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A 3-LOCAL CHARACTERIZATION OF Co_2

CHRISTOPHER PARKER AND PETER ROWLEY

Abstract. Conway's second largest simple group, Co_2, is characterized by the centralizer of an element of order 3 and certain fusion data.

1. Introduction

The vistas revealed by Goldschmidt in [12] inspired many investigations of amalgams, particularly in their application to finite groups and their geometries. One such was the fundamental work of Delgado and Stellmacher [7] in which weak BN pairs were classified. Later Parker and Rowley [25] determined the finite local characteristic p completions of weak BN pairs (when p is odd and excluding the amalgams of type $\mathrm{PSL}_3(p)$). However a number of exceptional configurations when $p \in \{3, 5, 7\}$ required further attention–all but one of them have been addressed in Parker and Rowley [24], [26], Parker [21] and Parker and Weidorn [27]. The last one is run to ground here in our main result which gives a characterization of Conway's second largest simple group, Co_2.

Theorem 1.1. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$, $Z = Z(S)$ and $C = C_G(Z)$. Assume that $O_3(C)$ is extraspecial of order 3^5, $O_2(C/O_3(C))$ is extraspecial of order 2^5 and $C/O_{3,2}(C) \cong \mathrm{Alt}(5)$. If Z is not weakly closed in S with respect to G, then G is isomorphic to Co_2.*

The hypothesis on the structure of C in Theorem 1.1 amounts to saying that C has shape $3^{1+4}.2^{1+4}.\mathrm{Alt}(5)$. Note that no assertion about the types of extension is included and the extraspecial groups could have either $+$- or $-$-type. We remark, as may be seen from [6], that Co_2 actually satisfies the hypothesis of Theorem 1.1. As a consequence of Theorem 1.1 and earlier work on the exceptional cases arising in [25], we can now see that part (ii) of [25, Theorem 1.5] does not occur. Theorem 1.1 investigates a more general configuration than required to settle [25, Theorem 1.5 (ii) (c)]. Though not immediately apparent,

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this configuration rather quickly gives rise to a subgroup M_* of shape $3^4.\mathrm{Alt}(6)$. This particular subgroup makes appearances in other simple groups such as $\mathrm{U}_4(3)$, $\mathrm{U}_6(2)$ and McL and is the root cause of the exceptional possibilities itemized in [25, Theorem 1.5 (ii)(a), (b) and (c)].

A number of the sporadic simple groups have been characterized in terms of 3-local data. The earliest being a characterization of J_1 by Higman [14, Theorem 12]. In [20], O'Nan determined the finite simple groups having an elementary abelian subgroup P of order 3^2 such that for $x \in P^{\#}$, $C_G(x)/\langle x\rangle$ is isomorphic to $\mathrm{PSL}_2(q)$, $\mathrm{PGL}_2(q)$ or $\mathrm{P\Sigma L}_2(q)$ (q odd). Thereby also characterizing the sporadic simple groups M_{22}, M_{23}, M_{24}, J_2, HS and Ru. For the remaining Janko groups, 3-local identifications for J_3 were obtained first by Durakov [9] and later by Aschbacher [1], and for J_4 by Stroth [34], Stafford [33] and Güloğlu [13]. The groups O'N and He were dealt with, respectively, by Il´inyh [15] and Borovik [4]. All of these results were obtained prior to 1990. Recently there has been a resurgence of interest and activity in 3-local characterizations of finite simple groups partly prompted by the revision project concerning groups of local characteristic p (see, for example, [19]). The sporadic simple groups studied in this renaissance period are Co_3 (Korchagina, Parker and Rowley [17]), Fi_{22} (Parker [21]), McL (Parker and Rowley[26]), M_{12} (Astill [3]), Th (Fowler [10]), and Co_1, Fi_{24}', M (Salarian[29, 30, 31]).

With a few exceptions, to date, characterization results for finite groups in terms of 3-local data ultimately rely upon identifying the target group(s) via 2-local information. This is the case here, F. Smith's Theorem [32] providing the final identification. Thus most of this paper is spent manoeuvering into a position where we can use this result. We begin in Section 2 giving background results– F. Smith's Theorem appearing as Theorem 2.1. Another characterization result appearing in Theorem 2.2, due to Prince, is employed in Lemma 5.4. Lemma 5.4, which is the bridge to the 2-local structure of G (G as in Theorem 1.1), states that $N_G(B) \cong \mathrm{Sym}(3) \times \mathrm{Aut}(\mathrm{U}_4(2))$ for a certain subgroup B of G of order 3. In $N_G(B)$ there is an involution t inverting B and centralizing $O^3(C_G(B)) \cong \mathrm{Aut}(\mathrm{U}_4(2))$. Not only does this lemma fill out our knowledge of the 3-local subgroups but it also gives us a toehold in $C_G(t)$. After Lemmas 2.3–2.8, results which play minor supporting roles, a compilation of GF(3)-module data for the groups Sym(4) and Alt(6) appear in Lemmas 2.9 and 2.10. From Lemma 2.10 we deduce Lemma 2.11 which concerns hyperplanes of the 4-dimensional permutation GF(3)Alt(6)-module–this plays an important role in Lemma 5.2 where we show that $3'$-signalizers for J are trivial. Here J is elementary abelian of order 3^4 and is the Thompson subgroup of S, $S \in \mathrm{Syl}_3(G)$. Various properties of groups of shape $2^{1+4}.\mathrm{Alt}(5)$ are given in Lemmas 2.12, 2.13 and 2.14. These results will be applied to bring the structure of $C_G(Z)$ into sharper focus, where $Z = Z(S)$. We conclude Section 2 with Lemmas 2.15 and 2.16 which concern the spin module for $\mathrm{Sp}_6(2)$, followed by an elementary result on $\mathrm{Aut}(\mathrm{U}_4(2))$ in Lemma 2.17.

The main result of Section 3, Theorem 3.1, anticipates the end game in our analysis of $C_G(t)$, t being the involution mentioned earlier. In fact, Theorem 3.1 will be applied to $C_G(t)/\langle t\rangle$.

Section 4 sees us start the proof of Theorem 1.1. After Lemma 4.1 in which the structure of $C_G(Z)$ is examined (where $Z = Z(S)$, $S \in \mathrm{Syl}_3(G)$), Lemmas 4.2 and4.3 look at centralizers and commutators of certain involutions in $C_G(Z)$. In Lemmas 4.4, 4.5 and 4.6 it is S and its subgroups that mostly occupy our attention. Two subgroups of S that will play central roles in the proof of Theorem 1.1 are Z and $J = C_S([Q,S])$ where $Q = O_3(N_G(Z))$. In Lemma 4.5 we learn that J is the Thompson subgroup of S, J is elementary abelian of order 3^4 and that all G-conjugates of Z in S are trapped inside J. Another important subgroup of S, namely B, along with the involution t, already noted earlier, make their entrance after Lemma 4.7. In the latter part of Section 4, our attention moves on to $N_G(Z)$, resulting in structural information about this subgroup in Lemmas 4.10, 4.11 and 4.12. Drawing upon the results in Section 4, in Section 5 we determine the structure of $N_G(B)$. Our last section brings to bear all the earlier results on $C_G(t)$ eventually yielding that $C_G(t)/\langle t\rangle$ satisfies the hypotheses of Theorem 3.1. Then using Theorem 3.1 we rapidly obtain the hypotheses of Theorem 2.1, whence we deduce that $G \cong \mathrm{Co}_2$.

Our main source of information for group structures is the ubiquitous ATLAS [6], and we also follow the notation and conventions there with a number of variations which we now mention. We shall use $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ to denote, respectively, the symmetric and alternating groups of degree n and $\mathrm{Dih}(n)$, $\mathrm{Q}(n)$ and $\mathrm{SDih}(n)$, respectively, to stand for the dihedral group, quaternion group and semidihedral group of order n. Finally $X \sim Y$ where X and Y are groups will indicate that X and Y have the same shape.

The remainder of our notation is standard as given, for example, in [2] and [18].

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2. Preliminary Results

Theorem 2.1 (F. Smith). *Suppose that X is a finite group with $Z(X) = O_{2'}(X) = 1$, and Y is the centralizer of an involution in X. If $Y/O_2(Y) \cong \mathrm{Sp}_6(2)$ and $O_2(Y)$ is a non-abelian group of order 2^9 such that the elements of order 5 in Y act fixed point freely on $O_2(Y)/Z(O_2(Y))$, then X is isomorphic to Co_2.*

Proof. See [32]. □

Theorem 2.2 (A. Prince). *Suppose that Y is isomorphic to the centralizer of a 3-central element of order 3 in $\mathrm{PSp}_4(3)$ and that X is a finite group with an element d such that $C_X(d) \cong Y$. Let $P \in \mathrm{Syl}_3(C_X(d))$ and E be the elementary abelian subgroup of P of order 27. If E does not normalize any non-trivial $3'$-subgroup of X and d is H-conjugate to its inverse, then either*

(i) $|X : C_X(d)| = 2$;
(ii) *X is isomorphic to* $\mathrm{Aut}(\mathrm{U}_4(2))$; *or*
(iii) *X is isomorphic to* $\mathrm{Sp}_6(2)$.

Proof. See [28, Theorem 2] □

Lemma 2.3. *Suppose that X is a group of shape $3_+^{1+2}.\mathrm{SL}_2(3)$, $O_2(X) = 1$ and a Sylow 3-subgroup of X contains an elementary abelian subgroup of order 3^3. Then X is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$.*

Proof. See [21, Lemma 6]. □

We will also use the following variation of Lemma 2.3.

Lemma 2.4. *Suppose that X is a group of shape $3_+^{1+2}.\mathrm{SL}_2(3)$, $O_2(X) = 1$ and the Sylow 3-subgroups of a centralizer of an involution in X are elementary abelian. Then X is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$.*

Proof. Let $S \in \mathrm{Syl}_3(X)$, $R = O_3(X)$, and $F \le R$ be a normal subgroup of S of order 9. Let $N = N_X(S)$. If F is not normal in N, then there exists $n \in N$ such that $R = F^nF$. But then S centralizes $FF^n/Z(R) = R/Z(R)$ and so $C_X(R/Z(R)) > R$ and this contradicts $O_2(X) \neq 1$. Hence F is normal in N. Let $E = C_S(F)(= C_N(F))$. Then E is abelian of order 27. Let u be an involution in N. Then u normalizes E and, as $[S, u] \le R$, $C_E(u) \not\le R$. Therefore $E = C_E(u)F$. Since F and $C_E(u)$ are elementary abelian by hypothesis, E is elementary abelian of order 3^3. Hence Lemma 2.3 applies and yields the result. □

Lemma 2.5. *Suppose that p is a prime, X is a finite group and $P \in \mathrm{Syl}_p(X)$. If $x, y \in Z(J(P))$ are X-conjugate, then x and y are $N_X(J(P))$-conjugate.*

Proof. See [2, 37.6]. □

Lemma 2.6. *Suppose that p is a prime, X is a finite group and $P \in \mathrm{Syl}_p(X)$. If $R \leq P$ is not weakly closed in P with respect to X, then there exists $x \in X$ such that $R \neq R^x$ and R and R^x normalize each other.*

Proof. Suppose that R is not normal in P. Let $N = N_P(R)$ and $M = N_P(N)$. Then $M > N$. Choose $x \in M \setminus N$. Then $R \neq R^x$ and, as R and R^x are both normal in N, we obtain the lemma. Hence we may assume that R is normal in P. Since R is not weakly closed in P with respect to X, there exists $y \in X$ such that $R^y \neq R$ and $R^y \leq P$. If R^y is normal in P, then R and R^y normalize each other and we take $x = y$. Otherwise, repeating the argument as for R, we find $z \in P$ such that R^y and R^{yz} normalize each other. Taking $x = yzy^{-1}$ completes the proof of the lemma. □

Lemma 2.7. *Suppose that X is a finite group, $x \in X$ an involution of X and V an elementary abelian normal 2-subgroup of X. Set $C = C_X(x)$. Then the map $(vx)^{VC} \mapsto (v[V,x])^C$ is a bijection between VC-orbits of the involutions in the coset Vx and the C-orbits of the elements of $C_V(x)/[V,x]$. Furthermore, for vx an involution in Vx, $|(vx)^{VC}| = |(v[V,x])^C|.|[V,x]|$.*

Proof. The given map is easily checked to be a bijection. □

Lemma 2.8. *Suppose that Q is an extraspecial p-group and $\alpha \in \mathrm{Aut}(Q)$. If A is a maximal abelian subgroup of Q and $[A, \alpha] = 1$, then α is a p-element.*

Proof. The Three Subgroup Lemma implies that $[Q, \alpha] \leq A$. Then $[Q, \alpha, \alpha] \leq [A, \alpha] = 1$ and so α is a p-element. □

Lemma 2.9. *Suppose that $X \cong \mathrm{Sym}(4)$ and V is a faithful 3-dimensional $\mathrm{GF}(3)X$-module. Then*

(i) *there is a set of 1-dimensional subspaces $\mathcal{B} = \{\langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle\}$ such that $X/O_2(X)$ acts as $\mathrm{Sym}(3)$ on $\mathcal{B}$ and each subspace in $\mathcal{B}$ is inverted by $O_2(X)$;*

(ii) *X has orbits of length 3, 4 and 6 on the 1-dimensional subspaces of V with representatives $\langle v_1\rangle$, $\langle v_1+v_2+v_3\rangle$ and $\langle v_1+v_2\rangle$ respectively; and*

(iii) *X has orbits of length 3, 4 and 6 on the 2-dimensional subspaces of V with representatives $\langle v_1, v_2\rangle$, $\langle v_1+v_2, v_2+v_3\rangle$ and $\langle v_1, v_1+v_2+v_3\rangle$ respectively.*

Proof. Let $Q = O_2(X)$ and $Q^\# = \{x_1, x_2, x_3\}$. Then, as V is a faithful irreducible $\mathrm{GF}(3)X$-module and X acts transitively on $Q^\#$ by conjugation, we have that $V = C_V(x_1)\oplus C_V(x_2)\oplus C_V(x_3)$ and that X permutes the subspace $\{C_V(x_i) \mid 1 \le i \le 3\}$ transitively. Setting $\langle v_i\rangle = C_V(x_i)$, we have that (i) holds.

Obviously $\{\langle v_i\rangle \mid 1 \le i \le 3\}$ is an orbit of length 3 on the 1-dimensional subspace of V. The subspaces $\langle v_1 \pm v_2 \pm v_3\rangle$ form an orbit of length 4 and the subspaces $\langle v_i \pm v_j\rangle$ with $i \ne j$ give an orbit of length 6. This proves part (ii). A similar calculation provides a proof of (iii). □

Lemma 2.10. *Suppose that $X = \mathrm{Alt}(6)$ and let V be the $\mathrm{GF}(3)$-permutation module for X with standard basis $\{v_1, \ldots, v_6\}$. Let $U_0 = \langle\sum_{i=1}^{6} v_i\rangle$ and $U = \langle v_i + 2v_j \mid 1 \le i, j \le 6\rangle$. Set $W = U/U_0$. Then W is 4-dimensional and the following hold.*

(i) *X has three orbits on the one-dimensional subspaces of W, $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, with representatives $\langle v_1+v_2+v_3+U_0\rangle$, $\langle v_1+2v_2+v_3+2v_4+U_0\rangle$ and $\langle v_1+2v_2+U_0\rangle$ respectively. Furthermore, $|\mathcal{O}_1| = 10$ and $|\mathcal{O}_2| = |\mathcal{O}_3| = 15$. The stabilizers of a member of $\mathcal{O}_2$ and of a member of $\mathcal{O}_3$ are not conjugate in X.*

(ii) *If t is an involution in X, then $\dim C_W(t) = 2$ and $C_W(t)$ contains two subspaces from $\mathcal{O}_1$ and one each from $\mathcal{O}_2$ and $\mathcal{O}_3$. Furthermore, $C_X(t) \cong \mathrm{Dih}(8)$ interchanges the two members of $\mathcal{O}_1$ in $C_W(t)$ and $|C_X(t)/C_{C_X(t)}(C_W(t))| = 4$.*

(iii) *If $g \in X$ has order 4, then $C_W(g) = 0$.*

(iv) *If $D \in \mathrm{Syl}_3(X)$, then $\dim C_W(D) = \dim W/[W,D] = 1$ and $C_W(D) \in \mathcal{O}_1$.*

(v) *If $d \in X$ has order 3, then $\dim C_W(d) = 2$.*

(vi) *If $D \in \mathrm{Syl}_3(X)$ and $t \in N_X(D)$ is an involution, then t centralizes $C_W(D)$ and $W/[W,D]$.*

Proof. This is an elementary calculation. □

We refer to the module appearing in Lemma 2.10 as the 4-dimensional permutation GF(3)-module for Alt(6)–we remark that this is in fact isomorphic to the $\Omega_4^-(3)$-module.

Lemma 2.11. *Suppose that X, W and $\mathcal{O}_1$ are as in Lemma 2.10 and assume that W_0 is a hyperplane of W. Then W_0 contains a member of $\mathcal{O}_1$.*

Proof. Let $D \in \mathrm{Syl}_3(X)$. Assume that W_0 contains no element from $\mathcal{O}_1$. Then, as $\dim C_W(D) = 1$ by Lemma 2.10(iv), W_0 does not contain any non-trivial D-invariant subspaces. So, as $\dim W_0 = 3$, $|W_0^D| = 9$ and no D-conjugate of W_0 contains any member of $\mathcal{O}_1$.

Now suppose that W_1, W_2, W_3, W_4 are distinct hyperplanes of W. Then, by the inclusion exclusion principle, $|\bigcup_{i=1}^4 W_i| \geq 63$. Therefore, using Lemma 2.10(i), $81 = |W| \geq |\bigcup_{x \in D} W_0^x| + 2|\mathcal{O}_1| \geq 63 + 20 = 83$, which is absurd. Thus the lemma holds. □

Lemma 2.12. *Suppose that V is a faithful 4-dimensional $\mathrm{GF}(3)X$-module and that X contains a normal subgroup Y with $Y \sim 2^{1+4}.\mathrm{Alt}(5)$. Then X is 2-constrained, $O_2(X) = O_2(Y)$ is extraspecial of $-$-type and either $X = Y$ or $X/O_2(X) \cong \mathrm{Sym}(5)$.*

Proof. Let $Q = O_2(Y)$. Then Q is normalized by X. Let $Z = C_X(Q)$. Then, as Q acts irreducibly on V and $\mathrm{GF}(3)$ is a splitting field for this action, $Z = Z(Q)$ by Schur's Lemma [2]. It follows that $\mathrm{Aut}(Q)$ contains a subgroup isomorphic to $2^4.\mathrm{Alt}(5)$ and so Q is extraspecial of $-$-type. Hence $\mathrm{Aut}(Q) \cong 2^4.\mathrm{Sym}(5)$ by [8, Theorems 20.8 and 20.9] and this proves the result. □

Lemma 2.13. *Suppose that $X \sim 2_-^{1+4}.\mathrm{Alt}(5)$ is 2-constrained. Let $Q = O_2(X)$ and $T \in \mathrm{Syl}_3(X)$.*

(i) *If $i \in Q$ is a non-central involution, then $|i^X| = 10$ and $C_X(i) \sim (\mathrm{Q}(8) \times 2).\mathrm{Alt}(4)$. In particular, $C_X(i)Q/Q \cong \mathrm{Alt}(4)$; and*
(ii) *$C_Q(T) \cong \mathrm{Dih}(8)$ and $N_X(T)Q/Q \cong \mathrm{Sym}(3)$.*

Proof. We know that Q is the central product of $\mathrm{Dih}(8)$ and $\mathrm{Q}(8)$ and so it is straightforward to calculate that there are 10 non-central involutions. They are conjugate in pairs in Q and the element of order 5 in X acts fixed point freely on $Q/Z(Q)$. It is now easy to confirm the details stated in (i). Since elements of order 3 in X centralize a non-central involution and since $C_Q(T)$ is extraspecial, we get $C_Q(T) \cong \mathrm{Dih}(8)$. The second part of (ii) follows from the Frattini Argument. □

Lemma 2.14. *Suppose that V is a faithful 4-dimensional $\mathrm{GF}(3)Y$-module and that $Y \sim 2_-^{1+4}.\mathrm{Alt}(5)$. Then the following hold.*

(i) *For $v \in V^\#$, we have $C_Y(v) \cong \mathrm{SL}_2(3)$. In particular, Y operates transitively on $V^\#$.*
(ii) *Every element of order 2 in Y is contained in $O_2(Y)$.*

Proof. Let $Q = O_2(Y)$, $s \in Z(Q)^\#$ and $v \in V^\sharp$. Then s negates v and so $C_Q(v)$ is a subgroup of Q which does not contain s. Since $Q \cong \mathrm{Dih}(8) \circ \mathrm{Q}(8)$, we get that $C_Q(v)$ has order dividing 2. Hence every

Conjugacy Classes	$\mathrm{Sp}_6(2)$	$\mathrm{Aut}(\mathrm{U}_4(2))$	$\|C_X(x)\|$	$\|C_Y(x)\|$	$\|C_V(x)\|$
A_1	2A	2C	$2^9.3^2.5$	$2^5.3^2.5$	2^4
A_2	2B	2A	$2^9.3^2$	$2^7.3^2$	2^6
A_3	2C	2B	$2^9.3$	$2^6.3$	2^4
A_4	2D	2D	$2^7.3$	$2^5.3$	2^4

TABLE 1. Involutions in $\mathrm{Aut}(\mathrm{U}_4(2))$ and $\mathrm{Sp}_6(2)$

orbit of Y on V has order divisible by 16. Since the elements of Y of order 5 centralize only the zero vector, the orbits of Y have length divisible by 5. As there are 80 non-zero vectors it follows that Y acts transitively on $V^{\#}$, $|C_Q(v)| = 2$ and $C_Y(v)Q/Q \cong \mathrm{Alt}(4)$. Since Y is perfect and is isomorphic to a subgroup of $\mathrm{SL}_4(3)$, the 2-rank of Y is at most 3. By considering $\langle s, C_Y(v)\rangle$ we see that $C_Y(v) \not\cong 2 \times \mathrm{Alt}(4)$ and therefore $C_Y(v)$ is isomorphic to the unique double cover of $\mathrm{Alt}(4)$, namely $\mathrm{SL}_2(3)$. This proves (i).

Now suppose that $y \in Y \setminus Q$ has order 2. Then as y is a noncentral involution in Y, $C_V(y) \neq 0$. But then (i) implies $y \in Q$, a contradiction. Hence (ii) holds. □

The group $\mathrm{Sp}_6(2)$ has a unique 8-dimensional irreducible module over $\mathrm{GF}(2)$ as can be seen for example in [16]. This module is usually called the *spin module* for $\mathrm{Sp}_6(2)$. On restriction to any subgroup of $\mathrm{Sp}_6(2)$ isomorphic $\mathrm{Aut}(\mathrm{U}_4(2))$ the spin module remains irreducible and is the unique irreducible module of dimension 8 over $\mathrm{GF}(2)$ for this group. For convenience is Section 3, we shall refer to this module as the *spin module* for $\mathrm{Aut}(\mathrm{U}_4(2))$. The next two lemmas collect information about the action of certain subgroups and elements of these two groups on the spin module.

Lemma 2.15. *Suppose that $X \cong \mathrm{Sp}_6(2)$, Y is a subgroup of X with $Y \cong \mathrm{Aut}(\mathrm{U}_4(2))$ and V is the $\mathrm{GF}(2)X$-spin module. Then the following hold.*

(i) *There are four conjugacy classes A_1, A_2, A_3 and A_4 of involutions in X and each of them has a representative in Y. For each conjugacy class A_i, $1 \le i \le 4$, and for x an involution in A_i, Table 1 gives the* ATLAS *class name for A_i in both X and Y, $|C_X(x)|$, $|C_Y(x)|$ and $|C_V(x)|$.*

(ii) *If P is a parabolic subgroup of shape $2^5.\mathrm{Sp}_4(2)$ in X, then $O_2(P)$ contains one involution from A_1 and fifteen involutions from each of A_2 and A_3. Furthermore, as a $P/O_2(P)$-module, $O_2(P)$ is an indecomposable extension of the trivial module by a natural module.*

(iii) *If $x \in A_2$, then $\langle x\rangle = Z(C_X(x))$ and $C_X(x)$ is a maximal subgroup of X.*
(iv) *If $f \in X$ has order five, then $C_V(f) = 0$.*
(v) *For $v \in V$, $|C_Y(v)|$ and $|C_X(v)|$ are divisible by 3.*
(vi) *For $S \in \mathrm{Syl}_2(Y)$, $|C_V(S)| = |C_{V/C_V(S)}(S)| = 2$.*
(vii) *If $S \in \mathrm{Syl}_2(X)$ and $x \in N_X(Z(S))$ has order 3, then x acts fixed point freely on V.*
(viii) *There are no subgroups of X of order 2^5 which have all non-trivial elements in class A_2.*

Proof. The facts in (i) regarding involutions classes and their centralizers in X and Y are taken from the ATLAS [6, pgs. 26 and 46]–we determine $|C_V(x)|$ later in the proof. We also immediately see that $C_X(x)$ is a maximal subgroup of X for $x \in A_2$. So (iii) holds.

Let $S \in \mathrm{Syl}_2(X)$ and P_1, P_2 and P_3 be the maximal parabolic subgroups of X containing S with $P_1 \sim 2^5.\mathrm{Sp}_4(2)$, $P_2 \sim 2^6.\mathrm{SL}_3(2)$ and $|P_3| = 2^9.3^2$. Then the restrictions of V to P_i, $i = 1, 2, 3$ are given in [22]. In particular, we have that $[V, O_2(P_1)] = C_V(O_2(P_1))$ has dimension 4 and, as $P/O_2(P_1)$ modules, $V/C_V(O_2(P_1)) \cong C_V(O_2(P_1))$ and both are natural $\mathrm{Sp}_4(2)$-modules. Therefore, the elements of order 5 in X act fixed point freely on V which gives (iv).

There are dihedral subgroups of X of order 10 which contain involutions from classes A_1, A_3 and A_4. Therefore $|C_V(x)| = 2^4$ for x in any of these classes. We have that V restricted to a Levi complement L of P_1 decomposes as a direct sum of two natural modules and so the transvections in L centralize a subspace of dimension 6 in V. These elements are therefore in class A_2. This completes the proof of (i).

Since $C_V(S)$ is normalized by P_2, we calculate that Y has two orbits on $V^\#$ one of length 135 and the other of length 120. In particular (v) holds.

Since $Z = Z(S)$ contains elements from classes A_1, A_2 and A_3 which we denote by z_a, z_b and z_c respectively, $N_X(Z) = C_X(Z) \le C_X(z_c) \le P_1 \cap P_3 \le C_X(z_a) \cap C_X(z_b) \le C_X(Z)$. It follows that $N_X(Z) \not\le P_2$ and thus the elements d of order 3 in $N_X(Z)$ have $C_V(d) = 0$. Thus (vii) holds.

From Table 1 we have that $Z(S) \le O_2(P_1)$ contains elements from each of the classes A_1, A_2 and A_3. As P_1 centralizes an element z of $Z(S)$ in class A_1 and since P_1 acts transitively on the non-trivial elements of $O_2(P_1)/\langle z\rangle$. The first part of (ii) holds. The final part of (ii) is well known and can be, for example, verified by using the Chevalley commutator formula to calculate that $|[O_2(P), S]| = 2^4$ where $S \in \mathrm{Syl}_2(P)$.

Suppose that B is an elementary abelian subgroup of X of order 2^5 in which every involution is in A_2. By considering the restriction of V to P_1, we see that $|BO_2(P_1)/O_2(P_1)| \leq 2$. Thus $B \cap O_2(P_1)$ contains all the A_2-involutions of $O_2(P_1)$ and is consequently P_1 invariant. This contradicts (ii), so proving part (viii).

We prove (vi). Let P be the parabolic subgroup of $\mathrm{Aut}(\mathrm{U}_4(2))$ of shape $2^4 : \mathrm{Sym}(5)$, $R = O_2(P)$ and $S \in \mathrm{Syl}_2(P)$. Then as the elements of order 5 in P act fixed point freely on V, $C_V(R) = [V, R]$ has dimension 4. Furthermore, $C_V(R)$ is an irreducible P/R-module and from this we obtain $C_V(S) = C_{C_V(R)}(S)$ and $C_{C_V(R)/C_V(S)}(S)$ have dimension 1. Since $[S, S] \cap R$ has order 2^3 and R contains only 5 elements in class A_2, we deduce that $[S, S]$ contains an involution that is not in class A_2. As the preimage of $C_{V/C_V(S)}(S)$ is centralized by $[S, S]$, we see that $C_{V/C_V(S)}(S) = C_{C_V(R)/C_V(S)}(S)$ and (vi) follows. □

Lemma 2.16. *Suppose that* $X \cong \mathrm{Sp}_6(2)$ *and* V *is the* $\mathrm{GF}(2)X$*-spin module. If* $F \leq X$, $[V, F, F] = 0$ *and* $|V/C_V(F)| \leq |F|$, *then there exists* $f \in F^{\#}$ *which is not in class* A_2.

Proof. First of all we note that, as V is self-dual, $|[V, F]| = |V/C_V(F)| \leq |F|$.

Assume that every non-trivial element of F is in class A_2. Then $2^4 \geq |F| > 2$ by Lemma 2.15 (i) and (viii). If $|F| = 2^2$, then for $f_1, f_2 \in F^{\#}$ with $f_1 \neq f_2$ we have $C_V(f_1) = C_V(f_2) = C_V(F)$. But then $C_V(F)$ is invariant under $\langle C_X(f_1), C_X(f_2)\rangle = X$ as $C_X(f_1)$ is a maximal subgroup of X by Lemma 2.15(iii). Therefore $|V : C_V(F)| \geq 2^3$ and $|F| \geq 2^3$.

Assume that P_1 is a parabolic subgroup of X of shape $2^5.\mathrm{Sp}_4(2)$ such that $F \leq P_1$. Set $E = F \cap O_2(P_1)$. Suppose that $|E| \geq 2^3$. If $|E| = 2^4$, then E contains all the A_2-elements of $O_2(P_1)$ and hence is invariant under the action of P_1. This contradicts Lemma 2.15(ii) and so we conclude that $|E| = 2^3$. Let $P \leq P_1$ be the parabolic subgroup of P_1 which normalizes $EZ(P_1)$. Since E contains all the A_2-elements of $EZ(P_1)$, P normalizes E. Also, since P normalizes $EZ(P_1)$, P normalizes $Z(S)$ for any $S \in \mathrm{Syl}_2(P)$. Hence P only normalizes subspaces of even dimension by Lemma 2.15(vii). Consequently, as P normalizes $C_V(E)$ and $|C_V(E)| \leq 2^5$, we deduce that $C_V(E) = C_V(O_2(P_1))$ has order 2^4. Since E acts quadratically on V, $[V, E] = C_V(E)$ and thus $C_V(F) = C_V(E)$. So $|F| = 2^4$ and hence, as $|E| = 2^3$, $F \not\leq O_2(P_1)$. But then $C_V(F) < C_V(E)$ which is a contradiction. Hence $|E| \leq 2^2$. Because $O^2(P_1) \setminus O_2(P_1)$ contains no A_2-elements, we have $|F| \leq 2^3$ and so $|F| = 2^3$. Finally, $[V, F] \geq [V, E] + [V, f]$ for some $f \in F \setminus O_2(P_1)$ and so, as $[V, f] \not\leq [V, O_2(P_1)]$ and $[V, E] \leq [V, O_2(P_1)]$ with $|[V, E]| \geq 2^3$, we have $|[V, F]| > |F|$, and this is our final contradiction. □

Lemma 2.17. *Suppose that* $X \cong \mathrm{Aut}(\mathrm{U}_4(2))$ *and* x *is an involution of* X *with* $C_X(x) \cong 2 \times \mathrm{Sym}(6)$*. Let* $F \in \mathrm{Syl}_3(C_X(x))$*. If* $T \in \mathrm{Syl}_3(X)$ *and* $F \leq T$*, then* $F \leq J(T)$*.*

Proof. Note that $J(T)$ is elementary abelian of order 3^3. If $Z(T) \leq F$, then $x \in C_X(Z(T)) \leq X'$ by [6, pg. 26] whereas $x \notin X'$. Thus $Z(T) \not\leq F$. Hence $Z(T)F$ is elementary abelian of order 3^3 and so $Z(T)F = J(T)$, and the lemma holds. □

3. A 2-local subgroup

As intimated in Section 1, the raison d'être for Theorem 3.1 is to assist in uncovering the structure of an involution centralizer in a group satisfying the hypothesis of Theorem 1.1. The main thrust of the proof of Theorem 3.1 is to show that Q is a strongly closed 2-subgroup of T with respect to G where $T \in \mathrm{Syl}_2(H)$. Goldschmidt's classification of groups with a strongly closed abelian 2-subgroup [11] quickly concludes the proof. We use the simultaneous notation for conjugacy classes in the groups $\mathrm{Sp}_6(2)$ and $\mathrm{Aut}(\mathrm{U}_4(2))$ given in Table 1.

Theorem 3.1. *Suppose that* G *is a finite group,* Q *is a subgroup of* G *and* $H = N_G(Q)$*. Assume that the following hold*

(i) $H/Q \cong \mathrm{Aut}(\mathrm{U}_4(2))$ *or* $\mathrm{Sp}_6(2)$*;*
(ii) $Q = C_G(Q)$ *is a minimal normal subgroup of* H *and is elementary abelian of order* 2^8*;*
(iii) H *controls fusion of elements of* H *of order* 3*; and*
(iv) *if* $g \in G \setminus H$ *and* $d \in H \cap H^g$ *has order* 3*, then* $C_Q(d) = 1$*.*

Then $G = HO_{2'}(G)$*.*

Proof. Let $T \in \mathrm{Syl}_2(H)$. To begin with we note that as a $\mathrm{GF}(2)H$-module, Q is isomorphic to the $\mathrm{GF}(2)H/Q$ spin-module (see the discussion before Lemma 2.15).

(3.1.1) Suppose that $g \in G$ and $y \in (Q^g \cap H) \setminus Q$. Then $C_H(y)$ is a $3'$-group.

Let $y \in (Q^g \cap H) \setminus Q$ and suppose that 3 divides $|C_H(y)|$, $S \in \mathrm{Syl}_3(C_H(y))$ and $x = y^{g^{-1}}$. Then $x \in Q$ and $|C_H(x)|$ is divisible by 3 by Lemma 2.15 (v). Let $P \in \mathrm{Syl}_3(C_H(x))$. If $P \notin \mathrm{Syl}_3(C_G(x))$, then $N_{C_G(x)}(P) \not\leq H$ and so there exists $n \in N_{C_G(x)}(P) \setminus H$ such that $P \leq H \cap H^n$. Since, for $d \in P$ of order 3, $x \in C_Q(d)$, this contradicts assumption (iv). Hence $P \in \mathrm{Syl}_3(C_G(x))$ and therefore $P^g \in \mathrm{Syl}_3(C_G(y))$.

Since S is a 3-subgroup of $C_G(y)$, there is an $h \in C_G(y)$ such that $P^{gh} \geq S$. By assumption (iii), H controls fusion of elements of order 3 in H. Hence, as each element of S is G-conjugate to an element of P, each element of S is H-conjugate to an element of P. Now, as $x \in C_Q(P)$ and Q is normal in H, for elements of $s \in S$ we have $C_Q(s) \neq 1$. Since $S \leq H \cap H^{gh}$, we then get $gh \in H$ by (iv). Thus $y = x^{gh} \in Q^{gh} = Q$ and we have a contradiction as $y \notin Q$. Therefore, 3 does not divide $|C_H(y)|$ as claimed. ♠

(3.1.2) Let $g \in G$ and suppose $y \in (Q^g \cap H) \setminus Q$. Then yQ is an A_2-involution in H/Q and $C_H(y)Q \in \mathrm{Syl}_2(H)$.

If yQ is not in the A_2-class of H/Q, then, by Lemma 2.15(i), $C_Q(y) = [Q, y]$ and so Lemma 2.7 gives $C_H(y)Q/Q = C_{H/Q}(y)$. Thus $C_H(y)$ is not a $3'$-group by Lemma 2.15(i) again, and this is contrary to (3.1.1). Hence yQ is in the A_2-class of H/Q. From Lemmas 2.15(i) and 2.7 we have $|C_Q(y)/[Q, y]| = 2^4$ and $|C_{H/Q}(yQ)|_3 = 3^2$. Since $|C_H(y)|$ is not divisible by 3 by (3.1.1), $C_H(y)$ must have an orbit of length divisible by 3^2 and hence of length exactly 3^2 on $C_Q(y)/[Q, y]$. It follows that $|C_H(y)| = 2^{15}$ if $H/Q \cong \mathrm{Sp}_6(2)$ and 2^{13} if $H/Q \cong \mathrm{Aut}(\mathrm{U}_4(2))$. Therefore, as $|Q : C_Q(y)| = 2^2$, $C_H(y)Q \in \mathrm{Syl}_2(H)$. So (3.1.2) holds. ♠

We note that (3.1.2) applies equally well to show that involutions in $(Q \cap H^g)Q^g/Q^g$ are in the A_2-class of H^g/Q^g.

(3.1.3) Q is weakly closed in H with respect to G. In particular, $T \in \mathrm{Syl}_2(G)$.

Suppose that (3.1.3) is false. Then, by Lemma 2.6, there exists $g \in G \setminus H$ such that Q^g and Q normalize each other. Hence we may assume that $|Q : C_Q(Q^g)| \leq |Q^gQ/Q|$. By (3.1.2) the non-trivial elements of Q^gQ/Q are all in H/Q class A_2. These two facts together contradict Lemma 2.16. Therefore Q is weakly closed in H with respect to G and $\mathrm{Syl}_2(H) \subseteq \mathrm{Syl}_2(G)$. ♠

Aiming for a contradiction we now suppose that Q is not strongly closed in T with respect to G.

(3.1.4) We can select $g \in G$ and $y \in (Q^g \cap H) \setminus Q$ so that $C_H(y) \leq H^g$.

Since Q is not strongly closed in T ($\leq H$), there exists $g \in G$ and $y \in (Q^g \cap H) \setminus Q$. Clearly $Q^g \leq C_G(y)$, and so we may select a Sylow 2-subgroup T_1 of $C_G(y)$ such that T_1 contains Q^g. Since $C_H(y)$ is a 2-group by (3.1.2), there exists a Sylow 2-subgroup T_2 of $C_G(y)$ which contains $C_H(y)$. Thus there is an $f \in C_G(y)$ such that $T_1^f = T_2$. Because

Q is weakly closed in H and $Q^{gf} \leq T_2$, $C_H(y) \leq T_2 \leq N_G(Q^{gf}) = H^{gf}$. Since $f \in C_G(y)$, $y \in (Q^{gf} \cap H) \setminus Q$. Thus we may replace g by gf and we have proved (3.1.4). ♠

Choosing g and y as in (3.1.4), we set $W = C_H(y)Q^g$.

(3.1.5) There exists a Sylow 2-subgroup T_0 of H^g which normalizes $Q \cap Q^g$ and contains W . Furthermore, $|T_0 : W| \leq 2$.

Since $C_H(y)Q \in \mathrm{Syl}_2(H)$ by (3.1.2), and $C_H(y)Q$ normalizes $Q \cap Q^g$ by (3.1.4), $N_H(Q \cap Q^g)$ contains a Sylow 2-subgroup of G by (3.1.3). Since W normalizes $Q \cap Q^g$, there is a $T_0 \in \mathrm{Syl}_2(N_G(Q \cap Q^g))$ with $T_0 \geq W$. Therefore, as Q^g is weakly closed in W, $T_0 \leq H^g$. Since $|Q : C_Q(y)| = 4$, we have $|T_0 : W| \leq 4$ by (3.1.2). If $|T_0 : W| = 4$, then we must have $Q^g \leq C_H(y)$ which contradicts Q being weakly closed in H and $Q \neq Q^g$. Hence $|T_0 : W| \leq 2$. ♠

Let $Z_2(T_0)$ be the second centre of T_0 where T_0 is as in (3.1.5). Then, as $|Z_2(T_0)| = 4$ by Lemma 2.15(vi) and $Q \cap Q^g$ is normal in T_0, we either have $|Q \cap Q^g| \leq 2$, or $Z_2(T_0) \leq Q \cap Q^g$. Since $|T_0 : W| \leq 2$, $C_{Q^g}(W) \leq Z_2(T_0)$. From $y \in C_{Q^g}(W) \leq Z_2(T_0)$ and $y \notin Q$, we must have $|Q \cap Q^g| \leq 2$. Since yQ is in H/Q class A_2, we have $|C_Q(y)| = 2^6$. Hence $|C_Q(y)Q^g/Q^g| = |C_Q(y) : Q \cap Q^g| \geq 2^5$ and, by (3.1.2), all the involutions of $C_Q(y)Q^g/Q^g$ are in H^g/Q^g class A_2, which contradicts Lemma 2.15 (viii). We have therefore shown that Q is strongly closed in T with respect to G.

Set $M = \langle Q^G \rangle$. If $M \neq QO_{2'}(G)$, then $|M : Q|$ is even and hence we have $T \cap M > Q$ by (3.1.3). But then $\langle (T \cap M)^H \rangle$ has index at most 2 in H and is contained in M. Finally, applying Goldschmidt's Theorem [11], we see that the possible composition factors of $M/O_{2',2}(M)$ do not involve either $\mathrm{U}_4(2)$ or $\mathrm{Sp}_6(2)$. Thus $M = QO_{2'}(G)$ and the Frattini Argument completes the proof of the theorem. □

4. Part of the 3-local structure

Having now gathered together our prerequisite results, we are ready to begin the proof of Theorem 1.1. Thus for the remainder of this article we assume that G is a finite group with S a Sylow 3-subgroup of G and $Z = Z(S)$. Additionally, we assume that Z is not weakly closed in S with respect to G and $C_G(Z)$ has shape $3^{1+4}.2^{1+4}.\mathrm{Alt}(5)$ as described in the hypothesis of Theorem 1.1. We set $L = N_G(Z)$, $L_* = C_G(Z)$, $Q = O_3(L)$ and let $P \in \mathrm{Syl}_2(O_{3,2}(L_*))$. So P and Q are extraspecial of order 2^5 and 3^5 respectively and $O_{3,2}(L_*) = PQ$. Furthermore, $O_3(L_*) = Q$. Let $\langle u \rangle = Z(P)$ and $U = C_L(u)$.

We begin by fleshing out the structure and embeddings of these groups. In the next proof we use the fact that $\mathrm{Sp}_4(3)$ contains no subgroup isomorphic to $\mathrm{Alt}(5)$. This is easy to see as the 2-rank of both $\mathrm{Sp}_4(3)$ and $\mathrm{Alt}(5)$ is 2 whereas $\mathrm{Alt}(5)$ has no non-trivial central elements.

Lemma 4.1. (i) *$Z = Z(Q)$ has order 3.*

(ii) *L_* and L are 3-constrained.*

(iii) *L_*/Q is 2-constrained, acts irreducibly on Q/Z and $P \cong 2_-^{1+4}$.*

(iv) *Q is extraspecial of $+$-type.*

Proof. From the given structure of L_*, we have $Z \le Q$ and so, as Q is extraspecial, $Z = Z(Q)$ has order 3. This is (i).

Suppose that $C_L(Q) \not\le Q$. Then $C_L(Q)Q/Q$ is a non-trivial normal subgroup of L_*/Q. Let $D \in \mathrm{Syl}_3(C_L(Q))$. Then $|D| \le 9$ and hence is abelian. If $D > Z$, then $DQ = S$ and hence $D \le Z(S) = Z$ which is a contradiction. Thus $D = Z \le Q$ by (i). The assumed structure of L_* now indicates that $C_L(Q) \le QP$. In particular, $L_*/C_L(Q)$ has a composition factor isomorphic to $\mathrm{Alt}(5)$. As Q is extraspecial, the commutator map defines a symplectic form on Q/Z and so $\mathrm{Out}(Q)$ is isomorphic to a subgroup of $\mathrm{GSp}_4(3)$. Since $\mathrm{Sp}_4(3)$ has no subgroups isomorphic to $\mathrm{Alt}(5)$, $C_L(Q) < QP$. If $C_L(Q)Q = \langle u\rangle Q$, then $PC_L(Q)Q/Q$ has 2-rank 4, contrary to the 2-rank of $\mathrm{Sp}_4(3)$ being 2. Thus $\langle u\rangle Q/Q < C_L(Q)Q/Q < PQ/Q$. In this case, $C_{L_*/Q}(PQ/Q)$ must contain a component L_1 isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{SL}_2(5)$. The former case being impossible, we get $L_1 \cong \mathrm{SL}_2(5)$. Since $L_1 \cap PQ/Q$ is normal of order 2 we deduce that $L_1 \ge \langle u\rangle Q/Q$, and once again we have $L_1C_L(Q)Q/Q \cong \mathrm{Alt}(5)$ which is our final contradiction. Hence $C_L(Q) = Z$ and (ii) holds.

Part (iii) follows from Lemma 2.12, since L_*/Q acts faithfully on Q/Z and PQ/Q is extraspecial.

Finally (iv) is a consequence of (iii) and [23, Lemma 2.8]. □

Lemma 4.2. *Suppose that s is an involution of L_* with $sQ \ne uQ$. Then the following hold.*

(i) *$s \in PQ$.*

(ii) *$C_{L_*}(s)PQ/PQ \cong \mathrm{Alt}(4)$.*

(iii) *$Q = C_Q(s)[Q,s]$, $[C_Q(s),[Q,s]] = 1$ and $C_Q(s) \cong [Q,s] \cong 3_+^{1+2}$.*

(iv) *$C_{PQ}(s) \sim 3_+^{1+2}.(\mathrm{Q}(8) \times 2)$.*

Proof. Part (i) follows from Lemma 2.14(ii) and part (ii) comes from Lemma 2.13 (i).

Let $s \in PQ$ be an involution with $sQ \ne uQ$. Then $Q = C_Q(s)[Q,s]$ and the Three Subgroup Lemma shows that $[C_Q(s),[Q,s]] = 1$. Thus, as $sQ \neq uQ$, $[Q,s] < Q$ and so, as s does not centralize Q, we deduce that $C_Q(s) \cong [Q,s] \cong 3_+^{1+2}$ from Lemma 4.1(iv).

Part (iv) follows from Lemma 2.13 (i) and part (iii). □

Lemma 4.3. *Suppose that s is an involution of L_* with $sQ \neq uQ$. Then the following hold.*

(i) $[O_2(C_{L_*}(s)), O_3(C_{L_*}(s))] = 1$.

(ii) $O_2(C_{L_*}(s)) = O_{3'}(C_{L_*}(s)) \cong \mathrm{Q}(8)$.

(iii) $C_{L_*}(s)/O_2(C_{L_*}(s)) \sim 3_+^{1+2}.\mathrm{SL}_2(3)$ *is isomorphic to the centralizer of a non-trivial 3-central element in* $\mathrm{PSp}_4(3)$.

(iv) *If $b \in C_{L_*}(s)$ has order 3 and $b \notin Q$, then $C_{O_{3'}(C_{L_*}(s))}(b) = \langle s \rangle$.*

Proof. Part (i) is trivial (and is included as it illuminates the structure of $C_{L_*}(s)$). Set $Y = C_{L_*}(s)$, $W = C_Q(s) = Q \cap Y$ and select an involution of Qu which centralizes s and, for convenience, call it u. Then, by Lemma 4.2 (iv), $W \cong 3_+^{1+2}$. Therefore $Y/C_Y(W)$ embeds into $\mathrm{Aut}(3_+^{1+2}) \sim 3^2.\mathrm{GL}_2(3)$. As W is extraspecial, $WC_Y(W)/C_Y(W) \cong 3^2$. Let $X = C_Y(W)$. Since $(QP \cap Y)Q/Q \cong \mathrm{Q}(8) \times 2$ by Lemma 4.2 (iv) and since u inverts W/Z, $C_{QP\cap Y}(W) = C_W(W)\langle s\rangle = Z\langle s\rangle$. Hence, as X is normal in Y, we have

$$[X, C_{QP}(s)] \leq X \cap C_{QP}(s) = Z\langle s\rangle.$$

As the elements of order 3 in $Y \setminus W$ act non-trivially on $(PQ \cap Y)Q/Q$, we get $X \leq C_{FQ}(s)$ where $F \in \mathrm{Syl}_2(Y)$. Additionally, as Y/Q is 2-closed, we have $Y/C_Y(W) \sim 3^2.\mathrm{SL}_2(3)$ and $C_Y(W)$ has order $2^3.3$. It follows that $|O_2(Y)| = 2^3$. Noting that $O_2(Y)$ and u are in a common Sylow 2-subgroup of Y, $[Q,s] = C_Q(su)$ and that $O_2(Y)$ acts faithfully on $[Q,s]$ by the 3-constraint of L_*, by applying the above conclusions to the involution su we obtain $O_2(Y) \cong \mathrm{Q}(8)$. As $O_2(Y) = O_{3'}(Y)$, (ii) holds.

Now we have $Y/O_2(Y) \sim 3_+^{1+2}.\mathrm{SL}_2(3)$ and, of course, $O_2(Y/O_2(Y)) = 1$. Now $C_{L_*}(u)/O_2(C_{L_*}(u))$ has shape $3.\mathrm{Alt}(5)$ and hence is isomorphic to $3 \times \mathrm{Alt}(5)$ as the Schur multiplier of $\mathrm{Alt}(5)$ has order 2. Hence $C_{L_*}(u)$ has elementary abelian Sylow 3-subgroups. It follows that the Sylow 3-subgroups of $C_{Y/O_2(Y)}(uO_2(Y))$ are elementary abelian. So, using Lemma 2.4 the conclusion in (iii) holds.

Finally assume we are given $b \in Y \setminus Q$ of order 3. If b centralizes $[Q,s]/Z$, then, as L_* is 3-constrained,

$$|[Q,b]Z/Z| = |[C_Q(s)[Q,s],b]Z/Z| = |[C_Q(s),b]Z/Z| = 3.$$

Select an L_*-conjugate b^* such that bb^* has order divisible by 5. Then $|[Q,bb^*]Z/Z| = 9$, but elements of order 5 which act faithfully on Q/Z in fact act fixed point freely on Q/Z and so we have a contradiction. It follows that b does not centralize $[Q,s]/Z$. Hence $O_2(Y)\langle b\rangle$ acts faithfully on $[Q,s]$ and so $O_2(Y)\langle b\rangle \cong \mathrm{SL}_2(3)$. Therefore $C_{O_2(Y)}(b) = \langle s\rangle$ as claimed in (iv). □

Another, less precise, way of recording Lemma 4.3 is to say that $C_{L_*}(s)$ has shape $(3_+^{1+2} \times \mathrm{Q}(8)).\mathrm{SL}_2(3)$.

Lemma 4.4. *$C_{Q/Z}(S) = [Q/Z, S]$ has order 3^2 and $[Q,S]$ is elementary abelian of order 3^3.*

Proof. Since $L_*/Q \sim 2_-^{1+4}.\mathrm{Alt}(5)$, Lemma 4.2 (ii) implies that there is an involution $s \in PQ$ which centralizes S/Q. Hence S normalizes $Q_1 = C_Q(s)$ and $Q_2 = [Q,s]$. Thus, by Lemma 4.2(iii), $C_{Q/Z}(S) = C_{Q_1/Z}(S)C_{Q_2/Z}(S)$. By Lemma 2.13 (ii), sQ and suQ are conjugate in $N_{PQ/Q}(S/Q) \cong \mathrm{Dih}(8)$ by an element fQ say. Since u inverts Q/Z by Lemma 4.1(iii), we get that $Q_1^f = Q_2$. Thus $|C_{Q_1/Z}(S)| = |C_{Q_2/Z}(S)|$. Therefore, as L_* is 3-constrained by Lemma 4.1 (ii), $|C_{Q/Z}(S)| = 3^2$. Since, for $i = 1, 2$, $[Q_i/Z, S] \le C_{Q_i/Z}(S)$, we get that $C_{Q/Z}(S) = [Q/Z, S]$ has order 3^2 as claimed. The Three Subgroup Lemma and Q being of exponent 3 shows that $[Q,S]$ is elementary abelian. Finally, noting that $Z \le [Q,S]$ we have the lemma. □

We now put $J = C_S([Q,S])$, and start the investigation of the 3-local subgroup $M = N_G(J)$. Set $M_* = O^{3'}(M)$.

Lemma 4.5. *The following hold.*

(i) *$J = J(S)$ is elementary abelian of order 3^4.*
(ii) *Z is weakly closed in Q with respect to G.*
(iii) *If $g \in G$ and $Z^g \le S$, then $Z^g \le J$.*

Proof. Let $s \in PQ$ be an involution such that $sQ \ne uQ$. Set $Q_1 = C_Q(s)$ and $Q_2 = [Q,s]$. Then we proceed exactly as in [21, Lemmas 13 and 15] first to show that $[S,S,S] \le Z$ and then to find that $J = J(S)$ is abelian of order 3^4 (but not necessarily elementary abelian). Arguing as in the proof of [21, Lemma 16], we get that Z is weakly closed in Q. So (ii) holds.

Suppose that $X = Z^g \le S$ and $X \ne Z$. Then $X \not\le Q$ by (ii). Assume that $[Q,S]X$ is non-abelian. Since $[[Q,S]X, Q] \le [Q,S]$ and $X \le [Q,S]X$, $S = QX$ normalizes $[Q,S]X$. Similarly, $[[Q,S],X] \le [[Q,S],S] \le Z$ by Lemma 4.4, and so $[Q,S]X$ normalizes ZX. Let $F = N_S(ZX)$. Then, as $S = QX$, $F = (F \cap Q)[Q,S]X$ and $[F \cap Q, X] \le ZX \cap Q = Z$. Therefore, applying Lemma 4.4, $(F\cap Q)/Z \le C_{Q/Z}(X) = C_{Q/Z}(S) = [Q,S]/Z$ and so $F = [Q,S]X$. Note that, as $[Q,S]$ does not centralize X, ZX contains three $[Q,S]$-conjugates of X. Hence every non-trivial element of XZ is conjugate to an element of Z. Since S

normalizes $[Q,S]X = N_S(ZX)$ and Z, there are nine S-conjugates of ZX contained in $[Q,S]X$ and these conjugates pairwise intersect in Z. Thus $[Q,S]X$ contains at least $9.3+1 = 28$ conjugates of Z. On the other hand, Z is the only conjugate of Z contained in $[Q,S]$ by (ii). Hence $[Q,S]$ contains twelve groups of order 3 which are not conjugate to Z. Since this accounts for all the subgroups of order 3 in $[Q,S]X$, we deduce that all the subgroups of $[Q,S]X$ which have order 3 and are not conjugate to Z are contained in $[Q,S]$. Now consider $C_{[Q,S]X}(X)$. Since $|[Q,S]X| = 3^4$ and $[[Q,S],X] = Z$, $|C_{[Q,S]X}(X)| = 3^3$. Therefore, setting $D = C_{[Q,S]X}(X) \cap Q^g$ (recall $X = Z^g$), we have $|D| \geq 3^2$. Since X is weakly closed in Q^g by (ii), X is the unique conjugate of Z in D. Therefore every element of D which is not in X is in fact in $[Q,S]$ and so

$$X \leq D = \langle d \mid d \in D \setminus X\rangle \leq [Q,S]$$

and we have a contradiction. Thus $[X,[Q,S]] = 1$ and so $X \leq J$ by the definition of J and hence (iii) holds.

Since Z is not weakly closed in S with respect to G, there exists a conjugate $X = Z^g \in S$ with $X \neq Z$. By (iii), $X \leq J$ and by (ii) $X \not\leq Q$. Since $[Q,S] = C_Q([Q,S])$, we conclude that $J = [Q,S]X$ is elementary abelian of order 3^4. Since the 3-rank of Q is 3, we now have that the 3-rank of S is 4 as $|S:Q| = 3$. Assume that B is an abelian subgroup of S of order 3^4 and that $B \neq J$. If $BJ = S$, then $Z = Z(S) \geq J \cap B$ which has order 9, a contradiction. Thus $|BJ| = 3^5$ and $|B \cap J| = 3^3$. If $B \cap J \leq Q$, then $B \cap J \leq Q \cap J = [Q,S]$. Since $|[Q,S]| = 3^3$, this gives $[Q,S] \leq B$ which implies that $B \leq C_S([Q,S]) = J$, another contradiction. Thus $B \cap J \not\leq Q$. If $x \in (B \cap J) \setminus Q$, then x centralizes $BJ \cap Q$ which has order 3^4. But then

$$(BJ \cap Q)/Z \leq C_{Q/Z}(x) = C_{Q/Z}(S) = [Q,S]/Z,$$

and so we conclude that no such B exists. Thus J is the unique abelian subgroup of S of maximal order and so $J = J(S)$. Thus (i) is true. □

Lemma 4.6. *The following hold:*

(i) $S = QJ$;
(ii) $L \cap M = N_G(S)$.
(iii) $C_G(J) = C_G([Q,S]) = J$.

Proof. Since $J \cap Q$ is elementary abelian of order 3^3, we have that $S = QJ$, giving (i). Also $N_G(S)$ normalizes $Z(S) = Z(Q)$ and $J = J(S)$. Hence $N_G(S) \leq L \cap M$. Since $L \cap M$ normalizes $S = QJ$, (ii) holds.

From $Z \leq [Q,S]$, $C_G([Q,S]) \leq L_*$ and so, by Lemma 2.8, $C_G([Q,S])$ is a 3-group. Since $C_G([Q,S]) \cap Q = [Q,S]$, we have $|C_G([Q,S])| \leq 3^4$ and hence $C_G([Q,S]) = J$ as claimed. □

Lemma 4.7. (i) *There are exactly ten G-conjugates of Z in J.*
(ii) $|L/L_*| = 2$, $L \sim 3_+^{1+4}.2_-^{1+4}.\mathrm{Sym}(5)$.
(iii) $M/O_{3,2}(M) \cong \mathrm{Aut}(\mathrm{Alt}(6))$, $M_*/J \cong \mathrm{Alt}(6)$ *and* $|Z(M/J)| = |O_{3,2}(M)/J| = 2$.

Proof. Since Z is not weakly closed in S, Lemma 4.5 (ii) and (iii) imply that there exists $g \in G$ such that $X = Z^g \le J$ and $X \ne Z$. Since J is abelian, J centralizes ZX and, as in the proof of Lemma 4.5(iii), $N_S(ZX) = [Q,S]X = J$. Thus there are nine S-conjugates of X in J. This shows that the number of G-conjugates of Z in J is congruent to 1 modulo 9. Since, by Lemmas 2.5 and 4.5 (i), M controls G-fusion in J, all the G-conjugates of Z in J are conjugate in M. Because there is a unique conjugate of Z in $J \cap Q$ by Lemma 4.5(ii), we deduce that $|Z^M| \le 28$. Since M/J acts faithfully on J by Lemma 4.6 (ii), we have that M/J is isomorphic to a subgroup of $\mathrm{GL}_4(3)$. Now $|\mathrm{GL}_4(3)|$ is not divisible by either 7 or 19 and so there is no choice other than $|Z^M| = 10$. Hence (i) holds.

Since J is characteristic in S, $N_{L_*}(S) \le M$. Thus, as $X^S = Z^M \setminus \{Z\}$ and $N_{L_*}(S)$ normalizes Z, $N_{N_{L_*}(S)}(X)S = N_{L_*}(S)$. In particular, X is normalized by a Sylow 2-subgroup T of $N_{L_*}(S)$. Since XQP/QP is inverted in L_*/QP, we must have that X is inverted by an element in T. Hence $L > L_*$ and now (ii) follows from Lemma 2.12.

From (ii) we have $|N_L(S)/J| = 2^5.3^2$. Therefore $|M/J| = 2^6.3^2.5$ by (i). Furthermore, M acts two transitively on Z^M which has order 10. Since $S/J = QJ/J$ has order 9 and is elementary abelian, we see that $C_{N_L(S)/J}(S/J)$ contains an involution i. Because S acts transitively on $Z^M \setminus \{Z\}$ we see that i normalizes every subgroup in Z^M and furthermore it either centralizes all or inverts all of the subgroups in X^S. Since J is self-centralizing we infer that i inverts every element of J. In particular, $i \in Z(M/J)$. Since M has order divisible by 5, M acts irreducibly on J. From $C_J(S) = Z$, we conclude that the splitting field for the action of M on J is GF(3). Therefore, $|Z(M/J)| = 2$ by Schur's Lemma. Let N be the kernel of the action of M on Z^M. Assume that $N > \langle i\rangle J$ and choose $n \in N \setminus \langle i\rangle J$. If n centralizes every element of Z^M or inverts every element of Z^M, then $n \in C_G(J)\langle i\rangle = J\langle i\rangle$ which is not the case. Thus n inverts some element in Z^M which, as $\langle i\rangle J \trianglelefteq N$, we may without loss take to be Z and centralizes some element $X \in Z^M$. From $S = QX$, we see that n normalizes S and then obtain that n centralizes all the members of $Z^M \setminus \{Z\}$. Since $\langle Z^M \setminus \{Z\}\rangle$ is normalized by S, $Z \le \langle Z^M \setminus \{Z\}\rangle$ and so n centralizes Z, which is a contradiction. Thus $N = J\langle i\rangle$. Using [5, Theorem XIII and Corollary, page 202] (Burnside says that this result is in fact due to Galois and is contained in his final letter of May 29th 1832), we get that M/N is an almost non-abelian simple group. It follows that $M/N \cong \mathrm{Aut}(\mathrm{Alt}(6))$ and that $N = O_{3,2}(M)$. If $O^2(M/J) \cong 2{}^{\cdot}\mathrm{Alt}(6) \cong \mathrm{SL}_2(9)$, then, as a Sylow 2-subgroup of the normalizer of a Sylow 3-subgroup in $\mathrm{Aut}(\mathrm{Alt}(6))$ is isomorphic to $\mathrm{SDih}(16)$, we infer that $N_L(S)$ has exponent 16. On the other hand, using (ii) it is straight forward to see that $N_L(S)$ has no elements of order 16. Therefore, $M_*/J = O^{3'}(M/J) \cong \mathrm{Alt}(6)$. □

Define $M_0 = M_*O_{3,2}(M)$ and let $t \in N_P(S)$ be an involution with $t \neq u$. Finally set $M_1 = \langle t\rangle M_0$ and $B = [J,t]$.

Lemma 4.8. $C_S(t) \in \mathrm{Syl}_3(C_{L_*}(t))$.

Proof. This follows directly from Lemma 4.3(iii).

□

Lemma 4.9. *B has order 3 and $|C_J(t)| = 3^3$.*

Proof. From the choice of t, we have that $B = [J,t] \leq J \cap Q = [Q,S]$. Since $t \in L_*$, t centralizes Z and so $|B| \leq 9$. If $|B| = 9$, then $[Q,S] = [[Q,S],u]Z = BZ$ and so $[[Q,S],ut] \leq Z$. Since ut centralizes Z and $J/[Q,S]$, we reason that ut centralizes J and, because $C_J(J) = J$ by Lemma 4.6 (iii), this means that $u = t$ contrary to the choice of t. Thus $|B| = 3$ and $|C_J(t)| = 3^3$ as claimed. □

Lemma 4.10. $M_1/J \cong 2 \times \mathrm{Sym}(6)$.

Proof. From Lemma 4.7(iii), we have that $M_0/J \cong 2 \times \mathrm{Alt}(6)$. Note that in its action on J, as M_* is perfect and $O_{3,2}(M)$ inverts J, every element of M_0 has determinant 1. Since, by Lemma 4.9, $|B| = 3$, we see that t has determinant -1 as an operator on J and so $t \notin M_0$. Again using Lemma 4.7, we have $N_M(S)/O_{3,2}(M)S \cong \mathrm{SDih}(16)$. Thus, as t is an involution, we deduce that $\langle t\rangle N_{M_0}(S)/O_{3,2}(M)S \cong \mathrm{Dih}(8)$. Hence $\langle t\rangle M_0/O_{3,2}(M) \cong \mathrm{Sym}(6)$. Since t is an involution and $M_0/O_{3,2}(M)$ contains normal subgroup isomorphic to $\mathrm{Alt}(6)$, we finally infer that $M_1/J \cong 2 \times \mathrm{Sym}(6)$. □

Lemma 4.11. *J is the 4-dimensional permutation $\mathrm{GF}(3)$-module for M_*/J.*

Proof. The proof of (i) is identical to the proof presented in [26, Lemma 5.4]. □

Lemma 4.12. *M has two orbits on the subgroups of J of order 3. One is Z^M and has length 10 and the other is B^M and has length 30. Furthermore, $N_M(Z)/J \sim (2 \times 3^2).\mathrm{SDih}(16)$ and $N_M(B)/J \cong 2 \times 2 \times \mathrm{Sym}(4)$.*

Proof. We have seen in Lemmas 4.6 (ii) and 4.7 that $|Z^M| = 10$ and $N_M(Z) = N_G(Z) = L \cap M$. The structure of $N_M(Z)/J$ can be extracted from Lemma 4.7 (ii).

Suppose that X is a subgroup of J of order 3 which is not in Z^M. Then X is not 3-central and, as the elements of order 5 in M act fixed point freely on J and $\mathrm{Aut}(\mathrm{Alt}(6))$ has no subgroups of index 15, we have that $|X^M| = 30$, as claimed. Furthermore, in $\mathrm{Aut}(\mathrm{Alt}(6))$, the subgroup of index 30 is contained in $\mathrm{Sym}(6)$. Thus Lemma 4.10 implies that $N_M(X)/J \cong 2 \times 2 \times \mathrm{Sym}(4)$. Finally we note that $B \le J \cap Q$ and $B \ne Z$ and so $B^M = X^M$ by Lemma 4.5(ii). □

Corollary 4.13. *$N_M(B)$ has two orbits on Z^M.*

Proof. We have that $|Z^M| = |Z^{M_*}| = 10$. Since $M^*/J \cong \mathrm{Alt}(6)$ and $\mathrm{Alt}(6)$ has a unique transitive permutation representation of degree 10, calculation in this permutation representation yields the statement. □

5. The centralizer of B

In this brief section we uncover the structure of $C_G(B)$. We maintain the notation of the previous section. So $t \in N_P(S)$ is an involution with $t \ne u$ and $B = [J, t]$.

Lemma 5.1. $\text{И}_{L_*}(J, 3') = \{1\}$.

Proof. Suppose that $R \in \text{И}_{L_*}(J, 3')$. Then, as R is normalized by J and normalizes Q, R centralizes $Q \cap J = [Q, S]$. Hence $R \le J$ by Lemma 4.6 (iii) and so $R = 1$. □

We now extend the scope of the last lemma to the whole of G.

Lemma 5.2. $\text{И}_G(J, 3') = \{1\}$.

Proof. Suppose that $R \in \text{И}_G(J, 3')$. Then $R = \langle C_R(H) \mid |J : H| = 3 \rangle$. By Lemmas 2.11, 4.11 and 4.12, each H with $|J : H| = 3$ contains a M-conjugate of Z. Thus

$$R = \langle C_R(Y) \mid Y \le J \text{ and } Y \text{ is } M\text{-conjugate of } Z \rangle.$$

Since, for each $Y \in Z^M$, $C_R(Y) \in \text{И}_{C_G(Y)}(J, 3')$, Lemma 5.1, implies that $C_R(Y) = 1$. Thus $R = 1$ and the lemma holds. □

Lemma 5.3. *We have that $C_{L_*}(B)/B$ is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$. Furthermore, $C_L(B)/B$ inverts ZB/B.*

Proof. Since Q is extraspecial of exponent 3, we have $C_Q(B) \cong 3 \times 3^{1+2}_+$. From Lemma 2.14 (i), we have that $C_{L_*}(B)Q/Q \cong \mathrm{SL}_2(3)$. Thus $C_{L_*}(B)/B \sim 3^{1+2}_+.\mathrm{SL}_2(3)$. Let $U = O_2(C_{L_*}(B))$. Then $[C_Q(U), U, Q] =$

$[Q, C_Q(U), U] = 1$ and hence the Three Subgroup Lemma implies that $C_Q(U)$ and $[Q, U]$ commute. If $U > 1$, then $Q \geq C_Q(U) \geq C_Q(B)$. Thus $|Q : C_Q(U)| \leq 3$ and hence $|[Q, U]| \leq 9$. In particular, $[Q, U]$ is abelian. Since $Q = C_Q(U)[Q, U]$ with $C_Q(U)$ and $[Q, U]$ commuting, we have $[Q, U] \leq Z$ which means that U centralizes Q and contradicts Lemma 4.1(ii). Therefore $U = 1$. Since $J \leq C_{L_*}(B)$ and J/B is elementary abelian of order 3^3, $C_{L_*}(B)/B$ satisfies the hypothesis of Lemma 2.3 and so $C_{L_*}(B)/B$ is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$.

By Lemma 2.14 (i), L_* acts transitively on $(Q/Z)^\#$ and so, as Q is extraspecial, L_* acts transitively on $Q \setminus Z$. Consequently $C_L(B) > C_{L_*}(B)$ and so ZB/B is inverted by $C_L(B)$. □

Lemma 5.4. *We have $C_G(B) \cong 3 \times \mathrm{Aut}(\mathrm{U}_4(2))$, $N_G(B) \cong \mathrm{Sym}(3) \times \mathrm{Aut}(\mathrm{U}_4(2))$ and t centralizes $O^3(C_G(B))$.*

Proof. Lemmas 5.2 and 5.3 imply that $C_G(B)/B$ satisfies the hypotheses of Theorem 2.2. Furthermore by Lemma 4.12, $N_M(B) \sim 3^4.(2 \times 2 \times \mathrm{Sym}(4))$ which is not a subgroup of L. Therefore $C_G(B) \neq C_L(B)$ and hence Theorem 2.2 gives $C_G(B)/B \cong \mathrm{Aut}(\mathrm{U}_4(2))$ or $\mathrm{Sp}_6(2)$. Set $E = O^3(C_G(B))$. Then as $\mathrm{Out}(\mathrm{Sp}_6(2)) = \mathrm{Out}(\mathrm{Aut}(\mathrm{U}_4(2)) = 1$ and the Schur multipliers of $\mathrm{Sp}_6(2)$ and $\mathrm{U}_4(2)$ have trivial 3-part, $E \cong \mathrm{Aut}(\mathrm{U}_4(2))$ or $\mathrm{Sp}_6(2)$ and, as t inverts B, $N_G(B) \cong \mathrm{Sym}(3) \times E$. Since t centralizes $E \cap J$ which is elementary abelian of order 3^3 and since this subgroup is self-centralizing in E, we infer that $B\langle t\rangle = C_{N_G(B)}(E) \cong \mathrm{Sym}(3)$. Thus the lemma will be proved once we have eliminated the possibility that $E \cong \mathrm{Sp}_6(2)$.

Suppose that $E \cong \mathrm{Sp}_6(2)$. Then E contains a subgroup F with $F \cong \mathrm{Sp}_2(2) \times \mathrm{Sp}_4(2) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(6)$. Since there is a unique conjugacy class of elementary abelian subgroups of order 27 in $\mathrm{Sp}_6(2)$, we may choose F so that $J \cap E \in \mathrm{Syl}_3(F)$. Note that t centralizes F. Let $R_1 \in \mathrm{Syl}_2(N_{\langle t\rangle F}(J \cap E))$. Then $R_1 \cong 2 \times 2 \times \mathrm{Dih}(8) \leq N_F(J)$ and R_1 contains t which inverts B. Let $x \in R_1'$. Then $x \leq F'' \cong \mathrm{Alt}(6)$ and x inverts $J \cap F''$ and centralizes $O_3(F)B$. On the other hand, by Lemma 4.10, $R_1 \leq M_1 \sim 3^4.(2 \times \mathrm{Sym}(6))$ and so $R_1' \leq M_*$. But then $C_J(x)$ contains 3-central elements of G by 2.10 (ii). Hence $O_3(F)B$ contains a 3-central element of G, say e. However this means that $\mathrm{Alt}(6) \cong F'' \leq C_G(e) \sim 3_+^{1+4}.2_-^{1+4}.\mathrm{Alt}(5)$, which is absurd. Hence $E \not\cong \mathrm{Sp}_6(2)$ and the lemma is proven. □

Now set $E = O^3(C_G(B))$, $E_L = E \cap L$ and $E_M = E \cap M$.

Lemma 5.5. *$E_L \sim 3_+^{1+2}.\mathrm{GL}_2(3)$ and $E_M = N_E(J_K) \sim 3^3.(2\times\mathrm{Sym}(4))$.*

Proof. We have that $E = C_G(\langle t, B\rangle)$ and so Z and J_K are contained in E. That Z is a 3-central subgroup of E follows from Lemma 5.3. Hence $E_L \sim 3_+^{1+2}.\mathrm{GL}_2(3)$ by [6, pg. 26]. Since a Sylow 3-subgroup of E contains a unique elementary abelian subgroup of order 27, we read that $E_M = N_E(J_K) \sim 3^3.(2 \times \mathrm{Sym}(4))$ from [6, pg. 26]. □

6. The centralizer of t

We now start our investigation of the centralizer of the involution t. We set $K = C_G(t)$. By Lemma 5.4, K contains $E = O^3(C_G(B)) \cong \mathrm{Aut}(\mathrm{U}_4(2))$. Our first lemma asserts that we already see the Sylow 3-subgroup of K in $C_L(t)$.

Lemma 6.1. *$C_S(t)$ is a Sylow 3-subgroup of K. In particular, $|K|_3 = 3^4$ and E contains a Sylow 3-subgroup of K.*

Proof. Let $F = C_S(t)$. Then Lemmas 4.3(iii) and 4.8 imply that $Z(F) = Z$ and $F \in \mathrm{Syl}_3(C_L(t))$. If $F_1 \in \mathrm{Syl}_3(K)$ and $F \le F_1$, then $N_{F_1}(F)$ normalizes Z and is consequently contained in L. Thus $N_{F_1}(F) = F$ and so $F = F_1$. □

Lemma 6.2. *The involutions t and u are not G-conjugate and $u \in M_*$.*

Proof. Choose an element s of order 2 in $N_{M_*}(S)$. Then s inverts S/J. Using Lemma 2.10(ii) and (vi) we see that s centralizes $J/(J \cap Q)$ and Z, and inverts $(Q \cap J)/Z$. Since s normalizes Q by Lemma 4.6 (ii), we deduce that $\langle s\rangle Q = \langle u\rangle Q$. In particular, $u \in M_*$ and so we have that $C_S(u) = C_J(u)$ contains exactly two 3-central subgroups by Lemma 2.10(ii). Let $F = C_S(u)$. Suppose that $F_1 \in \mathrm{Syl}_3(C_G(u))$ with $F \le F_1$. If $F_1 > F$, then $|Z^{N_{F_1}(F)}| = 3$ which is not the case. Thus $F_1 = F$ has order 9 and consequently, using Lemma 6.1, we see that t and u are not G-conjugate. □

Lemma 6.3. *Suppose that x is an involution of M with $|C_J(x)| = 3^3$. Then x is M-conjugate to t.*

Proof. Note that, as $|C_J(x)| = 3^3$, $[J, x]$ is cyclic of order 3. Suppose that $[J, x]$ is 3-central. Then, by Lemma 4.12, we may without loss suppose that $[J, x] = Z$ and so, as x inverts $[J, x]$, $x \in L \setminus L_*$. Because x centralizes J/Z, we have that J/Z preserves the decomposition of $Q/Z = C_{Q/Z}(x) \times [Q/Z, x]$. Since $C_{Q/Z}(x) \ge (J \cap Q)/Z$, we have $[[Q, x], J]Z/Z \le (J \cap Q)/Z \cap [Q/Z, x] \le C_{Q/Z}(x) \cap [Q/Z, x] = 1$. Thus J centralizes Q/Z which means that $[J, Q] \le Z$ and contradicts Lemma 4.1(ii). Thus we infer that $[J, x]$ is not 3-central. Therefore by Lemma 4.12 we may conjugate x in M so that $[J, x] = B = [J, t]$. Since x and t are involutions, xt centralizes B. But then xt centralizes J and

Lemma 4.6 (iii) implies that $xt \in J$. Therefore x and t are J-conjugate. Thus, in particular, x and t are M-conjugate. □

Set $J_K = J \cap K$.

Lemma 6.4. *We have that*

(i) $J_K = J(C_S(t))$;
(ii) $N_G(J_K) \leq M$;
(iii) $C_G(J_K) = J\langle t \rangle$;
(iv) $N_K(J_K)/C_K(J_K) \cong 2 \times \mathrm{Sym}(4)$; *and*
(v) $N_K(J_K) \leq \langle t \rangle E$

Proof. Since $C_S(t)$ is isomorphic to a Sylow 3-subgroup of $\mathrm{PSp}_4(3)$ by Lemma 4.3(iii), (i) holds.

Let $Y = N_G(J_K)$. Then Y normalizes $C_Y(J_K)$ which contains J. Hence the Frattini Argument implies that $Y = N_Y(J)C_Y(J_K)$. Since $Z \leq J_K$, $C_G(J_K) \leq L_*$. Because J_K centralizes t, $J_K \not\leq Q$ and so $C_G(J_K) \leq N_{L_*}(S)$ is 3-closed. It follows that $C_Y(J_K)$ normalizes $J = J(S)$. So (ii) holds.

Combining Lemma 5.4 with information about subgroups of $\mathrm{Aut}(\mathrm{U}_4(2))$ given in the ATLAS [6], we have $N_{N_G(B)}(J_K) \sim \mathrm{Sym}(3) \times 3^3.(2 \times \mathrm{Sym}(4))$. Since $N_G(J_K) \leq N_M(B)$, parts (iii), (iv) and (v) now follow from Lemma 4.12. □

Lemma 6.5. *K contains a subgroup isomorphic to* $\mathrm{Sym}(3) \times \mathrm{Sym}(6)$.

Proof. Let $t_1 \in E$ be such that $C_E(t_1) \cong 2 \times \mathrm{Sym}(6)$. Then $C_G(t_1) \geq B\langle t \rangle \times C_E(t_1) \cong \mathrm{Sym}(3) \times 2 \times \mathrm{Sym}(6)$ and so it suffices to show that t and t_1 are G-conjugate.

We make our initial choice of t_1 so that there exists $F \in \mathrm{Syl}_3(C_E(t_1))$ such that $F \leq C_S(B)$. Then by Lemma 2.17 F is contained in the Thompson subgroup of $S \cap E$ which is J_K. Hence $BF \leq J$.

Since BF is a maximal subgroup of J, BF contains a conjugate of Z by Lemma 2.11. Conjugating by a suitable element of M we may then suppose that $Z \leq BF \leq J$ and t_1 centralizes BF. Thus we may view the entire configuration in L_*. If $t_1 \in QP$, then either t_1 is conjugate to u or t_1 is conjugate to t. Since $|C_{L_*}(u)|_3 = 3^2$, we have that t_1 is conjugate to t in this case and are done. In the case when $t_1 \in L_* \setminus QP$ we have $C_{L_*/QP}(t_1QP)$ is a 2-group. As BF is centralized by t_1 we infer that $BF \leq QP$ and so $BF = Q \cap J$. Therefore $[[Q, \langle J, J^{t_1} \rangle], \langle J, J^{t_1} \rangle] \leq [BF, \langle J, J^{t_1} \rangle] = 1$. Hence $\langle J, J^{t_1} \rangle$ is a 3-group. Since J is the Thompson subgroup of any 3-group containing it, $J = J^{t_1}$. Finally, we deduce that t and t_1 are conjugate in M by Lemma 6.3. This completes the proof of the lemma. □

For $n \in \{0,1,2,3,4\}$, $\mathcal{Z}_n$ denotes the set of subgroups of J_K of order 9 containing precisely n subgroups which are G-conjugate to Z.

Lemma 6.6. (i) *J_K contains exactly 4 subgroups G-conjugate to Z and the remaining subgroups of J_K of order 3 are all G-conjugate to B.*

(ii) *The $N_K(J_K)$ orbits, under conjugation, of the subgroups of J_K of order 9 are $\mathcal{Z}_0$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Further, $|\mathcal{Z}_0| = 3$, $|\mathcal{Z}_1| = 4$ and $|\mathcal{Z}_2| = 6$.*

Proof. From Lemma 6.4 (iii), we have that $N_K(J_K)/C_K(J_K) \cong 2 \times Sym(4)$. Since J_K is irreducible as an $N_K(J_K)$-module, the centre of $N_K(J_K)/C_K(J_K)$ inverts J_K and thus has no effect on the orbits of $N_K(J_K)$ on subgroups of J_K. Using the notation from Lemma 2.9, we note that the subgroups corresponding to $N_K(J_K)$ conjugates of Z are the subgroups of the form $\langle v_1 \pm v_2 \pm v_3\rangle$. Thus $|Z^{N_K(J_K)}| = 4$ by Lemma 2.9 (ii). Since $N_M(B) = N_K(J_K)J$, we have that $N_K(J_K)$ has two orbits on Z^M by Corollary 4.13. Obviously one of the orbits is $Z^{N_K(J_K)}$ and, as J_K is not normalized by M, the other is an orbit of subgroups contained in J but not in J_K. Since M controls G-fusion in J, we conclude that J_K contains exactly four subgroups G-conjugate to Z. The second statement in (i) now follows from Lemma 4.12.

Now Lemma 2.9 (iii) gives part (ii) of the lemma. □

Lemma 6.7. *Let $A \in \mathcal{Z}_1$ and $a \in A^{\#}$ be 3-central. Then $A = J_K \cap O_3(C_G(a))$.*

Proof. By Lemma 4.5 (ii), we have that $J_K \cap O_3(C_G(a)) \in \mathcal{Z}_1$. The result is now verified as, by Lemma 6.6, there are exactly four $N_K(J_K)$-conjugates of $\langle a\rangle$ in J_K and $|\mathcal{Z}_1| = 4$. □

Lemma 6.8. *Suppose that $|J_K : A| = 3$. Then $J_K \in \mathrm{Syl}_3(C_K(A))$.*

Proof. Since J_K is abelian and $J_K \le K$, $J_K \le C_K(A)$. By Lemma 6.6, there exists $b \in A$ which is not 3-central. Now $C_G(b) \cong 3 \times \mathrm{Aut}(\mathrm{U}_4(2))$ by Lemma 5.4. Set $E_b = O^2(C_G(b))$. Then $t \in E_b$. From [6, pg. 26] we read that $|C_{E_b}(t)|_3 \le 3^2$ and so the result follows. □

Using the notation from Lemma 6.8 we see that either $C_{E_b}(t) \cong 2 \times \mathrm{Sym}(6)$ or $C_{E_b}(t) \sim 2^{1+4}_+.3^2.2^2$.

Lemma 6.9. *Suppose that $[J_K : A] = 3$.*

(i) *If $A \in \mathcal{Z}_1$, then $O_{3'}(C_K(A)) \cong \mathrm{Q}(8)$. Also, for $b \in A^{\#}$ with b not 3-central in G,*

$$O_{3'}(C_K(A)) \le O_{3'}(C_K(\langle b\rangle)) \cong 2^{1+4}_+.$$

(ii) *If* $A \in \mathcal{Z}_0 \cup \mathcal{Z}_2$, *then* $O_{3'}(C_K(A)) = \langle t \rangle$.

(iii) *If* $T \in$ И$_{C_G(A)}(J_K, 3')$, *then* $T \le O_{3'}(C_K(A))$.

Proof. Assume that $A \in \mathcal{Z}_1$. Let $a \in A^\#$ be a 3-central element and $b \in A \setminus \langle a \rangle$. Then $C_G(a) \sim 3_+^{1+4}.2_-^{1+4}.\mathrm{Alt}(5)$. Since every element of order 2 in $C_G(a)$ is contained in $O_{3,2}(C_G(a))$ by Lemma 2.14(ii), we have that $t \in O_{3,2}(C_G(a))$. As t is not conjugate to the elements in $Z(C_G(a)/O_3(C_G(a))$ by Lemma 6.2, we have $O_{3'}(C_{C_G(a)}(t)) \cong \mathrm{Q}(8)$ by Lemma 4.3. By Lemmas 4.3 and 6.7, $A = J_K \cap O_3(C_G(a)) \le C_{O_3(C_G(a)}(t)$. Thus Lemma 4.3 (i) and (ii) imply that $O_{3'}(C_K(A)) \cong \mathrm{Q}(8)$ which is the first claim in (i). We now focus on b. Using Lemmas 6.6 (i) and 5.4, we have $C_G(b) \cong 3 \times \mathrm{Aut}(\mathrm{U}_4(2))$. Let $E_b = O^3(C_G(b))$. Then, as t centralizes b, $t \in E_b$. Now $C_{C_G(b)}(t)$ contains $O_{3'}(C_K(A)) \cong \mathrm{Q}(8)$. Hence, as $2 \times \mathrm{Sym}(6)$ doesn't contain a subgroup isomorphic to $\mathrm{Q}(8)$, we may use [6, pg. 26] to deduce that $t \in E_b'$ and that $C_K(\langle b \rangle) \sim 3 \times 2_+^{1+4}.3^2.2$. Thus $O_{3'}(C_K(b)) \cong 2_+^{1+4}$. Now applying [18, 8.2.12, pg. 189] we have that (i) holds.

Assume that $A \in \mathcal{Z}_2$ and just as above let $a \in A^\#$ be a 3-central element. By Lemma 6.7, $A \not\le O_3(C_G(a))$. Let $b \in A \setminus O_3(C_G(a))$. Again by Lemma 6.2, t is not conjugate to an element of $Z(C_G(a)/O_3(C_G(a))$. Hence using Lemmas 2.14 (ii) and 4.3(iv) we get $C_{O_{3'}(C_K(a))}(b) = \langle t \rangle$. In particular, using [18, 8.2.12, pg. 189] yet again (ii) holds for $A \in \mathcal{Z}_2$.

Suppose that $A \in \mathcal{Z}_0$. Let $b \in A^\#$. Then $C_G(b) \cong 3 \times \mathrm{Aut}(\mathrm{U}_4(2))$ by Lemma 6.6 (i). Put $E_b = O^3(C_G(b))$. Then $t \in E$ and since J_K is centralized by t, using [6, pg. 26] we have $C_{C_G(b)}(t) \sim 3 \times 2_+^{1+4}.3^2.2^2$ or $C_{C_G(b)}(t) \cong 3 \times 2 \times \mathrm{Sym}(6)$. In the latter case the centralizer in $C_{C_G(b)}(t)$ of any further element of order 3 has shape $2 \times 3 \times \mathrm{Sym}(3)$ and so (ii) holds if this possibility arises. So assume the former possibility occurs. Then, as $O^2(C_{E_b}(t))$ is isomorphic to the central product $\mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)$, $C_{O_2(C_{E_b}(t))}(A \cap E_b)$ either has order 8 or 2. In the former case we deduce from centralizer orders that $A \cap E$ is 3-central in E and consequently 3-central in G, a contradiction. Thus $C_{O_2(C_E(t))}(A \cap E) = \langle t \rangle$ and so (ii) holds when $A \in \mathcal{Z}_0$.

By Lemma 6.8, $J_K \in \mathrm{Syl}_3(C_K(A))$ and so, as $C_K(A)$ is soluble, $J_K O_{3'}(C_K(A)) = O_{3',3}(C_K(A))$. Therefore any $3'$-subgroup of $C_K(A)$ which is normalized by J_K centralizes $J_K O_{3'}(C_G(A))/O_{3'}(C_G(A))$. Hence, as $C_K(A)$ is soluble, (iii) follows from [18, 6.4.4 pg. 134]. □

Define $R = \langle O_{3'}(C_K(A)) \mid A \in \mathcal{Z}_1 \rangle$. Notice, that by Lemma 6.9 (i) and (ii), we also have that $R = \langle O_{3'}(C_K(A)) \mid |J_K : A| = 3 \rangle$.

Lemma 6.10. $R \cong 2_+^{1+8}$ *and* И$^*_K(J_K, 3') = \{R\}$.

Proof. As $J_K \le C_K(A)$ for all $A \in \mathcal{Z}_1$, R as defined is normalized by J_K. Let $\mathcal{Z}_1 = \{A_1, A_2, A_3, A_4\}$. Then, by Lemma 6.9(i), for $1 \le i \le 4$, $O_{3'}(C_K(A_i)) \cong \mathrm{Q}(8)$. Additionally, for $1 \le i < j \le 4$, $A_i \cap A_j$ is a G-conjugate of B by Lemmas 6.6(i) and 6.7. Thus $O_2((C_K(A_i \cap A_j)) \cong 2_+^{1+4} \cong \mathrm{Q}(8) \circ \mathrm{Q}(8)$ by Lemma 6.9 (i). Note that 2_+^{1+4} contains exactly two subgroups isomorphic to Q(8) and that these subgroups commute. Assume that $O_{3'}(C_K(A_i)) = O_{3'}(C_K(A_j))$, then this subgroup is centralized by $\langle A_i, A_j\rangle = J_K$. Since $\mathcal{Z}_0 \cup \mathcal{Z}_2 \ne \emptyset$, this contradicts Lemma 6.9 (ii) and (iii). Thus $[O_{3'}(C_K(A_i)), O_{3'}(C_K(A_j))] = 1$. It follows now that R is a central product of 4 subgroups each isomorphic to Q(8) and so $R \cong 2_+^{1+8}$. In particular, $R \in И_K(J_K, 3')$.

Suppose that $R_0 \in И_K(J_K, 3')$. Then $R_0 = \langle C_{R_0}(A) \mid |J_K : A| = 3\rangle$. Since, for $|J_K : A| = 3$, $C_{R_0}(A) \in И_{C_G(A)}(J_K, 3')$, we have $C_{R_0}(A) \le O_{3'}(C_K(A)$ by Lemma 6.9 (iii). But then by Lemma 6.9 (i) and (ii), $R_0 \le R$. Hence $И^*_K(J_K, 3') = \{R\}$. □

Lemma 6.11. *Suppose that $A \in \mathcal{Z}_1$. Then $R = \langle O_{3'}(C_K(b)) \mid b \in A^{\#}, b$ not 3-central in $G\rangle$.*

Proof. We have $C_R(A) \cong \mathrm{Q}(8)$ by Lemma 6.9(i). By Lemma 6.10, $R/C_R(A)$ is elementary abelian of order 2^6. Since for $b \in A^{\#}$ such that b is not 3-central in G, we have $|O_{3'}(C_K(b))/C_R(A)| = 2^2$ by Lemma 6.9 (i), we infer that $R = \langle O_{3'}(C_K(b)) \mid b \in A^{\#}, b$ not 3-central in $G\rangle$. □

Lemma 6.12. $N_K(R) \ge RE$ *and* $C_K(R) \le R$.

Proof. By Lemma 5.5, $E_L \sim 3_+^{1+2}.\mathrm{GL}_2(3)$ and $E_M \sim 3^3.(2 \times \mathrm{Sym}(4))$. Furthermore, $O_3(E_M) = J_K$. Since R is the unique member of $И^*_K(J_K, 3')$, E_M normalizes R. Let $T = O_3(E_L)$. Then $T \cap J_K \in \mathcal{Z}_1$ by Lemma 4.5 (ii). Let $x \in E_L \setminus E_M$ and set $A = (T \cap J_K)^x$. Note that $A \le T = T^x$, so T normalizes R and as J_K normalizes R, A also normalizes R^x. Now, using Lemma 6.11, $R = \langle C_R(b) \mid b \in A^{\#}, b$ not 3-central in $G\rangle$. Suppose that $b \in A^{\#}$ is not 3-central in G. Then $C_R(b) = O_{3'}(C_K(b)) \cong 2_+^{1+4}$. Applying Lemma 6.9 (i) to R^x and $J_K^x \cap T$, we have $C_{R^x}(b) = O_{3'}(C_K(b)) \cong 2_+^{1+4}$. Thus

$$R^x \cap R \ge \langle O_{3'}(C_K(b)) \mid b \in A^{\#}, b \text{ not 3-central in } G\rangle = R.$$

Hence $R = R^x$. Therefore, R is normalized by $\langle E_M, x\rangle = E$.

Let $C = C_K(R)$. Then as E contains a Sylow 3-subgroup of K by Lemma 6.1 and E acts non-trivially on R, $C_K(R)$ is a $3'$-group which is normalized by E and hence by J_K. Thus $C_K(R) \le R$ by Lemma 6.10. □

We now set $H = N_G(R)$. Notice that as R is extraspecial, we have that H centralizes t and so $H = N_K(R)$. Our next goal is to show that G, H and R satisfy the hypothesis of Theorem 3.1.

Lemma 6.13. *$H/R \cong \mathrm{Aut}(\mathrm{U}_4(2))$ or $\mathrm{Sp}_6(2)$.*

Proof. We have that $Z \leq E \leq N_G(R)$ by Lemma 6.12. From the definition of R and Lemma 4.3 (iii), $O_2(C_{L_*}(t)) \leq R$. Thus $C_{L_*}(t)R/R \cong C_{L_*}(t)/O_2(C_{L_*}(t))$ is isomorphic to the centralizer of a 3-central element of order 3 in $\mathrm{PSp}_4(3)$. Since $ER/R \geq C_{L_*}(t)R/R$ we infer that ZR/R is inverted by its normalizer in H/R. Hence, using Theorem 2.2 and Lemma 6.10, we have that $H/R \cong \mathrm{Aut}(U_4(2))$ or $\mathrm{Sp}_6(2)$. □

Lemma 6.14. *$C_H(R) \leq R$ and $R/\langle t\rangle$ is a minimal normal subgroup of $H/\langle t\rangle$ of order 2^8.*

Proof. Lemma 6.12 ensures that $C_H(R) \leq R$. Also as R is extraspecial of order 2^9, $R/\langle t\rangle$ has order 2^8. Suppose that R_1 is a normal subgroup of H contained in R with $\langle t\rangle \leq R_1 \leq R$. Now J_KR/R is elementary abelian of order 27 and the 3-rank of $\mathrm{GL}_5(2)$ is 2, and therefore either R/R_1 or R_1 is centralized by $O^2(H/R)$ and hence by J_K. However $C_G(J_K) = J\langle t\rangle$ by Lemma 6.4(iii) and so we see that either $R = R_1$ or $R_1 = \langle t\rangle$. Thus $R/\langle t\rangle$ is a minimal normal subgroup of $H/\langle t\rangle$. □

Lemma 6.15. *The following hold.*

(i) *$C_K(Z) \leq H$.*
(ii) *ER controls fusion of elements of order 3 in K.*
(iii) *$B^G \cap K = B_1^K \cup B_2^K$ where B_1 is conjugate to a subgroup of J_K which together with Z forms a subgroup in $\mathcal{Z}_1$.*
(iv) *$C_K(B_1) \leq ER$.*

Proof. Looking in E, we see $C_E(Z) \sim 3^{1+2}.\mathrm{SL}_2(3)$. From Lemma 6.10, we have $C_R(Z) \cong \mathrm{Q}(8)$. Since $|C_K(Z)| = 2^6.3^4$ by Lemma 4.3 (iii), part (i) holds.

Using [6, pg. 26], we have that every element of order 3 in E is E-conjugate to an element of J_K. Since E contains a Sylow 3-subgroup of K and $N_K(J_K)$ controls K-fusion of 3-elements in J_K by Lemma 2.5, we have (ii).

From Lemmas 2.9 and 6.4 (iv), K has 3 conjugacy classes of elements of order 3 and just one 3-central classes. Thus (iii) follows from (ii).

Now consider the class B_1^K. We may suppose that $B_1Z \in \mathcal{Z}_1$. Then $C_R(B_1) \cong 2_+^{1+4}$ by Lemma 6.9 (i). It follows that t is an involution contained in $O^3(C_G(B_1))' \cong \mathrm{U}_4(2)$ with $C_{C_G(B_1)}(t) \sim 3 \times 2_+^{1+4}.3^2.2^2$. In particular, $(C_G(B_1) \cap K)R/R$ normalizes J_KR/R and so (iv) follows from Lemma 6.4 (v). □

Lemma 6.16. *Cyclic groups in the same H-class as B_2 (see Lemma 6.15) act fixed point freely on $R/\langle t\rangle$.*

Proof. Since B_2 is not contained in any member of $\mathcal{Z}_1$, we have that B_2 acts faithfully on $O_{3'}(C_G(A))$ for each $A \in \mathcal{Z}_1$. Thus, as $R = \prod_{A\in\mathcal{Z}_1} O_{3'}(C_G(A))$, we have that B_2 acts fixed point freely on $R/\langle t\rangle$. □

Lemma 6.17. *If $k \in K \setminus H$ and $d \in H \cap H^k$ has order 3, then $C_R(d) = \langle t\rangle$.*

Proof. We begin by noting that $R = O_2(H)$ and so $N_K(H) = H$. Hence if there exists $k \in K \setminus H$, then $H \cap H^k \neq H$

Suppose for a moment that a conjugate of J_K is contained in $H\cap H^k$. Then we may assume that $J_K \leq H^k$. Thus J_K and $J_K^{k^{-1}}$ are both contained in H. Hence there exists $h \in H$ such that $J_K = J_K^{k^{-1}h}$. But then $k^{-1}h \in N_K(J_K) \leq ER \leq H$ by Lemmas 6.4 (v) and 6.12, whence $k \in H$ and we have a contradiction.

Let $T \in \mathrm{Syl}_3(H \cap H^k)$ and assume $T \neq 1$. Suppose that T contains a K-conjugate Y of Z or B_1. Then, as H controls fusion of elements of order 3 in K by Lemma 6.15 (ii), we may suppose that either $Y = Z$ or $Y = B_1$. Hence Lemma 6.15 (i) and (iv) gives that $C_K(Y) \leq H$. However then $C_{H^k}(Y)$ contains a subgroup X of H^k which is conjugate to J_K as every element of order 3 in H is fused to an element of J_K in H. But this means $X \leq C_K(Y) \leq H$ by Lemma 6.15 (i) and (v) and this contradicts the observation in paragraph two of the proof. It follows that if $d \in H \cap H^k$ has order 3 and $k \notin H$, then d is conjugate to an element of B_2. The claim in the lemma now follows from Lemma 6.16. □

Proof of Theorem 1.1. Let $\overline{K} = K/\langle t\rangle$ and set $\overline{H} = N_{\overline{K}}(\overline{R})$. Lemmas 6.13, 6.14, 6.15 (ii) and 6.17 together show that the hypotheses of Theorem 3.1 are satisfied. Therefore $\overline{K} = O_{2'}(\overline{K})\overline{H}$. Now $\overline{H}$ contains a Sylow 3-subgroup of $\overline{K}$ and so $O_{2'}(\overline{K}) \leq O_{3'}(\overline{K})$. Since $И^*_K(J_K, 3') = \{R\}$, we infer that $O_{2'}(\overline{K}) \leq \overline{R}$. Thus $K = H$. Since, by Lemma 6.5, K contains a subgroup isomorphic $\mathrm{Sym}(3) \times \mathrm{Sym}(6)$ whereas $\mathrm{Aut}(\mathrm{U}_4(2))$ does not, we now get that $H/R \cong \mathrm{Sp}_6(2)$. Since $O_3(G) = 1$, Lemma 5.2 implies that $O_{2'}(G) = Z(G) = 1$. Since $R/\langle t\rangle$ is the spin-module for H/R, Lemma 2.15(iv) implies that the elements of order 5 in H act fixed point freely on $R/\langle t\rangle$. Hence, at last, Theorem 2.1 gives us that G is isomorphic to Co_2. □

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