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Localized Endomorphisms of Graph Algebras

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Abstract

Endomorphisms of graph C^*-algebras are investigated. A combinatorial approach to analysis of permutative endomorphisms is developed. Then invertibility criteria for localized endomorphisms are given. Furthermore, proper endomorphisms which restrict to automorphisms of the canonical diagonal MASA are analyzed. The Weyl group and the restricted Weyl group of a graph C^*-algebra are introduced and investigated. Criteria of outerness for automorphisms in the restricted Weyl group are found.

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1 Introduction

The main aim of this article is to carry out systematic investigations of a certain natural class of endomorphisms and, in particular, automorphisms of graph C^*-algebras. Namely, we focus on those endomorphisms which globally preserve the canonical AF-subalgebra and the diagonal MASA. This leads to the concept of the *Weyl group* of a graph algebra. We develop powerful novel techniques of both analytic and combinatorial nature for the study of automorphisms these groups comprise.

By analogy with the theory of semi-simple Lie groups, Cuntz introduced in [15] the Weyl group of the simple, purely infinite C^*-algebras $\mathcal{O}_n$. Quite similarly with the classical theory, it arises as the quotient of the normalizer of a maximal abelian subgroup of the automorphism group of the algebra. So defined Weyl group is discrete albeit infinite, and the abelian subgroup in question is an inductive limit of higher dimensional tori. Cuntz posed a problem of determining the structure of the important subgroup of the Weyl group, corresponding to those automorphisms in the Weyl group which globally preserve the canonical UHF-subalgebra of $\mathcal{O}_n$, [15]. After 30 years, this question has been finally answered in [8]. In the present paper, we take this programme one step further, expanding it from Cuntz algebras $\mathcal{O}_n$ to a much wider class of graph C^*-algebras.

The theory of graph algebras began in earnest in the late nineties, [26, 25], and since then it has developed into a fully fledged and very active area of research within operator algebras. In the case of finite graphs, the corresponding C^*-algebras essentially coincide with Cuntz-Krieger algebras, that were introduced much earlier, [17], in connection with topological Markov chains. The importance of graph algebras (or Cuntz-Krieger algebras) stems to large extent from numerous applications they have found. Not trying to be exhaustive in any way, we only mention: their role in classification of purely infinite, simple C^*-algebras, [32, 35], and related to that applications to the problem of semiprojectivity of Kirchberg algebras, [2, 36, 34]; their connection with objects of interest in noncommutative geometry and quantum group theory, [20, 6, 29]; their strong interplay with theory of symbolic dynamical systems, going back to the original paper of Cuntz and Krieger, [17, 3]. For a good general introduction to graph algebras we refer the reader to [31].

The analysis of endomorphisms of graph algebras, developed in the present article, owes a great deal to the close relationship between such algebras and the Cuntz algebras $\mathcal{O}_n$. C^*-algebras $\mathcal{O}_n$ were first defined and investigated by Cuntz in his seminal paper [14], and they bear his name ever since. The Cuntz algebras have been extensively used in many a diverse context, including classification of C^*-algebras, quantum field theory, self-similar sets, wavelet theory, coding theory, continuous fractions, spectral flow, subfactors and index theory.

Systematic investigations of endomorphisms of $\mathcal{O}_n$, $n < \infty$, were initiated by Cuntz in [15]. A fundamental bijective correspondence between unital $*$-endomorphisms and unitaries in $\mathcal{O}_n$ was established therein. Using this correspondence Cuntz proved a number of interesting results, in particular with regard to those endomorphisms which globally preserve either the core UHF-subalgebra $\mathcal{F}_n$ or the diagonal MASA $\mathcal{D}_n$. Likewise, investigations of automorphisms of $\mathcal{O}_n$ began almost immediately after the birth of the algebras in question, [15]. Endomorphisms of the Cuntz algebras played a role in certain aspects of index theory, both from the C^*-algebraic and von Neumann algebraic point of view, e.g. see [22], [11]. One of the most interesting applications of endomorphisms of $\mathcal{O}_n$, found by Bratteli and Jørgensen in [4, 5], is in the area of wavelets. In particular, permutative endomorphisms have been used in this context. These were further investigated by Kawamura, [24].

The present article builds directly on the progress made recently in the study of localized endomorphisms of $\mathcal{O}_n$ by Conti, Szymański and their collaborators. In particular, much better understanding of those endomorphisms of $\mathcal{O}_n$ which globally preserve the core UHF-subalgebra or the canonical diagonal MASA has been obtained in [12] and [19], respectively. In [37, 13, 10, 7, 9], a novel combinatorial approach to the study of permutative endomorphisms of $\mathcal{O}_n$ has been introduced, and subsequently significant progress in the investigations of such endomorphisms and automorphisms has been obtained. In particular, a striking relationship between permutative automorphisms of $\mathcal{O}_n$ and automorphisms of the full two-sided n-shift has been found in [8].

The main objective of the present paper is to extend this analysis of endomorphisms of the Cuntz algebras to the much larger class of graph C^*-algebras. Most of our results (but not all) are concerned with algebras corresponding to finite graphs without sinks, which may be identified with Cuntz-Krieger algebras of finite 0–1 matrices, [17]. From the very beginning, the theory of such algebras has been closely related to dynamical systems. In particular, endomorphisms of Cuntz-Krieger algebras have been studied in the context of index theory, [23]. Quasi-free automorphisms of Cuntz-Krieger algebras (and even more generally, Cuntz-Pimsner algebras) have been studied in [38] and [18]. An interesting connection between automorphisms of Cuntz-Krieger algebras and Markov shifts has been investigated by Matsumoto, [27, 28].

The present paper is organized as follows. In Section 2, we set up notation and present some preliminaries. In Section 3, an analogue of the Weyl group for graph algebras is introduced and investigated. Namely, let $\mathcal{F}_E$ be the core AF-subalgebra of the graph algebra $C^*(E)$, and let $\mathcal{D}_E$ be its canonical abelian subalgebra. Then, under a mild hypothesis, the group $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ of those automorphisms of $C^*(E)$ which fix $\mathcal{D}_E$ point-wise is a maximal abelian subgroup of $\mathrm{Aut}(C^*(E))$. The Weyl group of $C^*(E)$ is defined as the quotient of the normalizer of $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ by itself, and its structural properties are exhibited. The results of this section extend Cuntz's analysis of the Weyl group of $\mathcal{O}_n$, [15]. Moreover, a very convenient criterion of outerness of certain automorphisms of a graph algebra is obtained (Corollary 3.9). We also present a few technical facts about normalizers of the diagonal MASA and the core AF-subalgebra, and about automorphisms globally preserving one of these subalgebras. In Section 4, we obtain an algorithmic criterion of invertibility of localized endomorphisms of a graph algebra (Theorem 4.1), as well as a criterion of invertibility of the restriction of a localized endomorphism to the diagonal MASA (Theorem 4.3). These theorems extend the analogous results for Cuntz algebras obtained earlier in [13]. In Section

5, a special class of localized endomorphisms corresponding to permutation unitaries is investigated. Combinatorial invertibility criteria for permutative endomorphisms of graph algebras and their restrictions to the diagonal are given (Lemma 5.1, Lemma 5.3 and Theorem 5.4). Finally, a few examples are worked out in detail, illustrating applications of the combinatorial machinery developed in the present paper.

2 Notation and preliminaries

2.1 Directed graphs and their C^*-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph, where E^0 and E^1 are (countable) sets of vertices and edges, respectively, and $r, s : E^1 \to E^0$ are range and source maps, respectively. The C^*-algebra $C^*(E)$ corresponding to E is by definition the universal C^*-algebra generated by mutually orthogonal projections P_v, $v \in E^0$, and partial isometries S_e, $e \in E^1$, subject to the following relations

(GA1) $S_e^* S_e = P_{r(e)}$ and $S_e^* S_f = 0$ if $e \neq f \in E^1$,

(GA2) $S_e S_e^* \leq P_{s(e)}$ for $e \in E^1$,

(GA3) $P_v = \sum_{s(e)=v} S_e S_e^*$ if $v \in E^0$ emits finitely many and at least one edge.

A *path* μ of length $|\mu| = k \geq 1$ is a sequence $\mu = (\mu_1, \ldots, \mu_k)$ of k edges μ_j such that $r(\mu_j) = s(\mu_{j+1})$ for $j = 1, \ldots, k-1$. We also view the vertices as paths of length 0. The set of all paths of length k is denoted E^k. The range and source maps naturally extend from vertices E^1 to paths E^k. A *sink* is a vertex v which emits no edges, i.e. $s^{-1}(v) = \emptyset$. A *source* is a vertex w which receives no edges, i.e. $r^{-1}(v) = \emptyset$. By a *loop* we mean a path μ of length $|\mu| \geq 1$ such that $s(\mu) = r(\mu)$. We say that a loop $\mu = (\mu_1, \ldots, \mu_k)$ has an exit if there is a j such that $s(\mu_j)$ emits at least two distinct edges.

As usual, for a path $\mu = (\mu_1, \ldots, \mu_k)$ of length k we denote by $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$ the corresponding partial isometry in $C^*(E)$. It is known that each S_μ is non-zero, with the domain projection $P_{r(\mu)}$. Then $C^*(E)$ is the closed span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$, where E^* denotes the collection of all finite paths (including paths of length zero). Here we agree that $S_v = P_v$ for $v \in E^0$ viewed as path of length 0. Also note that $S_\mu S_\nu^*$ is non-zero if and only if $r(\mu) = r(\nu)$. In that case, $S_\mu S_\nu^*$ is a partial isometry with domain and range projections equal to $S_\nu S_\nu^*$ and $S_\mu S_\mu^*$, respectively.

The range projections $P_\mu = S_\mu S_\mu^*$ of all partial isometries S_μ mutually commute, and the abelian C^*-subalgebra of $C^*(E)$ generated by all of them is called the diagonal subalgebra and denoted $\mathcal{D}_E$. If E does not contain sinks and all loops have exits then $\mathcal{D}_E$ is a MASA (maximal abelian subalgebra) in $C^*(E)$ by [21, Theorem 5.2]. We set $\mathcal{D}_E^0 = \text{span}\{P_v : v \in E^0\}$ and, more generally, $\mathcal{D}_E^k = \text{span}\{P_\mu : \mu \in E^k\}$ for $k \geq 0$.

There exists a strongly continuous action γ of the circle group $U(1)$ on $C^*(E)$, called the gauge action, such that $\gamma_t(S_e) = tS_e$ and $\gamma_t(P_v) = P_v$ for all $e \in E^1$, $v \in E^0$ and $t \in U(1) \subseteq \mathbb{C}$. The fixed-point algebra $C^*(E)^\gamma$ for the gauge action is an AF-algebra, denoted $\mathcal{F}_E$ and called the core AF-subalgebra of $C^*(E)$. $\mathcal{F}_E$ is the closed span of

$\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ |\mu| = |\nu|\}$. For $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ we denote by $\mathcal{F}_E^k$ the linear span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ |\mu| = |\nu| = k\}$.

2.2 Endomorphisms determined by unitaries

We denote by $\mathcal{U}_E$ the collection of all those unitaries in the multiplier algebra $M(C^*(E))$ which commute with all vertex projections P_v, $v \in E^0$. That is

$$\mathcal{U}_E := \mathcal{U}((\mathcal{D}_E^0)' \cap M(C^*(E))). \tag{1}$$

These unitaries will play a crucial role throughout this paper. Let $u \in \mathcal{U}_E$. Then uS_e, $e \in E^1$, are partial isometries in $C^*(E)$ which together with projections P_v, $v \in E^0$, satisfy (GA1)–(GA3). Thus, by universality in the definition of $C^*(E)$, there exists a $*$-homomorphism $\lambda_u : C^*(E) \to C^*(E)$ such that[1]

$$\lambda_u(S_e) = uS_e \quad \text{and} \quad \lambda_u(P_v) = P_v, \quad \text{for } e \in E^1,\ v \in E^0. \tag{2}$$

Clearly, $\lambda_u(1) = 1$ whenever $C^*(E)$ is unital. In general, λ_u may be neither injective nor surjective. However, the following proposition is an immediate consequence of the gauge-invariant uniqueness theorem [1, Theorem 2.1].

Proposition 2.1. *If $u \in \mathcal{U}_E$ belongs to the minimal unitization of the core AF-subalgebra $\mathcal{F}_E$, then endomorphism λ_u is automatically injective.*

Note that $\{\lambda_u : u \in \mathcal{U}_E\}$ is a semigroup with the following multiplication law:

$$\lambda_u \circ \lambda_w = \lambda_{u*w}, \quad u * w = \lambda_u(w)u. \tag{3}$$

We say λ_u is *invertible* if λ_u is an automorphism of $C^*(E)$. For $K \subseteq \mathcal{U}_E$ we denote $\lambda(K)^{-1} := \{\lambda_u \in \mathrm{Aut}(C^*(E)) : u \in K\}$. It turns out that λ_u is invertible if and only if it is injective and u^* is in the range of its extension to $M(C^*(E))$. Indeed, if λ_u is injective and there exists a $w \in M(C^*(E))$ such that $\lambda_u(w) = u^*$ then w must belong to $\mathcal{U}_E$ and $\lambda_u \lambda_w = \lambda_{u*w} = \mathrm{id}$. Thus λ_u is surjective and hence an automorphism. In this case, we have $\lambda_u^{-1} = \lambda_w$.

In the present paper, we mainly deal with *finite graphs without sinks.* If E is such a graph then the association $u \mapsto \lambda_u$ establishes a bijective correspondence between $\mathcal{U}_E$ and the semigroup of those unital $*$-homomorphisms from $C^*(E)$ into itself which fix all the vertex projections P_v, $v \in E^0$. Indeed, if ρ is such a homomorphism then $u := \sum_{e \in E^1} \rho(S_e) S_e^*$ belongs to $\mathcal{U}_E$ and $\rho = \lambda_u$ (cf. [38]). If $u \in \mathcal{F}_E^1 \cap \mathcal{U}_E$ then λ_u is automatically invertible with inverse λ_{u^*} and the map

$$\mathcal{F}_E^1 \cap \mathcal{U}_E \ni u \mapsto \lambda_u \in \mathrm{Aut}(C^*(E)) \tag{4}$$

is a group homomorphism with range inside the subgroup of *quasi-free automorphisms* of $C^*(E)$, see [18, 38].

[1] The reader should be aware that in some papers (e.g. in [15], [37] and [13]) a different convention: $\lambda_u(S_e) = u^* S_e$ is used.

If λ_u is an endomorphism of $C^*(E)$ corresponding to a unitary u in the linear span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ |\mu| = |\nu|\}$ and the identity (i.e. in the minimal unitization of the algebraic part of the core AF-subalgebra), then we call λ_u *localized*, cf. [11].

Let E be a finite graph without sinks, and let

$$\varphi(x) = \sum_{e \in E^1} S_e x S_e^* \tag{5}$$

be the usual *shift* on $C^*(E)$, [17]. It is a unital, completely positive map. One can easily verify that the shift is an injective $*$-homomorphism when restricted to the relative commutant of $\{P_v : v \in E^0\}$. We will denote this relative commutant by B_E, i.e.

$$B_E := (\mathcal{D}_E^0)' \cap C^*(E). \tag{6}$$

In particular, $\mathcal{U}_E = \mathcal{U}(B_E)$. We have $\varphi(B_E) \subseteq B_E$ and thus $\varphi(\mathcal{U}_E) \subseteq \mathcal{U}_E$. It is also clear that $\varphi(\mathcal{F}_E) \subseteq \mathcal{F}_E$ and $\varphi(\mathcal{D}_E) \subseteq \mathcal{D}_E$. For $k \geq 1$ we denote

$$u_k := u\varphi(u) \cdots \varphi^{k-1}(u), \tag{7}$$

and agree that u_k^* stands for $(u_k)^*$. For each $u \in \mathcal{U}_E$ and for any two paths $\mu, \nu \in E^*$ we have

$$\lambda_u(S_\mu S_\nu^*) = u_{|\mu|} S_\mu S_\nu^* u_{|\nu|}^*. \tag{8}$$

The above equality is established with help of the identity $S_e u = \varphi(u) S_e$, which holds for all $e \in E^1$ because the unitary u commutes with all vertex projections P_v, $v \in E^0$, by hypothesis. Indeed,

$$\varphi(u) S_e = \Big(\sum_{f \in E^1} S_f u S_f^* \Big) S_e = S_e u S_e^* S_e = S_e u P_{r(e)} = S_e P_{r(e)} u = S_e u.$$

More generally, if $x \in C^*(E)$ commutes with all vertex projections then

$$S_\alpha x = \varphi^{|\alpha|}(x) S_\alpha \tag{9}$$

for each finite path α. Furthermore, we have

$$\mathrm{Ad}(u) = \lambda_{u\varphi(u^*)} \quad \text{for } u \in \mathcal{U}_E, \tag{10}$$

where $\mathrm{Ad}(u)(x) = uxu^*$, $x \in C^*(E)$.

3 The Weyl group

In this section, we recast some of the results from [15] in the setting of graph C^*-algebras. Throughout this section we assume that E is a *finite* graph.

3.1 Automorphisms globally preserving $\mathcal{D}_E$ or $\mathcal{F}_E$

For algebras $A \subseteq B$ we denote by $\mathcal{N}_B(A) = \{u \in \mathcal{U}(B) : uAu^* = A\}$ the normalizer of A in B, and by $A' \cap B = \{b \in B : (\forall a \in A)\ ab = ba\}$ the relative commutant of A in B.

Proposition 3.1. *Let E be a finite graph without sinks, and let $u \in \mathcal{U}_E$. Then the following hold:*

(i) If $u\mathcal{D}_E u^* \subseteq \mathcal{D}_E$ then $\lambda_u(\mathcal{D}_E) \subseteq \mathcal{D}_E$.

(ii) If $\lambda_u(\mathcal{D}_E) = \mathcal{D}_E$ then $u\mathcal{D}_E u^* \subseteq \mathcal{D}_E$.

(iii) If $u\mathcal{F}_E u^* \subseteq \mathcal{F}_E$ then $\lambda_u(\mathcal{F}_E) \subseteq \mathcal{F}_E$.

(iv) If $\lambda_u(\mathcal{F}_E) = \mathcal{F}_E$ then $u\mathcal{F}_E u^* \subseteq \mathcal{F}_E$.

Proof. Parts (i) and (iii) are established in the same way as the analogous statements about $\mathcal{O}_n$ in [15].

For part (iii), first note that $\mathcal{D}_E$ is a C^*-algebra generated by $\{\varphi^k(S_e S_e^*) : e \in E^1,\ k = 0, 1, 2, ...\}$, where $\varphi^0 = \mathrm{id}$. In particular, $\mathcal{D}_E$ is generated by $\mathcal{D}_E^1$ and $\varphi(\mathcal{D}_E)$. Now, if $u \in \mathcal{U}_E$ and $\lambda_u(\mathcal{D}_E) = \mathcal{D}_E$ then $u\mathcal{D}_E^1 u^* = \lambda_u(\mathcal{D}_E^1) \subseteq \mathcal{D}_E$ and $u\varphi(\mathcal{D}_E)u^* = u\varphi(\lambda_u(\mathcal{D}_E))u^* = \lambda_u(\varphi(\mathcal{D}_E)) \subseteq \mathcal{D}_E$. Hence $u\mathcal{D}_E u^* \subseteq \mathcal{D}_E$.

Part (iv) is proved in the same way. □

Further note that if E is a finite graph without sinks in which every loop has an exit then $\mathcal{D}_E$ is a MASA in $C^*(E)$ and in part (ii) of Proposition 3.1 we can further conclude that $u\mathcal{D}_E u^* = \mathcal{D}_E$, i.e. that $u \in \mathcal{N}_{C^*(E)}(\mathcal{D}_E)$. In that case, by virtue of [21, Theorem 10.1], every $u \in \mathcal{N}_{C^*(E)}(\mathcal{D}_E)$ can be uniquely written as $u = dw$, where $d \in \mathcal{U}(\mathcal{D}_E)$ and $w \in \mathcal{S}_E$. Here $\mathcal{S}_E$ denotes the group of all unitaries in $\mathcal{U}(C^*(E))$ of the form $\sum S_\alpha S_\beta^*$ (finite sum). Therefore, as in [13, Section 2], the normalizer of the diagonal in $C^*(E)$ is a semi-direct product

$$\mathcal{N}_{C^*(E)}(\mathcal{D}_E) = \mathcal{U}(\mathcal{D}_E) \rtimes \mathcal{S}_E. \tag{11}$$

Regarding the normalizer of the core AF-subalgebra $\mathcal{F}_E$ in $C^*(E)$, it turns out that in many instances this normalizer is trivial. The following theorem is essentially due to Mikael Rørdam, [33].

Theorem 3.2. *Let E be a directed graph with finitely many vertices and no sources. Suppose further that the relative commutant of $\mathcal{F}_E$ in $C^*(E)$ is trivial. Then $u \in \mathcal{U}(C^*(E))$ and $u\mathcal{F}_E u^* \subseteq \mathcal{F}_E$ imply that $u \in \mathcal{U}(\mathcal{F}_E)$.*

Proof. If $x \in \mathcal{F}_E$ then $uxu^* \in \mathcal{F}_E$ and thus $\gamma_t(u)x\gamma_t(u)^* = \gamma_t(uxu^*) = uxu^*$ for each $t \in U(1)$. Consequently, $u^*\gamma_t(u)$ belongs to $\mathcal{F}_E' \cap C^*(E) = \mathbb{C}1$. Thus, for each $t \in U(1)$ there exists a $z(t) \in \mathbb{C}$ such that $\gamma_t(u) = z(t)u$. Clearly, $t \mapsto z(t)$ is a continuous character of the circle group $U(1)$. Hence there is an integer m such that $z(t) = t^m$. If $m = 0$ then u is invariant under the gauge action and we are done. Otherwise, by passing to u^* if necessary, we may assume that $m > 0$.

For each vertex $v \in E^0$ choose one edge e_v with range v and let $T = \sum_{v \in E^0} S_{e_v}$. We have $\gamma_t(T) = tT$ for all $t \in U(1)$ and T is an isometry, since E has no sources. Furthermore, $TT^* \neq 1$, for otherwise each vertex would emit exactly one edge and consequently $\mathcal{F}_E' \cap C^*(E)$ would contain the non-trivial center of $C^*(E)$. Now $T^m u^*$ is fixed by the gauge action and thus it is an isometry in $\mathcal{F}_E$. Since $\mathcal{F}_E$ is an AF-algebra, $T^m u^*$ must be unitary. But then T^m and thus T itself would be unitary, a contradiction. This completes the proof. □

3.2 The full and the restricted Weyl groups

For algebras $A \subseteq B$, we denote by $\mathrm{Aut}(B, A)$ the collection of all those automorphisms α of B such that $\alpha(A) = A$, and by $\mathrm{Aut}_A(B)$ those automorphisms of B which fix A point-wise.

Proposition 3.3. *Let E be a finite graph without sinks in which every loop has an exit. Then the mapping $u \mapsto \lambda_u$ establishes a group isomorphism*

$$\mathcal{U}(\mathcal{D}_E) \cong \mathrm{Aut}_{\mathcal{D}_E}(C^*(E)).$$

Proof. It follows immediately from formula (3) that the mapping $u \mapsto \lambda_u$ is a group homomorphism from $\mathcal{U}(\mathcal{D}_E)$ into $\mathrm{Aut}(C^*(E))$. If μ is a finite path then $\lambda_u(P_\mu) = \mathrm{Ad}(u_{|\mu|})(P_\mu)$ by (8). Hence each λ_u fixes $\mathcal{D}_E$ point-wise. Consequently, the mapping $u \mapsto \lambda_u$ is a well-defined group homomorphism into $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$. The map is one-to-one, as already noted in Section 2. To see that the map is also onto, recall (again from Section 2) that every element of $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ is of the form λ_w for some $w \in \mathcal{U}_E$. Proceeding by induction on $|\mu|$ one shows that w commutes with all projections P_μ. Thus $w \in \mathcal{U}(\mathcal{D}_E)$, since $\mathcal{D}_E$ is a MASA in $C^*(E)$ by [21, Theorem 5.2]. □

We note that under the hypothesis of Proposition 3.3 the fixed point algebra for the action of $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ on $C^*(E)$ equals $\mathcal{D}_E$. Indeed, each element in this fixed point algebra is also fixed by all $\mathrm{Ad}(u)$, $u \in \mathcal{U}(\mathcal{D}_E)$, and thus belongs to $\mathcal{D}_E$.

Proposition 3.4. *Let E be a finite graph without sinks in which every loop has an exit. Then the normalizer of $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ in $\mathrm{Aut}(C^*(E))$ coincides with $\mathrm{Aut}(C^*(E), \mathcal{D}_E)$. If, in addition, the center of $C^*(E)$ is trivial, then $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ is a maximal abelian subgroup of $\mathrm{Aut}(C^*(E))$.*

Proof. Let $\alpha \in \mathrm{Aut}(C^*(E))$ be in the normalizer of $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$. Thus for each $w \in \mathcal{U}(\mathcal{D}_E)$ there is a $u \in \mathcal{U}(\mathcal{D}_E)$ such that $\alpha\lambda_u = \lambda_w\alpha$. Then $\alpha(d) = \alpha\lambda_u(d) = \lambda_w\alpha(d)$ for all $w \in \mathcal{U}(\mathcal{D}_E)$ and $d \in \mathcal{D}_E$. Whence $\alpha(\mathcal{D}_E) \subseteq \mathcal{D}_E$. Replacing α with α^{-1} we get the reverse inclusion, and thus $\alpha(\mathcal{D}_E) = \mathcal{D}_E$. This proves the first part of the proposition.

Now let $\alpha \in \mathrm{Aut}(C^*(E))$ commute with all elements of $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$. Then, in particular, $\mathrm{Ad}(u)\alpha(x) = \alpha\,\mathrm{Ad}(u)(x)$ for all $u \in \mathcal{U}(\mathcal{D}_E)$ and $x \in C^*(E)$. Thus $u^*\alpha(u)$ belongs to the center of $C^*(E)$ and hence it is a scalar. Therefore for each projection $p \in \mathcal{D}_E$ we have $\alpha(2p-1) = \pm(2p-1)$, and hence $\alpha(p)$ equals either p or $1-p$. The case $\alpha(p) = 1-p$ is impossible, since taking a projection $q \not\lneq p$, $q \neq 0$, we would get

$\alpha(q) \leq 1-p$ and thus $\alpha(q) \notin \{q, 1-q\}$. Hence α fixes all projections, and thus all elements of $\mathcal{D}_E$. The claim now follows from Proposition 3.3. $\square$

The quotient of $\mathrm{Aut}(C^*(E), \mathcal{D}_E)$ by $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ will be called the (full) *Weyl group* of $C^*(E)$ (cf. [15]), and denoted $\mathfrak{W}_E$. That is,

$$\mathfrak{W}_E := \mathrm{Aut}(C^*(E), \mathcal{D}_E)/\mathrm{Aut}_{\mathcal{D}_E}(C^*(E)). \tag{12}$$

If E has no sinks and all loops have exits, then each $\alpha \in \mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ automatically belongs to $\mathrm{Aut}(C^*(E), \mathcal{F}_E)$ by Proposition 3.3, above. Thus we may consider the quotient of

$$\mathrm{Aut}(C^*(E), \mathcal{F}_E, \mathcal{D}_E) := \mathrm{Aut}(C^*(E), \mathcal{D}_E) \cap \mathrm{Aut}(C^*(E), \mathcal{F}_E)$$

by $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$, which will be called the *restricted Weyl group* of $C^*(E)$ and denoted $\mathfrak{RW}_E$. That is,

$$\mathfrak{RW}_E := \mathrm{Aut}(C^*(E), \mathcal{F}_E, \mathcal{D}_E)/\mathrm{Aut}_{\mathcal{D}_E}(C^*(E)). \tag{13}$$

Just as in the case of the Cuntz algebras, the Weyl group of a graph algebra turns out to be countable.

Proposition 3.5. *Let E be a finite graph. Then the Weyl group $\mathfrak{W}_E$ is countable.*

Proof. For each coset in the quotient $\mathrm{Aut}(C^*(E), \mathcal{D}_E)/\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ choose a representative α and define a mapping from $\mathrm{Aut}(C^*(E), \mathcal{D}_E)/\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ to $\oplus^{|E^1|+|E^0|}C^*(E)$ by $\alpha \mapsto \oplus_e \alpha(S_e) \oplus_v \alpha(P_v)$. This mapping is one-to-one and the target space is separable. Thus it suffices to show that its image is a discrete subset of $\oplus^{|E^1|+|E^0|}C^*(E)$.

Let $\alpha \in \mathrm{Aut}(C^*(E), \mathcal{D}_E)$ be such that $||\alpha(x) - x|| < 1/2$ for all $x \in \{S_e : e \in E^1\} \cup \{P_v : v \in E^0\}$. We claim that $\alpha|_{\mathcal{D}_E} = \mathrm{id}$. To this end, we show by induction on $|\mu|$ that $\alpha(P_\mu) = P_\mu$ for each path μ. Indeed, if $v \in E^0$ then $||\alpha(P_v) - P_v|| < 1/2$ and thus $\alpha(P_v) = P_v$ since $\alpha(P_v) \in \mathcal{D}_E$ and $\mathcal{D}_E$ is commutative. This establishes the base for induction. Now suppose that $\alpha(P_\mu) = \mathcal{P}_\mu$ for all paths μ of length k. Let (e, μ) be a path of length $k+1$. Then, by the inductive hypothesis, we have

$$||\alpha(P_{(e,\mu)}) - P_{(e,\mu)}|| = ||\alpha(S_e)P_\mu\alpha(S_e^*) - S_eP_\mu S_e^*||$$

$$\leq ||\alpha(S_e)P_\mu\alpha(S_e^*) - \alpha(S_e)P_\mu S_e^*|| + ||\alpha(S_e)P_\mu S_e^* - S_eP_\mu S_e^*|| < \frac{1}{2} + \frac{1}{2} = 1.$$

Thus $\alpha(P_{(e,\mu)}) = P_{(e,\mu)}$, since $\alpha(P_{(e,\mu)})$ and $P_{(e,\mu)}$ commute. This yields the inductive step.

Now suppose that $\alpha, \beta \in \mathrm{Aut}(C^*(E), \mathcal{D}_E)$ are such that $||\alpha(x) - \beta(x)|| < 1/2$ for all $x \in \{S_e : e \in E^1\} \cup \{P_v : v \in E^0\}$. Then also $||\beta^{-1}\alpha(x) - x|| < 1/2$ for all such x, and hence $\beta^{-1}\alpha \in \mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ by the preceding paragraph. This completes the proof. $\square$

For the remainder of this section, we assume that E is a finite graph without sinks in which every loop has an exit. Clearly, each automorphism α of the graph E gives rise to an automorphism of the algebra $C^*(E)$, still denoted α, such that $\alpha(S_e) = S_{\alpha(e)}$, $e \in E^1$, and $\alpha(P_v) = P_{\alpha(v)}$, $v \in E^0$. With a slight abuse of notation, we are identifying automorphisms of graph E with the corresponding automorphisms of C^*-algebra $C^*(E)$, and denote this group $\mathrm{Aut}(E)$.

We denote by $\mathfrak{G}_E$ the subgroup of $\mathrm{Aut}(C^*(E))$ generated by automorphisms of the graph E and $\lambda(\mathcal{S}_E \cap \mathcal{U}_E)^{-1}$. That is,

$$\mathfrak{G}_E := \langle \lambda(\mathcal{S}_E \cap \mathcal{U}_E)^{-1} \cup \mathrm{Aut}(E) \rangle \subseteq \mathrm{Aut}(C^*(E)). \tag{14}$$

If $u \in \mathcal{S}_E \cap \mathcal{U}_E$ then $\lambda_u(\mathcal{D}_E) \subseteq \mathcal{D}_E$. Consequently, if $u \in \mathcal{S}_E \cap \mathcal{U}_E$ and λ_u is invertible then λ_u belongs to $\mathrm{Aut}(C^*(E), \mathcal{D}_E)$, since $\mathcal{D}_E$ is a MASA in $C^*(E)$. Since each graph automorphism gives rise to an element of $\mathrm{Aut}(C^*(E), \mathcal{D}_E)$ as well, we have $\mathfrak{G}_E \subseteq \mathrm{Aut}(C^*(E), \mathcal{D}_E)$.

Now, let $u \in \mathcal{S}_E \cap \mathcal{U}_E$ be such that λ_u is invertible. Then, as noted in Section 2 above, $\lambda_u^{-1} = \lambda_w$ for some $w \in \mathcal{U}_E$ and $w \in \mathcal{N}_{C^*(E)}(\mathcal{D}_E)$ by Proposition 3.1. Let $w = dz$ with $d \in \mathcal{U}(\mathcal{D}_E)$ and $z \in \mathcal{S}_E$. Then we have $z \in \mathcal{S}_E \cap \mathcal{U}_E$. Since $\lambda_{u*dz} = \lambda_u \lambda_{dz} = \mathrm{id} = \lambda_1$, we have $\lambda_u(d)\lambda_u(z)u = u * dz = 1$. Thus $d = 1$, by (11), and $\lambda_u^{-1} = \lambda_z$. This shows that $\lambda(\mathcal{S}_E \cap \mathcal{U}_E)^{-1}$ is a group. If α is an automorphism of E and $u \in \mathcal{S}_E \cap \mathcal{U}_E$ then $\alpha\lambda_u\alpha^{-1} = \lambda_w$ for some $w \in \mathcal{U}_E$, since $\alpha\lambda_u\alpha^{-1}$ fixes all the vertex projections. A short calculation shows that this w belongs to $\mathcal{S}_E$. It follows that $\lambda(\mathcal{S}_E \cap \mathcal{U}_E)^{-1}$ is a normal subgroup of $\mathfrak{G}_E$. Let Γ_E be the group of automorphisms of $C^*(E)$ corresponding to automorphisms of the underlying graph E, and let Γ_E^0 be its normal subgroup consisting of those automorphisms which fix all vertex projections. Then

$$\mathfrak{G}_E = \lambda(\mathcal{S}_E \cap \mathcal{U}_E)^{-1}\Gamma_E \quad \text{and} \quad \lambda(\mathcal{S}_E \cap \mathcal{U}_E)^{-1} \cap \Gamma_E = \Gamma_E^0. \tag{15}$$

Furthermore, it is easy to verify that $\Gamma_E^0 = \lambda(\mathcal{P}_E^1 \cap \mathcal{U}_E)$.

Proposition 3.6. *Let E be a finite graph without sinks in which every loop has an exit. Then there is a natural embedding of $\mathfrak{G}_E$ into the Weyl group $\mathfrak{W}_E$ of $C^*(E)$.*

Proof. Since $\mathfrak{G}_E \subseteq \mathrm{Aut}(C^*(E), \mathcal{D}_E)$, it suffices to show that $\mathfrak{G}_E \cap \mathrm{Aut}_{\mathcal{D}_E}(C^*(E)) = \{\mathrm{id}\}$. Indeed, if $\beta \in \mathfrak{G}_E$ then $\beta = \lambda_u\alpha$ for some $u \in \mathcal{S}_E \cap \mathcal{U}_E$ and $\alpha \in \mathrm{Aut}(E)$. If, in addition, $\beta \in \mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ then $\alpha \in \Gamma_E^0$ and thus $\alpha = \lambda_w$ for some $w \in \mathcal{P}_E^1 \cap \mathcal{U}_E$. Therefore $\beta = \lambda_u\lambda_w = \lambda_{u*w}$ and $u * w \in \mathcal{S}_E$. By Proposition 3.3 we also have $\beta = \lambda_d$ for some $d \in \mathcal{U}(\mathcal{D}_E)$. Consequently $u * w = d = 1$, since $\mathcal{S}_E \cap \mathcal{U}(\mathcal{D}_E) = \{1\}$, and thus $\beta = \mathrm{id}$. □

One of the more difficult issues arising in dealing with automorphisms of graph C^*-algebras is deciding if the automorphism at hand is outer or inner. The following theorem goes a long way towards providing such a criterion for automorphisms belonging to the subgroup $\mathfrak{G}_E$ of the Weyl group $\mathfrak{W}_E$.

Theorem 3.7. *Let E be a finite graph without sinks in which every loop has an exit. Then*

$$\mathfrak{G}_E \cap \mathrm{Inn}(C^*(E)) \subseteq \{\mathrm{Ad}(w) : w \in \mathcal{S}_E\}.$$

Proof. By (15), each element of $\mathfrak{G}_E$ is of the form $\lambda_u\alpha$, with $u \in \mathcal{S}_E \cap \mathcal{U}_E$ and $\alpha \in \Gamma_E$. If such an automorphism is inner and equals $\mathrm{Ad}(y)$ for some $y \in \mathcal{U}(C^*(E))$ then $y \in \mathcal{N}_{C^*(E)}(\mathcal{D}_E)$, and thus we have $\lambda_u\alpha = \mathrm{Ad}(d)\,\mathrm{Ad}(w)$ for some $d \in \mathcal{U}(\mathcal{D}_E)$ and $w \in \mathcal{S}_E$, by (11). Hence $\rho := \lambda_u\alpha\,\mathrm{Ad}(w^*) = \lambda_{d\varphi(d^*)}$. But then $d\varphi(d^*) = \sum_{e\in E^1}\rho(S_e)S_e^* \in \mathcal{S}_E$. Since $\mathcal{S}_E \cap \mathcal{U}(\mathcal{D}_E) = \{1\}$, we have $d\varphi(d^*) = 1$. Therefore $d = \varphi(d)$ and this implies (via a straightforward calculation) that d belongs to the center of $C^*(E)$. Hence $\mathrm{Ad}(u) = \mathrm{id}$ and thus $\lambda_u\alpha = \mathrm{Ad}(w)$, as required. □

Now, we turn our attention to the restricted Weyl group $\mathfrak{RW}_E$. We denote by $\mathcal{P}_E^k$ the collection of all unitaries in $\mathcal{U}(C^*(E))$ of the form $\sum S_\alpha S_\beta^*$ with $|\alpha| = |\beta| = k$. Clearly, each $\mathcal{P}_E^k$ is a finite subgroup of $\mathcal{U}(C^*(E))$, and we have $\mathcal{P}_E^k \subseteq \mathcal{P}_E^{k+1}$ for each k. We set $\mathcal{P}_E := \bigcup_{k=0}^\infty \mathcal{P}_E^k$. As shown in [30], every $u \in \mathcal{N}_{\mathcal{F}_E}(\mathcal{D}_E)$ can be uniquely written as $u = dw$, where $d \in \mathcal{U}(\mathcal{D}_E)$ and $w \in \mathcal{P}_E$. That is, the normalizer of the diagonal in $\mathcal{F}_E$ is a semi-direct product

$$\mathcal{N}_{\mathcal{F}_E}(\mathcal{D}_E) = \mathcal{U}(\mathcal{D}_E) \rtimes \mathcal{P}_E. \tag{16}$$

We denote by $\mathfrak{GR}_E$ the subgroup of $\mathrm{Aut}(C^*(E))$ generated by automorphisms of the graph E and $\lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1}$. That is,

$$\mathfrak{GR}_E := \langle \lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1} \cup \mathrm{Aut}(E)\rangle \subseteq \mathrm{Aut}(C^*(E)). \tag{17}$$

By Proposition 3.6, there is a natural embedding of $\mathfrak{G}_E$ into the Weyl group $\mathfrak{W}_E$ of $C^*(E)$. Its restriction yields an embedding of $\mathfrak{GR}_E$ into the restricted Weyl group $\mathfrak{RW}_E$ of $C^*(E)$. Furthermore, $\lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1}$ is a normal subgroup of $\mathfrak{GR}_E$. Hence we have

$$\mathfrak{GR}_E = \lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1}\Gamma_E \quad \text{and} \quad \lambda(\mathcal{P}_E \cap \mathcal{U}_E)^{-1} \cap \Gamma_E = \Gamma_E^0. \tag{18}$$

Similarly to Theorem 3.7, in the restricted case we have the following.

Proposition 3.8. *Let E be a finite graph without sinks and sources in which every loop has an exit, and such that the relative commutant of $\mathcal{F}_E$ in $C^*(E)$ is trivial. Then*

$$\mathfrak{GR}_E \cap \mathrm{Inn}(C^*(E)) \subseteq \{\mathrm{Ad}(w) : w \in \mathcal{P}_E\}.$$

Proof. By Theorem 3.7, every element of $\mathfrak{GR}_E \cap \mathrm{Inn}(C^*(E))$ is of the form $\mathrm{Ad}(w)$ with $w \in \mathcal{S}_E$. Since this $\mathrm{Ad}(w)$ globally preserves $\mathcal{F}_E$, by hypothesis, Theorem 3.2 implies that $w \in \mathcal{P}_E$. □

Corollary 3.9. *Let E be a finite graph without sinks and sources in which every loop has an exit, and such that the relative commutant of $\mathcal{F}_E$ in $C^*(E)$ is trivial. Then every element of infinite order in $\mathfrak{GR}_E$ has infinite order in $\mathrm{Out}(C^*(E))$ as well.*

In general, it is a non-trivial matter to verify outerness of an automorphism of a graph algebra. The preceding corollary solves this problem for a significant class of automorphisms. In the case of Cuntz algebras, an analogous result was proved in [37, Theorem 6], and provided a convenient outerness criterion for permutative automorphisms – probably, the most studied class of automorphisms of $\mathcal{O}_n$.

4 Localized automorphisms

Throughout this section, we assume that E is a *finite graph without sinks.* Recall that an endomorphism λ_u of $C^*(E)$ is called localized if the corresponding unitary u belongs to a finite dimensional algebra $\mathcal{F}_E^k \cap \mathcal{U}_E$ for some k. Our main aim in this section is to produce an invertibility criterion for localized endomorphisms, analogous to [13, Theorem 3.2].

Let u be unitary in $\mathcal{F}_E^k \cap \mathcal{U}_E$, $k \geq 1$. Recall that $\lambda_u(x) = \mathrm{Ad}(u_r)(x)$ for all $x \in \mathcal{F}_E^r$ and $r \geq 1$. Following [37], for each pair $e, f \in E^1$ we define a linear map $a_{e,f}^u : \mathcal{F}_E^{k-1} \to \mathcal{F}_E^{k-1}$ by

$$a_{e,f}^u(x) = S_e^* u^* x u S_f, \quad x \in \mathcal{F}_E^{k-1}. \tag{19}$$

Denote $V_k := \mathcal{F}_E^{k-1}/\mathcal{D}_E^0$, the quotient vector space, and let $\mathcal{L}(V_k)$ be the space of linear maps from V_k to itself. Since $a_{e,f}^u(\mathcal{D}_E^0) \subseteq \mathcal{D}_E^0$, there is an induced map $\tilde{a}_{e,f}^u : V_k \to V_k$. Now we define A_u as the subring of $\mathcal{L}(V_k)$ generated by $\{\tilde{a}_{e,f}^u \ : \ e, f \in E^1\}$.

We denote by H the linear span of the generators S_e's. Let u be as above. Following [11, p. 386], we define inductively

$$\Xi_0 = \mathcal{F}_E^{k-1}, \quad \Xi_r = \lambda_u(H)^* \Xi_{r-1} \lambda_u(H), \ r \geq 1 \ . \tag{20}$$

$\{\Xi_r\}_r$ is a nonincreasing sequence of finite dimensional, self-adjoint subspaces of $\mathcal{F}_E^{k-1}$ and thus it eventually stabilizes. If $\Xi_p = \Xi_{p+1}$ then we have $\Xi_u := \bigcap_{r=0}^{\infty} \Xi_r = \Xi_p$.

If α, β are paths of length r, then we denote by $T_{\alpha,\beta}$ the linear map from $\mathcal{F}_E^{k-1}$ to itself defined by $T_{\alpha,\beta} = a_{\alpha_r,\beta_r}^u \cdots a_{\alpha_1,\beta_1}^u$. We have $T_{\alpha,\beta}(x) = S_\alpha^* \mathrm{Ad}(u_r^*)(x) S_\beta$ for all $x \in \mathcal{F}_E^{k-1}$. It easily follows from our definitions that the space Ξ_r is linearly spanned by elements of the form $T_{\alpha,\beta}(x)$, for $\alpha, \beta \in E^r$, $x \in \mathcal{F}_E^{k-1}$.

Theorem 4.1. *Let E be a finite graph without sinks, and let $u \in \mathcal{U}_E$ be a unitary in $\mathcal{F}_E^k$ for some $k \geq 1$. Then the following conditions are equivalent:*

(1) λ_u is invertible with localized inverse;

(2) the sequence of unitaries $\{\mathrm{Ad}(u_m^*)(u^*)\}_{m \geq 1}$ eventually stabilizes;

(3) the ring A_u is nilpotent;

(4) $\Xi_u \subseteq \mathcal{D}_E^0$.

Proof. (1) $\Rightarrow$ (3): If the inverse of λ_u is localized then there exists an l such that $\lambda_u^{-1}(\mathcal{F}_E^{k-1}) \subseteq \mathcal{F}_E^l$. Let $\alpha = (e_1, e_2, \cdots, e_l)$ and $\beta = (f_1, f_2, \cdots, f_l)$ be paths of length l, and consider an element $T_{\alpha,\beta} = a_{e_l,f_l}^u \cdots a_{e_2,f_2}^u a_{e_1,f_1}^u$ of A_u^l. Let $b \in \mathcal{F}_E^{k-1}$ and let $x = \lambda_u^{-1}(b)$. Then $x \in \mathcal{F}_E^l$ and we have $b = \lambda_u(x) = \mathrm{Ad}(u_l)(x)$. Therefore

$$T_{\alpha,\beta}(b) = a_{e_l,f_l}^u \cdots a_{e_2,f_2}^u a_{e_1,f_1}^u(b) = S_\alpha^* \mathrm{Ad}(u_l^*)(b) S_\beta = S_\alpha^* x S_\beta.$$

Since x can be written as $\sum_{|\gamma|=|\rho|=l} c_{\gamma,\rho}(x) S_\gamma S_\rho^*$ for some $c_{\gamma,\rho}(x) \in \mathbb{C}$, we have

$$T_{\alpha,\beta}(b) = S_\alpha^* \Big(\sum_{|\gamma|=|\rho|=l} c_{\gamma,\rho}(x) S_\gamma S_\rho^* \Big) S_\beta = \begin{cases} c_{\alpha,\beta}(x) P_{r(\alpha)}, & \text{if } r(\alpha) = r(\beta) \\ 0, & \text{if } r(\alpha) \neq r(\beta) \end{cases}$$

because $S_\alpha^* S_\gamma = P_{r(\alpha)}$ if $\alpha = \gamma$, and $S_\alpha^* S_\gamma = 0$ otherwise. This implies that $T_{\alpha,\beta}(b) \in \mathcal{D}_E^0$, and hence we see that $A_u^l = 0$.

(3) $\Rightarrow$ (4): Let $A_u^l = 0$ for some positive integer l. Then $T_{\alpha,\beta}(b) \in \mathcal{D}_E^0$ for all $b \in \mathcal{F}_E^{k-1}$ and all α, β such that $|\alpha| = |\beta| = l$. But this immediately yields $\Xi_l \subseteq \mathcal{D}_E^0$ and, consequently, $\Xi_u \subseteq \mathcal{D}_E^0$.

(4) $\Rightarrow$ (2): Let $\Xi_u \subseteq \mathcal{D}_E^0$, and let l be a positive integer such that $\Xi_l = \Xi_u$. Let $b \in \mathcal{F}_E^{k-1}$ and let $\alpha, \beta \in E^l$. Then $T_{\alpha,\beta}(b)$ belongs to $\mathcal{D}_E^0$ and thus it commutes with $\varphi^m(u)$ for all m, since u commutes with the vertex projections. Consequently, for each $r \geq 1$ we have

$$\begin{aligned}
\mathrm{Ad}(u_{l+r}^*)(b) &= \mathrm{Ad}(\varphi^{l-1+r}(u^*)\cdots\varphi^l(u^*))\Big(\sum_{\alpha,\beta\in E^l} S_\alpha T_{\alpha,\beta}(b) S_\beta^*\Big) \\
&= \sum_{\alpha,\beta\in E^l} S_\alpha \,\mathrm{Ad}(\varphi^{r-1}(u^*)\cdots u^*)(T_{\alpha,\beta}(b)) S_\beta^* \\
&= \sum_{\alpha,\beta\in E^l} S_\alpha T_{\alpha,\beta}(b) S_\beta^* \ .
\end{aligned}$$

Thus for each $b \in \mathcal{F}_E^{k-1}$ the sequence $\mathrm{Ad}(u_m^*)(b)$ stabilizes from $m = l+1$. Write $u^* = \sum_{e,f\in E^1} S_e b_{e,f} S_f^*$, for some $b_{e,f} \in \mathcal{F}_E^{k-1}$. Then for each m we have

$$\begin{aligned}
\mathrm{Ad}(u_{m+1}^*)(u^*) &= \sum_{e,f\in E^1} \mathrm{Ad}(\varphi(\varphi^{m-1}(u^*)\cdots\varphi(u^*)u^*))(S_e b_{e,f} S_f^*) \\
&= \sum_{e,f\in E^1} S_e \,\mathrm{Ad}(\varphi^{m-1}(u^*)\cdots\varphi(u^*)u^*)(b_{e,f}) S_f^*
\end{aligned}$$

and, consequently, the sequence $\mathrm{Ad}(u_m^*)(u^*)$ stabilizes from $m = l+2$.

(2) $\Rightarrow$ (1): Suppose that the sequence $\mathrm{Ad}(u_m^*)(u^*)$ eventually stabilizes. Hence $\mathrm{Ad}(u_r^*)(u^*) = w$ for all sufficiently large r. It follows that $\lambda_u(w) = \mathrm{Ad}(u_r)(w) = u^*$. Thus $\lambda_u(w)u = u^*u = 1$ and, consequently, λ_u is invertible with inverse λ_w. This completes the proof.

We also include a different and much more direct proof of implication (1) $\Rightarrow$ (2), that sheds additional light on the equivalent conditions of the theorem and is interesting in its own right.

Let λ_u be invertible, and suppose that there exists an $l \in \mathbb{N}$ and a unitary $v \in \mathcal{U}_E$ in $\mathcal{F}_E^l$ such that $\lambda_u\lambda_v = \mathrm{id}$. Then we have $\lambda_u(v)u = 1$. Since $v \in \mathcal{F}_E^l$, $u^* = \lambda_u(v) = \mathrm{Ad}(u_l)(v)$, and hence $\mathrm{Ad}(u_l^*)(u^*) = v$. Now for $r \geq 1$ we have

$$\begin{aligned}
\mathrm{Ad}(u_{l+r}^*)(u^*) &= \mathrm{Ad}(\varphi^{l+r-1}(u^*)\cdots\varphi(u^*)u^*)(u^*) \\
&= \varphi^{l+r-1}(u^*)\cdots\varphi^l(u^*)\mathrm{Ad}(u_l^*)(u^*)\varphi^l(u)\cdots\varphi^{l+r-1}(u) \\
&= \varphi^{l+r-1}(u^*)\cdots\varphi^l(u^*)v\varphi^l(u)\cdots\varphi^{l+r-1}(u) \\
&= v
\end{aligned}$$

since v commutes with $\varphi^m(u)$ for every $m \geq l$. Thus we can conclude that $\mathrm{Ad}(u_m^*)(u^*)$ stabilizes at v from $m = l$. $\square$

Remark 4.2. Let $u \in \mathcal{F}_E^1 \cap \mathcal{U}_E$, so that λ_u is quasi-free. Then $\mathcal{F}_E^0 = \mathcal{D}_E^0$. Thus $V_1 = \mathcal{D}_E^0/\mathcal{D}_E^0 = \{0\}$ and consequently each $\tilde{a}^u_{e,f}$ is a zero map. Therefore $A_u = \{0\}$ and Theorem 4.1 trivially implies that λ_u is an automorphism of $C^*(E)$.

If $u \in \mathcal{U}_E$ normalizes $\mathcal{D}_E$ then $\lambda_u(\mathcal{D}_E) \subseteq \mathcal{D}_E$. It may well happen that such a restriction is an automorphism of $\mathcal{D}_E$ even though λ_u is not invertible. For unitaries in the algebraic part of $\mathcal{F}_E$ this can be checked in a way similar to [13, Theorem 3.4]. Indeed, let $u \in \mathcal{U}_E \cap \mathcal{N}_{\mathcal{F}_E^k}(\mathcal{D}_E^k)$. Then it follows from (9) that $u \in \mathcal{N}_{\mathcal{F}_E}(\mathcal{D}_E)$. Furthermore, the subspace $\mathcal{D}_E^{k-1}$ of $\mathcal{F}_E^{k-1}$ is invariant under the action of all maps $a^u_{e,f}$, $e, f \in E^1$. We denote by $b^u_{e,f}$ the restriction of $a^u_{e,f}$ to $\mathcal{D}_E^{k-1}$, and by $\tilde{b}^u_{e,f}$ the map induced on $V_k^D := \mathcal{D}_E^{k-1}/\mathcal{D}_E^0$. Let A_u^D be the subring of $\mathcal{L}(V_k^D)$ generated by $\{\tilde{b}^u_{e,f} : e, f \in E^1\}$. Also, we consider a nested sequence of subspaces Ξ_r^D of $\mathcal{D}_E^{k-1}$, defined inductively as

$$\Xi_0^D = \mathcal{D}_E^{k-1}, \quad \Xi_r^D = \lambda_u(H)^* \Xi_{r-1}^D \lambda_u(H), \ r \geq 1. \tag{21}$$

Each Ξ_r^D is finite dimensional and self-adjoint. We set $\Xi_u^D := \bigcap_r \Xi_r^D$.

Theorem 4.3. *Let E be a finite graph without sinks and let $u \in \mathcal{U}_E \cap \mathcal{N}_{\mathcal{F}_E^k}(\mathcal{D}_E^k)$. Then the following conditions are equivalent:*

(1) λ_u restricts to an automorphism of $\mathcal{D}_E$;

(2) the ring A_u^D is nilpotent;

(3) $\Xi_u^D \subseteq \mathcal{D}_E^0$.

Proof. (1) $\Rightarrow$ (3): Since the algebraic part $\bigcup_{t=0}^\infty \mathcal{D}_E^t$ of $\mathcal{D}_E$ coincides with the linear span of all projections in $\mathcal{D}_E$, every automorphism of $\mathcal{D}_E$ restricts to an automorphism of $\bigcup_{t=0}^\infty \mathcal{D}_E^t$. Thus, there exists an l such that $(\lambda_u|_{\mathcal{D}_E})^{-1}(\mathcal{D}_E^{k-1}) \subseteq \mathcal{D}_E^l$. Let $\alpha = (e_1, e_2, \cdots, e_l)$ and $\beta = (f_1, f_2, \cdots, f_l)$ be in E^l, and let $R_{\alpha,\beta} := b^u_{e_l,f_l} \cdots b^u_{e_2,f_2} b^u_{e_1,f_1}$ be in $(A_u^D)^l$. Then the same argument as in the proof of implication (1)$\Rightarrow$(3) in Theorem 4.1 yields that $R_{\alpha,\beta}(d) \in \mathcal{D}_E^0$ for all $d \in \mathcal{D}_E^{k-1}$. Thus $(A_u^D)^l = \{0\}$. But as in the proof of implication (3)$\Rightarrow$(4) in Theorem 4.1, this implies that $\Xi_u^D \subseteq \mathcal{D}_E^0$.

(3) $\Rightarrow$ (2): Let $\Xi_u^D \subseteq \mathcal{D}_E^0$ and let l be a positive integer such that $\Xi_l^D = \Xi_u^D$. Let $d \in \mathcal{D}_E^{k-1}$ and let $\alpha, \beta \in E^l$. Then $R_{\alpha,\beta}(d)$ belongs to $\mathcal{D}_E^0$, and this entails $(A_u^D)^l = \{0\}$, i.e. the ring A_u^D is nilpotent.

(2) $\Rightarrow$ (1): Suppose that A_u^D is nilpotent. We show by induction on $r \geq k$ that all $\mathcal{D}_E^r$ are in the range of λ_u restricted to $\bigcup_{t=0}^\infty \mathcal{D}_E^t$.

Firstly, let $r = k$ and $d \in \mathcal{D}_E^k$. Similarly to the argument in the implication (4)$\Rightarrow$(2) of the proof of Theorem 4.1 one shows that the sequence $\mathrm{Ad}(\varphi^m(u^*) \cdots \varphi(u^*)u^*)(d)$ eventually stabilizes at some $f \in \bigcup_{t=0}^\infty \mathcal{D}_E^t$. It then follows that $d = \lambda_u(f)$.

For the inductive step, suppose that $r \geq k$ and $\mathcal{D}_E^r \subseteq \lambda_u(\bigcup_{t=0}^\infty \mathcal{D}_E^t)$. Since $\mathcal{D}_E^{r+1}$ is generated by $\mathcal{D}_E^r$ and $\varphi^r(\mathcal{D}_E^1)$, it suffices to show that $\varphi^r(y)$ belongs to $\lambda_u(\bigcup_{t=0}^\infty \mathcal{D}_E^t)$ for all $y \in \mathcal{D}_E^1$. However, $\varphi^r(y)$ commutes with u and $\varphi^{r-1}(y) \in \mathcal{D}_E^r$ is in $\lambda_u(\bigcup_{t=0}^\infty \mathcal{D}_E^t)$ by the inductive hypothesis. Thus the sequence

$$\mathrm{Ad}(\varphi^m(u^*) \cdots \varphi(u^*)u^*)(\varphi^r(y)) = \varphi(\mathrm{Ad}(\varphi^{m-1}(u^*) \cdots \varphi(u^*)u^*)(\varphi^{r-1}(y)))$$

eventually stabilizes at $\lambda_u^{-1}(\varphi^r(y)) \in \bigcup_{t=0}^\infty \mathcal{D}_E^t$. $\square$

5 Permutative automorphisms

Throughout this section, we assume that E is a *finite graph without sinks.* Our main goal in this section is to find a combinatorial criterion for invertibility of λ_u, $u \in \mathcal{P}_E$, analogous to [8, Corollary 4.12].

It will be useful for us to look at collections of paths of a fixed length beginning or ending at th same vertex. Hence we introduce the following notation. For $v, w \in E^0$, let $E^k_{v,*} := \{\alpha \in E^k : r(\alpha) = v\}$, $E^k_{*,v} := \{\alpha \in E^k : s(\alpha) = v\}$ and $E^k_{v,w} := \{\alpha \in E^k : r(\alpha) = v,\ s(\alpha) = w\}$. Then $E^k = \bigcup_{v \in E^0} E^k_{*,v} = \bigcup_{v \in E^0} E^k_{v,*} = \bigcup_{v,w \in E^0} E^k_{v,w}$, disjoint unions. If $u \in \mathcal{P}^k_E$, $k > 0$, then there exist permutations $\sigma_v \in \mathrm{Perm}(E^k_{v,*})$ such that

$$u = \sum_{v \in E^0} \sum_{\alpha \in E^k_{v,*}} S_{\sigma_v(\alpha)} S^*_\alpha. \tag{22}$$

A unitary $u \in \mathcal{P}^k_E$, $k > 0$, commutes with all the vertex projections if and only if there exist permutations $\sigma_{v,w} \in \mathrm{Perm}(E^k_{v,w})$ such that

$$u = \sum_{v,w \in E^0} \sum_{\alpha \in E^k_{v,w}} S_{\sigma_{v,w}(\alpha)} S^*_\alpha. \tag{23}$$

If the unitary $u \in \mathcal{P}^k_E \cap \mathcal{U}_E$ is understood, as in equation (23), then we will denote by $\sigma = \cup_{v,w \in E^0} \sigma_{v,w}$ the corresponding permutation of E^k. In that case, we will also write $\lambda_u = \lambda_\sigma$.

Now let $u \in \mathcal{P}^k_E \cap \mathcal{U}_E$, $e, f \in E^1$, and consider the linear map $a^u_{e,f}$, as defined in (19). With respect to the basis $\{S_\mu S^*_\nu : \mu, \nu \in E^{k-1}\}$ of $\mathcal{F}^{k-1}_E$ so ordered that the initial vectors span $\mathcal{D}^{k-1}_E$, the matrix of $a^u_{e,f}$ has the block form

$$a^u_{e,f} = \begin{pmatrix} b^u_{e,f} & c^u_{e,f} \\ 0 & d^u_{e,f} \end{pmatrix}, \tag{24}$$

similarly to [13, Section 4]. The first block corresponds to the subspace $\mathcal{D}^{k-1}_E$ of $\mathcal{F}^{k-1}_E$. Thus, the map $\tilde{a}^u_{e,f} \in \mathcal{L}(V_k)$ has a matrix

$$\tilde{a}^u_{e,f} = \begin{pmatrix} \tilde{b}^u_{e,f} & * \\ 0 & d^u_{e,f} \end{pmatrix}, \tag{25}$$

with the first block corresponding to the subspace V^D_k of V_k. Note that the passage from the space $\mathcal{F}^{k-1}_E$ to its quotient V_k does not affect the matrix for $d^u_{e,f}$ and thus there is no tilde over it in formula (25). It is an immediate corollary to Theorem 4.1 that endomorphism λ_u of $C^*(E)$ is invertible if and only if the following two conditions are satisfied.

Condition (b): the ring generated by $\{\tilde{b}^u_{e,f} : e, f \in E^1\}$ is nilpotent.
Condition (d): the ring generated by $\{d^u_{e,f} : e, f \in E^1\}$ is nilpotent.

The remainder of this section is devoted to the description of a convenient combinatorial interpretation of these two crucial conditions, similar to the one appearing in [13] and used in the analysis of permutative endomorphisms of the Cuntz algebras. By virtue of Theorem 4.3, Condition (b) alone is equivalent to the restriction of λ_u to the diagonal $\mathcal{D}_E$ being an automorphism.

5.1 Condition (b)

We fix $u \in \mathcal{P}_E^k \cap \mathcal{U}_E$ and denote by σ the corresponding permutation, as above. If $e \neq f$ then $b^u_{e,f} = 0$. Thus, it suffices to consider the ring generated by maps $b^u_e := b^u_{e,e}$, $e \in E^1$. Since $b^u_e(1) = P_{r(e)}$, the matrix of b^u_e has exactly one 1 in the row corresponding to each $\alpha \in E^{k-1}_{*,r(e)}$, and 0's elsewhere. Consequently, each b^u_e may be identified with a mapping

$$f^u_e : E^{k-1}_{*,r(e)} \to E^{k-1}_{*,s(e)}, \quad f^u_e(\alpha) = \beta, \tag{26}$$

whenever b^u_e has 1 in the α–β entry. If the unitary u is given by a permutation σ then

$$f^u_e(\alpha) = \beta \ \Leftrightarrow\ \exists g \in E^1 \text{ s.t. } \sigma(e, \alpha) = (\beta, g). \tag{27}$$

The product $b^u_e b^u_g$ corresponds to the composition $f^u_g \circ f^u_e$ (in reversed order of e and g). Now Condition (b) may be phrased in terms of mappings $\{f^u_e\}$ rather than $\{b^u_e\}$, as follows:

There exists an m such that for all $e_1, \ldots, e_m \in E^1$ if $T = f^u_{e_1} \circ \ldots \circ f^u_{e_m}$ then for all $v \in E^0$ and $\alpha \in E^{k-1}$ either $E^{k-1}_{*,v} \cap T^{-1}(\alpha) = \emptyset$ or $E^{k-1}_{*,v} \subseteq T^{-1}(\alpha)$.

Taking into account (26) above, we arrive at the following:

Condition (b): There exists an integer m such that for all $e_1, \ldots, e_m \in E^1$ either: (i) $f^u_{e_1} \circ \ldots \circ f^u_{e_m}$ has the empty domain, or (ii) its domain equals $E^{k-1}_{*,r(e_m)}$ and its range consists of exactly one element.

In the remainder of this section, notation (α, β) indicates either a single path in E^* or an ordered pair in the cartesian product $E^* \times E^*$. This will be clear from context.

Lemma 5.1. *Let E be a finite graph without sinks in which every loop has an exit. Let $u \in \mathcal{P}_E^k \cap \mathcal{U}_E$. Then Condition (b) holds for u (and hence $\lambda_u|_{\mathcal{D}_E}$ is an automorphism of $\mathcal{D}_E$) if and only if there exists a partial order $\leq$ on $\bigcup_{v \in E^0} E^{k-1}_{*,v} \times E^{k-1}_{*,v}$ such that:*

1. *if $v \in E^0 \setminus r(E^1)$ then each element of $E^{k-1}_{*,v} \times E^{k-1}_{*,v}$ is minimal, each diagonal element (α, α) is minimal, and there are no other minimal elements;*

2. *if $e \in E^1$ and $\alpha \neq \beta \in E^{k-1}_{*,r(e)}$ then $(f^u_e(\alpha), f^u_e(\beta)) \leq (\alpha, \beta)$.*

Proof. At first suppose that Condition (b) holds for u. Define a relation $\leq$ as follows. For any α, $(\alpha, \alpha) \leq (\alpha, \alpha)$. If $\gamma \neq \delta$ then $(\alpha, \beta) \leq (\gamma, \delta)$ if and only if there exists a sequence $e_1, \ldots, e_d \in E^1$, possibly empty, such that $\alpha = f^u_{e_1} \circ \cdots \circ f^u_{e_d}(\gamma)$ and $\beta = f^u_{e_1} \circ \cdots \circ f^u_{e_d}(\delta)$. Reflexivity and transitivity of $\leq$ are obvious. To see that $\leq$ is also antisymmetric, suppose that $(\alpha,\beta) \le (\gamma,\delta)$ and $(\gamma,\delta) \le (\alpha,\beta)$. If $(\alpha,\beta) \neq (\gamma,\delta)$ then, by definition of $\le$, $\alpha \neq \beta$, $\gamma \neq \delta$ and there exist edges $e_1,\ldots,e_d$, $g_1,\ldots,g_h$ such that $(\alpha,\beta) = (f^u_{e_1} \circ \cdots \circ f^u_{e_d})(\gamma,\delta)$ and $(\gamma,\delta) = (f_{g_1} \circ \cdots \circ f_{g_h})(\alpha,\beta)$. Then $(\alpha,\beta) = (f^u_{e_1} \circ \cdots \circ f^u_{e_d} \circ f_{g_1} \circ \cdots \circ f_{g_h})(\alpha,\beta)$. That is, $f^u_{e_1} \circ \cdots \circ f^u_{e_d} \circ f_{g_1} \circ \cdots \circ f_{g_h}$ has two distinct fixed points, a contradiction with Condition (b). Thus $(\alpha,\beta) = (\gamma,\delta)$ and $\le$ is also antisymmetric. Hence $\le$ is a partial order. By the very definition of $\le$, if $v \in E^0 \setminus r(E^1)$ then each element of $E^{k-1}_{*,v} \times E^{k-1}_{*,v}$ is minimal, and likewise each diagonal element (α,α) is minimal. If any other element were minimal for $\le$ then there would exist $\alpha \neq \beta$ and $e \in E^1$ such that $f^u_e(\alpha) = \alpha$ and $f^u_e(\beta) = \beta$. Thus f^u_e would have two distinct fixed points, contradicting Condition (b).

Conversely, if a partial order $\le$ with the required properties exists, then counting shows that each sufficiently long composition product of mappings $\{f^u_e\}$ either has the empty domain or its range consists of a single element (and the domain is as required, due to (26)). This completes the proof. □

Remark 5.2. If the conditions of Lemma 5.1 are satisfied and $e \in E^1$ is such that $s(e) = r(e)$, then the diagram of f^u_e is a rooted tree with the root being the unique fixed point, cf. [13, Section 4.1]. See Example 5.5 below.

5.2 Condition (d)

Again, we fix a $u \in \mathcal{P}^k_E \cap \mathcal{U}_E$ and denote by σ the corresponding permutation. It is easy to verify that for $e,g \in E^1$ each row of the matrix $d^u_{e,g}$ either may have 1 in one place and 0 elsewhere or may consists of all zeros. This matrix has 1 in (α,β) row and (γ,δ) column if and only if there exists an $h \in E^1$ such that $S_\alpha S^*_\beta = S^*_e u^* S_\gamma S_h S^*_h S^*_\delta u S_g$. In turn, this takes place if and only if

$$\sigma(e,\alpha) = (\gamma,h) \quad \text{and} \quad \sigma(g,\beta) = (\delta,h). \tag{28}$$

For each $e,g \in E^1$ we now define a mapping $f^u_{e,g}$, as follows. The domain $D(f^u_{e,g})$ of $f^u_{e,g}$ consists of all $(\alpha,\beta) \in E^{k-1}_{*,r(e)} \times E^{k-1}_{*,r(g)}$ for which the (α,β) row of $d^u_{e,g}$ is non-zero, and the corresponding value is $f^u_{e,g}(\alpha,\beta) = (\gamma,\delta) \in E^{k-1} \times E^{k-1}$ for (γ,δ) satisfying (28). Note that, by the very definition of $d^u_{e,g}$, in such a case we must necessarily have $\alpha \neq \beta$ and $\gamma \neq \delta$. We denote $\Psi_u := E^{k-1} \times E^{k-1} \setminus \{(\alpha,\alpha) : \alpha \in E^{k-1}\}$. We also denote by Δ_u the subset of Ψ_u consisting of all those (α,β) for which there exist $e,g \in E^1$ such that (α,β) belongs to the domain $D(f^u_{e,g})$.

It is a simple matter to verify that in terms of mappings $\{f^u_{e,g}\}$ Condition (d) may be rephrased as follows (cf. [13, Section 4.3]).

Condition (d): There exists an m such that for all $(e_1,g_1),\ldots,(e_m,g_m) \in \Psi_u$ the domain of the map $f^u_{e_1,g_1} \circ \ldots \circ f^u_{e_m,g_m}$ is empty.

The proof of the following lemma is essentially the same as that of [13, Lemma 4.10] and thus it is omitted.

Lemma 5.3. *Let E be a finite graph without sinks in which every loop has an exit. Let $u \in \mathcal{P}_E^k \cap \mathcal{U}_E$. Then Condition (d) holds for u if and only if there exists a partial order $\leq$ on Ψ_u such that:*

1. *the set of minimal elements coincides with $\Psi_u \setminus \Delta_u$;*

2. *if $e, g \in E^1$ and $(\alpha, \beta) \in D(f_{e,g}^u)$ then $f_{e,g}^u(\alpha, \beta) \leq (\alpha, \beta)$.*

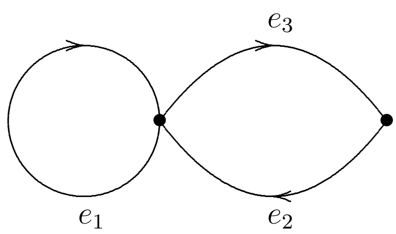
Combining Lemma 5.1 with Lemma 5.3 we obtain a combinatorial criterion of invertibility of permutative endomorphisms, similar to [13, Corollary 4.12].

Theorem 5.4. *Let E be a finite graph without sinks in which every loop has an exit, and let $u \in \mathcal{P}_E^k \cap \mathcal{U}_E$. Then the endomorphism λ_u is invertible if and only if conditions of Lemma 5.1 and Lemma 5.3 hold for u.*

5.3 Examples

We give two examples with small graphs illustrating the combinatorial machinery developed in the preceding section. In Example 5.5, we exhibit a proper permutative endomorphism of $C^*(E)$ which restricts to an automorphism of the diagonal MASA $\mathcal{D}_E$. On the other hand, in Example 5.6 we find an order 2 permutative automorphism of a Kirchberg algebra $C^*(E)$ with $K_0(C^*(E)) \cong \mathbb{Z} \cong K_1(C^*(E))$, which is neither quasi-free nor comes from a graph automorphism.

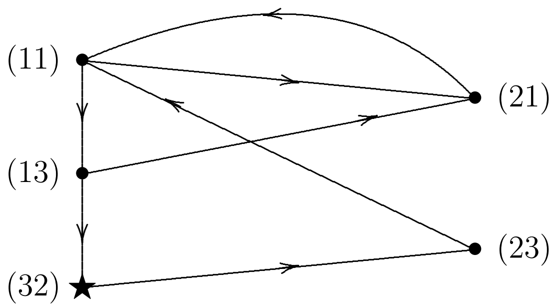
Example 5.5. Consider the following graph E:



At level $k = 2$, there are 2 permutations in $\mathcal{P}_E^2 \cap \mathcal{U}_E \cong \mathbb{Z}_2$. Denoting edge e_j simply by j, the non-trivial transposition is $(11, 32)$. The corresponding map $f_1 : \{1, 3\} \to \{1, 3\}$ is such that $f_1(1) = 3$ and $f_1(3) = 1$. Thus Condition (b) does not hold and, consequently, the corresponding endomorphism is surjective neither on $C^*(E)$ nor on $\mathcal{D}_E$.

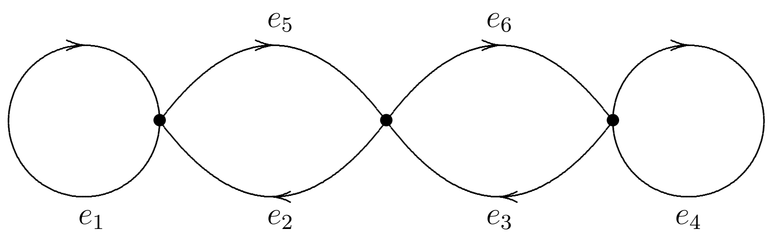
At level $k = 3$, there are 24 permutations in $\mathcal{P}_E^3 \cap \mathcal{U}_E \cong S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. These are generated by $\sigma = (132, 321)$, $\tau = (111, 132, 321)$, $\upsilon = (113, 323)$ and $\omega = (211, 232)$. Of these 24 permutations, only $\tau\upsilon$ and id satisfy Condition (b). However, $\tau\upsilon$ does not satisfy Condition (d). Thus, $\lambda_{\tau\upsilon}$ is a proper endomorphism of $C^*(E)$ (i.e., it is not surjective) which restricts to an automorphism of $\mathcal{D}_E$.

The maps $f_1 : \{(11), (13), (32)\} \to \{(11), (13), (32)\}$, $f_2 : \{(11), (13), (32)\} \to \{(21), (23)\}$, and $f_3 : \{(21), (23)\} \to \{(11), (13), (32)\}$ corresponding to permutation $\tau\upsilon$ and involved in verification of Condition (b) are illustrated in the following diagram.



Note that the diagram of the map f_1, corresponding to an edge whose source and range coincide, is a rooted tree. This is the left hand side of the diagram above, with the root (the unique fixed point for f_1) indicated by a star.

Example 5.6. Consider the following graph E:



At level $k = 2$, there are 8 permutations in $\mathcal{P}_E^2 \cap \mathcal{U}_E \cong \mathbb{Z}_2^3$. Denoting edge e_j simply by j, these are generated by transpositions $\sigma = (25, 63)$, $\tau = (11, 52)$ and $\upsilon = (36, 44)$. Of these 8 permutations, only σ and id satisfy Condition (b). Since σ satisfies Condition (d) as well, λ_σ is an automorphism of $C^*(E)$. We have

$$\begin{aligned}
\lambda_\sigma(S_2) &= S_6 S_3 S_5^* + S_2 S_1 S_1^*, \\
\lambda_\sigma(S_6) &= S_2 S_5 S_3^* + S_6 S_4 S_4^*, \\
\lambda_\sigma(S_j) &= S_j, \quad j = 1, 3, 4, 5,
\end{aligned}$$

and it follows immediately that $\lambda_\sigma^2 = \mathrm{id}$.

We note that in the present case $K_0(C^*(E)) \cong \mathbb{Z} \cong K_1(C^*(E))$. Thus $C^*(E)$ is not isomorphic to a Cuntz algebra, and hence this example is not covered in any way by the results of [13].

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