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Combinatorics of Vassiliev Invariants

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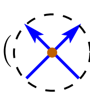
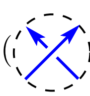
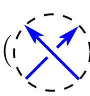
COMBINATORICS OF VASSILIEV INVARIANTS

SERGEI CHMUTOV

Abstract. This paper is an introductory survey of the combinatorial aspects of the Vassiliev theory of knot invariants following the lectures delivered at the Advanced School on Knot Theory and its Applications to Physics and Biology in the ICTP, Trieste (Italy), May 2009. The exposition is based on the forthcoming book [CDM], where the reader may find further details, examples, and developments.

1. Definition

Vassiliev knot invariants, also known as *finite type invariants*, were introduced independently by Victor Vassiliev (Moscow) [Vas] and by Mikhail Goussarov (St. Petersburg) [Gus] at the end of the 1980's. The principal idea of the combinatorial approach to the theory is to extend a knot invariant v to singular knots with double points according to the following rule, which we will refer to as *Vassiliev skein relation*:

(1) $v($  $) := v($  $) - v($  $)$.

The main definition is:

Definition 1.1. A knot invariant is said to be a *Vassiliev invariant* of order (or degree) $\leqslant n$ if its extension vanishes on all singular knots with more than n double points.

Denote by $\mathcal{V}_n$ the set of Vassiliev invariants of order $\leqslant n$ with values in the field of complex numbers $\mathbb{C}$. The definition implies that, for each n, the set $\mathcal{V}_n$ forms a complex vector space. Moreover, $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$, so we have an increasing filtration

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subseteq \mathcal{V} := \bigcup_{n=0}^{\infty} \mathcal{V}_n \ .$$

It will be shown that the spaces $\mathcal{V}_n$ have finite dimension, and that the quotients $\mathcal{V}_n/\mathcal{V}_{n-1}$ admit a nice combinatorial description. The study of these spaces is the main purpose of the combinatorial Vassiliev invariant theory. The exact dimension of $\mathcal{V}_n$ is known only for $n \leqslant 12$:

Key words and phrases. Vassiliev invariants, chord diagrams, weight systems, the Kontsevich integral.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{V}_n$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{V}_n/\mathcal{V}_{n-1}$	1	0	1	1	3	4	9	14	27	44	80	132	232

For higher values of n there are only asymptotic lower (O. Dasbach [Das]) and upper (D. Zagier [Zag]) bounds

$$e^{n/\log_a n} \lesssim \dim \mathcal{V}_n/\mathcal{V}_{n-1} \lesssim n!/a^n$$

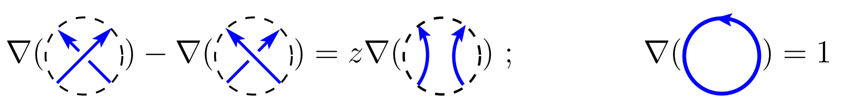
for any constant $a < \pi^2/6$.

Along with well known invariants of finite order, there are classical invariants which are not of finite order. Some of them belong to the *closure* of the Vassilev invariants $\bar{\mathcal{V}}$ which can be defied as follows (S. Duzhin). The space $\bar{\mathcal{V}}$ consists of all knot invariants which distinguish only the knots distinguishable by a finite order invariant. In other words, $v \in \bar{\mathcal{V}}$ if and only if for any two knots K_1 and K_2 with $v(K_1) \neq v(K_2)$ there is a Vassiliev invariant v_n (depending on K_1 and K_2) of finite order $\leqslant n$ which also distinguishes these knots, $v_n(K_1) \neq v_n(K_2)$. Below are some very important open problems of the theory:

- prove or disprove that any knot invariant belongs to the closure $\bar{\mathcal{V}}$,
- decide whether Vassiliev invariants detect the unknot,
- decide whether Vassiliev invariants detect orientation of knots.

2. Examples of Vassiliev invariants

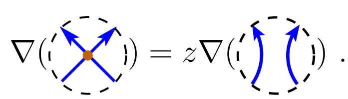
2.1. **The Conway polynomial.** The Conway polynomial may be defined by the skein relation and its normalization on the unknot:



For a knot K the Conway polynomial is always even

$$\nabla(K) = 1 + c_2(K)z^2 + c_4(K)z^4 + \dots \ .$$

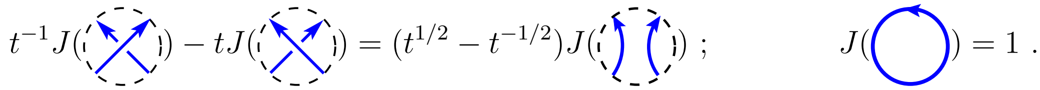
Its coefficients are Vassiliev invariants. Indeed, comparing its skein relation with the Vassiliev one (1) we conclude that the Conway polynomial of a knot with a singular point is divisible by z:



Consequently, the Conway polynomial of a knot with $k > n$ double points is divisible by z^k. Thus its coefficient $c_n(K)$ at z^n vanishes on singular knots with $> n$ double points. Therefore, it is a Vassiliev invariant of order $\leqslant n$.

Despite the fact that the coefficients are Vassiliev invariants, the degree of the Conway polynomial is not of finite type. Nevertheless, it belongs to the closure $\bar{\mathcal{V}}$.

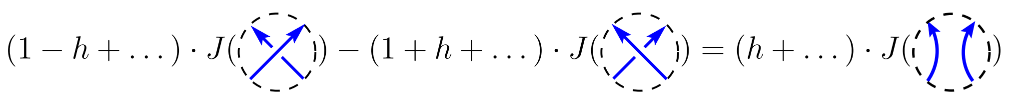
2.2. **The Jones polynomial.** The skein relation for the Jones polynomial with the same initial condition for the unknot are



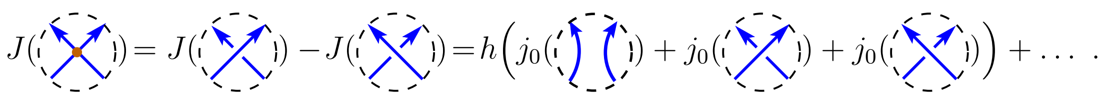
Making a substitution $t := e^h$ and then taking the Taylor expansion into a formal power series in h, we can represent the Jones polynomial of a knot K as a power series

$$J(K) = \sum_{n=0}^{\infty} j_n(K) h^n \ .$$

We claim that the coefficient $j_n(K)$ is a Vassiliev invariant of order $\leqslant n$. Indeed, substituting $t = e^h$ into the skein relation gives

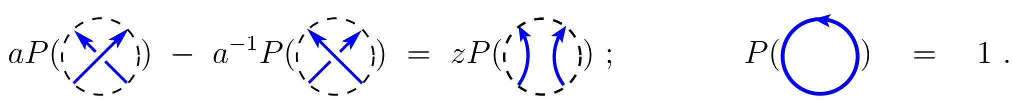


From which we get



This means that the value of the Jones polynomial on a knot with a single double point is congruent to 0 modulo h. Therefore, the Jones polynomial of a singular knot with $k > n$ double points is divisible by h^k, and thus its nth coefficient vanishes on such a knot.

2.3. **The HOMFLYPT polynomial.** The skein relation for the HOMFLYPT polynomial is



Making a substitution $a = e^h$ and taking the Taylor expansion in h, we represent $P(K)$ as a Laurent polynomial in z and a power series in h, $P(K) = \sum p_{k,l}(K)\ h^k z^l$. Similarly to the case of the Jones polynomial, one can show that $p_{k,l}(K)$ is a Vassiliev invariant of order $\leqslant k + l$.

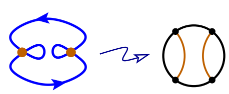
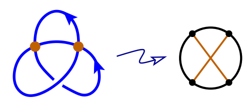
2.4. **Quantum knot invariants.** J. Birman and X.-S. Lin proved [BL] that all quantum invariants produce Vassiliev invariants in the way similar to the previous examples. Namely, making a substitution $q = e^h$, one can show that the coefficient of h^n in the Taylor expansion of a quantum invariant is a Vassiliev invariant of order $\leqslant n$. Thus all quantum invariants belong to the closure $\bar{\mathcal{V}}$.

3. Chord diagrams, symbols, and weight systems

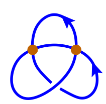
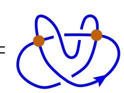
Let v be a Vassiliev invariant of order $\leqslant n$. It turns out that a value of v on a singular knot K with n double points does not depend on the specific knotedness of K. It depends only on the combinatorial arrangement of double points along the knot, which can be encoded by a *chord diagram* of K.

Definition 3.1. A *chord diagram* of order n (or *degree* n) is an oriented circle with a distinguished set of n disjoint pairs of distinct points, considered up to orientation preserving homeomorphisms of the circle.

This means that only the mutual combinatorial positions of the ends of chords are important. Their precise geometrical locations on the circle are irrelevant. By a chord diagram of a singular knot K we mean a circle parameterizing K with the two preimages of each double point connected by a chord. Here are two examples:

 ,  .

The value of $v(K)$ on a knot K with n double points depends only on the chord diagram of K. Indeed, let K_1 and K_2 be two singular knots with the same chord diagrams. If we place K_1, K_2 in $\mathbb{R}^3$ so that the corresponding double points (and both branches of the knot in their vicinity) coincide,

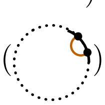
$K_1 =$  $K_2 =$  ,

then we can deform the arcs of K_1 into the corresponding arcs of K_2 so that the only singularities introduced or removed (one at a time) in the process are double points. By the Vassiliev skein relation, in each of these events the value of v does not change. So $v(K_1) = v(K_2)$.

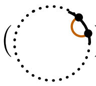
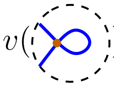
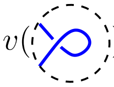
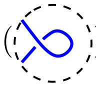
Definition 3.2. The *symbol* of v is a restriction of v to the set of knots with precisely n double points, considered as a function on the set of chord diagrams.

It is obvious that if v_1 and v_2 are two Vassiliev invariants of order $\leqslant n$ with the same symbols, then their difference $v_1 - v_2$ is an invariant of order $\leqslant (n-1)$. Thus the space $\mathcal{V}_n/\mathcal{V}_{n-1}$ coincides with the space of all symbols of Vassiliev invariants of order $\leqslant n$. The set of chord diagrams with n chords is finite. So the space of all ($\mathbb{C}$-valued) functions on this set, and hence $\mathcal{V}_n$, is finite dimensional.

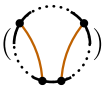
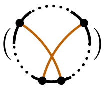
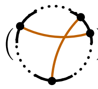
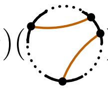
The symbol of an invariant is not an arbitrary function on chord diagrams; it satisfies certain relations. The so called *one-term relation* (1T) is very easy to see; the symbol of an invariant v always vanishes on a chord diagram with a short isolated chord (not intersecting other chords)

symb$(v)($  $) = 0$.

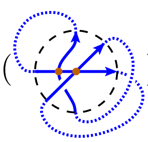
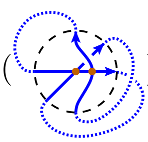
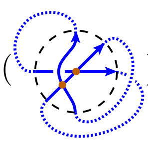
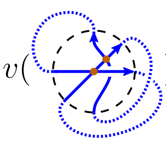
This follows from the fact that we can choose a singular knot representing such a chord diagram as having a small kink parameterized by the arc between the end-points of the isolated chord. Then the Vassiliev skein relation (1) gives

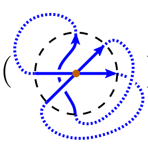
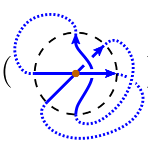
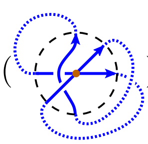
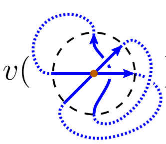
symb(v)() $=$ v() $=$ v() $-$ v() $= 0$,

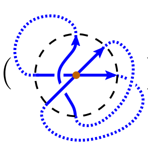
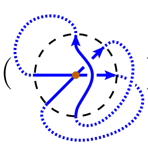
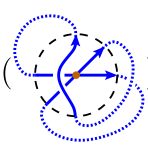
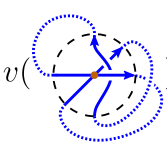
because the two knots on the right-hand side are isotopic. Another kind of relation is called the *four-term relation* (4T):

symb(v)() $-$ symb(v)() $+$ symb(v)() $-$ symb(v)() $= 0$.

Here it is assumed that the diagrams in the pictures may have other chords with end-points on the dotted arcs, while all the end-points of the chords on the solid portions of the circle are explicitly shown. For example, this means that in the first and second diagrams the two bottom end-points are adjacent. The chords omitted from the pictures should be the same in all four cases. To prove this relation we represent each chord diagram by a singular knot with a special fragment containing two double points. Then, resolving one of them for every knot, we will get eight terms which cancel out.

v() $-$ v() $+$ v() $-$ v()

$=$ v() $-$ v() $+$ v() $-$ v()

$-$ v() $+$ v() $-$ v() $+$ v()

$= 0$

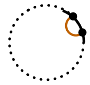
Definition 3.3. A *weight system* of order n is a function on the set of chord diagrams with n chords which satisfies one- and four-term relations.

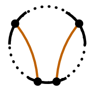
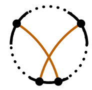
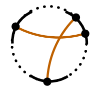
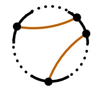
It turns out that all other relations for symbols of Vassiliev invariants are consequences of (1T) and (4T) relations. This is the content of a fundamental theorem of M. Kontsevich.

Theorem 3.4 (M. Kontsevich [Kon])**.** *Any weight system is a symbol of an appropriate Vassiliev invariant of order* $\leqslant n$.

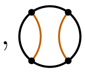
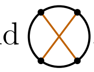
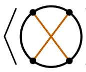
Together with the discussion above this theorem implies that the quotient space $\mathcal{V}_n/\mathcal{V}_{n-1}$ is isomorphic to the space of all ($\mathbb{C}$-valued) functions on the set of chord diagram with n chords satisfying the relations (1T) and (4T).

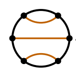
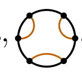
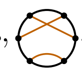
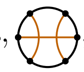
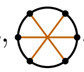
Let us introduce a $\mathbb{C}$-vector space $\mathcal{A}_n$ as the space spanned by chord diagrams modulo (1T) and (4T) relations:

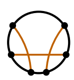
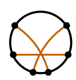
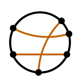
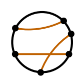
(1T)  $= 0$

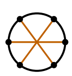
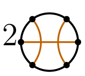
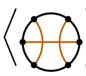
(4T)  $-$  $+$  $-$  $= 0$.

Then, $\mathcal{V}_n/\mathcal{V}_{n-1} \cong (\mathcal{A}_n)^*$.

Examples 3.5. For $n = 2$, among the two possible chord diagrams,  and , the first one is equal to zero by (1T). Thus $\mathcal{A}_2 = \Big\langle$  $\Big\rangle$ and has dimension 1.

For $n = 3$ there are five chord diagrams: , , , , . The first three diagrams are equal to zero by (1T). There is also a (4T) relation involving the last two

 $-$  $+$  $-$  $= 0$,

which gives  $= 2$  . Thus $\mathcal{A}_3 = \Big\langle$  $\Big\rangle$ and also has dimension 1.

4. Symbols of knot polynomials

The Kontsevich theorem reduces the study of Vassiliev knots invariants to the study of their symbols. Here we describe the symbols of coefficients of classical knot polynomials as functions on chord diagrams.

4.1. **The Conway polynomial.** For a chord diagram D with n chords, double every chord as shown

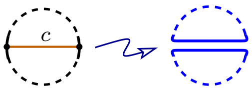
 .

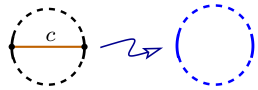
Let $|D|$ be equal to the number of components of the obtained (possibly self-intersecting) curve. Then

$$\mathrm{symb}(c_n)(D) = \begin{cases} 1, & \text{if } |D| = 1 \\ 0, & \text{otherwise} . \end{cases}$$

Note that doubling of a chord changes the number of component by 1 (either adding 1, or subtracting 1). Therefore $\mathrm{symb}(c_n)$ is automatically zero for odd n.

4.2. **The Jones polynomial.** For a chord diagram D with n chords, *a state* s is an arbitrary function on the set chords of D with values in the set $\{1, 2\}$. With each state s we associate an immersed plane curve obtained from D by resolving (either doubling or deleting) all its chords according to s:



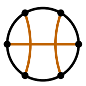
, if $s(c) = 1$; 

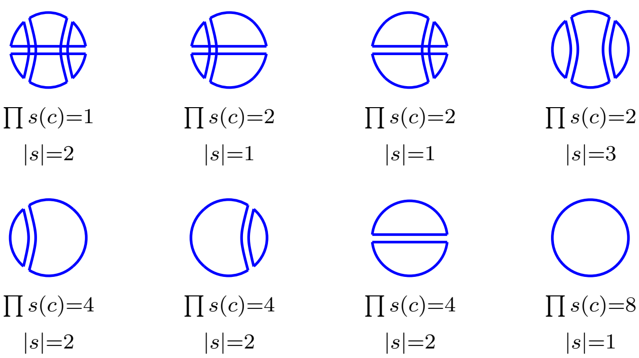
, if $s(c) = 2$.

Let $|s|$ denote the number of components of the curve obtained in this way. Then

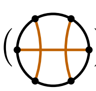
$$\mathrm{symb}(j_n)(D) = \sum_s \Big(\prod_c s(c)\Big)\ (-2)^{|s|-1}\ ,$$

where the product is taken over all n chords of D, and the sum is taken over all 2^n states for D.

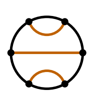
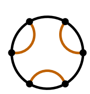
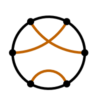
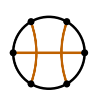
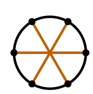
For example, to compute the value of the symbol of j_3 on the chord diagram , we must consider 8 states:



Therefore,

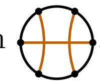
$\mathrm{symb}(j_3)($  $) = -2 + 2 + 2 + 2(-2)^2 + 4(-2) + 4(-2) + 4(-2) + 8 = -6\ .$

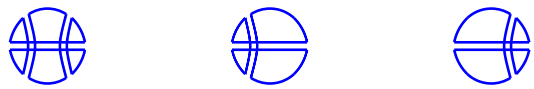
Similarly one can compute the values of $\mathrm{symb}(j_3)$ on all chord diagrams with three chords. Here is the result:

D					
$\mathrm{symb}(j_3)(D)$	0	0	0	-6	-12

4.3. **The HOMFLYPT polynomial.** In this case, for a chord diagram D with $k+l$ chords, *a state* s will be a subset of chords of D. As above, we use the notation $|s|$ for the number of curves obtained by doubling all chords in s. Denote by $||s||$ the number of chords of s. A state s is called *admissible* for $p_{k,l}$ if $||s|| - |s| + 1 = l$. Then

$$\mathrm{symb}(p_{k,l})(D) = (-1)^{k+l} 2^k \sum_{\substack{s \\ \text{admissible}}} (-1)^{||s||}\ .$$

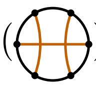
For example, if we would like to evaluate $\mathrm{symb}(p_{1,2})$ on the chord diagram , then out of the 8 states above only the first 3 are admissible:



$||s||$=3, $|s|$=2 $||s||$=2, $|s|$=1 $||s||$=2, $|s|$=1

Therefore,

$$\mathrm{symb}(p_{1,2})(\ \ )= -2\ .$$



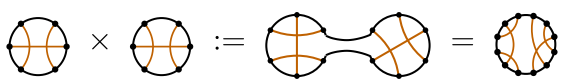
In fact, $j_3 = 3p_{1,2}$.

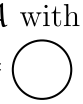
5. Bialgebra of chord diagrams

Before explaining the idea of a proof of the Kontsevich theorem we need some algebraic structure on the vector spaces $\mathcal{A}_n$.

The vector space $\mathcal{A} := \bigoplus\limits_{n\geqslant 0} \mathcal{A}_n$ has a natural algebra structure.

Definition 5.1. The *product* of two chord diagrams D_1 and D_2 is defined to be their connected sum:



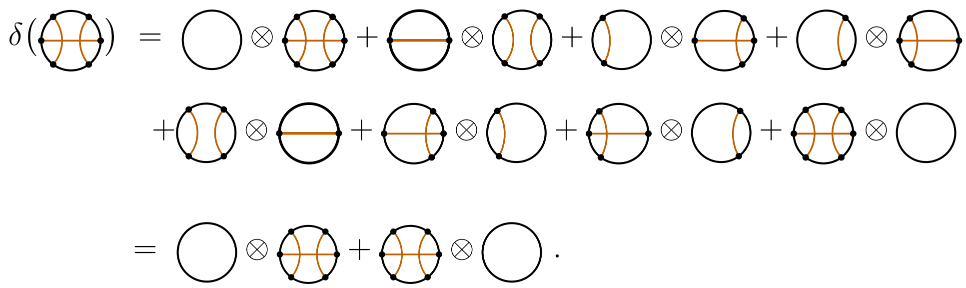
Though the connected sum itself depends on the places where the two diagrams are joined, modulo (4T), the result does not depend on them. Extended by linearity, this product endows $\mathcal{A}$ with the structure of graded commutative algebra with a unit $\iota : \mathbb{C} \to \mathcal{A}$, $\iota(1) :=$  $\in \mathcal{A}_0$ represented by a chord diagram without chords. For a proof of this fact we refer to [BN, CDM].

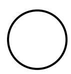
Definition 5.2. The *coproduct* $\delta : \mathcal{A}_n \to \bigoplus_{k+l=n} \mathcal{A}_k \otimes \mathcal{A}_l$ is defined on chord diagrams by the sum of all ways to split the set of chords into two disjoint parts. Namely, for $D \in \mathcal{A}_n$ we set

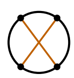
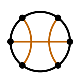
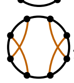
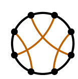
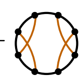
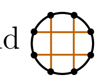
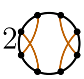
$$\delta(D) := \sum_{J\subseteq[D]} D_J \otimes D_{\overline{J}},$$

the summation taken over all subsets J of the set of chords of D. Here D_J is the diagram consisting of the chords that belong to J, and $\overline{J} = [D] \setminus J$ is the complementary subset of chords. The operator δ is extended by linearity to the entire space $\mathcal{A}$.

Here is an example:



This operation endows $\mathcal{A}$ with a structure of a cocommutative coalgebra with counit $\varepsilon : \mathcal{A} \to \mathbb{C}$, $\varepsilon\big(x$  $+ \ldots$ "higher order terms"' $\ldots\big) := x$. The *cocommutativity* here means that the image of δ belongs to the symmetric tensor subalgebra of the tensor square $\mathcal{A}^{\otimes 2}$.

The operations of multiplication and comultiplication together turn $\mathcal{A}$ to be a *bialgebra*. For bialgebras, the *primitive space* $\mathcal{P}$ plays an important role. By definition, $\mathcal{P}(\mathcal{A})$ is the space of elements $D \in \mathcal{A}$ with the property $\delta(D) = 1 \otimes D + D \otimes 1$. It is also a graded vector space $\mathcal{P}(\mathcal{A}) = \bigoplus\limits_{n \geqslant 1} \mathcal{P}_n$. For example, both basic diagrams in degree 2 and 3,  and  are primitive, however in degree 4 there is a non primitive element , which is the square of the primitive element of degree 2. A basis for $\mathcal{P}_4$ consists of two elements  $-$  and  $- 2$ .

A graded commutative, cocommutative bialgebra $\mathcal{A}$ with a unit, counit, and one-dimensional zero-degree space $\mathcal{A}_0$ may be turned into a *Hopf algebra* by defining the *antipode* $S : \mathcal{A} \to \mathcal{A}$ to be an antiautomorphism acting by multiplication by (-1) on the primitive space. For such Hopf algebras $\mathcal{A}$, the classical Milnor—Moore [MiMo] theorem states that $\mathcal{A}$ is isomorphic to the symmetric tensor algebra of the primitive space:

$$\mathcal{A} \cong \mathcal{S}(\mathcal{P}(\mathcal{A})) \ .$$

This means that if we choose a basis $p_1, p_2, \ldots$ for the primitive space $\mathcal{P}(\mathcal{A})$ then any element of $\mathcal{A}$ can be uniquely represented as a polynomial in commuting variables $p_1, p_2, \ldots$. Theoretically this implies that for the purposes of combinatorics of Vassiliev invariants it is enough to study the primitive spaces $\mathcal{P}_n$ only. In particular, here is the table of dimensions of these spaces up to $n \leqslant 12$ calculated by J. Kneissler [Kn]:

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{P}_n$	0	1	1	2	3	5	8	12	18	27	39	55

The table of dimensions $\dim(\mathcal{V}_n/\mathcal{V}_{n-1}) = \dim \mathcal{A}_n$ on page 2 can be derived from this table.

6. The Kontsevich integral

Roughly speaking, an idea of the proof of the Kontsevich theorem is to construct an element $Z(K) \in \mathcal{A}$ of the algebra of chord diagram for every knot K. Having a weight system w we can apply it to $Z(K)$ and prove that $w(Z(K))$ is a Vassiliev invariant whose symbol is w. However, when we try to realize this idea, several complications occur.

The first one is that $Z(K)$ is going to be an element of the *graded completion* $\widehat{\mathcal{A}}$ of the algebra $\mathcal{A}$, or in other words, it is going to be an infinite sum of elements of $\mathcal{A}_n$ for all values of n, like a formal power series. The second one is that $Z(K)$ is not quite an invariant of knots. We will have to correct it before applying the weight system to it.

6.1. **The construction.** Let $z \in \mathbb{C}$ and $t \in \mathbb{R}$ be coordinates (z, t) in $\mathbb{R}^3$. The planes $t = \text{const}$ are thought of being horizontal. We define the Kontsevich integral for strict Morse knots, i.e. knots with the property that the coordinate t restricted to the knot has only non-degenerate critical points with distinct critical values.

Definition 6.1. The *Kontsevich integral* $Z(K)$ of a strict Morse knot K is given by the following formula

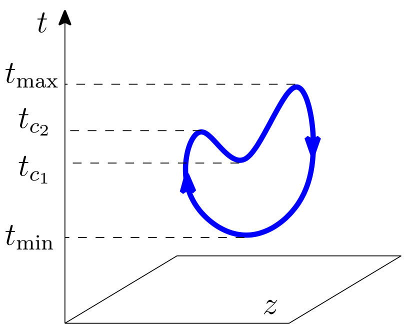
$$Z(K) := \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{\min}<t_1<\cdots<t_m<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j, z'_j)\}} (-1)^{\downarrow_P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \ ,$$

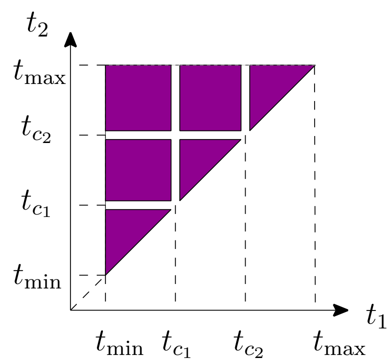
where

- the numbers $t_{\min}$ and $t_{\max}$ are the minimum and the maximum of the function t on K;
- the integration domain is the set of all points of the m-dimensional simplex $t_{\min} < t_1 < \cdots < t_m < t_{\max}$ none of whose coordinates t_i is a critical value of t; the m-simplex is divided by the critical values into several *connected components*;
- the number of summands in the integrand is constant in each connected component of the integration domain, but can be different for different components; in each plane $\{t = t_j\} \subset \mathbb{R}^3$ choose an unordered pair of distinct points (z_j, t_j) and (z'_j, t_j) on K, so that $z_j(t_j)$ and $z'_j(t_j)$ are continuous functions; we denote by $P = \{(z_j, z'_j)\}$ the set of such pairs for $j = 1, \ldots, m$ and call it a *pairing*; the integrand is the sum over all choices of the pairing P;
- for a pairing P, the symbol '$\downarrow_P$' denotes the number of points (z_j, t_j) or (z'_j, t_j) in P where the coordinate t decreases as one goes along K;
- for a pairing P, consider the knot K as an oriented circle and connect the points (z_j, t_j) and (z'_j, t_j) by a chord; we obtain a chord diagram with m chords (thus, intuitively, one can think of a pairing as a way of inscribing a chord diagram into a knot in such a way that all chords are horizontal and are placed on different levels);

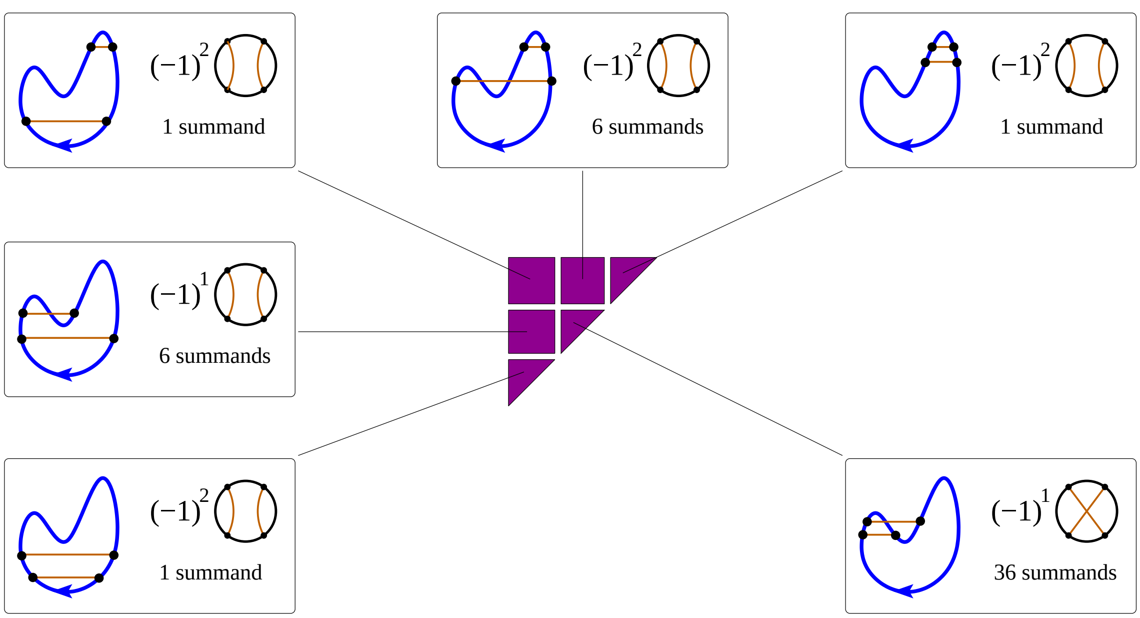
- over each connected component, z_j and z'_j are smooth functions in t_j; by $\bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j}$ we mean the pullback of this form to the integration domain of the variables $t_1, \dots, t_m$; the integration domain is considered with the orientation of the space $\mathbb{R}^m$ defined by the natural order of the coordinates $t_1, \dots, t_m$;
- by convention, the term in the Kontsevich integral corresponding to $m = 0$ is the (only) chord diagram of order 0 taken with coefficient one; it is the unit of the algebra $\widehat{\mathcal{A}}$.

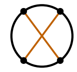
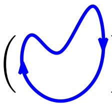
Example 6.2. We exemplify the integration domain for a strict Morse knot with two local maxima and two local minima for the case of $m = 2$ chords.



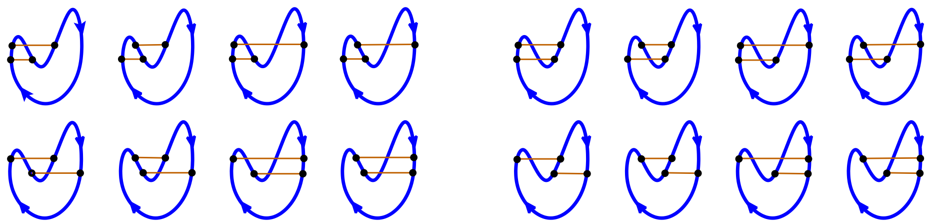


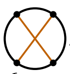
For each connected component of the integration domain, the number of summands corresponding to different choices of the pairing, a typical pairing P, and the corresponding chord diagram $(-1)^{\downarrow_P} D_P$ are shown in the picture.

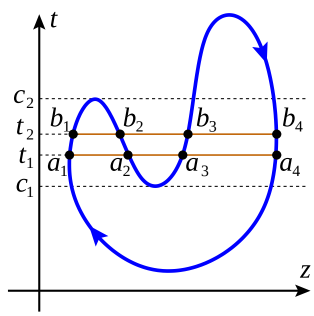


Now let us calculate the coefficient of the chord diagram  in $Z\Big($$\Big)$.

Out of the 51 pairings, the following 16 contribute to the coefficient:



All of them appear on the middle triangular component, $t_{c_1} < t_1 < t_2 < t_{c_1}$ of the integration domain. To handle the integral which appears as the coefficient at , we denote the z-coordinates of the four points in the pairings on the level $\{t = t_1\}$ by a_1, a_2, a_3, a_4. Correspondingly, we denote the z-coordinates of the four points in the pairings on the level $\{t = t_2\}$ by b_1, b_2, b_3, b_4:



Then each of the four possible pairings $z_1 - z_1'$ on the level $\{t = t_1\}$ will look like $a_{jk} := a_k - a_j$ for $(jk) \in A := \{(12), (13), (24), (34)\}$. Similarly, each of the four possible pairings $z_2 - z_2'$ on the level $\{t = t_2\}$ will look like $b_{lm} := b_m - b_l$ for $(lm) \in B := \{(13), (23), (14), (24)\}$. The integral we are interested in now can be written as

$$\frac{1}{(2\pi i)^2} \int\limits_{\Delta} \sum_{(jk)\in A} \sum_{(lm)\in B} (-1)^{j+k+l+m} d\ln a_{jk} \wedge d\ln b_{lm}$$

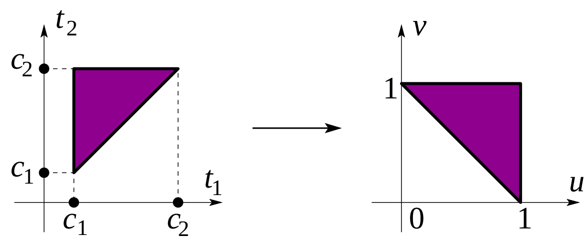
$$= -\frac{1}{4\pi^2} \int\limits_{\Delta} \left(\sum_{(jk)\in A} (-1)^{j+k+1} d\ln a_{jk} \right) \wedge \left(\sum_{(lm)\in B} (-1)^{l+m-1} d\ln b_{lm} \right)$$

$$= -\frac{1}{4\pi^2} \int\limits_{\Delta} d\ln \frac{a_{12}a_{34}}{a_{13}a_{24}} \wedge d\ln \frac{b_{14}b_{23}}{b_{13}b_{24}} \ .$$

The change of variables

$$u := \frac{a_{12}a_{34}}{a_{13}a_{24}}, \qquad v := \frac{b_{14}b_{23}}{b_{13}b_{24}}$$

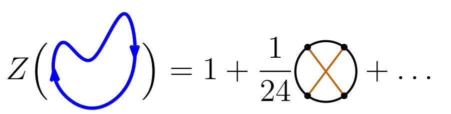
transforms the component Δ of the integration domain into the standard triangle Δ'

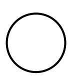


Since it changes the orientation of the triangle (has a negative Jacobian), our integral becomes

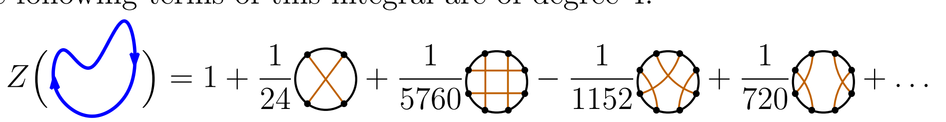
$$\begin{aligned}
\frac{1}{4\pi^2}\int\limits_{\Delta'} d\ln u \wedge d\ln v \;&=\; \frac{1}{4\pi^2}\int\limits_0^1 \Bigl(\int\limits_{1-u}^1 d\ln v\Bigr)\frac{du}{u} \\
&=\; -\frac{1}{4\pi^2}\int\limits_0^1 \ln(1-u)\frac{du}{u} \;=\; \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\int\limits_0^1 \frac{u^k}{k}\frac{du}{u} \\
&=\; \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \;=\; \frac{\zeta(2)}{4\pi^2} = \frac{1}{24}\ .
\end{aligned}$$

Therefore,



where the free term 1 stands for the unit in the algebra $\mathcal{A}$ of chord diagrams, $1 =$  $\in \mathcal{A}$.

The following terms of this integral are of degree 4:



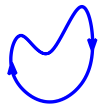
The problem of the exact calculation of all the terms of this integral remained opened for a long time. Finally it was solved in [BLT]; see also the explanation of this work in [CDM]. However even the formulation of the results requires a different language for representing elements of $\mathcal{A}$. The coefficients in this integral are related to the Bernoulli numbers.

6.2. **The universal Vassiliev invariant.** Here we deal with the second complication in the proof of the Kontsevich theorem. The Kontsevich integral possesses several basic properties:

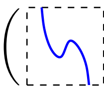
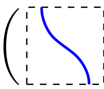
- $Z(K)$ converges for any strict Morse knot K.
- It is invariant under the deformations of the knot in the class of Morse knots with the same number of critical points.

- It behaves in a predictable way under the deformations that add a pair of new critical points to a Morse knot.

We will not prove these properties here referring the reader to [BN, CDM]. The last property may be specified as follows. Let H stand for the unknot from Example 6.2:

$H :=$  .

If a knot isotopy creates one extra local minimum and one local maximum, the Kontsevich integral is multiplied by $Z(H)$:

$Z\Big($  $\Big) = Z(H) \cdot Z\Big($  $\Big)$,

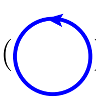
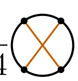
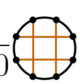
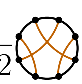
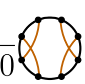
where the multiplication is understood in sense of the formal infinite series from the graded completion of the algebra of chord diagrams $\widehat{\mathcal{A}}$.

Definition 6.3. With the help of this formula, we can define the *universal Vassiliev invariant* by either

$$I(K) := \frac{Z(K)}{Z(H)^{c/2}} \qquad \text{or} \qquad I'(K) := \frac{Z(K)}{Z(H)^{c/2-1}},$$

where c denotes the number of critical points of K in an arbitrary strict Morse representation, and the quotient means division in the algebra $\widehat{\mathcal{A}}$: $(1+a)^{-1} = 1 - a + a^2 - a^3 + \dots$.

Both $I(K)$ and $I'(K)$ are invariants of topological knots with values in $\widehat{\mathcal{A}}$. The version $I'(K)$ has the advantage of being multiplicative with respect to the connected sum of knots; in particular, it vanishes (more precisely, takes the value 1) on the unknot. However, the version $I(K)$ is also used as it has a direct relationship with the quantum invariants. In particular,

$I($  $) := \frac{1}{Z(H)} = 1 - \frac{1}{24}$  $- \frac{1}{5760}$  $+ \frac{1}{1152}$  $+ \frac{1}{2880}$  $+ \dots$.

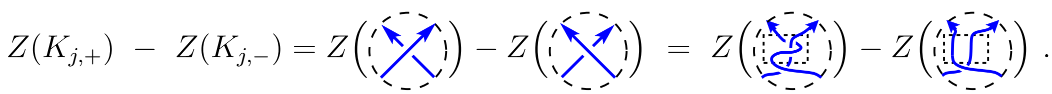
The central importance of $I(K)$ (as well as $I'(K)$) in the theory of finite-type invariants is that it is a universal Vassiliev invariant in the following sense. Consider a weight system w of order n (that is, a function on the set of chord diagrams with n chords satisfying one- and four-term relations). Applying w to the n-homogeneous part of the series $I(K)$, we get a numerical knot invariant $v := w(I(K))$. This invariant is a Vassiliev invariant of order $\leqslant n$ and $\mathrm{symb}(v) = w$. Moreover, any Vassiliev invariant is a sum of Vassiliev invariants obtained in this way. Indeed, any Vassiliev invariant v has a symbol $\mathrm{symb}(v)$ which is a weight system. So $v(K)$ and $\mathrm{symb}(v)(I(K))$ are two Vassiliev invariants with the same symbol. Their difference is of order $\leqslant (n-1)$ and we can repeat the same process by induction. This argument proves the Kontsevich theorem.

To prove that $v : K \longmapsto w(I(K))$ is a Vassiliev invariant with symbol w, it is enough to show that $I(K_D) = D +$ (terms of order $> n$) for a singular knot K_D representing a chord diagram D of order n. Let $p_1, \dots p_n$ be the double points of K_D.

Since the denominator of $I(K)$ starts with the unit of the algebra $\widehat{\mathcal{A}}$, it is sufficient to prove that

$$Z(K_D) = D + (\text{terms of order} > n)\ .$$

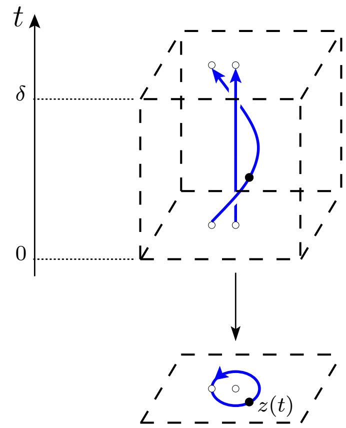
By definition, $Z(K_D)$ is the alternating sum of 2^n values of Z on the complete resolutions of the singular knot K_D at all its double points $p_1, \dots p_n$. To see what happens at a single double point p_j, let us look at the difference of Z on two knots $K_{j,+}$ and $K_{j,-}$ where an overpassing in $K_{j,+}$ is changed to an underpassing in $K_{j,-}$. By an isotopy we have:



These two knots differ only in that the first knot has one string that makes a small curl around another string. We assume that this curl is very thin and enclose it in a box of height δ to consider it as a tangle $T_{j,+}$. Let $T_{j,-}$ be the similar tangle for $K_{j,-}$ consisting of two parallel vertical strings:

The difference $Z(K_{j,+}) - Z(K_{j,-})$ comes only from the difference of the Kontsevich integrals of tangles $Z(T_{j,+}) - Z(T_{j,-})$. For the tangle $T_{j,-}$, both functions $z(t)$ and $z'(t)$ are constant, hence $Z(T_{j,-})$ consists of only one summand: the trivial chord diagram $D_-^{(0)}$ on $T_{j,-}$ (here 'trivial' means with an empty set of chords) with coefficient one. The zero degree term $D_+^{(0)}$ of $Z(T_{j,+})$ is also the trivial chord diagram on $T_{j,+}$ with coefficient one. So $D_-^{(0)} = D_+^{(0)}$ as chord diagrams. Thus the difference $Z(T_{j,+}) - Z(T_{j,-})$ starts with a first degree term which comes from the chord connecting the two strings of $T_{j,+}$.

Hence, $Z(K_D)$ starts from degree n. Moreover, the term of degree n is proportional to a chord diagram whose j-th chord connects the two strings of the tangle $T_{j,+}$ corresponding to the jth double point p_j in K_D. This chord diagram is precisely D. Now to calculate the order n part of $Z(K_D)$ we must compute the coefficient of D. It is equal to the product of coefficients of one chord terms in $Z(T_{j,+})$ over all j. We will show that all of them are equal to 1. Indeed, it is possible to choose the coordinates z and t is such a way that $z'(t) \equiv 0$ for one string, and for another one, the point $z(t)$ makes one complete turn around zero when t varies from 0 to δ:



So we have

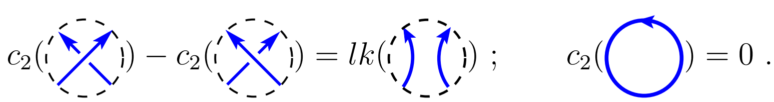
$$\frac{1}{2\pi i}\int_0^\delta \frac{dz-dz'}{z-z'} = \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z} = 1$$

by the Cauchy theorem.

7. Polyak—Viro formulas

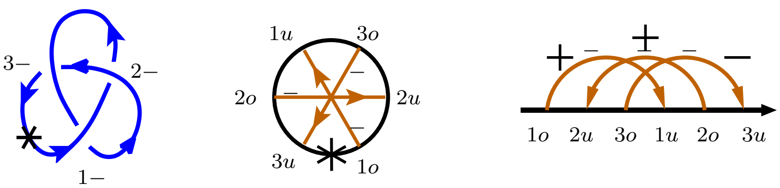
Another universal way of representing Vassiliev invariants was suggested by M. Polyak and O. Viro [PV]. Its universality was proved by M. Goussarov in [GPV]. In general, Polyak—Viro formulas are far reaching generalization of the classical expression of the linking number of two curves as the sum of local writhes of the crossings of these curves.

7.1. **Simplest example.** The simplest example of the Polyak—Viro formula is a formula for the second coefficient of the Conway polynomial, also known as the *Casson invariant*. The invariant can be defined by the skein relation using the linking number lk:



The Polyak—Viro formula for it is $c_2(K) = \sum_{\substack{i\,j\,i\,j\\ o\,u\,u\,o}} \varepsilon_i\varepsilon_j$, where the summation runs over all pairs of crossings (i,j) which appear in the order i,j,i,j when traveling along the knot and the first appearance of i is overpassing while the first appearance of j is underpassing; ε_i is the writhe of the ith crossing. For example, for the trefoil

3_1,

$3_1 =$ 

there is only one such pair $\boxed{\begin{smallmatrix}1\,2\\ ou\end{smallmatrix}}\begin{smallmatrix}3\\ o\end{smallmatrix}\boxed{\begin{smallmatrix}1\,2\\ uo\end{smallmatrix}}\begin{smallmatrix}3\\ u\end{smallmatrix}$. So $c_2(3_1) = 1$.

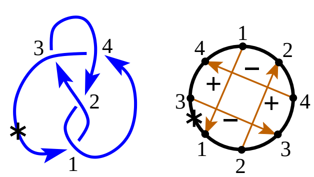
An appropriate language for expressing Polyak—Viro formulas is the language of Gauss diagrams.

Definition 7.1. A *Gauss diagram* is an oriented circle with a distinguished set of distinct points divided into ordered pairs, each pair carrying a sign ± 1.

Graphically, an ordered pair of points on a circle can be represented by a chord with an arrow connecting them and pointing, say, to the second point. Gauss diagrams are considered up to orientation-preserving homeomorphisms of the circle. Sometimes, an additional basepoint is marked on the circle and the diagrams are considered up to homeomorphisms that keep the basepoint fixed. In this case, we speak of *based Gauss diagrams.*

To a plane knot diagram one can associate a Gauss diagram as follows. Pairs of points on the circle correspond to the values of the parameter where the diagram has a self-intersection, each arrow points from the overcrossing to the undercrossing and its sign is equal to the local writhe at the crossing.

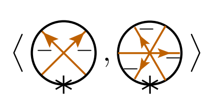
The Gauss diagram of the trefoil above is shown next to it. Here is an example of the figure eight knot:



Sometimes we will draw based Gauss diagrams as line diagrams assuming that the base point is at infinity; such a diagram for the trefoil is above.

Definition 7.2. For two Gauss diagrams A and G we denote by $\langle A, G\rangle$ the number of occurrences of A as a subdiagram of G.

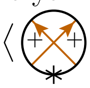
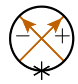
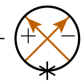
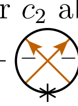
For example,


$$= 1 \ .$$

Define a pairing of a "non-signed" Gauss diagram A with a Gauss diagram G as the sum over all 2^n sign assignments $\varepsilon_1, \ldots, \varepsilon_n$ to the arrows of A, $A_{\varepsilon_1,\ldots,\varepsilon_n}$, of pairings $\langle A_{\varepsilon_1,\ldots,\varepsilon_n}, G\rangle$ with products of the signs $\varepsilon_1 \ldots \varepsilon_n$ as coefficients:

$$\langle A, G\rangle := \sum_{\varepsilon_1,\ldots,\varepsilon_n} (\varepsilon_1 \ldots \varepsilon_n)\langle A_{\varepsilon_1,\ldots,\varepsilon_n}, G\rangle \ .$$

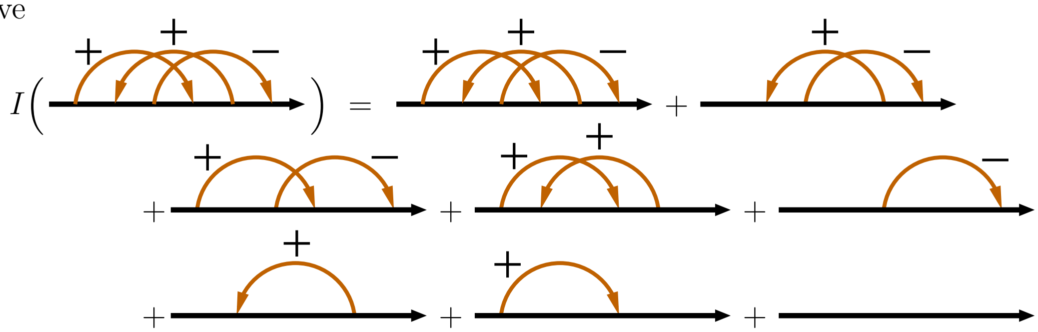
In this notation, for example, the Polyak—Viro formula for c_2 above will be

$c_2(K) = \langle$  $, G_K \rangle := \langle$  $-$  $-$  $+$  $, G_K \rangle$.

7.2. **The Goussarov theorem.** For a more formal definition, let us introduce a free $\mathbb{Z}$-module $\mathbb{Z}\mathbf{GD}$ spanned by the set of all Gauss diagrans $\mathbf{GD}$. We define the map $I : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}\mathbf{GD}$ by simply sending a diagram to the sum of its subdiagrams:

$$I(D) := \sum_{D' \subseteq D} D'$$

and continuing this definition to the whole of $\mathbb{Z}\mathbf{GD}$ by linearity. For example, we have



If we combine like terms in the expression $I(D)$ we get the sum over different Gauss diagrams,

$$I(D) = \sum_{A \in \mathbf{GD}} \langle A, D \rangle A.$$

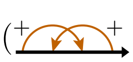
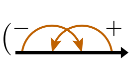
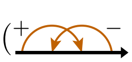
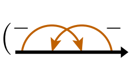
Having a linear function $c : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}$ we can construct a different function $I^*(c) : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}$ as follows:

$$I^*(c)(D) := (c \circ I)(D) = \sum_{A \in \mathbf{GD}} \langle A, D \rangle c(A). \tag{2}$$

Gauss diagrams that encode long classical knots, or *realizable* diagrams, form a subset $\mathbf{GD}^{re} \subset \mathbf{GD}$. Any integer-valued knot invariant v gives rise to a function $\mathbf{GD}^{re} \to \mathbb{Z}$. By linearity, it extends to a function, also denoted by $v : \mathbb{Z}\mathbf{GD}^{re} \to \mathbb{Z}$, on the free $\mathbb{Z}$-module $\mathbb{Z}\mathbf{GD}^{re}$ spanned by the set $\mathbf{GD}^{re}$.

Theorem 7.3 (Goussarov)**.** *For each integer-valued Vassiliev invariant v of classical knots of order $\leqslant n$ there exists a linear function $c_v : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}$ such that $v = I^*(c_v)\,|_{\mathbb{Z}\mathbf{GD}^{re}}$ and such that c_v is zero on each Gauss diagram with more than n arrows.*

We may think about the expression $I^*(c_v)$ in the equation (2) as the most general form of the Polyak—Viro formula for a Vassiliev invariant v. For our example of $v = c_2$, we get the following function c_v:

$c_v($  $) = 1,\quad c_v($  $) = -1,\quad c_v($  $) = -1\quad c_v($  $) = 1,$

and zero on the other Gauss diagrams.

There are two big parts in the proof of the Goussarov theorem: the construction of the function c_v and the proof of its vanishing on Gauss diagrams with more than n arrows. We explain the original idea for the first part, and refer to the paper [GPV] (and its explanation in [CDM]) for the second part.

7.3. **Construction of the map c_v.** One can easily check that the map I is an isomorphism with the inverse being

$$I^{-1}(D) = \sum_{D' \subseteq D} (-1)^{||D-D'||} D' \ ,$$

where $||D - D'||$ is the number of arrows of D not contained in D'.

Thus, in order to get $v = c_v \circ I$, we can obviously define

$$c_v := v \circ I^{-1} \ .$$

However, for this equation to make sense, we need to extend v from $\mathbb{Z}\mathbf{GD}^{re}$ to the whole of $\mathbb{Z}\mathbf{GD}$ since the image of I^{-1} contains all the subdiagrams of D and a subdiagram of a realizable diagram does not have to be realizable.

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