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RATIONAL APPROXIMATION ON PRODUCTS OF PLANAR DOMAINS

JAVIER FALCÓ AND VASSILI NESTORIDIS

Dedicated to Professor Manuel Maestre on the occasion of his 60th birthday.

ABSTRACT. We consider $A(\Omega)$, the Banach space of functions f from $\overline{\Omega} = \prod_{i \in I} \overline{U_i}$ to $\mathbb{C}$ that are continuous with respect to the product topology and separately holomorphic, where I is an arbitrary set and U_i are planar domains of some type. We show that finite sums of finite products of rational functions of one variable with prescribed poles off $\overline{U_i}$ are uniformly dense in $A(\Omega)$. This generalizes previous results where $U_i = \mathbb{D}$ is the open unit disc in $\mathbb{C}$ or $\overline{U_i}^c$ is connected.

1. INTRODUCTION

In [5] the set of uniform limits of polynomials (of finite type) on $\overline{\mathbb{D}}^I$, where $\overline{\mathbb{D}}$ denotes the closed unit disc in $\mathbb{C}$ and I is an arbitrary set was investigated. This set coincides with the set of functions $f : \overline{\mathbb{D}}^I \mapsto \mathbb{C}$ which are continuous with respect to the product topology on $\overline{\mathbb{D}}^I$ and separately holomorphic in $\mathbb{D}^I$ and is denoted by $A(\mathbb{D}^I)$. The space $A(\mathbb{D}^I)$ with respect to the supremum norm is a Banach algebra.

In [4], among other results, the above result was extended to the case where $\overline{\mathbb{D}}^I$ was replaced by $\Omega = \prod_{i \in I} \overline{U_i}$, where each U_i is a bounded planar domain satisfying $int(\overline{U_i}) = U_i$ and such that the complement of $\overline{U_i}$ is connected.

Now we consider the more general case where the complement of $\overline{U_i}$ has a finite number of connected components. In this case, approximating polynomials will be necessarily replaced by approximating rational functions. Gamelin and Garnett [3] were among the first to study approximation by rational functions of several complex variables. Here we improve their result, [3, Corollary 6.4], by showing that the approximating functions can be chosen to be finite sums of finite products of rational functions of one variable with prescribed poles, one in each complementary component of $\overline{U_i}$. This continues with the study of approximation of holomorphic functions in the spirit of Mergelyan. The expository papers [9] and [10] can be used for references and background of the study of uniform approximation by holomorphic functions on compact sets in spaces of one or more complex variables.

Our method will be to consider Cauchy transforms and Laurent decomposition in several variables, see [1]. First we treat the case where I is a finite set, and

thus, we are working in $\mathbb{C}^m$. For the general case where I is an arbitrary set we can approximate our functions by functions depending on finitely many complex variables taking advantage of the continuity of functions in the product topology, see [5]. Then, we can use the previous case.

These ideas connect the theory of rational approximation in finitely many variables with the theory of rational approximation in infinite dimension holomorphy. Even more, the approximating functions can be considered as functions depending on finitely many complex variables. Therefore, the techniques and results established for functions defined on domains in $\mathbb{C}^m$ can be applied to the approximating functions that form a dense set in $A(\prod_{i\in I} \overline{U_i})$.

With this approach, it is possible to study properties of infinite dimensional holomorphic functions from a different point of view. For more information on the theory of infinite dimensional holomorphy we recommend [2, 6, 7].

2. Notation and terminology

Let U_i, $i \in I$, be a family of bounded planar domains. We consider the products $\Omega = \prod_{i\in I} U_i$ and $\overline{\Omega} = \prod_{i\in I} \overline{U_i}$ endowed with the product topology. Thus, $\overline{\Omega}$ is a compact space by Tychonoff's theorem.

Definition 2.1. A function $f : \overline{\Omega} \mapsto \mathbb{C}$ belongs to the class $A(\Omega)$ if it is continuous on $\overline{\Omega}$ with respect the product topology and separately holomorphic on Ω. By separately holomorphic we mean that if all coordinates but one are fixed, say the coordinate i_0, then the restriction of f is a holomorphic function of the variable z_{i_0} in U_{i_0}. We endow the space $A(\Omega)$ with the supremum norm.

Remark 2.2. In Definition 2.1 when we distinguish the coordinate i_0 from the other coordinates i, the complex number z_i is fixed in the closure of U_i; in particular, it may belong to the boundary of U_i as well.

Notice that every function in $A(\Omega)$ is a continuous function on a compact set. Hence the functions in $A(\Omega)$ are bounded.

Remark 2.3. The space $A(\Omega)$ endowed with the supremum norm (on Ω or equivalently on $\overline{\Omega}$) is a Banach Algebra as is easily verified. It contains the functions which are finite sums of finite products of rational functions of one variable with poles in the complement in the Riemann Sphere of the closure of U_i. We shall show that the latter set of functions is in fact dense in $A(\Omega)$.

Since we will be working with domains Ω that are products of domains $\{U_i\}_{i\in I}$ it will be useful to define the following projection maps.

Given a set $F \subset I$, denote by π_F the projection map on the set of coordinates F defined by

$$\pi_F : \quad \Omega \quad \longrightarrow \quad \prod_{i\in F} U_i$$
$$\{z_i\}_{i\in I} \quad \rightsquigarrow \quad \{z_i\}_{i\in F}.$$

For a fixed point $w \in \Omega$, let the map $\pi_{w,F} : \Omega \mapsto \Omega$ be defined as

$$\left(\pi_{w,F}(z)\right)_i = \begin{cases} w_i & \text{if } i \notin F, \\ z_i & \text{if } i \in F. \end{cases}$$

Let us also denote an inverse of the projection map of π_F at the point w by $\pi_{w,F}^{-1} : \prod_{i \in F} U_i \mapsto \Omega$, where

$$\left(\pi_{w,F}^{-1}(z) \right)_i = \begin{cases} w_i & \text{if } i \notin F, \\ z_i & \text{if } i \in F. \end{cases}$$

The previous functions π_F (resp. $\pi_{w,F}$ and $\pi_{w,F}^{-1}$) can also be considered as functions defined on the closure of Ω (resp. $\overline{\Omega}$ and $\prod_{i \in F} \overline{U_i}$) with values in $\prod_{i \in F} \overline{U_i}$ (resp. $\overline{\Omega}$ and $\overline{\Omega}$).

3. Holomorphic functions defined on a product of planar domains

In this section we consider Ω to be the product $\Omega = \prod_{i \in I} U_i$ of planar domains U_i, $i \in I$. In section 4 we will add the condition that $int(\overline{U_i}) = U_i$. We deal with functions $f : \Omega \mapsto \mathbb{C}$ which are separately holomorphic. We consider Ω endowed with the product topology of $\mathbb{C}^I$. We say that a function $f : \Omega \mapsto \mathbb{C}$ *depends on a particular set of variables* $F \subset I$ on an open set $U \subset \Omega$ if for all $z, z' \in U$ with $\pi_F(z_i) = \pi_F(z_i')$, then $f(z) = f(z')$. Suppose that in a neighborhood of some fixed point of Ω, f depends on a particular subset F of the variables. Is it true that f depends globally on the same set of variables F?

This question is answered in the affirmative in [4] for functions f in $A(\Omega)$, but without proof. Here we sketch the proof of the more general result, for continuous and separately holomorphic functions on Ω, for the sake of completeness.

Proposition 3.1. *Let $f : \Omega \mapsto \mathbb{C}$ be a continuous and separately holomorphic function. Then, the following assertions are equivalent:*

(a) *The function f depends on the set of variables $F \subset I$ on Ω,*
(b) *There exists a continuous function $g : \prod_{i \in F} U_i \mapsto \mathbb{C}$, g holomorphic on $\prod_{i \in F} U_i$ with $g = f \circ \pi_{w,F}^{-1}$ for any point $w \in \Omega$ and $f = g \circ \pi_F$,*
(c) *For every $i \notin F$, $\frac{\partial f}{\partial z_i}(z) = 0$ at every point z in Ω.*

Proof. If a function depends on a set of variables F in Ω, then by definition, (a) and (b) are equivalent.

To prove that (c) implies (a) notice that for any point $w \in \Omega$ the function $f \circ \pi_{w,I \backslash F}^{-1}$ is a holomorphic function on $\prod_{i \in I \backslash F} U_i$ all of whose partial derivatives are zero whenever $i \in I \backslash F$. Hence $f \circ \pi_{w,I \backslash F}^{-1}$ is constant on the subset of $\prod_{i \in I \backslash F} U_i$, consisting of all points differing from a given point at almost finitely many coordinates. Since this set is dense, by continuity it follows that $f \circ \pi_{w,I \backslash F}^{-1}$ is constant on Ω. Hence if w and z are elements of Ω with $w_i = z_i$ for all i in $I \backslash F$, we have that $f(w) = f(z)$. Therefore f only depends on the set of variables F on Ω.

To prove (a) implies (c) notice that for fixed $i \notin F$ and a point $x \in \Omega$ we have $f(x) = f(z)$ for every $x \in \Omega$ with $x_j = z_j$ for all $j \neq i$. Hence the partial derivative of f in the direction z_i is zero at any point z in Ω. $\square$

Theorem 3.2. *Let $f : \Omega \mapsto \mathbb{C}$ be a continuous and separately holomorphic function. If for a fixed point $w \in \Omega$, there exists a subset F of I such that the restriction of f only depends on the variables in F in a neighborhood $U \subseteq \Omega$ of w, then f depends only on this set of variables in Ω.*

Proof. If f depends on a set of variables F in a neighborhood U of a point w, then there exists a basic open set $B = \{z = \{z_i\}_{i \in I} : |z_i - w_i| < \delta \text{ for } i \in \tilde{I}\} \subseteq U$, with $\tilde{I}$ a finite set and $\delta > 0$, where $\frac{\partial f}{\partial z_i} = 0$ for every point in B and every $i \notin F$.

By Proposition 3.1 we only need to check that $\frac{\partial f}{\partial z_i} = 0$ for every point in Ω and every $i \notin F$. If $\tilde{I} = \{i_1, \ldots, i_n\}$ then for every $x \in \mathbb{C}$ and every $z \in B$ with $\|x - w_{i_n}\| < \delta$ we have that $\frac{\partial f \circ \pi_{z,\{i_n\}}^{-1}}{\partial z_{i_n}}(x) = 0$. Hence by the theory of one complex variable, since the derivative of the function $f \circ \pi_{z,\{i_n\}}$ is zero on an open set, we have that $f \circ \pi_{z,\{i_n\}}$ is constant in U_{i_n}. Therefore if we denote by $B_{n-1} = \{z = \{z_i\}_{i \in I} : |z_i - w_i| < \delta \text{ for } i \in \tilde{I} \setminus \{i_n\}\}$ we have that $\frac{\partial f}{\partial z_i} = 0$ for every point z in B_{n-1} and every $i \notin F$. Repeating this argument n times we get that $\frac{\partial f}{\partial z_i} = 0$ for every point in Ω and every $i \notin F$. Therefore by Theorem 3.1, f only depends on the set of variables F in Ω. $\qquad\square$

4. Density of rational functions in the algebra of a product of finitely connected planar domains.

In this section we present our results about density of rational functions in $A(\Omega)$. In this section Ω is a product of planar domains U_i, $i \in I$. We will first assume that each U_i is defined by a finite set of disjoint rectifiable Jordan curves. Later we prove our approximation results under the more general assumption that the complement of $\overline{U_i}$ has a finite number of connected components. We will always assume that $int(\overline{U_i}) = U_i$.

The following lemma can be obtained as a consequence of the theory of one complex variable. The result is probably known but for completeness we present it here.

Lemma 4.1. *Let γ be a rectifiable Jordan curve in $\mathbb{C}$ and X be any metric space. Let $f : \gamma \times X \mapsto \mathbb{C}$ be a continuous function. For z in $\mathbb{C} \setminus \gamma$ and x in X we define*

$$F(z, x) = \frac{1}{2\pi i} \int_\gamma \frac{f(t, x)}{t - z} dt.$$

Then F is continuous on $(\mathbb{C} \setminus \gamma) \times X$. Even more, for every fixed x in X the function $F(\cdot, x)$ is holomorphic on $\mathbb{C} \setminus \gamma$ and its limit as z converges to infinity is zero.

Also, if f is bounded on $\gamma \times X$, then $\lim_{z \to \infty} \sup_{x \in X} |F(z, x)| = 0$.

In this case, F can be extended to be continuous on $((\mathbb{C} \cup \{\infty\}) \setminus \gamma) \times X$ and for every $x \in X$, $F(\cdot, x)$ can be extended to be holomorphic on $(\mathbb{C} \cup \{\infty\}) \setminus \gamma$.

Proof. First we show that F is continuous on $(\mathbb{C} \setminus \gamma) \times X$. Let (z_1, x_1) be an element of $(\mathbb{C} \setminus \gamma) \times X$ and U_1 a neighborhood of z_1 contained in $\mathbb{C} \setminus \gamma$. For every $z_2 \in U_1$ and every x_2 in X,

$$|F(z_1, x_1) - F(z_2, x_2)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(t, x_1)}{t - z_1} dt - \frac{1}{2\pi i} \int_\gamma \frac{f(t, x_2)}{t - z_2} dt \right|$$

$$\leq \left| \frac{1}{2\pi i} \int_\gamma \frac{f(t, x_1)}{t - z_1} - \frac{f(t, x_1)}{t - z_2} dt \right| + \left| \frac{1}{2\pi i} \int_\gamma \frac{f(t, x_1)}{t - z_2} - \frac{f(t, x_2)}{t - z_2} dt \right|.$$

Let $d := dist(z_1, \gamma)$. If z_2 is such that $|z_1 - z_2| < d/2$ we have that

$$\frac{|f(t, z_1)|}{|t - z_1||t - z_2|} \leq \frac{\sup_{t \in \gamma} |f(t, z_1)|}{d^2/2},$$

for all t in γ.

Hence,

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{f(t,x_1)}{t-z_1} - \frac{f(t,x_1)}{t-z_2} dt \right| \leq |z_1 - z_2| \frac{1}{2\pi} \int_\gamma \frac{|f(t,x_1)|}{|t-z_1||t-z_2|} dt$$

$$\leq |z_1 - z_2| \frac{\sup_{t\in\gamma} |f(t,z_1)|}{2\pi d^2/2} length(\gamma)$$

which converges to zero when z_2 converges to z_1.

Also, if $dist(z_1, z_2) < d/2$, then

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{f(t,x_1)}{t-z_2} - \frac{f(t,x_2)}{t-z_2} dt \right| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(t,x_1) - f(t,x_2)}{t-z_2} dt \right|$$

$$\leq \sup_{t\in\gamma} |f(t,x_1) - f(t,x_2)| \frac{length(\gamma)}{\pi d}.$$

To finish the proof of the continuity, we need to show that $\sup_{t\in\gamma} |f(t,x_1) - f(t,x_2)|$ converges to zero when x_2 converges to x_1. If this were not the case, we could find a number $\epsilon > 0$ and a sequence $\{(t_n, y_n)\} \subset \gamma \times X$ with $d(y_n, x_1) < 1/n$ and

$$(4.1) \qquad\qquad |f(t_n, y_n) - f(t_n, x_1)| > \epsilon$$

for all natural numbers n. By the compactness of γ, there exists a subsequence $\{t_{n_k}\}_{k=1}^\infty$ of $\{t_n\}_{n=1}^\infty$ convergent to a point $t_0 \in \gamma$.

By the continuity of f at the point (t_0, x_1) there exists a positive number δ such that if $|t - t_0| < \delta$ and $d(x, x_1) < \delta$ then $|f(t,x) - f(t_0, x_1)| < \epsilon$. But this is a contradiction with equation (4.1). Hence, $\sup_{t\in\gamma} |f(t,x_1) - f(t,x_2)|$ goes to zero when x_2 goes to x_1. Therefore the function F is continuous on $(\mathbb{C} \setminus \gamma) \times X$.

Now we show that for every fixed x in X the function $F(\cdot, x)$ is holomorphic on $\mathbb{C} \setminus \gamma$. Notice that for any point z in $\mathbb{C} \setminus \gamma$, and any w in $\mathbb{C} \setminus \gamma$ with $|z - w| < dist(z, \gamma)/2$,

$$F(w, x) = \frac{1}{2\pi i} \int_\gamma \frac{f(t,x)}{t-w} dt$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{1}{t-z} \frac{f(t,x)}{1 - \frac{w-z}{t-z}} dt$$

$$= \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^\infty (w-z)^n \frac{f(t,x)}{(t-z)^{n+1}} dt$$

$$= \sum_{n=0}^\infty \left(\frac{1}{2\pi i} \int_\gamma \frac{f(t,x)}{(t-z)^{n+1}} dt \right) (w-z)^n.$$

Therefore, $F(\cdot, x)$ is holomorphic on $\mathbb{C} \setminus \gamma$.

Also, since γ is compact and $f(\cdot, x)$ is continuous on γ, we have that $f(\cdot, x)$ is bounded on γ by some constant M_x. Therefore,

$$\lim_{z\to\infty} |F(z,x)| = \lim_{z\to\infty} \frac{1}{2\pi} \left| \int_\gamma \frac{f(t,x)}{t-z} dt \right| \leq \lim_{z\to\infty} \frac{M_x}{2\pi} \int_\gamma \left| \frac{1}{t-z} \right| dt = 0.$$

Also, for every $x \in X$, $F(\cdot, x)$ is holomorphic on $\mathbb{C} \setminus \gamma$ and continuous on $(\mathbb{C} \cup \{\infty\}) \setminus \gamma$. Hence, by the Riemman theorem on removable singularities, $F(\cdot, x)$ can be extended to be holomorphic on $(\mathbb{C} \cup \{\infty\}) \setminus \gamma$.

To finish, notice that if f is bounded by some constant M, then

$$\limsup_{z \to \infty} \sup_{x \in X} |F(z,x)| = \limsup_{z \to \infty} \sup_{x \in X} \frac{1}{2\pi} \left| \int_\gamma \frac{f(t,x)}{t-z} dt \right| \le \lim_{z \to \infty} \frac{M}{2\pi} \int_\gamma \left| \frac{1}{t-z} \right| dt = 0.$$

$\square$

Proposition 4.2. *Let U_i, $i = 1, \ldots, m$, be domains in $\mathbb{C} \cup \{\infty\}$ each one bounded by a finite set of disjoint rectifiable Jordan curves γ_i^j, $j = 1, \ldots, n_i$ with $\infty \notin \gamma_i^j$ for $j = 1, \ldots, n_i$, $i = 1 \ldots, m$. Let also α_i^j be an element of the interior of the component V_i^j of $(\mathbb{C} \cup \{\infty\}) \setminus U_i$ bounded by γ_i^j, $j = 1, \ldots, n_i$, $i = 1, \ldots, m$. If $\Omega = \prod_{i=1}^m U_i$ and $f \in A(\Omega)$ then f can be uniformly approximated on $\overline{\Omega}$ by functions which are finite sums of finite products of rational functions of one variable z_i with poles inside the set $\{\alpha_i^j : j = 1, \ldots, n_i\}$, $i = 1, \ldots, m$.*

Proof. We proceed by induction on the number $n = \prod_{i=1}^m n_i$. The least possible value for n is 1 and in this case $n_1 = \cdots = n_m = 1$. Hence, every U_i is bounded by only one Jordan curve and the fixed sets of points $\{\alpha_i^j\}$ contain only one point for each $i = 1, \ldots, m$. For simplicity we will denote the set $\{\alpha_i^j\}$ by $\{\alpha_i\}$ in this case.

If $\infty \notin U_i$ for each $i = 1, \ldots, m$, then according to [4] the function f can be approximated by polynomials of finite type. By Runge's theorem each monomial in one variable can be approximated by a rational function of the same variable with only one pole at $\{\alpha_i\}$, $i = 1, \ldots, m$, and the result holds.

For the case where $\infty \in U_i$ for one or more coordinates $\{r_1, \ldots, r_k\}$ we can consider a Mobius transformation T_{r_j} that sends α_{r_j} to ∞ and ∞ to zero. Then after applying this transformation to the domains U_i for the corresponding variables $\{r_1, \ldots, r_k\}$ we are led to the previous case where each domain does not contain ∞. Let us denote by $W_i = T_i(U_i)$ for $i = r_1, \ldots, r_k$, and by $W_i = U_i$ for $i \ne r_1, \ldots, r_k$. Obviously each function f in $A(U_1 \times \cdots \times U_m)$ can be naturally associated with a function in $A(W_1 \times \cdots \times W_m)$. Given a function $f \in A(U_1 \times \cdots \times U_m)$ we denote by $\tilde{f}$ the corresponding function defined on the transformed domains by composing f with the respective Mobius transformations on the variables with indices $r_1, \ldots, r_k$. This map is an isometric isomorphism between the spaces $A(U_1 \times \cdots \times U_m)$ and $A(W_1 \times \cdots \times W_m)$. By the previous case any transformed analytic function $\tilde{f}$ on this transformed domain can be approximated by a polynomial $\tilde{q} \in A(W_1 \times \cdots \times W_m)$ of finite type. Then these polynomials composed with the inverse of the previous Mobius transformations $T_{r_j}^{-1}$ on the corresponding variables with indices $r_1, \ldots, r_k$ give us a function $q \in A(U_1 \times \cdots \times U_m)$. This function q can be written as a finite sum of finite products of rational functions of one variable z_i with poles inside the set $\{\alpha_i\}$, $i = 1, \ldots, m$, and $\|g - f\| = \|\tilde{g} - \tilde{f}\|$. This concludes the case $n = 1$.

Let us now assume $k > 1$ and that our result is true for all $1 \le n < k$. We will show that the result holds for $n = k$. Without loss of generality we can assume that $n_1 > 1$. Thus, U_1 is bounded by n_1 rectifiable Jordan curves $\gamma^1, \ldots, \gamma^{n_1}$. We set $X = \prod_{i=2}^m U_i$. Then, for every $j = 1, \ldots, n_1$ the function f is continuous on $\gamma^j \times \overline{X}$. Thus, by Lemma 4.1, the function

$$F_j(z_1, x) = \frac{1}{2\pi i} \int_{\gamma^j} \frac{f(t,x)}{t - z_1} dt$$

is continuous on $\left((\mathbb{C} \cup \{\infty\}) \setminus \gamma^j \right) \times \overline{X}$ and holomorphic with respect to the variable z_1 in $\left((\mathbb{C} \cup \{\infty\}) \setminus \gamma^j \right) \times X$. Here we are assuming that the curves γ^j are oriented

in the appropriate direction. Also, if all but one of the variables $z_2, \ldots, z_n$ are fixed, using the power series expansion of $f(z_1, x)$ we can see that $F_j(z_1, x)$ is also holomorphic with respect to the variables $z_2, \ldots, z_n$ in $(\mathbb{C} \setminus \gamma^j) \times X$. Therefore, $F_j(z_1, x)$ is separately holomorphic on $\left((\mathbb{C} \cup \{\infty\}) \setminus \gamma^j\right) \times X$.

Furthermore, by the Cauchy formula in one variable, for every $x \in X$ we have

$$(4.2) \qquad f(z, x) = \sum_{j=1}^{n_1} F_j(z, x).$$

Let W_1^j be the connected component defined by γ^j on $(\mathbb{C} \cup \{\infty\}) \setminus \gamma^j$ on the Riemann sphere that contains U_1, for $j = 1, \ldots, n_1$. To prove that the functions F_j are in $A(W_1^j \times U_2 \cdots \times U_n)$ we only need to show that F_j is continuous on the boundary of $W_1^j \times U_2 \cdots \times U_n$, for $j = 1, \ldots, n_1$.

Given (z_1, x) in $\partial(W_1^j \times X)$, by equation (4.2) there exists a small neighborhood U of (z_1, x) where we can rewrite F_j as $F_j = f - \sum_{k \neq j} F_k$ for every point in $U \cap (\overline{W_1^j \times X})$. Then, f is continuous on $U \cap (\overline{W_1^j \times X})$ and F_k is holomorphic on U for $k \neq j$. Therefore F_j can be naturally extended to be continuous on $\overline{W_1^j \times X}$. We will continue to use the notation F_j for this extension. Thus, $F_j \in A(W_1^j \times X)$.

Notice that the number of connected components of $(\mathbb{C} \cup \{\infty\}) \setminus W_1^j$ is one for $j = 1, \ldots, n_1$. Therefore, since $\prod_{i=2}^m n_i < \prod_{i=1}^m n_i = n$ we can use the induction hypothesis for the functions F_j on $W_1^j \times U_2 \cdots \times U_n$, $j = 1, \ldots, n_1$. Hence F_j can be approximated by functions which are finite sums of finite products of rational functions of one variable z_i with poles inside the set $\{\alpha_i^j : j = 1, \ldots, n_i\}$ for $i = 1, \ldots, m$. Equation (4.2) yields the result for the function f and the proof is completed. $\qquad\square$

Next we consider the case where each U_i, $i = 1, \ldots, m$, is a bounded domain in $\mathbb{C}$ and has the properties that $int(\overline{U_i}) = U_i$ and $(\mathbb{C} \cup \{\infty\}) \setminus \overline{U_i}$ has finitely many components V_i^j, $j = 1, \ldots, n_i$, $i = 1, \ldots, m$.

Proposition 4.3. *Let U_i, $i = 1, \ldots, m$, be bounded planar domains such that $int(\overline{U_i}) = U_i$ and $(\mathbb{C} \cup \{\infty\}) \setminus \overline{U_i}$ has finitely many components V_i^j, $j = 1, \ldots, n_i$, $i = 1, \ldots, m$. Let $\alpha_i^j \in V_i^j$ be fixed. If $\Omega = \prod_{i=1}^m U_i$, then every function $f \in A(\Omega)$ can be uniformly approximated on $\overline{\Omega}$ by functions which are finite sums of finite products of rational functions of one variable z_i, $i = 1, \ldots, m$ with poles in the set $\{\alpha_i^j : j = 1, \ldots, n_i\}$, $i = 1, \ldots, m$.*

Proof. Let f be an analytic function on $\prod_{i=1}^m U_i$ which is continuous on the product of the closures of U_i, $i = 1, \ldots, m$. An extension of Mergelyan's theorem shows that for every compact planar set V whose complement $(\mathbb{C} \cup \{\infty\}) \setminus V$ has finitely many components, every function continuous on V and analytic in its interior is uniformly approximable by rational functions, see [8, Exercise 1, page 394]. Then by [3, Corollary 6.4] we obtain that f can be uniformly approximated on $\prod_{i=1}^m \overline{U_i}$ by rational functions in m variables that are analytic on an open set containing $\prod_{i=1}^m \overline{U_i}$. Let us fix one of these rational functions R. Each V_i^j is conformally equivalent to the open unit disc $\mathbb{D}$. We consider the Riemann mappings from each V_i^j to the unit disc. Thus, if $r < 1$ is close to one, the image by the Riemann mapping of $C(0, r) = \{z \in \mathbb{C} : |z| = r\}$ is a Jordan curve γ_i^j such that the bounded domains E_i, $i = 1, \ldots, m$, defined by γ_i^j, $j = 1 \ldots, n_i$, have the property that

$\prod_{i=1}^m \overline{U_i} \subset \prod_{i=1}^m \overline{E_i}$. Also the function R is analytic on $\prod_{i=1}^m \overline{E_i}$. Notice that the number of connected components of $\mathbb{C} \setminus \overline{E_i}$ is the same as the number of connected components of $\mathbb{C} \setminus \overline{U_i}$ and each α_i^j belongs to a different connected component of the complement of $\overline{E_i}$. Finally, every γ_i^j is an analytic Jordan curve, hence rectifiable. Thus, Proposition 4.2 implies that R can be approximated on $\prod_{i=1}^m \overline{E_i} \supseteq \prod_{i=1}^m \overline{U_i}$ by functions which are finite sums of finite products of rational functions of one variable z_i, $i = 1, \ldots, m$ with poles in the prescribed set $\{\alpha_i^j : j = 1, \ldots, n_i\}$. Therefore, by the triangle inequality the same approximation holds for f on $\prod_{i=1}^m \overline{U_i}$. $\qquad \square$

To prove the case in which Ω is an infinite product of domains, we will need the following lemma. This lemma states that even if a function in $A(\Omega)$ depends on an infinite number of variables, it can be approximated by a function depending only on a finite number of variables.

Lemma 4.4. *Let f be a function in $A(\Omega)$ and ϵ a positive number. Then, there exists a finite set $F \subseteq I$, such that*

$$\|f - f \circ \pi_{x,F}\| \leq \epsilon,$$

for every $x \in \Omega$.

The proof of this lemma is similar to the proof of [5, Proposition 2.9].

Now, we can conclude this section by proving our main result of density of rational functions in $A(\Omega)$.

Theorem 4.5. *Let U_i, $i \in I$ be an arbitrary family of bounded planar domains such that $int(\overline{U_i}) = U_i$ and $(\mathbb{C} \cup \{\infty\}) \setminus \overline{U_i}$ has finitely many components V_i^j, $j = 1, \ldots, n_i$, $i \in I$. Let $\alpha_i^j \in V_i^j$ be fixed. If $\Omega = \prod_{i \in I} U_i$ then, for every function $f \in A(\Omega)$ and every $\epsilon > 0$ there exists a function $g \in A(\Omega)$ depending on a finite number of coordinates $i \in F$, $F \subset I$ finite, such that $\|f - g\| < \epsilon$ and the function g is a finite sum of finite products of rational functions of one variable z_i, $i \in F$ finite, with poles in the set $S_i = \{\alpha_i^j : j = 1, \ldots, n_i\}$, $i \in F$.*

Proof. Let $f \in A(\Omega)$ and $\epsilon > 0$. Then by lemma 4.4 there exists a finite set F with

$$\|f - f \circ \pi_{x,F}\| \leq \epsilon/2,$$

for every $x \in \Omega$. In particular, if we fix $x \in \Omega$, the function $f \circ \pi_{x,F}$ only depends on the set of variables with indices in F. Then, by Proposition 4.3, there exists a function q that is a finite sum of finite products of rational functions of one variable in U_i, each with one pole in the set S_i, $i \in F$, with $\|f \circ \pi_{x,F} - q\| < \epsilon/2$. By the triangle inequality we have

$$\|f - q\| < \epsilon,$$

which concludes the proof. $\qquad \square$

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