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FIBONACCI-LIKE UNIMODAL INVERSE LIMIT SPACES

H. BRUIN AND S. ŠTIMAC

Abstract. We study the structure of inverse limit space of so-called Fibonacci-like tent maps. The combinatorial constraints implied by the Fibonacci-like assumption allows us to introduce certain chains that enable a more detailed analysis of symmetric arcs within this space than is possible in the general case. We show that link-symmetric arcs are always symmetric or a well-understood concatenation of quasi-symmetric arcs. This leads to simplification of some existing results, including the Ingram Conjecture for Fibonacci-like unimodal inverse limits.

1. Introduction

A unimodal map is called Fibonacci-like if it satisfies certain combinatorial conditions implying an extreme recurrence behavior of the critical point. The Fibonacci unimodal map itself was first described by Hofbauer and Keller [16] as a candidate to have a so-called wild attractor. (The combinatorial property defining the Fibonacci unimodal map is that its so-called *cutting times* are exactly the Fibonacci numbers $1, 2, 3, 5, 8, \dots$) In [13] it was indeed shown that Fibonacci unimodal maps with sufficiently large critical order possess a wild attractor, whereas Lyubich [21] showed that such is not the case if the critical order is 2 (or $\leq 2+\varepsilon$ as was shown in [20]). This answered a question in Milnor's well-known paper on the structure of metric attracts [23]. In [9] the strict Fibonacci combinatorics were relaxed to Fibonacci-like. Intricate number-theoretic properties of Fibonacci-like critical omega-limit sets were revealed in [22] and [14], and [10, Theorem 2] shows that Fibonacci-like combinatoric are incompatible with the Collet-Eckmann condition of exponential derivative growth along the critical orbit. This shows that Fibonacci-like maps are an extremely interesting class of maps in between the regular and the stochastic unimodal maps in the classification of [1].

One of the reasons for studying the inverse limit spaces of Fibonacci-like unimodal maps is that they present a toy model of invertible strange attractors (such as Hénon attractors) for which as of today very little is known beyond the Benedicks-Carleson parameters [4] resulting in strange attractors with positive unstable Lyapunov exponent. It is for example

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unknown if invertible wild attractors exist in the smooth planar context, or to what extent Hénon-like attractors satisfy Collet-Eckmann-like growth conditions. The precise recurrence and folding structure of Hénon-like attractors may be of crucial importance to answer such questions, and we therefore focus on these aspects of the structure of Fibonacci-like inverse limit spaces.

A second reason for this paper is to provide a better understanding and a potential simplification of the solution of the Ingram Conjecture. This conjecture was posed by Tom Ingram in 1992 for tent maps $T_s : [0,1] \to [0,1]$ with slope $\pm s$, $s \in [1,2]$, defined as $T_s(x) = \min\{sx, s(1-x)\}$:

> If $1 \leq s < s' \leq 2$, then the corresponding inverse limit spaces $\varprojlim([0, s/2], T_s)$ and $\varprojlim([0, s'/2], T_{s'})$ are non-homeomorphic.

The first results towards solving this conjecture were been obtained for tent maps with a finite critical orbit [18, 19, 26, 5]. Raines and Štimac [25] extended these results to tent maps with a possibly infinite, but non-recurrent critical orbit. Recently Ingram's Conjecture was solved completely (in the affirmative) in [3], but we still know very little of the structure of inverse limit spaces (and their subcontinua) for the case that $\mathrm{orb}(c)$ is infinite and recurrent, see [2, 6, 11].

The folding structure of a unimodal inverse limit space can be described by so-called p-points (where arc-components fold back on themselves) and their levels. The existence of such points is the reason why, contrary to (substitution) tiling spaces, unimodal inverse limits are locally not homeomorphic to a Cantor set of arcs. For Fibonacci-like maps, these p-points observe some hierarchical structure which allows us to introduce a special kind of chains in this paper. Using these chains, we are able to describe the symmetries and link-symmetries (w.r.t. chains) within the zero-composant $\mathfrak{C}$ in much more detail than is currently known for general unimodal inverse limits. In the proof of Ingram's Conjecture [3], such symmetric arcs are a crucial ingredient, especially those centered around so-called snappy points, see Definition 5.11. The methods developed here provide a more insightful proof for Fibonacci-like inverse limits that link-symmetric arcs, unless they are centered around a snappy point, can contain at most one snappy point.

The paper is organized as follows. In Section 2 we review the basic definitions of inverse limit spaces and tent maps and their symbolic dynamics. Section 3 is devoted to the construction of the chains $\mathcal{C}$ having special properties that allow us to prove desired properties of folding structure of the arc component $\mathfrak{C}$ in Section 4. In Section 5, we show that link-symmetric arcs are always symmetric or a well-understood concatenation of symmetric arcs. A simple and intuitive corollary of the revealed folding structure is the following very important property for the proof of the Ingram conjecture: Every p-link symmetric arc of $\mathfrak{C}$ that is not centered at a snappy point, contains at most one snappy p-point.

2. Preliminaries

Basic definitions: The tent map $T_s : [0,1] \to [0,1]$ with slope $\pm s$ is defined as $T_s(x) = \min\{sx, s(1-x)\}$. The critical or turning point is $c = 1/2$ and we write $c_k = T_s^k(c)$, so in particular $c_1 = s/2$ and $c_2 = s(1-s/2)$. We will restrict T_s to the interval $I = [0, s/2]$; this is larger than the *core* $[T_s^2(c), T_s(c)] = [s - s^2/2, s/2]$, but it contains the fixed point 0 on which the 0-composant $\mathfrak{C}$ is based.

The inverse limit space $\varprojlim([0,s/2], T_s)$ is

$$\{x = (\dots, x_{-2}, x_{-1}, x_0) : T_s(x_{i-1}) = x_i \in [0, s/2] \text{ for all } i \le 0\},$$

equipped with metric $d(x,y) = \sum_{n \leqslant 0} 2^n |x_n - y_n|$ and *induced* (or *shift*) *homeomorphism*

$$\sigma(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1}, x_0, T_s(x_0)).$$

Let $\pi_k : \varprojlim([0,s/2], T_s) \to I$, $\pi_k(x) = x_{-k}$ be the k-th projection map. Since $0 \in I$, the endpoint $(\dots, 0, 0, 0)$ is contained in $\varprojlim([0,s/2], T_s)$. The *composant* of $x \in X$ is defined as the union of all proper subcontinua of X containing x. The composant of $\varprojlim([0,s/2], T_s)$ of $(\dots, 0, 0, 0)$ will be denoted as $\mathfrak{C}$; it is a ray converging to, but disjoint from the core $\varprojlim([c_2, c_1], T_s)$ of the inverse limit space. We fix $s \in (\sqrt{2}, 2]$; for these parameters T_s is not renormalizable and $\varprojlim([c_2, c_1], T_s)$ is indecomposable.

Combinatorics of tent maps: Recall now some background on the combinatorics of unimodal maps, see *e.g.* [8]. The *cutting times* $\{S_k\}_{k \ge 0}$ are those iterates n (written in increasing order) for which the central branch of T_s^n covers c. More precisely, let $Z_n \subset [0, c]$ be the maximal interval with boundary point c on which T_s^n is monotone, and let $\mathfrak{D}_n = T_s^n(Z_n)$. Then n is a *cutting time* if $\mathfrak{D}_n \ni c$. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. There is a function $Q : \mathbb{N} \to \mathbb{N}_0$ called the *kneading map* such that

$$S_k - S_{k-1} = S_{Q(k)} \tag{2.1}$$

for all k. The kneading map $Q(k) = \{k-2, 0\}$ (with cutting times $\{S_k\}_{k \ge 0} = \{1, 2, 3, 5, 8, \dots\}$) belongs to the *Fibonacci map.* We call T_s *Fibonacci-like* if its kneading map is eventually non-decreasing, satisfying Condition (2.2) below as well.

$$Q(k+1) > Q(Q(k)+1) \qquad \text{for all } k \text{ sufficiently large.} \tag{2.2}$$

Remark 2.1. Condition (2.2) follows if the Q is eventually non-decreasing and $Q(k) \le k - 2$ for k sufficiently large. Geometrically, it means that $|c - c_{S_k}| < |c - c_{S_{Q(k)}}|$, see Lemma 2.2 and also [8].

Lemma 2.2. If the kneading map of T_s satisfies (2.2), then

$$|c_{S_k} - c| < |c_{S_{Q(k)}} - c| \quad \text{and} \quad |c_{S_k} - c| < \frac{1}{2}|c_{S_{Q^2(k)}} - c|. \tag{2.3}$$

for all k sufficiently large.

Proof. For each cutting time S_k, let $\zeta_k \in Z_{S_k}$ be the point such that $T_s^{S_k}(\zeta_k) = c$. Then ζ_k together with its symmetric image $\hat{\zeta}_k := 1 - \zeta_k$ are closest precritical points in the sense that $T_s^j((\zeta_k, c)) \not\ni c$ for $0 \leqslant j \leqslant S_k$. Consider the points ζ_{k-1}, ζ_k and c, and their images under $T_s^{S_k}$, see Figure 1. Note that $Z_{S_k} = [\zeta_{k-1}, c]$ and $T_s^{S_k}([\zeta_{k-1}, c]) = \mathfrak{D}_{S_k} = [c_{S_{Q(k)}}, c_{S_k}]$.

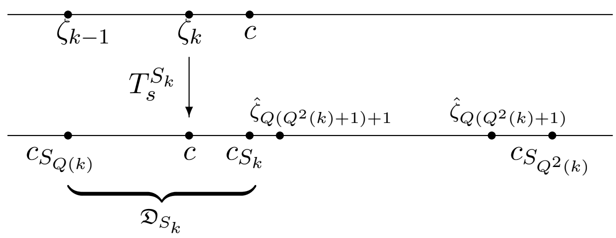

FIGURE 1. The points ζ_{k-1}, ζ_k and c, and their images under $T_s^{S_k}$.

Since $S_{k+1} = S_k + S_{Q(k+1)}$ is the first cutting time after S_k, the precritical point of lowest order on $[c, c_{S_k}]$ is $\zeta_{Q(k+1)}$ or its symmetric image $\hat\zeta_{Q(k+1)}$. Applying this to c_{S_k} and $c_{Q(k)}$, and using (2.2), we find

$$c_{S_k} \subset (\zeta_{Q(k+1)-1}, \hat\zeta_{Q(k+1)-1}) \subset (\zeta_{Q(Q(k)+1)}, \hat\zeta_{Q(Q(k)+1)}) \subset (c_{S_{Q(k)}}, \hat c_{S_{Q(k)}}).$$

Therefore $|c_{S_k} - c| < |c_{S_{Q(k)}} - c|$. Since $T_s^{S_k}|_{[\zeta_{k-1}, c]}$ is affine, also the preimages ζ_{k-1} and ζ_k of $c_{S_{Q(k)}}$ and c satisfy $|\zeta_k - c| < |\zeta_{k-1} - \zeta_k|$. Applying (2.2) twice we obtain

$$Q(k+1) > Q(Q^2(k)+1)+1, \tag{2.4}$$

for all k sufficiently large. Therefore there are at least two closest precritical points ($\hat\zeta_{Q(Q^2(k)+1)}$ and $\hat\zeta_{Q(Q^2(k)+1)+1}$ in Figure 1) between c_{S_k} and $c_{S_{Q^2(k)}}$. Therefore

$$|c_{S_k} - c| < |\hat\zeta_{Q(Q^2(k)+1)+1} - c| < \frac{1}{2}|\hat\zeta_{Q(Q^2(k)+1)} - c| < \frac{1}{2}|c_{S_{Q^2(k)}} - c|. \tag{2.5}$$

□

Not all maps $Q : \mathbb{N} \to \mathbb{N}_0$ nor all sequences of cutting times (as defined in (2.1)) correspond to a unimodal map. As was shown by Hofbauer [15], a kneading map Q belongs to a unimodal map (with infinitely many cutting times) if and only if

$$\{Q(k+j)\}_{j\geq 1} \geq_{lex} \{Q(Q^2(k)+j)\}_{j\geq 1} \tag{2.6}$$

for all $k \geq 1$, where $\geq_{lex}$ indicates lexicographical order. Clearly, Condition (2.2) is compatible with (and for large k implies) (2.6).

Remark 2.3. The condition $\{Q(k+j)\}_{j\geq 1} \geq_{lex} \{Q(l+j)\}_{j\geq 1}$ is equivalent to $|c - c_{S_k}| < |c - c_{S_l}|$. Therefore, because $c_{S_{k-1}} \in (\zeta_{Q(k)-1}, \zeta_{Q(k)})$, we find by taking the $T_s^{S_{Q(k)}}$-images, that $c_{S_k} \in [c_{S_{Q^2(k)}}, c]$ and (2.6) follows. The other direction, namely that (2.6) is sufficient for admissibility is much more involved, see [15, 8].

Let $\beta(n) = n - \sup\{S_k < n\}$ for $n \geq 2$ and find recursively the images of the central branch of T_s^n (the levels in the Hofbauer tower, see e.g. [8, 7]) as

$$\mathfrak{D}_1 = [0, c_1] \text{ and } \mathfrak{D}_n = [c_n, c_{\beta(n)}].$$

It is not hard to see that $\mathfrak{D}_n \subset \mathfrak{D}_{\beta(n)}$ for each n, see [8], and that if $J \subset [0, s/2]$ is a maximal interval on which T_s^n is monotone, then $T_s^n(J) = \mathfrak{D}_m$ for some $m \leqslant n$.

The condition that $Q(k) \to \infty$ has consequence on the structure of the critical orbit:

Lemma 2.4. If $Q(k) \to \infty$, then $|\mathfrak{D}_n| \to 0$ as $n \to \infty$, c is recurrent and $\omega(c)$ is a minimal Cantor set.

Proof. See *e.g.* [8]. □

Further definitions for inverse limit spaces: A point $x = (\dots, x_{-2}, x_{-1}, x_0) \in \mathfrak{C}$ is called a *p-point* if $x_{-p-l} = c$ for some $l \in \mathbb{N}_0$. The number $L_p(x) := l$ is the *p-level* of x. In particular, $x_0 = T_s^{p+l}(c)$. By convention, the endpoint $\bar{0} = (\dots, 0, 0, 0)$ of $\mathfrak{C}$ is also a p-point and $L_p(\bar{0}) := \infty$, for every p. The ordered set of all p-points of composant $\mathfrak{C}$ is denoted by E_p, and the ordered set of all p-points of p-level l by $E_{p,l}$. Given an arc $A \subset \mathfrak{C}$ with successive p-points $x^0, \dots, x^n$, the *p-folding pattern* of A is the sequence

$$FP_p(A) := L_p(x^0), \dots, L_p(x^n).$$

The *folding pattern of composant* $\mathfrak{C}$, denoted by $FP(\mathfrak{C})$, is the sequence $L_p(z^1), L_p(z^2), \dots,$ $L_p(z^n), \dots,$ where $E_p = \{z^1, z^2, \dots, z^n, \dots\}$ and p is any nonnegative integer. Let $q \in \mathbb{N}$, $q > p$, and $E_q = \{y^0, y^1, y^2, \dots\}$. Since σ^{q-p} is an order-preserving homeomorphism of $\mathfrak{C}$, it is easy to see that $\sigma^{q-p}(z^i) = y^i$ for every $i \in \mathbb{N}$, and $L_p(z^i) = L_q(y^i)$. Therefore the folding pattern of $\mathfrak{C}$ does not depend on p.

An arc A in $\varprojlim([0, s/2], T_s)$ is said to *p-turn at* c_n if there is a p-point $a \in A$ such that $a_{-(p+n)} = c$, so $L_p(a) = n$. This implies that $\pi_p : A \to [0, s/2]$ achieves c_n as a local extremum at a. If x and y are two adjacent p-turning points on the same arc-component, then $\pi_p([x, y]) = \mathfrak{D}_n$ for some n, so $\pi_p(x) = c_n$ and $\pi_p(y) = c_{\beta(n)}$ or vice versa. Let us call x and y (or $\pi_p(x)$ and $\pi_p(y)$) *β-neighbors* in this case. Notice, however, that there may be many post-critical points between $\pi_p(x)$ and $\pi_p(y)$. Obviously, every p-turning point has exactly two β-neighbors, except the endpoint $(\dots, 0, 0, 0)$ of $\mathfrak{C}$ whose β-neighbor (w.r.t. p) is by convention the first proper p-turning point in $\mathfrak{C}$, necessarily with p-level 1.

3. The Construction of Chains

A space is *chainable* if there are finite open covers $\mathcal{C} = \{\ell_i\}_{i=1}^N$, called *chains*, of arbitrarily small *mesh* ($\operatorname{mesh}\mathcal{C} = \max_i \operatorname{diam}\ell_i$) with the property that the *links* ℓ_i satisfy $\ell_i \cap \ell_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. The combinatorial properties of Fibonacci-like maps allow us to construct chains $\mathcal{C}_p$ such that whenever an arc A p-turns in $\ell \in \mathcal{C}_p$, *i.e.,* enters and exits ℓ through the same neighboring link, then the projections $\pi_p(x) = \pi_p(y)$ of the first and last p-point x and y of $A \cap \ell$ depend only on ℓ and not on A, see Proposition 3.3.

We will work with the chains which are the π_p^{-1} images of chains of the interval $[0, s/2]$. More precisely, we will define a finite collection of points $G = \{g_0, g_1, \dots, g_N\} \subset [0, s/2]$ such that $|g_m - g_{m+1}| \leq s^{-p}\varepsilon/2$ for all $0 \leq m < N$ and $|0 - g_0|$ and $|s/2 - g_N|$ positive but very small. From this one can make a chain $\mathcal{C} = \{\ell_n\}_{n=0}^{2N}$ by setting

$$
\begin{cases} \ell_{2m+1} = \pi_p^{-1}((g_m, g_{m+1})) & 0 \le m < N, \\ \ell_{2m} = \pi_p^{-1}((g_m - \delta, g_m + \delta) \cap [0, s/2]) & 0 \le m \le N, \end{cases} \tag{3.1}
$$

where $\min\{|0 - g_0|, |s/2 - g_N|\} < \delta \ll \min_m\{|g_m - g_{m+1}|\}$. Any chain of this type has link of diameter $< \varepsilon$.

Remark 3.1. We could have included all the points $\cup_{j \le p} T_s^{-j}(c)$ in G to ensure that $T_s^p|_{(g_m, g_{m+1})}$ is monotone for each m, but that is not necessary. Naturally, there are chains of $\varprojlim([0, s/2], T_s)$ that are not of this form.

For a component A of $\mathfrak{C} \cap \ell$, we have the following two possibilities:
(i) $\mathfrak{C}$ goes straight through ℓ at A, *i.e.,* A contains no p-point and $\pi_p(\partial A) = \partial \pi_p(\ell)$; in this case A enters and exits ℓ from different sides.
(ii) $\mathfrak{C}$ turns in ℓ: A contains (an odd number of) p-points $x^0, \dots, x^{2n+1}$ of which the middle one x^n has the highest level, and $\pi_p(\partial A)$ is a single point in $\partial \pi_p(\ell)$, in this case A enters and exits ℓ from the same side.

Before giving the details of the p-chains we will use, we need a lemma.

Lemma 3.2. *If the kneading map Q of T_s is eventually non-decreasing and satisfies Condition (2.4), then for all $n \in \mathbb{N}$ there are arbitrarily small numbers $\eta_n > 0$ with the following property: If $n' > n$ is such that $n \in \mathrm{orb}_\beta(n')$, then either $|c_{n'} - c_n| > \eta_n$ or $|c_{n''} - c_n| < \eta_n$ for all $n \le n'' \le n'$ with $n'' \in \mathrm{orb}_\beta(n')$.*

To clarify what this lemma says, Figure 2 shows the configuration of levels $\mathfrak{D}_k$ that should be avoided, because then η_n cannot be found.

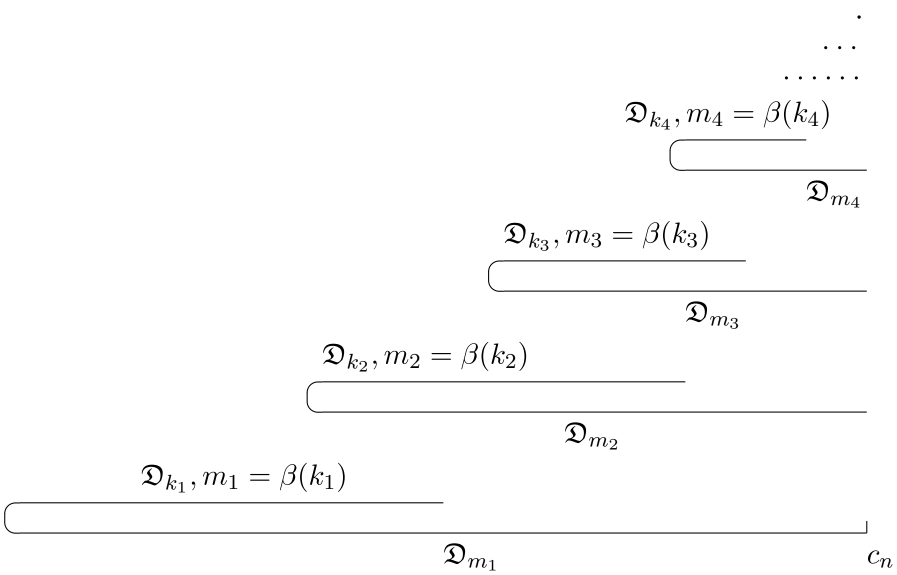

FIGURE 2. Linking of levels $\mathfrak{D}_{m_i}$ with $\beta(m_1) = \beta(m_2) = \beta(m_3) = \cdots = n$. The semi-circles indicates that two intervals have an endpoint in common.

Proof. We will show that the pattern in Figure 2 (namely with $c_{m_1} < c_{m_2} < c_{m_3} < \dots$ and $c_{m_{i-1}} < c_{k_i}$ for each i) does not continue indefinitely. To do this, we redraw the first few levels from Figure 2, and discuss four positions in $\mathfrak{D}_{m_1}$ where the precritical point $T_s^{-r}(c) \in \mathfrak{D}_{m_1}$ of lowest order r could be, indicated by points $a_1, \dots, a_4$, see Figure 3.

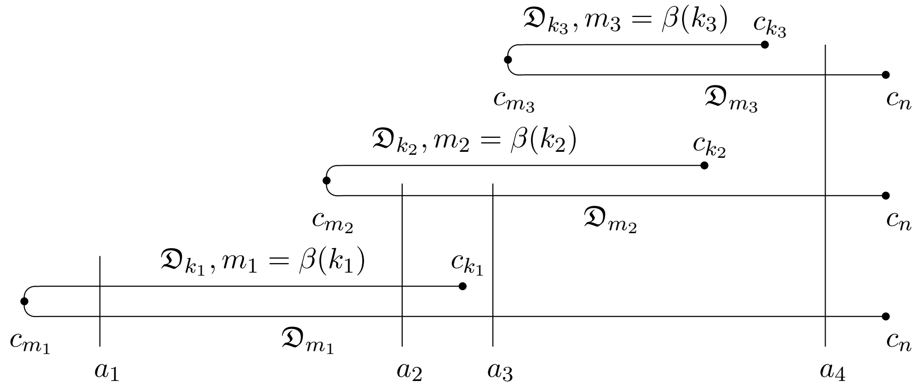

FIGURE 3. Linking of levels $\mathfrak{D}_{m_i}$, $i = 1, 2, 3$ and three possible positions of the precritical point $a_j = T_s^{-r}(c) \in \mathfrak{D}_{m_1}$ of lowest order r.

Case $a_1 \in (c_{m_1}, c_{m_2})$**:** Take the $r+1$-th iterate of the picture, which moves $\mathfrak{D}_{m_1}$ and $\mathfrak{D}_{k_1}$ to levels with lower endpoint c_1. then we can repeat the argument, until we arrive in one of the cases below.

Case $a_2 \in (c_{m_2}, c_{k_1})$**:** Take the r-th iterate of the picture, which moves $\mathfrak{D}_{m_1}$, $\mathfrak{D}_{k_1}$, $\mathfrak{D}_{m_2}$ and $\mathfrak{D}_{k_2}$ all to cutting levels and $c_{r+k_2} \in (c, c_{r+k_3})$. But $m_2 > m_1$, whence $k_2 > k_1$, and this contradicts that $|c_{S_{k_2}} - c| < |c_{S_{k_1}} - c|$. (If $a_2 \in (c_{m_3}, c_{k_2})$, then the same argument would give that $r + k_2 < r + k_3$ are both cutting times, but $|c - c_{r+k_2}| < |c - c_{r+k_3}|$.)

Case $a_3 \in (c_{k_1}, c_{m_3})$**:** Take the r-th iterate of the picture, which moves $\mathfrak{D}_{m_1}$, $\mathfrak{D}_{m_2}$ and $\mathfrak{D}_{k_2}$ to cutting levels, and $\mathfrak{D}_{m_3}$ to a non-cutting level $\mathfrak{D}_u$ with $u := m_3 + r$ such that

$$S_j := n + r = \beta(u) = \beta(m_2 + r) = \beta^2(k_2 + r).$$

The integer u such that c_u is closest to c is for $u = S_i + S_j$ where j is minimal such that $Q(i+1) > i$, and in this case, the itineraries of $T_s(c)$ and $T_s(c_u)$ agree for at most $S_{Q^2(i+1)+1} - 1$ iterates (if $Q(i+1) = j+1$) or at most $S_{Q(j+1)} - 1$ iterates (if $Q(i+1) > j+1$). Call $S_h := k_2 + r$, then $j = Q^2(h)$ and the itineraries of $T_s(c_{S_h})$ and c agree up to $S_{Q(h+1)} - 1$ iterates. By assumption (2.4), we have

$$Q(j+1) \leq Q^2(i+1) + 1 = Q(j+1) + 1 = Q(Q^2(h) + 1) + 1 < Q(h+1),$$

but this means that $\mathfrak{D}_u$ and $\mathfrak{D}_{S_h}$ cannot overlap, a contradiction.

Case $a_4 \in (c_{k_2}, c_n)$**:** Then take the $r+1$-st iterate of the picture, which has the same structure, with c_n replaced by $T_s^{r+1}(a_1) = c_1$. Repeating this argument, we will eventually arrive at Case a_2 or a_3 above.

Therefore we can find η_n such that $c_n - \eta_n$ separates c_n from all levels $\mathfrak{D}_{k_i}$, $\beta^2(k_i) = n$ that intersect $\mathfrak{D}_{m_1}$. Indeed, in Case a_2, we place $c_n - \eta_n$ just to the right of c_{k_1} and in Case a_3 (and hence $c_{k_1} \in \mathfrak{D}_{k_2}$), we place $c_n - \eta_n$ just to the right of c_{k_2}. □

Proposition 3.3. Under the assumption of Lemma 3.2, given $\varepsilon > 0$, there exists $p \in \mathbb{N}$ and a chain $\mathcal{C} = \mathcal{C}_p$ of $\varprojlim([0, s/2], T_s)$ with the following properties:

(1) The links of $\mathcal{C}$ have diameter $< \varepsilon$.
(2) For each $n \in \mathbb{N}$, there is exactly one link $\ell \in \mathcal{C}$ such that every $x \in \varprojlim([0, s/2], T_s)$ that p-turns at c_n belongs to ℓ.
(3) If $y \in \ell$ is a p-point not having the lowest p-level of p-points in ℓ, then both β-neighbors of y belong to ℓ.
(4) If $y \notin \ell$ is a β-neighbor of x above, then the other β-neighbor of y either lies outside ℓ, or has p-level n as well.

Proof. We will construct the chain $\mathcal{C}$ as outlined in the beginning of this section, see (3.1). So let us specify the collection G by starting with at least $\lceil 2s^p/\varepsilon \rceil$ approximately equidistant points $g_m \in [0, s/2]$ so that no g_m lies on the critical orbit, and then refining this collection inductively to satisfy parts 2.-4. of the proposition.

Start the induction with $n = 1$, *i.e.,* the point c_1. Note that $c_1 \notin G$, so there will be only one link $\ell \in \mathcal{C}$ with $c_1 \in \pi_p(\ell)$. Let $\eta_1 \in (0, s^{-p}\varepsilon/2)$ be as in Lemma 3.2. Then, since each k contains 1 in its β-orbit, each $\mathfrak{D}_k$ intersecting $(c_1 - \eta_1, c_1]$ is either contained in $(c_1 - \eta_1, c_1]$ or has c_1 as lower endpoint (*i.e.,* $\beta(k) = 1$). In the latter case, also $\mathfrak{D}_l \cap (c_1 - \eta_1, c_1] = \emptyset$ for each l with $\beta(l) = k$. Hence by inserting $c_1 - \eta_1$ into G, we can refine the chain $\mathcal{C}$ so that properties 3. and 4. holds for the link ℓ with $\pi_p(\ell) \ni c_1$.

Suppose we have refined the chain to accommodate links ℓ such that $\pi_p(\ell) \ni c_i$ for each $i < n$. Then c_n does not belong to the set G created so far, so there will be only one link $\ell \in \mathcal{C}$ with $\pi_p(\ell) \ni c_n$. Again, find $\eta_n \in (0, s^{-p}\varepsilon/2)$ as in Lemma 3.2 and extend G with $c_n + \eta_n$ if c_n is a local minimum of T_s^n or with $c_n - \eta_n$ if c_n is a local minimum of T_s^n.

We skip the induction step if $\mathfrak{D}_n$ already belongs to complementary interval to G extended with all point $c_i \pm \eta_i$ created so far. Since $|\mathfrak{D}_n| \to 0$, the induction will eventually cease altogether, and then the required set G is found. □

4. Symmetric and Quasi-Symmetric Arcs

From now on all chains $\mathcal{C}_p$ are as in Proposition 3.3. Also, we assume that the slope s is such that T_s is Fibonacci-like and we abbreviate $T := T_s$.

Definition 4.1. An arc $A \subset \mathfrak{C}$ such that $\partial A = \{u, v\}$ and $A \cap E_p = \{x^0, \dots, x^n\}$ is called *p-symmetric* if $\pi_p(u) = \pi_p(v)$ and $L_p(x^i) = L_p(x^{n-i})$, for every $0 \leqslant i \leqslant n$.

It is easy to see that if A is p-symmetric, then n is even and $L_p(x^{n/2}) = \max\{L_p(x^i) : x^i \in A \cap E_p\}$. The point $x^{n/2}$ is called the *center* or *midpoint* of A.

It frequently happens that $\pi_p(u) \neq \pi_p(v)$, but u and v belong to the same link $\ell \ni \mathcal{C}_p$. Let us call the arc components A_u, A_v of $\mathfrak{C} \cap \ell$ that contain u and v respectively the *link-tips* of A, see Figure 4. Sometimes we can make A p-symmetric by removing the link-tips. Let us denote this as $A \setminus \ell$-tips. Adding the closure of the link-tips can sometimes also produce a p-symmetric arc.

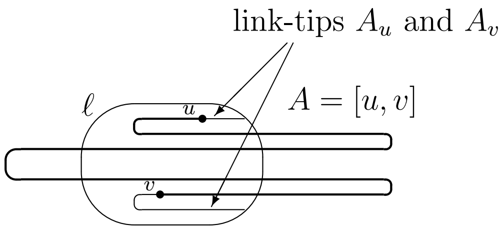

FIGURE 4. The arc A is neither p-symmetric, nor quasi-p-symmetric, but both arcs $A \setminus \ell$-tips and $A \cup \mathrm{Cl}(\ell\text{-tips})$ are p-symmetric.

Remark 4.2. (a) Let A be an arc and $m \in A$ be a p-point of maximal p-level, say $L_p(m) = L$. Then π_p is one-to-one of both components of $\sigma^{1-L}(A \setminus \{m\})$, so m is the only p-point of p-level L. It follows that between every two p-points of the same p-level, there is a p-point m of higher p-level.

(b) If $A \ni m$ is the maximal open arc such that m has the highest p-level on A, then we can write $\mathrm{Cl}\, A = [x, y]$ or $[y, x]$ with $L_p(x) > L_p(y) > L_p(m) =: L$, and π_p is one-to-one on $\sigma^{-L}(\mathrm{Cl}\, A)$. Here $L_p(x) = \infty$ is possible, but if $L_p(x) < \infty$, then $A' := \pi_p \circ \sigma^{-L}(A)$ is a neighborhood of c with boundary points $c_{S_k} = \pi_p \circ \sigma^{-L}(x)$ and $c_{S_l} = \pi_p \circ \sigma^{-L}(y)$ for some $k, l \in \mathbb{N}$ such that $l = Q(k)$. By Lemma 2.2 this means that the arc $[x, m]$ is shorter than $[m, y]$.

Definition 4.3. Let A be an arc of the composant $\mathfrak{C}$. We say that the arc A is *quasi-p-symmetric with respect to $\mathcal{C}_p$* if

(i) A is not p-symmetric;
(ii) ∂A belongs to a single link ℓ;
(iii) $A \setminus \ell$-tips is p-symmetric;
(iv) $A \cup \ell$-tips is not p-symmetric. (So A cannot be extended to a symmetric arc within its boundary link ℓ.)

Suppose $A = [u, v] \subset \mathfrak{C}$ is a quasi-p-symmetric arc with $u, v \in \ell$, and let A_u and A_v be arc components of ℓ that contain u and v respectively. We will sometimes say, for simplicity, that the arc $[A_u, A_v]$ between A_u and A_v, including A_u and A_v, is quasi-p-symmetric.

Definition 4.4. A quasi-p-symmetric arc $A = [u, v]$ with midpoint m is called *basic* if there is no p-point $w \in (u, v)$ such that either $[u, w] \subset [u, m]$ or $[w, v] \subset [m, v]$ is a quasi-p-symmetric arc.

Example 4.5. Let us consider the Fibonacci map and the corresponding inverse limit space. Then the composant $\mathfrak{C}$ contains the arc $A = [x^0, x^{33}]$ such that the folding pattern of A is as follows (see Figure 5):

$$27\;6\;\overbrace{\underbrace{\mathbf{1\;14\;1}\;6\;\mathbf{1}}_{\text{basic}}\;0\;3\;0\;1\;0\;2\;0\;1\;4\;1\;\mathbf{9}\;1\;4\;1\;0\;2\;0\;1\;0\;3\;0\;\underbrace{1\;6\;\mathbf{1}}_{\text{sym}}}^{\text{quasi-}p\text{-symmetric}}\;0\;3\;0 \tag{4.1}$$

We can choose a chain $\mathcal{C}_p$ such that the p-points with p-levels 1 and 14 belong to the same link. The arc $[x^2, x^6]$ with the folding pattern 1 14 1 6 1 is a basic quasi-p-symmetric

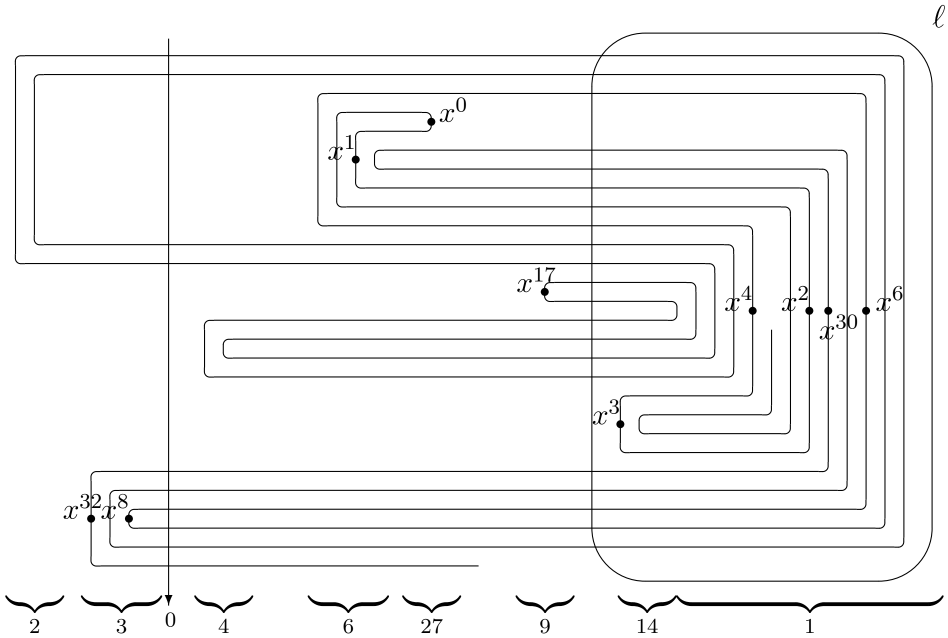

FIGURE 5. The arc A with folding pattern as in (4.1), with p-points of p-level 1 and 14 in a single link ℓ.

arc; the arc $[x^2, x^{30}]$ with the folding pattern as in (4.1) under the wide brace is also a quasi-p-symmetric but not basic, because it contains $[x^2, x^6]$. Notice also that the arc $[x^3, x^{30}]$ is a quasi-p-symmetric arc for which Proposition 4.12 and Proposition 4.10 do not work (see the folding patterns to the left of $[x^3, x^{30}]$ and to the right of $[x^3, x^{30}]$).

Lemma 4.6. Let $\mathcal{C}_p$ be a chain and $[x, y]$ a quasi-p-symmetric arc with respect to this chain (not contained in a single link) with midpoint m and such that $L_p(x) \geq L_p(m)$. Let A_x be the link-tip of $[x, y]$ which contains x. Then $L_p(m) > L_p(z)$ for all p-points $z \in [x, y] \setminus (\{m\} \cup A_x)$.

Proof. Let $A = [a, b] \ni m$ be the smallest arc with p-points a, b of higher p-level than $L_p(m)$, say $m \in [a, b]$ and $L_p(m) \leq L_p(a) \leq L_p(b)$. By part (a) of Remark 4.2 we obtain $L := L_p(m) < L_p(a) < L_p(b)$. Since $L_p(x) \geq L_p(m)$, $[x, m]$ contains one endpoint of A. We can assume that $[x, m] \setminus A$ is contained in a single link, because otherwise $[x, y] \setminus \ell$-tips is not p-symmetric. If $[y, m]$ does not contain the other endpoint of A, then the statement is proved.

Let us now assume by contradiction that $A \subset [x, y]$. Again, we can assume that $[y, m] \setminus A$ is contained in a single link, because otherwise $[x, y] \setminus \ell$-tips is not p-symmetric. By part

(a) of Remark 4.2 once more we have $\pi_{p+L}([a,b]) = [c_{S_l}, c_{S_k}] \ni c = \pi_{p+L}(m)$ for some k and $l = Q(k)$, and $|\pi_{p+L}(a) - c| > |\pi_{p+L}(b) - c|$, see the top line of Figure 6. It follows that $[a,b]$ contains a symmetric open arc (b', b) where $b' \in (a,b)$ is the unique point such that $T(\pi_{p+L}(b')) = T(\pi_{p+L}(b))$. Since $[x,y] \setminus \ell$-tips is p-symmetric, $L_p(b) > L_p(m)$ implies $b, b' \in \ell$-tips. Moreover, the arc $[a,b']$ is contained in the same link ℓ as b.

If k and l are relatively small, then $\pi_p^{-1}(c_{S_l})$ and $\pi_p^{-1}(c_{S_k})$ belong to different links of $\mathcal{C}_p$, so we can assume that they are so large that we can apply Condition (2.2). Let $r = Q(k+1)$

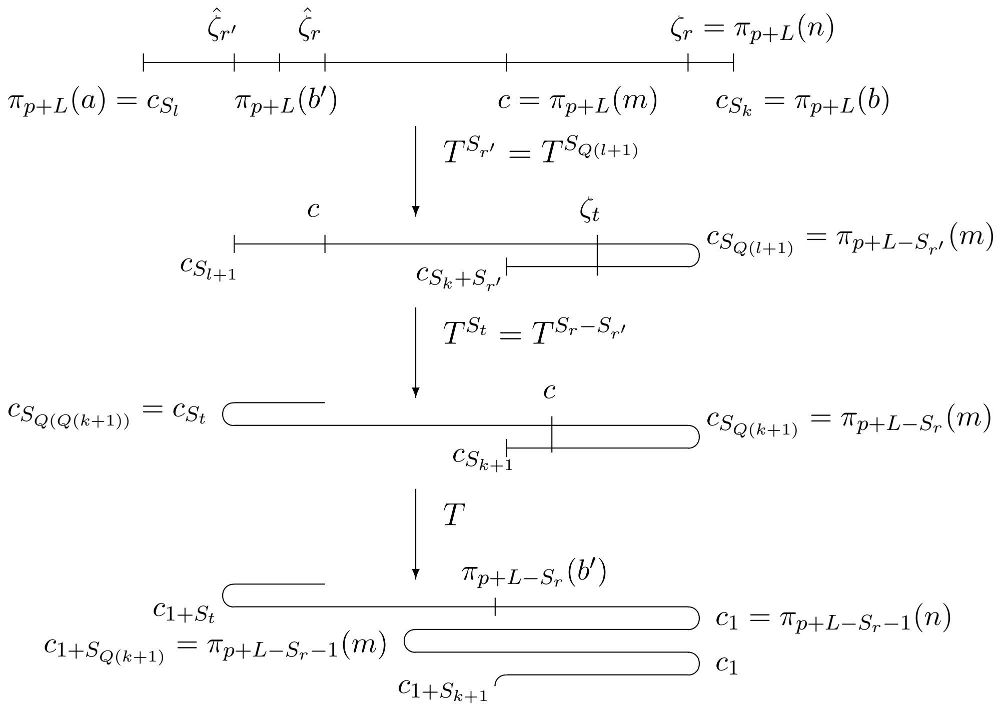

FIGURE 6. The image of $\pi_{p+L}([x,y]) \ni c = \pi_{p+L}(m)$ under appropriate iterates of T.

and $r' = Q(l+1)$ be the lowest indices such that the closest precritical points $\hat{\zeta}_{r'} \in [c_{S_l}, c]$ and $\zeta_r \in [c, c_{S_k}]$. By (2.2), $r' = Q(l+1) = Q(Q(k)+1) < Q(k+1) = r$. Consider the image of $[c_{S_l}, c_{S_k}]$ first under $T^{S_{r'}}$ and then under T^{S_r} (second and third level in Figure 6).

By the choice of r, we obtain $\pi_{p+L-S_r}([m,b]) = [c_{S_{k+1}}, c_{S_{Q(k+1)}}]$, and $\pi_{p+L-S_r}([a,b']) \ni c_{S_t}$ for $t = Q(Q(k+1))$. As in (2.5), $|c_{S_t} - c| > |c_{S_{Q(k+1)}} - c| > |c_{S_{k+1}} - c|$, and taking one more iterate, we see that $[c_{1+S_{k+1}}, c_1] \subset [c_{1+S_{Q(k+1)}}, c_1] \subset [1 + c_{S_t}, c_1]$ (last level in Figure 6).

Let $n \in [m,b]$ be such that $\pi_{p+L}(n) = \zeta_r$, see the first level in Figure 6. Since $[a,b']$ belongs to a single link $\ell \in \mathcal{C}_p$, $m \in \ell$ as well. Suppose that $[a,m]$ is not contained in ℓ. Then there is a maximal symmetric arc $[d', d]$ with midpoint n such that the points $d, d' \notin \ell$. Then the arcs $[d', a]$ and $[d, m]$ both enter the same link ℓ but they have different 'first' turning levels in ℓ, contradicting the properties of $\mathcal{C}_p$ from Proposition 3.3.

This shows that $[a, m] \subset \ell$. In the beginning of the proof we argued that the components of $[x, y] \setminus A$ belong to the same link, so that means that the entire arc $[x, y]$ is contained in a single link, contradicting the assumptions of the proposition. This concludes the proof. $\square$

Remark 4.7. In fact, this proof shows that the p-point $b \in \partial A$ of the highest p-level belongs to $[m, x]$. Indeed, if $a \in [m, x]$, then because $[m, b]$ has shorter arclength than $[m, a]$, either a and b, and therefore x and y do not belong to the same link ℓ (whence $[x, y]$ is not quasi-p-symmetric), or the arc $[a, b]$ itself is quasi-p-symmetric and contradicts Lemma 4.6.

Corollary 4.8. Let $[x, y] \subset \mathfrak{C}$ be a quasi-p-symmetric arc, not contained in a single link, such that $L_p(x) > L_p(m) > L_p(y)$ for the midpoint m. If $[m, x]$ is longer than $[y, m]$ measured in arc-length, then there exists a p-point $y' \in A_x$ such that $[y, y']$ is p-symmetric.

Proof. As in the previous proof, $b \in [x, m]$ and $y \in [m, b']$ and take $y' \in [m, b]$ such that $\pi_{p+L}(y') = \pi_{p+L}(y)$. $\square$

Remark 4.9. If $A_x \ni x$ and $A_y \ni y$ are maximal arc components of $\mathfrak{C} \cap \ell$ (with still $L_p(x) > L_p(m) > L_p(y)$), and m_y is the midpoint of A_y, then there is $y' \in A_x$ such that $[y', m_y]$ is p-symmetric.

In other words, when $\mathfrak{C}$ enters and turns in a link ℓ, then it folds in a symmetric pattern, say with levels $L_1, L_2, \dots, L_{m-1}, L_m, L_{m-1}, \dots, L_2, L_1$. The nature of the chain $\mathcal{C}_p$ is such that L_1 depends only on ℓ. The Corollary 4.8 does not say that the rest of the pattern is the same also, but only that if $A \subset \mathfrak{C}$ is such that $A \setminus \ell$-tips is p-symmetric, then the folding pattern at the one link-tip is a subpattern (stopping at a lower center level) of the folding pattern at the other link-tip.

Proposition 4.10 (Extending a quasi-p-symmetric arc at its higher level endpoint)**.** Let $A = [x, y] \subset \mathfrak{C}$ be a basic quasi-p-symmetric arc, not contained in a single link, such that the p-points $x, y \in \ell$ are the midpoints of the link-tips of A and $L_p(x) > L_p(y)$. Let m be the midpoint of A. Then there exists a p-point m' such that the arc $[m, m']$ is (quasi-)p-symmetric with x as its midpoint.

Remark 4.11. The conditions are all crucial in this lemma:

(*a*) It is important that y is a p-point. Otherwise, if $\mathfrak{C}$ goes straight through ℓ at y, then it is possible that x is the single p-point in A_x (where A_x is the arc components of $\mathfrak{C} \cap \ell$ containing x) and $[v, x]$ is shorter than $[x, m]$, and the lemma would fail.

(*b*) Without the assumption that $[x, y]$ is basic the lemma can fail. If Figure 5 the quasi-p-symmetric arc $[x, y] = [x^3, x^{30}]$ is not basic, and indeed there is no p-point $m' \in [x, v] = [x^3, x^0]$ with $L_p(m') = L_p(m) = L_p(x^{17}) = 9$.

Proof. Since $[u, y]$ is p-symmetric, $L_p(u) = L_p(y) < L_p(m)$ and $x \neq u$. Let w be the first p-point 'beyond' y such that $L_p(w) > L_p(x)$. Take $L = L_p(x)$; Figure 8 shows the configuration of the relevant points on $[w, v]$ and their images under $\pi_p \circ \sigma^{-L}$ denoted by $\tilde{}$-accents. Clearly $\tilde{x} = c$.

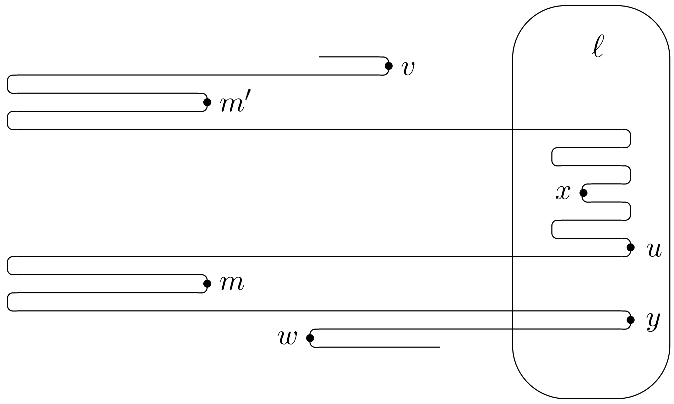

FIGURE 7. The configuration in Proposition 4.10 where the existence of p-point m' is proved. v is the first p-point 'beyond' x such that $L_p(v) > L_p(x)$ and u is such that $[u, y]$ is p-symmetric with the midpoint m.

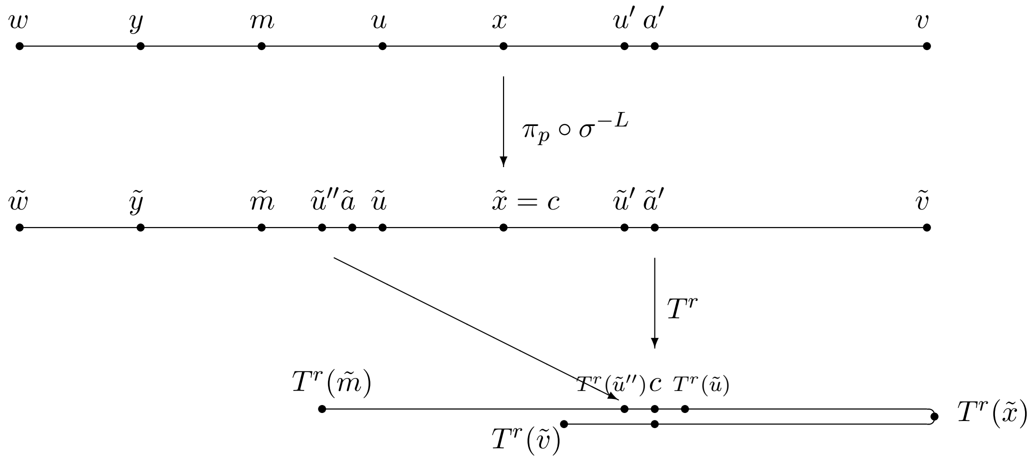

FIGURE 8. The configuration of points on $[w, v]$ and their images under $\pi_p \circ \sigma^{-L}$ and an additional T^r.

Case I: $|\tilde{w} - c| < |\tilde{v} - c|$. Then by Remark 4.2 (b), $\tilde{w} = c_{S_l}$ and $\tilde{v} = c_{S_k}$ with $k = Q(l)$. The points $\tilde{y}, \tilde{m}, \tilde{u}$ have symmetric copies $\tilde{y}', \tilde{m}', \tilde{u}'$ (*i.e.*, $T(\tilde{y}) = T(\tilde{y}')$, etc.) in reverse order on $[c, \tilde{v}]$, and the preimage under $\sigma^L \circ \pi_p^{-1}$ of the copy of $\tilde{m}'$ yields the required point m'.

Case II: $|\tilde{w} - c| > |\tilde{v} - c|$, so in this case, $l = Q(k)$. We can in fact assume that $|\tilde{m} - c| > |\tilde{v} - c|$ because otherwise we can find m' precisely as in Case I. Now take the p-point $a' \in (x, v)$ of maximal p-level, and let $a \in [m, x]$ be such that their $\pi_p \circ \sigma^{-L}$-images $\tilde{a}$ and $\tilde{a}'$ are each other symmetric copies. Let r be such that $T^r(\tilde{a}) = c$; the bottom part of Figure 8 shows the image of $[\tilde{m}, \tilde{v}]$ under T^r. The point $T^r(\tilde{x})$ and $T^r(v)$ are each others β-neighbors, and since $L_p(v) > L_p(x)$, and by (2.2), $|T^r(\tilde{x}) - c| > |T^r(v) - c|$. Therefore $[T^{r+j}(\tilde{x}), T^{r+j}(\tilde{a}')] \supset [T^{r+j}(\tilde{v}), T^{r+j}(\tilde{a}')]$ for all $j \geq 1$.

If $a, a' \in \ell$, then since $[x, a] \subset \ell$, this would imply that $[a', v] \subset \ell$ as well, contrary to the fact that x is the midpoint of A_x.

If on the other hand $a, a' \notin \ell$, then there is a point $u'' \in [m, a]$ such that $T^r(\tilde{u}'')$ and $T^r(\tilde{u})$ are each other symmetric copies. It follows that $[u'', x]$ is a quasi-p-symmetric arc properly contained in $[x, y]$, contradicting that $[x, y]$ is basic. ☐

Proposition 4.12 (Extending a quasi-p-symmetric arc at its lower level endpoint)**.** Let $A = [x, y] \subset \mathfrak{C}$ be a basic quasi-p-symmetric arc, not contained in a single link, such that x and y are the midpoints of the link-tips of A and $L_p(x) > L_p(y)$. Let m be the midpoint of A. Then there exists a point a such that $[m, a]$ is a quasi-p-symmetric arc with y as the midpoint.

Remark 4.13. The assumption that $[x, y]$ is basic is essential. Without it, we would have a counter-example in $[x, y] = [x^3, x^{30}]$ in Figure 5. The quasi-p-symmetric arc $[x^3, x^{30}]$ is indeed not basic, because $[x^3, x^6]$ is a shorter quasi-p-symmetric arc in the figure. There is a point $n = x^{32}$ beyond y with $L_p(n) = L_p(x^{32}) = 3 > 1 = L_p(y)$, making it impossible that y is the midpoint of a quasi-p-symmetric arc stretching unto m.

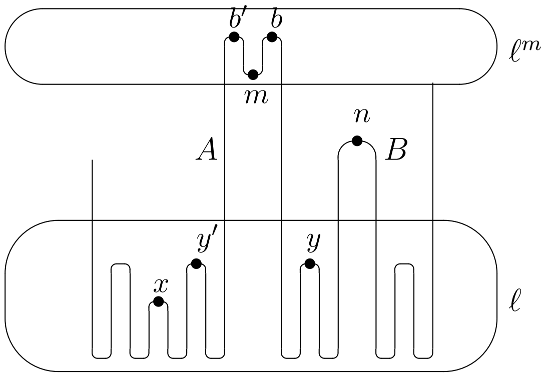

FIGURE 9. The arcs A and B and the relevant points for Proposition 4.12, which is meant to show that the point n does not exist in B.

Proof. A quasi-p-symmetric arc is not contained in a single link, so $[x, m] \not\subset \ell$. Let $H = [x, n] \supset A$ be the smallest arc containing a point n 'beyond' y with $L_p(n) > L_p(y)$.

Corollary 4.8 implies that the arc $[x, m]$ contains a p-point y' with $L_p(y') = L_p(y)$. Let b and b' be the p-points having the highest p-level on the arcs $[y, m)$ and $[y', m)$ respectively. By symmetry, $L_p(b) = L_p(b')$, and possibly $b = y$, $b' = y'$. Let $z \in [x, y']$ be the point closest to y' such that $L_p(z) > L_p(b)$; possibly $z = x$. Since $b' \in [y', m)$, we have

$$L_p(y) = L_p(y') \leqslant L_p(b) = L_p(b') < L_p(m).$$

Take $L := L_p(b)$ and let $\tilde{H} = \pi_p \circ \sigma^{-L}(H)$. Since y is the midpoint of its link-tip, $[y, n] \not\subset \ell$. Therefore $\pi_p^{-1}(c) \cap \sigma^{-L}(H) \supset \{\sigma^{-L}(b), \sigma^{-L}(b')\}$, and $\tilde{z} = \pi_p \circ \sigma^{-L}(z)$ and $\tilde{n} = \pi_p \circ \sigma^{-L}(n)$ have $\tilde{m} = \pi_p \circ \sigma^{-L}(m)$ as common β-neighbor, see Figure 10. Since $L_p(z) > L_p(b)$ there is k such that $\tilde{z} = c_{S_k}$. Also take l such that $\tilde{n} = c_{S_l}$ and j such that $\tilde{m} = c_{S_j}$. Let $\tilde{y} = \pi_p \circ \sigma^{-L}(y)$ and $\tilde{y}' = \pi_p \circ \sigma^{-L}(y')$.

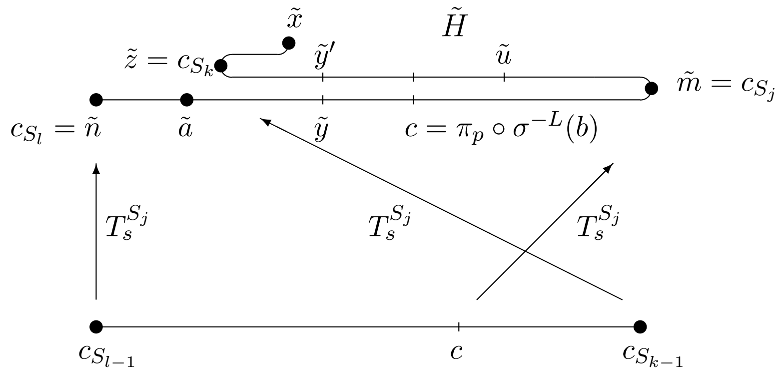

FIGURE 10. The arc $\tilde{H}$ drawn as multiple curve, its preimage under $T_s^{S_j}$ and the relevant points on them.

We claim that there is a point $a \in [n, m]$ such that

$$\tilde{a} := \pi_p \circ \sigma^{-L}(a) \in [c_{S_l}, \tilde{y}] \quad \text{and} \quad T_s(\tilde{a}) = T_s(\tilde{m}).$$

Since c_{S_j} is β-neighbor to both c_{S_l} and c_{S_k}, we have three cases:

(1) $j = Q(k)$ and $l = Q(j)$, so $l = Q^2(k)$. In this case, Equation (2.2) and Remark 2.1 imply that $|c - c_{S_l}| > |c - c_{S_{Q(k)}}|$, so $[c_{S_l}, c]$ contains the required point $\tilde{a}$ with $T_s(\tilde{a}) = T_s(\tilde{m})$. By the same token, $|c_{S_k} - c| < |c_{S_j} - c| = \frac{1}{2}|\tilde{a} - \tilde{m}|$. Since $|\tilde{y} - c| = |\tilde{y}' - c| < |c_{S_k} - c|$, we indeed obtain that $\tilde{a} \in [c_{S_l}, \tilde{y}]$.

(2) $j = Q(l)$ and $k = Q(j)$, so $k = Q^2(l)$. Then Remark 2.3 implies that $|c - c_{S_k}| > |c - c_{S_l}|$. But this would mean that the arc $[n, m]$ is shorter than $[z, m]$ and in particular that $[y, n] \subset \ell$, contradicting that y is the midpoint of its link-tip.

(3) $j = Q(k) = Q(l)$. In this case, we pull $\tilde{H}$ back for another S_j iterates, or more precisely, we look at the arc $\pi_p \circ \sigma^{-S_j - L}(H)$. The endpoints of this arc are $c_{S_{k-1}}$ and $c_{S_{l-1}}$ which are therefore β-neighbors. If $l - 1 = Q(k - 1)$, then we find

$$Q(k) = Q(l) = Q(Q(k-1) + 1)$$

which contradicts Condition (2.2) with k replaced by $k - 1$. If $k - 1 = Q(l - 1)$, then we find

$$Q(l) = Q(k) = Q(Q(l-1) + 1)$$

which contradicts Condition (2.2) with k replaced by $l - 1$.

This proves the claim.

Suppose now that $\tilde{y} \neq c$ (*i.e.*, $y \neq b$). Then $b, b' \notin \ell$ because y has the largest p-level in its link-tip. Since $|c_{S_k} - c| < |c - \tilde{m}|$, there is a point $u \in [z, m]$ such that $\tilde{u} = \pi_p \circ \sigma^{-L}(u) \in [c, \tilde{m}]$ and $T_s(\tilde{u}) = T_s(\tilde{y})$. This means that $[x, u]$ is a quasi-p-symmetric arc properly contained in $[x, m]$, contradicting the assumption that $[x, y]$ is a basic quasi-symmetric arc.

Therefore $y = b$, so there are no p-points between y and m of level higher than $L_p(y)$. Instead, the arc $[a, m]$ has midpoint y, and is the required quasi-p-symmetric arc, proving the lemma. ☐

Remark 4.14. Let $A = [x, y]$ be a basic quasi-p-symmetric arc such that x and y are the midpoints of the link-tips of A and $L_p(x) > L_p(y)$. Let ℓ^m be the link which contains the midpoint m of A, and let A_m be the arc component of ℓ^m containing m. Then, by the lemma above, $A \setminus (\ell\text{-tips } \cup A_m)$ does not contain any p-point z with $L_p(z) \geqslant L_p(y)$.

5. Concatenation of Quasi-p-symmetric Arcs

Definition 5.1. We say that the arc $[x, y]$ is *decreasingly (basic) quasi-p-symmetric* if it is the concatenation of (basic) quasi-p-symmetric arcs where the p-levels of the midpoints decrease. To be precise, if there are p-points $x = x^0, x^1, x^2, \dots, x^{n-1}$ and $x^n = y$ can be a p-point or not, such that the following hold:

(i) $[x^{i-1}, x^{i+1}]$ is a (basic) quasi-p-symmetric arc with midpoint x^i, for $i = 1, \dots, n-1$. (By definition of a (basic) quasi-p-symmetric arc, the points x^{2i} all belong to a single link, and the points x^{2i-1} belong to a single link as well.)

(ii) $L_p(x^i) > L_p(x^{i+1})$, for $i = 1, \dots, n-1$ (and if y is a p-point then also $L_p(x^{n-1}) > L_p(y)$).

Similarly, we say that the arc $[x, y]$ is *increasingly (basic) quasi-p-symmetric* if it is the concatenation of (basic) quasi-p-symmetric arcs where the p-levels of the midpoints increase.

Example 5.2. Consider the Fibonacci inverse limit space, and let our chain $\mathcal{C}_p$ be such that p-points with p-levels 1 and 14 belong to the same link ℓ, but p-points with the p-level 9 are not contained in ℓ. Since p-points with p-level 14 belong to the same link ℓ as p-points with p-level 1, also the p-points with p-levels 22, 35, 56 and 77 belong to ℓ. Let p-points with p-level 43 belong to the same link as p-points with p-level 9.

(1) **Example of a basic decreasingly quasi-p-symmetric arc.** Let $A = [y^0, y^{12}]$ be an arc with the following folding pattern:

$$\underbrace{\mathbf{1}\ \mathbf{22}\ \mathbf{77}\ \mathbf{22}\ \mathbf{1}\ 9\ 43\ 9\ \mathbf{1}}_{\text{basic}}\overbrace{\mathbf{22}\ \mathbf{1}\ 9\ \mathbf{1}}^{\text{basic}}$$

Let x^i be as in the above definition. Then $x^1 = y^2$, $x^2 = y^6$, $x^3 = y^9$, $x^4 = y^{11}$, and $x^5 = y^{12}$. So $[y^2, y^9]$ is basic quasi-p-symmetric with midpoint y^6, $[y^6, y^{11}]$ is basic quasi-p-symmetric with midpoint y^9, and $[y^9, y^{12}]$ is basic quasi-p-symmetric with midpoint y^{11}. Also $L_p(y^2) = 77 > L_p(y^6) = 43 > L_p(y^9) = 22 > L_p(y^{11}) = 9 > L_p(y^{12}) = 1$.

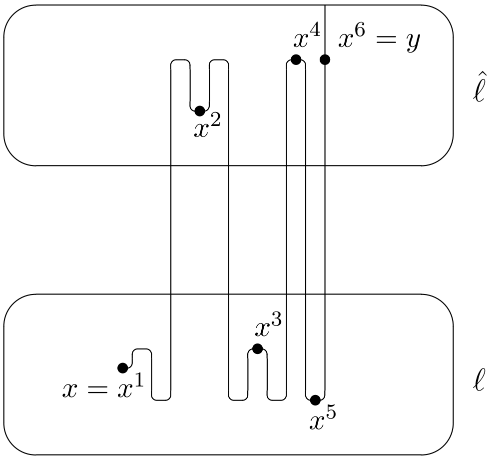

FIGURE 11. Illustration of a basic decreasingly quasi-p-symmetric arc. The point y is not a p-point here; instead, the arc A goes straight through $\hat{\ell}$ at y.

(2) **Example of a non-basic decreasingly quasi-p-symmetric arc.** Let $[y^0, y^{72}]$ be an arc with the following folding pattern:

$$\overbrace{\underbrace{\mathbf{1}\ \mathbf{22}\ \mathbf{1}\ \mathbf{56}\ \mathbf{1}\ \mathbf{22}\ \mathbf{1}\ 9\ \mathbf{1}}_{\text{basic}}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \mathbf{1}\ 6\ \mathbf{1}\ \mathbf{14}\ \mathbf{1}\ \mathbf{35}\ \mathbf{1}\ \mathbf{14}\ \mathbf{1}\ 6\ \mathbf{1}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \underbrace{\mathbf{1}\ 9\ \mathbf{1}}_{\text{basic}}}^{\text{quasi-}p\text{-symmetric}}$$

$$\underbrace{\overbrace{\mathbf{22}\ \mathbf{1}\ 9\ \mathbf{1}}^{\text{basic}}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \mathbf{1}\ 6\ \mathbf{1}\ \mathbf{14}\ \mathbf{1}\ 6\ \mathbf{1}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \overbrace{\mathbf{1}\ 9\ \mathbf{1}}^{\text{sym}}}_{\text{quasi-}p\text{-symmetric}}$$

Let x^i be again as in the above definition. Then $x^1 = y^3$, $x^2 = y^{23}$, $x^3 = y^{41}$, $x^4 = y^{57}$, and $x^5 = y^{72}$. So, arcs $[y^3, y^{41}]$, $[y^{23}, y^{57}]$ and $[y^{41}, y^{72}]$ are quasi-p-symmetric, and $L_p(y^3) = 56 > L_p(y^{23}) = 35 > L_p(y^{41}) = 22 > L_p(y^{57}) = 14 > L_p(y^{72}) = 1$.

(3) **Example of an arc that is the concatenation of two quasi-p-symmetric arcs (one of them is basic), but not decreasingly quasi-p-symmetric.** Let $[y^0, y^{40}]$ be an arc with the following folding pattern:

$$\underbrace{\mathbf{1}\ \mathbf{22}\ \mathbf{77}\ \mathbf{22}\ \mathbf{1}\ 9\ 43\ 9\ \mathbf{1}}_{\text{basic}}\underbrace{\mathbf{22}\ \mathbf{1}\ 9\ \mathbf{1}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \mathbf{1}\ 6\ \mathbf{1}\ \mathbf{14}\ \mathbf{1}\ 6\ \mathbf{1}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \mathbf{1}\ 9\ \mathbf{1}}_{\text{quasi-}p\text{-symmetric}}$$

Then $[y^2, y^9]$ is basic quasi-p-symmetric with midpoint y^6, $[y^6, y^{11}]$ is basic quasi-p-symmetric with midpoint y^9, and $[y^9, y^{12}]$ is basic quasi-p-symmetric with midpoint y^{11}. However, $[y^9, y^{40}]$ is quasi-p-symmetric with midpoint y^{25} and $[y^6, y^{25}]$ is neither basic quasi-p-symmetric, nor quasi-p-symmetric. So $A = [y^0, y^{40}]$ is not a decreasingly quasi-p-symmetric arc. Note that $[y^0, y^{12}]$ is a decreasingly quasi-p-symmetric arc.

Definition 5.3. Let $\ell_0, \ell_1, \dots, \ell_k$ be the links in $\mathcal{C}_p$ that are successively visited by an arc $A \subset \mathfrak{C}$, and let $A_i \subset \mathrm{Cl}(\ell_i)$ be the corresponding maximal subarcs of A. (Hence $\ell_i \neq \ell_{i+1}$,

$\ell_i \cap \ell_{i+1} \neq \emptyset$ but $\ell_i = \ell_{i+2}$ is possible if A turns in ℓ_{i+1}.) We call A *p-link-symmetric* if $\ell_i = \ell_{k-i}$ for $i = 0, \dots, k$. In this case, we say that A_i is p-link-symmetric to A_{k-i}.

Remark 5.4. Every p-symmetric and quasi-p-symmetric arc is p-link-symmetric by definition, but there are p-link-symmetric arcs which are not p-symmetric or quasi-p-symmetric. This occurs if A turns both at A_i and A_{k-i}, but the midpoint of A_i has a higher p-level than the midpoint of A_{k-i} and $i \notin \{0, k\}$. Note that for a p-link-symmetric arc A, if U and V are p-link-symmetric arc components which do not contain any boundary point of A, then U contains at least one p-point if and only if V contains at least one p-point.

Proposition 5.5. Let A be a non-basic quasi-p-symmetric arc. Then there are $k, n, m, d \in \mathbb{N}$, $d < k$, such that

$$A \cap E_p = \{x^0, \dots, x^k, \dots, x^{k+n}, \dots, x^{k+n+m}\},$$

$[x^0, x^k]$ is a basic quasi-p-symmetric arc with midpoint x^{k-d} and $[x^k, x^{k+n}]$ is p-symmetric. Moreover,

(i) If $[x^{k+n}, x^{k+n+m}]$ is p-symmetric, then $[x^{-k+m/2}, x^{k+n+3m/2}]$ is not p-link-symmetric.
(ii) If $[x^{k+n}, x^{k+n+m}]$ is a basic quasi-p-symmetric arc, then A is contained in a decreasingly quasi-p-symmetric arc consisting of at least two quasi-p-symmetric arcs. More precisely, $[x^{-k-n/2}, x^{k+n/2}]$ and $[x^{k+n/2}, x^{k+2m+3n/2}]$ are the quasi-p-symmetric arcs contained in the decreasingly quasi-p-symmetric arc $[x^{-k-n/2}, x^{k+2m+3n/2}]$ containing A.

Proof. Since A is a non-basic quasi-p-symmetric arc, there is a basic quasi-p-symmetric arc which we can label $[x^0, x^k]$. The arc $[x^k, x^{k+n}]$ in the middle is p-symmetric by definition of quasi-p-symmetry, and it has the same midpoint $x^{k+n/2}$ as A. The arc $[x^{k+n}, x^{k+n+m}]$ could be either p-symmetric or basic quasi-p-symmetric.

(i) Assume that $[x^{k+n}, x^{k+n+m}]$ is p-symmetric. Without loss of generality we can suppose that x^0 and x^{k+n+m} are the midpoints of the link-tips of A, and also that x^k and x^{k+n} are the midpoints of their arc components. Since the point $x^{k+n+m/2}$ is the midpoint of the p-symmetric arc $[x^{k+n}, x^{k+n+m}]$, and the symmetry of the arc $[x^k, x^{k+n}]$ can be extended to the midpoints of its neighboring (quasi-)symmetric arcs, we obtain that $d = m/2$ and the point $x^{k-m/2}$ is the midpoint of the basic quasi-p-symmetric arc $[x^0, x^k]$. Proposition 4.10 implies that we can extend $[x^0, x^{k-m/2}]$ beyond x^0 to obtain the arc $[x^{-k+m/2}, x^{k-m/2}]$ which is either p-symmetric, or quasi-p-symmetric, and hence p-link-symmetric.

First, let us assume that $L_p(x^{k+n+m}) = 1$. Let us consider the arc $[x^{k+n+m/2}, x^{k+n+3m/2}]$. Its midpoint x^{k+n+m} has p-level 1. If $L_p(x^{k+n+m-1}) = L_p(x^{k+n+m+1})$, then $L_p(x^{k+n+m-1}) = 0$. Furthermore $x^{k+n+m-1} \neq x^{k+n+m/2}$ since a midpoint cannot have p-level zero. It follows that $x^{k+n+m-2}$ and $x^{k+n+m+2}$ have different p-levels, and are not in the same link, since by Lemma 4.6 there is no quasi-p-symmetric arc whose both boundary points are p-points and whose midpoint has p-level 1.

If $L_p(x^{k+n+m-1}) \neq L_p(x^{k+n+m+1})$ then again $x^{k+n+m-1}$ and $x^{k+n+m+1}$ are not in the same link (by Lemma 4.6 there is no quasi-p-symmetric arc whose both boundary points are

p-points and whose midpoint has p-level 1). In either case, $[x^{k+n+m/2}, x^{k+n+3m/2}]$ is not p-link-symmetric and hence $[x^{-m/2}, x^{k+n+3m/2}]$ is not p-link-symmetric. This proves statement (i) in the case that $L_p(x^{k+n+m}) = 1$.

Now for the general case, let $L := L_p(x^{k+n+m})$. The basic idea is to shift $[x^0, x^{k+n+m}]$ back by L iterates, and use the above argument. Note that the arcs $[x^k, x^{k+n}]$ and $[x^{k+n}, x^{k+n+m}]$ are p-symmetric and hence $L_p(x^{k+n/2}) > L_p(x^{k+n}) = L_p(x^{k+n+m}) = L$. Then $\sigma^{-L+1}(A)$ is also a quasi-p-symmetric arc which is not basic, the arc $\sigma^{-L+1}([x^0, x^k])$ is a basic quasi-p-symmetric arc and $L_p(\sigma^{-L+1}(x^{k+n+m})) = 1$. Let

$$\sigma^{-L+1}(A) \cap E_p = \{u^0, \ldots, u^{\hat{k}}, \ldots, u^{\hat{k}+\hat{n}}, \ldots, u^{\hat{k}+\hat{n}+\hat{m}}\},$$

where $u^{\hat{i}} = \sigma^{-L+1}(x^i)$. (Note that $\hat{k} \leqslant k$, $\hat{n} \leqslant n$ and $\hat{m} \leqslant m$, since note every $\sigma^{-L+1}(x^i)$ needs to be a p-point.) Then $G = [u^{-\hat{k}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}]$ is an arc with 'boundary arcs' $[u^{-\hat{k}+\hat{m}/2}, u^{\hat{k}-\hat{m}/2}]$ and $[u^{k+n+\hat{m}/2}, u^{k+n+3\hat{m}/2}]$ and the midpoint of the latter has p-level 1. The above argument shows that this arc cannot be p-link-symmetric, and therefore the whole arc G is not p-link-symmetric with midpoint $u = \sigma^{-L+1}(x^{k+n/2})$.

We want to prove that $\sigma^j(G)$ is also not p-link-symmetric with the midpoint $\sigma^j(u)$ for $j = L - 1$.

Let us assume by contradiction that $\sigma^j(G)$ is p-link-symmetric. Since $[x^{-k+m/2}, x^{k-m/2}]$ is p-symmetric, also $\sigma^j([u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}])$ is p-link-symmetric. But $[u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}]$ has its midpoint at p-level 1, and hence is not p-link-symmetric. Therefore, there exists $l < j$ such that $\sigma^l([u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}])$ is not p-link-symmetric and $\sigma^{l+1}([u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}])$ is p-link-symmetric. By Proposition 3.3, and since $L_p(\sigma^l(u^{\hat{k}+\hat{n}+\hat{m}})) = l+1 \neq 0$, there exist $v \in \sigma^l([u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+\hat{m}}])$ and $w \in \sigma^l([u^{\hat{k}+\hat{n}+\hat{m}}, u^{\hat{k}+\hat{n}+3\hat{m}/2}])$ such that $L_p(v) = L_p(w) = 0$.

Since $\sigma^{l+1}(u^{\hat{k}+\hat{n}+\hat{m}/2})$ and $\sigma^{l+1}(u^{\hat{k}+\hat{n}+3\hat{m}/2})$ belong to the same link and $L_p(\sigma^{l+1}(u^{\hat{k}+\hat{n}+\hat{m}/2})) \neq L_p(\sigma^{l+1}(u^{\hat{k}+\hat{n}+3\hat{m}/2}))$, Proposition 3.3 implies that $\sigma^{l+1}(u^{\hat{k}+\hat{n}+\hat{m}/2})$ and $\sigma^{l+1}(u^{\hat{k}+\hat{n}+3\hat{m}/2})$ belong to the same link as $\sigma(v)$ and $\sigma(w)$. But then $\sigma^l(u^{\hat{k}+\hat{n}+\hat{m}/2})$ and $\sigma^l(u^{\hat{k}+\hat{n}+3\hat{m}/2})$ belong to the same link as v and w, contradicting the choice of l.

(ii) The rough idea of this proof is as follows: Whenever $[x^{k+n}, x^{k+n+m}]$ is not p-symmetric, there exists $N \in \mathbb{N}$ such that $\sigma^{-N}(A)$ is a basic quasi-p-symmetric arc and we can apply Propositions 4.10 and 4.12 to obtain the arc $B \supset \sigma^{-N}(A)$ which is decreasingly basic quasi-p-symmetric. Then $\sigma^N(B) \supset A$ is the required decreasingly quasi-p-symmetric arc.

Let us assume now that $[x^{k+n}, x^{k+n+m}]$ is basic quasi-p-symmetric. Let us denote by ℓ the link which contains x^0. Then $x^k, x^{k+n}, x^{k+n+m} \in \ell$. We can assume without loss of generality that x^k and x^{k+n} are the p-points in the link-tips of $[x^k, x^{k+n}]$ furthest away from the midpoint $x^{k+n/2}$ and, similarly, x^0 and x^{k+n+m} are the p-points in the link-tips of $[x^0, x^{k+n+m}]$ furthest away from the midpoint $x^{k+n/2}$. Then from the properties of the chain in Proposition 3.3 we conclude that $L_p(x^0) = L_p(x^k) = L_p(x^{k+n}) = L_p(x^{k+n+m})$. Let us denote by x^a and x^b the midpoints of arc components which contains x^0 and x^{k+n+m}

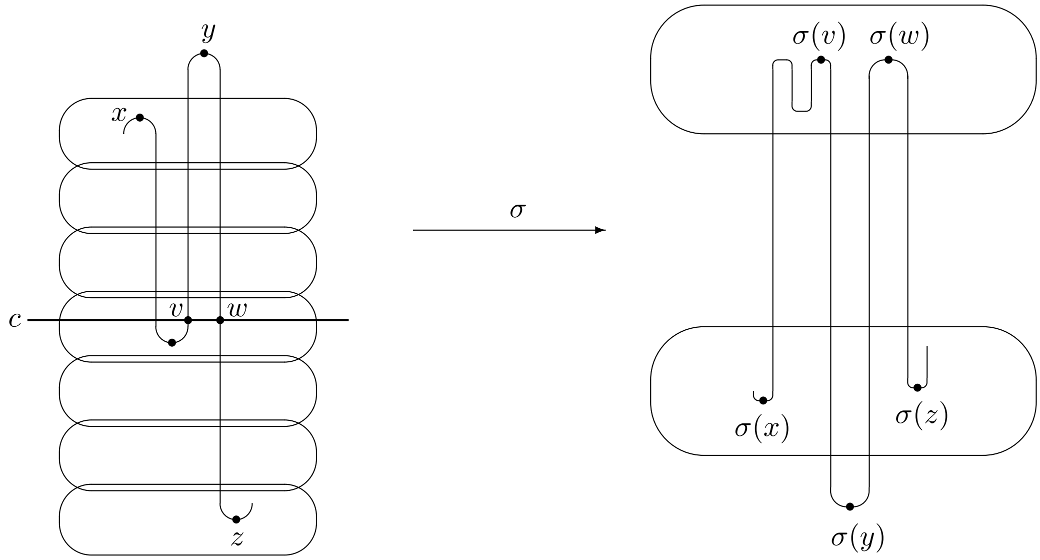

FIGURE 12. The configuration of p-levels that does not exist. Here $x = \sigma^l(u^{\hat{k}+\hat{n}+\hat{m}/2})$, $y = \sigma^l(u^{\hat{k}+\hat{n}+\hat{m}})$ and $z = \sigma^l(u^{\hat{k}+\hat{n}+3\hat{m}/2})$.

respectively. Then $x^a, x^b \in \ell$ and $x^b \neq x^{k+n+m}$. Without loss of generality we can assume that $L_p(x^a) > L_p(x^b)$.

Since x^{k-d} is the midpoint of $[x^0, x^k]$ and A is quasi-p-symmetric, x^{k+n+d} is the midpoint of $[x^{k+n}, x^{k+n+m}]$.

By Proposition 4.10, $L_p(x^{-d}) = L_p(x^{k-d})$ and $L_p(x^{k+n+d}) = L_p(x^{k+n+m+d})$, see Figure 13.

Let us denote by ℓ^d the link which contains x^{-d}, and by A_d the arc component of ℓ^d which contains x^{-d}.

Claim x^{-d} is the midpoint of its arc component A_d.

Consider the arc $\sigma^{-L+1}(A)$, where $L := L_p(x^b)$. Since $L_p(x^a) > L_p(x^{k+n/2}) > L_p(x^b) = L$, the preimage $\sigma^{-L+1}(A)$ contains the points $\sigma^{-L+1}(x^b)$ with $L_p(\sigma^{-L+1}(x^b)) = 1$, $\sigma^{-L+1}(x^a)$ and $\sigma^{-L+1}(x^{k+n/2})$ is the midpoint of $\sigma^{-L+1}(A)$.

By Corollary 4.8 the arc component containing x^a also contains p-points x' and x'' with the property that $[x', x'']$ is p-symmetric with midpoint x^a and $L_p(x') = L_p(x'') = L_p(x^b)$, Assume also that x' and x'' are furthest away from x^a with these properties. Therefore, $\sigma^{-L+1}(A) \cap E_p \supseteq \{u^0, u^{\hat{a}}, u^{2\hat{a}}, u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+2\hat{n}}\}$, where $u^{\hat{a}} = \sigma^{-L+1}(x^a)$, $u^{2\hat{a}+\hat{n}} = \sigma^{-L+1}(x^{k+n/2})$, $u^{2\hat{a}+2\hat{n}} = \sigma^{-L+1}(x^b)$, $u^0 = \sigma^{-L+1}(x')$, $u^{2\hat{a}} = \sigma^{-L+1}(x'')$ and $L_p(u^0) = L_p(u^{2\hat{a}}) = 1$.

Let us suppose that $\sigma^{-L+1}(A)$ is not contained in a single link. Since $\sigma^{-L+1}(x^a)$ and $\sigma^{-L+1}(x^b)$ are contained in the same link, $\sigma^{-L+1}(A)$ is a basic quasi-p-symmetric arc. Let ℓ^n be the link containing $u^{2\hat{a}+\hat{n}}$, and let A_{2a+n} be the arc component of ℓ^n containing

$u^{2\hat{a}+\hat{n}}$. Since $L_p(u^{2\hat{a}+2\hat{n}}) = 1$, by Remark 4.14, $(u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+2\hat{n}}) \setminus A_{2a+n}$ can contain at most one p-point and its p-level is 0. Therefore $(u^{2\hat{a}}, u^{2\hat{a}+\hat{n}}) \setminus A_{2a+n}$ can also contain at most one p-point and its p-level is 0. By Proposition 4.10, $[u^{-\hat{n}}, u^{2\hat{a}+\hat{n}}]$ is either a p-symmetric arc, or a basic quasi-p-symmetric arc, see Figure 13. Let us denote by A_n the arc component of ℓ^n containing $u^{-\hat{n}}$. Then $(u^{-\hat{n}}, u^0) \setminus A_n$ also does not contain any p-point with non-zero p-level.

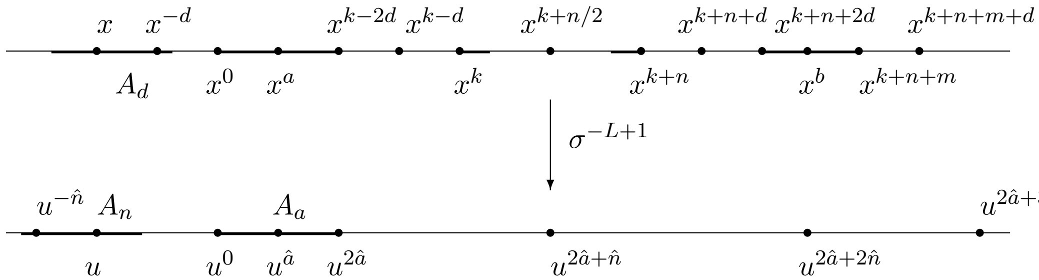

FIGURE 13. The configuration of points on $[x^{-d}, x^{k+n+m+2d}]$ and their images under σ^{-L+1} as in (ii).

Assume by contradiction that x^{-d} is not the midpoint of its arc component A_d. Let us denote the midpoint of A_d by x, and let $u := \sigma^{-L+1}(x)$. Since $L_p(x) > L_p(x^a)$, also $L_p(u) > L_p(u^{\hat{a}})$. Let ℓ^a be the link which contains $u^{\hat{a}}$, and let A_a be the arc component of ℓ^a containing $u^{\hat{a}}$. Then $u \in A_n$ and $[u^{-\hat{n}}, u^{2\hat{a}+\hat{n}}]$ is basic quasi-p-symmetric. But, since $u^{2\hat{a}+\hat{n}} \in \ell^n$ and $\sigma^{L-1}(u^{2\hat{a}+\hat{n}}) = x^{k+n/2}$, $x^{k+n/2} \in \ell^d$. Since the arc $[x, x^{k-d}]$ is quasi-p-symmetric, $[x^{k-d}, x^{k+n/2}]$ is also quasi-p-symmetric and $L_p(x^a) > L_p(x^{k-d})$ implies $L_p(x^{k-d}) > L_p(x^{k+n/2})$, a contradiction.

Let us assume now that $\sigma^{-L+1}(A)$ is contained in a single link. Since $L_p(u) > L_p(u^{\hat{a}})$ and $L_p(u^0) = 1$, we have $\pi_p([u, u^0]) \subset \pi_p([u^{\hat{a}}, u^0])$. Then $\sigma^{L-1}([u^{\hat{a}}, u^0]) \subset \ell$ implies $\sigma^{L-1}([u, u^{\hat{a}}]) \subset \ell$ and hence $[x^{-d}, x^{k-d}] \subset \ell$, a contradiction.

These two contradiction prove the claim.

In the same way we can prove that $x^{k+n+m+d}$ is the midpoint of its arc component, and by Proposition 4.12 the arc $[u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+3\hat{n}}]$ is either p-symmetric, or quasi-p-symmetric.

So we have proved that the arcs $[u^{-\hat{n}}, u^{2\hat{a}+\hat{n}}]$ and $[u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+3\hat{n}}]$ are both either p-symmetric, or quasi-p-symmetric. Since $[x^a, x^b] = \sigma^{L-1}([u^{\hat{a}}, u^{2\hat{a}+2\hat{n}}])$ is quasi-p-symmetric, the arcs $\sigma^{L-1}([u^{-\hat{n}}, u^{2\hat{a}+\hat{n}}])$ and $\sigma^{L-1}([u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+3\hat{n}}])$ are both either p-symmetric, or quasi-p-symmetric. This implies that $[x^{-2d-n/2}, x^{k+n/2}]$ and $[x^{k+n/2}, x^{k+n+m+2d+n/2}]$ are contained in the decreasingly quasi-p-symmetric arc $[x^{-2d-n/2}, x^{k+n+m+2d+n/2}]$ containing A. □

Example 5.6. (Example for (ii) of Proposition 5.5.) Let us consider the Fibonacci map and the corresponding inverse limit space. The composant $\mathfrak{C}$ contains an arc $A = [x^0, x^{77}]$

with the following folding pattern:

$$1\ 9\ \underbrace{\mathbf{1\ 22\ 1\ 56\ 1\ 22\ 1}\ 9\ \mathbf{1}}_{\text{basic}}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ 1\ 6$$

$$\overbrace{\underbrace{\mathbf{1\ 14\ 1\ 35\ 1\ 14\ 1}\ 6\ \mathbf{1}}_{\text{basic}}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \mathbf{1}\ 9\ \mathbf{1\ 22\ 1}\ 9\ \mathbf{1}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \underbrace{\mathbf{1}\ 6\ \mathbf{1\ 14\ 1}}_{\text{basic}}}^{\text{quasi-}p\text{-symmetric}}$$

$$6\ 1\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \underbrace{\mathbf{1}\ 9\ \mathbf{1}}_{\text{sym}}\ 4\ 1\ 0$$

We can choose a chain $\mathcal{C}_p$ such that the p-points with p-levels 1, 14, 22, 35 and 56 belong to the same link. Then the arc $[x^{22}, x^{60}]$ is quasi-p-symmetric, and it is not basic. The arc $\sigma^{-13}([x^{22}, x^{60}])$ is basic quasi-p-symmetric with the folding pattern 1 22 1 9 1. So we can apply Propositions 4.10 and 4.12 as in the above proof. The arc $[x^2, x^{74}]$ is decreasingly quasi-p-symmetric. Note that the arc $[x^1, x^{75}]$ is not p-link-symmetric.

Definition 5.7. We say that an arc $A = [x, y]$ is *maximal decreasingly (basic) quasi-p-symmetric* if it is decreasingly (basic) quasi-p-symmetric and there is no decreasingly (basic) quasi-p-symmetric arc $B \supset A$ that consists of more (basic) quasi-p-symmetric arcs than A.

Similarly we define a *maximal increasingly (basic) quasi-p-symmetric* arc.

Remark 5.8. Propositions 4.10 and 4.12 imply that $A = [x, y]$ is a maximal decreasingly basic quasi-p-symmetric arc if and only if A is decreasingly basic quasi-p-symmetric and for $x = x_1, x_2, \dots, x_{n-1}, x_n = y$ which satisfy (i) of Definition 5.3, there exists a point z such that $[z, x_2]$ is p-symmetric with midpoint x and y is not a p-point.

Lemma 5.9. Every (basic) quasi-p-symmetric arc A can be extended to a maximal decreasingly/increasingly (basic) quasi-p-symmetric arc $B \supset A$.

Proof. We take the largest decreasingly (basic) quasi-p-symmetric arc B containing A. The only thing to prove is that there really is a largest B. If this were not the case, then there would be an infinite sequence $(x_i)_{i\geqslant 0}$ with $x_0 \in \partial A$, $L_p(x_i) < L_p(x_{i+1})$ and $[x_i, x_{i+2}]$ is a (basic) quasi-p-symmetric arc for all $i \geqslant 0$. By definition of (basic) quasi-p-symmetric arc, there are two links ℓ and $\hat{\ell}$ containing x_i for all even i and odd i respectively. (Note that $\ell = \hat{\ell}$ is possible.) By Lemma 4.6 for the basic case, the p-points in $\bigcup_{i\geqslant 0}[x_0, x_i] \setminus (\ell \cup \hat{\ell})$ can only have finitely many different p-levels. By the construction in the proof of Proposition 5.5 *(ii)* the same conclusion is true for the non-basic case as well. But $\bigcup_{i\geqslant 0}[x_0, x_i]$ is a ray, and contains p-points of all (sufficiently high) p-levels. Since the closure of $\pi_p(\{x : L_p(x) \geqslant N\})$ contains $\omega(c)$ for all N, this set is not contained in the π_p-images of the two links ℓ and $\hat{\ell}$ only. So we have a contradiction. $\square$

Theorem 5.10. Let A be a p-link-symmetric arc with midpoint m and $\partial A = \{x, y\} \subset E_p$. Then A is either p-symmetric, or is contained in a maximal decreasingly/increasingly (basic) quasi-p-symmetric arc.

Proof. Let $A \cap E_p = \{x^{-l}, \dots, x^{-1}, x^0, x^1, \dots x^k\}$, where k and l are such that $x^0 = m$. If $k = l$ and $L_p(x^{-i}) = L_p(x^i)$, for $i = 1, \dots, k-1$, then the arc A is either p-symmetric, or (basic) quasi-p-symmetric. Hence in this case the theorem is true.

Let us assume that there exists $j < \min\{k, l\}$ such that $L_p(x^{-i}) = L_p(x^i)$, for $i = 1, \dots, j-1$, and $L_p(x^{-j}) \neq L_p(x^j)$. The arc $[x^{-j}, x^j]$ is (basic) quasi-p-symmetric and by Lemma 5.9, there exists the maximal decreasingly/increasingly (basic) quasi-p-symmetric arc which contains $[x^{-j}, x^j]$. Hence in this case the theorem is also true. □

Definition 5.11. Let $(s_i)_{i\in\mathbb{N}}$ be a sequence of p-points such that $0 \le L_p(x) < L_p(s_i)$ for every p-point $x \in (\bar{0}, s_i)$. We call p-points satisfying this property *snappy*.

Since for every slope $s > 1$ and $p \in \mathbb{N}_0$, the folding pattern of $\mathfrak{C}$ starts as ∞ 0 1 0 2 0 1 $\dots$, and since by definition $L_p(s_1) > 0$, we have $L_p(s_1) = 1$. Also, since $s_i = \sigma^{i-1}(s_1)$, $L_p(s_i) = i$, for every $i \in \mathbb{N}$. Note that the snappy p-points depend on p: if $p \ge q$, then the snappy p-point s_i equals the snappy q-point s_{i+p-q}.

For $i \in \mathbb{N}$, let A_i be the maximal p-link-symmetric arc with midpoint s_i.

Corollary 5.12. Fix $i \in \mathbb{N}$ and let ℓ^i and ℓ^{i-1} be the links of $\mathcal{C}_p$ containing s_i and s_{i-1} respectively. Let $y \in [s_{i-1}, s_i]$ be neither contained in the same arc-component of ℓ^i as s_i, nor in the same arc-component of ℓ^{i-1} as s_{i-1}. Then the maximal p-link-symmetric arc J with midpoint y contains at most one snappy p-point and $J \subset A_i$.

This was proved in more generality in [3] but the proof here is easier.

Proof. Let us suppose that J contains s_i. Then J is a maximal increasingly (basic) quasi-p-symmetric arc, and s_i is a boundary point of one of the (basic) quasi-p-symmetric arcs contained in J. Therefore, all p-points in $[\bar{0}, y) \cap J$ have p-levels less than $L_p(y)$ implying $s_{i-1} \notin J$. Also, $J \subset (s_{i-1}, x) \subset A_i$, where $x \in (s_i, s_{i+1})$ is a unique p-point with p-level $i-1$.

If J contains s_{i-1}, J is a decreasingly (basic) quasi-p-symmetric arc. Since $L_p(x) < L_p(s_{i-1})$ for every $x \in (\bar{0}, s_{i-1})$, $s_{i-2} \notin J$. Since all p-points in $J \setminus [\bar{0}, y]$ have p-levels less then $L_p(y)$, $s_{i+1} \notin J$. So, $J \subset (s_{i-2}, s_{i+1}) \subset A_i$. □

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