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EXTREMAL CONFIGURATIONS OF POLYGONAL LINKAGES

G.KHIMSHIASHVILI*, G.PANINA†, D.SIERSMA‡, A.ZHUKOVA✓

1. Introduction

The present paper is concerned with certain special configurations of polygonal linkages arising as solutions to various extremal problems on the moduli spaces of linkages in the spirit of [14], [16], [17]. Our approach is also naturally related to constrained optimization problems. Namely, we consider a geometrically significant target function on the moduli space of a polygonal linkage and aim at investigating its critical points. Some of the target functions we deal with can be interpreted as energies of the linkage endowed with some additional physically motivated structure, a typical example being the Coulomb energy of the equal charges placed at the vertices of linkage. However, we do not treat the Coulomb energy in the paper.

An important part of our study is concerned with the cyclic configurations of planar polygonal linkages considered as the critical points of oriented area [16]. Our approach yields, in particular, various enumerative and topological results about cyclic polygons (i.e., polygons which can be inscribed in a circle), which seems remarkable because cyclic polygons gained considerable attention in the last decade partially due to the results and conjectures of D.Robbins [20] (see also [3], [7], [21], [22]).

The aim of the present paper is to describe the state-of-the-art of these topics and present a number of essentially new results. We begin by describing the setting and recalling necessary results from [15], [16]. Throughout the paper we freely use basic results about moduli spaces of polygonal linkages, which can be found in [4], [12], and a few standard paradigms of differential topology and singularity theory for which we refer to [9], [1]. We also need several results from [15] which are reproduced below for the sake of the reader's convenience.

Specifically, we consider the oriented area as a function A on the planar moduli space $M_2(L)$ of a polygonal linkage L and embark on studying the critical points of A. As was revealed in [14] and proven in full generality in [16], for a generic polygonal linkage L with non-singular configuration space $M_2(L)$, the critical points of A on $M_2(L)$ are given by the cyclic configurations of linkage L. This fact is central for our exposition so we discuss its generalization

Key words and phrases. Mechanical linkage, polygonal linkage, robot arm, configuration space, moduli space, oriented area, oriented volume, Morse function, Morse index, cyclic polygon, Robbins formula.

applicable to arbitrary polygonal linkages and a version for open polygonal chains (or planar multiple penduli) (Theorem 2.2) obtained in [17].

Motivated by these results and conjectures of D.Robbins, in Sections 2, 3, and 9 we present a few remarks on the geometry of cyclic configurations which, to our mind, clarify a number of results of I.Sabitov, I.Pak and V.Varfolomeev concerned with conjectures of D.Robbins on computation of areas of cyclic polygons [20]. Our previous research witnesses that generically A is a Morse function on $M_2(L)$ and its Morse indices can be effectively calculated using the results of [19]. In the sequel we present an explicit formula for the Morse index which arises as a simplification and explication of [19]. We also present a version of this result in the case of spherical polygonal linkage.

An essentially new feature as compared with [16], [19], is that we also consider the case of an open polygonal chain or robot arm (Section 4). The main results include a formula for the Morse index of a critical configurations of polygon and robot arm in the plane (Sections 3 and 4), and a geometric description of the critical configurations in the case of oriented volume in 3D (Sections 6, 7, 8).

In addition to the aforementioned main topics we also present a number of by-product results. In particular, we give an algebraic proof of the main result of [16] (Theorem 11.4) and a self-contained proof of the stabilization theorem for moduli spaces of polygonal linkages (Theorem 10.3). When the report was completed, we learned that the stabilization theorem has already been proven in an even more general setting [18]. However, we keep our proof since in our case it comes as a simplification.

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2. Preliminaries and notation

An *n-linkage* is a sequence of positive numbers $l_1, \dots, l_n$. It should be interpreted as a collection of rigid bars of lengths l_i joined consecutively by revolving joints in a chain, either open or closed. Open linkages are sometimes called *robot arms*. We study the flexes of the both types of chain with allowed self-intersections. This is formalized in the following definitions.

Definition 2.1. (1) For an open linkage L, *a configuration* in the Euclidean space $\mathbb{R}^d$ is a sequence of points $R = (p_1, \dots, p_{n+1}),\ p_i \in \mathbb{R}^d$ with $l_i = |p_i, p_{i+1}|$ modulo the action of orientation preserving isometries.

We also call R *an open chain.*

The set $M_d^\circ(L)$ of all such configurations is *the moduli space, or the configuration space of the robot arm* L.

(2) For a closed polygonal linkage, we claim in addition that the last point coincides with the first point: a configuration of the linkage L in the Euclidean space $\mathbb{R}^d$ is a sequence of points $P = (p_1, \dots, p_n),\ p_i \in \mathbb{R}^d$ with $l_i = |p_i, p_{i+1}|$ for $i = 1, .., n-1$ and $l_n = |p_n, p_1|$. As above, the action of orientation preserving isometries is factored out.

We also call P *a closed chain* or a *polygon.*

The set $M_d(L)$ of all such configurations is *the moduli space, or the configuration space of the polygonal linkage L.*

In Sections 3–5 we deal with $d = 2$, that is, with planar configurations. It is convenient to use the following (equivalent) definition: the sequence of points $P = (p_1, \dots, p_n),\ p_i \in \mathbb{R}^2$ is called a *configuration of the linkage* L, if

(1) $l_i = |p_i, p_{i+1}|$, i.e. the lengths of the edges are fixed, and
(2) $p_1 = (0, 0)$, and $p_2 = (0, l_1)$. That is, a pair of consecutive vertices is pinned down.

A configuration carries a natural orientation which we indicate in figures by an arrow.

In the paper we treat the signed area function as the Morse function on the configuration space.

Definition 2.2. (1) The *signed area* of a polygon P with the vertices $p_i = (x_i, y_i)$ is defined by

$$2A(P) = (x_1y_2 - x_2y_1) + \ldots + (x_ny_1 - x_1y_n).$$

(2) The *signed area* of an open chain with the vertices $p_i = (x_i, y_i)$ is defined by

$$2A(P) = (x_1y_2 - x_2y_1) + \ldots + (x_ny_{n+1} - x_{n+1}y_n) + (x_{n+1}y_1 - x_1y_{n+1}).$$

In other words, we add one more edge that turns an open chain to a closed polygon and take the signed area of the polygon.

Definition 2.3. (1) A polygon P is called *cyclic* if all its vertices p_i lie on a circle.
(2) A robot arm R is called *diacyclic* if all its vertices p_i lie on a circle, and p_1p_{n+1} is the diameter of the circle.

Cyclic polygons and cyclic open chains arise in the framework of the paper as critical points of the signed area. They are also related to the critical points of the signed volume function (to be defined in Section 7).

Theorem 2.4. [16], [17].
(1) *Generically, a polygon P is a critical point of the signed area function A iff P is a cyclic configuration.*
(2) *Generically, an open robot arm R is a critical point of the signed area function A iff R is a diacyclic configuration.*

□

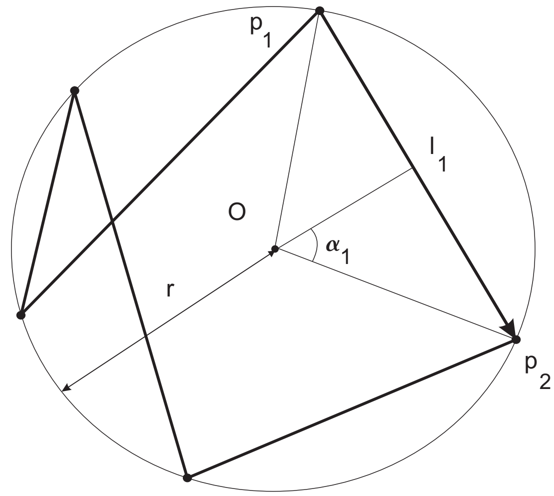


FIGURE 1. Basic notation for a pentagonal cyclic configuration with $E = (-1, -1, -1, 1, -1)$

The following notation (see Fig.1) is used throughout the paper for closed cyclic configurations:

r is the radius of the circumscribed circle.

α_i is the half of the angle between the vectors $\overrightarrow{Op_i}$ and $\overrightarrow{Op_{i+1}}$. The angle is defined to be positive, orientation is not involved.

ω_P is the winding number of P with respect to the center O.

$Hess_P(A)$ is the Hessian matrix of the function A at the point P.

$\mathcal{H}_P = \pm 1$ is the sign of the determinant of $Hess_P(A)$.

$\mu_P = \mu_P(A)$ is the Morse index of the function A in the point P. That is, $\mu_P(A)$ is the number of negative eigenvalues of the Hessian matrix $Hess_P(A)$.

A cyclic configuration is called *central* if one of its edges contains O.

For a non-central configuration, let ε_i be the orientation of the edge $p_i p_{i+1}$, that is,

$$\varepsilon_i = \begin{cases} 1, & \text{if the center } O \text{ lies to the left of } p_i p_{i+1}; \\ -1, & \text{if the center } O \text{ lies to the right of } p_i p_{i+1}. \end{cases}$$

$E(P) = (\varepsilon_1, \ldots, \varepsilon_n)$ is the string of orientations of all the edges.

$e(P)$ is the number of positive entries in $E(P)$.

Definition 2.5. For a non-central cyclic configuration P, we define

$$\delta P = \sum_{i=1}^{n} \varepsilon_i \tan \alpha_i;$$

$$d(P) = sign(\delta P).$$

Most of the above notation we will also use for open chains.

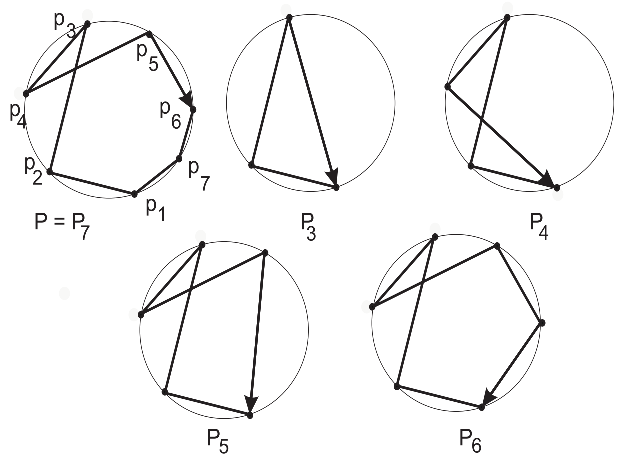


FIGURE 2. Subconfigurations $P_3, ..., P_n$

3. MORSE INDEX OF A PLANAR CYCLIC POLYGON

Lemma 3.1. [19] *Let two linkages L_1 and L_2 differ on a permutation, that is, $L_1 = \sigma L_2$ for $\sigma \in S_n$.*

Then the Morse critical points of the signed area A for L_1 and L_2 are in a natural bijection φ which preserves the Morse index, the value of A, the radius r, and the number of positively oriented edges e. □

Theorem 3.2. [19] *Let P be a cyclic configuration with a non-degenerate $Hess_P(A)$. Then*

$$\mathcal{H}_P = -d(P)(-1)^{e(P)} \quad \square$$

Theorem 3.3. [19] *Let $P = (p_1, ..., p_n)$ be a generic cyclic configuration. Introduce its subconfigurations $P_3, ..., P_n$ (see Fig. 2) as*

$$P_i = (p_1, ..., p_i), \quad i = 3, ..., n.$$

Put $\mathcal{H}_{P_3} = 1$ for the trigonal configuration P_3.

Then the Morse index $\mu_P(A)$ equals the number of the sign changes in the sequence

$$\mathcal{H}_{P_3}, \mathcal{H}_{P_4}, \mathcal{H}_{P_5}, ..., \mathcal{H}_{P_n},$$

where the values $\mathcal{H}_{P_i}$ can be computed via Theorem 3.2.

□

Lemma 3.4. *For a generic cyclic n-gonal configuration P with $e(P) = n$, we have $\mu_P(A) = n - 1 - 2\omega_P$.*

Proof. We will use induction by ω_P. Since all edges are positively oriented, ω_P is a positive number. If $\omega_P = 1$, then P is a convex configuration, which is known to be the maximum of A. Therefore, $\mu_P = n - 3$.

For the inductive step, assume that $\omega_P > 1$. Let k be the smallest number such that the open chain $(p_1, p_2...., p_{k+1})$ intersects itself (Fig. 3). The chain P splits into a homological sum of two closed chains:

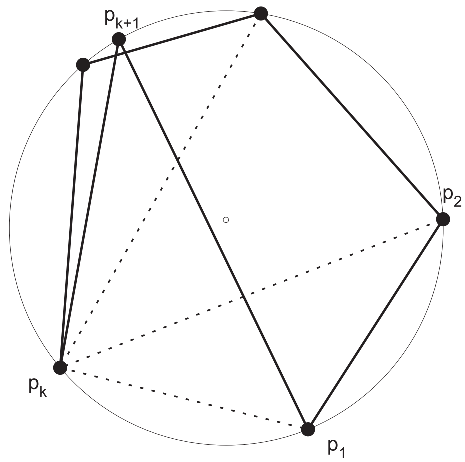


FIGURE 3. Configuration P'

$$P' = (p_1, p_2, ...p_{k+1}), \text{ and } P'' = (p_{k+1}, p_{k+2}, .., p_n, p_1)$$

(see Fig. 4).

The chains P' and P'' generate linkages L' and L''. Note that P'' is positively oriented and $\omega_{P"} = \omega_P - 1$, so $\mu_{P''}(A) = n - k - 2\omega_P + 2$.

The chain P' has exactly one negatively oriented edge (p_k, p_1), so $e(P') = k$. Besides, $\delta(P') > 0$ since $|p_k, p_1| < |p_k, p_{k+1}|$. By Theorem 3.2,

$$\mathcal{H}_{P'} = (-1)^k.$$

Theorem 3.2 applied to the polygon P' gives $\mu_{P'}(A) = k - 3$.

A neighborhood of the point $P' \times P''$ on the manifold $M_2(L') \times M_2(L'')$ admits a natural embedding on a neighborhood of P on the moduli space $M_2(L)$ as a codimension one submanifold. Indeed, given a configuration P_1' of the linkage L' and a configuration P_1" of the linkage L", we get a configuration P_1 by patching them by the edge (p_1, p_{k+1}). So, we have either

$$\mu_P = \mu_{P'} + \mu_{P"} = n - 1 - 2\omega_P,$$

or

$$\mu_{P'} + \mu_{P"} + 1 = n - 2\omega_P.$$

Besides, by Lemma 3.2, $\mathcal{H}_P = d(P)(-1)^{e(P)+1} = (-1)^{n-1}$. Therefore, $\mu_P(A) = n - 1 - 2\omega_P$. □

By symmetry reasons, we have the following lemma:

Lemma 3.5. *For a generic cyclic n-gonal configuration P with $e(P) = 0$, we have $\mu_P(A) = -2\omega_P + 2$.*

□

Theorem 3.6. *For a generic cyclic configuration P of a linkage L,*

$$\mu_P(A) = \begin{cases} e(P) - 1 - 2\omega_P & \textit{if } \delta(P) > 0; \\ e(P) - 2 - 2\omega_P & \textit{otherwise.} \end{cases}$$

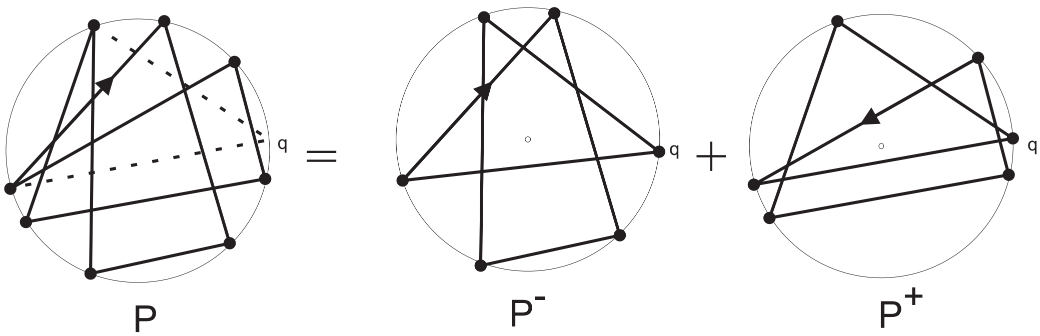


FIGURE 4. Splitting P into P^+ and P^-

Proof. By Lemmata 3.4 and 3.5, the statement is true for configurations with $e(P) = 0$ or $e(P) = n$.

Consider the configuration P that has both negative and positive entries in $E(P)$. We rearrange edges of linkage so that $\varepsilon_i > 0$ for $i \leq e(P)$ and $\varepsilon_i < 0$ for $i > e(P)$. By Lemma 3.1, this does not change μ_P. Add a new point q on circumscribed circle, such that vectors $(p_{e(P)+1}, q)$ and (q, p_1) are positively oriented with respect to O. Then P splits into the homological sum of a configuration $P^+ = (p_1, p_2, .., p_{e(P)+1}, q)$ of a linkage L^+ and a configuration $P^- = (p_{e(P)+1}, ...p_n, p_1, q)$ of a linkage L^-. The configuration P^+ fits the condition of Lemma 3.4, and P^- fits the condition of Lemma 3.5. Therefore,

$$\mu_{P^+} = e(P^+) - 1 - 2\omega(P^+) = e(P) + 1 - 2\omega(P^+)$$

and

$$\mu_{P^-} = -2\omega(P^-) - 2.$$

A neighborhood of P in the manifold $M_2(L)$ admits a natural embedding in a neighborhood of $P^+ \times P^-$ in the manifold $M_2(L^+) \times M_2(L^-)$ of codimension 1. Therefore, either

$$\mu_P = \mu_{P^+} + \mu_{P^-} = e(P) - 1 - 2\omega_P,$$

or

$$\mu_P = \mu_{P^+} + \mu_{P^-} - 1 = e(P) - 2 - 2\omega_P.$$

There are two possible cases:

(1) $\delta(P) > 0$. Then by Theorem 3.2, $\mu_P \equiv e(P) + 1 (mod\ 2)$. Therefore, $\mu_P = e(P) - 1 - 2\omega_P$.
(2) $\delta(P) < 0$. Then by Theorem 3.2, $\mu_P \equiv e(P) (mod\ 2)$. Therefore, $\mu_P = e(P) - 2 - 2\omega_P$.

$\square$

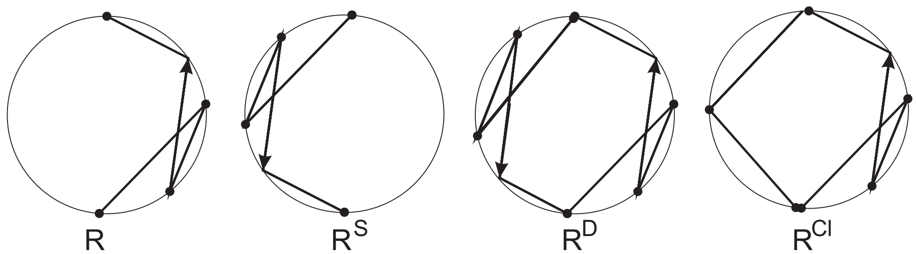


FIGURE 5. An open chain, its symmetry image, duplication and closure

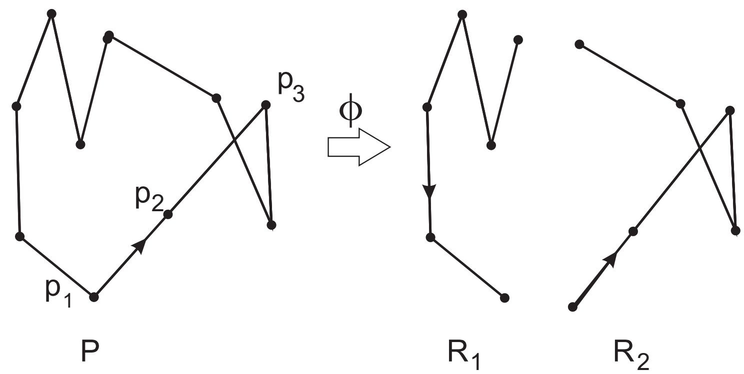


FIGURE 6. The mapping ϕ splits a closed chain into two open chains

4. MORSE INDEX OF A PLANAR ROBOT ARM

Let $R = (p_1, \dots, p_{n+1})$ be a critical configuration of a robot arm $(l_1, l_2, .., l_n)$. By Theorem 2.4, R is a diacyclic chain. Define its *closure* R^{Cl} as a closed cyclic polygon obtained from R by adding two positively oriented edges (see Fig. 5) and denote by ω_R the winding number of the polygon R^{Cl} with respect to the center O.

Theorem 4.1. *Let $L = (l_1, \dots, l_n)$ be a generic open linkage, and let R be one of its critical configuration. For the Morse index $\mu_R(A)$ of the signed area function A at the point R, we have*

$$\mu_R(A) = \begin{cases} e(R) - 2\omega_R + 1 & \textit{if } \delta(R) > 0, \\ e(R) - 2\omega_R & \textit{otherwise.} \end{cases}$$

Proof. Consider the manifold $M_2^\circ(L) \times M_2^\circ(L) = \{R_1 \times R_2 : \ R_1, R_2 \in M_2^\circ(L)\}$. Generically, the function $A(R_1 \times R_2) = A(R_1) + A(R_2)$ is a Morse function on $M_2^\circ(L) \times M_2^\circ(L)$.

Next, define the *duplication* of L as the closed linkage $L^D = (l_1, l_2, .., l_n, l_1, l_2, .., l_n)$.

Consider a mapping ϕ which splits a polygon $P \in L^D$ into two open chains, R_1 and R_2. The mapping ϕ embeds $M_2(L^D)$ as a codimension one submanifold of $M_2^\circ(L) \times M_2^\circ(L)$.

For a cyclic open chain R, define R^S as the symmetric image of R with respect to the center O. Define also $R^D \in M_2(L^D)$ as a closed polygon obtained by patching together R and R^S. The polygon R^D is cyclic. By Theorem 2.4, R^D is a critical point of the signed area.

On the one hand, the Morse index of its image $\phi(R^D) = R \times R^S$ on the manifold $M_2^\circ(L) \times M_2^\circ(L)$ equals $2\mu_R$. On the other hand, the Morse index of R^D on the manifold $M_2(L^D)$ is known by Theorem 3.6.

Since $M_2(L^D)$ embeds as a codimension one submanifold of $M_2^\circ(L) \times M_2^\circ(L)$, the Morse indices differ at most by one. More precisely, we have the following lemma:

Lemma 4.2. *Either* $\mu_{R^D} = 2\mu_R$, *or* $\mu_{R^D} = 2\mu_R - 1$. □

By Theorem 3.6,

$$\mu_{R^D} = \begin{cases} e(R^D) - 2\omega(R^D) - 1 & \text{if } \delta(R^D) > 0, \\ e(R^D) - 2\omega(R^D) - 2 & \text{otherwise.} \end{cases}$$

Clearly, we have $e(R^D) = 2e(R)$, $\delta(R^D) = 2\delta(R)$, and $\omega(R^D) = 2\omega(R) - 1$. This gives us

$$\mu_{R^D} = \begin{cases} 2e(R) - 4\omega(R)) + 1 & \text{if } \delta(R) > 0, \\ 2e(R) - 4\omega(R) & \text{otherwise.} \end{cases}$$

Assume that $\delta(R) > 0$. Then $\mu_{R^D} = 2e(R) - 4\omega(R)) + 1$ which is an odd number. The only possible choice in Lemma 4.2 is $2\mu_R = 2e(R) - 4\omega(R)) + 2$.

Analogously, if $\delta(R) < 0$ we conclude that $2\mu_R = 2e(R) - 4\omega(R))$. □

5. Critical points of spherical polygons

In the section we consider closed chains on the unit sphere $S^2 \subset \mathbb{R}^3$. We assume that the perimeter of a chain is less than 2π.

It is proven in [11] that the moduli space of a closed polygon on the sphere is diffeomorphic to the moduli space of a polygon with the same edge lengths in $\mathbb{R}^2$.

Theorem 5.1. *A spherical polygon P is a critical point of the signed area function A iff P is cyclic.*

Proof. The theorem is valid for 4-linkages by standard reasons. The proof for n-linkages repeats the inductive proof for Euclidean linkages, see [16]. □

Observe that all vertices of a cyclic spherical polygon P lie in some plane $a(P)$.

Theorem 5.2. *Generically, a critical point P of a spherical n-linkage is a Morse point of the spherical signed area function. Its Morse index equals the Morse index of the planar Euclidean polygon with the same vertices as the spherical polygon P.*

Proof. The proof for n-linkages repeats literally the proof of the same theorem for Euclidean linkages, see [19] and Section 3. □

6. 3-arm in $\mathbb{R}^3$

Before we treat in the next section open linkages with n arms in $\mathbb{R}^3$, we study here 3-arms in $\mathbb{R}^3$.
Let us fix some notation. The arm vectors are: $a = (1, 0, 0)$, b and c of length $|a|, |b|, |c|$.
A spatial arm is constructed as follows: we take the segments from O to the end points A, B, C of a, $a + b$, $a + b + c$. This yields a tetrahedron $OABC$.

Definition 6.1. We define the *signed volume* V of the 3-arm as the triple vector product:

$$V = [a, a + b, a + b + c] = [a, b, c].$$

We intend to study V on several parameter spaces:

- On $S^2 \times S^2$,
- On $S^1 \times S^2$, where we fix the vector b to lie in the xy plane,
- On the moduli space M_3^o (mod the $SO(3)$ action).

In each of these cases critical points may be different. We intend to compare the critical points and the Morse theory for the three cases.

6.1. On $S^2 \times S^2$.

Before starting we define some special positions of the 3-arm:

- Special diameter position or *tri-orthogonal*: The sphere with diameter OC contains also the points A and B. Equivalently : The vectors a,b, c are tri-orthogonal,
- *Degenerate*: The arm lies in a two-dimensional subspace,
- *Aligned*: The arm is contained in a line.

Proposition 6.2. *The signed area $V : S^2 \times S^2 \to \mathbb{R}$ has the following critical points:*

- *Tri-orthogonal arms (maximum, resp minimum). These are Bott-Morse critical points with transversal index* 3 *and critical value* $\pm|a||b||c|$.
- *Isolated points, corresponding to the aligned configurations. Here V has Morse index* 2 *and the critical value* 0.

Proof. We use coordinate systems on the spheres; we take partial derivatives with respect to all coordinates. We denote the partial derivatives of b by $\delta_1 b$ and $\delta_2 b$. Both are non-zero and orthogonal to b. We take partial derivatives of $V = [a, b, c]$ in the $(\delta_1 b, \delta_2 b)$ directions: $[a, \delta_1 b, c] = 0$ and $[a, \delta_2 b, c] = 0$.

We will shorten this to $[a, \dot{b}, c] = 0$ meaning that the equation holds for all vectors in the tangent space of b (which is orthogonal to b and spanned by $\delta_1 b$ and $\delta_2 b$). In this way we get:

$$[a, \dot{b}, c] = 0, \quad [a, b, \dot{c}] = 0.$$

For both equations we will consider two cases:

equation	ortho condition	parallel condition
$[a,\dot b,c]=0$	$a\times c\neq o$ equivalent to $b\perp a$ and $b\perp c$	$a\times c=o$ equivalent to $a\parallel c$
$[a,b,\dot c]=0$	$a\times b\neq o$ equivalent to $c\perp a$ and $c\perp b$	$a\times b=o$ equivalent to $a\parallel b$

The combination of the two ortho conditions gives the tri-orthogonal case of the proposition; combining the two parallel conditions is the aligned case. Combining one ortho condition with the other parallel condition gives a contradiction. □

Next we describe the type of the critical points. For the positively oriented tri-orthogonal case we get a maximum. Due to the remaining SO-action the singular set is an S^1, and its transversal Morse index is 3. The other orientation gives a minimum on S^1 with the transversal Morse index 0. The aligned configurations (4 cases) occur in isolated points. In all these cases we have index 2. We check the Bott-Morse formula:

$$\sum t^{\lambda(C)}P(C)-P(M)=(1+t)R(t)$$

where $R(t)$ must have non-negative coefficients. In our case we have

$$t^3(1+t)+(1+t)+(1+t)+4t^2-(t^4+2t^2+1)=t^3+2t^2+t=(1+t)(t^2+t),$$

so this is OK. □

6.2. **On $S^1\times S^2$.** After a rotation we can always assume that b lies in the xy-plane. We consider SO-action, that fixes this plane.

Proposition 6.3. *The signed volume $V: S^1\times S^2\to\mathbb{R}$ has the following critical points:*

- 4 *points, corresponding to tri-orthogonal arms (*2 *maxima, respectively* 2 *minima).*
  At these points V has critical value 0.
- *Two circles corresponding to degenerate configurations. where a and b are aligned and c is free to move in the xy-plane. At these points V has Bott-Morse critical points with transversal index* 1.

Proof. We use circle coordinate β on S^1 and spherical coordinates γ_1,γ_2 on S^2. The signed volume V is given by the determinant:

$$V=|a||b||c|\begin{vmatrix}1 & \cos\beta & \sin\gamma_1\cos\gamma_2\\ 0 & \sin\beta & \sin\gamma_1\sin\gamma_2\\ 0 & 0 & \cos\gamma_1\end{vmatrix}=|a||b||c|\sin\beta\cos\gamma_1 \tag{1}$$

Note that V does not depend on γ_2. Condition for critical points are:
$\partial V/\partial\beta=|a||b||c|\cos\beta\cos\gamma_1=0$,
$\partial V/\partial\gamma_1=-|a||b||c|\sin\beta\sin\gamma_1=0$,

$\partial V/\partial \gamma_2 = 0$.
There are two cases to consider:
(i) $\cos\beta = 0,\ \sin\gamma_1 = 0$.
The solution gives us four critical points; where we have the tri-orthogonal situation: two maxima (index 3):
$b = (0, 1, 0),\ c = (0, 0, 1)$, respectively $b = (0, -1, 0),\ c = (0, 0, -1)$, and two minima (index 0):
$b = (0, -1, 0),\ c = (0, 0, 1)$, respectively $b = (0, 1, 0),\ c = (0, 0, -1)$
(ii) $\sin\beta = 0,\ \cos\gamma_1 = 0$.
The solutions are $\beta = 0, \pi$ and $\gamma_1 = \pi/2$, which means that a and b are aligned, but c is allowed to move in the xy-plane. This gives us two S^1 (critical circles). Further analysis tells us, that the transversal Morse index is 1. At these points V has critical value 0. □

We check the result with Bott-Morse formula:
$2t^3 + 2 + 2t(1 + t) - (t^3 + t^2 + t + 1) = t^3 + t^2 + t + 1 = (t + 1)(t^2 + 1)$.

Note the difference between the situation on $S^2 \times S^2$ and on $S^1 \times S^2$.

6.3. **On the moduli space M_3^o .** This moduli space is homeomorphic to S^3. This is shown by [18]. We return to this later in this paper. An outline is as follows: first construct the non oriented moduli space and shows that this is a topological 3-ball. The sphere S^3 appears as a gluing of two such balls along their common boundary. This boundary consists of degenerate arms [those who are not the maximal dimension].

The function V will be studied separately on the two hemispheres, each of whom has exactly one Morse point. Near the common boundary one can show that V glues to a topologically regular function. In Section 9 we give details and prove the following:

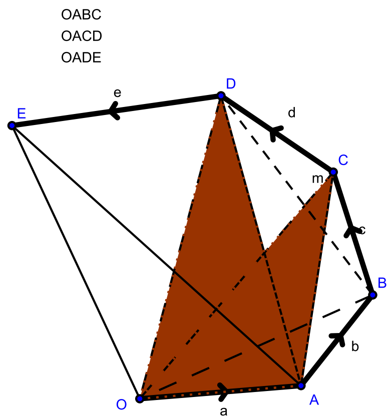
Theorem 6.4. *The oriented moduli space of 3-arms in $\mathbb{R}^3$ is a 3-sphere. V is an exact topological Morse function on this space with precisely two Morse critical points.* □

Note that the critical points with $V = 0$, which we got before in the cases with parametrization $S^2 \times S^2$ or $S^1 \times S^2$, disappear on the moduli space.

7. n-arms in $\mathbb{R}^3$

For a convex polyhedron there is a measure theoretic definition of its volume. On the one hand, the notion of volume is well-defined for non-convex (and even self-intersecting) polyhedra. On the other hand, it is unclear how to define the volume for a polygonal chain.

Therefore we decide to take one special situation as starting point for our definition of signed volume in case of a n-arm in $\mathbb{R}^3$. The following picture where all simplices contain $a = b_1$ illustrates the below definition.



Definition 7.1. Let an n-arm be given by the vectors $b_1, \cdots, b_n$. The vertices are $O, B_1, \cdots, B_n$. We fix $b_1 = a$ (as before). We denote $c_k = \sum_{i=1}^{k} b_i$ (the endpoint of this vector is B_k). The *signed volume function* is defined as

$$V = \sum_{k=1}^{n-1} [b_1, c_k, c_{k+1}],$$

which can be rewritten as:

$$V = [b_1, b_2, b_3] + [b_1, b_2 + b_3, b_4] + [b_1, b_2 + b_3 + b_4, b_5] + \cdots [b_1, b_2 + \cdots b_{n-1}, b_n].$$

N.B. Note that this signed volume is essentially the signed area of the projection onto the plane orthogonal to b_1.

Lemma 7.2. (Mirror lemma) *Let two arms differ on a permutation of the arms $2, \ldots, n$. Then there exits a bijection (by 'mirror-symmetry') between their "moduli spaces" which preserves the signed volume function. Consequently this bijection preserves critical points and their local (Morse) types.*

Proof. As in the planar case. $\square$

The condition for critical points are:

$$[b_1, \dot{b_2}, b_3] + [b_1, \dot{b_2}, b_4] + \cdots + [b_1, \dot{b_2}, b_n] = [b_1, \dot{b_2}, b_3 + \cdots + b_n] = 0.$$

$$[b_1, b_2, \dot{b_3}] + [b_1, \dot{b_3}, b_4] + \cdots [b_1, \dot{b_3}, b_n] = [b_1, b_2 - (b_4 + \cdots + b_n), \dot{b_3}] = 0.$$

The r^{th} -derivative gives the following:

$$[b_1, b_2 + \cdots + b_{r-1}, \dot{b_r}] + [b_1, \dot{b_r}, b_{r+1}] + \cdots + [b_1, \dot{b_r}, b_n] =$$

$$= [b_1, b_2 + \cdots + b_{r-1} - (b_{r+1} + \cdots + b_n), \dot{b_r}] = 0.$$

There are two cases for any $2 \le r \le n$: (which we call *ortho* and *parallel*)

- case O_r:

$$b_1 \times ((b_2 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n)) \neq 0.$$

Hence we have the following orthogonalities

$$b_r \perp b_1 \ \wedge \ \ b_r \perp (b_2 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n)$$

- case P_r:

$$b_1 \times ((b_2 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n)) = 0,$$

which means that $(b_2 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n) \in \mathbb{R}b_1$

Next we decompose vectors into their $\mathbb{R}b_1$-component and its orthogonal complement:

$$b_r = b'_r + b_r^{\perp}$$

Lemma 7.3. *For all $r = 2, \cdots, n$:*

$$b_r^{\perp} \perp (b_2^{\perp} + \cdots + b_{r-1}^{\perp}) - (b_{r+1}^{\perp} + \cdots + b_n^{\perp})$$

and also

$$(b_2^{\perp} + \cdots + b_{r-1}^{\perp}) \perp (b_r^{\perp} + \cdots b_n^{\perp}) \quad (*)$$

For any critical point of the signed volume function on n-arms in $\mathbb{R}^3$ one can consider the projection of the arm onto the hyperplane orthogonal to b_1.

Proposition 7.4. *The vertices of this planar $(n-1)$-arm $b_2^{\perp}, \ldots, b_n^{\perp}$ lie on a circle with diameter the interval $B_1B_n^{\perp}$ from the start point to the end point of this arm. This configuration corresponds to a critical point of such arms (but with fixed lengths) under the signed area function.* □

Note that in general we don't have fixed lengths of the projections and that projections can be "degenerate".

We next treat several cases of the spatial situations and after that state the general result in Theorem 7.4.

7.1. **Full ortho case:** O_r for all $r = 2, \ldots, n$
Now $b_r = b_r^{\perp}$. So we have:

Statement 1. The critical points of the signed volume function on n-arms in $\mathbb{R}^3$ are exactly those configurations, where all vertices (including the first O and the last B_r) are on a sphere with diameter OB_r; the first arm is perpendicular to the all other arms. Delete the first arm: the vertices of this planar $(n-1)$-arm lie on a circle with B_1B_r as the diameter. This configuration corresponds precisely to a critical point of such arms under the signed area function. Moreover,

$$V = |b_1| \cdot sA.$$

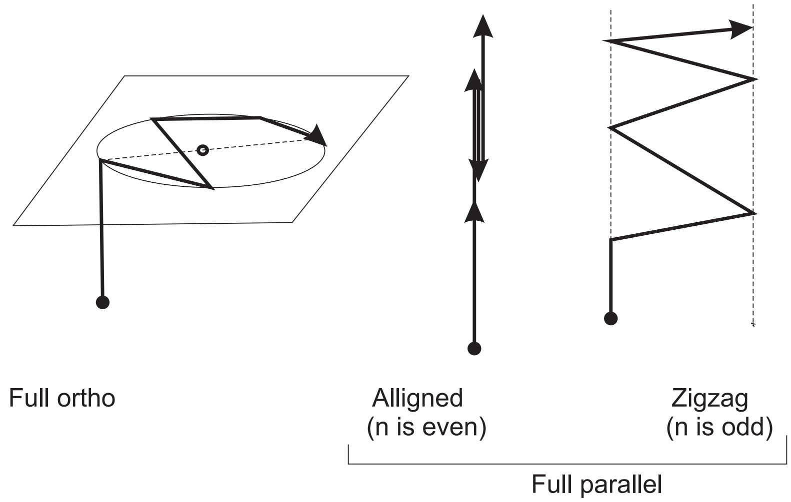


FIGURE 7

7.2. **Full parallel case:** P_r for all $r = 2, \ldots, n$.

If n is odd we find $b_r \in \mathbb{R}b_1$ $(r = 2, \ldots, n)$.
If n is even we find $b_r + b_{r+1} \in \mathbb{R}b_1$ $(r = 2, \ldots, n-1)$.

Statement 2. Critical points of V are aligned configurations if n is odd and 1-parameter families of zigzags if n is even. Zigzags are arms, which project all to the same interval (see Fig. 7, right).

Zigzags also contain the aligned configuration. In a zigzag the lengths of the projections can vary the from 0 to the minimum lengths of $b_2, \ldots, b_r$; in the zero case the zigzag contains an orthogonal arm.
Both full cases (see Fig. 7) have the property that solutions exists for all length vectors.

7.3. **General case:** $n-k$ **parallel conditions, and** $k-1$ **ortho conditions.** We can assume (due to the mirror lemma) that the last $n-k$ conditions are parallel. That is, we have

$$b_2 + \cdots + b_k + b_{k+1}^{\perp} + \cdots + b_{n-1}^{\perp} = 0$$

together with

$$b_{k+1} + b_{k+2} \in \mathbb{R}b_1, \cdots, b_{n-1} + b_n \in \mathbb{R}b_1.$$

So,

$$b_{k+1}^{\perp} + b_{k+2}^{\perp} = 0, \cdots, b_{n-1}^{\perp} + b_n^{\perp} = 0.$$

This has the following consequences:

- The $b_{k+1}^{\perp}, \cdots, b_n^{\perp}$ are diameters of the critical circle,

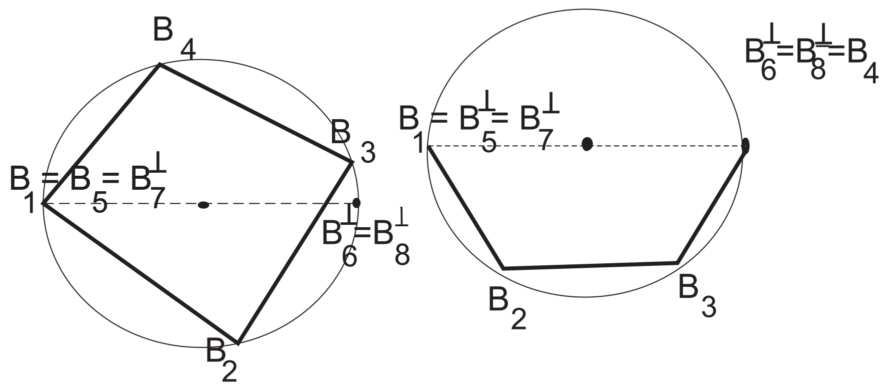


FIGURE 8. Projected vertices are on a circle.

- If $n-k$ is even, then $b_2+\cdots+b_k+b_{k+1}^{\perp}=0$.
  The $(k-1)$-arm $b_2,\cdots,b_k$ is an open planar diacyclic chain (*diameter condition*).
- If $n-k$ is odd, then $b_2+\cdots+b_k=0$. The k-1 arm $b_2,\cdots,b_{n-k-1}$ is a closed planar cyclic polygon (*closing condition*).

In both cases (odd and even) the projections of the vertices lie on a circle (see Fig. 8). There are only finite number of these circles possible. For a realization it is necessary that $|b_i|\geq R$ (radius of circle) if $k+1\leq i\leq n$.

The above discussion shows the following:

Theorem 7.5. *The critical points of V up to "mirror-symmetry" are as follows (see Fig. 9):*
There exits a division of the n-arm into a subarm b_1, a subarm $b_2,\ldots,b_k$ and a subarm $b_{k+1},\ldots,b_n$ such that:

- *b_1 is orthogonal to each of $b_2,\ldots,b_k$ (which lie in a plane $\mathbb{R}b_1^{\perp}$).*
- *The vertices of the arm $b_2,\ldots,b_k$ lie on a circle, satisfying*
  - *the closing condition if $n-k=$ odd,*
  - *the diameter condition if $n-k=$ even.*
- *The arm $b_{k+1},\ldots,b_n$ is a zigzag, which projects orthogonally to the diameter of the circle.* □

8. N-ARMS IN $\mathbb{R}^3$; PROJECTION ON PLANES

We consider in 3-space, a vector p, which is the direction of the orthogonal projection on a plane $\mathbb{R}p^{\perp}$

Let the n-arm be given by the vectors $b_1,\cdots,b_n$. The vertices are $O,B_1,\cdots,B_n$.

Define the signed Projected Area function as follows:

$$PA=[p,b_1,b_2]+[p,b_1+b_2,b_3]+[p,b_1+b_2+b_3,b_4]+$$
$$[p,b_1+b_2+b_3+b_4,b_5]+\cdots+[p,b_1+\cdots+b_{n-1},b_n].$$

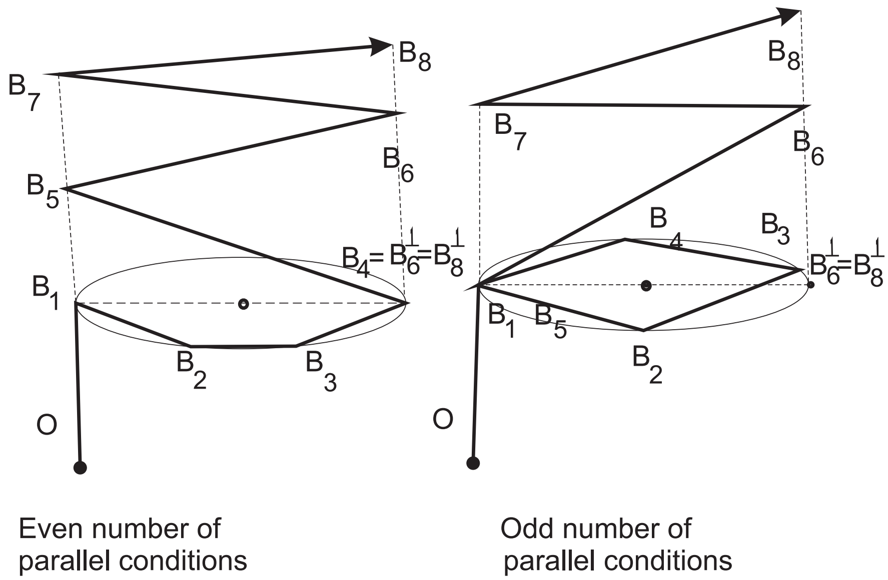


FIGURE 9. Solutions in the general case.

We fix first both the positions of p and b_1!.
We assume that $p \times b_1 \neq 0$.

Lemma 8.1. (Mirror lemma) *Let two open robot arms differ a permutation of the arms $2, \ldots, n$ then there exits a bijection (by 'mirror-symmetry') between their moduli spaces which preserves the signed projected area function. Consequently this bijection preserves critical points and their local (Morse) types.*

Proof. As in the planar case. □

The condition for critical points of PA is given by:

$$[p, b_1 + b_2 + \cdots + b_{r-1}, \dot{b_r}] + [p, \dot{b_r}, b_{r+1}] + \cdots + [p, \dot{b_r}, b_n] =$$
$$= [p, b_1 + \cdots + b_{r-1} - (b_{r+1} + \cdots + b_n), \dot{b_r}] = 0 \quad (r = 2, \cdots, n).$$

There are two cases for any $2 \leq r \leq n$: (which we call *ortho* and *parallel*)

- case O_r:

$$p \times ((b_1 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n)) \neq 0.$$

Then it follows that: we have the following orthogonalities

$$p \perp b_1 \;\wedge\; p \perp (b_1 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n)$$

- case P_r:

$$p \times ((b_1 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n)) = 0,$$

which means that $(b_1 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n) \in \mathbb{R}p$

Next we decompose vectors into their $\mathbb{R}p$-component and its orthogonal complement:

$$b_r = b_r' + b_r^\perp$$

Lemma 8.2. *For all $r = 2, \dots, n$:*

$$b_r^\perp \perp (b_1^\perp + \dots + b_{r-1}^\perp) - (b_{r+1}^\perp + \dots + b_n^\perp)$$

and also

$$(b_1^\perp + \dots + b_{r-1}^\perp) \perp (b_r^\perp + \dots b_n^\perp). \quad \square$$

Proposition 8.3. *The vertices of this planar n-arm $b_1^\perp, \dots, b_n^\perp$ lie on a circle with diameter the interval $OB_n^\perp$ from the start point to the end point of this arm. This configuration corresponds to a critical point of such arms (but with fixed lengths) under the signed area function.* $\square$

Note that in general we don't have fixed lengths of the projections and that projections can be "degenerate".

Theorem 8.4. ***(Projection with fixed p and b_1)*** *The critical points of PA up to "mirror-symmetry" are as follows:*
There exits a division of the n-arm into two subarms $b_1, \dots, b_k$ and $b_{k+1}, \dots, b_n$, such that:

- *The vertices of the arm $b_1^\perp, b_2, \dots, b_k$ lie on a circle in the projection plane, satisfying*
  - *the closing condition if $n - k = odd$,*
  - *the diameter condition if $n - k = even$.*
- *The arm $b_{k+1}, \dots, b_n$ is a zigzag, which projects orthogonally to the diameter of the circle.*

Proof. As in the signed volume case, see Theorem 7.3. $\square$

Remark 1. The special case that p is orthogonal to b_1 is included. In this case we obviously have $b_1^\perp = b_1$.
If p is parallel to b_1 we are in the case of signed volume studied before.

Remark 2. If we fix only p and not b_1 the study of the signed projected area of the n-arm $b_1, \dots, b_n$ is equivalent to that of the signed volume of the $(n+1)$-arm $p, b_1, \dots, b_n$. We state this:

Theorem 8.5. ***(General projection on plane)*** *The critical points of PA up to "mirror-symmetry" are as follows:*
There exits a division of the n-arm into two subarms $b_1, \dots, b_k$ and $b_{k+1}, \dots, b_n$, such that:

- *The vertices of the arm $b_1, b_2, \dots, b_k$ lie on a circle in the projection plane, satisfying*
  - *the closing condition if $n - k = odd$,*
  - *the diameter condition if $n - k = even$.*
- *The arm $b_{k+1}, \dots, b_n$ is a zigzag, which projects orthogonally to the diameter of the circle.* $\square$

9. Gram matrices and moduli space

One way to study the moduli space of n-arms in $\mathbb{R}^n$ is to use the Gram matrix. This has an advantage that there is a direct relation with the volume.

Given a set of vectors, the Gram matrix G is the matrix of all possible inner products. Let B be the matrix whose columns are the arm vectors $b_1, \dots, b_n$. Then the Gram matrix is $G = B^t B$. Its determinant is the square of the volume of the simplex spanned by these vectors:

$$\det G = (V)^2.$$

The Gram matrix is always a positive semi definite symmetric matrix. If G is positive definite it determines the vectors up to isometry. But not every positive semi definite matrix S (with $\det S = 0$) is a Gram matrix.

In case of n-arm in $\mathbb{R}^n$ the inner products $(b_i.b_i)$ are the fixed numbers b_i^2. The other entries of the Gram matrix we consider as variables x_{ij}. Its determinant is:

$$\begin{vmatrix} b_1^2 & x_{12} & x_{13} & & & x_{1n} \\ x_{12} & b_2^2 & x_{23} & & & x_{2n} \\ x_{13} & x_{23} & b_3^2 & & & x_{3n} \\ & & & & x_{ij} & \\ & & & x_{ij} & & \\ & & & & & \\ x_{1n} & & & & & b_n^2 \end{vmatrix}$$

For a given n-arm, Gram matrix is contained in a subspace of dimension $\frac{n(n-1)}{2}$.

Remark. Note that the equivalence is only up to isometry and not with respect to orientation. On the set GRAM of all Gram matrices we will consider $|V|$. In order to treat the oriented version we have to take two copies of GRAM and to glue it on the common boundary. The set *GRAM* is contained in a product of intervals $-b_i b_j \leq x_{ij} \leq b_i b_j$.

In [18] diagonals are used as coordinates of the moduli space. GRAM is related to that description by the cosine rule:

$$d_{ij} = b_i^2 + b_j^2 - 2x_{ij}.$$

Note that G is differentiable on the entire space $\mathbb{R}^{n(n-1)/2}$. In turn, $|V|$ is defined on GRAM, but is only differentiable on the interior $\{|V| > 0\}$. What happens on the boundary?

We consider the 3 dimensional case:

$$\det G = \begin{vmatrix} a^2 & z & y \\ z & b^2 & x \\ y & x & c^2 \end{vmatrix} = 2xyz - a^2x^2 - b^2y^2 - c^2z^2 + a^2b^2c^2 = 0$$

The compact component in the picture is contained in the cube. Its interior realizes the non-planar configurations.

The critical points of $\det G$ are given by the conditions
$\partial \det G/\partial x = 2(yz - a^2x) = 0$,
$\partial \det G/\partial y = 2(xz - b^2y) = 0$,

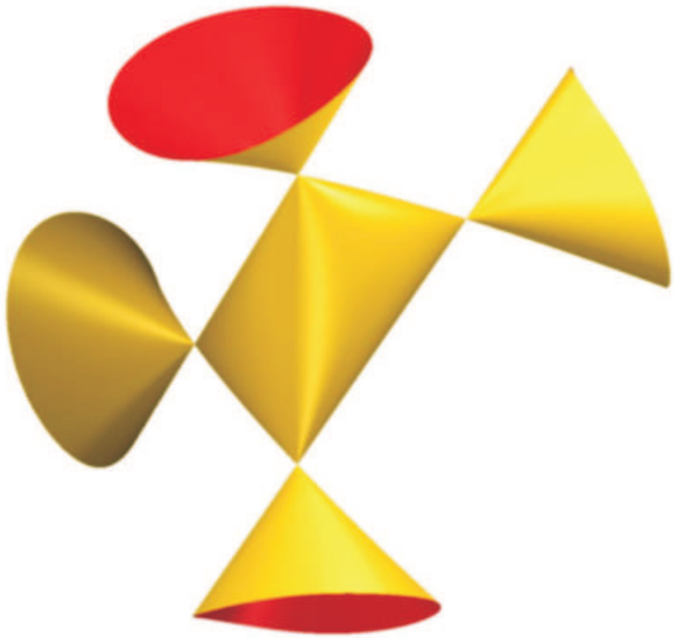


FIGURE 10. Zero locus of the determinant of G. The compact region corresponds to the set of Gram matrices. (The figure is produced by SINGULAR software.)

$\partial \det G / \partial z = 2(xy - c^2 z) = 0.$

We find the following critical points of $\det G$:

- $(x, y, z) = (0, 0, 0)$: maximum $a^2 b^2 c^2$ (index 3)
- $(x, y, z) = (bc, ac, ab), (-bc, ac, -ab), (-bc, -ac, ab)$ or $(bc, -ac, -ab)$

  The critical value is equal to 0. What are the types of these 4 critical points? We compute the Hessian matrix and its determinant:

$$\det H = \begin{vmatrix} -a^2 & z & y \\ z & -b^2 & x \\ y & x & -c^2 \end{vmatrix}$$

  Note that $\det H(x, y, z) = -\det G(-x, -y, -z)$.

  Each of our 4 critical points is non-degenerate; since $\det H \neq 0$. The Morse index is 2. Note also that they are related to aligned situations.

The local behavior of the level surfaces near the critical level can be studied with the local formula:

$$\det G = -\zeta_1^2 - \zeta_2^2 + \zeta_3^2.$$

Its zero level is a quadratic cone. We restrict ourselves by points inside the cube. Near the singular points we have a homeomorphism:

$$(\det G)^{-1}[0, \epsilon] = (\det G)^{-1}[\epsilon] \times [0, \epsilon]$$

For the non-critical points this is is guaranteed by the regular interval theorem; so the product structure is global. We have shown the following:

Proposition 9.1. *(Fig. 10) The closure of the component of $G^{-1}(0, a^2b^2c^2)$, which contains $(0,0,0)$ is a topological 3-ball. Its boundary is a topological 2-sphere (differentiable outside 4 critical points).* □

This component is exactly the set GRAM. Moreover, in this 3-dimensional case GRAM is equivalent (up to isometry) to the set of triples of arm vectors.

Since we have $\det G = |V|^2$, the both functions have the same level curves on the domain of common definition. So the above proposition tell us that the (unoriented) moduli space of 3-arm is a topological disc. By gluing two copies of GRAM along the common boundary we get:

Theorem 9.2. *The oriented moduli space of 3-arms in $\mathbb{R}^3$ is a 3-sphere. V is an exact topological Morse function on this space with precisely two Morse critical points.* □

10. Stabilization of moduli spaces

In this section we study moduli spaces of one and the same linkage and different $\mathbb{R}^d$. We prove that as d grows, the moduli space $M_d(L)$ stabilizes to a ball.

Definition 10.1. A configuration $P \in M_{n-1}(L)$ of an n-linkage is called *flat* if it fits in an $(n-2)$-plane.

Lemma 10.2. *The configuration spaces $M_d(L)$ and $M_n(L)$ are homeomorphic for all $d \geq n$.*

Proof. There is a natural homeomorphism $M_d(L) \to M_n(L)$ which sends a configuration P to its isometric image. □

Theorem 10.3. *Let $L = (l_1, ...l_n)$ be an closed n-linkage.*

(1) *The configuration space $M_n(L)$ is homeomorphic to the $\frac{n(n-3)}{2}$-dimensional ball B.*
(2) *A configuration $P \in M_n(L)$ is flat iff $P \in \partial M_n(L)$.*
(3) *The configuration space $M_{n-1}(L)$ is homeomorphic to the $\frac{n(n-3)}{2}$-dimensional sphere S.*

Proof. We first prove (1) and (2) using induction by n. For $n = 3$ the moduli space $M_3(l_1, l_2, l_3)$ is a one point set, whereas $M_2(l_1, l_2, l_3)$ consists of two points (these are two triangles with different orientation).

Assume that the theorem is proven for $n-1$. Let L be an n-linkage. Assume that l_1, l_2 are the shortest edges in L. Assume also that $l_1 \leq l_2$. Given a configuration $P \in M_n(L)$ denote by d the length of the diagonal $|p_1, p_3|$ (see Fig. 11). Denote by P' the truncated polygon $(p_1, p_3, .., p_n)$. Denote also by $L'(d)$ the induced linkage $(d, l_3, .., l_n)$.

We have a truncating mapping which sends a configuration $P = (p_1, p_2, p_3...p_n) \in M_n(L)$ with a diagonal d to the pair (P', d). Identifying by inductive assumption the moduli space $M_n(L'(d))$ with the $\frac{(n-1)(n-4)}{2}$-dimensional ball B, we get a mapping $\varphi : M_n(L) \to B \times [l_2 - l_1; l_1 + l_2]$.

Let us compute the preimage of a point $P' \in B \times [l_1 - l_2; l_1 + l_2]$. Two cases should be treated:

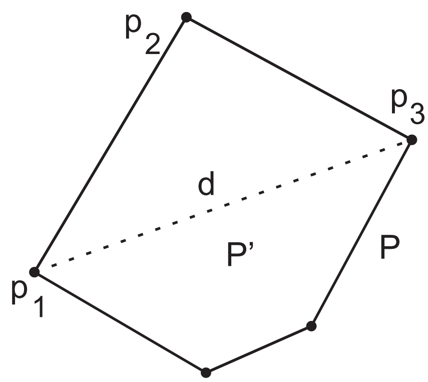


FIGURE 11. Truncation of a polygon P

(1) P' is an inner point, that is, $P' \in Int(B \times [l_1 - l_2; l_1 + l_2])$. By inductive assumption, the configuration P' is not flat. The vertex p_2 of a configuration $P \in \varphi^{-1}(P')$ lies on a closed hemisphere of dimension $n-3$ which topologically is a ball. Besides, P is flat if and only if p_2 lies on the boundary of the hemisphere.

(2) P' is a boundary point, that is, $P' \in \partial(B \times [l_2 - l_1; l_1 + l_2])$.
There are two subcases:
   (a) Either $P' \in B \times (l_1 - l_2)$ or $P' \in B \times (l_2 + l_1)$. Then for any configuration $P \in \varphi^{-1}(P')$ the points p_1, p_2 and p_3 are collinear. Therefore, $\varphi^{-1}(P')$ is a one-point set consisting of one flat configuration.
   (b) $P' \in \partial B \times [l_1 - l_2, l_1 + l_2]$. The points of $\varphi^{-1}(P')$ are parameterized by a hemisphere. Topologically, it is a disc, so (1) is proven. By inductive assumption, P' is flat. Therefore, any $P \in \varphi^{-1}(P')$ is flat as well.

Now prove (3).

There is a natural mapping $\pi : M_{n-1}(L) \to M_n(L)$. Namely, given a configuration in $\mathbb{R}^{n-1}$, we embed it in $\mathbb{R}^n$ using some embedding $\mathbb{R}^{n-1} \to \mathbb{R}^n$. For two different configurations $P, P' \in M_{n-1}(L)$, we have $\pi(P) = \pi(P')$ iff they differ on an isometry that does not preserve orientation in $\mathbb{R}^{n-1}$. Consequently, the preimage of a non-flat configuration from $M_n(L)$ consists of two configurations. Besides, a flat configuration has. a unique preimage. Together with (2) this directly gives the desired statement. ☐

Theorem 10.4. *Let $L = (l_1, ...l_n)$ be an open n-linkage.*

(1) *For all $d > n$ the moduli space $M_d^\circ(L)$ is homeomorphic to the $\frac{n(n-1)}{2}$-dimensional ball B.*
(2) *The moduli space $M_n^\circ(L)$ is homeomorphic to the $\frac{n(n-1)}{2}$-dimensional sphere S.*

Proof. (1) Consider the function $f : M_d^\circ \to \mathbb{R}$ defined as $f(P) = |p_0, p_n|$. The values of f vary from some l_0 to $l_1 + l_2 + ... + l_n$. By Theorem 10.3, for each value l such that $l_0 < l < l_1 + l_2 + .. + l_n$, the preimage $f^{-1}(l)$ is a ball. For $l = l_1 + l_2 + .. + l_n$, the preimage $f^{-1}(l)$ is a point. Analogously, for $l = l_0$, the preimage $f^{-1}(l)$ is either a ball or a point.

That is, M_d^0 is stratified into a family of balls (see Fig. 12).

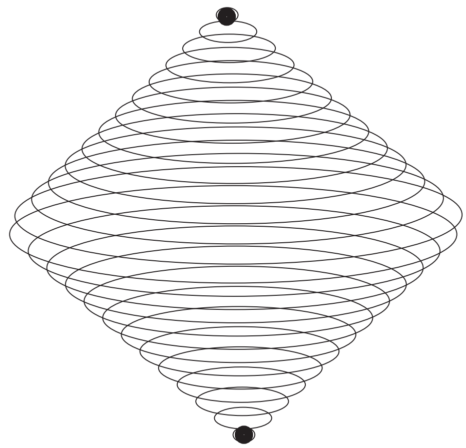


FIGURE 12. $M_d^\circ(L)$ stratifies into family of balls

(2) The same arguments give us a stratification of $M_n^\circ(L)$ into a family of spheres for $l \in (l_0, l_1 + l_2 + .. + l_n)$, a point for $l = l_1 + l_2 + .. + l_n$, and a ball for $l = l_0$.

□

11. Algebraic detection of critical points of a planar polygon

Let $L = (l_0, l_1, l_2, ..., l_{n+1})$ be a closed planar $(n+2)$-linkage. As is already mentioned, it was proven in [16], that generically, critical points of the signed area function are cyclic configurations. In the section we show that for each linkage (not only for a generic one) critical points of the signed area function are cyclic and aligned configurations. The proof goes as follows: we characterize the critical points as common roots of a polynomial ideal $I(L)$. Next, we characterize aligned and cyclic configuration as common roots of another ideal $J(L)$. Finally, we show that the two ideals coincide.

We assume that two vertices of its configuration P are pinned down. That is, $p_{n+1} = (x_{n+1}, y_{n+1})$ and $p_0 = (x_0, y_0)$, where x_{n+1}, y_{n+1}, x_0, y_0 are some fixed real numbers.

Denote by $g_i \in \mathbb{C}[x_1, x_2, y_1, y_2, ..., x_n, y_n]$ the polynomial that fixes the length l_i, that is,

$$g_i(x_1, x_2, y_1, y_2, ..., x_n, y_n) = (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 - l_i^2.$$

The set of real common zeroes of the polynomials $\{g_i\}_{i=1}^{n+1}$ has a natural identification with the moduli space $M_2(L)$. That is, each configuration is encoded by the list of the coordinates $(x_1, x_2, y_1, y_2, \ldots, x_n, y_n)$.

Denote by $A(x_1, x_2, y_1, y_2, \ldots, x_n, y_n) \in \mathbb{C}[x_1, x_2, y_1, y_2, \ldots, x_n, y_n]$ the signed area polynomial (see Definition 2.2).

Let G_n be a $(n+2) \times 2n$ matrix whose rows are the following gradient vectors:

$$G_n = \begin{pmatrix} \nabla g_1 \\ \nabla g_2 \\ \dots \\ \nabla g_{n+1} \\ \nabla A \end{pmatrix} = \begin{pmatrix} x_1 - x_0 & y_1 - y_0 & 0 & 0 & \dots & 0 \\ x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & \dots & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_2 - y_0 & x_0 - x_2 & y_3 - y_1 & x_1 - x_3 & \dots & \dots \end{pmatrix}.$$

Example 11.1. *For a 4-linkage, we have*

$$G_2 = \begin{pmatrix} x_1 - x_0 & y_1 - y_0 & 0 & 0 \\ x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 \\ y_2 - y_0 & x_0 - x_2 & y_3 - y_1 & x_1 - x_3 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1 - x_0 & y_1 - y_0 & 0 & 0 \\ x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 \\ 0 & 0 & x_2 & y_2 \\ y_2 - y_0 & x_0 - x_2 & -y_1 & x_1 \end{pmatrix}.$$

Denote by m the determinant of G_2:

$$m(x_1, x_2, y_1, y_2) = det\ G_2 \in \mathbb{C}[x_1, x_2, y_1, y_2].$$

For $i = 1, \dots, n-1$ we define minor matrices Min_i of the matrix G_n by the following rule. We eliminate from the matrix G_n its $2(i-1)$ first columns, $(i-1)$ first rows, $2(n-i-1)$ last columns and $(n-i-1)$ last rows (the very last row ∇A is not taken into account), see Fig. 13.

Define polynomials

$$m_i(x_1, x_2, y_1, y_2, \dots, x_n, y_n) = det\ Min_i \in \mathbb{C}[x_1, x_2, y_1, y_2, \dots, x_n, y_n].$$

Lemma 11.2. *Each m_i minor is a polynomial that equals the determinant of G_2 for a quadrilateral linkage. Namely,*

$$m_i = m(x_i, y_i, x_{i+1}, y_{i+1}). \quad \square$$

Lemma 11.3.

$$rank\ G_n(x_1, x_2, y_1, y_2, \dots, x_n, y_n) \leq n+1$$

if and only if

$$\forall\ i\ m_i(x_1, x_2, y_1, y_2, \dots, x_n, y_n) = 0. \quad \square$$

Denote by $I(L) \subset \mathbb{C}[x_1, x_2, y_1, y_2, \dots, x_n, y_n]$ the ideal generated by polynomials $g_1, ..., g_{n+1}, m_1, ...m_{n-1}$:

$$I(L) = \langle g_1, ..., g_{n+1}, m_1, ...m_{n-1} \rangle.$$

By the above lemma, the real common roots of the ideal correspond to critical configurations of L. This is automatically the case for non-degenerated configuration space. In the singular case the critical points are just by definition those that correspond to the common roots of I.

For $i = 1, \dots, n-1$ denote the matrix

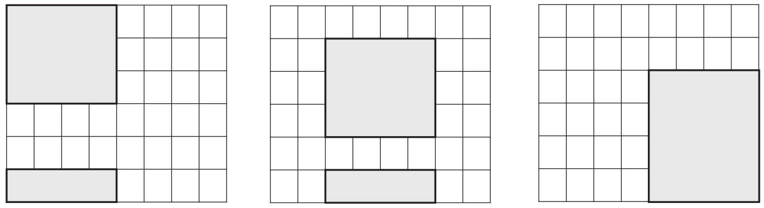


FIGURE 13. Minors Min_1, Min_2, and Min_3

$$C_i = \begin{pmatrix} x_i^2 + y_i^2 & x_i & y_i & 1 \\ x_{i+1}^2 + y_{i+1}^2 & x_{i+1} & y_{i+1} & 1 \\ x_{i+2}^2 + y_{i+2}^2 & x_{i+2} & y_{i+2} & 1 \\ x_{i+3}^2 + y_{i+3}^2 & x_{i+3} & y_{i+3} & 1 \end{pmatrix}$$

Denote also the polynomials

$$c_i = det\ C_i \in \mathbb{C}[x_1, x_2, y_1, y_2, \dots, x_n, y_n].$$

Denote by $J(L) \subset \mathbb{C}[x_1, x_2, y_1, y_2, \dots, x_n, y_n]$ the ideal

$$J(L) = \langle g_1, \dots, g_{n+1}, c_1, \dots c_{n-1} \rangle.$$

The real common roots of the ideal J_n correspond to aligned or cyclic configurations of L. Indeed, $c_i = 0$ is equivalent to the condition that the points $p_i, p_{i+1}, p_{i+2}, p_{i+3}$ are either aligned or cyclic.

Theorem 11.4.

$$I(L) = J(L).$$

Proof. For $n = 2$, that is, for a quadrilateral linkage, there is just one polynomial c_1 and just one polynomial m_1. An easy check shows that they coincide up to a constant multiplier. By Lemma 11.2, we also have $c_i = m_i$ up to a constant. ☐

REFERENCES

[1] Arnold V., Varchenko A., Gusein-Zade S., *Singularities of differentiable mappings* (Russian). Nauka, Moscow, 2005.

[2] Cerf J., *La stratification naturelle des espaces de fonctions differentiables reelles et le theoreme de la pseudo-isotopie.* Inst. Hautes Etudes Sci. Publ. Math., 1970, 39, 169, 5-173.

[3] Connelly R., *Comments on generalized Heron polynomials and Robbins' conjectures.* Discr. Math. 309(2009), 4192-4196.

[4] Connelly R., Demaine E., *Geometry and topology of plygonal linkages, Handbook of discrete and computational geometry, 2nd ed.* CRC Press, Boca Raton, 2004, 197-218.

[5] Elerdashvili E., Jibladze M., Khimshiashvili G., *Cyclic configurations of pentagon linkages.* Bull. Georgian Acad. Sci., 2008, 4, 4, 13-16.

[6] Farber M., Schütz D., *Homology of planar polygon spaces.* Geom. Dedicata, 2007, 125, 18, 75-92.

[7] Fedorchuk M., Pak I., *Rigidity and polynomial invariants of convex polytopes.* Duke Math. J. 129(2005), No.2, 371-404.

[8] Gibson C., Newstead P., O*n the geometry of the planar 4-bar mechanism*. Acta Appl. Math., 1986, 7, 23, 113-135.
[9] Hirsch M., *Differential topology*. Springer, Berlin-Heidelberg-New York, 1976.
[10] Kapovich M., Millson J., *The symplectic geometry of polygons in Euclidean space*. J. Differ. Geom. 44, No.3, 479-513 (1996). ISSN 0022-040X
[11] Kapovich M., Millson J., *On the moduli space of polygons in the Euclidean plane*. J. Differential Geom. Volume 42, Number 1 (1995), 133-164.
[12] Kapovich M., Millson J., *Universality theorems for configuration spaces of planar linkages*. Topology 41(2002), 1051-1107.
[13] Klyachko A., *Spatial polygons and stable configurations of points in the projective line.* Tikhomirov, Alexander (ed.) et al., Algebraic geometry and its applications. Proceedings of the 8th algebraic geometry conference, Yaroslavl', Russia, August 10-14, 1992. Braunschweig: Vieweg. Aspects Math. E 25, 67-84 (1994).
[14] Khimshiashvili G., *Cyclic polygons as critical points*. Proc. I.Vekua Inst. Appl. Math. 58(2008), 74-83.
[15] Khimshiashvili G., *Configuration spaces and signature formulae*. J. Math. Sci. 160(2009), No.6, 727-736.
[16] Khimshiashvili G., Panina G., *Cyclic polygons are critical points of area*. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 2008, 360, 8, 238–245.
[17] Khimshiashvili G., Siersma D., Preprint ICTP, IC/2009/047. 11 p.
[18] Mermoud O., Steiner M., *Configuration spaces of weighted graphs in high dimensional Euclidean spaces.* Beitr. Algebra Geom. 43, No.1, 27-31 (2002).
[19] Panina G., Zhukova A.,*Morse index of a cyclic polygon*,Cent. Eur. J. Math., 9(2) (2011), 364-377.
[20] Robbins D., *Areas of polygons inscribed in a circle*. Discrete Comput. Geom., 1994, 12, 14, 223-236.
[21] Sabitov I., *The volume as a metric invariant of polyhedra*. Discr. Comp. Geom. 20(1998), 405-425.
[22] Varfolomeev V., *Inscribed polygons and Heron polynomials*(Russian). Math. Sb. 194(2003), 3-24.
[23] Zvonkine D., *Configuration spaces of hinge constructions*. Russian J. of Math. Phys., 1997, 5, 20, 247-266.

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