epsilon]$. Hence, $H(t) \geq 0, t \in \mathbb{R}$, and so

$$\|H_\epsilon\|_1 = h_\epsilon(0) = 1.$$

This gives

$$\|(\Phi * H_\epsilon)^{(k)}\|_\infty \leq \|\Phi^{(k)}\|_\infty \|H_\epsilon\|_1 = \|\Phi^{(k)}\|_\infty.$$

It is also easy to see that $H_\epsilon * \Phi \in S_{d+N\epsilon}$. Therefore, by (4),

$$|\langle G_\epsilon, \Phi(t-n)\rangle| = |\langle g_\epsilon, e_n\varphi\rangle| = |\langle g, e_n\varphi h_\epsilon\rangle| = |\langle G, (\Phi * H_\epsilon)(t-n)\rangle|$$
$$\leq C\|\Phi^{(k)}\|_\infty e^{-C|n|-\frac{C}{N\epsilon+d}},\ n \in \mathbb{Z}. \tag{5}$$

Set

$$R_\epsilon(t) := \sum_{n\in\mathbb{Z}} g_\epsilon(n)e^{2\pi int}.$$

Let us calculate the product $\langle R, \Phi\rangle$. By the Poisson formula, we have

$$R_\epsilon(t) = \sum_{n\in\mathbb{Z}} g_\epsilon(n)e^{2\pi int} = \sum_{n\in\mathbb{Z}} G_\epsilon(t+n).$$

Therefore, by (5),

$$|\langle R_\epsilon, \Phi\rangle| = |\langle \sum_{n\in\mathbb{Z}} G_\epsilon(t+n), \Phi(t)\rangle| = |\langle G_\epsilon(t), \sum_{n\in\mathbb{Z}} \Phi(t-n)\rangle|$$
$$\leq C\|\Phi^{(k)}\|_\infty e^{-\frac{C}{N\epsilon+d}}.$$

Letting $\epsilon \to 0$, we conclude that

$$|\langle R, \Phi\rangle| \leq C\|\Phi^{(k)}\|_\infty e^{-\frac{C}{d}}, \quad \varphi = \hat{\Phi} \in S_d(\mathbb{R}). \tag{6}$$

Now, the Main Lemma follows from

Lemma 2. *Condition (6) implies property (B).*

The proof of Lemma 2 is standard, and we omit it.

2. Proof of Theorem 5. Write $g = g_r + ig_i$, where $g_r(x) := \Re g(x)$ and $g_i(x) := \Im g(x)$. Then $g_r(x)$ and $g_i(x)$ are analytic, real for real x, vanish on Λ, and by Lemma 1, $g_r, g_i \in \hat{K}$. It follows from the Main Lemma that g_r and g_i vanish on $\mathbb{Z}$.

Let us show that $g_r = 0$. Since g_r vanishes both on $\mathbb{Z}$ and on $\Lambda = \{n + r_n, n \in \mathbb{Z}\}, r_n \neq 0$, we see that g_r' vanishes on some set $\Lambda_1 := \{n + r_1^{(1)}\}$, where each point $r_n^{(1)}$ lies inside the open interval between 0 and r_n. Since Λ satisfies (1) and (2), so does Λ_1. By Lemma 1, we have $g_r' \in \hat{K}$. We may now apply the Main Lemma above to this function to get $g_r'(n) = 0, n \in \mathbb{Z}$. Using this argument j times, we prove that $g_r^{(j)}$ vanishes on $\mathbb{Z}$, for all $j \in \mathbb{N}$. Since g_r is analytic, then $g_r = 0$. Similarly, we prove that $g_i = 0$. Hence, $g = 0$.

4 Proof of Theorem 4

Choose any function $G \in \mathcal{S}(\mathbb{R})$ satisfying (3). We may assume also that $G(-t) = G(t) > 0$ for $t \notin \mathbb{Z}$. Let Λ satisfy (1) and (2).

Suppose the set of translates $\{g(x - \lambda), \lambda \in \Lambda\}$ does not span $L^p(\mathbb{R})$. In this case there is a non-trivial function $f \in L^{p'}(\mathbb{R}), 1/p + 1/p' = 1$, such that $(g * f)(\lambda) = 0, \lambda \in \Lambda$. Clearly, $f = \hat{F}$ for some $F \in \mathcal{S}'(\mathbb{R})$ and so $g * f = \hat{GF} \in \hat{K}$. Theorem 5 yields $g * f = 0$. This means that all translates $\{g(x - s), s \in \mathbb{R}\}$ do not span the space $L^p(\mathbb{R})$. However, this contradicts to Beurling's theorem (see [B51]), which states that if $g = \hat{G} \in L^p(\mathbb{R}) \cap L^1(\mathbb{R}), 1 < p < 2$, is such that the Hausdorff measure of the zero set of G is zero, then all translates of g span the space $L^p(\mathbb{R})$. Theorem 4 is proved.

5 Acknowledgments

This research was supported through the program "Research in Pairs" by the Mathematisches Forschungsinstitut Oberwolfach in April 2014. The authors would like to thank the MFO for providing a stimulating and pleasant atmosphere.

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