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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel   +49 7834 979 50
Fax   +49 7834 979 55
Email   admin@mfo.de
URL   www.mfo.de

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AN IDENTIFICATION THEOREM FOR $\mathrm{PSU}_6(2)$ AND ITS AUTOMORPHISM GROUPS

CHRIS PARKER AND GERNOT STROTH

Abstract. We identify the groups $\mathrm{PSU}_6(2)$, $\mathrm{PSU}_6(2){:}2$, $\mathrm{PSU}_6(2){:}3$ and $\mathrm{Aut}(\mathrm{PSU}_6(2))$ from the structure of the centralizer of an element of order 3.

1. Introduction

When classifying finite simple groups G one is sometimes confronted with the following situation. For a prime p, some (but perhaps not all) of the p-local subgroups of G containing a given Sylow p-subgroup S of G generate a subgroup H which is known to be isomorphic to a Lie type group in characteristic p. The expectation (or rather hope) is that $G = H$. In the case that H is a proper subgroup of G, one usually tries to prove that H contains *all* the p-local subgroups of G which contain S and then in a next step to prove that H is strongly p-embedded in G. This then leads to the conclusion that $G = H$. The last two steps are well understood, at least for groups with mild extra assumptions. However it might be that the first step cannot be made. One example of this phenomenon occurs with $F^*(H) \cong \Omega_7(3)$ and H embedded in the way just described in both the groups ${}^2\mathrm{E}_6(2)$ and $\mathrm{M}(22)$. In this case the normalizer of some root subgroup R of H is not contained in H. A similar example occurs with $F^*(H) \cong \Omega_8^+(3)$ embedded in F_2, the baby monster sporadic simple group. In a series of papers [14, 15] we will establish 3-local characterisations for all these groups, where we will forget the group H and just use information about $N_G(R)$ and thereby obtain more general theorems. We will finally identify the target groups by the centralizer of a certain involution or by the action on an appropriate building. In this paper we focus on an identification theorem that is required in both the identifications of $\mathrm{M}(22)$ and ${}^2\mathrm{E}_6(2)$. That is a 3-local characterisation theorem of $\mathrm{U}_6(2)$ and its automorphism groups. Indeed the centralizers of involutions in both $\mathrm{M}(22)$ and ${}^2\mathrm{E}_6(2)$ feature these groups prominently.

In earlier work [12] the first author proved the following result: let G be a finite group, S be a Sylow 3-subgroup of G and $Z = Z(S)$. Assume that $N_G(Z)$ is similar to a 3-normalizer in $\mathrm{PSU}_6(2)$. Then either Z is weakly closed in S or $G \cong \mathrm{PSU}_6(2)$. However, for our intended applications of such results as outlined above, we also need to identify the groups $\mathrm{PSU}_6(2){:}3$, $\mathrm{PSU}_6(2){:}2$ and $\mathrm{PSU}_6(2){:}\mathrm{Sym}(3)$ from their

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3-local data (here and throughout this work we use the Atlas [3] notation for group extensions). The addition of these automorphisms cause numerous difficulties.

Definition 1.1. *We say that X is similar to a 3-normalizer in a group of type* $\mathrm{PSU}_6(2)$ *provided the following conditions hold.*

(i) $Q = F^*(X)$ *is extraspecial of order* 3^5*;*
(ii) X/Q *is isomorphic to a subgroup of index at most* 6 *in the subgroup of* $\mathrm{GSp}_4(3)$ *which preserves a decomposition of the natural* 4*-dimensional symplectic space over* $\mathrm{GF}(3)$ *into a perpendicular sum of two non-degenerate* 2*-spaces; and*
(iii) $Q/Z(Q)$ *is an* X*-chief factor.*

A precise description of the possibilities for the group X/Q will be given in Section 3. Our theorem is as follows.

Theorem 1.2. *Suppose that G is a group, $Z \le G$ has order* 3 *and set $M = N_G(Z)$. If M is similar to a 3-normalizer in a group of type* $\mathrm{PSU}_6(2)$ *and Z is weakly closed in $F^*(M)$ but not in M, then* $G \cong \mathrm{PSU}_6(2)$, $\mathrm{PSU}_6(2)$:2, $\mathrm{PSU}_6(2)$:3 *or* $\mathrm{PSU}_6(2)$:Sym(3).

In the case that Z is weakly closed in M, then G could be a nilpotent group extended by a group similar to a 3-normalizer of type $\mathrm{PSU}_6(2)$. Thus the hypothesis that Z is not weakly closed in M is necessary to have an identification theorem. On the other hand, the hypothesis that Z is weakly closed in $F^*(M)$ is there to prevent further examples related to $\mathrm{F}_4(2)$ arising. We expect that the methods that we use here will also be applicable to this type of configuration, however the investigation of such a possibility would take a rather different road at the very outset of our proof and so the analysis of this possibility is not included here.

We now describe the layout of the paper and highlight a number of interesting features of the article. We begin in Section 2 with preliminary lemmas and background material. Noteworthy results in this section are Lemma 2.5 where we embellish the statement of Hayden's Theorem [9] to give the structure of the normal subgroup of index 3 and Lemma 2.11 where we use transfer theorems to show that a group with a certain specified 2-local subgroup has a subgroup of index 2. The relevance of such results to our proof is apparent as a look at the list of groups in the conclusion of our theorem shows. Let G, M and Z be as in the statement of Theorem 1.2 and let $S \in \mathrm{Syl}_3(M)$. In Section 3, we tease out the structure of M and establish much of the notation that is used throughout the proof of Theorem 1.2.

In Section 4, we determine the structure the normalizer of a further 3-subgroup which we call J and turns out to be the Thompson subgroup of S. The fact that $N_G(J)$ is not contained in M is a consequence of the hypothesis that Z is not weakly closed in M. We find in Lemma 4.6 that $N_G(J)/J \cong 2 \times \mathrm{Sym}(6)$ or $\mathrm{Sym}(6)$. With this information, after using a transfer theorem, we are able to apply [12] and do so in Theorem 4.7 to get that $G \cong \mathrm{PSU}_6(2)$ or $\mathrm{PSU}_6(2)$:3

if $N_M(S)/S \cong \mathrm{Dih}(8)$. Thus from this stage on we assume that $N_M(S)/S \cong 2 \times \mathrm{Dih}(8)$ and $N_M(J)/J \cong 2 \times \mathrm{Sym}(6)$. With this assumption, our target groups all have a subgroup of index 2. Our plan is to determine the structure of a 2-central involution r, apply Lemma 2.11 and then apply Theorem 4.7 to the subgroup of index 2. The involution we focus on is contained in M and centralizes a subgroup of $F^*(M)$ isomorphic to 3_+^{1+2}. But before we can make this investigation we need to determine the centralizers of another subgroup (for now we will call it X) which has order either 3 or 9. It turns out we may apply the theorems of Hayden [9] and Prince [16] to get $E(C_G(X)) \cong \mathrm{SU}_4(2)$. At this juncture, given the 3-local information that we have gathered, we can construct an extraspecial 2-subgroup Σ of order 2^9 in $K = C_G(r)$. In Theorem 5.5 we show that $N_K(\Sigma)/\Sigma \cong \mathrm{Aut}(\mathrm{SU}_4(2))$, $(\mathrm{SU}_4(2) \times 3){:}2$ or $\mathrm{Sp}_6(2)$. In our target groups the possibility $\mathrm{Sp}_6(2)$ does not arise and we will say more about this shortly.

In Section 6 we show that Σ is strongly closed in $N_K(\Sigma)$ with respect to K and then we apply Goldschmidt's Theorem to get that $K = N_K(\Sigma)$. At this stage we know the centralizer of a 2-central involution and so we prove the theorem in Section 6. We mention here that when $K/\Sigma \cong \mathrm{Sp}_6(2)$ we apply [17] to obtain $G \cong \mathrm{Co}_2$ and then eliminate this group as it does not satisfy our hypothesis on the structure of M. One should wonder if the configuration involving $\mathrm{Sp}_6(2)$ could be eliminated at an earlier stage. However, as Co_2 contains $\mathrm{PSU}_6(2){:}2$ as a subgroup of index 2300, these groups are intimately related. A 3-local identification of Co_2 can be found in [13].

Our notation follows that in [1], [6] and [7]. As mentioned earlier we use Atlas [3] notation for group extensions. We also use [3] as a convenient source for information about subgroups of almost simple groups. Often this information can be easily gleaned from well-known properties of classical groups. For odd p, the extraspecial groups of exponent p and order p^{2n+1} are denoted by p_+^{1+2n}. The extraspecial 2-groups of order 2^{2n+1} are denoted by 2_+^{1+2n} if the maximal elementary abelian subgroups have order 2^{1+n} and otherwise we write 2_-^{1+2n}. We hope our notation for specific groups is self-explanatory. In addition, for a subset X of a group G, X^G denotes that set of G-conjugates of X. If $x, y \in H \leq G$, we often write $x \sim_H y$ to indicate that x and y are conjugate in H. All the groups in this paper are finite groups.

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2. Preliminaries

In this section we gather preliminary results for our proof of Theorem 1.2. For a group G with Sylow p-subgroup P and $u \in P$, a G-conjugate v of u is said to be *extremal* in P if $C_P(v)$ is a Sylow p-subgroup of $C_G(v)$.

Lemma 2.1. *Suppose that p is a prime and G is a group. Let P a Sylow p-subgroup of G and Q be a proper normal subgroup of P such that P/Q is cyclic. Assume there is $u \in P \setminus Q$ such that*

(a) *no conjugate of u^p is contained in $P \setminus Q$; and*
(b) *any extremal conjugate of u in P is contained in $Q \cup Qu$.*

Then either G has a normal subgroup N with G/N cyclic and $u \notin N$ or there is $g \in G$ such that

(i) *$u^g \in Q$;*
(ii) *u^g is extremal in P; and*
(iii) *$C_P(u)^g \le C_P(u^g)$.*

Proof. See [7, Proposition 15.15] or [18, Corollary 5.3.1]. □

Lemma 2.2. *Suppose that p is a prime, G is a group and $P \in \mathrm{Syl}_p(G)$.*

(i) *Assume that there is a normal subgroup Q of P such that P/Q is cyclic and that $y \in P \setminus Q$ has order p. If every extremal conjugate of y in P is contained in Qy, then G has a normal subgroup N with $y \notin N$ and G/N cyclic.*
(ii) *Assume that $P \le M \le G$, $y \in P \setminus M'$ has order p and that, if $x \in G$ with $y^x \in P$ extremal, then there is $g \in M$ such that $y^x = y^g$. Then $y \notin G'$.*
(iii) *Assume that $J = J(P)$ is the Thompson subgroup of P. If J is elementary abelian and $J \not\le N_G(J)'$, then $J \not\le G'$.*

Proof. (i) This follows from 2.1.

(ii) As M/M' is abelian, there is $N \le M$ such that $M' \le N$, $y \notin N$, $M = NP$ and $P/(P \cap N)$ is cyclic. Set $Q = P \cap N$. Now for $g \in M$ with $y^g \in P$ we have that $y^g \in Qy$. Hence by assumption $y^x \in Qy$ for all $x \in G$ such that y^x is extremal in P. Now (ii) follows from (i).

(iii) Set $M = N_G(J)$ and pick $y \in J \setminus M'$. Assume that $g \in G$ and y^g is extremal in P. Then $C_P(y^g) \in \mathrm{Syl}_p(C_G(y^g))$. Since $C_G(y)$ contains J, we have $C_P(y^g)$ contains a G-conjugate of J. Since J is weakly closed in P, we have $J \le C_P(y^g)$. But then $y^g \in C_P(J) \le J$. Since M controls fusion in J, we now have that $y^g = y^m$ for some $m \in M$. Now (iii) follows from (ii). □

Lemma 2.3. *Suppose that F is a field, V is an n-dimensional vector space over F and $G = \mathrm{GL}(V)$. Assume that q is quadratic form of Witt index at least 1 and $\mathcal{S}$ is the set of singular 1-dimensional subspaces of V with respect to q. Then the stabiliser in G of $\mathcal{S}$ preserves q up to similarity.*

Proof. See [13, Lemma 2.10]. □

Lemma 2.4. *Suppose that p is an odd prime, $X = \mathrm{GL}_4(p)$ and V is the natural $\mathrm{GF}(p)X$-module. Let $A = \langle a, b\rangle \le X$ be elementary abelian of order p^2 and assume that $[V, a] = C_V(b)$ and $[V, b] = C_V(a)$ are distinct and of dimension 2. Let $v \in V \setminus [V, A]$. Then A leaves invariant a non-degenerate quadratic form with respect to which v is a singular vector. In particular, X contains exactly two conjugacy classes of subgroups such as A. One is conjugate to a Sylow p-subgroup of $\mathrm{GO}_4^+(p)$ and the other to a Sylow p-subgroup of $\mathrm{GO}_4^-(p)$.*

Proof. See [13, Lemma 2.11]. □

Lemma 2.5. *Suppose that X is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$ and that H is a group with an element d such that $C_H(d) \cong X$. Let $P \in \mathrm{Syl}_3(C_H(d))$ and E be the elementary abelian subgroup of P of order 27. Assume that E does not normalize any non-trivial $3'$-subgroup of H, that d is not H-conjugate to its inverse and X has a normal subgroup of index 3. Then $X = C_H(d)$.*

Proof. Notice first of all that $P \in \mathrm{Syl}_3(H)$. Let H_1 be a normal subgroup of H of index 3 and set $E_1 = E \cap H$. So $C_{H_1}(d) \cong 3_+^{1+2}{:}\mathrm{Q}_8$ and E_1 has order 9. Suppose that $x \in E_1 \setminus \langle d\rangle$. Then, as x is conjugate to its inverse and d is not, d is the unique conjugate of d in E_1. Furthermore, d is not conjugate to any element of $E \setminus H'$ and so d is the unique conjugate of d in E. Since x is not conjugate to d, we have that $E_1 = \langle d, x\rangle$ is a Sylow 3-subgroup of $C_{H_1}(x)$. As $E_1/\langle x\rangle$ is self-normalizing in $C_{H_1}(x)/\langle x\rangle$, $C_{H_1}(x)$ has a normal 3-complement T by Burnside's Theorem. However $C_{H_1}(x)$ is normalized by E and so $T = 1$ by hypothesis. It follows that $C_H(x) = E$ for all $x \in E_1 \setminus \langle d\rangle$.

Let $y \in E \setminus H_1$. Then, as before, E_1 is a Sylow 3-subgroup of $C_{H_1}(y)$. Since d is not conjugate to any non-trivial element of $E_1 \setminus \{d\}$, we have $N_H(E_1) \le X$. So $N_{C_{H_1}(y)}(E_1) = \langle E_1, s\rangle$ where s is an element of order at most two in X. Since $[E_1, s] < E_1$, Grün's Theorem [6, Theorem 4.4] implies that $C_{H_1}(y)$ has a subgroup L of index at least $|E_1 : [E_1, s]|$ with Sylow 3-subgroup $[E_1, s]$. Since L is normalized by E, we also have $O_{3'}(L) = 1$. Hence, if $s = 1$, then $C_H(y) \le X$ which means that $C_H(y) = E$. So suppose that $[E_1, s]$ has order 3. Then, as $C_H([E_1, s]) = E$, we have $[E_1, s]$ is self-centralizing in L. Applying the other Feit-Thompson Theorem [5] to L and using $O_{3'}(L) = 1$, we now have that either $L \cong \mathrm{Sym}(3)$ with $L = N_{X\cap H_1}([E_1, s])$ or $L \cong \mathrm{PSL}_3(2)$ or $\mathrm{Alt}(5)$. The latter two cases are eliminated as L is normalized by E_1 and the centralizers of all of the non-trivial elements of E_1 are soluble. Therefore, $C_H(y) = C_X(y) \le X$ for all $y \in E \setminus E_1$.

Now let $R \in \mathrm{Syl}_2(X)$ and $r \in R$ be an involution. Then $C_X(r) = R\langle d, y\rangle$ for some $y \in E \setminus E_1$. Furthermore, as d is the unique conjugate of $d \in \langle d, y\rangle$,

$$N_{C_H(r)}(\langle d, y\rangle) = N_X(\langle d, y, r\rangle) = \langle d, y, r\rangle$$

and so $C_H(r)$ has a normal 3-complement U by Burnside's Theorem. Finally

$$U = \langle C_U(w) \mid w \in \langle d, y\rangle^{\#}\rangle \leq X$$

as $C_H(w) \leq X$ for each $w \in \langle d, y\rangle^{\#}$. It follows that $U = R$. But then $R \in \mathrm{Syl}_2(H)$ and $r \in Z^*(H)$ by [2]. As $[O_3(X), r] = O_3(X)$, we conclude $O_3(X) \leq O_{2'}(X)$ and deduce $H = X$ from the Frattini Argument. This completes the proof of the lemma. □

Theorem 2.6 (Hayden)**.** *Suppose that X is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$ and that H is a group with an element d such that $C_H(d) \cong X$. Let $P \in \mathrm{Syl}_3(C_H(d))$ and E be the elementary abelian subgroup of P of order 27. If E does not normalize any non-trivial $3'$-subgroup of H and d is not H-conjugate to its inverse, then either $H = X$ or $H \cong \mathrm{PSp}_4(3)$.*

Proof. By [9] either $H \cong \mathrm{PSp}_4(3)$ or H has a normal subgroup of index 3. The result now follows from Lemma 2.5. □

Theorem 2.7 (A. Prince)**.** *Suppose that Y is isomorphic to the centralizer of 3-central element of order 3 in $\mathrm{PSp}_4(3)$ and that X is a finite group with a non-trivial element d such that $C_X(d) \cong Y$. Let $P \in \mathrm{Syl}_3(C_X(d))$ and E be the elementary abelian subgroup of P of order 27. If E does not normalize any non-trivial $3'$-subgroup of X and d is X-conjugate to its inverse, then either*

(i) $|X : C_X(d)| = 2$;
(ii) *X is isomorphic to $\mathrm{Aut}(\mathrm{SU}_4(2))$; or*
(iii) *X is isomorphic to $\mathrm{Sp}_6(2)$.*

Proof. See [16, Theorem 2]. □

Lemma 2.8. *Let G be a finite group and S be a Sylow 3-subgroup of G. Set $Z = Z(S)$ and $M = N_G(Z)$. Suppose that G^* is a normal subgroup of G and set $M^* = M \cap G^*$. Assume that the following hold:*

(i) $|M^*| = 2^7.3^6$;
(ii) *$M^* \geq QR = O_{3,2}(M^*)$, where Q is extraspecial of order 3^5;*
(iii) *$O^2(M^*) = (S \cap M^*)R$ has index 2 in M^*; and*
(iv) *Q/Z is a M^*-chief factor.*

If $N_{G^*}(J \cap G^*) \not\leq M^*$, then $G^* \cong \mathrm{PSU}_6(2)$ and G is a subgroup of $\mathrm{Aut}(\mathrm{PSU}_6(2))$ such that $G/G^* \cong M/M^*$.

Proof. Since $N_{G^*}(J \cap G^*) \not\leq M^*$, Z is not weakly closed in $S \cap G^*$. The conditions imposed on the structure of M^* mean that M^* is similar to a 3-normalizer in $\mathrm{PSU}_6(2)$ [12, Definition 1]. Hence [12, Theorem 1] gives the result. □

Lemma 2.9. *Suppose that E is an extraspecial 2-group and $x \in \mathrm{Aut}(E)$ is an involution. If $C_E(x) \geq [E, x]$, then $[E, x]$ is elementary abelian.*

	$\mathrm{Aut}(\mathrm{SU}_4(2))$	$\mathrm{Sp}_6(2)$	$\dim C_V(u_j)$
u_1	$2_+^{1+4}.(\mathrm{Sym}(3)\times\mathrm{Sym}(3))$	$2^9.(\mathrm{Sym}(3)\times\mathrm{Sym}(3))$	2
u_2	$2^6.3$	$2^7.3$	4
u_3	$2\times\mathrm{Sym}(6)$	$2^5.\mathrm{Sym}(6)$	4
u_4	$2\times(\mathrm{Sym}(4)\times 2)$	$2^9.3$	4

TABLE 1. Involutions in $\mathrm{Sp}_6(2)$ and $\mathrm{Aut}(\mathrm{SU}_4(2))$. The involutions in the first row are the *unitary transvections.* The involutions in the last two rows are those which are in $\mathrm{Aut}(\mathrm{SU}_4(2))\setminus\mathrm{SU}_4(2)$.

Proof. Let $\langle e\rangle = Z(E)$. We show that every element of $[E,x]$ has order 2. Let $f\in[E,x]$. Then fe has the same order as f. Thus we may suppose that $f=[h,x]$ for some $h\in E$. As $x[h,x]=[h,x]x$ by hypothesis, we have

$$\begin{aligned} f^2 &= [h,x][h,x] = h^{-1}xhx[h,x] = h^{-1}xh[h,x]x \\ &= h^{-1}xhh^{-1}xhxx = 1 \end{aligned}$$

as required. This proves the lemma. □

For use in Lemma 2.11 and Section 6, we collect some facts about the action of $\mathrm{Sp}_6(2)$ and $\mathrm{Aut}(\mathrm{SU}_4(2))$ on their irreducible 8-dimensional module V over GF(2). Recall that $\mathrm{Aut}(\mathrm{SU}_4(2))\cong \mathrm{O}_6^-(2)$ is a subgroup of $\mathrm{Sp}_6(2)$ [3, page 46]. We will frequently use that fact that as $\mathrm{SU}_4(2)$-module, V is the natural 4-dimensional $\mathrm{GF}(4)\mathrm{SU}_4(2)$-module regarded as a module over GF(2). We will often refer to this as the *natural* $\mathrm{SU}_4(2)$-module.

Proposition 2.10. *Let $X\cong\mathrm{Sp}_6(2)$ and $Y\cong\mathrm{Aut}(\mathrm{SU}_4(2))$. Assume that V is the 8-dimensional irreducible module for X (and hence Y) over* GF(2). *Then the following hold:*

(i) *X and Y both possess exactly four conjugacy classes of involutions. In Table 1 we list the four classes of involutions and give structural information about the centralizers in both groups as can be found in* [3, page 46,page 26].

(ii) *X and Y have orbits of length* 135 *and* 120 *on the non-zero elements of V. We call elements of the orbits non-singular and singular vectors respectively. Suppose that x is singular and y is non-singular. Then*

$$|C_Y(x)| = 2^7\cdot 3,\qquad |C_X(x)| = 2^9\cdot 3\cdot 7.$$
$$C_Y(y)\cong 3_+^{1+2}.\mathrm{SDih}(16),\qquad C_X(y)\cong \mathrm{G}_2(2).$$

(iii) *X and Y both have exactly three conjugacy classes of elements of order* 3. *They are distinguished by their action on V. They have centralizers of dimension* 0, 2 *and* 4. *The elements with centralizer of dimension* 2 *are* 3*-central and centralize only non-singular vectors in $V^\#$.*

(iv) *For* $u \in Y$ *an involution,* $\dim C_V(u)$ *is given in column* 4 *of Table 1.*
(v) *Let* u *be a unitary transvection. Then* $C_{Y'}(u)$ *acts on* $C_V(u)/[V,u]$ *with orbits of length* 1*,* 6 *and* 9*.*
(vi) *If* u *is a unitary transvection,* $S_2 \le C_Y(u)$ *has order* 3 *and* $C_{C_V(u)/[V,u]}(S_2) \neq 0$*, then* $\dim C_V(S_2) = 2$*.*
(vii) *For* $S \in \mathrm{Syl}_2(Y)$*,* V *has a unique subspace of each dimension which is* S*-invariant.*
(viii) Y *does not contain a fours group all of whose non-trivial elements are unitary transvections.*
(ix) $C_V(u_4)$ *is generated by non-singular vectors.*

Proof. (i) From [3, page 27, page 47], we see that $\mathrm{Aut}(\mathrm{SU}_4(2))$ and $\mathrm{Sp}_6(2)$ both possess exactly four conjugacy classes of involutions.

(ii) By Witt's lemma Y has exactly two orbits on the non-zero elements of $V^\#$ and they correspond to the singular and the non-singular vectors. Since $2^8 - 1$ does not divide $|X|$, these orbits are also orbits under the action of X. Since the lengths of the orbits are 135 and 120, using [3, page 26, page 46] we get the given structure of the stabilizers.

(iii) As Y contains a Sylow 3-subgroup of X, we find representatives of all X-conjugacy classes of elements of order 3 in Y. By [3, page 27] there are exactly three conjugacy classes of elements of order 3 in Y, which we easily distinguish by their action on V. We have elements, which are fixed point free, which have centralizer of dimension 2 and those which have centralizer of dimension 4. In particular, these elements are not fused in X.

Let $d \in Y$ have 2-dimensional fixed space on V. Then as $C_V(d)$ is perpendicular to $[V,d]$ we deduce that $C_V(d)$ is non-singular (a 1-dimensional non-singular GF(4)-space).

(iv) For the unitary transvection u we have that $\dim[V,u] = 2$. Suppose that u is not a unitary transvection but $u \in Y'$. Then, as V supports the structure of a vector space over GF(4), we have that $[V,u]$ is 2-dimensional and so $\dim[V,u] = 4$. If u is an involution in $Y \setminus Y'$, then we see that it induces a field automorphism on Y' and so again $\dim[V,u] = 4$.

(v) Let u be a unitary transvection. Then $C_{Y'}(u)$ acts on $C_V(u)/[V,u]$ as the group $\mathrm{GU}_2(2) \cong \mathrm{Sym}(3) \times 3$ and has three orbits one of length 1, one of length 6 and one of length 9.

(vi) From (v), a Sylow 3-subgroup S_1 of $C_{Y'}(u)$ contains two subgroups of order 3 whose centralizer in $C_V(u)/[V,u]$ is of order 4 and two which are fixed point free. As the elements of order three in $C_{Y'}(u)$ act the same way on $[V,u]$ as on $V/C_V(u)$, the elements with fixed points on $C_V(u)/[V,u]$ have centralizer in V of dimension 2, as by (iii) there are no elements of order three which centralize a subspace of dimension 6. Now by coprime action we get that one subgroup of order three in S_1 centralizes in V a subspace of dimension 4 and acts fixed point freely on $C_V(u)/[V,u]$, one acts fixed point freely V and the other two centralize a subspace of dimension 2 in V.

(vii) Let $S \in \mathrm{Syl}_2(Y)$ and $S_1 = S \cap Y'$. Then, as V is the natural 4-dimensional unitary module for Y', we have that S_1 fixes unique subspace of GF(2)- dimension 2, 4 and 6. Since S contains a field automorphism, we now get the result.

(viii) Suppose that $F = \langle x_1, x_2\rangle$ is a fours group with all non-trivial elements unitary transvections. Then, as $x_3 = x_1x_2$, is also a unitary transvection, we get that $C_V(x_1) = C_V(x_2)$. But then $C_V(x_1)$ is normalized by $\langle C_Y(x_1), C_Y(x_2)\rangle = Y$, which is impossible.

(xi) Let y be a non-singular vector. By (ii), we have that $C_Y(y) \cong 3_+^{1+2}.\mathrm{GL}_2(3)$. This group contains an involution u in $Y \setminus Y$. If u is conjugate to u_3 (in Table 1), then $C_{Y'}(u) \cong \mathrm{Sym}(6)$ acts transitively on $C_V(u)^\#$ and so $C_V(u)^\#$ contains only non-singular vectors. Since $\dim C_V(u) = 4$, this is impossible. Therefore v is conjugate to u_4 and $y \in C_V(u) = [V,u]$. Since $C_{C_{Y'}(y)}(u)$ has order 6, there are eight conjugates of y in $C_V(u)$. Hence $C_V(u)$ is generated by non-singular elements. □

In the next lemma the group denoted by $(\mathrm{SU}_4(2)\times 3){:}2$ the subgroup of index 2 in $\mathrm{Aut}(\mathrm{SU}_4(2)) \times \mathrm{Sym}(3)$ which is not expressible as a direct product.

Lemma 2.11. *Assume that G is a group, $t \in G$ is an involution, $H = C_G(t)$ and $Q = F^*(H)$ is extraspecial of order 2^9. If $H/Q \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $(\mathrm{SU}_4(2)\times 3){:}2$ and $Q/\langle t\rangle$ is the natural $F^*(H/Q)$–module, then G has a subgroup of index 2.*

Proof. We let $S \in \mathrm{Syl}_2(H)$ and note that, as $Z(S) = Z(Q) = \langle t\rangle$, we have $S \in \mathrm{Syl}_2(G)$. Let $\overline{H} = H/\langle t\rangle$. We first show that

(2.11.1) $t^G \cap Q = \{t\}$.

Assume that $u \sim_G t$ with $u \in Q \setminus \langle t\rangle$. Then $\overline{u}$ is singular in $\overline{Q}$ and so we may suppose that $\langle \overline{u}\rangle = Z(\overline{S})$. Now $C_Q(u)$ contains an extraspecial group of order 2^7. As there is no such subgroup in H/Q, we have that $t \in Q_u = O_2(C_G(u))$. Note that $\Phi(Q_u \cap Q) \le \langle u\rangle \cap \langle t\rangle = 1$. Hence $Q_u \cap Q$ is elementary abelian. As Q is extraspecial of order 2^9, we deduce that $|Q\cap Q_u| \le 2^5$. Since the 2-rank of H/Q is 4 and $|C_{Q_u}(t)| = 2^8$, we infer that $|Q\cap Q_u|$ is either 2^4 or 2^5. Furthermore, because $C_H(u)Q \ge S$, we have that $Q\cap Q_u$ is a normal subgroup of S. We know that $\overline{Q}$ is a GF(4)-module for $F^*(H/Q)$. Let $\overline{U}$ be the one-dimensional GF(4)-space in $\overline{Q}$ containing $\overline{u}$, U be its preimage in H and set $R = C_G(U)$. Since U, $Q_u \cap Q$ and R are normalized by S, Proposition 2.10 (viii) implies $U \le Q_u \cap Q \le R$. Assume that $|Q_u \cap Q| = 2^5$. Then, as $(Q_u \cap H)Q$ is a normal subgroup of $C_H(u)Q$ and the subgroups of Q containing $Q \cap Q_u$ are non-abelian, there exists an involution $w \in Q_u$ such that $\langle wQ\rangle = Z(S/Q)$ is the unitary transvection group centralizing $\overline{R}$ (again using Proposition 2.10 (viii)). Thus we have

$$[Q_u \cap Q, w] \le [R,w] \cap \langle u\rangle = \langle t\rangle \cap \langle u\rangle = 1,$$

which is impossible as $Q_u \cap Q$ is a maximal abelian subgroup of Q_u. Thus $|Q_u \cap Q| = 2^4$. Since $|(Q_u \cap Q)/U| = 2$, we now have a contradiction to the fact that $C_{R/U}(C_H(u)) = 1$ by Proposition 2.10 (v). Thus (2.11.1) holds.

By Proposition 2.10 (i), H/Q has exactly two conjugacy classes of involutions not in H'/Q. We choose representatives $\widetilde{x}, \widetilde{y} \in S/Q$ for these conjugacy classes and fix notation so that $C_{F^*(H/Q)}(\widetilde{x}) \cong \mathrm{Sp}_4(2)$ and $C_{F^*(H/Q)}(\widetilde{y}) \cong 2 \times \mathrm{Sym}(4)$. We have that $|[\overline{Q}, \widetilde{x}]| = |[\overline{Q}, \widetilde{y}]| = 2^4$ by Proposition 2.10 (iv). Let $z \in H$ with $z^2 \in \langle t \rangle$ be such that zQ is either $\widetilde{x}$ or $\widetilde{y}$. Then $C_H(z)Q/Q = C_{H/Q}(zQ)$. Let $T \in \mathrm{Syl}_2(C_H(z))$. Then $T' \cap Z(T) \le T \cap H'$ and $Z(T) \cap H' \le Q$ as $Z(T) = \langle z, C_Q(z) \rangle$. Thus, by (2.11.1), we have $t^G \cap T' \cap Z(T) = \{t\}$. In particular, $T \in \mathrm{Syl}_2(C_G(z))$. It follows that z is not conjugate to t in G and that $t^G \cap Z(T) = \{t\}$. We record these observations as follows:

(2.11.2) Let $z \in S \setminus (S \cap H')$ be such that $z^2 \in \langle t \rangle$ and $T \in \mathrm{Syl}_2(C_H(z))$. Then $T \in \mathrm{Syl}_2(C_G(z))$, $t^G \cap Z(T) = \{t\}$ and $t^G \cap H \subset H'$.

Now let $z_1 \in S$ be such that $z_1 Q = \widetilde{x}$. Since $C_{H/Q}(z_1 Q)$ contains an element fQ of order 5 with f of order 5 acting fixed point freely on $\overline{Q}$, we see that $C_{Q\langle z \rangle}(f)$ has order 4. Let $z \in C_Q(f)$ have minimal order so that $zQ = z_1 Q$. Then $z^2 \in \langle t \rangle$. Suppose that $g \in G$ and $z^g \in S \cap H'$ is extremal in S. Then $C_S(z^g) \in \mathrm{Syl}_2(C_G(z^g))$. Now let $T \in \mathrm{Syl}_2(C_H(z))$. Then $T \in \mathrm{Syl}_2(C_G(z))$ by (2.11.2). Hence $T^g \in \mathrm{Syl}_2(C_G(z^g))$ and there is a $w \in C_G(z^g)$ such that $T^{gw} = C_S(z^g)$. Now, by (2.11.2), $t^G \cap Z(T^{gw}) = \{t^{gw}\}$ and of course $t^G \cap C_S(z^g) = \{t\}$ as $t \in Z(H)$. Thus $gw \in H$, which is impossible as $z \in H \setminus H'$, $z^g \in H'$ and $z^{gw} = z^g$. Hence there are no extremal conjugates of z in $S \cap H'$. Since also $z^2 \in \langle t \rangle$ and $t^G \cap H \subset H'$, Theorem 2.1 implies that G has a subgroup of index 2 as claimed. □

3. The finer structure of M

Suppose that G is a group, $Z \le G$ has order 3 and set $M = N_G(Z)$. Assume that M is similar to a 3-normalizer in a group of type $\mathrm{PSU}_6(2)$. Let $S \in \mathrm{Syl}_3(M)$ and $Q = F^*(M) = O_3(M)$.

Lemma 3.1. *The following hold.*

(i) $Z = Z(S) = Z(Q)$, $N_G(S) \le M$ *and* $S \in \mathrm{Syl}_3(G)$;
(ii) $3 \le |S/Q| \le 3^2$;
(iii) Q *has exponent* 3; *and*
(iv) *the commutator map from* $Q/Z \times Q/Z$ *to* Z *is an* M/Z*-invariant non-degenerate symplectic form.*

Proof. (i) Since $C_M(Q) \le Q$, we have that $Z = Z(Q) = Z(S)$. Therefore $N_G(S) \le N_G(Z) = M$ and, in particular, $S \in \mathrm{Syl}_3(N_G(S)) \subseteq \mathrm{Syl}_3(G)$.

(ii) This follows straight from the definition of a 3-normalizer in a group of type $\mathrm{PSU}_6(2)$.

(iii) Since Q/Z is a chief factor, Q has exponent 3.
(iv) See [10, III(13.7)].

□

By definition $\overline{M} = M/Q$ is isomorphic to a subgroup of index at most 6 in the subgroup of $\mathrm{GSp}_4(3)$ which preserves a decomposition of the natural 4-dimensional symplectic space into a perpendicular sum of two non-degenerate 2-spaces. We first describe this subgroup of $\mathrm{GSp}_4(3)$. We denote it by $\overline{\mathbf{M}}$ the boldface type is supposed to indicate that this is a subgroup of $\mathrm{GSp}_4(3)$ which contains (the image of) $\overline{M}$ but may be greater than it. Similarly $\overline{\mathbf{S}}$ is a Sylow 3-subgroup of $\overline{\mathbf{M}}$ which contains $\overline{S}$.

We have $\overline{\mathbf{M}}$ contains a subgroup of index 2 which is contained in $\mathrm{Sp}_4(3)$ and is isomorphic to the wreath product of $\mathrm{Sp}_2(3) \cong \mathrm{SL}_2(3)$ by a group of order 2. For $i = 1, 2$, we let $\overline{\mathbf{M_i}} \cong \mathrm{SL}_2(3)$, $\overline{\mathbf{R_i}} = O_2(\overline{\mathbf{M}}_i) \cong \mathrm{Q}_8$ and $\overline{\mathbf{S_i}} = \overline{\mathbf{S}} \cap \overline{\mathbf{M_i}}$. We let $\overline{\mathbf{t}_1}$ be an involution in $\overline{\mathbf{M}}$ which negates the symplectic form and normalizes $\overline{\mathbf{S}_1}$ and $\overline{\mathbf{S}_2}$. Note that, for $i = 1, 2$, $\overline{\mathbf{M}}_i\langle\overline{\mathbf{t}_1}\rangle \cong \mathrm{GSp}_2(3) \cong \mathrm{GL}_2(3)$. Next select an involution $\overline{\mathbf{t}_2}$ which commutes with $\overline{\mathbf{t}_1}$, preserves the symplectic form, normalizes $\overline{\mathbf{S}}$ and conjugates $\overline{\mathbf{M_1}}$ to $\overline{\mathbf{M_2}}$. With this notation we have

$$\overline{\mathbf{M}} = \overline{\mathbf{M_1 M_2}}\langle\overline{\mathbf{t}_1}, \overline{\mathbf{t}_2}\rangle.$$

Now $\overline{M}$ is a subgroup of $\overline{\mathbf{M}}$ which has index at most 6. In particular, $\overline{S}$ has index at most 3 in $\overline{\mathbf{S}}$. Since $\overline{\mathbf{R}_1\mathbf{R}_2}$ is contained in all subgroups of $\overline{\mathbf{M}}$ of index 6, M contains subgroups R_1 and R_2 isomorphic to Q_8 such that $[R_1, R_2] = 1$ and $\overline{R_i} = \overline{\mathbf{R}}_i$ for $i = 1$ and 2. Set $R = R_1R_2$. Let $T \in \mathrm{Syl}_2(M)$ with $T \geq R$. Since $\overline{M}$ acts irreducibly on Q/Z, there is an involution t in T which maps to either $\overline{\mathbf{t}_2}$ or $\overline{\mathbf{t_1 t_2}}$. We denote this involution by t_2 or t_1t_2 as appropriate. If $T > \langle t\rangle R$, then T contains an involution t_1 which maps to $\overline{\mathbf{t}_1\mathbf{M_1M_2}}$. Finally, note that if $|T| = 2^7$, $N_M(S) = Z(R)S\langle t\rangle$ and if $|T| = 2^8$, then $N_M(S) = Z(R)S\langle t_1, t_2\rangle$. This discussion proves the following lemma.

Lemma 3.2. *There are exactly three possibilities for a Sylow 2-subgroup T of M. Moreover, the following hold. are as follows:*

(i) $T = R\langle t_2\rangle$, $N_M(S) = SZ(R)\langle t_2\rangle$ *and* $N_M(S)/S \cong \mathrm{Dih}(8)$*;*
(ii) $T = R\langle t_1t_2\rangle$, $N_M(S) = SZ(R)\langle t_1t_2\rangle$ *and* $N_M(S)/S \cong \mathrm{Dih}(8)$*; and*
(iii) $T = R\langle t_1, t_2\rangle$, $N_M(S) = SZ(R)\langle t_1, t_2\rangle$ *and* $N_M(S)/S \cong 2 \times \mathrm{Dih}(8)$. □

For $i = 1, 2$, let $r_i \in Z(R_i)^\#$ and set $Q_i = [Q, r_i] = [Q, R_i]$. Note that, as $\overline{r_1r_2} \in Z(\overline{M})$ and Q/Z is irreducible as an $\overline{M}$-module, r_1r_2 inverts Q/Z. Let A be the preimage of $C_{Q/Z}(S)$. So A is the second centre of S.

Lemma 3.3. *The following hold.*

(i) $Q_1 = [Q, R_1] = C_Q(R_2)$, $Q_2 = [Q, R_2] = C_Q(R_1)$ *and both are normal in* S*;*
(ii) $Q_1 \cong Q_2 \cong 3^{1+2}_+$, $[Q_1, Q_2] = 1$ *and* $Q = Q_1Q_2$*;*

(iii) $A = [Q, S] = [Q_1, S][Q_2, S]$ *is elementary abelian of order* 3^3*; and*

Proof. (i) This follows directly from the action of M on Q as $\overline{R_1}$ and $\overline{R_2}$ are normalized by $\overline{S}$.

(ii) We have that $C_Q(r_1)$ and $[Q, r_1]$ commute by the Three Subgroup Lemma. Since, for $i = 1, 2$, $[Q, r_i] = [Q, R_i]$ has order 3^3 it follows that $Q_1 \cong 3_+^{1+2}$. As r_1r_2 inverts Q/Z, r_2 inverts $C_{Q/Z}(r_1)$ and so $C_Q(r_1) = Q_2$. In particular, Q_1 and Q_2 commute and $Q = Q_1Q_2$.

(iii) From the description of M/Q, we have $A = [Q_1, S][Q_2, S]$. Since $[Q_1, S]$ and $[Q_2, S]$ have order 9, they are elementary abelian. Hence A is elementary abelian of order 3^3 by (ii). □

Because, for $i = 1, 2$, r_i inverts Q_i/Z, if M happens to contain the involution t_1, we may and do adjust t_1 by multiplying by elements from $Z(R)$ so that t_1 inverts A/Z. Therefore

Lemma 3.4. *If* $|M|_2 = 2^8$*, then* t_1 *inverts* A *and centralizes* Q/A. □

We now define a subgroup which will play a prominent role in all the future investigations. Set

$$J = C_S(A).$$

It will turn out that J is the Thompson subgroup of S.

Lemma 3.5. *The following hold:*

(i) $|S : J| = 3^2$, $J \cap Q = A$ *and* $S = JQ$*;*
(ii) *if* $|M|_2 = 2^8$, $N_M(J)/J \cong 2 \times 3^2{:}\mathrm{Dih}(8)$, t_1 *inverts* J *and* J *is abelian; and*
(iii) *if* $|M|_2 = 2^7$, $N_M(J)/J \cong 3^2{:}\mathrm{Dih}(8)$.

Proof. By Lemma 3.3(iii), A is elementary abelian of order 3^3. Furthermore, by the definition of J, J is a normal subgroup of $N_M(S)$. Since $[S, A] = Z$, the 3-structure of $\mathrm{GL}_3(3)$ shows that $|S/J| \le 3^2$. As $J \cap Q = C_Q(A) = A$, we infer that $|S : J| = 3^2$ and $S = JQ$. Thus (i) holds.

As $N_M(J) = N_M(J \cap Q) = N_M(A)$ and $[S, A] = Z$, we see $N_M(S) = N_M(J)$ and so (iii) and the first assertion of (ii) follow from Lemma 3.2. Suppose that $|M|_2 = 2^8$. Then $t_1 \in M$. Now t_1 inverts S/Q, centralizes Q/A and inverts A by Lemma 3.4. Thus t_1 inverts J and so J is abelian. This concludes the proof of (ii) and completes the verification of the lemma. □

Note that $|J| = 3^4$ if $|S/Q| = 3$ and $|J| = 3^5$ if $|S/J| = 3^2$.

Lemma 3.6. *We have* $C_G(J) = J$.

Proof. As $Z \le J$, we have $C_G(J) = C_M(J)$. Then, as $JQ = S$ by Lemma 3.5(i), we have $C_M(J)Q/Q \le C_{M/Q}(S/Q)$ and the result follows from Lemma 3.5 (ii) and (iii) and the definition of J.

□

Lemma 3.7. *Every element of Q is conjugate in M to an element of A.*

Proof. It suffices to prove that every element of Q/Z is conjugate to an element of A/Z. Let $w \in Q/Z$. Then $w = x_1x_2$ where $x_i \in Q_i/Z$ by Lemma 3.3 (ii). Since, by Lemma 3.3 (iii), for $i = 1, 2$, $(A \cap Q_i)/Z$ has order 3 and R_i acts transitively on Q_i/Z, there exists $u_i \in R_i$ such that $w^{u_1u_2} = x_1^{u_1}x_2^{u_2} \in A/Z$. This proves the claim. □

4. The structure of the normalizer in G of J

For the remainder of the paper assume the hypothesis of the Theorem 1.2. Thus we have M, Q, S and Z as in Section 3 and additionally we have that Z is weakly closed in Q and not in M. In this section we determine the possible structures of $N_G(J)$.

Lemma 4.1. *If Z is not weakly closed in J, then J is elementary abelian and coincides with the Thompson subgroup of S. In particular, $N_G(J)$ controls fusion in J.*

Proof. Choose $X \in Z^G$ with $X \neq Z$ and $X \leq J$. Set $K = AX$. As Z is weakly closed in Q and $J = C_S(A)$, we have that K is elementary abelian of order 3^4. In particular, if $|J| = 3^4$, then $K = J$ is elementary abelian.

Suppose that $|J| = 3^5$, then $|J : K| = 3$ and $|S/Q| = 3^2$. We claim that J is abelian. Set $Q_X = O_3(N_G(X))$. As K has index 3 in J, K is normal in J and, as $[Q, X] \leq A$, K is normalized by Q. Therefore K is normal in $S = JQ$ by Lemma 3.5 (i). If $C_S(X) = K$, then $|X^S| = 3^3$ and, in particular, every element of K which is not conjugate to an element of Z is contained in A. Now $K \cap Q_X$ has order either 3^2 or 3^3 and, so, as X is weakly closed in Q_X, $K \cap Q_X$ is generated by elements which are not conjugate to elements of Z. It follows that $X \leq K \cap Q_X \leq A$ and this contradicts $X \not\leq Q$. Therefore $C_S(X) \neq K$. If $C_S(X) \not\leq J$, then $Z = [A, C_S(X)] \leq C_S(X)' \leq Q_X$ and this contradicts the fact that X is weakly closed in Q_X. So $C_S(X) \leq J$. But then we have $K \leq Z(J)$ and so J is abelian as claimed.

Suppose that $B \leq S$ is abelian and $|B| \geq |J|$. Then, as $|B \cap Q| \leq 3^3$, we have $BQ = S$ and then $(B \cap Q)/Z \leq C_{Q/Z}(S) = A/Z$. Thus $B \leq C_S(A) = J$. Hence J is the Thompson subgroup of S. It follows that $N_M(J)$ controls fusion in J. In particular, X and Z are conjugate in $N_M(J)$. Since $\Phi(J) \leq A$, $X \not\leq \Phi(J)$ and hence $Z \not\leq \Phi(J)$. Therefore $Z(S) \cap \Phi(J) = 1$. As $\Phi(J)$ is normal in S, we get $\Phi(J) = 1$ and J is elementary abelian. This completes the proof of the lemma. □

Lemma 4.2. *Assume that Z is not weakly closed in J and set $J_0 = \langle Z^{N_G(J)} \rangle$. Then*

(i) $|Z^{N_G(J)}| = 10$ *and, if $X \in Z^{N_G(J)}$ with $X \neq Z$, $|X^Q| = 3^2$;*
(ii) $N_G(J)$ *acts two transitively on $Z^{N_G(J)}$; and*

(iii) $|J_0Q/Q| = 3$ *and* J_0Q/Q *is normalized by* $N_M(S)/Q$.

Proof. Let $\mathcal{Y} = Z^{N_G(J)}$ and $X \in \mathcal{Y}$ with $X \neq Z$. Of course $X \not\leq Q$ as Z is weakly closed in Q. If $C_Q(X) \not\leq J$, then, as X centralizes A, $C_Q(X)$ has order 81 and consequently is non-abelian and we have $Z = C_Q(X)' \leq O_3(C_G(X))$. However X is weakly closed in $O_3(C_G(X))$ with respect to G and so this is impossible. Thus $C_Q(X) = A$ has order 3^3 and, in particular, $X^S = X^{JQ} = X^Q$ has order 3^2 and so $|\mathcal{Y}| \equiv 1 \pmod 9$. Since $N_G(J)$ controls fusion in J by Lemma 4.1, $N_G(J)$ acts transitively on $\mathcal{Y}$ and $|N_G(J)/J| = |N_M(J)/J||\mathcal{Y}|$ by Lemma 3.6. As $|J| = 3^4$ or 3^5 and J is self-centralizing and elementary abelian by Lemmas 3.6 and 4.1, $|N_G(J)/J|$ divides $|\mathrm{GL}_5(3)|$. If $|J| = 3^4$, then, as no subgroup of order three in A which is not Z is conjugate to Z, J contains at most 28 conjugates of Z. This means that $|\mathcal{Y}| = 10, 19$ or 28. On the other hand, $|\mathrm{GL}_4(3)|_{3'} = 2^9 \cdot 5 \cdot 13$ and so in this case $|\mathcal{Y}| = 10$. So assume from now on that $|J| = 3^5$. Then J contains 121 subgroups of order 3 and 12 of these are contained in A and are not conjugate to Z as Z is weakly closed in Q. Since $|\mathrm{GL}_5(3)|_{3'} = 2^{10} \cdot 5 \cdot 11^2 \cdot 13$ and $|\mathcal{Y}| \equiv 1 \pmod 9$, the only candidates for $|\mathcal{Y}|$ are 10, 55 and 64. We recall from Lemma 3.5 that $|N_M(J)/J| = 2^i \cdot 3^2$ where $i = \{3, 4\}$ and, if $|N_M(J)/J| = 2^4 \cdot 3^2$, then $t_1J \in Z(N_G(J)/J)$ by 3.5(ii) and therefore t_1 normalizes every member of $\mathcal{Y}$.

Suppose that $|\mathcal{Y}| = 55$. Then $|N_G(J)/J| = 2^i \cdot 3^2 \cdot 5 \cdot 11$ where $i \in \{3, 4\}$. Let $E \in \mathrm{Syl}_{11}(N_G(J)/J)$. Then, as the normalizer of a cyclic subgroup of order 11 in $\mathrm{GL}_5(3)$ has order $2 \cdot 5 \cdot 11^2$, the normalizer in $N_G(J)/J$ of E has order dividing 110. In particular, E is not normal in $N_G(J)/J$. If $|N_M(J)|_2 = 2^4$, then t_1J normalizes E. So in any case the number of conjugates of E in $N_G(J)/J$ divides $2^3 \cdot 3^2 \cdot 5$ and is divisible by $2^2 \cdot 3^2$ and this is impossible as it must also be equivalent to 1 mod 11.

Suppose that $|\mathcal{Y}| = 64$. Then $|N_G(J)/J| = 2^{10} \cdot 3^2$ or $2^9 \cdot 3^2$. In particular, $N_G(J)$ is soluble. Since $|\mathcal{Y}| = 64$, we have that $J = \langle \mathcal{Y} \rangle$. If $1 \neq K \leq J$ is normal in $N_G(J)$, then K is normal in S and consequently $Z \leq K$. But then $\mathcal{Y} \subseteq K$ and so $K = J$. Thus $N_G(J)$ acts irreducibly on J. Since $J = 3^5$ and $N_G(J)/J$ is not abelian, Schur's Lemma implies that $|Z(N_G(J)/J)|$ divides 2 and, additionally, $O_3(N_G(J)/J) = 1$. Let $L = O_{3,2}(N_G(J))$. By Clifford's Theorem [6, Theorem 4.3.1], J is completely irreducible as an L-module and $N_G(J)$ acts transitively on the homogeneous summands of J restricted to L. Since J has dimension 5 as a $\mathrm{GF}(3)N_G(J)$-module, and 5 does not divide $|N_G(J)|$, we have that J is homogeneous as an L-module. It follows that J is either a direct sum of five 1-dimensional L-modules or is irreducible as an L-module. It the first case, we get that $[L, N_G(J)] \leq J$ and this contradicts $O_3(N_G(J)) = Q$. Thus J is an irreducible L-module. However, the degrees of irreducible L/Q-modules over the algebraic closure of $\mathrm{GF}(3)$ are all powers of 2 [11, 15.13] and this again implies that L is cyclic and $O_3(N_G(J)) > Q$. Since $|\mathcal{Y}| \neq 55$ or 64, we must have $|\mathcal{Y}| = 10$

as claimed in the first part of (i). Since we have also shown that $C_Q(X) = A$ the remaining parts of (i) also hold.

Part (ii) follows directly from (i).

Now with $J_0 = \langle Z^{N_G(J)} \rangle$, we have $\langle X^Q \rangle Q = XQ$ is normalized by $N_M(S)$ and $|XQ/Q| = 3$. This is (iii). □

Lemma 4.3. *Suppose that $X \in Z^G \setminus \{Z\}$ and $X \leq S$. Then, for $i = 1, 2$, $[X, R_i] \not\leq Q$.*

Proof. We suppose that $[X, R_1] \leq Q$ and seek a contradiction. Let $Q_X = O_3(N_G(X))$ and W be the full preimage of $C_{Q/Z}(X)$. Then $|W| = 3^4$. Since R_1 acts irreducibly on Q_1/Z and $[Q_1, QX]$ is R_1-invariant, we have $Q_1 \leq W$. Hence $W = Q_1 A \cong 3 \times 3_+^{1+2}$ and $Z(W) = A \cap Q_2$.

If $C_W(X)$ is non-abelian, then, as $C_W(X)Q_X/Q_X$ is abelian, $Z = C_W(X)' \leq Q_X$. Since X is weakly closed in Q_X by assumption and $Z \neq X$, we have a contradiction.

Thus $C_W(X)$ is abelian. Since W is non-abelian and XZ is normalized by W, we get that $|C_W(X)| = 3^3$. Because $C_W(X)$ is abelian and W is not, it follows that $A \cap Q_2 \leq C_W(X)$. Furthermore, we have $|C_W(X) \cap Q_1| = 3^2$ and thus, as R_1 acts transitively on the subgroups of order 9 in Q_1, we may adjust X by conjugating by an element of R_1 and arrange for $W \cap Q_1 = A \cap Q_1$. But then $W = A$ and $X \leq J$. Put $J_0 = \langle X^{N_G(J)} \rangle$. Then by Lemma 4.2 (ii) $J_0 Q = XQ$ is normalized by $N_M(S)$. Since $N_M(S)$ does not normalize R_1, we have $[X, R_1 R_2] \leq Q$, and this contradicts the structure of M. Therefore $[X, R_i] \not\leq Q$ for both $i = 1$ and 2. □

Lemma 4.4. *Assume that $X \in Z^G$ with $X \leq S$. Then $X \leq J$. In particular, Z is not weakly closed in J.*

Proof. Suppose that $X \leq S$ and $X \not\leq J$. Then $[A, X] = Z$ and $|C_A(X)| = 3^2$. By Lemma 4.3 , XQ acts non-trivially on both $R_1 Q/Q$ and $R_2 Q/Q$ and so $C_A(X) = C_Q(X)$. On the other hand AX is normalized by Q and so AX contains at least, and hence exactly, 28 conjugates of Z. In particular, $C_A(X)X$ contains 10 conjugates of Z and three subgroups of order 3 which are not conjugate to Z. Set $Q_X = O_3(N_G(X))$. Then the only conjugate of Z contained in $C_A(X)X \cap Q_X$ is X. Since the subgroups of order 3 in $C_A(X)$ which are not conjugate to Z generate $C_A(X)$, we get $C_A(X)X \cap Q_X = X$. So $|C_A(X)Q_X/Q_X| = 3^2$. By Lemma 4.3 two of the non-trivial cyclic subgroups $C_A(X)Q_X/Q_X$ do not have representatives from Z^G. Since $C_A(X)X$ contains only three subgroups of order 3 which are not conjugate to Z, we have a contradiction. Therefore, if $X \in Z^G$ and $X \leq S$, $X \leq J$ as claimed. □

Set

$$J_0 = \langle Z^{N_G(J)} \rangle.$$

By Lemmas 4.2, 4.3 and 4.4, we have $|J_0 Q/Q| = 3$, $J_0 \cap Q = A$ and $J_0 Q/Q$ does not centralize either $R_1 Q/Q$ or $R_2 Q/Q$. In particular, $|J_0| = 3^4$. We record these facts in the first part of the next lemma.

Lemma 4.5. *The following hold.*

(i) $|J_0| = 3^4$, $|J_0Q/Q| = 3$, $J_0 \cap Q = A$ *and* J_0Q/Q *acts non-trivially on both* R_1Q/Q *and* R_2Q/Q*;*
(ii) $N_G(J) = N_G(J_0)$*; and*
(iii) $C_G(J_0) = C_G(J) = J$.

Proof. From the construction of J_0 we have $N_G(J_0) \geq N_G(J)$. Since $N_G(J)$ is transitive on the subgroups of J which are G-conjugate to Z, we get that $N_G(J_0) = N_G(J)N_M(J_0)$. Hence, as $N_M(J_0Q) = N_M(S) \leq N_G(J)$, (ii) holds. Obviously $C_G(J_0) \leq C_M(J_0) \leq C_M(A) = J$ so (iii) also holds. □

Define $F = O^2(N_G(J))\langle r_2\rangle$. Then

Theorem 4.6. *The following hold:*

(i) *The action of* $N_G(J)$ *on* J_0 *preserves a non-degenerate quadratic form* q *of* $-$*-type;*
(ii) $Z^{N_G(J)}$ *is the set of singular one-dimensional subspaces with respect to* q*;*
(iii) $N_G(J)/J \cong 2 \times \mathrm{Sym}(6)$ *or* $\mathrm{Sym}(6)$*; and*
(iv) $F/J \cong \mathrm{Sym}(6)$ *and* $|[J, r_2]| = 3$*. Furthermore* $[r_2, J] \leq J_0$ *and* $[J, F] \leq J_0$*.*

Proof. Let $X \leq J$ be conjugate to Z but not equal Z. For $i = 1, 2$, using Lemma 4.5 (i), we have that $|[J_0, Q_i]| = 3^2$ and $[J_0, Q_i, Q_i] = Z$. Furthermore, $[J_0, Q_i]$ is centralized by Q_{3-i}. Hence we have $[J_0, Q_i] = C_{J_0}(Q_{3-i})$. By Lemma 2.4, there exists a non-degenerate quadratic form q on J_0 which is preserved by Q and such that the elements of X are singular vectors. It follows that with respect to q, the elements of $\bigcup X^Q$ are singular. Furthermore, as $Z = C_{J_0}(Q)$, Z also consists of singular vectors. Now with respect to the form bilinear for f associated with q, none of the non-trivial elements of $\bigcup X^Q$ are perpendicular to the non-trivial elements of Z. It follows that XZ contains exactly two singular subspace, namely X and Z. Since $N_G(J)$ acts two transitively on $Z^{N_G(J)}$ by Lemma 4.2 (ii), we infer that if $X, Y \in Z^{N_G(J)}$ with $X \neq Y$, then XY contains exactly two members of $Z^{N_G(J)}$. Now suppose that $a \in Q \setminus J$ is such that aJ acts quadratically on J_0. Then, for $X \in Z^{N_G(J)} \setminus Z$, X centralizes $[J_0, a]$ and normalizes $[J_0, a]\langle a\rangle$ and so $|[X, a]| = 3$ as $[X, a] \neq 1$. It follows that $X[X, a]$ contains three members of $Z^{N_G(J)}$ namely X, X^a and X^{a^2}. This contradiction shows that no non-trivial element of S/J acts quadratically on J_0. If q was of $+$-type, this would not be the case. Hence q is of $-$-type. We now have that $Z^{N_G(J)}$ is the set of singular one spaces in J_0 with respect to q. Since $N_G(J)$ preserves this set, we have that $N_G(J)/J$ is isomorphic to a subgroup of $\mathrm{GO}_4^-(3)$ from Lemma 2.3. Because $N_{N_G(J)}(Z)$ has index 10 in $N_G(J)$, we deduce that $|N_G(J)| = 2^4.5.3^2$ if either Lemma 3.2 (i) or (ii) holds and $|N_G(J)| = 2^5.5.3^2$ if Lemma 3.2 (iii) holds. In particular, $O^2(N_G(J)/J) \cong \Omega_4^-(3) \cong \mathrm{Alt}(6)$. Now using the structure of $N_M(S)$ given in Lemma 3.2 we infer that $N_G(J)/J \cong \mathrm{Sym}(6)$ or $\mathrm{GO}_4^-(3) \cong 2 \times \mathrm{Sym}(6)$. We have now established (i), (ii) and (iii).

We know that $S/Q = JQ/Q$ is centralized by r_2 and that $[Q, r_2] = Q_2$. It follows that $[J, r_2] \le Q_2 \cap J = A \cap Q_2$ and, as $[A \cap Q_2, r_2]$ has order 3, we now have that $[J, r_2] = [A, r_2]$ is a non-central cyclic subgroup of Q. In particular, $[J, r_2] \le A \le J_0$. Since $|[J_0, r_2]| = 3$ we get that r_2 has determinant -1 on J_0. Hence we have $r_2 \notin O^2(N_G(J))$ and so we conclude that $F/J \cong \mathrm{Sym}(6)$ and that all the parts of (iv) hold. □

Theorem 4.7. *If $N_M(S)/S \cong \mathrm{Dih}(8)$, then $G \cong \mathrm{PSU}_6(2)$ or $\mathrm{PSU}_6(2){:}3$.*

Proof. Since $N_M(S)/S \cong \mathrm{Dih}(8)$, we have that $N_G(J)/J \cong \mathrm{Sym}(6)$ from Theorem 4.6 (ii). Since $[J, r_2] \le J_0$, we infer that J/J_0 is centralized by $N_M(J)$. If $J > J_0$, then, by Lemma 2.2, G has a normal subgroup G^* at index 3. If $J = J_0$, then set $G = G^*$. Now $M \cap G^*$ satisfies the hypothesis of Theorem 2.8. Hence $G^* \cong \mathrm{PSU}_6(2)$ and this proves the theorem. □

In light of Theorem 4.7 and Lemma 3.2, from here on we may assume that $N_M(S) = SZ(R)\langle t_1, t_2\rangle$. In particular from Theorem 4.6, we have

$$\begin{aligned} N_M(S)/S &\cong 2 \times \mathrm{Dih}(8);\\ N_G(J)/J &\cong 2 \times \mathrm{Sym}(6);\ \text{and}\\ C_{F/J}(r_2J) &\cong 2 \times \mathrm{Sym}(4). \end{aligned}$$

Furthermore, as t_1 inverts J, we have $t_1 J \in Z(N_G(J)/J)$.

Lemma 4.8. *We have*

$$C_S(Q_1) = C_S(R_1) = C_S(Q_1R_1),$$
$$C_J(Q_1) = C_J(R_1) = C_J(Q_1R_1)$$

and $|J : C_J(Q_1)| = 3^2$.

Proof. We have that $[Q_1, C_S(R_1)]$ is R_1-invariant and is a proper subgroup of Q_1. Therefore $[Q_1, C_S(R_1)] \le Z$. Hence $[Q_1, C_S(R_1), R_1] = 1$ and $[C_S(R_1), R_1, Q_1] = 1$ and thus the Three Subgroups Lemma implies that $[Q_1, R_1, C_S(R_1)] = 1$. Since $Q_1 = [Q_1, R_1]$, we have $C_S(R_1) \le C_S(Q_1)$. Now, as Q_1 is normal in S and extraspecial of order 3^3, $|S : C_S(Q_1)Q_1| = 3$, and so $|C_S(Q_1)| = 3^4$ if $|S| = 3^7$ and $|C_S(Q_1)| = 3^3$ if $|S| = 3^6$. Since R_1 centralizes Q_2, we have $C_S(R_1) = C_S(Q_1) = Q_2$ if $|S| = 3^6$. If $|S| = 3^7$, then, as R_1Q is normalized by R_1S, we have $|S/C_S(R_1)Q| = 3$ and hence the $C_S(Q_1) = C_S(R_1)$ holds in this case as well. Of course we now have $C_J(Q_1) = C_J(R_1) = C_J(Q_1R_1)$.

Since J normalizes R_1Q and does not centralize R_1Q/Q by Lemma 4.3, Q_1 is normalized by J. Since J is abelian and $J \cap Q_1 = A \cap Q_1$, we now have that $|J : C_J(Q_1)| = 3^2$. □

Notice that r_1J and r_2J are conjugate in $N_G(J)/J$ (by t_2J for example) and

$$\langle r_1, r_2, Q_1\rangle J/J \cong 2 \times \mathrm{Sym}(3).$$

In particular, we have $r_1 \in F$.

Let $U \le F$ be chosen so that $\langle r_1, r_2, Q_1\rangle J \le U$ and $U/J \cong \mathrm{Sym}(5)$. Suppose that $J \ne J_0$. Since $O^2(U)$ is generated by two conjugates of Q_1J, and $|J : C_J(Q_1)| = 3^2$ by Lemma 4.8, we have that $|C_J(O^2(U))| = 3$. Note, furthermore, that the elements of order 5 act fixed-point-freely on J_0 and therefore $C_J(O^2(U)) \not\le J_0$. Thus, as r_2 centralizes J/J_0 and normalizes $C_J(O^2(U))$, we get that $C_J(O^2(U)) = C_J(U)$. We have proven

Lemma 4.9. *If $J \ne J_0$, then $|C_J(U)| = 3$ and $|C_J(U)^F| = |C_J(U)^{N_G(J)}| = 6$.* □

Lemma 4.10. *Suppose that $B \le J_0$ with $|B| = 3^3$. Then B contains a conjugate of Z.*

Proof. Recall that J_0 is a non-degenerate quadratic space by Theorem 4.6(i). Hence this result follows because every subgroup of order 3^3 in the J_0 contains a singular vector and the singular one-spaces in J_0 are G-conjugate to Z. □

We now fix some further notation. First let $W = C_F(r_2)$. So $WJ/J \cong 2 \times \mathrm{Sym}(4)$ and $J \cap W$ has index 3 in J by Theorem 4.6 (iv).

If $J = J_0$, set $\tau = 1$, whereas, if $J > J_0$, select $\tau \in C_J(U)^{\#}$.

Suppose that $J > J_0$. Then $\tau \ne 1$. Let

$$\mathcal{T} = \tau^F = \{\tau_1 = \tau, \ldots, \tau_6\}$$

be the six F-conjugates of τ. Then, as $[J, r_2]$ has order 3 by Theorem 4.6 (iv), r_2 acts as a transposition on $\mathcal{T}$ and r_2 centralizes τ (as $r_2 \in U$). Since $W/J \cong 2 \times \mathrm{Sym}(4)$ and W has orbits of length 2 and 4 on $\mathcal{T}$. It follows that, after adjusting notation if necessary, $\tau^W = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ and $\tau_5^{r_2} = \tau_6$. We further fix notation so that Q_1 acts as $\langle(\tau_2, \tau_3, \tau_4)\rangle$ and, since r_1 is conjugate to r_2 in $N_G(J)$ and inverts QJ/J, we may suppose that r_1 induces the transposition (τ_2, τ_3) on τ^W.

For $1 \le i \le 4$, let

$$J_i = \langle \tau_j \mid 1 \le j \le 4, i \ne j\rangle.$$

Then each J_i is centralized by r_2 and is a hyperplane of $C_J(r_2)$. Further

$$J_i \cap J_j = \langle \tau_k \mid 1 \le k \le 4, k \notin \{i, j\}\rangle.$$

Let $\rho \in [J, r_2]^{\#}$. Then $\rho \in (A \cap Q_2) \setminus Z$. Since $[J, r_1] \le A \cap Q_1$, we know $[\rho, r_1] = 1$. From the choice of τ and ρ, we have that $\langle Q_1, r_1\rangle$ and $\langle \tau, \rho\rangle$ commute.

For $J_0 = J$ we have to define the groups J_1, J_2, J_3 and J_4 differently. Set $J_1 = C_A(r_2) = A \cap Q_1$. So J_1 is normalized by $\langle r_1, r_2, Q_1, J\rangle$ which has index 4 in W. Since W is not contained in M and Z is the unique element of Z^G contained in J_1, we have $J_1^W = \{J_1, J_2, J_3, J_4\}$ and W acts two transitively on J_1^W. As $r_1 \sim_M r_2$, all the elements in $J_1 \setminus Z$ are conjugate to ρ. Therefore, as all the subgroup J_i are centralized by r_2, we have that $|J_i \cap J_j| = 3$ for all $i \ne j$ and these intersections are conjugate to $\langle \rho\rangle$. We capture some of the salient properties of these subgroups in the next lemma.

Lemma 4.11. *For $1 \le i \le 4$, $J_i \le C_G(r_2)$ and $N_{N_G(J)}(J_i)$ contains a Sylow 3-subgroup of $N_G(J)$.*

Proof. If $J > J_0$, this is transparent from the construction of the subgroups. In the case that $J = J_0$, we have already mentioned that the subgroups commute with r_2. Also we have $J_1 = A \cap Q_1$ is normalized by S and as J_i, $2 \le i \le 4$ are conjugates of J_1 in $N_G(J)$, we have $N_{N_G(J)}(J_i)$ contains a Sylow 3-subgroup of $N_G(J)$. □

Note also that when $|J| = 3^5$, $\rho \in \langle [\tau_5, r_2] \rangle$. It follows that $\langle \tau_5, \tau_6 \rangle$ contains ρ in this case. When $J = J_0$, of course we have $\tau_i = 1$. Thus to handle the two possible cases simultaneously we will consider the group $\langle \tau_5, \rho \rangle$.

Lemma 4.12. *$\langle \tau_5, \rho \rangle$ is centralized by JQ_1R_1. In particular, $C_G(\langle \tau_5, \rho \rangle) \not\le M$.*

Proof. Set $X = \langle \tau_5, \rho \rangle$. If $|J| = 3^4$, then $X = \langle \rho \rangle \le A \cap Q_2$ and the lemma holds. So suppose that $|J| = 3^5$. Then $X = \langle \tau_5, \tau_6 \rangle$ is centralized by J. Further, as $\{\tau_5, \tau_6\}$ is a W-orbit and $Q_1 \le C_F(r_2) \le W$, Q_1 centralizes X. Since $C_S(Q_1) = C_S(R_1)$ by Lemma 4.8 we now have $[X, R_1] = 1$ and this completes the proof.

Notice that $\langle \tau_5, \rho \rangle$ is centralized by a subgroup of index 2 in W and so $C_G(\langle \tau_5, \rho \rangle)$ is not contained in M. □

Lemma 4.13. *The following hold.*

(i) $C_M(\rho) = JQ_1R_1\langle r_2t_1 \rangle$
(ii) *If $J > J_0$, $C_M(\langle \tau_5, \rho \rangle) = JQ_1R_1$; and*

Proof. We calculate that $C_M(\rho)$ contains $JQ_1R_1\langle r_2t_1 \rangle$. So (i) holds.

By Lemma 4.12, $\langle \tau_5, \rho \rangle$ is centralized by JQ_1R_1. Since r_2t_1 conjugates τ_5 to τ_6, part (ii) follows from (i). □

Lemma 4.14. *Z is the unique G-conjugate of Z in $\langle \tau_5, \rho, Z \rangle$.*

Proof. Since Z is weakly closed in Q, Z is the unique conjugate of Z in $\langle Z, \rho \rangle$. Also, as τ_5 is not contained in J_0 and all the G-conjugates of Z in J are contained in J_0, there are no G-conjugates of Z in $\langle \tau_5, \rho, Z \rangle \setminus \langle \rho, Z \rangle$. This proves the claim. □

Lemma 4.15. *Assume that $J > J_0$. Then $N_G(\langle r_1, r_2 \rangle)/C_G(\langle r_1, r_2 \rangle) \not\cong \mathrm{Sym}(3)$.*

Proof. Let $U = \langle r_1, r_2 \rangle$. As $J > J_0$, using the structure of M, we have $|C_M(U)|_3 = 3^3$ and so $D = C_J(U)$ is a Sylow 3-subgroup of $C_M(U)$. Since $Z \le D$, we have $C_G(D) = C_M(Z) = JU$ which is 3-closed. Therefore, $N_G(D) \le N_G(J)$. Since r_1 and r_2 act as transpositions on $\mathcal{T}$, $|N_{N_F(J)}(DU)/J| = 32$ and so we deduce that $D \in \mathrm{Syl}_3(C_G(U))$. Let $P = N_{N_G(U)}(D)$. Then by the Frattini Argument $PC_G(U) = N_G(U)$. Therefore, if $N_G(U)/C_G(U) \cong \mathrm{Sym}(3)$, then r_2 and r_1r_2 are conjugate in P. But $P \le N_G(J)$, $r_2 \in F \setminus F'$ and $r_1r_2 \in F'$ which is a contradiction. Hence $N_G(U)/C_G(U) \not\cong \mathrm{Sym}(3)$. □

5. A further 3-local subgroup and a 2-local subgroup in the centralizer of an involution

In this section we study the normalizer of $\langle \tau_5, \rho\rangle$ and construct a 2-local subgroup of $C_G(r_2)$.

Lemma 5.1. *We have* $\text{И}_G(J_0, 3') = \{1\}$.

Proof. Suppose that $1 \neq Y \in \text{И}_G(J_0, 3')$. Then, as every hyperplane of J_0 contains a conjugate of Z by Lemma 4.10, we may assume that $X = C_Y(Z) \neq 1$. So $X \in \text{И}_M(J_0, 3')$. As X is normalized by $A = J_0 \cap Q$ and X normalizes Q, $[A, X] \le Q \cap X = 1$ and hence, as $X \neq 1$, $X \le C_M(A) = J\langle t_1\rangle$ and X is conjugate to $\langle t_1\rangle$. Therefore, J centralizes X. Since $C_G(J_0) = J$ by Lemma 4.5 (iii), this is impossible. Hence $\text{И}_G(J_0, 3') = \{1\}$. □

Lemma 5.2. *Assume $J = J_0$. Then $C_G(\rho) \not\cong \langle\rho\rangle \times \mathrm{Sp}_6(2)$.*

Proof. Suppose that $C_G(\rho) \cong \langle\rho\rangle \times \mathrm{Sp}_6(2)$. Set $E = E(C_G(\rho))$. Then $E \cong \mathrm{Sp}_6(2)$. We have that r_2 inverts ρ and centralizes in $J/\langle r_2\rangle$, so as $J \cap E$ has order 3^3 and $C_E(J \cap E) = J \cap E$, r_2 induces the trivial automorphism on E. Hence $N_G(\langle\rho\rangle) \cong \mathrm{Sym}(3) \times E$ and $[E, r_2] = 1$. In $E \cap J$ there is an element $\tilde\rho$ with $N_E(\langle\tilde\rho\rangle) \cong \mathrm{Sp}_2(2) \times \mathrm{Sp}_4(2)$. Hence $N_{N_G(J)}(\langle\rho\rangle) \cap N_{N_G(J)}(\langle\tilde\rho\rangle)$ contains a Sylow 2-subgroup T of $N_G(J)$. Now $\langle\rho, \tilde\rho\rangle = C_J(i)$, where $i \in T' \le F'$. But such involutions centralize some conjugate of Z (as $C_J(i)$ is a $+$-space with respect to the quadratic form from Theorem 4.6 (i)), and so $\langle\rho, \tilde\rho\rangle$ contains a conjugate of Z. This then contradicts the fact that M does not involve $\mathrm{Alt}(6)$. □

Lemma 5.3. *Let B be a maximal subgroup of $\langle \tau_5, \rho, Z\rangle$ and assume that $C_G(B) \not\le M$. Then $B \in \langle\tau_5, \rho\rangle^{Q_2}$ and either*

(i) *$J > J_0$ and $C_G(B) \cong B \times \mathrm{SU}_4(2)$; or*
(ii) *$J = J_0$ and $C_G(\rho) \cong \langle\rho\rangle \times \mathrm{Aut}(\mathrm{SU}_4(2))$.*

Proof. Set $U = \langle Z, \tau_5, \rho\rangle$, let B be a maximal subgroup of U, $X = C_G(B)$ and $\widetilde{X} = X/B$. Assume that $X \not\le M$. By Lemma 4.14, Z is the unique conjugate of Z in U and so, as $C_G(B) \not\le M$, $U = ZB$ and $\widetilde{N_X(Z)} = N_{\widetilde{X}}(\widetilde{Z})$.

Assume that $J > J_0$. Then, by Lemma 4.13, $N_X(Z) = X \cap M = JQ_1R_1$ and so $\widetilde{N_X(Z)} = N_{\widetilde{X}}(\widetilde{Z}) = \widetilde{JR_1Q_1} \cong 3_+^{1+2}.\mathrm{SL}_2(3)$ which is isomorphic to the centralizer of a 3-central element in $\mathrm{SU}_4(2)$. As $z \in Z^{\#}$ is not X-conjugate to its inverse by Lemma 4.13, $\text{И}_G(J_0, 3') = \{1\}$ by Lemma 5.1 and $C_G(B) \not\le M$, we may apply Hayden's Theorem 2.6 to get that $\widetilde{X} \cong \mathrm{SU}_4(2)$. Finally, as JQ_1, splits over B, X splits over B by Gaschütz's Theorem [7, 9.26]. Hence X has the structure described in (i).

Assume that $J = J_0$. In this case B is Q_2-conjugate to ρ. By Lemma 4.13, $C_X(Z) = X \cap M = JQ_1R_1$ and so $\widetilde{C_X(Z)}$ is isomorphic to the centralizer of a 3-central element in $\mathrm{SU}_4(2)$. Since r_2t_1 inverts z, we may use Prince's Theorem

2.7 to obtain $\widetilde{X} \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$. Again Gaschütz's Theorem implies that $X \cong \langle \rho \rangle \times E$ where $E \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$. Therefore, by Lemma 5.2, X has the structure claimed in (ii).

Now we consider the possibilities for B when $J > J_0$. We have $B \le U$ and $C_G(B) \not\le M$. Thus, by (i), $C_G(B) \cong B \times E$ where $E \cong \mathrm{SU}_4(2)$. Consequently, $N_{C_G(B)}(J) \cong 3^2 \times (3^3{:}\mathrm{Sym}(4))$. Since $N_{C_G(B)}(J) \ge Q_1$ and since there are exactly 3-subgroups isomorphic to $\mathrm{Alt}(4)$ which contain a given 3-cycle in $\mathrm{Sym}(6)$, we see that B is Q_2-conjugate to $\langle \tau_5, \rho \rangle$ as claimed. □

We now set $r = r_2$ and aim to determine

$$K = C_G(r).$$

We will frequently use the following observation.

Lemma 5.4. *$C_J(r)Q_1$ is a Sylow 3-subgroup of K.*

Proof. Certainly $C_J(r)Q_1 \le K$ by Lemma 3.3 (i). Because $[Q_1, C_J(r), Q_1] = [A \cap Q_1, Q_1] = Z$, we have that Z is a characteristic subgroup of $C_J(r)Q_1$ and so it follows that $N_K(C_J(r)Q_1) \le C_M(r)$. As $C_J(r)Q_1 \in \mathrm{Syl}_3(C_M(r))$, the lemma holds. □

Define $E = E(C_G(\langle \tau_5, \rho \rangle))$. Then $E \cong \mathrm{SU}_4(2)$ by Lemma 5.3.

Lemma 5.5. *We have $E\langle t_1, \tau_5\tau_6 \rangle \le K$ and $E\langle t_1 \rangle \cong \mathrm{Aut}(\mathrm{SU}_4(2))$.*

Proof. We know that r inverts ρ and exchanges τ_5 and τ_6. Hence r normalizes $B = \langle \tau_5, \rho \rangle$ and consequently r normalizes E. Furthermore, r centralizes $J \cap E$ and since no non-trivial automorphism of E acts in this way, we have that r centralizes E. Therefore $E \le K$.

Since t_1 inverts J, t_1 normalizes $\langle \tau_5, \rho \rangle$ and t_1 therefore normalizes E. Since t_1 inverts $J \cap E$, we have $E\langle t_1 \rangle \cong \mathrm{Aut}(\mathrm{SU}_4(2))$. □

From Lemmas 4.13 and 5.3 we have $Q_1R_1 \le E$. Furthermore, as $W (= C_F(r))$ normalizes $[J, r] = \langle \rho \rangle$, we also have that $C_W(\rho) \le E$. In particular, we have

Lemma 5.6. $\langle \tau_5\tau_6 \rangle E = \langle C_W(\rho), Q_1R_1C_J(r) \rangle$.

Proof. We have that $Y = Q_1R_1C_J(r)$ is a maximal subgroup of $E\langle \tau_5\tau_6 \rangle$ and $C_W(\rho) \not\le Y$.

□

When $J > J_0$, as $N_G(J)$ acts 2-transitively on $\mathcal{T}$, $\langle \tau_5, \tau_6 \rangle$ is G-conjugate to each subgroup $J_i \cap J_j$ for $1 \le i < j \le 4$. When $J = J_0$ we have the same result from the construction of J_1, J_2, J_3 and J_4 in Section 4. Hence we may apply Lemma 5.3 to obtain the following conclusion.

Lemma 5.7. *Assume that $1 \le i < j \le 4$.*

(i) *If $J > J_0$, then $C_G(J_i \cap J_j) \cong (J_i \cap J_j) \times \mathrm{SU}_4(2)$; and*
(ii) *If $J = J_0$, then $C_G(J_i \cap J_j) \cong (J_i \cap J_j) \times \mathrm{Aut}(\mathrm{SU}_4(2))$.*

□

For $1 \le i < j \le 4$, define

$$E_{ij} = E(C_G(J_i \cap J_j)).$$

Lemma 5.8. *For $1 \le i < j \le 4$ and $k \in \{i, j\}$, $E_{ij} \cap J_k$ is conjugate to Z and is 3-central in E_{ij}. In particular, $C_G(J_i) \cong (J_i \cap J_j) \times 3_+^{1+2}{:}\mathrm{SL}_2(3)$ if $J > J_0$ and $C_G(J_i) \cong (J_i \cap J_j) \times 3_+^{1+2}{:}\mathrm{SL}_2(3).2$ if $J = J_0$.*

Proof. Let $1 \le i \le 4$. Then by Lemma 4.11, J_i is normalized by a Sylow 3-subgroup T_i of $N_G(J)$ and $C_{T_i}(J_i)$ has index 3 in T_i. In particular, as $|C_G(J_i \cap J_j)|_3 = 3|J|$, we see that $C_{T_i}(J_i) \in \mathrm{Syl}_3(C_G(J_i \cap J_j))$. Therefore $J_i \cap E_{ij}$ is normalized by a Sylow 3-subgroup of E_{ij}. As $|J_i \cap E_{ij}| = 3$, we have that $J_i \cap E_{ij}$ is 3-central in E_{ij} as J_i is normal in T_i, we see that this subgroup is also normal in a Sylow 3-subgroup of G. □

Define

$$\Sigma = \langle O_2(C_K(J_k)) \mid 1 \le k \le 4\rangle.$$

In the next lemma we use the fact that if $x \in \mathrm{SU}_4(2)$ is an involution which centralizes a subgroup of order 9, then x is 2-central and

$$C_X(x) \cong 2_+^{1+4}.(3 \times \mathrm{Sym}(3)) \cong (\mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)).2$$

where $\circ$ denotes a central product (see [3, page 26]).

Lemma 5.9. *Assume that $1 \le i < j \le 4$. Then*

(i) *$O_2(C_K(J_i)) \cong O_2(C_K(J_j)) \cong \mathrm{Q}_8$, $[O_2(C_K(J_i)), O_2(C_K(J_j))] = 1$ and $O_2(C_K(J_i \cap J_j)) = O_2(C_K(J_i))O_2(C_K(J_j)) \cong 2_+^{1+4}$; and*
(ii) *Σ is extraspecial of $+$-type and order 2^9.*

Proof. Suppose that $1 \le i < j \le 4$. Then $J_i \le C_G(r)$ by Lemma 4.11. If $J > J_0$, we have $r \in E_{ij}$ by Lemma 5.3. If $J = J_0$, then $r \in Z(R_2) \le C_G(J_1)$ and so $r \in E_{12}$ and consequently $r \in E_{ij}$ as W acts 2-transitively on $\{J_1, J_2, J_3, J_4\}$.

Since $r \in E_{ij}$ and $|C_J(r) \cap E_{ij}|_3 \ge 9$, r is a 2-central involution in E_{ij}. It follows that $K \cap E_{ij}$ has shape $2_+^{1+4}.(3 \times \mathrm{Sym}(3))$ and, in particular, $O_2(C_K(J_i \cap J_j)) \cong 2_+^{1+4}$. Furthermore, as $J_i \cap E_{ij}$ is 3-central by Lemma 5.8, we get $O_2(C_K(J_i)) \cong \mathrm{Q}_8$ and $O_2(C_K(J_i \cap J_j)) = O_2(C_K(J_i))O_2(C_K(J_j))$. Since $O_2(C_K(J_i \cap J_j))$ contains exactly two subgroups isomorphic to Q_8, we have that $[O_2(C_K(J_i)), O_2(C_K(J_j))] = 1$. This completes the proof of (i).

Part (i) shows that Σ is isomorphic to a central product of 4 quaternion groups. Hence Σ is extraspecial of $+$-type and order 2^9. So (ii) holds. □

Recall from Lemmas 3.2 and 4.7, $t_2 \in N_G(S) \le M \cap N_G(J)$ and $R_1^{t_2} = R_2$.

Lemma 5.10. *We have J_1 is centralized by R_2, $R_2 \le \Sigma$ and $R_2 = C_\Sigma(Z)$.*

Proof. Suppose first that $J = J_0$. Then $J_1 = C_A(r) \le Q_1 = C_Q(R_2)$ by Lemma 3.3 (i). So $[J_1, R_2] = 1$. Hence $R_2 = O_2(C_K(J_1)) \le \Sigma$.

Assume that $J > J_0$. We have that τ_1 commutes with Q_1 and $[\langle \tau_5, \tau_6\rangle, Q_1] = 1$ by Lemma 4.13. Hence $C_J(Q_1) = \langle \tau_1, \tau_5, \tau_6\rangle = \langle \tau_5, A \cap Q_2\rangle$. Thus $C_J(Q_2) = C_J(Q_1)^{t_2} = \langle \tau_2, \tau_3, \tau_4\rangle = \langle \tau_2, A \cap Q_1\rangle$. By Lemma 4.13 $C_J(Q_1)$ is centralized by R_1, thus $J_1 = \langle \tau_2, \tau_3, \tau_4\rangle$ is centralized by $R_2 = R_1^{t_2}$. Hence $R_2 = O_2(C_K(J_1)) \le \Sigma$.

Since R_2 commutes with Z, we have $R_2 \le C_\Sigma(Z)$ and, as $C_\Sigma(Z)$ is extraspecial we have that $R_2 = C_\Sigma(Z)$ from the structure of M. □

Lemma 5.11. *We have* $W\langle t_1\rangle \le N_K(\Sigma)$.

Proof. Since $W\langle t_1\rangle$ permutes $\{J_1, J_2, J_3, J_4\}$ and is contained in K, $W\langle t_1\rangle \le N_K(\Sigma)$ by the definition of Σ. □

Lemma 5.12. *We have* $N_K(C_J(r)) = N_{N_K(\Sigma)}(C_J(r))$. *In particular* $N_{N_K(\Sigma)}(C_J(r))$ *controls* K*-fusion in* $C_J(r)$.

Proof. We have that $C_G(C_J(r)) = J\langle r\rangle$. Hence J is normal in $N_G(C_J(r))$. Now we have that $W = N_K(C_J(r))$. By Lemma 5.11 we have $W \le N_K(\Sigma)$ and so $N_K(C_J(r)) = N_{N_K(\Sigma)}(C_J(r))$. Further by Lemma 4.1 we have that $N_G(J)$ controls fusion in J and so $N_K(C_J(r))$ controls fusion in $C_J(r)$. As $N_K(C_J(r)) = N_{N_K(\Sigma)}(C_J(r))$ this fusion takes place in $N_K(\Sigma)$. □

Lemma 5.13. *Every* J_1*-signalizer in* K *is contained in* Σ. *In particular,* $N_K(J_1) \le N_K(\Sigma)$.

Proof. Let $\Sigma_1 \le K$ be a J_1-signalizer. Let X_1 be a hyperplane in J_1 such that $C_G(X_1) \le M$. Then $C_{\Sigma_1}(X_1) \le M$ is normalized by J_1 and so $\Sigma_1 \le R_2 \le \Sigma$ by Lemma 5.10. In particular $[C_{\Sigma_1}(Z), J_1] = 1$.

Suppose next that X_1 is a hyperplane such that $C_G(X_1) \not\le M$. Then, by Lemma 5.3, we may assume that $X_1 = J_1 \cap J_2$. Since r is 2-central in E_{12}, $O_2(C_K(J_1 \cap J_2))$ is the unique maximal J_1-signalizer in $C_G(X_1)$. Hence by Lemma 5.9 (i) we have that $C_{\Sigma_1}(X_1) \le \Sigma$ in this case as well. Because

$$\Sigma_1 = \langle C_{\Sigma_1}(X_1) \mid |J_1 : X_1| \le 3\rangle \le \Sigma,$$

we have that every J_1-signalizer is contained in Σ. Thus Σ is the unique maximal member of $И_K(J_1, 3')$ and so $N_K(J_1) \le N_K(\Sigma)$. □

Lemma 5.14. $C_K(\Sigma) = \langle r\rangle$.

Proof. If $C_K(\Sigma)$ is a $3'$-group, then $C_K(\Sigma)$ is normalized by J_1 and so $C_K(\Sigma) \le Z(\Sigma) = \langle r\rangle$ by Lemma 5.13. So suppose that $C_K(\Sigma)$ has order divisible by 3. As $C_J(r)Q_1 \in \mathrm{Syl}_3(K)$ and $C_J(r)Q_1 \le W \le N_K(\Sigma)$ by Lemma 5.11, we have $C_J(r)Q_1 \cap C_G(\Sigma)$ is a Sylow 3-subgroup of $C_G(\Sigma)$. As Z does not centralize Σ, we have $C_J(r)Q_1 \cap C_G(\Sigma) \le C_J(r)$. Now, for $1 \le i < j \le 4$

$$C_{C_J(r)}(O_2(C_K(J_i \cap J_j))) = J_i \cap J_j$$

and consequently $C_{C_J(r)}(\Sigma) \le J_1 \cap J_2 \cap J_3 \cap J_4 = 1$ which is a contradiction. □

Lemma 5.15. *$\Sigma/\langle r\rangle$ is a minimal normal subgroup of $N_K(\Sigma)/\langle r\rangle$.*

Proof. Suppose that $U \le \Sigma$ and $U/\langle r\rangle$ is a minimal normal subgroup of $N_K(\Sigma)/\langle r\rangle$ of minimal order. Aiming for a contradiction, assume that $U \ne \Sigma$. Then either $|\Sigma : U| \le 2^4$ or $|U/\langle r\rangle| \le 2^4$. In particular, as Q_1 normalizes Σ and $\mathrm{GL}_4(2)$ has elementary abelian Sylow 3-subgroups, Z centralizes one of U or Σ/U. By Lemma 5.10, either $U \le R_2$ or $|\Sigma : U| \le 2^2$ and $U \ge [\Sigma, Z]$.

Since $C_J(r)$ acts non-trivially on R_2, we get $U = R_2$ or $U = [\Sigma, Z]$. In the latter case, we have $U_1 = C_\Sigma(U)$ is normalized by $N_K(\Sigma)$ and has order smaller than U. Hence the minimal choice of U implies that $U = R_2$. However $W \le N_G(\Sigma)$ by Lemma 5.11 and W does not normalize R_2 and so we have a contradiction. □

Theorem 5.16. *One of the following holds.*

(i) *$J = J_0$ and $N_G(\Sigma)/\Sigma \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$; or*
(ii) *$J > J_0$ and $N_G(\Sigma)/\Sigma \cong (3 \times \mathrm{SU}_4(2)){:}2$.*

Furthermore, $E\langle \tau_5\tau_6, t_1\rangle \le N_K(\Sigma)$ and $\Sigma/\langle r\rangle$ is isomorphic to the natural $E\Sigma/\Sigma$-module.

Proof. From Lemma 5.11 we have that $W\langle t_1\rangle \le N_G(\Sigma)$. Set $L = J_1Q_1$. Then $L \le W$ and so $L \le N_G(\Sigma)$. By Lemma 5.13 we have that Σ is a maximal signalizer in K for L and for $C_J(r)$. Hence $N_K(L)$ and $N_K(C_{J_1}(r))$ both normalize Σ.

Suppose that $J = J_0$.Then $J_1Q_1 = (A \cap Q_1)Q_1 \le Q_1$ and so $R_1 \le N_K(Q_1) \le N_K(\Sigma)$. Therefore Lemma 5.6 implies that $\langle E, t_1\rangle \le N_K(\Sigma)$. In particular, we have $C_{N_K(\Sigma)/\Sigma}(Z\Sigma/\Sigma)$ is isomorphic to the centralizer of a 3 element in $\mathrm{SU}_4(2)$ and is inverted by $t_1\Sigma$. Hence Theorem 2.7 shows that (i) holds.

Suppose that $J > J_0$. This time $N_K(J_1Q_1)$ does not contain R_1. On the other hand $N_K(\Sigma) \ge N_K(C_J(r))\Sigma = W\Sigma$ and $W\Sigma/\Sigma$ has shape $3^4{:}(\mathrm{Sym}(4)\times 2)$. By the Frattini argument, $N_{N_K(\Sigma)/\Sigma}(C_J(r)\Sigma/\Sigma) = N_{N_K(\Sigma)}(C_J(r))$. Since $N_K(C_J(r)) = W$, we now have $N_{N_K(\Sigma)/\Sigma}(C_J(r)\Sigma/\Sigma) = W\Sigma/\Sigma$.

Since $C_G(\Sigma) = \langle r\rangle$ by Lemma 5.14, we have that $N_K(\Sigma)/\Sigma$ is isomorphic to a subgroup of $\mathrm{O}_8^+(2)$. Because $N_{N_K(\Sigma)/\Sigma}(C_J(r)\Sigma/\Sigma) = W\Sigma/\Sigma$, we infer from the list of maximal subgroups of $\mathrm{O}_8^+(2)$ given in [3, page 85] that either $N_K(\Sigma) = W\Sigma$ or $N_K(\Sigma)/\Sigma \cong (3 \times \mathrm{SU}_4(2)){:}2$. In the latter case we have (ii) so suppose that $N_K(\Sigma) = W\Sigma$. Let $T \in \mathrm{Syl}_2(N_K(\Sigma))$. We claim that $T \in \mathrm{Syl}_2(K)$. Assume that $x \in N_K(T) \setminus N_K(\Sigma)$. Then, as $\Sigma^x \ne \Sigma$, $J(T/\langle r\rangle) \not\le \Sigma/\langle r\rangle$. Hence, setting $H = \langle J(T)^{N_K(\Sigma)}\rangle$ and noting that $|O_3(N_K(\Sigma)/\Sigma)| = 3^4$, we may apply [1, (32.5)] to get that H/Σ is a direct product of four subgroups isomorphic to $\mathrm{SL}_2(2)$. But then the 2-rank of W/Σ is at least 4 contrary to $T/\Sigma \cong \mathrm{Dih}(8) \times 2$. Hence $N_K(T) \le N_K(\Sigma)$ and, in particular, $T \in \mathrm{Syl}_2(K)$.

From Lemma 5.5, we have $E \le K$. Since $T \in \mathrm{Syl}_2(K)$, $T/\Sigma \cong \mathrm{Dih}(8) \times 2$ and E contains an extraspecial subgroup of order 2^5 with centre $\langle r_1\rangle$, we have that r_1 is K-conjugate to an element of Σ. Thus there is some $x \in K$ such that $\langle r_1, r\rangle \le \Sigma^x$. Since $r_1^{t_2} = r$ and since r_1 and rr_1 are Σ^x-conjugate, we have $N_G(\langle r_1, r\rangle)/C_G(\langle r_1, r\rangle) \cong \mathrm{Sym}(3)$. This contradicts Lemma 4.15. Hence (ii) holds.

We have already seen that $E \le N_K(\Sigma)$ if $J = J_0$. If $J > J_0$, then we have $N_{N_K(\Sigma)}(Z)$ contains a subgroup $(3 \times 3^{1+2}_+).\mathrm{SL}_2(3).2$. Since $N_K(Z) = C_M(r) = Q_1R_1R_2C_J(r)\langle t_1\rangle$, we have $C_M(r) \le N_K(\Sigma)$. Now $E\langle \tau_5\tau_r, t_1\rangle \le N_K(\Sigma)$ by Lemma 5.6. Finally, as E acts irreducibly on $\Sigma/\langle r\rangle$ by Lemma 5.15, we have that $\Sigma/\langle r\rangle$ is the natural E-module. □

We need just two final details before we can move on to determine the structure of K.

Lemma 5.17. *The following hold.*

(i) $N_K(Z) \le N_K(\Sigma)$*; and*
(ii) $N_K(J_i \cap J_j) \le N_K(\Sigma)$*, for* $1 \le i < j \le 4$.

Proof. For (i) we note that $N_K(Z) = C_M(r) \le E\langle \tau_5\tau_6, t_1\rangle\Sigma \le N_K(\Sigma)$ by Theorem 5.16.

By Lemma 5.9 (i) we have that $O_2(C_K(J_i \cap J_j)) \le \Sigma$ and, as r is a 2-central element in E_{ij}, $C_J(r) \in \mathrm{Syl}_3(C_K(J_i \cap J_j))$. Hence

$$N_K(J_i \cap J_j) = N_{N_K(J_i\cap J_j)}(C_J(r))O_2(C_K(J_i \cap J_j)) \le N_K(\Sigma)$$

by Lemma 5.13. □

6. The structure of K

In this section we prove Theorem 6.11 which asserts that $K = N_K(\Sigma)$. We continue the notation introduced in the previous sections. We further set $K_1 = N_K(\Sigma)$ denote by $\widetilde{\ }$ the natural homomorphism from K onto $K/\langle r\rangle$.

By Lemma 5.15, the subgroup $\widetilde{\Sigma}$ can be regarded as the 8-dimensional irreducible GF(2)-module for $\widetilde{K}_1/\widetilde{\Sigma}$. Thus we may employ the results of Proposition 2.10 to obtain information about various centralizers of elements of order 2 and 3 in $\widetilde{\Sigma}$. Using Proposition 2.10(ii), we have $\widetilde{K}_1$ has two orbits on $\widetilde{\Sigma}$. We pick representatives $\tilde{x}$ and $\tilde{y}$ of these orbits with $\widetilde{x}$ singular and $\widetilde{y}$ non-singular. It follows that x is an involution and y has order 4.

Our aim is to show that $\widetilde{\Sigma}$ is strongly closed in $\widetilde{K}$ and then use Goldschmidt's Theorem [4] to show that $K = K_1$. We now begin the proof of Theorem 6.11.

Lemma 6.1. *We have* $\widetilde{K}_1$ *contains a Sylow 2-subgroup of* $C_{\widetilde{K}}(\tilde{y})$*. In particular* $|C_{\widetilde{K}}(\tilde{y})|_2 = 2^{12}$ *if* $E(\widetilde{K}_1/\widetilde{\Sigma}) \cong \mathrm{SU}_4(2)$ *and* $|C_{\widetilde{K}}(\tilde{y})|_2 = 2^{14}$ *if* $\widetilde{K}_1/\widetilde{\Sigma} \cong \mathrm{Sp}_6(2)$.

Proof. Let T be a Sylow 2-subgroup of $C_{\widetilde{K}_1}(\tilde{y})$ and assume that T_1 is a 2-group with $|T_1 : T| = 2$. Choose $u \in T_1 \setminus T$. If $|\widetilde{\Sigma}^u\widetilde{\Sigma}/\widetilde{\Sigma}| \le 2$, then $|\widetilde{\Sigma}^u \cap \widetilde{\Sigma}| \ge 2^7$. But by Proposition 2.10 (iv), $\widetilde{K_1}$ has no 2-elements not in $\widetilde{\Sigma}$ which centralize a subgroup of index two in $\widetilde{\Sigma}$. Therefore $\widetilde{\Sigma} = \widetilde{\Sigma}^u$ and so $u \in T_1 \cap K_1 = T$ which is a contradiction. Hence $|\widetilde{\Sigma}^u\widetilde{\Sigma}/\widetilde{\Sigma}| \ge 4$.

If $E(\widetilde{K}_1/\widetilde{\Sigma}) \cong \mathrm{SU}_4(2)$, then $T/\widetilde{\Sigma}$ is a semidihedral group of order 16 by Proposition 2.10(ii). Since $\widetilde{\Sigma}^u\widetilde{\Sigma}/\widetilde{\Sigma}$ is a normal elementary abelian subgroup of $T/\widetilde{\Sigma}$ of order at least 4, we have a contradiction. Hence $\widetilde{K}_1/\widetilde{\Sigma} \cong \mathrm{Sp}_6(2)$. Now Proposition 2.10(ii), gives

$$C_{\widetilde{K}_1}(\tilde{y})/\widetilde{\Sigma} \cong \mathrm{G}_2(2).$$

Since, by [8, Table 3.3.1], $\mathrm{G}_2(2)$ does not contain elementary abelian subgroups of order 16, $2^6 \geq |\widetilde{\Sigma}^u \cap \widetilde{\Sigma}| \geq 2^5$. But then all involutions in $\widetilde{\Sigma}^u$ centralize a subgroup of order at least 2^5 in $\widetilde{\Sigma}$, and so Proposition 2.10 (i) and (iv) shows that all the involutions in $\widetilde{\Sigma}^u\widetilde{\Sigma}/\widetilde{\Sigma}$ are unitary transvections and are conjugate in $\widetilde{K_1/\Sigma}$. Since the two classes of involutions in $C_{\widetilde{K}_1}(\tilde{y})/\widetilde{\Sigma} \cong \mathrm{G}_2(2)$ are not fused in $\widetilde{K_1/\Sigma}$, we infer that

$$\widetilde{\Sigma}^u\widetilde{\Sigma}/\widetilde{\Sigma} \leq (C_{\widetilde{K}_1}(\tilde{y})/\widetilde{\Sigma})' \cong \mathrm{G}_2(2)' \cong \mathrm{SU}_3(3).$$

Since, by [8, Table 3.3.1], $\mathrm{SU}_3(3)$ has no elementary abelian groups of order 8, we have $|\widetilde{\Sigma}^u\widetilde{\Sigma}/\widetilde{\Sigma}| = 4$. This means that $|\widetilde{\Sigma}^u \cap \widetilde{\Sigma}| = 2^6$ and consequently all the involutions in $\widetilde{\Sigma}^u\widetilde{\Sigma}/\widetilde{\Sigma}$ have the same centralizer. As centralizers of involutions in $\mathrm{G}_2(2)'$ are maximal subgroups [3, page 14], we conclude that $\widetilde{\Sigma}^u \cap \widetilde{\Sigma}$ is normalized by $(C_{\widetilde{K}_1}(\tilde{y})/\widetilde{\Sigma})'$. Thus $(C_{\widetilde{K}_1}(\tilde{y})/\widetilde{\Sigma})'$ centralizes $\widetilde{\Sigma}$ which is impossible. This contradiction proves the lemma. The order of T is calculated from Proposition 2.10(iii). □

Lemma 6.2. *Let S_1 be a Sylow 3-subgroup of $C_{\widetilde{K}_1}(\tilde{x})$ or $C_{\widetilde{K}_1}(\tilde{y})$. Then $N_{\widetilde{K}}(S_1) \leq \widetilde{K}_1$. In particular, for $z \in \widetilde{\Sigma}^\#$, $C_{\widetilde{K_1}}(z)$ contains a Sylow 3-subgroup of $C_{\widetilde{K}}(z)$.*

Proof. We consider $\tilde{y}$ first. By Proposition 2.10(iii), S_1 has centre of order 3 and, as faithful GF(2)-representations of extraspecial groups of type 3^{1+2}_+ have dimension 6, we have $|C_{\widetilde{\Sigma}}(Z(S_1))| = 4$. Hence we may assume that $Z = Z(S_1)$. On the other hand by Lemma 5.17 (i) gives $C_M(r) \leq K_1$, hence we have that $N_{\widetilde{K}}(S_1) \leq \widetilde{K}_1$.

Now we consider $\tilde{x}$. By Lemma 5.9 (i), we have $O_2(C_K(J_1 \cap J_2)) \leq \Sigma$. Hence we may assume that $S_1 = J_1 \cap J_2$. But then by Lemma 5.17 (ii) $N_{\widetilde{K}}(S_1) \leq \widetilde{K}_1$. □

Let $\tilde{E} \leq \widetilde{K}_1$ such that $\tilde{E}/\tilde{\Sigma} = E(\widetilde{K}_1/\tilde{\Sigma})$. We have that $\tilde{E}/\tilde{\Sigma} \cong \mathrm{SU}_4(2)$ or $\mathrm{Sp}_6(2)$. By Proposition 2.10(iii) there are exactly three classes of elements of order three in $\tilde{E}$. As a Sylow 3-subgroup of $\tilde{E}$ is isomorphic $3 \wr 3$, there is a unique elementary abelian subgroup of order 27, and this subgroup contains elements from each of the conjugacy classes of elements of order 3. As $C_J(r) \cap \tilde{E}$ is elementary abelian of order 27, there are representatives of these elements in $\widetilde{C_J(r)} \cap \tilde{E}$. It follows that every element of order 3 in $\widetilde{K}$ is conjugate to an element of $\widetilde{C_J(r)}$. So using Lemma 5.12 get the following lemma.

Lemma 6.3. *Two elements of order three in $\tilde{K}$ are conjugate in $\tilde{K}$ if and only if they are conjugate in $\widetilde{K}_1$. If $J > J_0$, then $\widetilde{K}_1/\widetilde{\Sigma} \cong (\langle\sigma\rangle \times \mathrm{SU}_4(2)).2$, σ is inverted and σ is not conjugate to any element in $\tilde{E}$.* □

Lemma 6.4. *Suppose that $\tilde{u} \in \widetilde{K}_1 \setminus \widetilde{\Sigma}$ is an involution which is $\widetilde{K}$ -conjugate to some involution in $\widetilde{\Sigma}$. Assume that $\nu \in C_{\widetilde{K_1}}(\tilde{u})$ is an element of order three. Then we have*

(i) $C_{\widetilde{\Sigma}}(\nu) \neq 1$;
(ii) $\langle\nu\rangle \not\sim Z$ *in* $\widetilde{K}$;
(iii) *if* $J = J_0$, *then* $\nu \not\sim \rho$ *in* $\widetilde{K}$; *and*
(iv) $|C_{\widetilde{E}}(\tilde{u})|$ *is not divisible by* 9.

Proof. Let $\tilde{a} \in \widetilde{\Sigma}$ with $\tilde{a} \sim_{\widetilde{K}} \tilde{u}$. By Lemma 6.2, $\widetilde{K}_1$ contains a Sylow 3-subgroup of $C_{\widetilde{K}}(\tilde{a})$. By Lemma 6.3, ν is conjugate to an element μ of $C_{\widetilde{K_1}}(\tilde{a})$ inside of $\widetilde{K}_1$. Now obviously $C_{\widetilde{\Sigma}}(\mu) \neq 1$ and so the same holds for ν which is (i).
If $\langle\nu\rangle$ is conjugate to Z in $\widetilde{K}$ or to $\langle\rho\rangle$ in case of $\tau = 1$, this happens also in $\widetilde{K}_1$ by Lemma 6.3. Hence we may assume that $\tilde{a}$ is conjugate to $\tilde{u}$ in $M \cap K$, or $N_K(\langle\rho\rangle)$, which both are contained in K_1 by Lemma 5.17, a contradiction. Hence also (ii) and (iii) hold.
Assume now that $S_1 \leq C_{\widetilde{E}}(\tilde{u})$, $|S_1| = 9$. Then S_1 is conjugate into a Sylow 3-subgroup S_2 of $C_{\widetilde{E}}(\tilde{a})$. So by Lemma 6.2 and Proposition 2.10(ii) we may assume that $\tilde{a} = \tilde{y}$ and thus S_2 is extraspecial of order 27. Hence S_1 contains some element which is conjugate into $Z(S_2)$. But $Z(S_2)$ is conjugate to Z, and this contradicts (ii). This finishes the proof. □

Lemma 6.5. *Suppose that $\tilde{u} \in \widetilde{K}_1 \setminus \tilde{\Sigma}$ is an involution which is $\tilde{K}$-conjugate to some involution in $\tilde{\Sigma}$. Then either*

(i) $\tilde{u} \in \tilde{E}$, $|[\tilde{\Sigma}, \tilde{u}]| = 4$ *and* $C_{\tilde{E}}(\tilde{u})$ *has order* 2^{13} *if* $E(\widetilde{K}_1/\widetilde{\Sigma}) \cong \mathrm{SU}_4(2)$ *and order* 2^{15} *if* $\widetilde{K}_1/\widetilde{\Sigma} \cong \mathrm{Sp}_6(2)$; *or*
(ii) $J > J_0$, $\sigma^{\tilde{u}} = \sigma^{-1}$ *and* $C_{\tilde{E}/\tilde{\Sigma}}(\tilde{u}) \cong 2 \times \mathrm{Sym}(4) \leq \mathrm{Sym}(6)$.

Proof. If $|[\tilde{u}, \tilde{\Sigma}] = 16$, then all involutions in $\tilde{\Sigma}\tilde{u}$ are conjugate by elements of $\tilde{\Sigma}$. Hence, by Proposition 2.10(i), $\tilde{u}$ centralizes some 3-element $\nu \in \tilde{E}$. By Lemma 6.4(i), $C_{\tilde{\Sigma}}(\nu) \neq 1$. If $J = J_0$, then by Proposition 2.10(ii) $\langle\nu\rangle$ is conjugate to Z or $\langle\rho\rangle$, which contradicts Lemma 6.4 (ii),(iii). So assume that $J > J_0$. If $\tilde{u} \notin \tilde{E}$, we have the assertion with Proposition 2.10(i) and Lemma 6.4(iv). So assume $\tilde{u} \in \tilde{E}$. Then $C_{\tilde{E}/\tilde{\Sigma}}(\tilde{u})$ is contained in a parabolic subgroup of $\widetilde{E}/\widetilde{\Sigma}$ of shape 2^4:Alt(5) and so ν acts fixed point freely on $\tilde{\Sigma}$, contradicting Lemma 6.4 (i).
So assume that $|[\tilde{u}, \tilde{\Sigma}]| = 4$. Then, by Proposition 2.10 (v), $C_{\tilde{E}/\tilde{\Sigma}}(\tilde{u}\widetilde{\Sigma})$ has orbits of length 1,6 and 9 on $C_{\widetilde{\Sigma}}(\tilde{u})/[\widetilde{\Sigma}, \tilde{u}]$. Hence there are exactly three conjugacy classes of involutions in $\widetilde{\Sigma}\tilde{u}$ two of them have representatives centralized by an

element of order three. Assume that $\tilde{u}$ is one of these. Let $\hat{u}$ be the involution, which is centralized by S_1, a Sylow 3-subgroup of $C_{\widetilde{K}_1/\widetilde{\Sigma}}(\tilde{u}\widetilde{\Sigma})$. Set $S_2 = C_{S_1}(\tilde{u})$. Then, using Lemmas 6.3, 6.2 and 6.4(iv), we see that $|S_2| = 3$. Therefore $\tilde{u} \not\sim \hat{u}$. In particular we have $\tilde{u} = \hat{u}\tilde{s}$ where $\tilde{s} \in C_{\widetilde{\Sigma}}(\tilde{u})\setminus[\widetilde{\Sigma}, \tilde{u}]$. Hence $C_{C_{\widetilde{\Sigma}}(\tilde{u})/[\widetilde{\Sigma},\tilde{u}]}(S_2) \neq 1$. By Proposition 2.10 (vi), we get $|C_{\widetilde{\Sigma}}(S_2)| = 4$. So S_2 does not centralize involutions in Σ. Thus we may assume that $S_2 = Z$. But this contradicts Lemma 6.4 (iii) and proves the lemma. $\square$

Lemma 6.6. *We have $\tilde{y}^{\widetilde{K}} \cap \widetilde{E} \subseteq \widetilde{\Sigma}$.*

Proof. Assume $\tilde{y} \sim_{\widetilde{K}} \tilde{u}$ for some involution $\tilde{u} \in \widetilde{E} \setminus \widetilde{\Sigma}$. By Lemma 6.1, we have that $|C_{\widetilde{K}}(\tilde{y})|_2 = 2^{12}$ if $\widetilde{E}/\widetilde{\Sigma} \cong \mathrm{SU}_4(2)$ or 2^{14} if $\widetilde{E}/\widetilde{\Sigma} \cong \mathrm{Sp}_6(2)$. This conflicts with that information given in Lemma 6.5. Hence no such elements exist. $\square$

Lemma 6.7. *$\widetilde{\Sigma}$ is weakly closed in $\widetilde{K}_1$. In particular, $\widetilde{K}_1$ contains a Sylow 2-subgroup of $\widetilde{K}$.*

Proof. Assume that $T \in \mathrm{Syl}_2(\widetilde{K}_1)$, $w \in \widetilde{K}$ and $\widetilde{\Sigma}^w \leq T$ with $\widetilde{\Sigma} \neq \widetilde{\Sigma}^w$. Then $\widetilde{\Sigma}^w \cap \widetilde{E}$ has order at least 2^7 and therefore is generated by conjugates of $\tilde{y}$. Thus Lemma 6.6 implies that $\widetilde{\Sigma}^w \cap \widetilde{E} \leq \widetilde{\Sigma}$. But then $|\widetilde{\Sigma}^w\widetilde{\Sigma}/\widetilde{\Sigma}| = 2$ and $|\Sigma \cap \Sigma^w| = 2^7$. Since $\widetilde{K}_1$ does not contain transvections, we have a contradiction. $\square$

Lemma 6.8. *No element of $\widetilde{\Sigma}$ is $\widetilde{K}$-conjugate to an involution $\tilde{u} \in \widetilde{K}_1$ with $|[\tilde{\Sigma}, \tilde{u}]| = 4$.*

Proof. Assume the statement is false. Then, by Lemma 6.6 $\tilde{u} \sim_{\widetilde{K}} \tilde{x}$. Let T_1 be a Sylow 2-subgroup of $C_{\widetilde{K}_1}(\tilde{u})$ and T_2 be a Sylow 2-subgroup of $C_{\widetilde{K}}(u)$ with $T_1 \leq T_2$. By Lemma 6.5 and 6.7, $|T_2 : T_1| = 4$. Let $\widetilde{\Sigma}_u$ be the group corresponding $\tilde{\Sigma}$ in T_2. Then $|\widetilde{\Sigma}_u \cap T_1| \geq 2^6$. As any subgroup of $\tilde{\Sigma}$ of order at least 2^6 is generated by conjugates of $\tilde{y}$, we have that $\widetilde{\Sigma}_u \cap T_1 \not\leq \widetilde{E}$ by Lemma 6.6. In particular, by Lemma 6.5, $J > J_0$. Therefore, we may suppose that there is a $\widetilde{w} \in \widetilde{\Sigma}_u \cap T_1$ such that $\widetilde{w}$ inverts σ. Notice that $(\widetilde{\Sigma}_u \cap T_1)\widetilde{\Sigma}$ is normal in $T_1\widetilde{\Sigma} \in \mathrm{Syl}_2(\widetilde{K}_1)$. In particular, if $|(\widetilde{\Sigma}_u \cap T_1)\widetilde{\Sigma}/\widetilde{\Sigma}| = 2^2$, then $\widetilde{w}\widetilde{\Sigma}$ is centralized by a maximal subgroup of $T_1\widetilde{\Sigma}/\widetilde{\Sigma}$, which it impossible. Hence

$$|(\widetilde{\Sigma}_u \cap T_1)\widetilde{\Sigma}/\widetilde{\Sigma}| \geq 2^3.$$

In particular, we have $|(\widetilde{\Sigma}_u \cap T_1 \cap \widetilde{E})\widetilde{\Sigma}/\widetilde{\Sigma}| \geq 2^2$ and all the non-trivial elements are unitary transvections. This, however, contradicts Proposition 2.10 (vii) and proves the lemma. $\square$

Lemma 6.9. *We have $\tilde{y}^{\widetilde{K}} \cap \widetilde{K}_1 \subseteq \widetilde{\Sigma}$. In particular, $\widetilde{\Sigma}$ is strongly closed in $\widetilde{E}$.*

Proof. Suppose that $\tilde{u} \in \tilde{y}^{\widetilde{K}} \cap \widetilde{K}_1 \setminus \widetilde{\Sigma}$. Then by Lemmas 6.6 and 6.5, we get that $\tau \neq 1$ and $\tilde{u}$ inverts σ. Furthermore, all involutions in $\widetilde{\Sigma}\tilde{u}$ are conjugate. Hence, for $T_1 \in \mathrm{Syl}_2(C_{\widetilde{K}_1}(\tilde{u}))$, we have $|T_1| = 2^9$. Let T_2 be a Sylow 2-subgroup of $C_{\widetilde{K}}(\tilde{u})$ with $T_1 \leq T_2$ and $\widetilde{\Sigma}_u \leq T_2$ be a $\widetilde{K}$-conjugate of $\tilde{\Sigma}$ in T_2. Then $\widetilde{\Sigma}_u \cap T_1 \subseteq \langle\widetilde{u}\rangle\widetilde{\Sigma}$ by Lemmas 6.5, 6.6 and 6.8. Since $|\widetilde{\Sigma}_u \cap T_1| \geq 2^5$, we now have that $\widetilde{\Sigma}_u \cap T_1 = \langle u\rangle C_{\widetilde{\Sigma}}(\widetilde{u})$ has order 2^5. Hence $T_2 = T_1\widetilde{\Sigma}_u$ and $T_2/\widetilde{\Sigma}_u \cong T_1/\langle u\rangle C_{\widetilde{\Sigma}}(\widetilde{u}) \cong 2\times \mathrm{Dih}(8)$. But $T_2/\Sigma_u \cong \mathrm{SDih}(16)$ by Proposition 2.10 (ii) and we thus have a contradiction. Hence $\tilde{y}^{\widetilde{K}} \cap \widetilde{K}_1 \subseteq \tilde{\Sigma}$. □

Lemma 6.10. *We have that $\widetilde{\Sigma}$ is strongly closed in $\widetilde{K}_1$.*

Proof. Assume by way of contradiction that there is some involution $\tilde{u} \in \widetilde{K}_1 \setminus \tilde{\Sigma}$, which is conjugate in $\tilde{K}$ to some element in $\tilde{\Sigma}$. By Lemma 6.9 we have $\tilde{u} \sim_{\widetilde{K}} \tilde{x}$. By Lemmas 6.8 and 6.5 we have that $\tau \neq 1$ and we may assume that $\tilde{u}$ inverts σ. Furthermore we have

$$C_{\widetilde{E}/\tilde{\Sigma}}(\widetilde{u}) \cong 2 \times \mathrm{Sym}(4).$$

Let T_1 be a Sylow 2-subgroup of $C_{\widetilde{K}_1}(\tilde{u})$ and T_2 be a Sylow 2-subgroup of $C_{\widetilde{K}}(\tilde{u})$, which contains T_1. Further let $\widetilde{\Sigma}_u$ be the normal subgroup of T_2 which is $\widetilde{K}$-conjugate to $\widetilde{\Sigma}$. Since, by Proposition 2.10 (viii), $C_{\widetilde{\Sigma}}(\widetilde{u})$ is generated by conjugates of $\widetilde{y}$, we have $C_{\widetilde{\Sigma}}(\tilde{u}) \leq \widetilde{\Sigma}_u$ by Lemma 6.9. Since $(\widetilde{\Sigma}_u \cap T_1)\widetilde{\Sigma}/\widetilde{\Sigma} = \langle\widetilde{u}\rangle\widetilde{\Sigma}/\widetilde{\Sigma}$, we get

$$T_3 = \widetilde{\Sigma}_u \cap T_1 = C_{\widetilde{\Sigma}}(\tilde{u})\langle\tilde{u}\rangle.$$

Therefore T_3 is normalized by $\widetilde{\Sigma}$ and is centralized by $\widetilde{\Sigma}_u$.This is impossible as $\widetilde{\Sigma}$ and $\widetilde{\Sigma}_u$ are conjugate in $\langle\widetilde{\Sigma}, \widetilde{\Sigma}_u\rangle$ by Lemma 6.7. □

Theorem 6.11. *We have $K = K_1$.*

Proof. Let $T \in \mathrm{Syl}_2(K)$. By Lemmas 6.7 and 6.10 we have that $\widetilde{\Sigma}$ is strongly closed in $\widetilde{T}$ with respect to $\widetilde{K}$. Hence an application of [4] yields that $\tilde{L} = \langle\widetilde{\Sigma}^{\widetilde{K}}\rangle$ is an extension of a group of odd order by a product of a 2-group and a number of Bender groups. Furthermore $\widetilde{\Sigma}$ is the set of involutions in some Sylow 2-subgroup of $T \cap \tilde{L}$. By Lemma 5.13 we have that $O(\tilde{L}) = 1$. As $\widetilde{K}_1$ acts primitively on $\widetilde{\Sigma}$, either $L = \widetilde{\Sigma}$ and we are done, or $\widetilde{L}$ is a simple group. So suppose that $\widetilde{L}$ is a simple group. Then $N_{\widetilde{L}}(\widetilde{\Sigma})$ acts transitively on $\widetilde{\Sigma}$, which is not possible as Σ is extraspecial. This proves that $K = K_1$. □

7. Proof of the Theorem 1.2

We continue with all the notation established in previous sections. If $N_M(S)/S \cong \mathrm{Dih}(8)$, Theorem 1.2 follows with Theorem 4.7. So we may assume that $N_M(S)/S \cong 2\times\mathrm{Dih}(8)$. Using Theorem 6.11 and Lemma 5.17 we get that $K/\Sigma \cong \mathrm{Aut}(\mathrm{SU}_4(2))$, $(3 \times \mathrm{SU}_4(2))$:2 or $\mathrm{Sp}_6(2)$.

Suppose that $K/\Sigma \cong \mathrm{Sp}_6(2)$. Then [17] implies that $G \cong \mathrm{Co}_2$ and consequently $M = N_G(Z)$ has order $2^8.3^6.5$ and shape $3^{1+4}_+.2^{1+4}_-.\mathrm{Sym}(5)$, which is not similar to a normalizer of type $\mathrm{PSU}_6(2)$. This contradicts our initial hypothesis. So suppose

$K/\Sigma \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $(3 \times \mathrm{SU}_4(2))$:2. Then Lemma 2.11 shows that G possesses a subgroup G_0 of index two. In particular we get $C_{G_0}(r)/\Sigma \cong \mathrm{SU}_4(2)$ or $3\times\mathrm{SU}_4(2)$. Now we see that $N_{G_0\cap M}(S)/S \cong \mathrm{Dih}(8)$. Hence Theorem 4.7 gives $G_0 \cong \mathrm{PSU}_6(2)$ or $\mathrm{PSU}_6(2)$:3 and so Theorem 1.2 is proved.

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CHRIS PARKER, SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, EDGBASTON, BIRMINGHAM B15 2TT, UNITED KINGDOM

E-mail address: c.w.parker@bham.ac.uk

GERNOT STROTH, INSTITUT FÜR MATHEMATIK, UNIVERSITÄT HALLE - WITTENBERG, THEORDOR LIESER STR. 5, 06099 HALLE, GERMANY

E-mail address: gernot.stroth@mathematik.uni-halle.de