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ON THE COMPLEMENT OF THE RICHARDSON ORBIT

KARIN BAUR AND LUTZ HILLE

ABSTRACT. We consider parabolic subgroups of a general algebraic group over an algebraically closed field k whose Levi part has exactly t factors. By a classical theorem of Richardson, the nilradical of a parabolic subgroup P has an open dense P-orbit. In the complement to this dense orbit, there are infinitely many orbits as soon as the number t of factors in the Levi part is ≥ 6. In this paper, we describe the irreducible components of the complement. In particular, we show that there are at most $t-1$ irreducible components. We are also able to determine their codimensions.

CONTENTS

1. INTRODUCTION AND NOTATIONS

Let P be a parabolic subgroup of a reductive algebraic group G over an algebraically closed field k. Let $\mathfrak{p}$ be its Lie algebra and let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be the Levi decomposition of $\mathfrak{p}$, i.e. $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$. A classical result of Richardson [R] says that P has an open dense orbit in the nilradical. We will call this P-orbit the *Richardson orbit for P*. However, in general there are infinitely many P-orbits in $\mathfrak{n}$.

For classical G, the cases where there are finitely many P-orbits in $\mathfrak{n}$ have been classified in [HR1]. Also, the P-action on the derived Lie algebras of $\mathfrak{n}$ have been studied in a series of papers, and the cases with finitely many orbits have been classified, cf. [BrH1], [BrH2], [BrH3], [BrHR].

If G is a general linear group, $G = \mathrm{GL}_n$, then the parabolic subgroup P can be described by the lengths of the blocks in the Levi factor: Write $P = LN$ where L

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is a Levi factor and N is the unipotent radical of P. Then we can assume that L consists of matrices which have non-zero entries in square blocks on the diagonal. Similarly, the Levi factor $\mathfrak{l}$ of $\mathfrak{p}$ consists of the $n \times n$-matrices with non-zero entries lying in squares of size $d_i \times d_i$ ($i = 1, \dots, t$) on the diagonal and $\mathfrak{n}$ are the matrices which only have non-zero entries above and to the right of these square blocks.

Let t be the number of such blocks and $d_1, \dots, d_t$ the lengths of them, $\sum d_i = n$ (with $d_i > 0$ for all i). So d is a composition of n. We will call such a $d = (d_1, \dots, d_t)$ a *dimension vector*. We write $P(d)$ for the corresponding parabolic subgroup and $\mathfrak{n}(d)$ for the nilpotent radical of $P(d)$, the Richardson orbit of $P(d)$ is denoted by $\mathcal{O}(d)$. Its partition will be $\lambda(d)$. Once d is fixed, we will often just use P, $\mathfrak{n}$ and λ if there is no ambiguity. Recall that the nilpotent GL_n-orbits are parametrised by partitions of n. We will use $C(\mu)$ to denote the nilpotent GL_n-orbit for the partition μ (μ a partition of N). And we will usually denote P-orbits in $\mathfrak{n}$ by a calligraphic O, i.e. we will write $\mathcal{O}$ or $\mathcal{O}(\mu)$ if μ is the partition of the nilpotency class of the P-orbit.

Now, the nilradical $\mathfrak{n}$ is a disjoint union of the intersections $\mathfrak{n} \cap C(\mu)$ of the nilradical with all nilpotent GL_n-orbits. By Richardsons result, $\mathfrak{n} \cap C(\lambda) = \mathcal{O}(\lambda)$ is a single P-orbit. In particular, the Richardson orbit consists exactly of the elements of the nilpotency class λ. However, for $\mu \leq \lambda$, the closure $\mathfrak{n}(\mu) := \overline{\mathfrak{n} \cap C(\mu)}$ might be reducible (cf. Proposition 3.3).

In the case where $\mathfrak{n}$ is a Borel subalgebra of the Lie algebra of a simple algebraic group G, Spaltenstein has first studied the varieties $\mathfrak{n} \cap (G \cdot e)$ for $G \cdot e$ a nilpotent orbit under the adjoint action ([S]). In [GHR], the authors study the action of a Borel subgroup B of a simple algebraic group on the closure $\mathfrak{n} \cap C(\mu)$ for the subregular nilpotency class $C(\mu)$ and characterise the cases where B has only finitely many orbits under the adjoint action.

The main goal of this article is to describe the irreducible components of the complement $Z := \mathfrak{n} \setminus \mathcal{O}(d)$ of the Richardson orbit in $\mathfrak{n}$. They occur in the closures $\mathfrak{n}(\mu)$ of the intersections of the nilradical with nilpotent GL_n-orbits $C(\mu)$ lying under $C(\lambda)$.

We have two descriptions of the irreducible components of Z. On one hand, we give rank conditions on the matrices of $\mathfrak{n}$, on the other hand, we use tableaux $T(i,j)$ for certain pairs (i,j) with $1 \leq i < j \leq t$ and associate closures $\mathfrak{n}(T(i,j))$ of P-orbits to them. Before we can state the two results we now introduce the necessary notation.

Let $d = (d_1, \dots, d_t)$ be a dimension vector, $\mathfrak{n}$ the nilradical of the corresponding parabolic subalgebra. For $A \in \mathfrak{n}$ and $1 \leq i, j \leq t$ we write A_{ij} to describe the matrix formed by taking the entries of A lying in the rectangle formed by rows $d_1 + \cdots + d_{i-1} + 1$ up to $d_1 + \cdots + d_i$ and columns $d_1 + \cdots + d_{j-1} + 1$ up to $d_1 + \cdots + d_j$ and with zeroes everywhere else. For $i \geq j$, this is just the zero matrix. Figure 1 shows the blocks A_{ij} for $d = (2, 4, 7)$.

We set $A[i,j]$ to be the matrix formed by entries of the $(A_{k,l})_{i \leq k < j, i < l \leq j}$, i.e. by the rectangles right to and below of $A_{i,i}$ and left to and above of $A_{j,j}$. For instance, $A[i, i+1]$ is just $A_{i,i+1}$. On the other hand, $A[1,t]$ has the same non-zero entries as A.

We are now ready to explain the rank conditions. For the rest of this section, we will always assume that a pair (i,j) satisfies $1 \leq i < j \leq t$. We write $X(d)$ for an element of $\mathcal{O}(d)$. For $k \geq 1$ define

$$\begin{aligned} r_{ij}^k &:= \mathrm{rk}(X(d)[i,j]^{\,k}) \\ \kappa(i,j) &:= 1 + \#\{l \mid i < l < j,\ d_l \geq \min(d_i, d_j)\} \end{aligned}$$

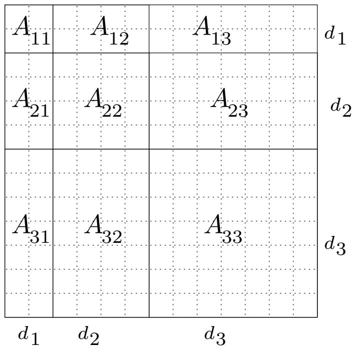


FIGURE 1. The block decomposition of the matrix A for $d = (2, 4, 7)$

to be the rank of the kth power of $X(d)[i, j]$ respectively to be one more than the number of indices l between i and j such that d_l is at least as large as the minimum of d_i and d_j. Observe that the numbers r_{ij}^k are independent of the choice of an element of the Richardson orbit. With this, we can define two subsets of $\mathfrak{n}$ as our candidates for irreducible components of Z.

Definition 1.1. *Let $d = (d_1, \dots, d_t)$ be a dimension vector and $\mathfrak{n}$ the nilradical of the parabolic subgroup P of GL_n. We set*

$$\begin{aligned} Z_{ij}^k &:= \{A \in \mathfrak{n} \mid \operatorname{rk} A[ij]^k < r_{ij}^k\} \\ Z_{ij} &:= Z_{ij}^{\kappa(i,j)} \end{aligned}$$

to be the elements A of $\mathfrak{n}$ for which the rank of kth power of the matrix $A[ij]$ is defective, respectively the A for which the rank of the $\kappa(i, j)$th power is defective.

To any dimension vector $d = (d_1, \dots, d_t)$ we associate subsets $\Gamma(d)$ and $\Lambda(d)$ of the pairs $\{(i, j) \mid 1 \leq i < j \leq t\}$. In Section 2 we will show that Z_{ij} is irreducible for any $(i, j) \in \Gamma(d)$ and that the Z_{ij} with $(i, j) \in \Lambda(d)$ are the irreducible components of Z.

$$\begin{aligned} \Gamma(d) &:= \{(i,j) \mid d_l < \min(d_i, d_j) \text{ or } d_l > \max(d_i, d_j) \ \forall\ i < l < j\}, \\ \Lambda(d) &:= \{(i,j) \in \Gamma(d) \mid d_i = d_j\} \cup \\ &\quad\ \{(i,j) \in \Gamma(d) \mid d_i \neq d_j \text{ and } \\ &\quad\ \left.\begin{array}{ll} \bullet & \forall\ k \leq t: \ d_k \leq \min(d_i, d_j) \text{ or } d_k \geq \max(d_i, d_j) \\ \bullet & \text{for } k < i: \ d_k \neq d_j \\ \bullet & \text{for } k > j: \ d_k \neq d_i \end{array}\right\} \end{aligned}$$

Let us describe $\Gamma(d)$ and $\Lambda(d)$ in words: the pairs (i, j) in $\Gamma(d)$ are such that for all l lying between i and j, the entries d_l are smaller than d_i and d_j or larger than d_i and d_j. For (i, j) to be in $\Lambda(d)$, we require furthermore that either $d_i = d_j$ or that there is no index $1 \leq k \leq t$ such that d_k strictly lies between d_i and d_j. In the case $d_i \neq d_j$, if k is smaller than i, we want $d_k \neq d_j$ and if k is larger than j, we require $d_k \neq d_i$. In general, $\Gamma(d)$ is different from $\Lambda(d)$ as is illustrated here.

Example 1.2. (a) *If $d = (1, 3, 4, 2)$ then $\Gamma(d) = \{(1, 2), (2, 3), (3, 4), (2, 4), (1, 4)\}$ and $\Lambda(d) = \{(2, 3), (2, 4), (1, 4)\}$.*

(b) *For $d = (1, 2, 3, 2)$, $\Gamma(d) = \{(1, 2), (2, 3), (3, 4), (2, 4)\}$, $\Lambda(d) = \{(1, 2), (2, 4)\}$.*

(c) *If $d = (d_1, \dots, d_t)$ is increasing or decreasing, then* $\Gamma(d) = \Lambda(d) = \{(1, 2), (2, 3), \dots, (t-1, t)\}$.

We claim that the irreducible components of $Z = \mathfrak{n} \setminus \mathcal{O}(d)$ are the Z_{ij} with (i,j) from the parameter set $\Lambda(d)$:

Theorem. *(Theorem 4.1) Let $d = (d_1, \dots, d_t)$ be a composition of n, $\lambda = \lambda(d)$ the partition of the Richardson orbit corresponding to d. Then*

$$Z = \bigcup_{(i,j)\in\Lambda(d)} Z_{ij}$$

is the decomposition of Z into irreducible components.

For the second description of the irreducible components we let $T(d)$ be the unique Young tableau obtained by filling the Young diagram of λ with d_1 ones, d_2 twos, etc. (for details, we refer to Subsection 3.1). Now for each pair (i,j) we write $s(i,j)$ for the last row of $T(d)$ containing i and j and we let $T(i,j)$ be the tableau obtained from $T(d)$ by removing the box containing the number j from row $s(i,j)$ and inserting it at the next possible position in order to obtain another tableau. The tableau $T(i,j)$ corresponds to an irreducible component of the intersection of $\mathfrak{n}$ with a nilpotent GL_n-orbit as is explained in Section 3 (Proposition 3.3). We write $\mathfrak{n}(T(i,j)) \subseteq \mathfrak{n}$ for the closure of the intersection of the nilradical with the nilpotency class of $T(i,j)$. We claim that they correspond to irreducible components of Z exactly for the $(i,j) \in \Lambda(d)$.

Theorem. *(Corollary 4.4) Let $d = (d_1, \dots, d_t)$ be a dimension vector, $\lambda = \lambda(d)$ the partition of the Richardson orbit corresponding to d Then*

$$Z = \bigcup_{(i,j)\in\Lambda(d)} \mathfrak{n}(T(i,j))$$

is the decomposition of Z into irreducible components.

As a consequence, we obtain that Z has at most $t-1$ irreducible components (cf. Corollary 4.2) and we can describe their codimensions in $\mathfrak{n}$ (Corollary 4.3). To be more precise, if d is increasing or decreasing or if all the d_i are different, then Z has $t-1$ irreducible components. In particular, this applies to the Borel case where $d = (1, \dots, 1)$. An example with $t = 9$ and where we only have four irreducible components is given in Example 3.7.

Note that the techniques we use are similar to the ones of [BaH] where we describe the complement to the generic orbit in a representation space of a directed quiver of type A_t. However, the indexing sets are different and cannot be derived from each other.

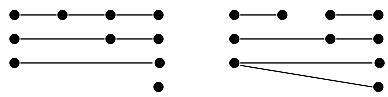
The paper is organised as follows: in Section 2 we explain how to obtain the rank conditions. We first describe line diagrams associated to a composition d of n. Line diagrams will be used to describe elements of the corresponding nilradical $\mathfrak{n}$. In Subsection 2.3 we prove that the elements of $\Lambda(d)$ give the irreducible components. For this, we show that if (i,j) does not belong to $\Gamma(d)$ then the variety Z_{ij} is contained in a union of $Z_{k_s l_s}$ for a subset of pairs (k_s, l_s) of $\Gamma(d)$ (Lemma 2.11). Next, if (i,j) is in $\Gamma(d) \setminus \Lambda(d)$, then we can find $(k,l) \in \Lambda(d)$ such that Z_{ij} is contained in Z_{kl} (Corollary 2.13). In Section 3, we recall Young diagrams and their fillings. Then we consider Young tableaux associated to a composition d of n and a nilpotency class $\mu \leq \lambda(d)$. In a next step, we consider Young tableaux $T(i,j)$ associated to the elements of the parameter set $\Lambda(d)$. To each of these tableaux $T(i,j)$ we associate an irreducible variety $\mathfrak{n}(T(i,j))$ defined as the closure of $\mathfrak{n} \cap C(\mu(i,j))$ where $\mu(i,j)$ is the nilpotency class of the diagram of $T(i,j)$. By showing that the $\mathfrak{n}(T(i,j))$ corresponds to the Z_{ij} from Section 2 we obtain the two descriptions of the decomposition of the Richardson orbit in $\mathfrak{n}$ into irreducible components.

2. Components via rank conditions

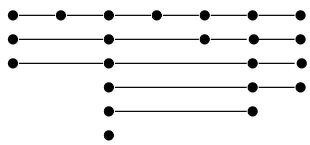
2.1. **Line diagrams.** Let $d = (d_1, \ldots, d_t)$ be a dimension vector for a parabolic subalgebra of $\mathfrak{gl}_n$, $\mathfrak{n}$ the corresponding nilradical. We recall a pictorial way to represent elements of $\mathfrak{n}$ and in particular, to obtain an element of the Richardson orbit $\mathcal{O}(d)$. This can be found in [BrHRR, Section 2] and in [Ba, Section 3]. We draw t top-adjusted columns of d_1, d_2, $\ldots, d_t$ vertices. The vertices are connected using edges between vertices of different columns. The *complete line diagram* for d, $L_R(d)$, is the diagram with horizontal edges between all neighboured vertices. A *line diagram* $L(d)$ for d is a diagram with arbitrary edges between different columns (possibly with branching). The *length of a chain of edges* in a line diagram is the number of edges the chain contains. A chain of length 0 is a vertex that is not connected to any other vertex.

In Example 2.1, we show two complete and a branched line diagram for $d = (3, 1, 2, 4)$ resp. for $d = (3, 1, 6, 1, 2, 5, 4)$.

Example 2.1. *a) The complete line diagram $L_R(d)$ and a line diagram with branching for $d = (3, 1, 2, 4)$ are here*



b) Let $d = (3, 1, 6, 1, 2, 5, 4)$. Its complete line diagram is



We will see in the next subsection that the line diagram $L_R(d)$ determines an element of the Richardson orbit of $\mathfrak{n}$. In general, partial line diagrams give rise to elements of the nilradical of nilpotency class smaller than $\lambda = \lambda(d)$ with respect to the Bruhat order.

Any line diagram (complete or partial) gives rise to an element A of $\mathfrak{n}$: The sizes of the columns of a line diagram correspond to the sizes of the square blocks in the Levi factor of $\mathfrak{p}$.

An edge between column i and column j (with $i < j$) of the diagram corresponds to a non-zero entry in the block A_{ij} of the matrix A. A chain of two joint edges between three columns $i_0 < i_1 < i_2$ gives rise to a non-zero entry in block $A^2_{(i_0,i_2)}$ of the matrix A^2, etc. This can be made explicit, as we explain in the next subsection.

2.2. **From line diagrams to the nilradical.** The elements of the nilradical $\mathfrak{n}$ for the dimension vector $d = (d_1, \ldots, d_t)$ are nilpotent endomorphisms of k^n, for $n = \sum d_i$. In particular, if we write $e_1, \ldots, e_n$ for a basis of k^n, then the elements of $\mathfrak{n}$ are sums $\sum_{i<j} a_{ij} E_{ij}$ with certain $a_{ij} \in k$ where the elementary matrix E_{ij} sends e_j to e_i.

We now describe a map associating an element of the nilradical to a given line diagram. We view the vertices of a line diagram $L(d)$ as labelled by the numbers $1, 2, \ldots, n$, starting at the top left vertex, with $1, 2, \ldots, d_1$ in the first column, $d_1 + 1, \ldots, d_1 + d_2$ in the second column, etc. Now if two vertices i and j (with $i < j$) are joint by an edge, we associate to this edge the matrix E_{ij}.

We denote an edge between two vertices i and j ($i < j \leq n$) of the diagram by $e(i, j)$. Then we associate to an edge $e(i, j)$ of $L(d)$ the elementary matrix $E_{ij} \in \mathfrak{n}$.

This can be extended to a map from the set of line diagrams for d to the nilradical $\mathfrak{n}$ by linearity.

For later use, we denote this map by Φ:

$$\Phi : \{\text{line diagrams for } d\} \longrightarrow \mathfrak{n},\ L(d) \mapsto \sum_{e(i,j)\in L(d)} E_{ij}\,.$$

If $L(d)$ is a line diagram without branching, then the partition of the image under Φ of the line diagram $L(d)$ can be read off from it directly as follows: if $L(d)$ has s chains of lengths $c_1, c_2, \dots, c_s$ (all ≥ 0), i.e. a chain of length c_i connetcs $c_i + 1$ vertices. Then $\sum_{j=1}^{s}(c_j + 1) = \sum_{i=1}^{t} d_i = n$.

Remark 2.2. *Let $L(d)$ be a line diagram without branching and let $c_1, \dots, c_s$ be the lengths of the chains of $L(d)$. If $\mu = (\mu_1, \dots, \mu_s)$ is the partition obtained by ordering the numbers $c_j + 1$ by size. Then μ is the partition of $\Phi(L(d))$.*

In particular, $\Phi(L_R(d))$ is an element of the Richardson orbit $\mathcal{O}(d)$ since the partition of $L_R(d)$ is just the dual of the dimension vector d and this is equal to $\lambda(d)$ (cf. Section 3 in [Ba]). It is straightforward to see that for any other line diagram $L(d)$, the partition of $\Phi(L(d))$ is smaller than or equal to the partition of $\Phi(L_R(d))$ under the Bruhat order as the number of chains of any given length k in $L(d)$ is always bounded by the number of chains of length k in $L_R(d)$.

To summarize, we have the following:

Lemma 2.3. *Let d be a dimension vector. Then, $\Phi(L(d))$ is an element of the nilradical $\mathfrak{n}$ of nilpotency class $\mu \leq \lambda(d)$. In other words, $\Phi(L(d))$ lies in $\mathfrak{n} \cap C(\mu)$.*

Example 2.4. *a) Let $d = (3, 1, 2, 4)$ as in Example 2.1, (a). The Richardson orbit $\mathcal{O}(d)$ has partition $\lambda = (4, 3, 2, 1)$. Let $X(d) := \Phi(L_R(d))$ Then $X(d)$ and its powers are*

$$\begin{aligned}
X(d) &= E_{14} + E_{45} + E_{57} + E_{26} + E_{68} + E_{39}\\
X(d)^2 &= E_{15} + E_{47} + E_{28}\\
X(d)^3 &= E_{17}\\
X(d)^k &= 0 \text{ for } k > 3.
\end{aligned}$$

b) Let $d = (3, 1, 6, 1, 2, 5, 4)$ as in Example 2.1, (b). It has partition $\lambda = (7, 5, 4, 3, 2, 1)$. The line diagram $L_R(d)$ gives rise to the following matrix $X(d)$ and its powers, written in groups given by the five chains of positive length in $L_(d)$:

$$\begin{aligned}
X(d) &= \overbrace{E_{1,4} + E_{4,5} + E_{5,11} + E_{11,12} + E_{12,14} + E_{14,19}}^{1^{st}\ chain}\\
&\quad + \overbrace{E_{2,6} + E_{6,13} + E_{13,15} + E_{15,20}}^{2^{nd}\ chain} \quad + \overbrace{E_{3,7} + E_{7,16} + E_{16,21}}^{3^{rd}\ chain}\\
&\quad + \overbrace{E_{8,17} + E_{17,22}}^{4^{th}\ chain} \quad + \quad \overbrace{E_{9,18}}^{5^{th}\ chain}\\
X(d)^2 &= E_{1,5} + E_{4,11} + E_{5,12} + E_{11,14} + E_{12,19}\\
&\quad + E_{2,13} + E_{6,15} + E_{13,20} \qquad + E_{3,16} + E_{7,21} \qquad + E_{8,22}\\
X(d)^3 &= E_{1,11} + E_{4,12} + E_{5,14} + E_{11,19} \qquad + E_{2,15} + E_{6,20} \qquad + E_{3,21}\\
X(d)^4 &= E_{1,12} + E_{4,14} + E_{5,19} \qquad + E_{2,20}\\
X(d)^5 &= E_{1,14} + E_{4,19}\\
X(d)^6 &= E_{1,19}\\
X(d)^k &= 0 \text{ for } k > 6.
\end{aligned}$$

Recall that we have defined the varieties Z_{ij}^k by comparing the ranks of certain submatrices of elements in the nilradical $\mathfrak{n}$ to the corresponding rank r_{ij}^k of a Richardson element, cf. Definition 1.1. We thus need to be able to compute the rank of the submatrix $X(d)[ij]$ of an element $X(d)$ of the Richardson orbit $\mathcal{O}(d)$ and of its powers. For this, we can use the line diagram $L_R(d)$. Let $X(d) = \sum_{e(k,l)\in L_R(d)} E_{kl}$ be the Richardson element given by $L_R(d)$.

To compute the rank r_{1t}^k of $X(d)^k$, it is enough to count the chains of length $\geq k$ in the line diagram $L_R(d)$. Analogously, to find the rank r_{ij}^k of the kth power of the submatrix $X(d)[ij]$, one has to count the chains of length $\geq k$ between the ith and jth column in $L_R(d)$:

Let $1 \leq k < l \leq n$ be such that the image $\Phi(e(k,l))$ of the edge $e(k,l)$ is in $X(d)[ij]$. That means we are considering edges $e(k,l)$ starting in some column $i_1 \geq i$ and ending in some column $i_2 \leq j$. Thus, in computing r_{ij}^k, we really consider the kth power of the matrix which arises from columns $i, i+1, \ldots, j$ of $L_R(d)$. We now introduce the notation to refer to the subdiagram consisting of these columns. We denote by $L_R(d)[ij]$ subdiagram of $L_R(d)$ of all vertices from the ith up to the jth column and of all edges starting strictly after the $(i-1)$st column resp. ending strictly before the $(j+1)$st column. In other words, we remove columns $1, 2, \ldots, i-1$ and columns $j+1, \ldots, t$ together with all edges incident with them.

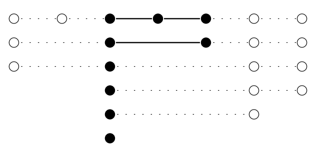
With this notation we have

$$r_{ij}^k = \#\{\text{chains in } L_R(d)[ij] \text{ with at least } k \text{ edges}\} \tag{2.1}$$

for $1 \leq i < j \leq t$, $k \geq 1$.

Similarly, if $L(d)$ is a partial line diagram for d, we write $L(d)[ij]$ to denote the subdiagram of $L(d)$ of rows i to j.

Example 2.5. *The subdiagram $L_R(d)[35]$ for $d = (3, 1, 6, 1, 2, 5, 4)$ of the diagram $L_R(d)$ from (b) of Example 2.1 is shown here (dotted lines and empty circles are thought to be removed):*



2.3. **The varieties Z_{ij}.** As explained earlier, the irreducible components of Z are indexed by the parameter set $\Lambda(d)$. With this in mind, we now discuss the properties of the varieties Z_{ij}^k. We will show that for $l \neq \kappa(i,j)$, Z_{ij}^l is either empty or contained in Z_{ij} or in the union $Z_{ij_0} \cup Z_{i_0 j}$ for some $i_0 \leq j_0$.

The following notations will be useful:

$$\begin{aligned} d_<[ij] &:= \{l \mid i < l < j,\ d_l < \min(d_i, d_j)\} \subseteq \{i+1, \ldots, j-1\} \\ d_\geq[ij] &:= \{l \mid i < l < j,\ d_l \geq \min(d_i, d_j)\} \subseteq \{i+1, \ldots, j-1\}\,. \end{aligned}$$

They denote the indices l between i and j such that the corresponding d_l is strictly smaller than d_i and d_j, respectively the indices l between i and j such that d_l is at least as large as the minimum of d_i and d_j.

Remark 2.6. *Observe that*

$$\begin{aligned} \kappa(i,j) &= 1 + \#d_\geq[ij] \\ &= j - i - \#d_<[ij]\,. \end{aligned}$$

In particular, $\kappa(i,j) = j - i$ if and only if $d_<[ij] = \emptyset$. Figure 2 illustrates this.

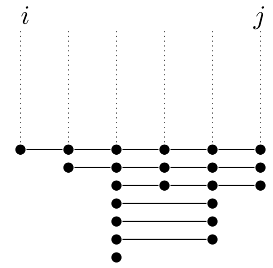


FIGURE 2. The case $d_<[ij] = \emptyset$ with $\kappa(i,j) = j - i = 5$

Lemma 2.7. *Let $d = (d_1, \dots, d_t)$ be a dimension vector and $1 \leq i < j \leq t$. Then for $k > 0$ we have*

$$Z_{ij}^k = \emptyset \text{ if and only if } k > j - i\,.$$

Proof. One has $r_{ij}^k = \operatorname{rk} X(d)[ij]^k > 0$ exactly for $k \leq j - i$ and $0 \in Z_{ij}^k$ if and only if $r_{ij}^k > 0$. □

It remains to consider the cases where l is smaller than $\kappa(i,j)$ or when l lies between $\kappa(i,j)$ and $j - i$. This is covered by the next two statements.

Lemma 2.8. *For $1 \leq l < \kappa(i,j)$ the following holds:*

$$Z_{ij}^l \subsetneq Z_{ij}\,.$$

Proof. We may assume $d_i \leq d_j$. For any $B \in \mathfrak{n}$ the rank of $B[ij]^l$ is independent of the order of $d_i, d_{i+1}, \dots, d_j$ and we may reorder them to obtain $d_{s_1}, \dots, d_{s_{j-i+1}}$ with $d_{s_k} \leq d_{s_{k+1}}$ for $k = 1, \dots, j - i$. One computes $r_{ij}^l = \operatorname{rk} X(d)[ij]^l$ as the sum $\sum_{k=1}^{j-i-l+1} \min\{d_{s_k}, \dots, d_{s_{k+l}}\}$.
Let A belong to Z_{ij}^l for some $l < \kappa(i,j)$. Thus $\operatorname{rk} A[ij]^l < r_{ij}^l = \operatorname{rk} X(d)[ij]^l$. But then also the rank of $A[ij]^k$ is smaller than r_{ij}^k for $k = l+1, \dots, \kappa(i,j)$. In particular, $A \in Z_{ij}$. The inequality is clear. □

Lemma 2.9. *For $\kappa(i,j) < l \leq j - i$ the following holds: there exist $i_0 \leq j_0 \in d_<[ij]$, d_{i_0}, $d_{j_0} < \min(d_i, d_j)$ maximal, such that*

$$Z_{ij}^l \subseteq Z_{ij_0} \cup Z_{i_0 j}\,.$$

Proof. We first observe that for elements of the Richardson orbit, the rank r_{ij}^l is just the maximum over subsets of cardinality $l + 1$ of $d_i, \dots, d_j$ of the minimum among such a subset,

$$r_{ij}^l = \max_{\substack{d_{i_1}, \dots, d_{i_{l+1}} \\ \subset\ d_i, \dots, d_j}} \min\{d_{i_1}, \dots, d_{i_{l+1}}\}$$

(1) Let us first consider the case where $d_<[ij]$ only has one element, say $d_<[ij] = \{i_0\}$, see Figure 3). Then $\kappa(i,j) = j - i - 1$ and so $l = j - i$.
For $A \in \mathfrak{n}$ to be an element of Z_{ij}^l, the rank of $A[ij]^l$ is smaller than r_{ij}^l. Since the entry d_{i_0} is minimal among all $d_i, \dots, d_j$, this implies $\operatorname{rk} A[ii_0]^l < r_{ij}^l$ or $\operatorname{rk} A[i_0 j]^l < r_{ij}^l$ and we are done.
(2) The case where $d_<[ij]$ has at least two elements only needs a slight modification of the argument. Take i_0, j_0 from $d_<[ij]$ with d_{i_0}, d_{j_0} maximal with i_0 being the smallest among these indices, j_0 the largest one (we do not distinguish between the

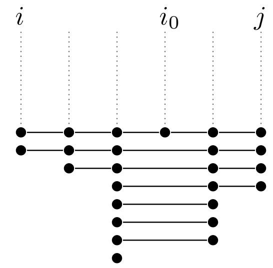


FIGURE 3. The case $d_<[ij] = \{i_0\}$ with $\kappa(i, j) = 4$

two possibilities $d_{i_0} = d_{j_0}$ and $d_{i_0} \neq d_{j_0}$), see Figure 4. With a similar reasoning as in part (1) of the proof, A then lies in Z_{i,j_0} or in $Z_{i_0,j}$.

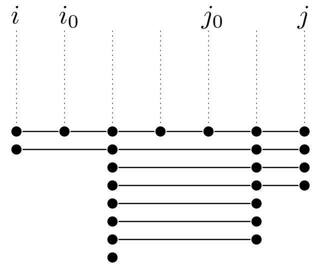


FIGURE 4. The case $i_0 \neq j_0 \in d_<[ij]$ with $\kappa(i, j) = 3$.

□

Lemma 2.10. *The complement Z decomposes as follows:*

$$Z = \cup_{1 \leq i < j \leq t} Z_{ij} = \cup_{ij} \cup_{k \geq 1} Z_{ij}^k \,.$$

Proof. The inclusion $\supseteq$ of the second equality is clear. To obtain the inclusion $\subseteq$, one uses Lemmata 2.7, 2.8 and 2.9. Consider the first equality: by definition, $A \in Z$ if and only if $A \notin \mathcal{O}(d)$. The latter is the case if and only if there exist $1 \leq i < j \leq t$, $k \leq j - i$, such that $A \in Z_{ij}^k$. □

It now remains to see that the $(i, j) \in \Lambda(d)$ are enough to describe the irreducible components of Z. In a first step (Lemma 2.11), we start with $(i, j) \notin \Gamma(d)$ and show that for such a pair, Z_{ij} is contained in a union of Z_{kl}'s with the pairs (k, l) lying in $\Gamma(d)$.
Then we consider a pair (i, j) in $\Gamma(d) \setminus \Lambda(d)$ and show that we can find an element (k, l) of $\Lambda(d)$ with $Z_{ij} \subseteq Z_{kl}$ (Lemma 2.12 and Corollary 2.13). As always, $1 \leq i < j \leq t$ and $1 \leq k < l \leq t$.

Lemma 2.11. *Assume that (i, j) does not belong to $\Gamma(d)$. Then there exists $\Gamma'(d) \subseteq \Gamma(d)$ such that*

$$Z_{ij} \subseteq \bigcup_{(k,l) \in \Gamma'(d)} Z_{kl} \,.$$

Proof. It is enough to show that we can find an l, $i < l < j$, with $\min(d_i, d_j) \leq d_l \leq \max(d_i, d_j)$, such that

$$Z_{ij} \subseteq Z_{il} \cup Z_{lj} \,.$$

By doing this iteratedly, we will eventually end up with a subset $\Gamma'(d) \subset \Gamma(d)$ as in the statement of the lemma. So choose an l, $i < l < j$, with $\min(d_i, d_j) \leq d_l \leq \max(d_i, d_j)$ (such an l exists since $(i,j) \notin \Gamma(d)$). Take $A \in Z_{ij}$ arbitrary. Consider the line diagram $L(A)$ obtained from A by drawing an edge between the rth and the sth vertex whenever the entry A_{rs} is non-zero. Now $\operatorname{rk} A[ij]^{\kappa(i,j)}$ is strictly smaller than $r_{ij}^{\kappa(i,j)}$. Thus, (at least) one chain of length $\kappa(i,j)$ present in $L_R(d)$ cannot appear in the diagram $L(A)$. So at least one edge of such a chain has been removed when going from $L_R(d)$ to $L(A)$. If this edge ends before the $l+1$st column, then the rank of $A[il]^{\kappa(i,l)}$ is smaller than $r_{il}^{\kappa(i,l)}$. Hence $A \in Z_{il}$. If the removed edge originates after the $l-1$st column, $A \in Z_{lj}$ accordingly. This proves the claim. See Figure 5 for an example.

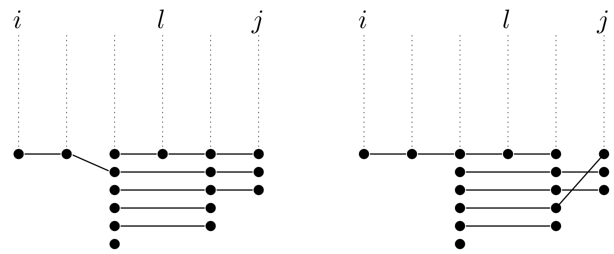


FIGURE 5. Examples for $A \in Z_{il}$ resp. for $A \in Z_{lj}$.

□

The following lemma states that for any (i,j) from $\Gamma(d) \setminus \Lambda(d)$ there exists (k,l) from $\Lambda(d)$ with $k \leq i < j \leq l$ such that $Z_{ij} \subseteq Z_{kl}$.

Lemma 2.12. *Assume that $(i,j) \in \Gamma(d) \setminus \Lambda(d)$. Then one of the following holds:*

$$\begin{array}{rl} & \textit{there exists } k > j \textit{ with } Z_{ij} \subseteq Z_{ik} \\ \textit{or} & \textit{there exists } l < i \textit{ with } Z_{ij} \subseteq Z_{lj}. \end{array}$$

Proof. First observe that $d_i \neq d_j$ since (i,j) belongs to $\Lambda(d)$ otherwise. Without loss of generality, we assume $d_i < d_j$. We have three cases to consider:

(i) There is $k_1 \in \{1, \dots, i-1\} \cup \{j+1, \dots, t\}$ with $d_i < d_{k_1} < d_j$.
(ii) There exists $k_2 < i$ with $d_{k_2} = d_j$.
(iii) There exists $k_3 > j$ with $d_{k_2} = d_i$.

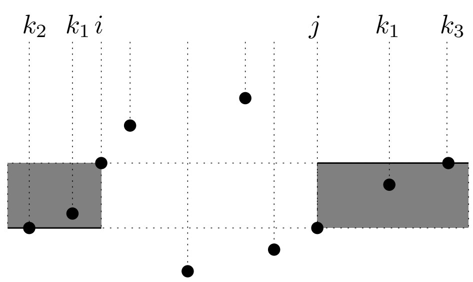


FIGURE 6. Reasons for (i,j) to be in $\Gamma(d) \setminus \Lambda(d)$ (with $\kappa(i,j) = 3$).

The three cases are illustrated in Figure 6: if $(i,j) \in \Gamma(d)$ but not in $\Lambda(d)$ then one of the following has to occur: there has to be a k with d_k inside the shaded area or with d_k lying on one of the highlighted lines (on the same row as d_j if $k < i$ respectively, on the same row as d_i if $k > j$).

Case (i) with $k_1 > j$: Among the $k_1 > j$ with $d_i < d_{k_1} < d_j$ choose one with $d_{k_1} - d_i$ minimal, and k_1 minimal (i.e. as close to j as possible). Note that we have $\kappa(i,j) \leq \kappa(i,k_1)$. Now $A \in Z_{ij}$ means that at least one chain of length $\kappa(i,j)$ of $L_R(d)$ cannot appear in the line diagram $L(A)$ of A (as in the proof of Lemma 2.11). But then, a chain of length $\kappa(i,k_1)$ of the diagram $L_R(d)[i,k_1]$ is not present in $A[i,k_1]$ and hence $Z_{ij} \subseteq Z_{i,k_1}$.

Case (i) with $k_1 < i$: here, we choose k_1 accordingly to be such that $d_j - d_{k_1}$ is minimal and $k_1 < i$ maximal among those (i.e. as close to i as possible). One checks that $\kappa(i,j) \leq \kappa(k_1,j)$. Similarly as before, one gets $Z_{ij} \subseteq Z_{k_1,j}$.

Case (ii) : Among the $k_2 < i$ with $d_{k_2} = d_j$, choose the maximal one (i.e. the one closest to i). We have $\kappa(i,j) \leq \kappa(k_2,j)$ and we get $Z_{ij} \subseteq Z_{k_2,j}$. Case (iii) is completely analogous to case (ii). □

Observe that (k_2,j) and (i,k_3) from cases (ii) and (iii) above are elements of $\Lambda(d)$.

Corollary 2.13. *For any $(i,j) \in \Gamma(d) \setminus \Lambda(d)$ there exists $(k,l) \in \Lambda(d)$ such that*

$$Z_{ij} \subseteq Z_{kl} \,.$$

Proof. Without loss of generality, we can assume $d_i < d_j$. By the observation after the proof of Lemma 2.12, we are done if there exists $k' < i$ with $d_{k'} = d_j$ or $k'' > j$ with $d_{k''} = d_i$. Using similar arguments, one sees that if there exist $k' < i$ and $k'' > j$ with $d_i < d_{k'} = d_{k''} < d_j$ then $(k',k'') \in \Lambda(d)$ and $Z_{ij} \subseteq Z_{k',k''}$. Thus, assume that there exists $k \in \{1,\dots,i-1\} \cup \{j+1,\dots,t\}$ with $d_i < d_k < d_j$ and such that there is no $k' < i$ with $d_{k'} = d_j$ and no $k'' > j$ with $d_{k''} = d_i$.
If $k > j$, we choose k such that $d_k - d_i$ is minimal and take the minimal $k > j$ among these (i.e. k is as close to j as possible). There are two possibilities:
Either we have $d_{k'} > d_k$ for all $k' < i$. Then, $(k',k) \in \Lambda(d)$ and one checks that $Z_{ij} \subseteq Z_{k',k}$.
Or there exists is $k' < i$ with $d_i < d_{k'} < d_k$. In that case, among the $k' < i$ with this property, we choose one with $d_k - d_{k'}$ minimal and such that $k' < i$ is maximal (i.e. k' is as close to i as possible). Again, we get $(k',k) \in \Lambda(d)$ and $Z_{ij} \subseteq Z_{k',k}$.
The case $k < i$ is analogous. □

3. Irreducible components via tableaux

Let $d = (d_1,\dots,d_t)$ be a composition of n and $\mathcal{O}(d)$ be the corresponding Richardson orbit in $\mathfrak{n}$, let $\lambda = \lambda(d)$ be the partition of the Richardson orbit. The second description of the irreducible components of $Z = \mathfrak{n} \setminus \mathcal{O}(d)$ uses partitions μ_{ij}, for $(i,j) \in \Lambda(d)$ and tableaux corresponding to them. Observe that $\lambda_1 = t$, that λ_2 is the number of $d_i \geq 2$ appearing in d, $\lambda_3 = \#\{d_i \mid d_i \geq 3\}$, and so on.

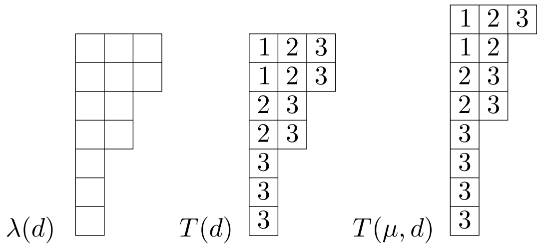
Let us introduce the necessary notation. If $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 1$ is a partition of n we will also use λ to denote the Young diagram of shape λ. It has s rows, with λ_1 boxes in the top row, λ_2 boxes in the second row, etc., up to λ_s boxes in the last row. That means that we view Young diagrams as a number of right adjusted rows of boxes, attached to the top left corner, and decreasing in length from top to bottom. A standard reference for this is the book [F] by Fulton.

3.1. **The Young tableaux $T(\mu, d)$.** Let $\mu \leq \lambda(d)$ be a partition of n (unless mentioned otherwise, we will always deal with partitions of n).

Definition 3.1. *We define a* Young tableau $T(\mu, d)$ of shape μ and of dimension vector d *to be a filling of the Young diagram of μ with d_1 ones, d_2 twos, etc. We write $\mathcal{T}(\mu, d)$ for the set of all Young tableaux $T(\mu, d)$ of shape μ and for d.*

Recall that the rules for fillings of a Young diagram are that the numbers in a row strictly increase from left to right and that the numbers in a column increase from top to bottom.There is exactly one Young tableau of shape $\lambda = \lambda(d)$ and for d. To abbreviate, we will just call it $T(d)$. The boxes of its first row has the entries $1, 2, \ldots, t$.

Example 3.2. *Let $d = (2, 4, 7)$ be a composition of 13. Then $\lambda(d) = (3, 3, 2, 2, 1, 1, 1)$ and $\mathcal{T}(d)$ is as below. The partition $\mu = (3, 2, 2, 2, 1, 1, 1, 1)$ is smaller than $\lambda(d)$ and $\mathcal{T}(\mu, d)$ only has one element $T(\mu, d)$:*



In order to understand the irreducible components of the complement $Z = \mathfrak{n} \setminus \mathcal{O}(d)$, we have to consider the intersections $\mathfrak{n} \cap C(\mu)$ for $\mu < \lambda(d)$. Each irreducible component of Z corresponds to an irreducible component in such an intersection. Here, we can use a result of the second author (cf. Section 4.2 of [H]). First, one observes that the irreducible components of $\mathfrak{n} \cap C(\mu)$ are given by sequences $\mu^1, \ldots, \mu^t$ where μ^i is a partition of $\sum_j^i d_j$ where $\mu^t = \mu$ and such that $0 \leq \mu_j^{i+1} - \mu_j^i \leq 1$ (for all j, for $1 \leq i < t$). And the latter correspond to tableaux of shape μ with d_i entries i, i.e. the elements of $\mathcal{T}(\mu, d)$ in our notation.

Proposition 3.3. *Let $\mu \leq \lambda(d)$ be a partition of n. Then the irreducible components of $\mathfrak{n} \cap C(\mu)$ are in natural bijection with with the tableaux in $\mathcal{T}(\mu, d)$.*

Proof. This is Satz 4.2.8 in [H]. □

Example 3.4. *Let $d = (d_1, \ldots, d_t)$ be a dimension vector and $\lambda = \lambda(d)$. We know that $\mathfrak{n} \cap C(\lambda) = \mathcal{O}(d)$ is the Richardson orbit. On the other hand, $\mathcal{T}(\lambda, d) = T(d)$ has exactly one tableau. We now explain how to relate the complete line diagram $L_R(d)$ to the tableau $T(d)$. The latter can be obtained from the line diagram $L_R(d)$ by writing an entry i for each vertex of column i. And if two columns i and j are joint by an edge in $L_R(d)$ then there are two neighboured boxes with entries i and j in a row of $T(d)$.*

From this connection between the line diagram $L_R(d)$ and $T(d)$ one deduces the following useful observation. Every pair (i, j) with $1 \leq i < j \leq t$ determines a unique row of $T(d)$ namely the last row of $T(d)$ containing i and j. Such a row always exists as the first row just consists of the boxes with numbers $1, 2, 3, \ldots, t$. We denote this row by $s(i, j)$.

Lemma 3.5. *The number of boxes between i and j in row $s(i, j)$ of $T(d)$ is equal to $\kappa(i, j) - 1$.*

Proposition 3.3 describes the irreducible components of the intersections $\mathfrak{n} \cap C(\mu)$ for $\mu \leq \lambda$: They are given by the Young tableaux in $\mathcal{T}(\mu, d)$, i.e. by all possible fillings of the diagram μ by the numbers given by d.

Clearly, not all irreducible components of the different intersections $\mathfrak{n} \cap C(\mu)$ give rise to an irreducible component of Z. If $\mu_2 \leq \mu_1$ and $T_i \in \mathcal{T}(\mu_i, d)$ are tableaux such that T_2 can be obtained from T_1 by moving down boxes successively, then the irreducible component corresponding to T_2 is already contained in the irreducible component corresponding to T_1 and thus does not give rise to a new irreducible component of the complement Z of the Richardson orbit. This is in particular the case, if T_1 is obtained from the tableau $T(d)$ of the Richardson orbit by moving down a single box and T_2 is a degeneration of T_1 (obtained by moving down boxes from T_1). Thus, the only candidates for irreducible components are the ones given by tableaux which can be obtained from $T(d)$ by moving down a single box to the closest possible row. We call such a degeneration a *minimal movement*.

3.2. **The Young tableaux $T(i,j)$.**

To describe minimal movements, we now define certain tableaux $T(i,j)$.

Definition 3.6. *The tableau $T(i,j)$ is the tableau obtained from $T(d)$ by removing the box containing the number j from row $s(i,j)$ and inserting it at the next possible position in order to obtain another tableau. We denote the partition of $T(i,j)$ by $\mu(i,j)$.*

For a tableau $T(i,j)$ we define $\mathfrak{n}(T(i,j)) \subseteq \mathfrak{n}$ to be the closure of the intersection of the nilradical with the nilpotency class of $\mu(i,j)$,

$$\mathfrak{n}(T(i,j)) := \overline{\mathfrak{n} \cap C(\mu(i,j))}\,.$$

The $\mathfrak{n}(T(i,j))$ are the candidates for the irreducible components of Z. The goal is now to show that such a $\mathfrak{n}(T(i,j))$ gives rise to an irreducible component exactly when (i,j) belongs to the parameter set $\Lambda(d)$.

By definition, the tableau $T(i,j)$ is obtained from $T(d)$ through a minimal movement. Its partition $\mu(i,j)$ is clearly smaller than $\lambda = \lambda(d)$ as the lengths of the rows of a tableau are the parts of the corresponding partition. In particular, these lengths form a decreasing sequence of positive numbers. Thus, moving down a box from a row of length k to a lower row (of strictly smaller length) results in a partition which is smaller than the original partition. Note, however, that different pairs (i,j) and (k,l) can lead to the same partition $\mu(i,j) = \mu(k,l)$, e.g. $\mu(2,5) = \mu(5,9)$ in Example 3.7 below.

Example 3.7. *Let $d = (7,5,2,3,5,1,2,6,5)$ be a dimension vector, $n = 36$. To illustrate the construction of $T(i,j)$ we compute these tableaux for (i,j) from $\Lambda(d) = \{(1,8),(2,5),(3,7),(5,9)\}$. They are presented in Figure 7. In the picture showing the line diagram $L_R(d)$ we have indicated by full lines the connections between the columns i and j for all pairs $(i,j) \in \Lambda(d)$.*

Lemma 3.8. *Let $d = (d_1, \dots, d_t)$ be a dimension vector, $(i,j) \in \Gamma(d)$. Then*

$$\mathfrak{n}(T(i,j)) = Z_{ij}\,.$$

Proof. The elements of $\mathfrak{n}(T(i,j))$ are exactly the A with $\operatorname{rk} A[ij]^{\kappa(i,j)} \leq r_{ij}^{\kappa(i,j)} - 1$. □

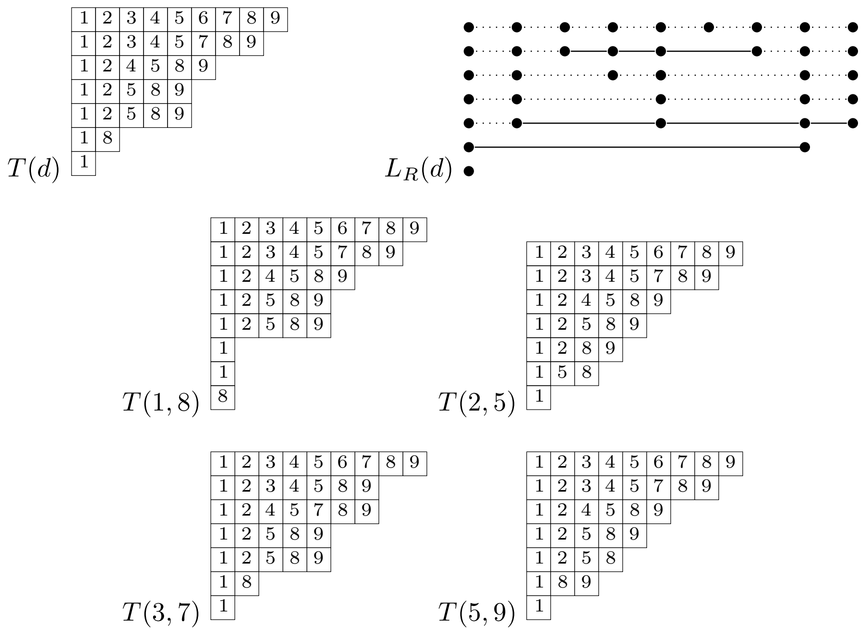


FIGURE 7. The tableaux $T(d)$, $T(i,j)$ and $L_R(d)$ for Example 3.7.

4. The irreducible components of Z

We are now ready to finish the proof of the descriptions of the decomposition of the complement $Z = \mathfrak{n} \setminus \mathcal{O}(d)$ of the Richardson orbit into irreducible components. Again, let $d = (d_1, \dots, d_t)$ be a dimension vector, $\lambda = \lambda(d)$ the partition of the Richardson orbit and (i,j) a pair with $1 \le i < j \le t$. Recall that the $T(i,j)$ are elements of $\mathcal{T}(\mu(i,j), d)$. By Proposition 3.3 the $T(i,j)$ correspond to irreducible components of $\mathfrak{n} \cap C(\mu(i,j))$. So the corresponding $\mathfrak{n}(T(i,j))$ are irreducible.

Theorem 4.1.

$$Z = \bigcup_{(i,j)\in\Lambda(d)} Z_{ij}$$

is the decomposition of Z into irreducible components.

Proof. We know that Z is the union of all Z_{ij} over all (i,j) with $1 \le i < j \le t$ by Lemma 2.10. By Lemma 2.11,

$$Z = \bigcup_{(k,l)\in\Gamma'(d)} Z_{kl}$$

for some subset $\Gamma'(d) \subseteq \Gamma(d)$. And finally, Corollary 2.13 tells us that for each (k,l) in this subset $\Gamma'(d)$, there exists $(i,j) \in \Lambda(d)$ such that Z_{kl} is contained in Z_{ij}.

It remains to see that $Z_{ij} \subsetneq Z_{kl}$ and $Z_{ij} \supsetneq Z_{kl}$ for all $(i,j) \neq (k,l) \in \Lambda(d)$. This follows as for $(i,j) \neq (k,l)$ from $\Lambda(d)$, one can find matrices $A[ij]$ in Z_{ij} which do not satisfy the conditions for Z_{kl} and vice versa.

The irreducibility follows now since $Z_{ij} = \mathfrak{n}(T(i,j))$ (Lemma 3.8). $\square$

Corollary 4.2. *The complement $Z = \mathfrak{n} \setminus \mathcal{O}(d)$ has at most $t-1$ irreducible components.*

Proof. If d is increasing or decreasing then clearly, $\Lambda(d)$ has size $t-1$, cf. Example 1.2. The same is true if the d_i are all different. In all other cases there are $d_i = d_j$ with $|j-i| > 1$, and such that there exists an index $i < l < j$ with $d_l \neq d_i$. If $d_l > d_i$ is minimal among these, then neither (i,l) nor (l,j) belong to $\Lambda(d)$ and thus $\Lambda(d)$ has at most $t-2$ elements. The same is true for $d_l < d_i$, d_l maximal among such. □

Furthermore, we can describe the codimension of Z_{ij} in $\mathfrak{n}$ as follows. Recall that $T(i,j)$ is obtained from $T(d)$ through a minimal movement (see Subsection ss:youngtab). Let $c(i,j)$ be the number of rows the box with label j moves down, i.e. j goes from row $s(i,j)$ to row $s(i,j)+d(i,j)$. Since the resulting $\mathfrak{n}(T(i,j))$ then has codimension $d(i,j)$ in the nilradical $\mathfrak{n}$ we get:

Corollary 4.3. *For $(i,j) \in \Gamma(d)$, Z_{ij} has codimension $c(i,j)$ in $\mathfrak{n}$.*

The second description of the irreducible components of Z is now an immediate consequence of Theorem 4.1 and Lemma 3.8:

Corollary 4.4.
$$Z = \bigcup_{(i,j)\in\Lambda(d)} \mathfrak{n}(T(i,j))$$
is the decomposition of Z into irreducible components.

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