Oberwolfach Preprints

OWP 2009 - 03

KRZYSZTOF KURDYKA AND ADAM PARUSIŃSKI

On the Non-analyticity Locus of an Arc-analytic Function

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

Starting in 2007, the MFO publishes a preprint series which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

Of course, the full copy right is left to the authors. The MFO only needs the right to publish it on its website *www.mfo.de* as a documentation of the research work done at the MFO, which you are accepting by sending us your file.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

ON THE NON-ANALYTICITY LOCUS OF AN ARC-ANALYTIC FUNCTION

KRZYSZTOF KURDYKA AND ADAM PARUSIŃSKI

ABSTRACT. A function is called arc-analytic if it is real analytic on each real analytic arc. In real analytic geometry there are many examples of arc-analytic functions that are not real analytic. Arc analytic functions appear while studying the arc-symmetric sets and the blow-analytic equivalence. In this paper we show that the non-analyticity locus of an arc-analytic function is arc-symmetric. We discuss also the behavior of the non-analyticity locus under blowings-up. By a result of Bierstone and Milman a big class of arc-analytic function, namely those that satisfy a polynomial equation with real analytic coefficients, can be made analytic by a sequence of global blowings-up with smooth centers. We show that these centers can be chosen, at each stage of the resolution, inside the non-analyticity locus.

1. INTRODUCTION.

Let X be a real analytic manifold. A function $f : X \to \mathbb{R}$ is called *arc-analytic*, cf. [12], if for every real analytic $\gamma : (-1, 1) \to X$ the composition $f \circ \gamma$ is analytic. The arc-analytic functions are closely related to blow-analytic functions of Kuo, cf. [10]. In particular, we have the following result, conjectured for the functions with semi-algebraic graphs in [12], and shown in [2].

Theorem 1.1. *Let X be a nonsingular real analytic manifold and let $f : X \to \mathbb{R}$ be an arc-analytic function on X. Suppose that*

$$G(x, f(x)) = 0,$$

where

$$G(x, y) = \sum_{i=0}^{p} g_i(x) y^{p-i}$$

is a nonzero polynomial in y with coefficients $g_i(x)$ which are analytic functions on X. Then there is a mapping $\pi : X' \to X$ which is a composite of a locally finite sequence of blowings-up with nonsingular closed centers, such that $f \circ \pi$ is analytic.

Let $f : X \to \mathbb{R}$ be an arc-analytic subanalytic function. In this paper we study *the set $S(f)$ of non-analyticity of f*. By definition, $S(f)$ is the complement of the set $R(f)$ of points $p \in X$, such that f as a germ is real analytic at p. It is known (cf. [17], [11], [1]) that $S(f)$ is closed and subanalytic. It follows from [2] or [16], that $\dim S(f) \leq \dim X - 2$. As we show in Theorem 3.1 below, $S(f)$ is arc-symmetric in the sense of [12]. Theorem 3.1 is shown in section 3.

Date: 17 December, 2008.

1991 *Mathematics Subject Classification.* 14Pxx, 32S45, 32B20.

This research was done at the Mathematisches Forschungsinstitut Oberwolfach during a stay within the Research in Pairs Programme from January 20 to February 2, 2008. We would like to thank the MFO for excellent working conditions.

We also study how the set of non-analyticity behaves under blowings-up with smooth centers. This depends on whether the center is entirely contained in $S(f)$ or not. If it is not then the non-analyticity lifts to the entire fiber, see Proposition 3.10. Note that Theorem 1.1 can be also derived from [16]. Using the method of [16] and Proposition 3.10 we show the following refinement of Theorem 1.1.

Theorem 1.2. *In Theorem 1.1 we may require that the mapping $\pi : X' \to X$, that is a locally finite composite $\pi = \cdots \circ \pi_k \circ \cdots \circ \pi_0$ of blowings-up with smooth centers, satisfies additionally:*

for every k the center of π_{k+1} is contained in the locus of non-analyticity of $f \circ \pi_0 \circ \cdots \circ \pi_k$.

1.1. **Algebraic case.** Theorem 1.1 can be stated in the real algebraic version, see [2]. In this case if we assume that X is a nonsingular real algebraic variety and that the coefficients g_i are regular then we may require that π is a finite composite of blowings-up with nonsingular algebraic centers.

In the algebraic case we cannot require that the centers of blowings-up are entirely contained in the non-analyticity loci as Example 1.5 shows.

An analytic function on X is called *Nash* if its graph is semialgebraic. It is called *blow-Nash* if it can be made Nash after composing with a finite sequence of blowing-ups with smooth nowhere dense regular centers. Thus the algebraic version of Theorem 1.1, cf. [2], says that the function with semi-algebraic graph is arc-analytic if and only if it is blow-Nash. Nash morphisms and manifolds form a natural category that contains the algebraic one, cf. [4]. We note that our refinement of the statement of Theorem 1.1 holds in the Nash category.

Theorem 1.3. *Let X be a Nash manifold and let $f : X \to \mathbb{R}$ be an arc-analytic function on X. Suppose that*

$$G(x, f(x)) = 0,$$

where

$$G(x, y) = \sum_{i=0}^{p} g_i(x) y^{p-i}$$

is a nonzero polynomial in y with coefficients $g_i(x)$ which are Nash functions on X. Then there is a finite composite $\pi = \cdots \circ \pi_k \circ \cdots \circ \pi_0$ of blowings-up of nonsingular Nash submanifolds, such that for every k the center of π_{k+1} is contained in the locus of non-analyticity of $f \circ \pi_0 \circ \cdots \circ \pi_k$, and $f \circ \pi$ is Nash.

1.2. **Subanalytic case.** Less is known for an arc-analytic function with subanalytic graph if it does not satisfy an equation (1.1). It is known that an arc-analytic subanalytic function has to be continuous and can be made real analytic by composing with finitely many local blowings-up with smooth centers, see [2] or [16] (we refer the reader to these papers for a precise statement). It is not known whether these blowings-up can be made global that is whether the arc-analytic subanalytic functions coincide with the family of blow-analytic functions of T.-C.Kuo, see e.g. [10], [6], [7]. It is also not known, whether the centers of such blowings-up can be chosen in the locus of non-analyticity of the function.

We present below in Example 1.6 a subanalytic arc-analytic function that cannot be made analytic, even locally, by a blowing-up of a coherent ideal. In particular, it cannot satisfy an equation of type (1.1).

1.3. **Examples.**

Examples 1.4. The function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \frac{x^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$, is arc-analytic but not differentiable at the origin.

The function $g(x,y) = \sqrt{x^4 + y^4}$ is arc-analytic but not C^2. This example is due to E. Bierstone and P.D. Milman.

The function $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = \frac{xy^5}{x^4+y^6}$ for $(x,y) \neq (0,0)$ and $h(0,0) = 0$ is arc-analytic but not lipschitz. This example is due to L. Paunescu.

We generalize the first example as follows. Fix a real analytic Riemannian metric on X and let Y be a nonsingular real analytic subset of X. Then $d_Y^2 : X \to \mathbb{R}$, the square of the distance to Y, is a real analytic function on X. Suppose that Y is of codimension ≥ 2 in X and let $f : X \to \mathbb{R}$ be an analytic function vanishing on Y and not divisible by d^2. Then, $\frac{f^3}{d^2}$ vanishes on Y, is arc-analytic and not analytic at the points of Y. Note that $\frac{f^3}{d^2}$ composed with the blowing-up of Y is analytic.

Example 1.5. Let $g(x,y) = y^2 + x(x-1)(x-2)(x-3)$. Then $g^{-1}(0) \subset \mathbb{R}^2$ is irreducible and has two connected compact components, denoted by X_1 and X_2. These connected components that can be separated by $h(x,y) = x - 1.5$, that is $h < 0$ on X_1 and $h > 0$ on X_2. For $\varepsilon > 0$ sufficiently small, $h^2 + \varepsilon g$ is strictly positive on $\mathbb{R}^2$. Define

$$g_1(x,y) = \sqrt{h^2 + \varepsilon g} + h.$$

Then g_1 is analytic, 0 is a regular value of g_1 and $g_1^{-1}(0) = X_1$. Moreover, g_1 is Nash. Then $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = \frac{z^3}{z^2 + g_1^2(x,y)}$$

for $(x,y,z) \neq 0$ and $f(0) = 0$, is arc-analytic and $S(f) = X_1 \times \{0\}$. The function f becomes analytic after blowing-up of $S(f)$.

Example 1.6. Let $\pi_0 : \widetilde{\mathbb{R}}^3 \to \mathbb{R}^3$ be the blowing-up of the origin and let E be the exceptional divisor of π_0. Let $C \subset E$ be a transcendental (the smallest algebraic subset of E that contains C is E itself) non-singular analytic curve and let $\pi_C : M \to \widetilde{\mathbb{R}}^3$ be the blowing-up of C. Let f be an arc-analytic function on $\mathbb{R}^3$ such that the set of non-analyticity of $f \circ \pi_0$ is C and $f \circ \pi_0 \circ \pi_C$ is analytic. Such a function can be constructed as follows. Using the last remark of Examples 1.4 we may construct an arc-analytic function $g : \widetilde{\mathbb{R}}^3 \to \mathbb{R}$ such that $S(g) = C$. Then we may set $f(x,y,z) = (x^2 + y^2 + z^2)\, g(\pi_0^{-1}(x,y,z))$.

Such f, as a germ at 0, cannot be made analytic by a single blowing-up of an ideal. Indeed, suppose contrary to our claim that there exists an ideal $\mathcal{I}$ of $\mathbb{R}\{x_1, x_2, x_3\}$ such that $f \circ \pi_{\mathcal{I}}$ is analytic, where $\pi_{\mathcal{I}}$ denotes the blowing-up of $\mathcal{I}$. Multiplying $\mathcal{I}$ by the maximal ideal at 0 we may assume that $\pi_{\mathcal{I}}$ factors through π_0, i.e. $\pi_{\mathcal{I}} = \pi_{\mathcal{J}} \circ \pi_0$, where $\mathcal{J}$ is a sheaf of coherent ideals centered on an algebraic subset Y of E. We may assume that $\dim Y \leq 1$. Thus the blowing-up of $\mathcal{J}$, $\pi_{\mathcal{J}} : M_{\mathcal{J}} \to \widetilde{R}^3$ is an isomorphism over the complement of Y that contradicts the construction of f.

2. Arc-meromorphic mappings.

In this section *subanalytic* mean subanalytic at infinity. Let us recall, [17], [11], that a subset A of $\mathbb{R}^n$ is called *subanalytic at infinity* if A is subanalytic in some algebraic compactification of $\mathbb{R}^n$. (Then in fact it is subanalytic in every algebraic compactification of $\mathbb{R}^n$.) All functions and mappings are supposed to be subanalytic, that is their graphs are subanalytic at infinity.

Definition 2.1. Let U be an open subanalytic subset of $\mathbb{R}^n$. An everywhere defined subanalytic mapping $f : U \to \mathbb{R}^m$ is called *arc-meromorphic* if for any analytic arc $\gamma : (-1,1) \to U$ there exists a discrete set $D \subset (-1,1)$ and φ an meromorphic function on $(-1,1)$ with poles contained in D and such that $f \circ \gamma = \varphi$ on $(-1,1) \setminus D$. Note that it may happen that $f \circ \gamma$ does not coincide with φ at some points of D and may be at these points discontinuous.

Example 2.2. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ can be extended to an arc-meromorphic function on $\mathbb{R}^2$ by assigning any value at the origin. Then it becomes discontinuous at $(0,0)$ even if for every analytic arc $\gamma : (-1,1) \to \mathbb{R}^2$, $\gamma(0) = (0,0)$, $f \circ \gamma$ extends to an analytic function.

Remark 2.3. If f is an arc-meromorphic and continuous function on an open connected set $U \subset \mathbb{R}^n$, then f is arc-analytic.

Remark 2.4. Let f and g be arc-meromorphic functions on an open connected set of U. Assume that $f = g$ on an open non-empty subset $U \subset \mathbb{R}^n$, then $f = g$ except on a nowhere dense subanalytic subset of U.

Lemma 2.5. *Let U be an open bounded subanalytic subset in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ be an arc-meromorphic mapping. Then there exists $\Gamma \subset \mathbb{R}^n$ a closed nowhere dense subanalytic set, $N \in \mathbb{N}$ and $C > 0$ such that*

$$|f(x)| \leq C \operatorname{dist}(x,\Gamma)^{-N},\ x \in U \setminus \Gamma. \tag{1}$$

In particular we can take as U the complement of the non-analyticity locus of f.

Proof. It is well-known (cf. e.g. [9], [15]) that there exists a stratification of $\mathbb{R}^n$ which is compatible with $\overline{U}$ and such that f is analytic on each stratum contained in U. We take as Γ the union of all strata contained in $\overline{U}$ of dimension less than n. Let us consider the function defined as follows: $g(x) = |f(x)|$ if $|f(x)| \leq 1$, and $g(x) = |f(x)|^{-1}$ if $|f(x)| \geq 1$. Then $h(x) := dist(x,\Gamma)g(x)$ is a subanalytic and continuous function on $\overline{U}$ which is compact. Moreover, if $\operatorname{dist}(x,\Gamma) = 0$ then $h(x) = 0$. Therefore, by the classical Łojasiewicz's inequality (cf. e.g. [9], [1]) for subanalytic functions, there exist $N \in \mathbb{N}$ and $c > 0$ such that

$$h(x) \geq c \operatorname{dist}(x,\Gamma)^{N+1},\ x \in U. \tag{2}$$

Thus inequality (1) follows with $C = \max\{1/c, M\}$, where $M = \sup_{x \in U} dist(x,\Gamma)^N$. □

We state now an auxiliary lemma on arc-meromorphic functions in two variables.

Lemma 2.6. *Let U be an open subanalytic subset in $\mathbb{R}^2$ and let $f : U \to \mathbb{R}^m$ be an arc-meromorphic mapping. Then for any $a \in U$ there exists a neighborhood V of a and an analytic function $\varphi : V \to \mathbb{R}$, $\varphi \not\equiv 0$, such that φf is arc-analytic.*

Proof. Let Γ be the subanalytic set associated to f by Lemma 2.5. Clearly we may assume that $a \in \Gamma$, otherwise f is analytic at a and the statement is trivial. Since $\dim \Gamma = 1$, by a result of Łojasiewicz's [14] (see also [13]), the set Γ is actually semianalytic. Then there exists a neighborhood V of a and an analytic function $\psi : V' \to \mathbb{R}$, $\psi \not\equiv 0$, which vanishes on $V' \cap \Gamma$. Hence for some compact neighborhood $V \subset V'$ of a there exists $c > 0$ such that

$$|\psi(x)| \leq c \operatorname{dist}(x, \Gamma),\ x \in V.$$

(This is a consequence of the main value theorem). Put $\varphi = \psi^{N+1}$, then by Lemma 2.5 the function φf is continuous on V. Clearly φf is arc-meromorphic, so by Remark 2.3 this function is arc-analytic. □

Proposition 2.7. *Let $f : U \to \mathbb{R}$ be an arc-meromorphic function, where U is an open subset in $\mathbb{R}^n$. Assume that f is analytic with respect to the variable x_1. Then the function $\frac{\partial f}{\partial x_1} : U \to \mathbb{R}$ is again arc-meromorphic.*

Proof. First observe that by [11] the function $\frac{\partial f}{\partial x_1}$ is (globally) subanalytic. To prove that $\frac{\partial f}{\partial x_1}$ is arc-meromorphic let us fix an analytic arc $\gamma : (-1, 1) \to U$. We define an arc-meromorphic function $g : V \to \mathbb{R}$ by $g(s, t) = f(\gamma(t) + se_1)$, where $e_1 = (1, 0, \dots 0)$ and V is an open neighborhood of $\{0\} \times (-1, 1)$ in $\mathbb{R}^2$. Clearly

$$\frac{\partial f}{\partial x_1}(\gamma(t)) = \frac{\partial g}{\partial s}(0, t).$$

By Lemma 2.6 there exist a neighborhood V of $(0, 0)$ and an analytic function $\varphi : V \to \mathbb{R}$ such that $h := \varphi g$ is arc-analytic on V. Since $\dim S(h) \leq 0$, for any $t \neq 0$ sufficiently small h is analytic at $(0, t)$, but of course also φ is analytic at $(0, t)$. Since $g(s)$ is analytic with respect to s it follows that $g = h/\varphi$ is actually analytic at $(0, t)$ for any $t \neq 0$ sufficiently small. By [2] there exists a map $\pi : M \to \mathbb{R}^2$, which is a finite composition of blowing-ups of points, such that $h \circ \pi$ is analytic. Consider the arc $\eta(t) := (0, t)$ and let $\tilde{\eta}(t) \in M$ be the unique analytic arc such that $\pi \circ \tilde{\eta} = \eta$. The chain rule gives

$$d_{\tilde{\eta}(t)} h \circ \pi = (d_{\eta(t)} h) \circ (d_{\tilde{\eta}(t)} \pi). \tag{3}$$

Note that $d_{\tilde{\eta}(t)} \pi$ is invertible for $t \neq 0$, moreover the map $t \mapsto (d_{\tilde{\eta}(t)} \pi)^{-1}$ is meromorphic. It follows that $t \mapsto d_{\eta(t)} h$ is meromorphic. In particular $t \mapsto \frac{\partial h}{\partial s}(0, t)$ is meromorphic. We have

$$\frac{\partial h}{\partial s}(0, t) = \varphi \frac{\partial g}{\partial s}(0, t) + g \frac{\partial \varphi}{\partial s}(0, t).$$

Since $\varphi(0, t) \neq 0$ for $t \neq 0$, the map $t \mapsto \frac{\partial g}{\partial s}(0, t)$ is meromorphic and Proposition 2.7 follows. □

Remark 2.8. A repeated application of Proposition 2.7 shows that for every $k \in \mathbb{N}$,

$$\frac{\partial^k f}{\partial x_1^k} : U \to \mathbb{R}$$

is arc-meromorphic. Moreover, there exists a subanalytic stratification $\mathcal{S}$ of U such that for every stratum $S \in \mathcal{S}$ and every $x \in S$ there is $\varepsilon > 0$ and a neighborhood V of x in S such that $f(x + se_1)$ is an analytic function of $(x, s) \in V \times (-\varepsilon, \varepsilon)$. In particular, for every $k \in \mathbb{N}$, $\partial^k f / \partial x_1^k : U \to \mathbb{R}$ is analytic on the strata of $\mathcal{S}$.

3. The non-analyticity locus of an arc-analytic function is arc-symmetric.

Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be arc-analytic with subanalytic graph. We denote by $S(f)$ the non-analyticity set of f and by $R(f)$ its complement in U. Then $S(f)$ is closed in U and by [17] (see also [11], [2]) it is a subanalytic set. It follows from [2] or [16] that $\dim S(f) \le n-2$.

Theorem 3.1. *Let $\gamma : (-\varepsilon, \varepsilon) \to U$ be an analytic arc such that $\gamma(t) \in R(f)$ for $t < 0$. Then $\gamma(t) \in R(f)$ for $t > 0$ and small. In other words, $S(f)$ is arc-symmetric subanalytic in the sense of* [12].

For the proof we need some basic properties of Gateaux differentials. For each $k \in \mathbb{N}$ we consider

$$h_k(x,v) = \frac{1}{k!}\partial_v^k f(x) = \frac{1}{k!}\frac{d^k}{dt^k} f(x+tv)_{|t=0}. \tag{4}$$

Proposition 3.2. *Let $f : U \to \mathbb{R}$ be an arc-analytic function. Then for any $k \in \mathbb{N}$ the function $h_k(x,v) : U \times \mathbb{R}^n \to \mathbb{R}$ is arc-meromorphic.*

Proof. Let $(x(t), v(t))$ be an analytic arc in $U \times \mathbb{R}^n$. Define an arc-analytic function $g(s,t) = f(x(t) + sv(t))$. Then

$$h_k(x(t), v(t) = \frac{1}{k!}\frac{\partial^k}{\partial s^k} g(t,s)_{|s=0}$$

that is meromorphic by Proposition 2.7. □

For $x \in U$, $k \in \mathbb{N}$ we denote

$$h_{x,k}(v) = h_k(x,v) = \frac{1}{k!}\partial_v^k f(x)$$

Note that $h_{x,k}$ is k-homogeneous function. If f is analytic at x, then $h_{x,k}$ is polynomial. We have also the inverse which is Bochnak-Siciak Theorem, see [5], which states that if $h_{x,k}$ is polynomial for each $k \in \mathbb{N}$, then f is analytic at x. Traditionally if $h_{x,k}$ is polynomial then it is called the Gateaux differential of f at x of order k.

We call $h_{x,k}$ *generically polynomial* if it is equal to a polynomial except on a nowhere dense subanalytic (and homogenous) subset of $\mathbb{R}^n$. Note that, by Remark 2.4, $h_{x,k}$ is generically polynomial if it coincides with a polynomial on an open nonempty set.

Proposition 3.3. *Let $f : U \to \mathbb{R}$ be an arc-analytic function, where U is an open subset in $\mathbb{R}^n$. Let $\gamma : (-\varepsilon, \varepsilon) \to U$ be an analytic arc and $k \in \mathbb{N}$. If $h_{\gamma(t),k}$ is generically polynomial for $t \in (-\varepsilon, 0)$, then there exists a finite set $F_k \subset (0, \varepsilon)$ such that $h_{\gamma(t),k}$ is generically polynomial for each $t \in (0,\varepsilon) \setminus F_k$.*

Proof. Let $\mathbb{R}_k[x_1, \dots, x_n]$ denote the space of homogenous polynomials of degree k and let $d_k = \binom{n+k-1}{n}$ denote its dimension. We need the classical multivariate interpolation.

Lemma 3.4. *There exists an algebraic nowhere dense subset $\Delta \subset (\mathbb{R}^n)^{d_k}$ such that for $V = (v^1, \dots, v^{d_k}) \in (\mathbb{R}^n)^{d_k} \setminus \Delta$ the map $\Psi_V : \mathbb{R}_k[x_1, \dots, x_n] \to \mathbb{R}^{d(k)}$ given by*

$$\Psi_V(P) = (P(v^1), \dots, P(v^{d_k})).$$

is a linear isomorphism. □

Fix $V = (v^1, \dots, v^{d_k}) \in (\mathbb{R}^n)^{d_k} \setminus \Delta$ generic and denote $\Phi_V = \Psi_V^{-1} : \mathbb{R}^{d(k)} \to \mathbb{R}_k[x_1, \dots, x_n]$. We define an arc-meromorphic map $P_k : (-\varepsilon, \varepsilon) \to \mathbb{R}_k[x_1, \dots, x_n]$ by

$$P_k(t) := \Phi_V(h_k(\gamma(t), v^1), \dots, h_k(\gamma(t), v^{d(k)})).$$

The map $p_k; (-\varepsilon, \varepsilon) \times \mathbb{R}^n \to \mathbb{R}$, where $p_k(t, v) = P_k(t)(v)$ is arc-meromorphic. If V is sufficiently generic then, for $t \in (-\varepsilon, 0) \setminus \{\text{finite set}\}$, $p_k(t)$ coincides with $h_{\gamma(t),k}$. Since they both are arc-meromorphic, by Remark 2.4 they coincide on $(-\varepsilon, \varepsilon) \times \mathbb{R}^n \setminus Z_k$, where Z_k is a closed subanalytic set with $\dim Z_k \leq n$. Hence there exists a finite set $F_k \subset (0, \varepsilon)$ such that for $t \in (0, \varepsilon) \setminus F_k$ the intersection $Z_k \cap (\{t\} \times \mathbb{R}^n)$ is of dimension less than n. Thus, for each $t \in (0, \varepsilon) \setminus F_k$ the function $h_{\gamma(t),k}$ is generically polynomial, as claimed. □

The following proposition is a version of the mentioned above Bochnak-Siciak Theorem, [5].

Proposition 3.5. *If for every k there is a nonempty open subset $V_k \subset \mathbb{R}^n$ and a homogeneous polynomial P_k of degree k such that $h_{x,k} \equiv P_k$ on V_k, then f is analytic at x.*

Proof. We first show that $\sum_k P_k(v)$ is convergent in a neighborhood of $0 \in \mathbb{R}^n$.

We may assume that x is the origin. Let π_0 be the blowing up of the origin, $\pi_0(y, s) = (sy, s), s \in \mathbb{R}, y \in \mathbb{R}^{n-1}$, in a chart. The function $\tilde{f}(y, s) := f(\pi(y, s))$, defined in a neighborhood U' of the exceptional divisor $E : s = 0$, is arc-analytic. The set of non-analyticity of $\tilde{f}$, denoted by $\tilde{S}$, is closed subanalytic and of codimension at least 2. For $y \notin \tilde{S}$, $\tilde{f}$ is analytic in a neighborhood of $(0, y)$ and, moreover, by analytic continuation,

$$h_{x,k}(v) = P_k(v) \qquad \text{for} \quad v = t(y, 1),\ t \in \mathbb{R},\ y \notin \tilde{S}. \tag{5}$$

Fix A' an open non-empty subset of E such that the closure of A' does not intersect $\tilde{S}$. Let $A \subset \mathbb{R}^n$ be the cone over A'. Then, by (5), $\sum_k P_k(v)$ is convergent in any compact subset of A. The convergence in a neighborhood of 0 in $\mathbb{R}^n$ follows from the following lemma.

Lemma 3.6. *Let $V \subset \mathbb{R}^n$ be starlike with respect to the origin, $a \in V$, and suppose that*

$$|P_k(v)| \leq L \quad \text{on } V' = a + V.$$

Then

$$|P_k(v)| \leq L \quad \text{on } \frac{1}{2e}V.$$

Proof. Since P_k is homogeneous of degree k

$$P_k(v) = \frac{1}{k!} \sum_{s=0}^{s=k} (-1)^{k-s} \binom{k}{s} P_k(a + sv). \tag{6}$$

Indeed, (6) can be shown recursively on k using Euler's formula as follows. First note (6) holds for $a = 0$ and the derivative of the RHS of (6) with respect to a equals

$$0 = \frac{1}{k!} \sum_{s=0}^{s=k} (-1)^{k-s} \binom{k}{s} Q(a + sv), \tag{7}$$

where $Q(x) = \sum_{i-1}^{n} a_i \frac{\partial P_k}{\partial x_i}(x)$ is a homogeneous polynomial of degree $k-1$. By the inductive assumption

$$\sum_{s=0}^{s=k}(-1)^{k-s}\binom{k}{s}Q(a+sv) = \sum_{s=0}^{s=k-1}(-1)^{k-1-s}\binom{k-1}{s}Q(a+sv) +$$
$$+\sum_{s=1}^{s=k}(-1)^{k-s}\binom{k-1}{s-1}Q(a+sv) = -Q(v)+Q(v) = 0$$

This shows (6). Thus, if $v \in \frac{1}{k}V$, $|P_k(v)| \le \frac{1}{k!}L\sum_{s=0}^{k}\binom{k}{s} = L\frac{2^k}{k!}$, that means that for $v \in \frac{1}{2e}V$

$$|P_k(v)| \le L\frac{(2k)^k}{k!}\frac{1}{(2e)^k} \le L.$$

This ends the proof of lemma 3.6. □

Then $\sum_k P_k(v)$ is an analytic function in a neighborhood of the origin that coincides with f on a set with non-empty interior. Hence $f(v) = \sum_k P_k(v)$ in a neighborhood of the origin. This shows proposition 3.5. □

Proof of theorem 3.1. We may assume that γ is injective otherwise the image of $t > 0$ equals the image of $t < 0$ and the statement is obvious. Let $F := \bigcup F_k$, where F_k are finite subsets of $(0,\varepsilon)$ given by Proposition 3.3. Clearly the complement of F is dense in $(0,\varepsilon)$, so by Proposition 3.5 our function f is analytic at $\gamma(t)$ for $t \in G$, where G is an open dense subset of $(0,\varepsilon)$. Hence theorem 3.1 follows. □

Consider the subanalytic sets

$$\tilde{R}_{k_0}(f) = \{x \in U; \forall k \le k_0,\ h_{x,k} \text{ is generically polynomial }\},$$
$$R_{k_0}(f) = \{x \in U; \forall k \le k_0,\ h_{x,k} \text{ is polynomial }\}.$$

Clearly $\tilde{R}_{k+1}(f) \subset \tilde{R}_k(f)$ and $R_{k+1}(f) \subset R_k(f)$. We recall from [11] the following result

Proposition 3.7. [[11], Proposition 4.4] *Let $f : U \to \mathbb{R}$ be a subanalytic (not necessarily arc-analytic) function on an open bounded $U \subset \mathbb{R}^n$. Then for any compact $K \subset U$ there is $k \in \mathbb{N}$ such that $R(f) \cap K = R_k(f) \cap K$.*

Proposition 3.8. *For any compact $K \subset U$ there is $k \in \mathbb{N}$ such that $R(f) \cap K = \tilde{R}_k(f) \cap K$.*

Proof. By Remark 2.8 there exists a stratification $\mathcal{S}$ of $U \times S^{n-1}$ such that for every k, h_k is analytic on the strata. Refining the stratification, if necessary, we may suppose that for every stratum $S \subset U \times S^{n-1}$ its projection to U has all fibers of the same dimension. In the proof we use only these strata for which all the fibers of projection to U are of maximal dimension $n-1$. We denote the collection of them by $\mathcal{S}_n$ and their union as Z. Now it is easy to adapt the proof of Lemma 6.1 of [11] (based on multivariate interpolation) and show the following lemma.

Lemma 3.9. *There are analytic subanalytic functions*

$$w_i : U \times S^{n-1} \to \mathbb{R}, \quad i \in \mathbb{N},$$

analytic on each stratum of $\mathcal{S}$ such that $h_{x,i}$ is generically polynomial if and only if $w_i \equiv 0$ generically on $\{x\} \times S^{n-1}$. □

Now Proposition 3.8 follows from Lemma 2.5 of [11] that shows that for every stratum there exist k such that

$$\bigcap_{i=1}^{\infty}\{w_i = 0\} = \bigcap_{i=1}^{k}\{w_i = 0\}.$$

□

We complete this section with two results, one that controls the change of non-analyticity locus by blowings-up. This result will be crucial in the next section. The last result of this section, Proposition 3.11, though not used in this paper, indicates a possible analogy between our approach and the theory of complex analytic functions.

Proposition 3.10. *Let $T = \{x_k = x_{k+1} = \cdots = x_n = 0\}$ and let π_T be the blowing-up of T. Suppose that the origin is in the closure of $R(f) \cap T$ and that $f \circ \pi_T$ is analytic at least at one point of $\pi_T^{-1}(0)$ (hence on a neighborhood of this point). Then f is analytic at 0.*

Proof. Let $\Pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be given by $\Pi(x, t, v) = x + tv$ and let $\Pi_T : T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the restriction of Π. First we show that if $f \circ \Pi_T$ is analytic at some points of $\Pi_T^{-1}(0) \cap \{t = 0\}$ and 0 is in the closure of $R(f) \cap T$ then f is analytic at 0. Indeed, suppose that $A' \subset \mathbb{R}^n$ has non-empty interior and suppose that $f \circ \Pi_T$ is analytic in a neighborhood $\{0\} \times \{0\} \times A'$. Let $h_k(x, v)$, $x \in T, v \in \mathbb{R}^n$, be defined by (4). Then h_k is arc-meromorphic and analytic on $A = U' \times A'$, where U' is a small neighborhood of 0 in T. For each k, we define by Lemma 3.4,

$$P_k(x, v) = \Psi_V^{-1}(h_k(x, v^1), \ldots, h_k(x, v^{d(k)}))(v), \tag{8}$$

where $v^1, \ldots, v^{d_k} \in A'$ are generic. Each P_k is analytic on A and equals h_k for $x \in R(f) \cap T$. Therefore $h_k(0, v) = P_k(0, v)$ for $v \in A'$ and the claim follows from proposition 3.5.

Thus it remains to show that $f \circ \Pi_T$ is analytic at some points of $\Pi_T^{-1}(0) \cap \{t = 0\}$. For this we factor Π_T restricted to $\{v_n \neq 0\}$ through π_T and use the assumption on π_T. Write π_T in an affine chart $\pi_T(\tilde{x}, y, s) = (\tilde{x}, sy, s)$, where $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{k-1})$, $y = (y_k, \ldots, y_{n-1})$ and $s \in \mathbb{R}$. Then on these charts $\Pi_T = \pi_T \circ \varphi$, where

$$(\tilde{x}, y, s) = \varphi(x, t, v) = (x + tv', \frac{1}{v_n} v'', tv_n),$$

where $v' = (v_1, \ldots, v_{k-1})$, $v'' = (v_k, \ldots, v_{n-1})$. Restricted to $t = 0$, φ is a surjective projection $(x, v) \to (x, \frac{1}{v_n} v'')$ onto $s = 0$. Hence $R(f \circ \Pi_T) \cap \Pi_T^{-1}(0) \cap \{t = 0\} \supset \varphi^{-1}(R(f \circ \pi_T) \cap \pi^{-1}(0))$ is non-empty. □

Proposition 3.11. *Let $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and suppose that for every $x_1 > 0$ and small, $f(x_1, x')$ is analytic at $(x_1, 0)$ as a function of x'. Moreover, suppose that for $x_1 > 0$ and small we have a uniform bound*

$$|h_k((x_1, 0), v')| \leq c^k, \qquad \text{for } \|v'\| \leq \varepsilon, k \in \mathbb{N},$$

where $v' = (v_2, \ldots, v_n)$. Then f is analytic at the origin.

Proof. The function $h_k((x_1, 0), v')$ is arc-meromorphic as a function of x_1, v'. Moreover, since continuous arc-meromorphic functions of one variable are analytic, using polynomial interpolation lemma, Lemma 3.4, we may show that each $h_k((x_1, 0), v')$ extends to an analytic function $\Psi(x_1, v')$ defined in a neighborhood of $(0, 0)$, such that for each x, $v' \to \Psi(x_1, v')$ is a homogeneous polynomial in v'. Moreover, for $x_1 > 0$ and $\|x'\| < \varepsilon/c$

$$f(x_1, x') = \sum_k h_k((x_1, 0), x')$$

and the series on the right-hand side is convergent.

Fix any $k \in \mathbb{N}$ and $\|v'\| < \varepsilon/c$. Then for $v = (1, v')$, $0 < t < 1$,

$$f(tv) = \sum_{j=0}^{\infty} h_j((t,0), tv') = \sum_{j=0}^{\infty} t^j h_j((t,0), v') = \sum_{j=0}^{k} t^j h_j((t,0), v') + \varphi(t, v'),$$

where φ is subanalytic and $O(t^{k+1})$. Therefore for such v

$$H_k(0,v) := \frac{1}{k!}\frac{d^k}{dt^k} f(tv)_{|t=0} = \frac{1}{k!}\frac{d^k}{dt^k} \sum_{j=0}^{k} h_j((t,0), tv')_{|t=0}. \tag{9}$$

Note that the right-hand side, and hence $H_k(0,v)$ as well, is a polynomial in v. Indeed, this follows from the fact that $x \to \sum_{j=0}^{k} h_j((x_1,0), x')$ is an analytic function of x and $H_k(0,v)$ coincides with its Gateaux differential. Thus proposition 3.11 follows from proposition 3.5. □

4. Proof of Theorem 1.2.

We may suppose that U is connected. We suppose also that the coefficients g_0 and g_p of G and the discriminant $\Delta(x)$ of G are not identically equal to zero. By the resolution of singularities [8], [3], [18], there is a locally finite sequence of blowings-up $\pi : U' \to U$ with nonsingular centers such that $(g_0 g_p \Delta) \circ \pi$ is normal crossings. Thus Theorem 1.1 follows from the following.

Proposition 4.1. *Let an arc-analytic function $f(x)$ satisfy the equation* (1.1) *with analytic coefficients g_i. If g_0, g_p and $\Delta(x)$ are simultaneously normal crossings (and hence not identically equal to zero) then f is real analytic.*

Proposition 4.1 was proven in [16] under an additional assumption $g_0 \equiv 1$, see the proof of Theorem 3.1 of [16]. It is easy to reduce the proof to this case by replacing f by $g_p f$. Then, an argument of [16] shows that locally f can be expand as a fractional power series. Finally, an arc-analytic fractional power series is analytic, see the proof of Theorem 3.1 of [16]. If the discriminant of G vanishes identically then we replace it by the first non-vanishing higher order discriminant.

To show Theorem 1.2 we follow, for the product $h(x) = g_0(x) g_p(x) \Delta(x)$, the monomialisation procedure of Włodarczyk or Bierstone-Milman. In this procedure the centre of blowing-up is defined as a the locus of points where a local invariant is maximal. Thus suppose that we have the following data described in a local system of coordinates $x_1, \cdots, x_n$ at the origin. The function $h \circ \pi$, where $\pi = \pi_k \circ \cdots \circ \pi_0$, is of the form $h \circ \pi = x^A h_k$, where h_k is the controlled transform by the preceding blowings-up. Let $m = \operatorname{ord}_x h_k$. We may assume that $H = \{x_n = 0\}$ is a hypersurface of maximal contact. Then, using the notation $x = (x', x_n)$,

$$h_k(x) = x_k^m + \sum_{j=0}^{m-2} c_j(x') x_k^j,$$

and $\operatorname{mult}_0 c_i \geq m - i$.

Let C be the next centre given by the procedure and denote by π_C the blowing-up of C. We show that it cannot happen that $0 \in S(h \circ \pi)$ and $0 \in \overline{C \setminus S(f \circ \pi)}$. Suppose, contrary to our claim, that this is the case. Then, by Proposition 3.10, the fibre over the origin of the blowing-up $\pi_C = \pi_{k+1}$ of C is contained in $S(f \circ \pi \circ \pi_C)$. Since C is contained in the equimultiplicity locus of h_k, at the generic point $\pi_C^{-1}(0)$ the strict transform of h_k is nonzero, and hence $h \circ \pi \circ \pi_C$ is normal crossing. This contradicts Proposition 4.1.

Let C' denote the connected component of C containing 0. Then either $C' \subset S(h \circ \pi)$ or $C' \cap S(h \circ \pi) = \emptyset$. Thus Theorem 1.2 proven. □

References

[1] E. Bierstone and P. D. Milman, *Semianalytic and Subanalytic sets*, Publ. I.H.E.S., **67** (1988), 5-42.

[2] E. Bierstone, P. D. Milman, *Arc-analytic functions,* Invent. math. **101** (1990), 411–424.

[3] E. Bierstone, P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128 (1997), 207-302.

[4] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*, E.M.G vol. 36 (1998) Springer.

[5] J. Bochnak, J. Siciak, *Analytic functions in topological vector spaces*, Studia Math., **XXXIX** (1971), 77–112

[6] T. Fukui, S. Koike, T.-C. Kuo, *Blow-analytic equisingularities, properties, problems and progress*, Real Analytic and Algebraic Singularities (T. Fukuda, T. Fukui, S. Izumiya and S. Koike, ed),Pitman Research Notes in Mathematics Series, **381** (1998), pp. 8–29.

[7] T. Fukui, L. Paunescu, *On Blow-analytic Equivalence*, in "Arc Spaces and Additive Invariaants in Real Algebraic and Analytic Geometry", Panoramas et Synthèses, S.M.F., **24**, 2007, 87–125

[8] H. Hironaka, *Resolution of Singularities of an algebraic variety over a field of characteristic zero, I-II* Ann. of Math., **97** (1964).

[9] H. Hironaka, *Subanalytic sets*, in Number Theory, Algebraic Geometry and Commuta- tive Algebra (Kinokuniya, Tokyo), 1973, volume in honor of Yasuo Akizuki, 453–493.

[10] T.-C. Kuo, *On classification of real singularities,* Invent. math. **82** (1985), 257–262.

[11] K. Kurdyka, *Points réguliers d'un ensemble sous-analytique*, Ann. Inst. Fourier, Grenoble, **38** (1988), 133-156

[12] K. Kurdyka, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann. **281** no.3 (1988), 445–462

[13] K. Kurdyka, S.Łojasiewicz, M. Zurro *Stratifications distinguées comme un util en géométrie semi-analytique*, Manuscripta Math. 86, 81–102, (1995).

[14] S. Łojasiewicz, *Ensembles semi-analytiques,*, preprint, I.H.E.S. (1965) available on the web page "http://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf".

[15] S. Łojasiewicz, *Sur la géométrie semi- et sous-analytique*, Ann. Inst. Fourier (Grenoble) 43 (1993), 1575–1595.

[16] A. Parusiński, *Subanalytic functions*, Trans. Amer. Math. Soc. **344**, 2 (1994), 583–595.

[17] M. Tamm, *Subanalytic sets in the calculus of variation*, Acta Math. 146 (1981), no. 3-4, 167–199.

[18] J. Włodarczyk, *Simple Hironaka resolution in characteristic zero*, J. Amer. Math. Soc., **18** (2005), no. 4, 779–822

Laboratoire de Mathématiques, UMR 5175 du CNRS, Université de Savoie, Campus Scientifique, 73 376 Le Bourget–du–Lac Cedex, France,

E-mail address: `kurdyka@univ-savoie.fr`

Laboratoire Angevin de Recherche en Mathématiques, UMR 6093 du CNRS, Université d'Angers, 2, bd Lavoisier, 49045 Angers cedex, France

E-mail address: `adam.parusinski@univ-angers.fr`