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OWP 2010 - 16

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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SHAPE THEORY AND EXTENSIONS OF C^* -ALGEBRAS

VLADIMIR MANUILOV AND KLAUS THOMSEN

ABSTRACT. Let A, A' be separable C^* -algebras, B a stable σ -unital C^* -algebra. Our main result is the construction of the pairing $[[A', A]] \times \text{Ext}^{-1/2}(A, B) \rightarrow \text{Ext}^{-1/2}(A', B)$, where $[[A', A]]$ denotes the set of homotopy classes of asymptotic homomorphisms from A' to A and $\text{Ext}^{-1/2}(A, B)$ is the group of semi-invertible extensions of A by B . Assume that all extensions of A by B are semi-invertible. Then this pairing allows us to give a condition on A' that provides semi-invertibility of all extensions of A' by B . This holds, in particular, if A and A' are shape equivalent. A similar condition implies that if $\text{Ext}^{-1/2}$ coincides with E -theory (via the Connes-Higson map) for A then the same holds for A' .

1. INTRODUCTION

The theory of extensions of C^* -algebras is presently experiencing an unprecedented level of activity aiming to improve our understanding of extensions of non-nuclear C^* -algebras. One line of research was sparked by the examples of non-invertible extensions by the reduced group C^* -algebra of a free group obtained by Haagerup and Thorbjørnsen in [HT] and the subsequent applications of their result by Hadwin and Shen in [HS] which has resulted in a wealth of examples of C^* -algebras with non-invertible extensions by the compact operators \mathbb{K} . As pointed out in [MT5] each such example gives rise to a non-invertible extension of the same C^* -algebra by any C^* -algebra of the form $B \otimes \mathbb{K}$ with B unital. Although it may still be a pre-mature to conclude that the presence of non-invertible extensions is more of a rule than an exception in the non-nuclear case, the new wealth of examples has made it more urgent to find a way to handle non-invertible extensions.

In our previous work we have proposed an approach towards an analysis of C^* -extensions in which many non-invertible extensions can be handled in a way analogous to how invertible extensions are dealt with and classified in the theories developed by Brown, Douglas and Fillmore, [BDF], and Kasparov, [K1]. Specifically, in a series of papers, beginning with [MT3] and culminating in [MT5], it has been shown that many of the non-invertible extensions are invertible in a slightly weaker sense, called *semi-invertibility*. Recall that an extension of a C^* -algebra A by a stable C^* -algebra B is invertible when there is another extension, the inverse, with the property that the direct sum extension of the two is a split extension. Semi-invertibility requires only that the sum is *asymptotically split*, in the sense that there is an asymptotic homomorphism as defined by Connes and Higson, [CH], consisting of right-inverses of the quotient map. What has been shown is that many classes of (non-nuclear) C^* -algebras, including suspensions, certain full and reduced group C^* -algebras and certain amalgamated free products, have the property that all extensions of the algebra by a stable (σ -unital) C^* -algebra are semi-invertible. For some of these algebras it is known, thanks to the development mentioned above, that there exist non-invertible extensions, but for many or most it is simply not

known if all extensions are invertible or not. Intriguingly it has also been shown, in [MT4], that non-semi-invertible extensions exist.

The main reason why semi-invertibility is easier to establish, and a good reason why it can appear to be more natural in the homology and co-homology theories that are based on extensions of C^* -algebras is that it is homotopy invariant, in the sense that if the Busby invariant of two C^* -extensions are homotopic as $*$ -homomorphisms then one of the extensions is semi-invertible if and only if the other is. This is in glaring contrast to invertibility; it is e.g. known that there are contractible C^* -algebras with non-invertible extensions by \mathbb{K} , cf. [Ki]. All but one of the methods used so far to establish automatic semi-invertibility use the homotopy invariance property; the exception being Theorem 3.3 of [MT5]. However, it is shown in [MT4] that there are C^* -extensions which are not even invertible up to the more natural and much weaker notion of homotopy usually applied in connection with C^* -extensions, and it becomes therefore a natural problem to identify the borderline between the C^* -algebras for which semi-invertibility of extensions is automatic, and the rather mysterious algebras with non-semi-invertible extensions. Presently such an identification seems out of reach, although one may be slightly more optimistic about the possibility of finding the right separating conditions than for doing the analogous thing concerning invertibility.

The main purpose with the present paper is to show that automatic semi-invertibility of extensions, as a property of C^* -algebras, is not only invariant under homotopy equivalence, but also under shape equivalence. This allows us to identify a large natural class of C^* -algebras which have this property, namely the class of C^* -algebras whose shape is dominated by a nuclear C^* -algebra. To make this more precise recall that shape theory of C^* -algebras was introduced by Effros and Kaminker in [EK] as a generalisation of shape theory for topological spaces. It was developed further by Blackadar in [B] before it was tied together with the E-theory of Connes and Higson by Dadarlat in [D]. Roughly speaking what Dadarlat showed was that shape theory of C^* -algebras can be described by the homotopy category of asymptotic homomorphisms which suitably suspended becomes the E-theory of Connes and Higson, [CH]. In short, shape theory is unsuspended E-theory. It is in this guise that we use shape theory here. As shown by Dadarlat a morphism in the shape category, say from the C^* -algebra A to the C^* -algebra B , is given by an element in $[[A, B]]$, the homotopy classes of asymptotic homomorphism from A to B . Our main result says that if A has the property that there is another C^* -algebra A' and asymptotic homomorphisms $\psi : A \rightarrow A \otimes \mathbb{K}$, $\lambda : A \rightarrow A' \otimes \mathbb{K}$ and $\mu : A' \otimes \mathbb{K} \rightarrow A \otimes \mathbb{K}$ such that

$$[\text{id}_A] + [\psi] = [\mu] \bullet [\lambda]$$

in $[[A, A \otimes \mathbb{K}]]$, where id_A is the identity map on A , considered as a map $A \rightarrow A \otimes \mathbb{K}$ in the standard way and \bullet denotes the composition product of Connes and Higson, then all extensions of A by a stable σ -unital C^* -algebra B are semi-invertible if all extensions of A' by B are. When ψ can be taken to be zero the assumption means that A is shape dominated by A' in a sense generalising the notion of homotopy domination introduced by Voiculescu, [V], and when A' can be taken to be zero the assumption is that A is homotopy symmetric in the sense defined by Dadarlat and Loring in [DL].

We consider also the relation between the group of semi-invertible extensions and E-theory proper. As we showed in [MT4] the Connes-Higson construction introduced

in [CH] does not give an isomorphism between E-theory and the homotopy classes of extensions in general, but we show that it does for C^* -algebras that are homotopy symmetric or shape dominated by a nuclear C^* -algebra, or any other C^* -algebra for which it does.

Acknowledgement. The main part of this work was done during a stay of both authors at the Mathematisches Forschungsinstitut in Oberwolfach in January 2010 in the framework of the ‘Research in Pairs’ programme. We want to thank the MFO for the perfect working conditions.

2. PAIRING EXTENSIONS WITH ASYMPTOTIC HOMOMORPHISMS

2.1. Asymptotic homomorphisms. Let A and B be C^* -algebras, A separable. As in [CH] we define an *asymptotic homomorphism* $\alpha : A \rightarrow B$ to be a path of maps $\alpha_t : A \rightarrow B$, $t \in [1, \infty)$, such that

- $t \mapsto \alpha_t(a)$ is continuous,
- $\lim_{t \rightarrow \infty} \alpha_t(a + \lambda b) - \alpha_t(a) - \lambda \alpha_t(b) = 0$,
- $\lim_{t \rightarrow \infty} \alpha_t(ab) - \alpha_t(a)\alpha_t(b) = 0$, and
- $\lim_{t \rightarrow \infty} \alpha_t(a^*) - \alpha_t(a)^* = 0$

for all $a, b \in A$ and all $\lambda \in \mathbb{C}$. It follows from these conditions that $\limsup_t \|\alpha_t(a)\| \leq \|a\|$, and hence in particular that $\sup_{t \in [1, \infty)} \|\alpha_t(a)\| < \infty$ for all $a \in A$.

We say that an asymptotic homomorphism $\alpha : A \rightarrow B$ is *equi-continuous* when $\alpha_t, t \in [1, \infty)$, is an equi-continuous family of maps. By a standard argument any asymptotic homomorphism α is asymptotic to an equi-continuous asymptotic homomorphism α' , i.e. α' is equi-continuous and $\lim_{t \rightarrow \infty} \alpha_t(a) - \alpha'_t(a) = 0$ for all $a \in A$. Hence we may assume, as we shall, that all asymptotic homomorphisms under consideration are equi-continuous. It will also be convenient for us, if only as a tool, to deal with asymptotic homomorphisms α which are both equi-continuous and *uniformly continuous* in the sense that $t \mapsto \alpha_t(a)$ is uniformly continuous for all $a \in A$. We shall need the following lemma in order to fully exploit this additional property.

Lemma 2.1. *Let D be a C^* -algebra containing a σ -unital ideal D_0 , and let $q : D \rightarrow D/D_0$ be the quotient map. Let $\varphi = (\varphi_t)_{t \in [1, \infty)} : A \rightarrow D/D_0$ be a uniformly continuous asymptotic homomorphism. There is then a family $\bar{\varphi}_t : A \rightarrow D, t \in [1, \infty)$, of maps such that*

- i) $q \circ \bar{\varphi}_t = \varphi_t$ for all $t \in [1, \infty)$,
- ii) $\bar{\varphi}_t : A \rightarrow D, t \in [1, \infty)$, is equi-continuous,
- iii) $t \mapsto \bar{\varphi}_t(a)$ is uniformly continuous for all $a \in A$, and
- iv) $\sup_{t \in [1, \infty)} \|\bar{\varphi}_t(a)\| < \infty$ for all $a \in A$.

Proof. Let $\psi = (\psi_t)_{t \in [1, \infty)} : A \rightarrow D$ be an equi-continuous lift of φ such that $\sup_{t \in [1, \infty)} \|\psi_t(a)\| < \infty$ for all $a \in A$. ψ exists by Lemma 2.1 of [MT2]. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ be a sequence of finite sets with dense union in A . Let $u_1 \leq u_2 \leq u_3 \leq \dots$ be an approximate unit in D_0 such that

$$\|(1 - u_n)(\psi_t(a) - \psi_{t'}(a))\| \leq \|\varphi_t(a) - \varphi_{t'}(a)\| + \frac{1}{n}$$

for all $a \in F_n$ and all $t, t' \in [1, n+1]$. Such an approximate unit exists because D_0 is σ -unital. For $t \in [n, n+1]$, set $v_t = (t - n)u_{n+1} + (n + 1 - t)u_n$, and define

$\bar{\varphi}_t : A \rightarrow M(D)$ such that

$$\bar{\varphi}_t(a) = (1 - v_t)\psi_t(a).$$

It is obvious that $(\bar{\varphi}_t)_{t \in [1, \infty)}$ is equi-continuous since $(\psi_t)_{t \in [1, \infty)}$ is and that i) and iv) hold. To check that $\bar{\varphi}$ is uniformly continuous, let $a \in A$ and $\epsilon > 0$ be given. By equi-continuity there is a $b \in F_k$ such that $\frac{1}{k} \leq \epsilon$ and $\|\bar{\varphi}_t(a) - \bar{\varphi}_t(b)\| \leq \epsilon$ and $\|\psi_t(a) - \psi_t(b)\| \leq \epsilon$ for all $t \in [1, \infty)$. Let $t \geq k$. If $|t' - t| \leq 1$ we find that

$$\begin{aligned} \|\bar{\varphi}_t(a) - \bar{\varphi}_{t'}(a)\| &\leq \|\bar{\varphi}_t(b) - \bar{\varphi}_{t'}(b)\| + 2\epsilon \\ &\leq \|(1 - v_t)(\psi_t(b) - \psi_{t'}(b))\| + \|\psi_{t'}(b)(v_{t'} - v_t)\| + 2\epsilon \\ &\leq \|\varphi_t(b) - \varphi_{t'}(b)\| + \frac{1}{k} + |t - t'| \sup_{s \in [1, \infty)} \|\psi_s(b)\| + 2\epsilon \\ &\leq \|\varphi_t(a) - \varphi_{t'}(a)\| + 5\epsilon + |t - t'| \sup_{s \in [1, \infty)} \|\psi_s(a)\| + |t - t'|\epsilon. \end{aligned}$$

By uniform continuity of $t \mapsto \varphi_t(a)$ this shows there is a $\delta > 0$ such that

$$\|\bar{\varphi}_t(a) - \bar{\varphi}_{t'}(a)\| \leq 6\epsilon + \epsilon \sup_{s \in [1, \infty)} \|\psi_s(a)\|$$

when $t \geq k$ and $|t - t'| \leq \delta$. Since $[1, k]$ is compact, we see that $t \mapsto \bar{\varphi}_t(a)$ is uniformly continuous on $[1, \infty)$. \square

2.2. Folding. Let A and B be C^* -algebras, A separable, B σ -unital. Let $M(B)$ be the multiplier algebra of B and $q_B : M(B) \rightarrow Q(B)$ the quotient map onto the generalised Calkin algebra $Q(B) = M(B)/B$. As in [MT2] an asymptotic homomorphism $\varphi : A \rightarrow Q(B)$ will be called an *asymptotic extension*. In [MT2] we used a construction called *folding* which produces a genuine extension out of an asymptotic one. To introduce it here, let $\varphi : A \rightarrow Q(B)$ be an equi-continuous asymptotic extension. A *lift* of φ is an equi-continuous family of maps $\bar{\varphi}_t : A \rightarrow M(B)$, $t \in [1, \infty)$, such that $\sup_t \|\bar{\varphi}_t(a)\| < \infty$ for all $a \in A$ and $q_B \circ \bar{\varphi}_t = \varphi_t$ for all t . The existence of such a lift follows from Lemma 2.1 of [MT2].

Let b be a strictly positive element in B , $0 \leq b \leq 1$, which exists because we assume that B is σ -unital. As in [MT2] a *unit sequence* is a sequence $u_0 \leq u_1 \leq u_2 \leq \dots$ of elements in B such that

- $u_n = f_n(b)$ for some $f_n \in C[0, 1]$, $0 \leq f_n \leq 1$, which is zero in a neighbourhood of 0,
- $u_{n+1}u_n = u_n$ for all n , and
- $\lim_{n \rightarrow \infty} u_n b = b$.

The existence of a unit sequence, with some important additional properties that we shall need is a consequence of the following well-known lemma.

Lemma 2.2. *Let $K \subseteq B$ and $L \subseteq M(B)$ be compact in the norm topology, and let $\delta > 0$ and $\epsilon > 0$ be arbitrary. It follows that there is a continuous function $f : [0, 1] \rightarrow [0, 1]$ such that*

- i) f is zero in an open neighbourhood of 0,
- ii) $f(t) = 1$, $t \geq \delta$,

and $u = f(b) \in B$ has the property that

- iii) $\|um - mu\| \leq \epsilon \forall m \in L$ and
- iv) $\|uk - k\| \leq \epsilon \forall k \in K$.

Proof. See for example Lemma 7.3.1 in [BO]. \square

Given a unit sequence $\{u_n\}$ we set $\Delta_0 = \sqrt{u_0}$ and $\Delta_n = \sqrt{u_n - u_{n-1}}$, $n \geq 1$. Then

- a1) $\Delta_i \Delta_j = 0$ when $|i - j| \geq 2$, and
- a2) $\sum_{j=0}^{\infty} \Delta_j^2 = 1$, with convergence in the strict topology.

In particular, it follows that

$$\text{a3) } \sum_{j=0}^{\infty} \Delta_i \Delta_j^2 = \sum_{l=i-1}^{i+1} \Delta_i \Delta_l^2 = \Delta_i$$

for all i , including $i = 0$ when we set $\Delta_{-1} = 0$.

A *discretization* (of $[1, \infty)$) is an increasing sequence $t_0 \leq t_1 < t_2 < \dots$ in $[1, \infty)$ such that

- a4) $\lim_{n \rightarrow \infty} t_n = \infty$,
- a5) $\lim_{n \rightarrow \infty} t_{n+1} - t_n = 0$, and
- a6) $t_n \leq n$ for all $n \geq 1$.

When $\varphi : A \rightarrow Q(B)$ is an asymptotic extension and $\bar{\varphi} : A \rightarrow M(B)$ is a lift of φ a pair $(\{u_n\}, \{t_n\})$, where $\{u_n\}$ is a unit sequence and $\{t_n\}$ a discretization, is said to be *compatible* with $\bar{\varphi}$ when

$$\lim_{n \rightarrow \infty} \sup_{t \in [1, n+2]} \|u_n \bar{\varphi}_t(a) - \bar{\varphi}_t(a) u_n\|, \quad (2.1)$$

and

$$\lim_{t \rightarrow \infty} \sup_{t \in [t_n, t_{n+1}]} \|\bar{\varphi}_t(a) - \bar{\varphi}_{t_n}(a)\| = 0 \quad (2.2)$$

for all $a \in A$, and

$$\lim_{n \rightarrow \infty} \sup_{t \in [1, n+2]} [|(1 - u_n)f(t)| - \|q_B(f(t))\|] = 0 \quad (2.3)$$

for all $f \in C_b([1, \infty), M(B))$ of the form

- $f(t) = \bar{\varphi}_t(a)\bar{\varphi}_t(b) - \bar{\varphi}_t(ab)$,
- $f(t) = \bar{\varphi}_t(a) + \lambda\bar{\varphi}_t(b) - \bar{\varphi}_t(a + \lambda b)$, and
- $f(t) = \bar{\varphi}_t(a^*) - \bar{\varphi}_t(a)^*$

for any elements $a, b \in A$, $\lambda \in \mathbb{C}$. The existence of compatible pairs $(\{u_n\}, \{t_n\})$ was established in [MT1] and [MT2]. Note that condition (2.2) is automatically fulfilled when $\bar{\varphi}$ is uniformly continuous; it follows then from a5).

Assume that $(\{u_n\}, \{t_n\})$ is a pair compatible with $\bar{\varphi}$. The combined triple $f = (\bar{\varphi}, \{u_n\}, \{t_n\})$ will be called *folding data* for the asymptotic extension φ . We can then define $\bar{\varphi}_f : A \rightarrow M(B)$ such that

$$\bar{\varphi}_f(a) = \sum_{j=0}^{\infty} \Delta_j \bar{\varphi}_{t_j}(a) \Delta_j,$$

cf. Lemma 3.1 of [MT2]. By Lemmma 3.5 in [MT2],

$$\varphi_f = q_B \circ \bar{\varphi}_f$$

is an extension $\varphi_f : A \rightarrow Q(B)$ which we call a *folding* of φ .

A *re-parametrisation* is a non-decreasing continuous function $r : [1, \infty) \rightarrow [1, \infty)$ such that $\lim_{t \rightarrow \infty} r(t) = \infty$. If there is a constant K such that $|r(s) - r(t)| \leq K|s - t|$ for all $s, t \in [1, \infty)$ we say that r is *Lipschitz*. Note that when $\varphi : A \rightarrow Q(B)$ is an asymptotic extension and $\bar{\varphi} : A \rightarrow M(B)$ is a lift of φ , we can define a new asymptotic extension φ^r with a lift $\bar{\varphi}^r$ such that $\varphi_t^r = \varphi_{r(t)}$ and $\bar{\varphi}_t^r = \bar{\varphi}_{r(t)}$. Both φ^r and $\bar{\varphi}^r$ remain uniformly continuous if $\bar{\varphi}$ is uniformly continuous and r is Lipschitz.

Lemma 2.3. *Let $\varphi : A \rightarrow Q(B)$ be an asymptotic extension and let $f = (\overline{\varphi}, \{u_n\}, \{t_n\})$ be folding data for φ . There is then a Lipschitz re-parametrisation r and a discretization $\{s_n\}$ such that*

- i) φ^r and $\overline{\varphi^r}$ are both uniformly continuous, and
- ii) $r(s_n) = t_n$ for all n .

Furthermore, $f' = (\overline{\varphi^r}, \{u_n\}, \{s_n\})$ is folding data for φ^r and $\varphi_{f'}^r = \varphi_f$.

Proof. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ be an increasing sequence of finite sets with dense union in A . It follows from (2.2) that there is an increasing sequence $1 < m_1 < m_2 < m_3 < \dots$ in \mathbb{N} such that

$$\sup_{t \in [t_n, t_{n+k}]} \|\overline{\varphi}_t(a) - \overline{\varphi}_{t_n}(a)\| \leq \frac{1}{k} \quad (2.4)$$

for all $a \in F_k$ and all $n \geq m_k$. By increasing the m_k 's we can arrange that $m_{k+1} - m_k = kl_k$ for some $l_k \in \mathbb{N}$. Set $l_0 = m_1$. Thanks to a5) we can also arrange that

$$t_{n+k} - t_n \leq \frac{1}{k} \quad (2.5)$$

for all $n \geq m_k$. Set

$$s_{m_k+i} = \sum_{j=0}^{k-1} l_j + \frac{i}{k} \quad (2.6)$$

for all $i \in \{0, 1, 2, \dots, kl_k\}$ and all $k = 1, 2, 3, \dots$. Then $s_n \leq n$ for all $n \geq m_1$. For $j \in \{0, 1, \dots, m_1 - 1\}$ we choose $s_j \in [1, m_1]$ such that s_j increases with j and $s_j \leq j$ for all $j \in \{1, 2, \dots, m_1 - 1\}$. Then $\{s_n\}$ is a discretization. Define $r : [1, \infty) \rightarrow [1, \infty)$ such that ii) holds and r is linear on $[s_n, s_{n+1}]$ for all n . Then r is a re-parametrisation and

$$r \left(\sum_{j=0}^{k-1} l_j + i \right) = t_{m_k+ik},$$

for all $i \in \{0, 1, \dots, kl_k\}$ and all k . It follows then from (2.4), by use of the equicontinuity of $\overline{\varphi}$, that

$$\lim_{t \rightarrow \infty} \sup_{v \in [0, 1]} \|\overline{\varphi}_{r(t+v)}(a) - \overline{\varphi}_{r(t)}(a)\| = 0 \quad (2.7)$$

for all $a \in A$. (2.7) implies that $\overline{\varphi^r}$, and hence also φ^r are uniformly continuous, i.e. i) also holds. It follows from (2.5) and (2.6) that

$$\frac{r(s_{m_k+i+1}) - r(s_{m_k+i})}{s_{m_k+i+1} - s_{m_k+i}} \leq 1$$

when $i \in \{0, 1, 2, \dots, kl_k - 1\}$. It follows that there is a $K > 0$ such that $r(s_{j+1}) - r(s_j) \leq K(s_{j+1} - s_j)$ for all $j \geq 0$, proving that r is Lipschitz. Finally, it is now straightforward to check that $f' = (\overline{\varphi^r}, \{u_n\}, \{s_n\})$ is folding data for φ^r and that $\varphi_{f'}^r = \varphi_f$. \square

There is an alternative picture of the folding operation which we shall need. Let $l^2(B)$ denote the standard Hilbert B -module of 'square-summable' sequences (b_0, b_1, b_2, \dots) of elements from B . The C^* -algebra of adjoint-able operators on $l^2(B)$ can be identified with $M(B \otimes \mathbb{K})$ and the 'compact' operators on $l^2(B)$ is then identified with $B \otimes \mathbb{K}$, cf. [K2]. By using the standard matrix units $\{e_{ij}\}_{i,j=0}^{\infty}$ which

act on $l^2(B)$ in the obvious way, we can use a set of folding data $f = (\bar{\varphi}, \{u_n\}, \{t_n\})$ to define a map $\bar{\varphi}^f : A \rightarrow M(B \otimes \mathbb{K})$ such that

$$\bar{\varphi}^f(a) = \sum_{i=0}^{\infty} \sum_{j=i-1}^{i+1} \Delta_i \bar{\varphi}_{t_i}(a) \Delta_j \otimes e_{ij}.$$

The sum converges in the strict topology because $\sup_{i,j} \|\Delta_i \bar{\varphi}_{t_i}(a) \Delta_j\| < \infty$. $\bar{\varphi}^f$ is continuous by equi-continuity of $\bar{\varphi}_t, t \in [1, \infty)$, and a direct check, as in the proof of Lemma 3.5 of [MT2], shows that $\bar{\varphi}^f$ is a $*$ -homomorphism modulo $B \otimes \mathbb{K}$, i.e. $\varphi^f = q_{B \otimes \mathbb{K}} \circ \bar{\varphi}^f$ is an extension of A by $B \otimes \mathbb{K}$. In order to see the relation between φ^f and φ_f , observe that

$$V = \sum_{j=0}^{\infty} \Delta_j \otimes e_{j0}$$

is a partial isometry in $M(B \otimes \mathbb{K})$ such that

- $V(\bar{\varphi}_f(a) \otimes e_{00})V^* - \bar{\varphi}^f(a) \in B \otimes \mathbb{K}$, and
- $V^*V = 1 \otimes e_{00}$.

Since $(1 - VV^*)(l^2(B)) \oplus l^2(B)$ and $(1 - 1 \otimes e_{00})(l^2(B)) \oplus l^2(B)$ are isomorphic Hilbert B -modules by Kasparov's stabilisation theorem, cf. [K2], it follows that there is a unitary dilation U of V , acting on $l^2(B) \oplus l^2(B)$, such that

$$U \begin{pmatrix} \bar{\varphi}_f(a) \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} U^* - \begin{pmatrix} \bar{\varphi}^f(a) & 0 \\ 0 & 0 \end{pmatrix} \in M_2(B \otimes \mathbb{K}).$$

In this way we obtain the following conclusion.

Lemma 2.4. *Assume that B is stable, and identify $B \otimes \mathbb{K}$ with B .*

Then $\varphi_f \oplus 0$ is unitarily equivalent to $\varphi^f \oplus 0$.

2.3. A key lemma. Two asymptotic extensions $\varphi, \varphi' : A \rightarrow Q(B)$ are *strongly homotopic* when they define the same element in $[[A, Q(B)]]$. This means that there is an asymptotic homomorphism $\alpha : A \rightarrow C[0, 1] \otimes Q(B)$ such that $\text{ev}_0 \circ \alpha_t = \varphi_t$ and $\text{ev}_1 \circ \alpha_t = \varphi'_t$ for all t , where $\text{ev}_s : C[0, 1] \otimes Q(B) \rightarrow Q(B)$ is evaluation at $s \in [0, 1]$. In this subsection we will relate a particular folding of φ to a folding of φ' , assuming that the strong homotopy α connecting φ to φ' is uniformly continuous. By Lemma 2.1, applied with $D = C[0, 1] \otimes M(B)$ and $D_0 = C[0, 1] \otimes B$, there is an equi-continuous and uniformly continuous lift $\bar{\alpha} : A \rightarrow C[0, 1] \otimes M(B)$ of α such that $\sup_{t \in [1, \infty)} \|\bar{\alpha}_t(a)\| < \infty$ for all $a \in A$. Let $\{u_n\}$ be a unit sequence in B such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [1, n+2]} \left[\sup_{s \in [0, 1]} \|(1 - u_n) f(t)(s)\| - \|q_{C[0, 1] \otimes B}(f(t))\| \right] = 0 \quad (2.8)$$

when $f \in C_b([1, \infty), C[0, 1] \otimes M(B))$ is any of the following functions:

- $f(t) = \bar{\alpha}_t(a) \bar{\alpha}_t(b) - \bar{\alpha}_t(ab)$,
- $f(t) = \bar{\alpha}_t(a) + \lambda \bar{\alpha}_t(b) - \bar{\alpha}_t(a + \lambda b)$, or
- $f(t) = \bar{\alpha}_t(a^*) - \bar{\alpha}_t(a)^*$

for any $a, b \in A, \lambda \in \mathbb{C}$. Furthermore, we require also that

$$\lim_{n \rightarrow \infty} \sup \{ \|u_n \bar{\alpha}_t(a)(s) - \bar{\alpha}_t(a)(s) u_n\| : t \in [1, n+2], s \in [0, 1] \} = 0 \quad (2.9)$$

for all $a \in A$. That such a unit sequence exists follows from the separability of A and the equi-continuity of $\bar{\alpha}$ by use of Lemma 2.2.

Let $\{t'_j\}$ and $\{t_j\}$ be discretizations. Then $f_0 = (\text{ev}_0 \circ \bar{\alpha}, \{u_n\}, \{t'_n\})$ is folding data for $\text{ev}_0 \circ \alpha$ and $f_1 = (\text{ev}_1 \circ \bar{\alpha}, \{u_n\}, \{t_n\})$ is folding data for $\text{ev}_1 \circ \alpha$. The key lemma referred to in the title of this section is

Lemma 2.5. *In the above setting, assume that $\kappa : A \rightarrow Q(B)$ is an extension such that $(\text{ev}_0 \circ \alpha)_{f_0} \oplus \kappa$ is asymptotically split. It follows that $(\text{ev}_1 \circ \alpha)_{f_1} \oplus \kappa$ is asymptotically split.*

For the proof we need the following

Lemma 2.6. *Let $\{t'_n\}$ and $\{t_n\}$ be two discretizations and $a_1 < a_2 < \dots$ a strictly increasing sequence in \mathbb{N} .*

There is a sequence $h_0 \leq h_1 \leq h_2 \leq \dots$ of continuous functions $h_j : [1, \infty) \rightarrow [1, \infty)$ such that

- i) $h_j(t) = t'_j$, $j \leq a_k$, $t \in [k, k+1]$,*
- ii) $h_{j+1}(t) - h_j(t) \leq \max\{\frac{1}{k}, t'_{j+1} - t'_j\} \forall j, t \in [k, k+1]$,*
- ii) for all $n \in \mathbb{N}$ there is an $N_n \in \mathbb{N}$ such that $h_j(t) = t_j$ when $t \in [1, n]$, $j \geq N_n$,
and*
- iv) $h_j(t) \leq j$ for all $j \geq 1$ and all t .*

Proof. Let $k \in \mathbb{N}$. Since $\lim_{j \rightarrow \infty} t'_{j+1} - t'_j = 0$ and $\lim_{j \rightarrow \infty} t_{j+1} - t_j = 0$ there is a $b_k \geq a_k$ such that $\max\{t'_{j+1} - t'_j, t_{j+1} - t_j\} \leq \frac{1}{k}$ for all $j \geq b_k$. We arrange that $b_{k+1} > b_k$. On the interval $[k, k+1]$ we set $h_j(t) = t'_j$ when $j \leq b_k$. Since $\lim_{j \rightarrow \infty} t_{j+1} - t_j = 0$ and $\lim_{j \rightarrow \infty} t_j = \infty$ there is an $m_k > b_k$ such that $\frac{n}{k} + t'_{b_k} \geq t_n > t'_{b_k}$ for all $n \geq m_k$. We set

$$h_j(t) = \max \left\{ \min \left\{ \frac{j}{k} + t'_{b_k}, t_j \right\}, t'_{b_k} \right\}$$

when $j > b_k$ and $t \in [k, k + \frac{1}{2}]$. Then $h_j(t) = t_j$ when $j \geq m_k$. With these choices we have defined the h_j 's on all the intervals $[k, k + \frac{1}{2}]$, $k = 1, 2, 3, \dots$, but it remains to define the h_j 's on $[k + \frac{1}{2}, k+1]$ when $j > b_k$. For this note that $h_{j+1}(k + \frac{1}{2}) - h_j(k + \frac{1}{2}) \leq \frac{1}{k}$ and

$$h_{j+1}(k+1) - h_j(k+1) \leq \max \left\{ \frac{1}{k+1}, t'_{j+1} - t'_j \right\} \leq \frac{1}{k}$$

for all $j \geq b_k$. Hence by defining h_j , $j > b_k$, to be the linear function on $[k + \frac{1}{2}, k+1]$ which connects $h_j(k + \frac{1}{2})$ to $h_j(k+1)$ we have obtained what we wanted. \square

Proof of Lemma 2.5. Set $\alpha_t^s = \text{ev}_s \circ \bar{\alpha}_t$, $\Delta_0 = \sqrt{u_0}$ and $\Delta_n = \sqrt{u_n - u_{n-1}}$, $n \geq 1$. We define $\psi^0 : A \rightarrow M(B)$ such that

$$\psi^0(a) = \sum_{j=0}^{\infty} \Delta_j \alpha_{t'_j}^0(a) \Delta_j.$$

Note that ψ^0 is continuous thanks to the equi-continuity of $\text{ev}_0 \circ \bar{\alpha}_t$, $t \in [1, \infty)$, cf. Lemma 3.1 of [MT2], and that $q_B \circ \psi^0 = (\text{ev}_0 \circ \alpha)_{f_0}$. It follows from our assumption that there is an equi-continuous asymptotic homomorphism $\mu : A \rightarrow M_2(M(B))$ such that

$$\mu_t = \begin{pmatrix} \mu_t^{11} & \mu_t^{12} \\ \mu_t^{21} & \mu_t^{22} \end{pmatrix},$$

where $q_B \circ \mu_t^{11} = q_B \circ \psi^0$, $q_B \circ \mu_t^{12} = q_B \circ \mu_t^{21} = 0$ and $q_B \circ \mu_t^{22} = \kappa$ for all $t \in [1, \infty)$. It follows from Lemma 2.2 that we can choose a continuous path $v_t, t \in [1, \infty)$, in the C^* -subalgebra of B generated by the strictly positive element b such that $t \mapsto v_t$ is norm-continuous, $0 \leq v_t \leq 1$ for all t , and

- a7) $v_t \Delta_i = \Delta_i, i \leq t$,
- a8) $\lim_{t \rightarrow \infty} \|v_t \mu_t^{11}(a) - \mu_t^{11}(a) v_t\| = 0$ for all $a \in A$,
- a9) $\lim_{t \rightarrow \infty} \|(1 - v_t) \mu_t^{12}(a)\| = \lim_{t \rightarrow \infty} \|(1 - v_t) \mu_t^{21}(a)\| = 0$ for all $a \in A$,
- a10) $\lim_{t \rightarrow \infty} \|v_t \psi^0(a) - \psi^0(a) v_t\| = 0$ for all $a \in A$, and
- a11) $\lim_{t \rightarrow \infty} \|(1 - v_t) [\mu_t^{11}(a) - \psi^0(a)]\| = 0$ for all $a \in A$.

Since $\lim_{i \rightarrow \infty} v_t \Delta_i = 0$ for all t there is an increasing function $i : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\lim_{k \rightarrow \infty} \sup_{j \geq i(k)} \sup_{t \in [k, k+1]} \|v_t \Delta_j\| = 0. \quad (2.10)$$

Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ be a sequence of finite subsets with dense union in A . For each $n \in \mathbb{N}$ there is an $\epsilon_n > 0$ such that

$$\left\| \alpha_t^s(a) - \alpha_t^{s'}(a) \right\| \leq \frac{1}{n} \quad (2.11)$$

when $|s - s'| \leq \epsilon_n, t \in [1, n + 2], a \in F_n$. Choose then a sequence of continuous non-increasing functions $g_k : [1, \infty) \rightarrow [0, 1], k = 0, 1, 2, 3, \dots$, such that

- a12) for each $t \in [1, \infty), g_k(t) = 1$ for all but finitely many k ,
- a13) $g_i(t) = 0$ for all $i = 1, 2, \dots, i(k)$, when $t \geq k$,
- a14) $g_k \leq g_{k+1}$ for all k , and
- a15) $g_{k+1}(t) - g_k(t) \leq \epsilon_n$ when $t \in [1, n + 2]$, for all k, n .

Since the g_k 's are non-increasing it follows from a12) that there are numbers $a_1 < a_2 < a_3 < \dots$ in \mathbb{N} such that

$$a_k \geq i(k) \quad (2.12)$$

and

$$g_j(t) = 1, t \in [k, k + 1], j \geq a_k - 1. \quad (2.13)$$

We can then use Lemma 2.6 to obtain continuous functions $h_j : [1, \infty) \rightarrow [1, \infty), j = 0, 1, 2, 3, \dots$, such that $h_0 \leq h_1 \leq h_2 \leq \dots$ and

- a16) $h_j(t) = t'_j, j \leq a_k, t \in [k, k + 1]$,
- a17) $h_{j+1}(t) - h_j(t) \leq \max \left\{ \frac{1}{k}, t'_{j+1} - t'_j \right\} \forall j, t \in [k, k + 1]$,
- a18) for all $n \in \mathbb{N}$ there is an $N_n \in \mathbb{N}$ such that $h_j(t) = t_j$ when $t \in [1, n]$ and $j \geq N_n$, and
- a19) $h_j(t) \leq j$ for all j, t .

Now we set

$$\psi_t(a) = \sum_{j=0}^{\infty} \Delta_j \alpha_{h_j(t)}^{g_j(t)}(a) \Delta_j.$$

It follows from Lemma 3.1 of [MT2] and the equi-continuity of $\bar{\alpha}$ that $\psi_t : A \rightarrow M(B), t \in [1, \infty)$, is an equi-continuous family. Furthermore, it follows from a12) and a18) that for each $n \in \mathbb{N}$ there is an $N'_n \in \mathbb{N}$ such that

$$\psi_t(a) = \sum_{j=0}^{N'_n} \Delta_j \alpha_{h_j(t)}^{g_j(t)}(a) \Delta_j + \sum_{j=N'_n+1}^{\infty} \Delta_j \alpha_{t_j}^1(a) \Delta_j \quad (2.14)$$

for all $t \leq n$ and all $a \in A$. In particular, this shows that $q_B \circ \psi_t = (\text{ev}_1 \circ \alpha)_{f_1}$ and that $t \mapsto \psi_t(a)$ is norm-continuous.

Let

$$\lambda_t^{11}(a) = v_t \mu_t^{11}(a) v_t + (1 - v_t^2)^{\frac{1}{2}} \psi_t(a) (1 - v_t^2)^{\frac{1}{2}}$$

and set $\lambda_t^{ij}(a) = \mu_t^{ij}(a)$ for $(i, j) \neq (1, 1)$. Finally, set

$$\Lambda_t(a) = \begin{pmatrix} \lambda_t^{11}(a) & \lambda_t^{12}(a) \\ \lambda_t^{21}(a) & \lambda_t^{22}(a) \end{pmatrix}.$$

Then $\Lambda_t : A \rightarrow M_2(M(B))$, $t \in [1, \infty)$, is an equi-continuous family of maps and since $q_B \circ \lambda_t^{11} = (\text{ev}_1 \circ \alpha)_{f_1}$, $q_B \circ \lambda_t^{12} = q_B \circ \lambda_t^{21} = 0$ and $q_B \circ \lambda_t^{22} = \kappa$, it suffices to show that $(\Lambda_t)_{t \in [1, \infty)}$ is an asymptotic homomorphism.

Each μ^{ij} is asymptotically linear and $\mu_t^{ij}(a^*)$ agrees asymptotically with $\mu_t^{ji}(a)^*$ since μ is an asymptotic homomorphism. Furthermore, it follows from a7) and (2.8) that $a \mapsto (1 - v_t^2)^{\frac{1}{2}} \psi_t(a) (1 - v_t^2)^{\frac{1}{2}}$ is asymptotically linear and asymptotically commutes with the involution as t tends to infinity. It is then clear that the same is true for λ^{11} and hence for Λ .

We check that Λ is asymptotically multiplicative. For this we write $A(t) \sim B(t)$ between t -dependent elements from $M(B)$ when $\lim_{t \rightarrow \infty} \|A(t) - B(t)\| = 0$. Let $a, b \in A$. It suffices to show that

$$\lambda_t^{11}(a) \lambda_t^{12}(b) + \lambda_t^{12}(a) \lambda_t^{22}(b) \sim \lambda_t^{12}(ab) \quad (2.15)$$

and

$$\lambda_t^{11}(a) \lambda_t^{11}(b) + \lambda_t^{12}(a) \lambda_t^{21}(b) \sim \lambda_t^{11}(ab). \quad (2.16)$$

To handle (2.15) observe that $v_t \mu_t^{11}(a) v_t \mu_t^{12}(b) \sim \mu_t^{11}(a) \mu_t^{12}(b)$ by a8) and a9), and that $(1 - v_t^2)^{\frac{1}{2}} \psi_t(a) (1 - v_t^2)^{\frac{1}{2}} \mu_t^{12}(b) \sim 0$ by a9). Since μ is an asymptotic homomorphism we have also that $\mu_t^{11}(a) \mu_t^{12}(b) + \mu_t^{12}(a) \mu_t^{22}(b) \sim \mu_t^{12}(ab) = \lambda_t^{12}(ab)$. (2.15) follows from this.

It remains to verify (2.16). For this we prove first that

$$v_t (\psi_t(c) - \psi^0(c)) \sim (\psi_t(c) - \psi^0(c)) v_t \sim 0 \quad (2.17)$$

for all $c \in A$. To establish (2.17) observe that the following estimate is valid when $t \geq k$:

$$\begin{aligned} \|v_t (\psi_t(c) - \psi^0(c))\|^2 &= \left\| v_t \left(\sum_{j>i(k)} \Delta_j \left(\alpha_{h_j(t)}^{g_j(t)}(c) - \alpha_{t_j}^0(c) \right) \Delta_j \right) \right\|^2 \\ &\quad \text{(by a13), (2.12) and a16))} \\ &= \left\| \sum_{j>i(k)} \sum_{l=j-1}^{j+1} \Delta_j \left(\alpha_{h_j(t)}^{g_j(t)}(c) - \alpha_{t_j}^0(c) \right)^* \Delta_j v_t^2 \Delta_l \left(\alpha_{h_l(t)}^{g_l(t)}(c) - \alpha_{t_l}^0(c) \right) \Delta_l \right\|^2 \\ &\quad \text{(using a1))} \end{aligned}$$

$$\leq 12 \sup_{s \in [1, \infty)} \|\bar{\alpha}_s(c)\|^2 \sup_{j \geq i(k)} \|\Delta_j v_t\| \quad \text{(by Lemma 3.1 of [MT2]).}$$

In this way it follows from (2.10) that $v_t (\psi_t(c) - \psi^0(c)) \sim 0$. The same arguments work to show that also $(\psi_t(c) - \psi^0(c)) v_t \sim 0$, giving us (2.17).

Note that (2.17) combined with a10) implies that

$$[v_t, \psi_t(c)] \sim 0 \quad (2.18)$$

and combined with a11) that

$$v_t(1 - v_t)\psi_t(c) \sim v_t(1 - v_t)\mu_t^{11}(c) \quad (2.19)$$

for all $c \in A$. Using (2.18), (2.19) and a8) we find that

$$\begin{aligned} & \lambda_t^{11}(a)\lambda_t^{11}(b) \\ & \sim v_t^4\mu_t^{11}(a)\mu_t^{11}(b) + 2v_t^2(1 - v_t^2)\mu_t^{11}(a)\mu_t^{11}(b) + (1 - v_t^2)^2\psi_t(a)\psi_t(b) \\ & = (2v_t^2 - v_t^4)\mu_t^{11}(a)\mu_t^{11}(b) + (1 - v_t^2)^2\psi_t(a)\psi_t(b). \end{aligned} \quad (2.20)$$

To continue we show next that

$$(1 - v_t)\psi_t(a)\psi_t(b) \sim (1 - v_t)\psi_t(ab) \quad (2.21)$$

for all $a, b \in A$. To this end let $t \in [k, k + 1]$ and $a, b \in F_k$. By using Lemma 3.1 from [MT2] several times we find that

$$\begin{aligned} & (1 - v_t)\psi_t(a)\psi_t(b) \\ & = (1 - v_t) \sum_{j>t} \left[\sum_{l=j-1}^{j+1} \Delta_j \alpha_{h_j(t)}^{g_j(t)}(a) \Delta_j \Delta_l \alpha_{h_l(t)}^{g_l(t)}(b) \Delta_l \right] \\ & \hspace{25em} \text{(using a7))} \\ & = (1 - v_t) \sum_{t < j \leq a_k} \left[\sum_{l=j-1}^{j+1} \Delta_j \alpha_{h_j(t)}^{g_j(t)}(a) \Delta_j \Delta_l \alpha_{h_l(t)}^{g_l(t)}(b) \Delta_l \right] \\ & \quad + (1 - v_t) \sum_{j>a_k} \left[\sum_{l=j-1}^{j+1} \Delta_j \alpha_{h_j(t)}^1(a) \Delta_j \Delta_l \alpha_{h_l(t)}^1(b) \Delta_l \right] \\ & \hspace{25em} \text{(using (2.13))} \\ & \sim (1 - v_t) \sum_{t < j \leq a_k} \left[\sum_{l=j-1}^{j+1} \Delta_j \alpha_{h_j(t)}^{g_j(t)}(a) \Delta_j \Delta_l \alpha_{h_l(t)}^{g_l(t)}(b) \Delta_l \right] \\ & \quad + (1 - v_t) \sum_{j>a_k} \left[\sum_{l=j-1}^{j+1} \Delta_j \alpha_{h_j(t)}^1(a) \Delta_j \Delta_l \alpha_{h_l(t)}^1(b) \Delta_l \right] \\ & \hspace{25em} \text{(using (2.11), a19) and a15) } \end{aligned}$$

$$\begin{aligned}
& \sim (1 - v_t) \sum_{t < j \leq a_k} \left[\sum_{l=j-1}^{j+1} \Delta_j \alpha_{h_j(t)}^{g_j(t)}(a) \Delta_j \Delta_l \alpha_{h_j(t)}^{g_j(t)}(b) \Delta_l \right] \\
& \quad + (1 - v_t) \sum_{j > a_k} \sum_{l=j-1}^{j+1} \Delta_j \alpha_{h_j(t)}^1(a) \Delta_j \Delta_l \alpha_{h_j(t)}^1(b) \Delta_l \\
& \hspace{15em} \text{(using a17) and the uniform continuity of } \bar{\alpha} \text{)} \\
& \sim (1 - v_t) \sum_{t < j \leq a_k} \Delta_j \alpha_{h_j(t)}^{g_j(t)}(a) \alpha_{h_j(t)}^{g_j(t)}(b) \Delta_j + (1 - v_t) \sum_{j > a_k} \Delta_j \alpha_{h_j(t)}^1(a) \alpha_{h_j(t)}^1(b) \Delta_j \\
& \hspace{15em} \text{(using (2.9), a19) and a3)} \\
& \sim (1 - v_t) \sum_{t < j \leq a_k} \Delta_j \alpha_{h_j(t)}^{g_j(t)}(ab) \Delta_j + (1 - v_t) \sum_{j > a_k} \Delta_j \alpha_{h_j(t)}^1(ab) \Delta_j \quad \text{(using a7)} \\
& = (1 - v_t) \psi_t(ab) \quad \text{(using (2.3)),}
\end{aligned}$$

giving us (2.21). Inserting (2.21) into (2.20) we find that

$$\begin{aligned}
\lambda_t^{11}(a) \lambda_t^{11}(b) & \sim (2v_t^2 - v_t^4) \mu_t^{11}(a) \mu_t^{11}(b) + (1 - v_t^2)^2 \psi_t(ab) \\
& \sim (2v_t^2 - v_t^4) (\mu_t^{11}(ab) - \mu_t^{12}(a) \mu_t^{21}(b)) + (1 - v_t^2)^2 \psi_t(ab) \\
& \hspace{15em} \text{(since } \mu \text{ is an asymptotic homomorphism)} \\
& \sim (2v_t^2 - v_t^4) \mu_t^{11}(ab) - \mu_t^{12}(a) \mu_t^{21}(b) + (1 - v_t^2)^2 \psi_t(ab) \\
& \hspace{15em} \text{(using a9)} \\
& = v_t^2 \mu_t^{11}(ab) + (v_t^2 - v_t^4) \mu_t^{11}(ab) - \mu_t^{12}(a) \mu_t^{21}(b) + (1 - v_t^2)^2 \psi_t(ab) \\
& \sim v_t^2 \mu_t^{11}(ab) + (v_t^2 - v_t^4) \psi_t(ab) - \mu_t^{12}(a) \mu_t^{21}(b) + (1 - v_t^2)^2 \psi_t(ab) \\
& \hspace{15em} \text{(using (2.19))} \\
& = v_t^2 \mu_t^{11}(ab) + (1 - v_t^2) \psi_t(ab) - \mu_t^{12}(a) \mu_t^{21}(b) \\
& \sim v_t \mu_t^{11}(ab) v_t + (1 - v_t^2)^{\frac{1}{2}} \psi_t(ab) (1 - v_t^2)^{\frac{1}{2}} - \mu_t^{12}(a) \mu_t^{21}(b) \\
& \hspace{15em} \text{(using a8) and (2.18)),}
\end{aligned}$$

which gives us (2.16). \square

2.4. On the dependence of the folding on the folding data. We can now show that if a folding φ_f of φ is semi-invertible then the same is true for any other folding of φ and that the unitary equivalence class of φ_f , modulo asymptotically split extensions does not depend on the folding data. The main step is the following lemma.

Lemma 2.7. *Let A and B be C^* -algebras, A separable, B stable and σ -unital. Let $\varphi : A \rightarrow Q(B)$ be an asymptotic extension, and $f = (\bar{\varphi}, \{u_n\}, \{t_n\})$, $f' = (\hat{\varphi}, \{u'_n\}, \{t'_n\})$ two sets of folding data for φ . Assume that $\psi : A \rightarrow Q(B)$ is an extension such that $\varphi_f \oplus \psi \oplus 0$ is asymptotically split.*

It follows that $\varphi_{f'} \oplus \psi \oplus 0$ is asymptotically split.

Proof. Thanks to Lemma 2.3 we may assume that φ , $\bar{\varphi}$ and $\hat{\varphi}$ are all uniformly continuous. Since $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} t'_n = \infty$ we can use Lemma 2.2 recursively

to obtain a unit sequence $\{u_n''\}$ and increasing functions $m, m' : \mathbb{N} \rightarrow \mathbb{N}$ such that $m(0) = m'(0) = 0$,

- a20) $t_{m(n)} \geq n$,
- a21) $u_n'' u_{m(n)} = u_{m(n)}''$,
- a22) $u_{m(n)}'' u_{n-1}'' = u_{n-1}''$ for all $n \geq 1$, and
- a23) $t_n - t_{n-1} \leq t_{m(n)} - t_{m(n-1)}$, $n \geq 1$,

and also

- a24) $t'_{m'(n)} \geq n$,
- a25) $u_n'' u'_{m'(n)} = u'_{m'(n)}''$,
- a26) $u'_{m'(n)}'' u_{n-1}'' = u_{n-1}''$, $n \geq 1$, and
- a27) $t'_n - t'_{n-1} \leq t'_{m'(n)} - t'_{m'(n-1)}$, $n \geq 1$.

In addition we arrange that $g = (\bar{\varphi}, \{u_n''\}, \{t_n\})$ is folding data for φ and that

$$\lim_{n \rightarrow \infty} \sup_{1 \leq t \leq n+3} \|(1 - u_n'')(\bar{\varphi}_t(a) - \hat{\varphi}_t(a))\| = 0 \quad (2.22)$$

for all $a \in A$. Then also $g' = (\bar{\varphi}, \{u_n''\}, \{t'_n\})$ is folding data for φ . We claim that there are Lipschitz re-parametrisations $r, r' : [1, \infty) \rightarrow [1, \infty)$ such that

- a28) $r(n) \leq n$ for all $n \in \mathbb{N}$ and
- a29) $\varphi_g \oplus 0$ is unitarily equivalent to $\varphi_h \oplus 0$, where $h = (\bar{\varphi}, \{u_n\}, \{r(t_n)\})$,

and

- a30) $r'(n) \leq n$ for all $n \in \mathbb{N}$ and
- a31) $\varphi_{g'} \oplus 0$ is unitarily equivalent to $\varphi_{h'} \oplus 0$, where $h' = (\hat{\varphi}, \{u_n'\}, \{r'(t'_n)\})$.

The construction of r and r' are almost identical, but slightly more demanding for r' since we pass from $\bar{\varphi}$ to $\hat{\varphi}$. We describe therefore only the construction of r' .

Define $r' : [1, \infty) \rightarrow [1, \infty)$ to be the continuous function such that r' is linear on $[t'_{m(n-1)}, t'_{m(n)}]$, $r'(t'_{m(n-1)}) = t'_{n-1}$ and $r'(t'_{m(n)}) = t'_n$. Then r' is Lipschitz thanks to a27) and $\lim_{t \rightarrow \infty} r'(t) = \infty$ since $\lim_{n \rightarrow \infty} t'_n = \infty$. Using a24) we find that

$$r'^{-1}([1, n]) \supseteq r'^{-1}([1, t'_n]) \supseteq [1, t'_{m(n)}] \supseteq [1, n].$$

It follows that $r'(n) \leq n$ for all n . Furthermore, it is straightforward to combine (2.22) with the fact that g is folding data for φ to verify that the same is true for g' . (Recall that $\hat{\varphi}$ is uniformly continuous.) It remains now only to show that $\varphi_{g'} \oplus 0$ is unitarily equivalent to $\varphi_{h'} \oplus 0$. For this purpose note first that by Lemma 2.4 we may as well show that $\varphi^{g'} \oplus 0$ is unitarily equivalent to $\varphi^{h'} \oplus 0$. This is done as follows.

Set $\Delta'_0 = \sqrt{u'_0}$, $\Delta'_n = \sqrt{u'_n - u'_{n-1}}$, $n \geq 1$, $\Delta''_0 = \sqrt{u''_0}$, $\Delta''_n = \sqrt{u''_n - u''_{n-1}}$. It follows from a25) and a26) that

$$m'(n-1) < k \leq m'(n) \Rightarrow \Delta'_k \Delta''_j = 0, \quad j \notin \{n-1, n\}. \quad (2.23)$$

Set $V = \sum_{i,j} \Delta''_i \Delta'_j \otimes e_{ij}$ which is a partial isometry in $M(B \otimes \mathbb{K})$. By using (2.23) and Lemma 3.1 of [MT2] we find that

$$\begin{aligned} V \widehat{\varphi}^{h'}(a) V^* &= \sum_{i,j} \sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \widehat{\varphi}_{r'(t'_k)}(a) \Delta'_l{}^2 \Delta''_j \otimes e_{ij} \\ &= \sum_{i=0}^{\infty} \sum_{j=i-3}^{i+3} \sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \widehat{\varphi}_{r'(t'_k)}(a) \Delta'_l{}^2 \Delta''_j \otimes e_{ij} \\ &= \sum_{i=0}^{\infty} \sum_{j=i-3}^{i+3} \sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \widehat{\varphi}_{t'_i}(a) \Delta'_l{}^2 \Delta''_j \otimes e_{ij} \quad \text{modulo } B \otimes \mathbb{K}. \end{aligned}$$

It follows from (2.22) that

$$\begin{aligned} &\sum_{i=0}^{\infty} \sum_{j=i-3}^{i+3} \sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \widehat{\varphi}_{t'_i}(a) \Delta'_l{}^2 \Delta''_j \otimes e_{ij} \\ &= \sum_{i=0}^{\infty} \sum_{j=i-3}^{i+3} \sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \overline{\varphi}_{t'_i}(a) \Delta'_l{}^2 \Delta''_j \otimes e_{ij} \quad \text{modulo } B \otimes \mathbb{K}. \end{aligned}$$

Note that

$$\sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \overline{\varphi}_{t'_i}(a) \Delta'_l{}^2 = \sum_{k \geq m'(i-1)} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \overline{\varphi}_{t'_i}(a) \Delta'_l{}^2,$$

thanks to (2.23). Since

$$\begin{aligned} &\left\| \sum_{k \geq m'(i-1)} \sum_{l=k-1}^{k+1} \Delta'_k{}^2 \left[\overline{\varphi}_{t'_i}(a), \Delta'_l{}^2 \right] \right\| \\ &\leq \left\| \sum_{k \geq m'(i-1)} \Delta'_k{}^2 \right\|^{\frac{1}{2}} \left\| \sum_{k \geq m'(i-1)} \Delta'_k \left[\overline{\varphi}_{t'_i}(a), \sum_{l=k-1}^{k+1} \Delta'_l{}^2 \right] \left[\overline{\varphi}_{t'_i}(a), \sum_{l=k-1}^{k+1} \Delta'_l{}^2 \right]^* \Delta'_k \right\|^{\frac{1}{2}} \\ &\leq \left\| \sum_{k \geq m'(i-1)} \Delta'_k \left[\overline{\varphi}_{t'_i}(a), \sum_{l=k-1}^{k+1} \Delta'_l{}^2 \right] \left[\overline{\varphi}_{t'_i}(a), \sum_{l=k-1}^{k+1} \Delta'_l{}^2 \right]^* \Delta'_k \right\|^{\frac{1}{2}} \end{aligned}$$

it follows from the compatibility of $(\{u'_n\}, \{t'_n\})$ with $\overline{\varphi}$ and Lemma 3.1 of [MT2] that

$$\begin{aligned} &\sum_{i=0}^{\infty} \sum_{j=i-3}^{i+3} \sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \overline{\varphi}_{t'_i}(a) \Delta'_l{}^2 \Delta''_j \otimes e_{ij} \\ &= \sum_{i=0}^{\infty} \sum_{j=i-2}^{i+3} \sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} \Delta''_i \Delta'_k{}^2 \Delta'_l{}^2 \overline{\varphi}_{t'_i}(a) \Delta''_j \otimes e_{ij} \quad \text{modulo } B \otimes \mathbb{K} \\ &= \sum_{i=0}^{\infty} \sum_{j=i-2}^{i+3} \Delta''_i \overline{\varphi}_{t'_i}(a) \Delta''_j \otimes e_{ij} \quad (\text{using a1) - a3) for } \{u'_n\}) \\ &= \overline{\varphi}^{g'}(a). \end{aligned}$$

Since $V^*V\widehat{\varphi}^{h'}(a) = \widehat{\varphi}^{h'}(a)V^*V = \widehat{\varphi}^{h'}(a)$ modulo $B \otimes \mathbb{K}$ it follows that $V^*\widehat{\varphi}^{g'}(a)V = \widehat{\varphi}^{h'}(a)$ modulo $B \otimes \mathbb{K}$. Thus an application of Kasparov's stabilisation theorem as in the proof of Lemma 2.4 shows that V can be dilated to give a unitary equivalence between $\varphi^{g'} \oplus 0$ and $\varphi^{h'} \oplus 0$. Hence a31) follows from Lemma 2.4.

Using a28) - a31) we can now complete the proof as follows: By assumption $\varphi_f \oplus \psi \oplus 0$ is asymptotically split. Since f and h only differ in the discretization sequences, it follows from Lemma 2.5 that $\varphi_h \oplus \psi \oplus 0$ is asymptotically split. Then (2.4) implies that so is $\varphi_g \oplus \psi \oplus 0$. g and g' differ also only in the discretization sequence so another application of Lemma 2.5 shows that also $\varphi_{g'} \oplus \psi \oplus 0$ is asymptotically split. Hence a31) shows that $\varphi_{h'} \oplus \psi \oplus 0$ is asymptotically split and a final application of Lemma 2.5 implies then that the same is true for $\varphi_{f'} \oplus \psi \oplus 0$. \square

We can now combine Lemma 2.7 with Lemma 2.5 to obtain the following:

Proposition 2.8. *Let A and B be C^* -algebras, A separable, B stable and σ -unital. Let $\varphi, \varphi' : A \rightarrow Q(B)$ be asymptotic extensions which are strongly homotopic. Let f and f' be any folding data for φ and φ' , respectively. Assume that $\psi : A \rightarrow Q(B)$ is an extension such that $\varphi_f \oplus \psi \oplus 0$ is asymptotically split.*

It follows that $\varphi'_{f'} \oplus \psi \oplus 0$ is asymptotically split.

Proof. Let $f = (\overline{\varphi}, \{u_n\}, \{t_n\})$ be folding data for φ and $f' = (\overline{\varphi'}, \{u'_n\}, \{t'_n\})$ for φ' . It follows from Lemma 2.3 that we can assume that φ and φ' are uniformly continuous. Let $\alpha : A \rightarrow C[0, 1] \otimes Q(B)$ be a strong homotopy connecting φ and φ' . By Lemma 4.3 of [MT2] (or Lemma 2.3 above) there is a Lipschitz re-parametrisation $r : [1, \infty) \rightarrow [1, \infty)$ such that α^r is uniformly continuous. Since r is Lipschitz there are strong homotopies consisting of uniformly continuous asymptotic homomorphisms $A \rightarrow C[0, 1] \otimes Q(B)$ connecting φ to φ^r and φ' to φ'^r . Concatenation with α^r gives us a strong homotopy which connects φ to φ' and consists of a uniformly continuous asymptotic homomorphism $A \rightarrow C[0, 1] \otimes Q(B)$. The desired conclusion follows then by combining Lemma 2.7 with Lemma 2.5. \square

2.5. The pairing. To obtain the desired pairing between extensions and asymptotic homomorphisms we must review the composition product of asymptotic homomorphisms, as defined by Connes and Higson in [CH], in a form suitable for the present purpose.

Lemma 2.9. *Let A, A' and D be C^* -algebras, A, A' separable. Let $\varphi : A \rightarrow D$ and $\lambda : A' \rightarrow A$ be equi-continuous asymptotic homomorphisms. Let $X \subseteq A'$ be a σ -compact subset with dense span in A' .*

There is a re-parametrisation $s : [1, \infty) \rightarrow [1, \infty)$ and an equi-continuous family of maps $\kappa_{t,x} : A' \rightarrow D, t, x \in [1, \infty), t \geq x$, such that

- i) $\lim_{x \rightarrow \infty} \sup_{t \geq x} \|\varphi_t \circ \lambda_{s(x)}(a) - \kappa_{t,x}(a)\| = 0$ for all $a \in X$, and*
- ii) $\lim_{x \rightarrow \infty} \sup_{t \geq x} \|\kappa_{t,x}(a)\kappa_{t,x}(b) - \kappa_{t,x}(ab)\| = 0$,*
- iii) $\lim_{x \rightarrow \infty} \sup_{t \geq x} \|\kappa_{t,x}(a) + z\kappa_{t,x}(b) - \kappa_{t,x}(a + zb)\| = 0$,*
- iv) $\lim_{x \rightarrow \infty} \sup_{t \geq x} \|\kappa_{t,x}(a^*) - \kappa_{t,x}(a)^*\| = 0$*
- v) $\sup_{t,x} \|\kappa_{t,x}(a)\| < \infty$*

for all $a, b \in A'$ and all $z \in \mathbb{C}$.

Proof. By the method used to define the composition product of φ and λ in [CH] we get a re-parametrisation $r : [1, \infty) \rightarrow [1, \infty)$ such that $r(1) = 1$ and

- a32) $\limsup_{t \rightarrow \infty} \sup_{y \geq r(t)} \|\varphi_y \circ \lambda_t(a) - \varphi_y \circ \lambda_t(b)\| \leq \|a - b\|,$
a33) $\lim_{t \rightarrow \infty} \sup_{y \geq r(t)} \|\varphi_y \circ \lambda_t(a) \varphi_y \circ \lambda_t(b) - \varphi_y \circ \lambda_t(ab)\| = 0,$
a34) $\lim_{t \rightarrow \infty} \sup_{y \geq r(t)} \|\varphi_y \circ \lambda_t(a) + z \varphi_y \circ \lambda_t(b) - \varphi_y \circ \lambda_t(a + zb)\| = 0,$
a35) $\lim_{t \rightarrow \infty} \sup_{y \geq r(t)} \|\varphi_y \circ \lambda_t(a^*) - \varphi_y \circ \lambda_t(a)^*\| = 0$

for all $a, b \in X$ and all $z \in \mathbb{C}$. Then $s = r^{-1} : [1, \infty) \rightarrow [1, \infty)$ is a re-parametrisation such that

- a36) $\limsup_{x \rightarrow \infty} \sup_{t \geq x} \|\varphi_t \circ \lambda_{s(x)}(a) - \varphi_t \circ \lambda_{s(x)}(b)\| \leq \|a - b\|,$
a37) $\lim_{x \rightarrow \infty} \sup_{t \geq x} \|\varphi_t \circ \lambda_{s(x)}(a) \varphi_t \circ \lambda_{s(x)}(b) - \varphi_t \circ \lambda_{s(x)}(ab)\| = 0,$
a38) $\lim_{x \rightarrow \infty} \sup_{t \geq x} \|\varphi_t \circ \lambda_{s(x)}(a) + z \varphi_t \circ \lambda_{s(x)}(b) - \varphi_t \circ \lambda_{s(x)}(a + zb)\| = 0,$
a39) $\lim_{x \rightarrow \infty} \sup_{t \geq x} \|\varphi_t \circ \lambda_{s(x)}(a^*) - \varphi_t \circ \lambda_{s(x)}(a)^*\| = 0$

for all $a, b \in X$ and all $z \in \mathbb{C}$. Set $Z = \{(t, x) \in [1, \infty)^2 : t \geq x\}$, $\mathcal{A} = C_b(Z, D)$ and

$$\mathcal{J} = \left\{ f \in \mathcal{A} : \lim_{x \rightarrow \infty} \sup_{t \geq x} \|f(t, x)\| = 0 \right\}.$$

Then \mathcal{J} is an ideal in \mathcal{A} and we let $q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$ be the quotient map. It follows from a36)-a39) that there is a $*$ -homomorphism $\Phi : A' \rightarrow \mathcal{A}/\mathcal{J}$ such that $\Phi(a) = q(f_a)$ for each $a \in X$, where $f_a \in \mathcal{A}$ is defined such that $f_a(t, x) = \varphi_t \circ \lambda_{s(x)}(a)$. By the Bartle-Graves selection theorem there is a continuous right-inverse $S : \mathcal{A}/\mathcal{J} \rightarrow \mathcal{A}$ for q . Set $\kappa_{t,x}(a) = S \circ \Phi(a)(t, x)$. Then i)-iv) hold. \square

With the re-parametrisation s from Lemma 2.9 at hand we can now introduce the composition product $\bullet : [[A, D]] \times [[A', A]] \rightarrow [[A', D]]$ of Connes and Higson, [CH], such that

$$[\varphi] \bullet [\lambda] = [\Phi],$$

where $\Phi : A' \rightarrow D$ is any equi-continuous asymptotic homomorphism with the property that

$$\lim_{t \rightarrow \infty} \Phi_t(a) - \kappa_{t,s'(t)}(a) = 0$$

for all $a \in X$ and s' is any re-parametrisation for which $s' \leq s$.

Lemma 2.10. *Let A, A' be separable C^* -algebra, B stable and σ -unital. Let $\varphi : A \rightarrow Q(B)$ be a semi-invertible extension and $\lambda : A' \rightarrow A$ an asymptotic homomorphism.*

It follows that any folding $(\varphi \circ \lambda)_f$ of the asymptotic extension $\varphi \circ \lambda$ is semi-invertible.

Proof. Let $\varphi' : A \rightarrow Q(B)$ be an extension such that $\varphi \oplus \varphi'$ is asymptotically split. By considering the composition product between λ and an asymptotic lift of $\varphi \oplus \varphi'$ it follows that there is a re-parametrisation s such that $(\varphi \circ \lambda^s) \oplus (\varphi' \circ \lambda^s)$ is an asymptotic extension which is asymptotically split in the sense of [MT2]. It follows therefore from Lemma 4.4 of [MT2] that $(\varphi \circ \lambda^s)_f$ is semi-invertible for any folding $(\varphi \circ \lambda^s)_f$ of $\varphi \circ \lambda^s$. Since $\varphi \circ \lambda^s$ is strongly homotopic to $\varphi \circ \lambda$, it follows from Proposition 2.8 that the same is true for any folding of $\varphi \circ \lambda$. \square

As in [MT3],[MT4] and [MT5] we denote $\text{Ext}^{-\frac{1}{2}}(A, B)$ the group of semi-invertible extensions of A by B modulo unitary equivalence and addition by asymptotically split extensions. By combining Proposition 2.8 and Lemma 2.9 with Lemma 4.5 of [MT2] we get the desired pairing:

Theorem 2.11. *Let A, A' be separable C^* -algebra, B stable and σ -unital. There is map \star*

$$\star : \text{Ext}^{-\frac{1}{2}}(A, B) \times [[A', A]] \rightarrow \text{Ext}^{-\frac{1}{2}}(A', B)$$

such that $[\varphi] \star [\lambda] = [(\varphi \circ \lambda)_f]$ where $(\varphi \circ \lambda)_f$ is an arbitrary folding of the asymptotic extension $\varphi \circ \lambda$.

In the construction of the pairing \star we have fixed the strictly positive element $b \in B$ which we used to define the unit sequences. It follows from Lemma 4.4 of [MT2] that \star is independent of the choice of b . The freedom in the choice of strictly positive element makes it easy to show that

$$([\varphi] + [\psi]) \star [\lambda] = ([\varphi] \star [\lambda]) + ([\psi] \star [\lambda]). \quad (2.24)$$

As should be expected it is somewhat more tricky to establish the natural associativity involving \star and the composition product \bullet for asymptotic homomorphisms.

Lemma 2.12. *Let A', A, B be C^* -algebras, A', A separable and B stable and σ -unital. Let $\varphi : A \rightarrow Q(B)$ be an asymptotic extension and $\lambda : A' \rightarrow A$ an asymptotic homomorphism. There is a folding φ_f of φ , a re-parametrisation s , a folding $(\varphi_f \circ \lambda^s)_{f'}$ of $\varphi_f \circ \lambda^s$ and an asymptotic extension $\mu : A' \rightarrow Q(B)$ such that*

- i) $[\mu] = [\varphi] \bullet [\lambda]$ in $[[A', Q(B)]]$ where \bullet denotes the composition product $\bullet : [[A', A]] \times [[A, Q(B)]] \rightarrow [[A', Q(B)]]$, and*
- ii) $\mu_{f''} = (\varphi_f \circ \lambda^s)_{f'}$ for some folding $\mu_{f''}$ of μ .*

Proof. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ be a sequence of finite subsets with dense union in A' and set $X = \bigcup_n F_n$. By construction of the composition product there is a parametrisation s and an equi-continuous family of maps $\kappa_{t,x} : A \rightarrow Q(B)$, $t, x \in [1, \infty)$, $t \geq x$, such that i)-v) of Lemma 2.9 hold. The composition product $[\varphi] \bullet [\lambda]$ is then represented by any asymptotic extension μ with the property that $\lim_{t \rightarrow \infty} \mu_t(a) - \varphi_t \circ \lambda_{s(r(t))}(a) = 0$ for all $a \in X$, where r can be any re-parametrisation such that $r(t) \leq t$ for all t .

From the Bartle-Graves selection theorem we get an equi-continuous family of maps $\bar{\kappa}_{t,x} : A \rightarrow M(B)$, $t, x \in [1, \infty)$, $t \geq x$, such that $q_B \circ \bar{\kappa}_{t,x} = \kappa_{t,x}$ for all t, x . Let $f = (\bar{\varphi}, \{u_n\}, \{t_n\})$ be folding data for φ which have a series of additional properties which we now describe. For each $k \in \mathbb{N}$ there is a $\delta_k > 0$ such that

$$\max\{|s - s'|, |t - t'|\} \leq \delta_k \Rightarrow \|\bar{\kappa}_{t,s}(a) - \bar{\kappa}_{t',s'}(a)\| \leq \frac{1}{k} \quad (2.25)$$

when $s, s', t, t' \in [1, k+2]$ and $a \in F_k$. We shall require of the discretization $\{t_n\}$ that

$$|t_i - t_{i+1}| \leq \delta_k \quad (2.26)$$

when $t_i \leq k$. Concerning the unit sequence $\{u_n\}$ we will require that

$$\lim_{n \rightarrow \infty} \sup_{t,x \in [1, n+2]} [|(1 - u_n)f(t, x)| - \|q_B(f(t, x))\|] = 0 \quad (2.27)$$

when f is any of the $M(B)$ -valued functions

- $\cdot f(t, x) = \bar{\kappa}_{t,x}(a)\bar{\kappa}_{t,x}(b) - \bar{\kappa}_{t,x}(ab),$
- $\cdot f(t, x) = \bar{\kappa}_{t,x}(a^*) - \bar{\kappa}_{t,x}(a)^*,$
- $\cdot f(t, x) = \bar{\kappa}_{t,x}(a + \lambda b) - \bar{\kappa}_{t,x}(a) - \lambda\bar{\kappa}_{t,x}(b)$

for any $a, b \in A'$, $\lambda \in \mathbb{C}$, or

$$\cdot f(t, x) = \bar{\varphi}_t \circ \lambda_{s(x)}(a) - \bar{\kappa}_{t,x}(a)$$

for any $a \in X$, and also that

$$\lim_{n \rightarrow \infty} \sup_{t,x \in [1, n+2]} \|u_n \bar{\kappa}_{t,x}(a) - \bar{\kappa}_{t,x}(a) u_n\| = 0 \quad (2.28)$$

for all $a \in A'$.

We choose next a discretization $\{t'_n\}$ of $[1, \infty)$ such that

$$\lim_{n \rightarrow \infty} \sup_{v \in [1, \infty)} \sup_{t \in [t'_n, t'_{n+1}]} \|\bar{\varphi}_v \circ \lambda_{s(t)}(a) - \bar{\varphi}_v \circ \lambda_{s(t'_n)}(a)\| = 0 \quad (2.29)$$

for all $a \in X$. This is done as follows: Let $n \geq 2$. By compactness of

$$\{\lambda_{s(t)}(a) : t \in [1, n+2], a \in F_n\}$$

and equi-continuity of $\bar{\varphi}$ there is a $\delta > 0$ such that $\sup_{v \in [1, \infty)} \|\bar{\varphi}_v(y) - \bar{\varphi}_v(z)\| \leq \frac{1}{n}$ when $z, y \in \{\lambda_{s(t)}(a) : t \in [1, n+2], a \in F_n\}$ and $\|y - z\| \leq \delta$. Choose then a $\delta' \in]0, 1]$ such that $\|\lambda_{s(t)}(a) - \lambda_{s(t')}(a)\| \leq \delta$ when $t, t' \in [1, n+1], a \in F_n$ and $|t - t'| \leq \delta'$. Arrange that $|t'_{i+1} - t'_i| \leq \delta'$ when $t_i \in [n, n+1]$. Then (2.29) holds.

Subsequently we choose folding data $f' = (\bar{\psi}, \{u'_n\}, \{t'_n\})$ for $\varphi_f \circ \lambda^s$ with the additional properties that

$$\lim_{n \rightarrow \infty} \sup_{t \in [1, n+2]} \left\| \Delta'_{n+j} \left[\sum_{k=0}^{\infty} \Delta_k \bar{\varphi}_{t_k} \circ \lambda_{s(t)}(a) \Delta_k - \bar{\psi}_t(a) \right] \right\| = 0 \quad (2.30)$$

and

$$\sum_{n=1}^{\infty} \sup_{t \in [1, n+2]} \left\| \Delta'_{n+j} \left[\sum_{k=0}^{\infty} \Delta_k \bar{\varphi}_{t_k} \circ \lambda_{s(t)}(a) \Delta_k \right] - \left[\sum_{k=0}^{\infty} \Delta_k \bar{\varphi}_{t_k} \circ \lambda_{s(t)}(a) \Delta_k \right] \Delta'_{n+j} \right\| < \infty \quad (2.31)$$

for all $j \in \{-1, 0, 1\}, a \in X$. It follows from (2.30) and (2.31) that

$$(\varphi_f \circ \lambda)_{f'}(a) = q_B \left(\sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} \Delta_j \bar{\varphi}_{t_j} \circ \lambda_{s(t'_n)}(a) \Delta_j \right] \Delta_n'^2 \right). \quad (2.32)$$

for all $a \in X$. As in the proof of Lemma 2.7 we can also require of $\{u'_n\}$ that there is a strictly increasing function $m : \mathbb{N} \rightarrow \mathbb{N}$ such that $m(0) = 0$ and

- a40) $t_{m(n)} \geq t'_{n+1}, n \geq 1,$
- a41) $u'_n u_{m(n)} = u_{m(n)},$
- a42) $u_{m(n)} u'_{n-1} = u'_{n-1}, n \geq 1,$ and
- a43) $t'_n - t'_{n-1} \leq t_{m(n)} - t_{m(n-1)}, n \geq 1.$

Define $r : [1, \infty) \rightarrow [1, \infty)$ such that r is linear on $[t_{m(n-1)}, t_{m(n)}], r(t_{m(n-1)}) = t'_{n-1}$ and $r(t_{m(n)}) = t'_n$ for all n . It follows from a40) that $r(t) \leq t$ for all t and from a43) that

$$|r(t) - r(t')| \leq |t - t'| \quad (2.33)$$

for all $t, t' \in [1, \infty)$. From a41) and a42) we deduce that

$$\begin{aligned} & \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} \Delta_j \bar{\varphi}_{t_j} \circ \lambda_{s(t'_n)}(a) \Delta_j \right] \Delta_n'^2 = \\ & \sum_{n=1}^{\infty} \sum_{m(n-1) \leq k < m(n)} \sum_{j \in \{-1, 0\}} \Delta_k \bar{\varphi}_{t_k} \circ \lambda_{s(t'_{n+j})}(a) \Delta_k \Delta_{n+j}'^2. \end{aligned}$$

Now we combine (2.29) and Lemma 3.1 of [MT2] to conclude that

$$\begin{aligned} (\varphi_f \circ \lambda)_{f'}(a) &= q_B \left(\sum_{n=1}^{\infty} \sum_{m(n-1) \leq k < m(n)} \sum_{j \in \{-1, 0\}} \Delta_k \bar{\varphi}_{t_k} \circ \lambda_{s(r(t_k))}(a) \Delta_k \Delta_{n+j}'^2 \right) \\ &= q_B \left(\left[\sum_{k=0}^{\infty} \Delta_k \bar{\varphi}_{t_k} \circ \lambda_{s(r(t_k))}(a) \Delta_k \right] \sum_{n=0}^{\infty} \Delta_n'^2 \right) \\ &= q_B \left(\sum_{k=0}^{\infty} \Delta_k \bar{\varphi}_{t_k} \circ \lambda_{s(r(t_k))}(a) \Delta_k \right) \end{aligned}$$

for all $a \in X$. It follows from (2.27) that

$$(\varphi_f \circ \lambda)_{f'}(a) = q_B \left(\sum_{k=0}^{\infty} \Delta_k \bar{\kappa}_{t_k, r(t_k)}(a) \Delta_k \right) \quad (2.34)$$

for all $a \in X$. Now note that it follows from (2.25), (2.26) and (2.33) that

$$\lim_{k \rightarrow \infty} \sup_{t \in [t_k, t_{k+1}]} \left\| \bar{\kappa}_{t_k, r(t_k)}(a) - \bar{\kappa}_{t, r(t)}(a) \right\| = 0$$

for all $a \in A'$. Combining this with (2.27) and (2.28) we can conclude that $f'' = (\bar{\kappa}_{t, r(t)}, \{u_n\}, \{t_n\})$ is folding data for the asymptotic extension $\{\kappa_{t, r(t)}\}_{t \in [1, \infty)}$. Since (2.34) implies that $(\varphi_f \circ \lambda)_{f'} = \mu_{f''}$, where $\mu_t = q_B \circ \bar{\kappa}_{t, r(t)}$, this completes the proof. \square

Theorem 2.13. *Let A'', A', A be a separable C^* -algebras and B a stable σ -unital C^* -algebra. Let $\nu : A'' \rightarrow A'$ and $\lambda : A' \rightarrow A$ asymptotic homomorphisms and $\varphi : A \rightarrow Q(B)$ a semi-invertible extension. Then*

$$([\varphi] \star [\lambda]) \star [\nu] = [\varphi] \star ([\lambda] \bullet [\nu])$$

in $\text{Ext}^{-\frac{1}{2}}(A'', B)$.

Proof. Apply Lemma 2.12 with $\varphi \circ \lambda$ in the role of φ and ν in the role of λ . \square

3. SEMI-INVERTIBILITY

Lemma 3.1. *Let A', A, B be C^* -algebras, A', A separable and B stable and σ -unital. Let $\varphi : A \rightarrow Q(B)$ be an asymptotic extension and $\lambda : A' \rightarrow A$ an asymptotic homomorphism. Let $\nu : A' \rightarrow Q(B)$ be an asymptotic extension such that $[\nu] = [\varphi] \bullet [\lambda]$ in $[[A', Q(B)]]$. Assume that φ_g is semi-invertible for some folding φ_g of φ . Then $\nu_{g'}$ is semi-invertible for every folding $\nu_{g'}$ of ν .*

Proof. Let φ_f, s, μ and f'' be as in Lemma 2.12. By assumption φ_g is semi-invertible for some folding φ_g of φ and it follows then from Proposition 2.8 that also φ_f is semi-invertible. Thus $\mu_{f''}$ is semi-invertible by Lemma 2.12. Since μ is strongly homotopic to ν it follows from Proposition 2.8 that $\nu_{g'}$ is semi-invertible for any folding of ν . \square

With the following definition we try to cover the most general result about automatic semi-invertibility which can be obtained from the pairing of $\text{Ext}^{-1/2}$ with asymptotic homomorphisms. It is inspired by three sources. One is the paper by Dadarlat and Loring on unsuspended E-theory, [DL], where homotopy symmetric C^* -algebras are introduced. Another is the paper [V] of Voiculescu where the notion of homotopy domination is introduced and the third is the work of Dadarlat [D] where it is shown that shape-equivalence of separable C^* -algebras is the same thing as equivalence in the asymptotic homotopy category of Connes and Higson.

Definition 3.2. Let A and A' be C^* -algebras. Following the notation of [DL] we denote by $[\text{id}_A]$ the element of $[[A, A \otimes \mathbb{K}]]$ represented by the $*$ -homomorphism $s(a) = a \otimes e$ for some minimal non-zero projection e in \mathbb{K} . We say that A is *homotopy symmetric relative to A'* when there are asymptotic homomorphisms $\lambda : A \rightarrow A'$, $\mu : A' \rightarrow A \otimes \mathbb{K}$ and $\psi : A \rightarrow A \otimes \mathbb{K}$ such that

$$[\text{id}_A] + [\psi] = [\mu] \bullet [\lambda]$$

in $[[A, A \otimes \mathbb{K}]]$. When ψ can be taken to be zero, we say that A is *shape dominated* by A' .

Thus A is homotopy symmetric in the sense of Dadarlat and Loring if and only if it is homotopy symmetric relative to 0, and shape domination generalises homotopy domination in the sense of Voiculescu.

Theorem 3.3. *Let A', A, B be C^* -algebras, A', A separable and B stable and σ -unital. Assume that A is homotopy symmetric relative to A' and that all extensions of A' by B are semi-invertible.*

It follows that all extensions of A by B are semi-invertible.

Proof. By Lemma 4.3 of [MT5] it suffices to show that all extensions of $A \otimes \mathbb{K}$ by B are semi-invertible, i.e. we may assume that A is stable. Then our assumptions imply that there are asymptotic homomorphisms $\lambda : A \rightarrow A' \otimes \mathbb{K}$, $\mu : A' \otimes \mathbb{K} \rightarrow A$ and $\psi : A \rightarrow A$ such that $[\varphi \oplus (\varphi \circ \psi)] = [\varphi \circ \mu] \bullet [\lambda]$ in $[[A, Q(B)]]$ for any extension $\varphi : A \rightarrow Q(B)$. By assumption any folding of $\varphi \circ \mu$ is semi-invertible and hence Lemma 3.1 implies that $\varphi \oplus (\varphi \circ \mu)_f$ is semi-invertible for any folding $(\varphi \circ \mu)_f$ of $\varphi \circ \mu$. It follows that φ is semi-invertible. \square

4. RELATION TO E-THEORY

Recall that the E -theory of Connes and Higson, [CH], depends on a fundamental construction, *the Connes-Higson construction*, which produces asymptotic homomorphisms out of extensions. Since the asymptotic homomorphism obtained from an asymptotically split extension is homotopic to 0, the Connes-Higson construction gives rise to a group homomorphism

$$CH : \text{Ext}^{-\frac{1}{2}}(A, B) \rightarrow [[SA, B]]. \quad (4.1)$$

As shown in [DL] the group $[[SA, B]]$ is isomorphic to the E -theory group $E(A, SB)$. Thus CH gives a direct relation between $\text{Ext}^{-1/2}$ and E -theory. It is unknown if

CH is always an isomorphism, but we can now show that it is when A is shape dominated by another C^* -algebra A' , for example a nuclear C^* -algebra, for which $CH : \text{Ext}^{-1/2}(A', B) \rightarrow [[SA', B]]$ is an isomorphism.

Theorem 4.1. *Let A', A, B be separable C^* -algebras, B stable. Let $\varphi : A \rightarrow Q(B)$ be a semi-invertible extension and $\lambda : A' \rightarrow A$ an asymptotic homomorphism. It follows that*

$$CH([\varphi] \star [\lambda]) = CH[\varphi] \bullet [S\lambda]$$

in $[[SA', B]]$, where $S\lambda : SA' \rightarrow SA$ is the suspension of λ .

Proof. We refer to [CH] for the description of the Connes-Higson construction we shall use here. Let ψ be a lift of $\varphi \circ \lambda$. Applying the same re-parametrisation to both ψ and λ we can arrange that ψ is uniformly continuous, and still have that $q_B \circ \psi_t = \varphi \circ \lambda_t$ for all t , cf. Lemma 2.3. Let $f = (\psi, \{u_n\}, \{t_n\})$ be folding data defining the folding $(\varphi \circ \lambda)_f$, and let $\bar{\varphi} : A \rightarrow M(B)$ be a continuous lift of φ . Let R be a countable dense subset of $C_0(0, 1)$ and X a countable dense subset of A' . Define $u_t \in B$, $t \in [n, n+1]$, such that $u_t = (t-n)u_{n+1} + (n+1-t)u_n$. Let r be a re-parametrisation of $[1, \infty)$ such that

$$\text{a44) } r(t) \leq t \text{ for all } t,$$

$$\text{a45) } \lim_{n \rightarrow \infty} r(n+1) - r(n) = 0,$$

$$\text{a46) } \lim_{t \rightarrow \infty} (1 - u_t) (\psi_{r(t)}(a) - \bar{\varphi} \circ \lambda_{r(t)}(a)) = 0 \text{ for all } a \in X, \text{ and}$$

$$\text{a47) } [CH(\varphi) \bullet (S\lambda)] \text{ is represented in } [[SA', B]] \text{ by an asymptotic homomorphism } \Phi : SA' \rightarrow B \text{ such that } \lim_{t \rightarrow \infty} g(u_t) \bar{\varphi}(\lambda_{r(t)}(a)) - \Phi_t(g \otimes a) = 0 \text{ for all } g \in R \text{ and all } a \in X.$$

It follows from a44) and a45) that $f' = (\psi, \{u_n\}, \{r(n)\})$ is folding data for $\varphi \circ \lambda$ since $(\psi, \{u_n\}, \{t_n\})$ is. If we let $A(t) \sim B(t)$ mean that $\lim_{t \rightarrow \infty} A(t) - B(t) = 0$, we have for any $g \in R$ and $a \in X$ that

$$g(u_t) \left(\sum_{n=0}^{\infty} \Delta_n \psi_{r(n)}(a) \Delta_n \right) = \sum_{j=n-4}^{n+4} g(u_t) \Delta_j \psi_{r(j)}(a) \Delta_j \quad \text{where } t \in [n, n+1]$$

(by a1) and the definition of $\{u_t\}$)

$$\sim \sum_{j=n-4}^{n+4} g(u_t) \Delta_j \psi_{r(t)}(a) \Delta_j$$

(using a45) and the uniform continuity of ψ)

$$\sim \sum_{j=n-4}^{n+4} g(u_t) \Delta_j^2 \psi_{r(t)}(a) = g(u_t) \psi_{r(t)}(a)$$

(thanks to a44) and the properties of folding data)

$$\sim g(u_t) \bar{\varphi}(\lambda_{r(t)}(a))$$

(thanks to a46))

$$\sim \Phi_t(g \otimes a)$$

(thanks to a47)).

It follows that $CH([\varphi] \star [\lambda])$ is represented by an asymptotic homomorphism which asymptotically agrees with Φ . \square

Corollary 4.2. *Let A', A, B be C^* -algebras, A', A separable and B stable and σ -unital. Assume that $CH : \text{Ext}^{-\frac{1}{2}}(A', B) \rightarrow [[SA', B]]$ is an isomorphism, and assume that A is shape dominated by A' . It follows that $CH : \text{Ext}^{-\frac{1}{2}}(A, B) \rightarrow [[SA, B]]$ is an isomorphism.*

Proof. This follows from a simple diagram chase in the commuting diagram

$$\begin{array}{ccc}
 \text{Ext}^{-\frac{1}{2}}(A, B) & \xrightarrow{CH} & [[SA, B]] \\
 \lambda^* \quad \mu^* & & (S\lambda)^* \quad (S\mu)^* \\
 \text{Ext}^{-\frac{1}{2}}(A', B) & \xrightarrow{CH} & [[SA', B]]
 \end{array}$$

□

Corollary 4.3. *Let A be a separable C^* -algebra which is shape dominated by a separable nuclear C^* -algebra. It follows that $CH : \text{Ext}^{-\frac{1}{2}}(A, B) \rightarrow [[SA, B]]$ is an isomorphism for every stable σ -unital C^* -algebra B .*

Corollary 4.4. *Let A be a separable C^* -algebra and B a stable σ -unital C^* -algebra. Assume that A is homotopy symmetric in the sense of Dadarlat and Loring, [DL]. It follows that all extensions of A by B are semi-invertible and that $CH : \text{Ext}^{-\frac{1}{2}}(A, B) \rightarrow [[SA, B]]$ is an isomorphism.*

Proof. The first assertion follows from Theorem 3.3. To establish the second we assume without loss of generality that A is stable. It follows from [DL] that because A is homotopy symmetric the E -theory inverse of the canonical asymptotic homomorphism $S^2A \rightarrow A$ (arising from the Toeplitz extension) has an inverse $A \rightarrow S^2A$ giving us a shape equivalence between A and S^2A . Since $CH : \text{Ext}^{-1/2}(S^2A, B) \rightarrow [[S^3A, B]]$ is an isomorphism for every stable σ -unital B by [MT3], it follows from Corollary 4.2 that also $CH : \text{Ext}^{-\frac{1}{2}}(A, B) \rightarrow [[SA, B]]$ is an isomorphism. □

It remains an open question if (4.1) is always an isomorphism.

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