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Spectral Sequences in Combinatorial Geometry: Cheeses, Inscribed Sets, and Borsuk-Ulam Type Theorems

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Spectral sequences in combinatorial geometry: Cheeses, Inscribed sets, and Borsuk–Ulam type theorems

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Abstract

Algebraic topological methods are especially suited to determining the nonexistence of continuous mappings satisfying certain properties. In combinatorial problems it is sometimes possible to define a mapping from a space X of configurations to a Euclidean space $\mathbb{R}^m$ in which a subspace, a discriminant, often an arrangement of linear subspaces $\mathcal{A}$, expresses a desirable condition on the configurations. Add symmetries of all these data under a group G for which the mapping is equivariant. Removing the discriminant leads to the problem of the existence of an equivariant mapping from X to $\mathbb{R}^m-$ the discriminant. Algebraic topology may be applied to show that no such mapping exists, and hence the original equivariant mapping must meet the discriminant.

We introduce a general framework, based on a comparison of Leray–Serre spectral sequences. This comparison can be related to the theory of the Fadell–Husseini index. We apply the framework to:

- solve a mass partition problem (antipodal cheeses) in $\mathbb{R}^d$,
- determine the existence of a class of inscribed 5-element sets on a deformed 2-sphere,
- obtain two different generalizations of the theorem of Dold for the nonexistence of equivariant maps which generalizes the Borsuk–Ulam theorem.

1 Introduction

Mass partition and transversal problems have drawn the interest in combinatorial circles for a century. Ham Sandwich Theorem and Tverberg theorems stand as examples of the use of topological methods in solving such problems.

The classical Borsuk–Ulam theorem which treats mappings of the form $f\colon S^n \to \mathbb{R}^n$, for which $f(-x) = -f(x)$, is best formulated in terms of equivariant topology: let $\mathbb{Z}/2$ act on S^n by the antipodal action, and on $\mathbb{R}^n$ by $x \mapsto -x$. Then any such map must meet the origin. Generalizations of the Borsuk-Ulam theorem abound and their applications include some of the most striking results in some fields (see, for example, [14]). One of the general formulations of Borsuk-Ulam type is the theorem of Dold [8]: For an n-connected G-space X and a free G-space Y of dimension at most n, there are no G-equivariant mappings $X \to Y$.

Nonexistence theorems for equivariant mappings are the most delicate steps in combinatorial arguments and they are applied as follows: One wishes to show that a certain configuration of elements

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achieves a condition usually given by linear equations in associated quantities. Furthermore there is a symmetry group acting on the space of all configurations and on the linear equations. If no configuration achieves the condition, then the mapping associating the quantities will land in the target linear space away from the subspace of points that meet the linear equations. This gives rise to an equivariant mapping from the configuration space to the linear space minus the subspace of points that meet the test. However, if no such mapping can exist, then the equivariant mapping that we began with must meet the test subspace, and a configuration exists meeting the conditions. The ingredients of such an application are a configuration space on which a group acts, a test space, usually $\mathbb{R}^n$, a test subspace of $\mathbb{R}^n$, and a mapping from the configuration space to the test space that is seen to be equivariant.

This paper introduces a general framework using the Leray–Serre spectral sequence (Section 2) as a main method for the study of the nonexistence of equivariant maps. This method is the backbone of the ideal valued Fadell–Husseini index theory. Using the framework we present four different results which share a common bond as consequences of the nonexistence of appropriate equivariant maps:

- the solution of a mass partition problem (antipodal cheeses in $\mathbb{R}^d$), Section 3.1, Theorem 4;
- the existence of a class of inscribed 5-element sets on a deformed 2-sphere, Section 4, Theorem 8;
- a generalization of Dold's theorem where the range space is a complement of an a arrangement of linear subspaces, Section 5.2, Theorem 14;
- a generalization of Dold's theorem for elementary abelian groups, Section 6, Theorem 16.

The results are obtained through the study of Leray–Serre spectral sequences associated to particular Borel constructions.

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2 A general framework

To any G-space X we associate the *Borel construction* given by $EG \times_G X$ where EG is a free, acyclic G-space and G acts on the product by $g \cdot (e, x) = (e \cdot g^{-1}, g \cdot x)$. The *equivariant cohomology of X, with coefficients in the ring R*, is defined by $H^*_G(X, R) := H^*(EG \times_G X, R)$.

Sending the space X to a point gives a fibration

$$X \hookrightarrow EG \times_G X \xrightarrow{f} EG \times_G * = BG.$$

The Borel construction and the associated fibration are functorial. Suppose that $f\colon X \to Y$ is an equivariant mapping of G-spaces X and Y. Then this induces a mapping of fibrations:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
EG \times_G X \arrow[r] \arrow[d, "\pi_X"'] & EG \times_G Y \arrow[d, "\pi_Y"] \\
BG \arrow[r, "="] & BG
\end{tikzcd}$$

In turn, the mapping of fibrations determines a morphism of the associated Leray–Serre spectral sequences

$$E_r^{p,q}(f) : E_r^{p,q}\left(EG \times_G Y\right) \longrightarrow E_r^{p,q}\left(EG \times_G X\right).$$

The key property of this morphism, that allow us to formulate our framework can be stated as follows.

Proposition 1 *Let X and Y be connected, G-spaces and $f\colon X \to Y$ a G-equivariant map. The morphism*

$$E_2^{p,0}(f) : E_2^{p,0}(EG \times_G Y) \longrightarrow E_2^{p,0}(EG \times_G X).$$

of Leray–Serre spectral sequences induced by f is the identity.

Proof. Notice that $E_2^{p,0}(EG \times_G Y) = E_2^{p,0}(EG \times_G X) = H^p(BG; R)$. Then the claim follows from the definition of Leray–Serre spectral sequences, connectivity of X and Y, and the fact that f is a mapping of fibrations. ■

Definition A **spectral sequence witness** of a pair of G-spaces X and Y, with coefficients in R, is any nonzero element $l \in H^{n+1}(BG; R) = E_2^{n+1,0}(EG \times_G X) = E_2^{n+1,0}(EG \times_G Y)$, for some fixed integer $n \geq 2$, satisfying

(A) In the spectral sequence for Y, for $2 \leq i < n$, l is an i-cocyle, and l is in the image of the transgression, that is,

$$l \in \mathrm{im}(d_n \colon E_n^{0,n}(EG \times_G Y) \longrightarrow E_n^{n+1,0}(EG \times_G Y).$$

(B) In the spectral sequence for X, l survives to E_∞, that is,

$$l \notin \mathrm{im}(d_i \colon E_i^{n-i,i}(EG \times_G X) \to E_i^{n+1,0}(EG \times_G X)),$$

for all $2 \leq i \leq n$.

Here d_s denotes the sth differential in the spectral sequence. The set of all spectral sequence witnesses is denoted by $\mathrm{W}(X, Y; R)$. We call the set of spectral sequence witnesses for X and Y and R the **W-invariant**.

From the definition there is no reason to expect, *a priori*, that the W-invariant for particular mapping is nonempty. Moreover, $\mathrm{W}(X, Y; R)$ depends crucially on the coefficient ring R. For example, if the coefficient ring is a field of characteristic relatively prime to the order of a finite group G, then $\mathrm{W}(X, Y; R) = \emptyset$ for any mapping of G-spaces X and Y. But when $\mathrm{W}(X, Y; R) \neq \emptyset$ we show in the next theorem that any element of $\mathrm{W}(X, Y; R)$ is a witness of the NON-existence of a G-equivariant map $X \longrightarrow Y$.

Theorem 2 *Let X and Y be connected G-spaces. If, for any ring R, $\mathrm{W}(X, Y; R) \neq \emptyset$, then there is no G-equivariant map $X \to Y$.*

Proof. Let $f\colon X \to Y$ be a G-equivariant map and $l \in \mathrm{W}(X, Y; R)$. The condition (A) of the definition of the W-invariant implies that

$$l \notin \mathrm{im}\left(d_i \colon E_i^{n-i,i}(EG \times_G X) \longrightarrow E_i^{n+1,0}(EG \times_G X)\right)$$

for fixed $n > 2$ and every $2 \leq i < n$. By Proposition 1 we have that the morphism $E_i^{*,0}(f)$ induced by the G-equivariant map f is the identity on $l \in \mathrm{W}(X, Y; R)$ for $2 \leq i \leq n$. By condition (B) in the definition of' W-invariants, we have the relation

$$E_n^{n+1,0}(EG \times_G Y) \ni l \xrightarrow{E_n^{*,0}(f)} l \in E_n^{n+1,0}(EG \times_G X).$$

Accounting for the differentials for X and Y, we find on the E_{n+1}-page:

$$E_{n+1}^{*,0}(EG \times_G Y) \ni 0 \xrightarrow{E_{n+1}^{*,0}(f)} l \in E_{n+1}^{*,0}(EG \times_G X).$$

Since $l \neq 0$, we obtain a contradiction to the existence of a G-equivariant mapping $X \to Y$. Thus, there cannot be a G-equivariant map $X \to Y$ and l is a witness of this fact. ■

We next relate the W-invariant $\mathrm{W}(X, Y; R)$ and the Fadell–Husseini ideal-valued indices $\mathrm{Ind}_G(X)$ and $\mathrm{Ind}_G(Y)$. Recall the definition of the *Fadell-Husseini index* of a G-space X, with the coefficients in the ring R: Let

$$\mathrm{Ind}_G(X) := \ker(\pi_X^* \colon H^*(BG, R) \to H_G^*(X, R)).$$

This index is an ideal contained in $H^*(BG, R)$. If we apply ordinary cohomology to the diagram of fibrations given by the Borel constructions we obtain the commutative square:

$$\begin{tikzcd} H^*_G(X,R) & H^*_G(Y,R) \arrow[l, "f^*"'] \\ H^*(BG,R) \arrow[u, "\pi^*_X"] & H^*(BG,R) \arrow[l, "\mathrm{id}"'] \arrow[u, "\pi^*_Y"'] \end{tikzcd}$$

It follows that

$$\ker \pi^*_X = \ker(f^* \circ \pi^*_Y) = (\pi^*_Y)^{-1}(\ker f^*) \supset \ker \pi^*_Y.$$

Thus, a necessary condition for the existence of an equivariant mapping $f \colon X \to Y$ is that $\mathrm{Ind}_G Y \subset \mathrm{Ind}_G X$. For further details consult initial paper of Fadell and Husseini [9] and for treatment of the ring coefficients [5]. From this observation the connection between the Fadell–Husseini index and the W-invariant becomes apparent:

Proposition 3 *Let X and Y be connected G-spaces. Then*

$$\mathrm{Ind}_G Y \not\subseteq \mathrm{Ind}_G X \implies \mathrm{ls}(X, Y; R) \neq \emptyset. \tag{1}$$

Moreover,

$$\mathrm{Ind}_G Y - \mathrm{Ind}_G X \subseteq \mathrm{W}(X, Y; R)$$

Implication (1) holds only in one direction. Therefore, the W-invariant framework can give results in the situations when Fadell–Husseini method does not.

We illustrate our framework and some methods of computation of W-invariants in proofs of the following well known theorems. First, let us consider Dold's theorem [8].

Example Let G be a nontrivial finite group, X an n-connected G-space, and Y a free, at most n-dimensional G-space. Then there is no G-equivariant map $X \to Y$.

Let p be a prime that divides the order of the group G and G_p denote a Sylow p-subgroup. There exists a subgroup of G_p isomorphic to the cyclic group $\mathbb{Z}/p$. Therefore, spaces X and Y can be considered as $\mathbb{Z}/p$-spaces. Let $R = \mathbb{F}_p$ and $H^*(B\mathbb{Z}/p, \mathbb{F}_p) = \mathbb{F}_p[e, t]/e^2$, where $\deg(e) = 1$ and $\deg(t) = 2$. We are prove that $\mathrm{W}(X, Y; \mathbb{F}_p) \cap \bigcup_{0 \le i \le n+1} H^i(B\mathbb{Z}/p, \mathbb{F}_p) \neq \emptyset$. In particular, we show that $H^{n+1}(B\mathbb{Z}/p, \mathbb{F}_p) - \{0\} \subseteq \mathrm{W}(X, Y; \mathbb{F}_p)$.

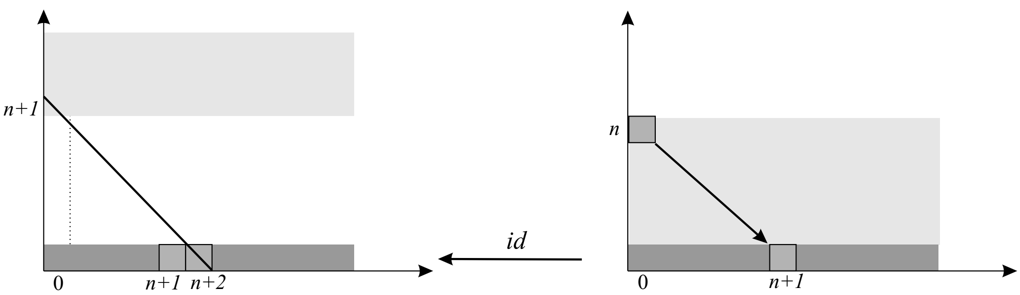

Figure 1: E_2-terms of $E\mathbb{Z}/p \times_{\mathbb{Z}/p} X$ and $E\mathbb{Z}/p \times_{\mathbb{Z}/p} Y$

The E_2-terms of both spectral sequences are pictured in Figure 1 with the spectral sequence for X on the left and for Y on the right. From the connectivity of X it follows that

$$l \notin \mathrm{im}\left(d_i : E_i^{n-i,i}(E\mathbb{Z}/p \times_{\mathbb{Z}/p} X) \longrightarrow E_i^{n+1,0}(E\mathbb{Z}/p \times_{\mathbb{Z}/p} X)\right)$$

for all $l \in H^{n+1}(B\mathbb{Z}/p, \mathbb{F}_p) - \{0\}$ and all $i \geq 2$. Hence, requirement (B) of the definition of a spectral sequence witness is satisfied. The behavior of the differentials in the spectral sequence $E_i^{*,*}(E\mathbb{Z}/p \times_{\mathbb{Z}/p} Y)$

are determined by geometric reasons. Since Y is a free $\mathbb{Z}/p$-space and a finite complex of dimension at most n, we have that the orbit space $Y/(\mathbb{Z}/p) \simeq E\mathbb{Z}/p \times_{\mathbb{Z}/p} Y$ and its cohomology is zero above degree n. Thus there must be nonzero differentials in the spectral sequence to leave behind a finite-dimensional cohomology. Hence, there exists k, $2 \leq k \leq n$, such that

$$l \in \text{im}\left(d_k : E_k^{p,q}(E\mathbb{Z}/p \times_{\mathbb{Z}/p} Y) \longrightarrow E_k^{p+k+1,q-k}(E\mathbb{Z}/p \times_{\mathbb{Z}/p} Y)\right)$$

for every $l \in H^{n+1}(B\mathbb{Z}/p, \mathbb{F}_p) - \{0\}$. Consequently, $H^{n+1}(B\mathbb{Z}/p, \mathbb{F}_p) - \{0\} \subseteq \mathrm{W}(X, Y; \mathbb{F}_p) \neq \emptyset$. Therefore, there cannot be $\mathbb{Z}/p$-equivariant (G-equivariant) mapping $X \to Y$.

The next example is a detailed proof of the critical lemma in the proof of the Topological Tverberg theorem for powers of a prime [18].

Example Let $p = q^n$ where q is a prime and $n > 0$. Set $N = (d+1)(p-1)$ with $d > 0$. Let $[p] = \{0, 1, 2, \ldots, p-1\}$, a discrete space, and $G = (\mathbb{Z}/q)^n$. The topological Tverberg theorem for prime powers is the consequence of the following fact (for further details consult [14, Chapter 6.4]): There are no $(\mathbb{Z}/q)^n$-equivariant maps

$$[p]^{*(N+1)} \to S\left(W_p^{\oplus(d+1)}\right).$$

Here $[p]^{*(N+1)}$ is the $N+1$st iterated join of $[p]$ with itself and $W_p = \{(x_1, .., x_p) \in \mathbb{R}^p \mid \sum x_i = 0\}$ is the standard $(\mathbb{Z}/q)^n$-representation and $(\mathbb{Z}/q)^n$ acts on $[p]$ by left translation by identifying $[p]$ with the group $(\mathbb{Z}/q)^n$.

We prove the claim by showing that

$$H^N(B(\mathbb{Z}/q)^n, \mathbb{F}_q) \cap \mathrm{W}\left([p]^{*(N+1)}, S(W_p^{\oplus(d+1)}); \mathbb{F}_q\right) \neq \emptyset.$$

Since $[p]^{*(N+1)}$ is $(N-1)$-connected, condition (B) of the definition of a spectral sequence witness is satisfied, that is,

$$l \notin \text{im}\left(d_i : E_i^{N-i-1,i}(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} [p]^{*(N+1)}) \longrightarrow E_i^{N,0}(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} [p]^{*(N+1)})\right)$$

for all $l \in H^N(B(\mathbb{Z}/q)^n, \mathbb{F}_q) - \{0\}$ and all $i \geq 2$.

The sphere $S(W_p^{\oplus(d+1)})$ is a fixed point free, but not a free $(\mathbb{Z}/q)^n$-space. A consequence of a localization theorem for elementary abelian groups [11, Corollary 1, page 45] implies that the natural projection of the Borel construction

$$\pi : E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)}) \to B(\mathbb{Z}/q)^n$$

induces a noninjective morphism

$$\pi^* : H^*(B(\mathbb{Z}/q)^n, \mathbb{F}_q) \to H^*\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)}), \mathbb{F}_q\right).$$

This means that one of the differentials in the associated Leray–Serre spectral sequence,

$$d_i : E_i^{N-i-1,i}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right) \longrightarrow E_i^{N,0}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right),$$

must be nonzero. Because $S(W_p^{\oplus d+1})$ is a $(p-1)(d+1)-1$ sphere, the associated Borel is an $(N-1)$-dimensional sphere bundle. Therefore, the only possible nonzero differential is given by

$$d_{N-1} : E_{N-1}^{p,N-1}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right) \longrightarrow E_{N-1}^{N+p,0}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right).$$

Since $(\mathbb{Z}/q)^n$ acts trivially on the cohomology of the fiber $H^*(S(W_p^{\oplus(d+1)}), \mathbb{F}_q)$, there is an isomorphsim

$$\begin{aligned} E_{N-1}^{*,*}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right) &= E_2^{*,*}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right) \\ &\cong H^*(B(\mathbb{Z}/q)^n, \mathbb{F}_q) \otimes H^*(S(W_p^{\oplus(d+1)}), \mathbb{F}_q) \end{aligned}$$

which determines the $H^*(B(\mathbb{Z}/q)^n, \mathbb{F}_q)$-module structure on $E_*^{*,*}$. This implies that the differential d_{N-1} is different from zero (for some $p>0$) if and only if the transgression

$$d_{N-1} : E_{N-1}^{0,N-1}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right) \longrightarrow E_{N-1}^{N,0}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right)$$

is different from zero. Consequently, there is an element

$$0 \neq l \in \operatorname{im}(d_{N-1}) \subset E_{N-1}^{N,0}\left(E(\mathbb{Z}/q)^n \times_{(\mathbb{Z}/q)^n} S(W_p^{\oplus(d+1)})\right) = H^N(B(\mathbb{Z}/q)^n, \mathbb{F}_q).$$

Thus, $l \in \mathrm{W}\left([p]^{*(N+1)}, S(W_p^{\oplus(d+1)}); \mathbb{F}_q\right)$ and we conclude from Theorem 2 that there are no $(\mathbb{Z}/q)^n$-equivariant mappings $[p]^{*(N+1)} \to S\left(W_p^{\oplus(d+1)}\right)$.

3 Cheeses

In this section we consider the antipodal cheese problem. Development of a particular configuration test map scheme relates the problem with the nonexistence of D_{2n}-equivariant mappings from the Stiefel manifold of 2-frames in $\mathbb{R}^d$ to the complements of arrangements. In this way, a solution of the antipodal cheese problem, Theorem 4, becomes a consequence of a general Dold type result (Theorem 7).

Fan and 3-plane mass partition problems in the plane and on the sphere S^2 were introduced by Kaneko and Kano [12] and developed in [1], [2], [3] and [4]. They motivate a study of various types of partitions in higher dimensions.

3.1 Antipodal Cheese Problem

Suppose that an even number $k = 2l$ of people are sitting around a circle table in such a way that everyone has an antipodal friend. On the table there is a pile of j many (high-dimensional) cheese pieces (in $\mathbb{R}^d$), all of different shapes, mass, density, and flavor. A knife is available and the cheese can only be cut all j pieces at once. There are two types of cuts allowed

- the half-straight cut: pick a point on the table as a center and make k straight cuts beginning at the center and continuing in one direction;
- the straight cut: pick a point on the table as center and make l straight cuts through the chosen point in both directions.

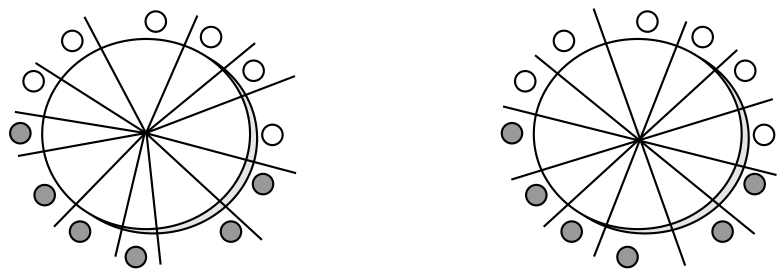

Figure 2: Half-straight and straight cheese cuts

The objective is to divide the cheeses in one of these manners in such a way that every member of an antipodal pair get the same, nonnegative, part of each of the j pieces of cheese. The vocabulary for a mathematical translation of the problem is as follows:

half-straight cut	$\to$	fan,
straight cut	$\to$	arrangement in fan position,
piece of cheese	$\to$	a measure.

Let H be an affine hyperplane in $\mathbb{R}^d$ given by a choice of $v \in \mathbb{R}^d$ and $r \in \mathbb{R}$, and

$$H = \{x \in \mathbb{R}^d \mid \langle x, v\rangle = r\}.$$

A hyperplane divides $\mathbb{R}^d$ into two subsets, H^+ and H^- determined by $\langle x, v \rangle \geq r$ or $\leq r$, respectively. Another such hyperplane H' meeting H transversally determines a codimension one subspace of H denoted by $L = H \cap H'$. Then $H \backslash L$ has two connected components $F_+ = H \cap (H'^+$ and $F_- = H \cap (H'^-$ called *half-hyperplanes* whose common boundary is L.

Definition A k**-fan** in $\mathbb{R}^d$ is a collection $(L; F_1, \dots, F_k)$ consisting of
(A) a $(d-2)$-dimensional oriented linear subspace L, and
(B) different half-hyperplanes $F_1, \dots, F_k$ with the common boundary L, oriented by a compatible orientation on the plane $L^\perp$.

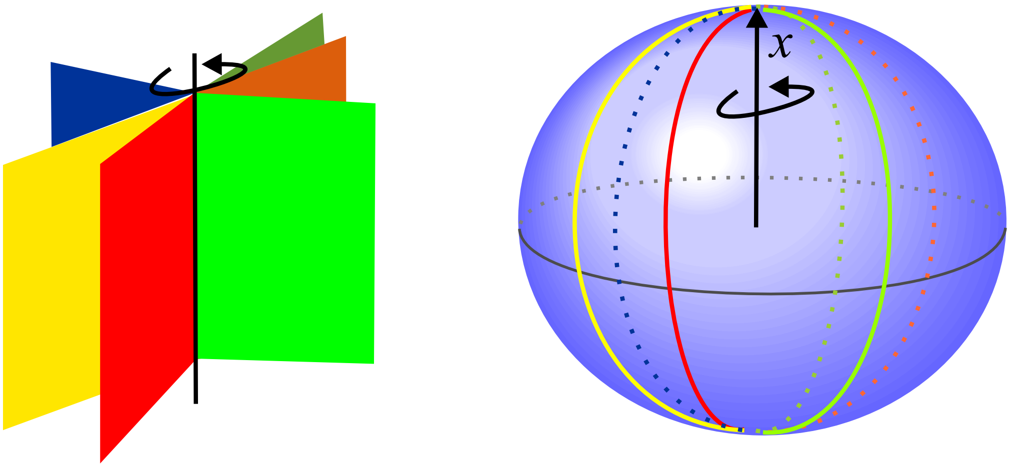

Figure 3: Two models for fan

Suppose $x_0 \in L$ is a choice of point, a center of the k-fan. In the plane $L^\perp$ passing through x_0 there is a unit circle $S(L^\perp, x_0)$. Then $F_1 \cap S(L^\perp, x_0)$ $F_2 \cap S(L^\perp, x_0)$, ..., $F_k \cap S(L^\perp, x_0)$ are consecutive points on the circle oriented by the given orientation on $L^\perp$.

Let $\mathcal{F}_k$ denote the space of all k-fans in $\mathbb{R}^d$. There are several equivalent descriptions of $\mathcal{F}_k$ which provide some flexibility.

(1) Let $S^{d-1} = S^{d-1}(x_0)$ denote the unit sphere centered at x_0 in $\mathbb{R}^d$ and let $l_i = F_i \cap S^{d-1}(x_0)$ denote the *half-great circle* determined by the half-hyperplane F_i on the sphere. The k-fan is determined by the data $(L; l_1, \dots, l_k)$.

(2) The space bounded by the half-hyperspaces F_i and F_{i+1} is called an *orthant* $\mathcal{O}_i$, and so we can express the k-fan as a k-tuple of orthants, $(L; \mathcal{O}_1, \dots, \mathcal{O}_k)$. If we focus on the sphere $S^{d-1}(x_0)$, then we let $(L; \mathcal{O}_1, \dots, \mathcal{O}_k)$ denote the subdivision by the regions $\mathcal{O}_i$ of the sphere between l_i and l_{i+1}.

(3) A third model for a k-fan is the collection $(L; v_1, \dots, v_k)$ where we move x_0 to the origin in $\mathbb{R}^d$ and let $v_i \in S(L^\perp)$ be a unit vector in the direction of the half-hyperplane F_i. If ϕ_i is the angle from v_i to v_{i+1} (ϕ_k the angle between v_k and v_1), then

$$\phi_1 + \phi_2 + \cdots + \phi_k = 2\pi.$$

We can also consider an arrangement of hyperplanes $\mathcal{A} = \{H_1, \dots, H_k\}$. Then we say that $\mathcal{A}$ **is in fan position** if the intersection $L = H_1 \cap \cdots \cap H_k$ is a subspace of codimension one inside each H_i. In other words, $\mathcal{A}$ is in fan position if there is a $2k$-fan $(L; F_1, \dots, F_{2k})$ such that $F_i \cup F_{i+k} = H_i$ for $1 \leq i \leq k$.

As in the case of a fan, an arrangement $\mathcal{A} = \{H_1, \dots, H_k\}$ in fan position inherits a natural orientation from $L^\perp$. The orientation on $L^\perp$ induces an orientation of the connected components (orthants) of the complement $M_{\mathcal{A}} = \mathbb{R}^d \backslash \bigcup H_i$. The orientation is determined up to a cyclic permutation. If $(H_1, \dots, H_k)$ is the induced ordering and $H_{k+1} = H_1$, then we denote by

- $\mathcal{O}_i^+$ the orthant between H_i and H_{i+1}, that is, the orthant between the associated F_i and F_{i+1}, and
- $\mathcal{O}_i^-$ the orthant between H_{i+1} and H_i, that is, the orthant between F_{k+i} and F_{k+i+1}.

By analogy with the Ham Sandwich Theorem, the k-fans represent slices of a knife in $\mathbb{R}^d$. The *cheeses* to be cut are given by proper Borel probability measures on $\mathbb{R}^d$ or on S^{d-1}. A measure μ on a sphere S^{d-1} is a **proper** measure if for every hyperplane $H \subset \mathbb{R}^d$, $\mu(H \cap S^{d-1}) = 0$ and for every nonempty open set $U \subseteq S^{d-1}$, $\mu(U) > 0$. From now on a measure on a sphere S^{d-1} or on $\mathbb{R}^d$ will mean a proper Borel probability measure. Let $\mathcal{M} = \{\mu_1, \dots, \mu_j\}$ be a collection of measures on S^{d-1}.

The partition problems of interest to us can be pictured by viewing a party of k gourmands (or industries) around a circular table. Each desires (requires) a certain portion of any cheese (raw material) to eat (to use). There are j cheeses (resources) to sample and a k-fan is a division of the cheeses to each gourmand to achieve their required portions. We represent the portions by a **ration** which is vector $\beta = (\beta_1, \dots, \beta_k)$ of positive rational numbers for which $\beta_1 + \cdots + \beta_k = 1$.

If each gourmand brings his or her spouse, then the subdivision problem is carried out with an arrangement in fan position (the spouse sits across the table). Here a ration takes the form

$$\alpha = (\alpha_1^+, \dots, \alpha_k^+; \alpha_1^-, \dots, \alpha_k^-) \text{ with } \alpha_i^+, \alpha_i^- \in \mathbb{Q}, \alpha_i^+, \alpha_i^- > 0, \sum_{i=1}^{k} (\alpha_i^+ + \alpha_i^-) = 1.$$

When $k = 2l$, a ration $\beta = (\beta_1, \dots, \beta_k)$ is **symmetric** (each gourmand brings a friend) if $\beta_i = \beta_{i+l}$, for all $1 \leq i \leq l$.

Definition A k-fan $(L, \mathcal{O}_1, \dots, \mathcal{O}_k)$ is a β**-partition** of $\mathcal{M}$ if

$$\text{for all } q \in \{1, \dots, k\} \text{ and all } r \in \{1, \dots, j\},\ \mu_r(\mathcal{O}_q) = \beta_q.$$

An arrangement $\mathcal{A} = \{H_1, \dots, H_k\}$ in fan position is an α**-partition** of $\mathcal{M}$ if

$$\text{for all } q \in \{1, \dots, k\} \text{ and all } r \in \{1, \dots, j\},\ \mu_r(\mathcal{O}_q^+) = \alpha_q^+ \text{ and } \mu_r(\mathcal{O}_q^-) = \alpha_q^-.$$

The goal of this section is to prove the following theorem as a corollary of the main tools.

Theorem 4
(A) *If k is even and $\beta = (\beta_1, \dots, \beta_k)$ is a symmetric ration and if $k(j-1) < d-1$, then for every collection of j measures $\mathcal{M}$ on a sphere S^{d-1}, there exists a β-partition of $\mathcal{M}$ by a k-fan.*
(B) *If $\alpha = (\alpha_1^+, \dots, \alpha_k^+; \alpha_1^-, \dots, \alpha_k^-)$ satisfies $\alpha_i^+ = \alpha_i^-$ for all i (that is, α is a symmetric ration), and if $kj < d-1$, then for every collection of j measures $\mathcal{M}$ on a sphere S^{d-1}, there exists an α-partition of $\mathcal{M}$ by an arrangement $\mathcal{A} = \{H_1, \dots, H_k\}$ in fan position.*

To apply our general framework, we need to introduce a group action which is not obvious for an arbitrary choice of ration. Suppose $\beta = (\beta_1, \dots, \beta_k)$ is a given ration. Choosing a common denominator we can write

$$\beta_1 + \cdots + \beta_k = \frac{b_1}{n} + \cdots \frac{b_k}{n} = 1.$$

Thus the k-tuple $(b_1, \dots, b_k)$ of positive integers satisfies $b_1 + b_2 + \cdots + b_k = n$. Suppose we can find an n-fan $(L; \mathcal{O}_1, \dots, \mathcal{O}_n)$ which satisfies for all $i = 1, \dots, j$

$$\mu_i(\mathcal{O}_1) = \cdots = \mu_i(\mathcal{O}_n) = \frac{1}{n}.$$

Then the following unions of orthants

$$\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{b_1},\ \mathcal{O}_{b_1+1} \cup \cdots \cup \mathcal{O}_{b_1+b_2},\ \dots, \mathcal{O}_{b_1+\cdots+b_{k-1}+1} \cup \cdots \cup \mathcal{O}_n$$

determines a k-fan that is a β-partition. A similar construction works for the arrangement in fan position.

Consider the subspace of all n-fans given by

$$X_{\mu_1,n} = \{(L; \mathcal{O}_1, \dots, \mathcal{O}_n) \in \mathcal{F}_n \mid \mu_1(\mathcal{O}_i) = 1/n, i = 1, \dots, n\}.$$

That is, this is the space of n-fans that achieve the equipartition of μ_1. To analyze this configuration space write an n-fan $(L;\mathcal{O}_1,\dots,\mathcal{O}_n)=(L;v_1,v_2,\dots,v_n)$, where the v_i are the directions of the fans in $S(L^\perp)$. Since our measures are continuous (the Borel condition), the choice of $v_1\in S(L^\perp)$ allows us to determine $(v_2,\dots,v_n)$ by the property that $\mu_1(\mathcal{O}_i)=1/n$ for each i. Thus, we sweep out the first orthant in a chosen direction and beginning at v_1 until we get measure $1/n$ and this determines v_2; continue in this manner until the rest of the vectors are chosen.

We identify $X_{\mu_1,n}$ with $V_2(\mathbb{R}^d)$, the Stiefel manifold of 2-frames in $\mathbb{R}^d$, by choosing $[u,w]\in V_2(\mathbb{R}^d)$. Let $L=(\operatorname{span}\{u,w\})^\perp$ with $u=v_1$ and w determining the orientation of $S(L^\perp)$.

There is a natural action of the dihedral group $D_{2n}=\langle\varepsilon,\sigma\mid\varepsilon^n=\sigma^2=1,\ \varepsilon^{n-1}\sigma=\sigma\varepsilon\rangle$ on $X_{\mu_1,n}$ given by

$$\begin{aligned}\varepsilon\cdot(L;v_1,\dots,v_n)&=(L;v_n,v_1,\dots,v_{n-1}),\\ \sigma\cdot(L;v_1,\dots,v_n)&=(L;v_n,v_{n-1},\dots,v_1).\end{aligned}$$

Let $W_n=\{(x_1,\dots,x_n)\in\mathbb{R}^n\mid x_1+\dots+x_n=0\}$. There is a D_{2n}-action on $\mathbb{R}^n$ and on W_n given by

$$\begin{aligned}\varepsilon\cdot(x_1,\dots,x_n)&=(x_n,x_1,\dots,x_{n-1}),\\ \sigma\cdot(x_1,\dots,x_n)&=(x_n,x_{n-1},\dots,x_1).\end{aligned}$$

The group D_{2n} acts diagonally on the sum $(W_n)^{\oplus l}$.

(A) We associate a test map $F\colon X_{\mu_1,n}\to(W_n)^{\oplus(j-1)}$ which detects a solution to the β-partition k-fan problem:

$$F(L,\mathcal{O}_1,\dots,\mathcal{O}_n)=\big(\mu_i(\mathcal{O}_1)-\tfrac{1}{n},\dots,\mu_i(\mathcal{O}_n)-\tfrac{1}{n}\big)_{i=2}^{j}\in(W_n)^{\oplus(j-1)}.$$

(B) A test map for the α-partition fan position arrangement problem is given by $H\colon X_{\mu_1,n}\to W_n\oplus(W_n)^{\oplus(j-1)}$:

$$H(L,\mathcal{O}_1,\dots,\mathcal{O}_n)=\big(\phi_r-\tfrac{2\pi}{n}\big)_{r=1}^{n}\times\Big(\big(\mu_i(\mathcal{O}_t)-\tfrac{1}{n}\big)_{t=1}^{n}\Big)_{i=2}^{j}\in W_n\oplus(W_n)^{\oplus(j-1)}$$

where $(L,\mathcal{O}_1,\dots,\mathcal{O}_n)=(L;v_1,\dots,v_n)$ and ϕ_r denotes the angle between v_r and v_{r+1}.

Both maps are defined in such a way that the following proposition holds:

Proposition 5 *The maps*

$$F\colon X_{\mu_1,n}\to(W_n)^{\oplus(j-1)}\ \text{ and }\ H\colon X_{\mu_1,n}\to W_n\oplus(W_n)^{\oplus(j-1)}$$

are D_{2n}-equivariant maps.

Natural discriminants for both problems are given by arrangements of linear subspaces of $(W_n)^{\oplus(j-1)}$ and $W_n\oplus(W_n)^{\oplus(j-1)}$, defined in the following way. If $\mathcal{A}$ is an arrangement of linear subspaces in $\mathbb{R}^d$ and G a group that acts linearly on $\mathbb{R}^d$, then by $G\mathcal{A}$ we denote the *minimal G-invariant arrangement* containing the arrangement $\mathcal{A}$, namely,

$$G\mathcal{A}=\{gL\mid g\in G\text{ and }L\in\mathcal{A}\}.$$

An arrangement $\mathcal{A}$ is G-invariant if and only if $G\mathcal{A}=\mathcal{A}$.

(A) Suppose that $k=2l$ and β is a symmetric ration, $\beta=(\beta_1,\dots,\beta_k)$. Then

$$\beta_1+\dots+\beta_k=2(\beta_1+\dots+\beta_l)=2\left(\frac{b_1}{n}+\dots+\frac{b_l}{n}\right)=1.$$

So we find that $n=2m$, and $(b_1,\dots,b_k)$ satisfies $b_i=b_{i+l}$ for $i=1,\dots,l$. Denote a point in $(W_n)^{\oplus(j-1)}$ by

$$x=(x_{1,2},x_{2,2},\dots,x_{n,2};x_{1,3},\dots,x_{n,3};\cdots;x_{1,j},\dots,x_{n,j}).$$

Here $x_{q,r}$ denotes the q-th coordinate in the $(r-1)$-st copy of W_n for $r=2,\dots,j$. Let $\mathcal{B}$ be the minimal D_{2n}-invariant arrangement in $(W_n)^{\oplus(j-1)}$ containing the subspace $L_{\mathcal{B}}$ given by following $l\times(j-1)$ equations:

$$\begin{aligned}&x_{1,i}+\dots+x_{b_1,i} &&=x_{b_l+1,i}+\dots+x_{b_l+b_{l+1},i}\\ &x_{b_1+1,i}+\dots+x_{b_1+b_2,i} &&=x_{b_l+b_{l+1}+1,i}+\dots+x_{b_l+b_{l+1}+b_{l+2},i}\\ &\vdots\\ &x_{b_1+\dots+b_{l-1}+1,i}+\dots+x_{m,i} &&=x_{b_l+\dots+b_{2l-1}+1,i}+\dots+x_{2m,i}\end{aligned}$$

for all $i \in \{2, \dots, j\}$.
(B) Suppose we have a symmetric α-partition given by $\alpha = (\alpha_1^+, \dots, \alpha_k^+; \alpha_1^-, \dots, \alpha_k^-)$ and we write

$$\alpha_1^+ + \cdots + \alpha_k^+ + \alpha_1^- + \cdots + \alpha_k^- = 2\alpha_1^+ + \cdots + 2\alpha_k^+ = \frac{2a_1}{n} + \cdots + \frac{2a_k}{n} = 1.$$

Here we get an k-tuple $(a_1, \dots, a_k)$ with $a_1 + \cdots + a_k = m = n/2$. Let $\mathcal{A}$ be the minimal D_{2n}-invariant arrangement in $W_n \oplus (W_n)^{\oplus(j-1)}$ containing the subspace $L_{\mathcal{A}}$ described by $k + k \times (j-1) = kj$ equations:

$$\begin{array}{lcl} x_{1,1} + \cdots + x_{m,1} & = & 0 \\ x_{a_1+1,1} + \cdots + x_{a_1+m,1} & = & 0 \\ \vdots & & \\ x_{a_1+\cdots+a_{k-1}+1,1} + \cdots + x_{a_1+\cdots+a_{k-1}+m,1} & = & 0 \end{array} \tag{2}$$

and

$$\begin{array}{ll} x_{1,i} + \cdots + x_{a_1,i} & = x_{a_l+1,i} + \cdots + x_{a_l+a_{l+1},i} \\ x_{a_1+1,i} + \cdots + x_{a_1+a_2,i} & = x_{a_l+a_{l+1}+1,i} + \cdots + x_{a_l+a_{l+1}+a_{l+2},i} \\ \vdots & \\ x_{a_1+\cdots+a_{l-1}+1,i} + \cdots + x_{m,i} & = x_{a_l+\cdots+a_{2l-1}+1,i} + \cdots + x_{2m,i} \end{array} \tag{3}$$

for all $i \in \{2, \dots, j\}$. The set of equations (2) test whether the fan ascends from the arrangemenet for the hyperplanes in the fan position, illustration in the Figure 4.

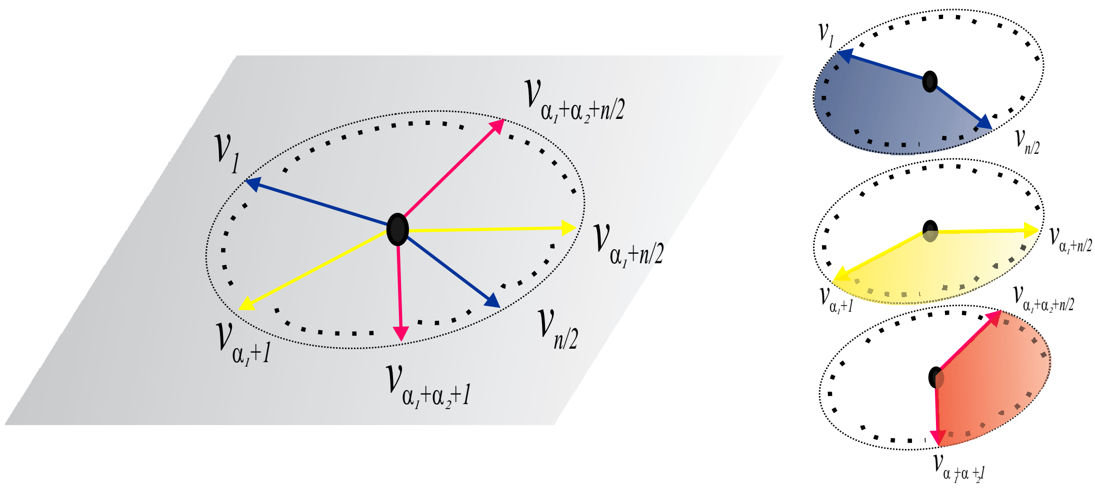

Figure 4: Equations that test ascendancy from the arrangement in fan position

The discriminants are defined in such a way that the following proposition holds.

Proposition 6 *With all the previously made assumptions:*
(A) *If there is no D_{2n}-equivariant mapping*

$$V_2(\mathbb{R}^d) \to (W_n)^{\oplus(j-1)} - \bigcup\nolimits_{L \in \mathcal{B}} L,$$

then the statement of Theorem 4 (A) *is true.*
(B) *If there are no D_{2n}-equivariant mappings*

$$V_2(\mathbb{R}^d) \to W_n \oplus (W_n)^{\oplus(j-1)} - \bigcup\nolimits_{L \in \mathcal{A}} L,$$

then the statement of Theorem 4 (B) *is true.*

Thus, the solution of the mass partition antipodal cheese problem, Theorem 4, is a direct consequence of the following generalization of Dold's theorem.

Theorem 7 *With all the previously made assumptions: There are no D_{2n}-equivariant mappings*

$$V_2(\mathbb{R}^d) \to (W_n)^{\oplus(j-1)} - \bigcup\nolimits_{L\in\mathcal{B}} L \quad \textit{and} \quad V_2(\mathbb{R}^d) \to W_n \oplus (W_n)^{\oplus(j-1)} - \bigcup\nolimits_{L\in\mathcal{A}} L$$

The proof of the theorem is postponed to Section 5.2 where it is a direct application of Theorem 14, a general Dold type theorem.

4 Inscribed pentagons on deformed 2-spheres

In [6] Blagojević and Ziegler study the existence of a class of tetrahedra inscribed on deformed 2-spheres, that is, on the continuous injective image of a sphere in $\mathbb{R}^3$. To prove the main result of [6] the authors compared Fadell–Husseini index theory with coefficients in $\mathbb{Z}$ versus in the field $\mathbb{F}_2$. This was the first example in combinatorics where the use of ring coefficients for Fadell–Husseini index theory gave a result in a situation where use of the appropriate field coefficient failed. In this section we continue the study of configurations of points on deformed 2-spheres and add two new results.

Let $f\colon S^2 \to \mathbb{R}^3$ be an injective continuous map. A collection of distinct points $\{f(x_1), ..., f(x_n)\} \subset f(S^2)$ determines an n-gon inscribed in $f(S^2)$. We call such a set an **inscribed n-set**. Such a set may have metric properties that determine a tetrahedron in the case of an inscribed 4-set, or a polygon in $\mathbb{R}^2$. The principal property for us is the incidence relation $\{f(x_1), \dots, f(x_n)\} \subset f(S^2)$. In this section we prove the following theorem.

Theorem 8 *Let $f\colon S^2 \to \mathbb{R}^3$ be an injective continuous map. Then its image contains an inscribed 5-set $\{f(x_1), ..., f(x_5)\}$ with the following metric properties*

$$d(f(x_1), f(x_2)) = d(f(x_2), f(x_3)) = d(f(x_3), f(x_4)) = d(f(x_4), f(x_5)) = d(f(x_5), f(x_1)), \tag{4}$$

$$d(f(x_1), f(x_3)) = d(f(x_2), f(x_4)) = d(f(x_3), f(x_5)) = d(f(x_4), f(x_1)) = d(f(x_5), f(x_2)). \tag{5}$$

These metric requirements of the theorem do not force the inscribed 5-set to be planar. Thus, we cannot expect to have a regular pentagon inscribed on the deformed 2-sphere. The proof of the theorem does not rely on any particular properties of the Euclidean space $\mathbb{R}^3$. The argument can be made in any metric space instead of $\mathbb{R}^3$. A direct consequence of the theorem is the following claim.

Corollary 9 *Let $f\colon S^2 \to \mathbb{R}^3$ be an injective continuous map. Then its image contains a tetrahedron or a quadrilateral $\{f(x_1), ..., f(x_4)\}$ with the following metric properties*

$$d(f(x_1), f(x_2)) = d(f(x_2), f(x_3)) = d(f(x_3), f(x_4)),$$

$$d(f(x_1), f(x_3)) = d(f(x_2), f(x_4)) = d(f(x_4), f(x_1)).$$

We recast the claim of Theorem 8 into a question of the non-existence of an equivariant map. For a configuration space consider:

$$X = (S^2)^5 - \{(x, x, x, x, x) \mid x \in S^2\}.$$

The group $\mathbb{Z}/5 = \langle\varepsilon\rangle$ acts freely on the configuration space by

$$\varepsilon \cdot (x_1, x_2, x_3, x_4, x_5) = (x_2, x_3, x_4, x_5, x_1).$$

The test map $\tau\colon X \to W_5 \oplus W_5$ is defined by

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) \quad \overset{\tau}{\longmapsto} \quad & \Big(d(f(x_1), f(x_2)) - \frac{\Delta_1}{5}, d(f(x_2), f(x_3)) - \frac{\Delta_1}{5}, d(f(x_3), f(x_4)) - \frac{\Delta_1}{5},\\ & d(f(x_4), f(x_5)) - \frac{\Delta_1}{5}, d(f(x_5), f(x_1)) - \frac{\Delta_1}{5}\Big) \oplus\\ & \Big(d(f(x_1), f(x_3)) - \frac{\Delta_2}{5}, d(f(x_2), f(x_4)) - \frac{\Delta_2}{5}, d(f(x_3), f(x_5)) - \frac{\Delta_2}{5},\\ & d(f(x_4), f(x_1)) - \frac{\Delta_2}{5}, d(f(x_5), f(x_2)) - \frac{\Delta_2}{5}\Big)\end{aligned}$$

where $W_5 = \{(y_1, ..., y_5) \in \mathbb{R}^5 \mid y_1 + ... + y_5 = 0\}$ is the regular $\mathbb{Z}/5$-representation and

$$\begin{aligned}\Delta_1 &= d(f(x_1), f(x_2)) + d(f(x_2), f(x_3)) + d(f(x_3), f(x_4)) + d(f(x_4), f(x_5)) + d(f(x_5), f(x_1)),\\ \Delta_2 &= d(f(x_1), f(x_3)) + d(f(x_2), f(x_4)) + d(f(x_3), f(x_5)) + d(f(x_4), f(x_1)) + d(f(x_5), f(x_2)).\end{aligned}$$

With the $\mathbb{Z}/5$ action on W_5 given by $\varepsilon \cdot (y_1, y_2, y_3, y_4, y_5) = (y_2, y_3, y_4, y_5, y_1)$, the test map $\tau\colon X \to W_5 \oplus W_5$ is a $\mathbb{Z}/5$-equivariant map. The test subspace or discriminant in this situation is the one-point set $\{\mathbf{0}\} \subset W_5 \oplus W_5$.

Proposition 10 *With the actions above, if there are no $\mathbb{Z}/5$-equivariant maps $X \to W_5 \oplus W_5 - \{\mathbf{0}\}$, or equivalently, $X \to S(W_5 \oplus W_5)$, then the claim of Theorem 8 holds, that is, there exists an inscribed 5-set $\{f(x_1), ..., f(x_5)\}$ satisfying the metric properties (4) and (5).*

Proof. Let $\tau(x_1, x_2, x_3, x_4, x_5) = \mathbf{0}$. Then collection $\{f(x_1), ..., f(x_5)\}$ has metric properties (4) and (5). It remains to prove that the points $f(x_1), ..., f(x_5)$ are distinct. Without loss of generality we may assume that $f(x_1) = f(x_j)$ for some $j \in \{2, 3, 4, 5\}$. If $j \in \{2, 5\}$ and $d(f(x_1), f(x_j)) = 0$, then

$$d(f(x_1), f(x_2)) = d(f(x_2), f(x_3)) = d(f(x_3), f(x_4)) = d(f(x_4), f(x_5)) = d(f(x_5), f(x_1)) = 0.$$

Consequently $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5)$. But this cannot be since f is injective and all entries in $(x_1, x_2, x_3, x_4, x_5) \in X$ are not equal. On the other hand, if $j \in \{3, 4\}$, then

$$d(f(x_1), f(x_3)) = d(f(x_2), f(x_4)) = d(f(x_3), f(x_5)) = d(f(x_4), f(x_1)) = d(f(x_5), f(x_2)) = 0.$$

Again $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5)$ gives the contradiction. Thus, $\{f(x_1), ..., f(x_5)\}$ is an inscribed 5-set. ■

The previous proposition implies that Theorem 8 is a direct consequence of the following theorem. All the actions are as introduced.

Theorem 11 *There are no $\mathbb{Z}/5$-equivariant maps*

$$(S^2)^5 - \{(x, x, x, x, x) \mid x \in S^2\} \to S(W_5 \oplus W_5).$$

The proof of the theorem is presented in Section 6. It is an application of the Dold type theorem for elementary abelian groups, Theorem 16.

5 Dold's theorem for the complements of arrangements

This section, and the results in it, are motivated by the problem of the existence of equivariant maps in the complements of arrangements. In order to approach this problem from the W-invariant point of view we have to understand the cohomology of complements of arrangements. The key insight comes from the intersection poset of the arrangement [10] and work of de Longueville and Schultz [13]. In our case, as a first step, we introduce an auxiliary construction of independent interest in the use of arrangements that simplifies our computations.

5.1 Blowups

By the **codimension** of an arrangement $\mathcal{A}$, denoted $\mathrm{codim}_{\mathbb{R}^m}\mathcal{A}$, we understand

$$\mathrm{codim}_{\mathbb{R}^m}\mathcal{A} = \min_{L\in\mathcal{A}} \{\mathrm{codim}_{\mathbb{R}^m} L\}.$$

Recall from [13] that an arrangement $\mathcal{A}$ satisfies the **codimension condition** if for any pair L_1 and L_2 of subspaces in $\mathcal{A}$ we have

$$\mathrm{codim}(L_1 \cap L_2) = \mathrm{codim}(L_1) + \mathrm{codim}(L_2).$$

Furthermore, $\mathcal{A}$ is a c-**arrangement** if it satisfies the following conditions:

- For every maximal element L in $\mathcal{A}$, $\mathrm{codim}_{\mathbb{R}^m} L = c$.

- For all pairs $L_1 \subset L_2$ of elements in $\mathcal{A}$, c divides $\mathrm{codim}_{L_2} L_1$.

For a linear subspace $L \subset \mathbb{R}^m$, one can choose a linearly independent family of *forms*, $\xi_1, \ldots, \xi_t$, given by

$$\xi_j(x_1, \ldots, x_m) = a_{1j}x_1 + \cdots + a_{mj}x_m$$

for which $L = \{\mathbf{x} \in \mathbb{R}^m \mid \xi_1(\mathbf{x}) = \cdots = \xi_t(\mathbf{x}) = 0\}$.

Definition Let $\mathcal{A}$ be an arrangement of linear subspaces in $\mathbb{R}^m$, $\{L_1, \ldots, L_w\}$ the set of maximal elements of $\mathcal{A}$ and $k_i = \mathrm{codim}_{\mathbb{R}^m} L_i$, for $i \in \{1, \ldots, w\}$. For each maximal element L_i, choose a linearly independent family $\{\xi_{i,1}, \ldots, \xi_{i,k_i}\}$ of forms defining L_i. The **blow up** of the arrangement $\mathcal{A}$ is the arrangement $\mathfrak{B}(\mathcal{A})$ in

$$(\mathbb{R}^m)^{k_1+\cdots+k_w} = (\mathbb{R}^m)^{k_1} \times \cdots \times (\mathbb{R}^m)^{k_w} = E_1 \times \cdots \times E_w$$

(where $(\mathbb{R}^m)^{k_i} = E_i$) defined by w maximal elements $\bar{L}_1, \ldots, \bar{L}_w$ introduced in the following way: The subspace $\bar{L}_i$, $i = 1, \ldots, w$, is defined by forms:
$\xi_{i,1} = 0$ seen as a form on the 1-st copy of $\mathbb{R}^m$ in E_i;
$\xi_{i,2} = 0$ seen as a form on the 2-nd copy of $\mathbb{R}^m$ in E_i;
$\vdots$
$\xi_{i,k_i} = 0$ seen as a form on the k_i-th copy of $\mathbb{R}^m$ in E_i.

The blow up $\mathfrak{B}(\mathcal{A})$ depends on the choice of the linear forms $\xi_{*,*}$. Observe that we do not allow any extra dependent forms. Note also that the arrangement operations $\mathfrak{B}(\cdot)$ and $G(\cdot)$ need not commute.

Remark For an arrangement $\mathcal{A}$ inside (a G-invariant) subspace $V \subset \mathbb{R}^m$, the blow up may be taken as an arrangement inside $(V)^{k_1} \times \cdots \times (V)^{k_w}$ defined analogously as in the definition for blow-ups in $\mathbb{R}^m$.

Example Let $L \subset \mathbb{R}^2$ denote the trivial subspace $L = \{(0,0)\}$, and $\mathcal{A} = \{L\}$. Then the blow up $\mathfrak{B}(\mathcal{A})$ is an arrangement in $\mathbb{R}^4$ with one element defined by $x_1 = x_4 = 0$.

Here is a list of significant properties of the blow up of arrangement.

Proposition 12 *Let $\mathcal{A}$ be an arrangement of linear subspaces in $\mathbb{R}^m$ and $\mathfrak{B}(\mathcal{A})$ its associated blow up in $(\mathbb{R}^m)^{k_1+\cdots+k_w}$.*
(A) $\mathrm{codim}_{\mathbb{R}^m}\mathcal{A} = \mathrm{codim}_{\mathbb{R}^{m(k_1+\cdots+k_w)}}\mathfrak{B}(\mathcal{A})$.
(B) *If $L_1, \ldots, L_w$ are the maximal elements in $\mathcal{A}$, and* $\mathrm{codim}_{\mathbb{R}^m} L_1 = \cdots = \mathrm{codim}_{\mathbb{R}^m} L_w$*, then $\mathfrak{B}(\mathcal{A})$ is a* $(\mathrm{codim}_{\mathbb{R}^m}\mathcal{A})$*-arrangement.*
(C) *The identity map $\mathbb{R}^m \to \mathbb{R}^m$ induces a diagonal map $D\colon \mathbb{R}^m \to (\mathbb{R}^m)^{k_1+\cdots+k_w}$ which restricts to a map of complements*

$$D\colon \mathbb{R}^m - \bigcup\nolimits_{L\in\mathcal{A}} L \longrightarrow (\mathbb{R}^m)^{k_1+\cdots+k_w} - \bigcup\nolimits_{\bar{L}\in\mathfrak{B}(\mathcal{A})} \bar{L}.$$

Proof. The statements follow because the distribution of forms to separate copies of $\mathbb{R}^m$ obtains the codimension condition. The rest follows easily. ∎

If G acts on an arrangement $\mathcal{A}$ by linear isomorphisms for which $G\mathcal{A} = \mathcal{A}$, then we can choose the underlying 1-forms that define a subspace L, $\{\xi_t(\vec{x})\}$ to reflect the G-action. A 1-form can be written as dot products, $\xi_t(x_1, \ldots, x_n) = \vec{a}_t^T \vec{x}$, and so if $g \in G$, then $\vec{a}_t^T(g^{-1}g) \cdot \vec{x} = 0$, from which it follows that $(g^{-1})^T \vec{a}_t$ determines another form, written $\xi_t^g(\vec{x})$. The subspace determined by $\{\xi_t^g(\vec{x})\}$ is $gL \in \mathcal{A}$ and so we can construct a choice of defining 1-forms that is invariant under the G-action. Using these forms in the blow up implies that $G\mathfrak{B}(\mathcal{A}) = \mathfrak{B}(\mathcal{A})$ with these choices. More general choices need not conserve this property.

Proposition 13 *Let $R = \Bbbk$ be a field. Consider a G-action on $\mathbb{R}^m$, which we extend diagonally to the product $(\mathbb{R}^m)^{k_1+\cdots+k_w}$. Let $\mathcal{A}$ be a G-invariant arrangement in $\mathbb{R}^m$ and construct $\mathfrak{B}(\mathcal{A})$ to be a G-invariant arrangement. Suppose further that G acts trivially on $H^*(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$. Then*
(1) *The diagonal map $D\colon \mathbb{R}^m \to (\mathbb{R}^m)^{k_1+\cdots+k_w}$ is a G-map. The diagonal map restricts to a G-map of complements*

$$D\colon \mathbb{R}^m - \bigcup\nolimits_{L\in\mathcal{A}} L = M_{\mathcal{A}} \longrightarrow (\mathbb{R}^m)^{k_1+\cdots+k_w} - \bigcup\nolimits_{\bar{L}\in\mathfrak{B}(\mathcal{A})} \bar{L} = M_{\mathfrak{B}(\mathcal{A})};$$

(2) *If the maximal elements in* $\mathcal{A}$, $\{L_1, \ldots, L_w\}$ *have codimensions* $k_1 = \cdots = k_w = k$, *then the blow up* $\mathfrak{B}(\mathcal{A})$ *is a* k*-arrangement and the cohomology ring* $\tilde{H}^*(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$ *is generated as an algebra by* $H^{k-1}(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$.
(3) *If for all* $L \in \mathcal{A}$, $L \supseteq (\mathbb{R}^m)^G$, *then, for all* $L \in \mathfrak{B}(\mathcal{A})$, *we have* $L \supseteq ((\mathbb{R}^m)^{k_1+\cdots+k_w})^G$.
(4) *If the map* $H^*(BG, \Bbbk) \to H^*(EG \times_G M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$ *is not a monomorphism, then the same will be true for the map* $H^*(BG, \Bbbk) \to H^*(EG \times_G M_{\mathcal{A}}, \Bbbk)$. *Moreover,*

$$H^k(BG, \Bbbk) \to H^k(EG \times_G M_{\mathcal{A}}, \Bbbk)$$

is not a monomorphism.

Proof. Statement (1) is a consequence of the definition of the diagonal action on $(\mathbb{R}^m)^{k_1+\cdots+k_w}$ and the careful choice of defining 1-forms. Statement (2) follows by the definition of a blow up, and Corollary 5.6 in [13]. The equality $((\mathbb{R}^m)^{k_1+\cdots+k_w})^G = ((\mathbb{R}^m)^G)^{k_1+\cdots+k_w}$ implies (3).

To prove statement (4) we consider the mapping induced by the G-equivariant diagonal mapping $M_{\mathcal{A}} \to M_{\mathfrak{B}(\mathcal{A})}$ on the Borel constructions,

$$D\colon EG \times_G M_{\mathcal{A}} \to EG \times_G M_{\mathfrak{B}(\mathcal{A})}.$$

By assumption, the edge homomorphism $H^*(BG, \Bbbk) \to H^*(EG \times_G M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$ is not a monomorphism and this is equivalent to the fact that there is a nonzero differential in the Leray-Serre spectral sequence for the fibration

$$M_{\mathfrak{B}(\mathcal{A})} \hookrightarrow EG \times_G M_{\mathfrak{B}(\mathcal{A})} \to EG \times_G \{\mathrm{pt}\} = BG.$$

By the assumption that G acts trivially on $H^*(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$, the E_2-term may be written $E_2^{p,q} \cong H^p(BG, \Bbbk) \otimes H^q(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$. By the result in [13], $\tilde{H}^*(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$ is generated as an algebra in dimension $k-1$. Since the cohomology Leray-Serre spectral sequence is a spectral sequence of algebras, the first differential must be $d_k\colon \tilde{H}^{k-1}(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk) \to H^k(BG, \Bbbk)$. Furthermore, if $d_k = 0$, then $d_{k+l} = 0$ for $l \geq 1$, and the spectral sequence collapses at E_2, which contradicts the assumption that the edge homomorphism is not a monomorphism. Thus, $d_k \neq 0$.

Suppose $1 \otimes v = d_k(u \otimes 1)$ for $v \in H^*(BG, \Bbbk)$ and $u \in H^*(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$. The diagonal mapping induces a mapping of spectral sequences that is given on the E_2-term by the identity on $E_2^{*,0}$ and the induced mapping on cohomology on $E_2^{0,*}$. Since the differential commutes with this induced mapping, we have

$$0 \neq 1 \otimes v = E_2(D)(1 \otimes v) = E_2(D)(d_k(u \otimes 1)) = d_k(D^*(u) \otimes 1).$$

If $D^*(u) = 0$, then $d_k(D^*(u) \otimes 1) = 0$, which contradicts the choice of v. Thus, $D^*(u) \neq 0$ and the transgressive differential $d_k \neq 0$ in the spectral sequence for $EG \times_G M_{\mathcal{A}} \to BG$. Statement (4) follows immediately. ∎

5.2 A General theorem for the complements of arrangements

Having prepared all the details to apply the general framework to equivariant mappings from spaces to complements of arrangements, we summarize our work in a general theorem.

Theorem 14 *Let* G *denote a finite or a compact Lie group and* $\Bbbk$ *a field. Let* X *be a* G*-space satisfying* $H^i(X, \Bbbk) = 0$ *for* $1 \leq i \leq n$ *for some* $n \geq 2$. *Consider a* G*-invariant arrangement* $\mathcal{A}$ *in (some subspace* V *of)* $\mathbb{R}^m$ *and its* G*-invariant blow up* $\mathfrak{B}(\mathcal{A})$ *such that*

(A) *the codimension of all maximal elements in* $\mathcal{A}$ *is* $n+1$;
(B) G *acts trivially on the cohomology* $H^*(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$;
(C) *the map* $H^*(BG, \Bbbk) \to H^*(EG \times_G M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$, *induced by the natural projection* $EG \times_G M_{\mathfrak{B}(\mathcal{A})} \to BG$, *is not a monomorphism, and*
(D) *for all* $L \in \mathcal{A}$, $L \supseteq (\mathbb{R}^m)^G$.

Then there are no G*-equivariant mappings* $X \to M_{\mathcal{A}}$.

Proof. We have prepared the argument for the proof of Theorem 14 in Propositions 12 and 13. It suffices to notice that we have a composite of G-maps,

$$X \xrightarrow{f} M_{\mathcal{A}} \xrightarrow{D} M_{\mathfrak{B}(\mathcal{A})}. \tag{6}$$

The induced mappings of Borel constructions and of the associated fibrations give a mapping of Leray-Serre spectral sequences:

$$E_*^{*,*}(X \times_G EG; \Bbbk) \overset{E^*(f\circ D)}{\longleftarrow} E_*^{*,*}(M_{\mathfrak{B}(\mathcal{A})} \times_G EG; \Bbbk).$$

However, this is the comparison done to prove Proposition 13 and so we conclude that the element $1 \otimes v \in H^{n+1}(BG, \Bbbk)$, from the proof of Proposition 13, is a spectral sequence witness. Thus, there are no G-equivariant mappings $f\colon X \to M_{\mathcal{A}}$. ■

In the case of the mass partition problem posed before, we need to look more closely.

Proof of Theorem 7. The D_{2n}-action that we consider requires that we expand the arrangements $\mathcal{B}$ and $\mathcal{A}$ to become D_{2n}-invariant, and at this point we may lose the condition that D_{2n} acts trivially on $H^*(M_{\mathfrak{B}(\mathcal{A})}, \Bbbk)$. However, the dihedral group contains several interesting subgroups, and we choose the subgroup $G = \langle \varepsilon^m \rangle \cong \mathbb{Z}/2$.

- Since the rations are symmetric, with this choice of a subgroup the blow ups can be constructed in such a way that G acts trivially on the $\mathbb{F}_2$ cohomology of the complement of the blown up arrangements.
- The Stiefel manifold $V_2(\mathbb{R}^d)$ is $(d-3)$-connected and so $H^i(V_2(\mathbb{R}^d), \mathbb{F}_2) = 0$ for $1 \le i \le d-3$.
- The codimension of the maximal elements of the arrangement $\mathcal{B}$ inside $(W_n)^{\oplus(j-1)}$ is $(k-1)(j-1)$ and the codimension of the maximal elements in $\mathcal{A}$ inside $W_n \oplus (W_n)^{\oplus(j-1)}$is $k + (k-1)(j-1)$.
- The effect of ε^m on $\mathbb{R}^n$ is to interchange the first m entries with the last m entries of an n-vector. The fixed set under this exchange has $x_1 = x_{m+1}, \ldots, x_m = x_{2m}$. However, this set lies in zero set of the forms defining our arrangements so we have $L_{\mathcal{B}} \supset ((W_n)^{\oplus(j-1)})^G$ and $L_{\mathcal{A}} \supset (W_n \oplus (W_n)^{\oplus(j-1)})^G$.
- The G-action on the complements

  $$(W_n)^{\oplus(j-1)} - \bigcup\nolimits_{L\in\mathcal{B}} L \quad \text{and} \quad W_n \oplus (W_n)^{\oplus(j-1)} - \bigcup\nolimits_{L\in\mathcal{A}} L$$

  is free. This implies that the index of G for the complements of arrangements is nontrivial.

Theorem 14 implies that if $k(j-1) - 1 < d-2$, there are no G- equivariant mappings

$$V_2(\mathbb{R}^d) \longrightarrow (W_n)^{\oplus(j-1)} - \bigcup\nolimits_{L\in\mathcal{B}} L$$

and if $k + k(j-1) - 1 < d-2$, then there are no G-equivariant mappings

$$V_2(\mathbb{R}^d) \longrightarrow W_n \oplus (W_n)^{\oplus(j-1)} - \bigcup\nolimits_{L\in\mathcal{A}} L.$$

■

6 Dold's theorem for elementary abelian groups

In this section let $G = (\mathbb{Z}/p)^n$, for a prime p and $R = \mathbb{F}_p$. The cohomology of BG is well known and given by:

$$\begin{array}{rcll} H^*(B(\mathbb{Z}/2)^n, \mathbb{F}_2) & = & \mathbb{F}_2[t_1, ..., t_n], & \deg t_j = 1 \\ H^*(B(\mathbb{Z}/p)^n, \mathbb{F}_p) & = & \mathbb{F}_p[t_1, ..., t_n] \otimes \Lambda[e_1, ..., e_n], & \deg t_j = 2,\ \deg e_i = 1 \end{array}$$

In the case of odd p there is a connection between generators via Bockstein homomorphism $\beta(e_j) = t_j$. These cohomology algebras each contain a canonical "maximal" multiplicative set in $H^*(BG)$:

$$S_G := (\text{polynomial part of } H^*(BG)) - \{0\} = \begin{cases} \mathbb{F}_2[t_1, ..., t_n] - \{0\}, & \text{for } G = (\mathbb{Z}/2)^n \\ \mathbb{F}_p[t_1, ..., t_n] - \{0\}, & \text{for } G = (\mathbb{Z}/p)^n. \end{cases}$$

Let Sub_G denotes the collection of all *proper* subgroups of the group G. The essential lemma that allows us to obtain a general Dold type result (Theorem 16) is the following:

Lemma 15 *Let $G = (\mathbb{Z}/p)^n$ and $\Bbbk = \mathbb{F}_p$. Then*

(A) $\bigcap_{H\in \mathrm{Sub}_G} \ker(\mathrm{res}^G_H\colon H^*(BG,\Bbbk)\to H^*(BH,\Bbbk))\neq\emptyset$,
(B) $\bigcap_{H\in \mathrm{Sub}_G} \ker(\mathrm{res}^G_H\colon H^*(BG,\Bbbk)\to H^*(BH,\Bbbk))\cap S_G\neq\emptyset$.

Proof. The groups we are considering have the following unique property: For every proper subgroup H of G an inclusion $H\subset(\mathbb{Z}/p)^n$, after "change of coordinates," can be presented as the natural inclusion

$$(x_1,\ldots,x_m)\mapsto(x_1,\ldots,x_m,0,\ldots,0)\qquad(\mathbb{Z}/p)^m\subset(\mathbb{Z}/p)^n\,,$$

for $m<n$. At the level of cohomology of classifying spaces this means that $(\mathrm{res}^G_H)^*$ is given by natural projection

$$\begin{array}{l}\mathbb{F}_2[t_1,...,t_n]\to\mathbb{F}_2[t_1,...,t_m],\\ \mathbb{F}_p[t_1,...,t_n]\otimes\Lambda[e_1,...,e_n]\to\mathbb{F}_p[t_1,...,t_m]\otimes\Lambda[e_1,...,e_m].\end{array}$$

This implies that

$$\bigcap_{H\in\mathrm{Sub}_G}\ker(\mathrm{res}^G_H:H^*(BG,\Bbbk)\to H^*(BH,\Bbbk))\cap S_G\neq\emptyset.$$

More explicitly, if $x_H\in\ker(\mathrm{res}^G_H\colon H^*(BG,\Bbbk)\to H^*(BH,\Bbbk))\cap S_G$, for $H\in\mathrm{Sub}_G$, then

$$0\neq\prod_{H\in\mathrm{Sub}_G}x_H\in\bigcap_{H\in\mathrm{Sub}_G}\ker(\mathrm{res}^G_H\colon H^*(BG,\Bbbk)\to H^*(BH,\Bbbk))\cap S_G\neq\emptyset.$$

■

Let $\mathcal{N}_G$ denotes the following family of G-modules

$$\mathcal{N}_G:=\begin{cases}\{\mathbb{F}_2[G/H]\mid H\in\mathrm{Sub}_G\}, & \text{for } G=(\mathbb{Z}/2)^n\\ \{\mathbb{F}_p[G/H]\mid H\in\mathrm{Sub}_G\}, & \text{for } G=(\mathbb{Z}/p)^n\,.\end{cases}$$

By $\mathcal{M}_G$ we denote the family of G-modules obtained from $\mathcal{N}_G$ by taking finite direct sums and include the trivial G-module $\mathbf{0}$. We can now formulate a general Dold type result for the elementary abelian groups. In this case, the cohomology of the G-spaces X and Y are considered as $H^*(BG,\Bbbk)$-modules.

Theorem 16 *Let $G=(\mathbb{Z}/p)^n$ and $\Bbbk=\mathbb{F}_p$. Let X and Y be connected G-spaces. If*
(1) $H^i(X;\Bbbk)\in\mathcal{M}_G$, for every $1\leq i\leq n-1$, and
(2) $\pi_j^*:H^j(BG,\Bbbk)\to H^j(EG\times_G Y,\Bbbk)$ is not injective for some $1\leq j\leq n$,
then there are G-equivariant mappings $X\to Y$.

Proof. From condition (2) it follows that there is an integer $j\in\{1,...,n\}$ and $l\in H^j(BG,\Bbbk)$ such that $\pi_j^*(l)=0$. Then

$$l\in\mathrm{im}\left(d_k\colon E_k^{j-k-1,k}(EG\times_G Y)\longrightarrow E_k^{j,0}(EG\times_G Y)\right)$$

for some $k\in\{2,...,j\}\subset\{1,...,n\}$.

By induction on $s\in\{2,...,n\}$, we prove that for every $q\geq0$ differential

$$d_s:E_s^{q,s-1}(EG\times_G X)\longrightarrow E_s^{q+s,0}(EG\times_G X)$$

is zero. This implies that $H^r(BG,\Bbbk)\to H^r(EG\times_G X,\Bbbk)$ is injective for all $r\in\{1,...,n\}$ and consequently

$$l\notin\mathrm{im}\left(d_k\colon E_k^{j-k-1,k}(EG\times_G X)\longrightarrow E_k^{j,0}(EG\times_G X)\right)$$

for any k. Then $l\in\mathrm{W}(X,Y;\Bbbk)$ and so there are G-equivariant mappings $X\to Y$ and the theorem is proved.

(Induction basis) Let $s=2$. Since $H^1(X,\mathbb{F}_p)\in\mathcal{M}_G$, then $H^1(X,\mathbb{F}_p)=\bigoplus_{\alpha\in\Lambda}\Bbbk[G/H_\alpha]^{\oplus m_\alpha}$, where $H_\alpha\in\mathrm{Sub}_G$ and $m_\alpha\geq0$. The E_2-term of the spectral sequence $E_2^{*,*}(EG\times_G X)$ on the 1-row is given by

$$E_2^{*,1}=H^*(BG,H^1(X,\mathbb{F}_p))\cong H^*\left(BG,\bigoplus_{\alpha\in\Lambda}\Bbbk[G/H_\alpha]^{\oplus m_\alpha}\right)\cong\bigoplus_{\alpha\in\Lambda}H^*\left(BG,\Bbbk[G/H_\alpha]\right)^{\oplus m_\alpha}.$$

Shapiro's lemma [7, Proposition 6.2, page 73] implies that

$$E_2^{*,1} \cong \bigoplus_{\alpha \in \Lambda} H^* (BH_\alpha, \mathbb{k})^{\oplus m_\alpha}$$

and so the $H^*(BG, \mathbb{k})$-module structure on $E_2^{*,1}$ is given by the restrictions $\mathrm{res}^G_{H_\alpha}$. Since $H_\alpha \neq G$, for all $\alpha \in \Lambda$, there exists an element $0 \neq x_1 \in S_G \subset H^*(BG, \mathbb{k})$ such that $x_1 \cdot E_2^{*,1} = 0$ and $x_1 \cdot (H^*(BG, \mathbb{k}) - \{0\}) \subset H^*(BG, \mathbb{k}) - \{0\}$. Here $\cdot$ denotes the $H^*(BG, \mathbb{k})$-module multiplication. Let us assume that for some $q \geq 0$ there exists $y \in E_s^{q,1}(EG \times_G X)$ such that $d_2(y) \neq 0$. Since differentials are $H^*(BG, \mathbb{k})$-module morphisms we obtain a contradiction:

$$0 = d_2(x_1 \cdot y) = x_1 \cdot d_2(y) \neq 0.$$

Thus, $d_2 \colon E_s^{q,1}(EG \times_G X) \longrightarrow E_s^{q+1,0}(EG \times_G X)$ is zero.

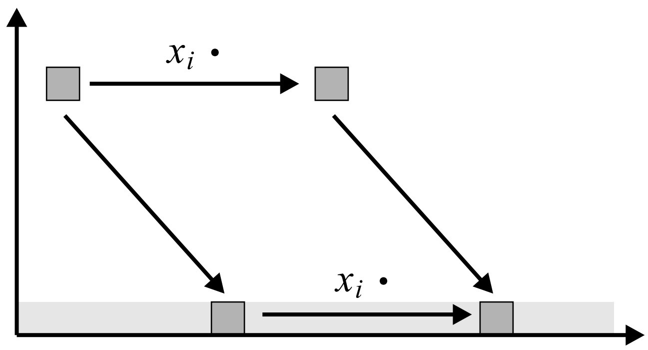

Figure 5: Action of x_i on the spectral sequence

(Induction step) Let the differentials $d_2, ..., d_{s-1}$ are all be zero. Then $E_s^{*,0} = E_2^{*,0} = H^*(BG, \mathbb{k})$. Since $H^{s-1}(X, \mathbb{F}_p) \in \mathcal{M}_G$, then $H^{s-1}(X, \mathbb{F}_p) = \bigoplus_{\beta \in \Omega} \mathbb{k}[G/H_\beta]^{\oplus v_\beta}$, where $H_\beta \in \mathrm{Sub}_G$ and $v_\beta \geq 0$. Again, the E_2-term of the spectral sequence $E_2^{*,*}(E\mathbb{Z}/p \times_{\mathbb{Z}/p} X)$ on the $(s-1)$-row is given by Shapiro's lemma

$$E_2^{*,s-1} = H^*(BG, H^{s-1}(X, \mathbb{F}_p)) \cong \bigoplus_{\beta \in \Omega} H^*(BH_\beta, \mathbb{k})^{\oplus v_\beta}$$

and $H^*(BG, \mathbb{k})$-module structure on $E_2^{*,s-1}$ is determined by the restrictions $\mathrm{res}^G_{H_\beta}$. There exists an element $0 \neq x_{s-1} \in S_G \subset H^*(BG, \mathbb{k})$ such that $x_{s-1} \cdot E_2^{*,1} = 0$, consequently $x_{s-1} \cdot E_s^{*,1} = 0$, and $x_{s-1} \cdot (H^*(BG, \mathbb{k}) - \{0\}) \subset H^*(BG, \mathbb{k}) - \{0\}$. Assume that for some $q \geq 0$ there is $z \in E_s^{q,s-1}(EG \times_G X)$ such that $d_s(z) \neq 0$. Then a contradiction is obtained in the same way:

$$0 = d_s(x_{s-1} \cdot z) = x_{s-1} \cdot d_s(z) \neq 0.$$

Thus, $d_s : E_s^{q,s-1}(EG \times_G X) \longrightarrow E_s^{q+s,0}(EG \times_G X)$ is zero. It is important in the induction that $E_s^{*,0} = E_2^{*,0} = H^*(BG, \mathbb{k})$.

We have proved that for all $s \in \{2, ..., n\}$ and all $q \geq 0$ all the differentials

$$d_s \colon E_s^{q,s-1}(EG \times_G X) \longrightarrow E_s^{q+s,0}(EG \times_G X)$$

are zero. Thus we have found a spectral sequence witness which proves the theorem. ■

As an application of the theorem we prove Theorem 11.

Proof of Theorem 8. We verify the conditions (1) and (2) of Theorem 16 for the space $X = (S^2)^5 - \{(x,x,x,x,x) \mid x \in S^2\}$ and $Y = S(W_5 \oplus W_5)$, with group $G = \mathbb{Z}/5$ and $n = 8$.

(1) Let $A = (S^2)^5$ and $B = \{(x,x,x,x,x) \mid x \in S^2\}$. The long exact sequence of the pair (A, B)

$$\cdots \to H_i(B, \mathbb{F}_5) \to H_i(A, \mathbb{F}_5) \to H_i(A, B, \mathbb{F}_5) \to H_{i-1}(B, \mathbb{F}_5) \to \cdots$$

implies that for $i > 3$

$$H_i(A,B,\mathbb{F}_p) \cong H_i(A,\mathbb{F}_p) = \begin{cases} \mathbb{F}_5[\mathbb{Z}/5]^{\oplus 10}, & \text{for } i = 4,6 \\ \mathbb{F}_5[\mathbb{Z}/5]^{\oplus 5}, & \text{for } i = 8 \\ \mathbb{F}_5, & \text{for } i = 10 \\ \{0\}, & \text{for } i > 3 \text{ and } i \notin \{4,6,8,10\} \end{cases}.$$

For $2 \leq i \leq 3$ the long exact sequence becomes

$$0 \to H_3(A,B,\mathbb{F}_5) \to H_2(B,\mathbb{F}_5) \xrightarrow{\phi} H_2(A,\mathbb{F}_5) \to H_2(A,B,\mathbb{F}_5) \to 0,$$

where the map $\phi\colon\ (H_2(B,\mathbb{F}_5) = \mathbb{F}_5) \longrightarrow (H_2(A,\mathbb{F}_5) = \mathbb{F}_5[\mathbb{Z}/5])$ is given by $1 \mapsto 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1$. Thus

$$H_3(A,B,\mathbb{F}_5) = 0 \quad \text{and} \quad H_2(A,B,\mathbb{F}_5) \cong \mathbb{F}_5[\mathbb{Z}/5]/_{(1+\varepsilon+\varepsilon^2+\varepsilon^3+\varepsilon^4)\mathbb{F}_5}$$

Finally, for $0 \leq i \leq 1$ the sequence

$$0 \to H_1(A,B,\mathbb{F}_5) \to H_0(B,\mathbb{F}_5) \overset{1-1 \text{ and } onto}{\longrightarrow} H_0(A,\mathbb{F}_5) \to H_0(A,B,\mathbb{F}_5) \to 0$$

implies that $H_1(A,B,\mathbb{F}_5) = H_0(A,B,\mathbb{F}_5) = 0$.
The Poincaré–Lefschetz duality isomorphism $H^{10-i}(X,\mathbb{F}_p) \cong H_i(A,B,\mathbb{F}_p)$ yields

$$H^i(X,\mathbb{F}_5) \cong \begin{cases} \mathbb{F}_5[\mathbb{Z}/5]^{\oplus 10}, & \text{for } i = 4,6 \\ \mathbb{F}_5[\mathbb{Z}/5]^{\oplus 5}, & \text{for } i = 2 \\ \mathbb{F}_5, & \text{for } i = 0 \\ \mathbb{F}_5[\mathbb{Z}/5]/_{(1+\varepsilon+\varepsilon^2+\varepsilon^3+\varepsilon^4)\mathbb{F}_5}, & \text{for } i = 8 \\ \{0\}, & \text{otherwise.} \end{cases}$$

Since $\mathbb{F}_5[\mathbb{Z}/5] \in \mathcal{N}_{\mathbb{Z}/5}$ then $H^i(X,\mathbb{F}_5) \in \mathcal{M}_{\mathbb{Z}/5}$ for $1 \leq i \leq 7$. The first condition of Theorem 16 is satisfied.

(2) We prove that $\pi_8^* : H^8(B\mathbb{Z}/5,\mathbb{F}_5) \to H^8(E\mathbb{Z}/5 \times_{\mathbb{Z}/5} Y)$ is not a monomorphism. The sphere $Y = S(W_5 \oplus W_5)$ is a free $\mathbb{Z}/5$-space and so $E\mathbb{Z}/5 \times_{\mathbb{Z}/5} Y \simeq Y/(\mathbb{Z}/5)$. Consequently all the entries $E_\infty^{p,q}$ in the E_∞-term of the Leray–Serre spectral sequence of the Borel construction $E\mathbb{Z}/5 \times_{\mathbb{Z}/5} Y$ above the diagonal $p+q > 7$ must vanish. This can only be achieved if the differential $d_8 \neq 0$. In particular, this requires that $d_8 : \left(E_2^{0,7} = E_8^{0,7}\right) \longrightarrow \left(E_2^{8,0} = E_8^{8,0}\right)$ is different from zero. Thus, π_8^* is not a monomorphism. Both conditions of Theorem 16 are satisfied. Therefore there cannot be $\mathbb{Z}/5$-equivariant maps

$$(S^2)^5 - \{(x,x,x,x,x) \mid x \in S^2\} \longrightarrow S(W_5 \oplus W_5)$$

■

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