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Steinberg Groups for Jordan Pairs

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Steinberg groups for Jordan pairs

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Abstract. We introduce categories of groups with commutator relations with respect to root groups and Weyl elements, permuting the root groups. This allows us to view the classical Steinberg groups, for example the Steinberg group of a ring, as an initial object in an appropriate category.

The general framework is then specialized to groups associated to Jordan pairs, first for arbitrary Jordan pairs and then later for Jordan pairs with Peirce gradings or more general gradings by root systems, for example Jordan pairs covered by grids.

Introduction

Background. Throughout its history, the theory of Jordan algebras and Jordan pairs was heavily influenced by its connection to Lie algebras and groups. This started with the work of Chevalley-Schafer [**11**], it continued with the fundamental contributions by Jacobson [**19**, **20**, **21**], Kantor [**27**, **28**], Koecher [**29**, **30**, **31**, **32**], Springer [**53**], Tits [**57**, **58**], and was later taken up by others. We explore this relation further by studying categories of groups associated to Jordan pairs.

The model for the groups considered here comes from the theory of semisimple algebraic groups over an algebraically closed field. Let G be such a group and R its root system (with respect to a maximal torus). Then G is generated by the family of root groups $(U_\alpha)_{\alpha\in R}$, isomorphic to the additive group of the base field. Moreover, G has commutator relations with respect to these subgroups in the sense defined below. Generalizations in different directions are possible. The base field can be arbitrary (Chevalley groups [**55**]) or can be replaced by a commutative ring (classical groups over rings [**17**], group schemes over rings [**13**, **12**]). Even more general coordinate systems appear in Faulkner's work on groups with Steinberg relations [**14**] and in the theory of Moufang polygons [**61**]. For $R = \mathrm{A}_1$, the most general coordinate system considered up to now is a Jordan pair, and the corresponding groups are the groups over V, studied in Chapter II.

One can also replace the finite root system R by a more general root system, for example the set of real roots of a Kac-Moody algebra. In fact, in [**59**] (see also [**49**]) Tits constructs a Steinberg group functor $\mathfrak{St}$ for every Kac-Moody algebra with root system R with the property that for any ring A, the group $\mathfrak{St}(A)$ has R-commutator relations.

We now give a detailed description of the contents.

Chapter I: Groups with commutator relations. These are groups with a distinguished family of subgroups indexed by a root system R of some sort. In the examples mentioned before, R is a finite root system or the set of real roots of a Kac-Moody algebra. However, it turns out that a good deal of the theory can be developed under minimal assumptions on R. In essence, it suffices to have a subset R of some vector space over a field of characteristic zero.

The key notion is that of a nilpotent pair in R. This was developed in [**42**] and is reviewed in Section 1. For non-zero elements $\alpha, \beta \in R$ and putting $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ we define

$$\big(\alpha,\beta\big) = R \cap (\mathbb{N}_+\alpha + \mathbb{N}_+\beta), \quad \big[\alpha,\beta\big] = \big(\alpha,\beta\big) \cup (R\cap \mathbb{N}_+\alpha) \cup (R \cap \mathbb{N}_+\beta),$$

called the *open and closed root interval from α to β*, respectively. We say (α, β) is a *nilpotent pair* if $\big[\alpha,\beta\big]$ is finite and does not contain zero.

The central concept of Chapter I is introduced in Section 2. Let G be a group and let $(U_\alpha)_{\alpha\in R}$ be a family of subgroups of G. We say that G has *R-commutator relations with root groups U_α* if $U_0=\{1\}$ and, for all $\alpha,\beta\in R$,

$$\begin{aligned} U_\alpha &\subset U_\beta && \textit{if } \alpha\in\mathbb{N}\beta,\\ \big(U_\alpha,U_\beta\big) &\subset U_{(\alpha,\beta)} && \textit{if } (\alpha,\beta) \textit{ is a nilpotent pair.}\end{aligned}$$

Here $\big(U_\alpha,U_\beta\big)$ denotes the subgroup generated by all group commutators $\big(a,b\big)=aba^{-1}b^{-1}$ for $a\in U_\alpha$, $b\in U_\beta$, and for any subset A of R, U_A is the subgroup generated by all U_γ, $\gamma\in A$.

Groups with R-commutator relations form a category $\mathbf{gc}_R$, the morphisms being group homomorphisms $\varphi\colon G\to G'$ preserving root groups: $\varphi(U_\alpha)\subset U'_\alpha$ for all α. Let us now fix a group $\bar G$ with subgroups $\bar U_\alpha$ in $\mathbf{gc}_R$. In Section 3 we introduce the *Steinberg category* $\mathbf{st}(\bar G)$ as follows. Its objects are the morphisms $\pi\colon G\to\bar G$ of $\mathbf{gc}_R$ which induce *isomorphisms*

$$\pi\colon U_\alpha \xrightarrow{\cong} \bar U_\alpha \quad\text{and}\quad \pi\colon U_{[\alpha,\beta]} \xrightarrow{\cong} \bar U_{[\alpha,\beta]},$$

for all α and all nilpotent pairs (α,β). We show in Theorem 3.13 that $\mathbf{st}(\bar G)$ has an initial object, called the *Steinberg group of $\bar G$* and denoted $\mathrm{St}(\bar G)$. It is unique up to unique isomorphism and is constructed as an inductive limit, following Tits' approach to Steinberg groups in the Kac-Moody setting. Under a mild assumption on $\bar G$, it can also be described in more down-to-earth fashion by generators and relations (Theorem 3.17). As we show in 3.18, this notion of Steinberg group specializes to the well-known Steinberg group $\mathrm{St}_n(A)$ of a ring A in case $n\geqslant 3$, with $\bar G$ being the elementary linear group $\mathrm{E}_n(A)$ of A. Similarly, taking for $\bar G$ the elementary unitary group $\mathrm{EU}_{2n}(A,\Lambda)$, we obtain the usual unitary Steinberg group.

The remainder of Chapter I contains the beginning of a theory of Weyl elements in groups with commutator relations. To do so, we must assume that R is a reflection system in the sense of [**42**, §2], reviewed in Section 4. Essentially, this means that, for all α in a suitable subset of R^{re} of R, reflections s_α of R are defined which have properties similar to the well-known reflections of ordinary root systems.

Weyl elements and Weyl triples are introduced in Section 5 and studied further in Section 6. A *Weyl element* for the root $\alpha\in R^{\mathrm{re}}$ in a group $G\in\mathbf{gc}_R$ is an element $w=xyz$ where $x,z\in U_\alpha$ and $y\in U_{-\alpha}$ such that conjugation by w in G corresponds to the reflection s_α in the sense that $wU_\beta w^{-1}=U_{s_\alpha(\beta)}$ for all $\beta\in R$. We then say that (x,y,z) is a *Weyl triple* for α. It is well known that Weyl elements exist in the standard examples mentioned above, e.g., in $\mathrm{St}_n(A)$, $\mathrm{E}_n(A)$, semisimple algebraic groups and Kac-Moody groups.

Let $\bar G\in\mathbf{gc}_R$ and fix a set $\bar{\mathfrak{X}}$ of Weyl triples for $\bar G$. We define a full subcategory $\mathbf{st}(\bar G,\bar{\mathfrak{X}})$ of $\mathbf{st}(\bar G)$ whose objects have Weyl triples projecting onto the prescribed Weyl triples $\bar{\mathfrak{X}}$ of $\bar G$. In Theorem 5.10 we show that $\mathbf{st}(\bar G,\bar{\mathfrak{X}})$ is a reflective subcategory of $\mathbf{st}(\bar G)$. Hence (Corollary 5.11) it has an initial object as well, called the Steinberg group $\mathrm{St}(\bar G,\bar{\mathfrak{X}})$. For example, when $\bar G=\mathrm{E}_2(A)$ and $\bar{\mathfrak{X}}$ is the set of all Weyl triples, we show in 5.12 that $\mathrm{St}(\bar G,\bar{\mathfrak{X}})$ is the usual Steinberg group $\mathrm{St}_2(A)$.

Section 6 contains further material on Weyl elements and Weyl triples and in particular studies the connection with rank one groups in the sense of Timmesfeld

[**56**]. We give a characterization of rank one groups in terms of Weyl triples in Proposition 6.7. A Weyl triple (x, y, z) is called balanced if $x = z$ and $xyx = yxy$. We show that a rank one group is special if and only if its Weyl triples are balanced (Proposition 6.8).

Chapter II: Groups associated to Jordan pairs. Here we begin to specialize the theory developed in Chapter I to groups related to Jordan pairs. In an attempt to make this book accessible to readers unfamiliar with Jordan theory, we give in Section 7 an introduction to Jordan pairs, following [**34**] and [**38**]. In particular, we introduce fundamental notions like quasi-inverses, inner automorphisms, structural transformations, and idempotents. We continue with Section 8 on Peirce gradings which form an important tool for our work.

Let $V = (V^+, V^-)$ be a Jordan pair. Section 9 introduces the Tits-Kantor-Koecher algebra

$$\mathfrak{L}(V) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{L}_i(V),$$

a $\mathbb{Z}$-graded Lie algebra with $\mathfrak{L}_i(V) = 0$ for $|i| > 1$, $\mathfrak{L}_{\pm1}(V) \cong V^\pm$ and $\mathfrak{L}_0(V)$ isomorphic to a suitable subalgebra of the derivation algebra of V. From the $\mathbb{Z}$-graded structure it is clear that $(\operatorname{ad} x)^3 = 0$ for $x \in V^\pm$. Hence, the exponentials

$$\exp_\pm(x) = \exp(\operatorname{ad} x) = \mathrm{Id} + \operatorname{ad} x + \frac{1}{2}(\operatorname{ad} x)^2$$

for $x \in V^\pm$ define automorphisms of $\mathfrak{L}(V)$ (due to the definition of Jordan pairs involving quadratic operators, the last term makes sense even when 2 is not a unit in the base ring). The projective elementary group $\mathrm{PE}(V)$ is then defined as the subgroup of $\mathrm{Aut}(\mathfrak{L}(V))$ generated by $\bar{U}^\pm = \exp_\pm(V^\pm)$. These subgroups are abelian, so that $\mathrm{PE}(V)$ is a group with A_1-commutator relations. For a quasi-invertible pair $(x, y) \in V^+ \times V^-$, the inner automorphism $\beta(x, y)$ of V is contained (after a natural identification) in $\mathrm{PE}(V)$. We extend the notion of a quasi-invertible pair to Faulkner's notion of higher order quasi-inverses and use it to determine the centre of $\mathrm{PE}(V)$ in Theorem 9.9. For an idempotent e of V we define the element

$$\omega_e = \exp_+(e_+) \exp_-(e_-) \exp_+(e_+) \in \mathrm{PE}(V)$$

and prove formulas for the action of ω_e on $\mathfrak{L}(V)$ and $\mathrm{PE}(V)$.

In the final Section 10 of Chapter II we define the category $\mathbf{st}(V)$ of *groups over* V as the Steinberg category of $\mathrm{PE}(V)$, viewed as a group with A_1-commutator relations. Thus an object of $\mathbf{st}(V)$ can be considered as a quadruple $(G, U^\pm, \pi)$ consisting of a group G, two abelian subgroups U^+ and U^-, and a group homomorphism $\pi\colon G \to \mathrm{PE}(V)$ which induces an isomorphism from $U^\pm \subset G$ onto the subgroups $\bar{U}^\pm$ of $\mathrm{PE}(V)$. The elements $\beta(x, y)$ and ω_e of $\mathrm{PE}(V)$ have canonical lifts, denoted $\mathrm{b}(x, y)$ and w_e, to any group G over V. In general, they do not satisfy the relations enjoyed by $\beta(x, y)$ and ω_e. This gives rise to relations $\mathfrak{B}(x, y)$ and $\mathfrak{W}(e)$ in G which will later be used to define subcategories of $\mathbf{st}(V)$ in case V has additional structure, for example, a Peirce grading or a suitable family of idempotents.

This preprint is a preliminary version of the first two chapters of a book which will contain material on groups over Jordan pairs with Peirce gradings and, more generally, root gradings, as well as a description of groups over Jordan pairs with a root grading. For example, we will describe the universal central extension of $\mathrm{PE}(V)$ in case V has a root grading of infinite rank by generators and relations. Some of these results have been announced in [**41**].

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CHAPTER I

GROUPS WITH COMMUTATOR RELATIONS

§1. Nilpotent sets of roots

1.1. $\mathbb{N}$-free subsets. In this section, X is a vector space over a field $\Bbbk$ of characteristic 0. We identify $\mathbb{Q}$ with the prime field of $\Bbbk$. For a subset A of X we use the notation

$$\mathbb{Z}[A] = \operatorname{span}_{\mathbb{Z}}(A) \quad \text{and} \quad \mathbb{N}[A]$$

for the subgroup and the submonoid of $(X, +)$ generated by A. Moreover, we let $\mathbb{N}^{(A)}$ be the free abelian monoid generated by A, i.e., the set of all maps $v\colon A \to \mathbb{N}$ with finite support. Depending on the context, it may be more convenient to think of an element of $\mathbb{N}^{(A)}$ as a family $(n_\alpha)_{\alpha \in A}$, where $n_\alpha \in \mathbb{N}$ and $n_\alpha = 0$ except for finitely many α. We denote by $\kappa\colon \mathbb{N}^{(A)} \to X$ the canonical map sending v to $\sum_{\alpha \in A} v(\alpha)\alpha$ and put

$$\mathbb{N}_+[A] := \kappa\big(\mathbb{N}^{(A)} \setminus \{0\}\big) = \bigcup_{n=1}^{\infty} \underbrace{(A + \cdots + A)}_{n}.$$

Note that $\mathbb{N}[A] = \kappa\big(\mathbb{N}^{(A)}\big)$.

A subset A of X is called $\mathbb{N}$-*free* [**5**] if $0 \notin \mathbb{N}_+[A]$; in other words, if for all $(n_\alpha)_{\alpha \in A} \in \mathbb{N}^{(A)}$, the relation $\sum_{\alpha \in A} n_\alpha \cdot \alpha = 0$ implies $n_\alpha = 0$ for all α. Clearly subsets of $\mathbb{N}$-free sets are $\mathbb{N}$-free, and $\mathbb{N}$-free sets do not contain 0. An $\mathbb{N}$-free set A defines a partial order $\succcurlyeq_A$ on X by

$$x \succcurlyeq_A y \iff x - y \in \mathbb{N}[A]. \tag{1}$$

This is easily verified. The notation $x \succ_A y$ means $x \succcurlyeq_A y$ and $x \neq y$.

The following fact will be useful.

1.2. Lemma. *Let V be a finite-dimensional real vector space and let $V_{\mathbb{Q}} \subset V$ be a rational form of V, i.e., a $\mathbb{Q}$-vector subspace such that $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong V$ under the natural map sending $x \otimes r$ to xr . We endow V with its natural topology and let $U \subset V$ be an open subset. Then*

$$U \neq \emptyset \quad \Longrightarrow \quad U \cap V_{\mathbb{Q}} \neq \emptyset. \tag{1}$$

Proof. After choosing a basis of V contained in $V_{\mathbb{Q}}$, we may identify V with $\mathbb{R}^n$ and $V_{\mathbb{Q}}$ with $\mathbb{Q}^n$, so the claim follows from density of $\mathbb{Q}^n$ in $\mathbb{R}^n$.

1.3. Lemma. *Let $A \subset X$ be a finite non-empty $\mathbb{N}$-free subset. We denote by $V_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}}(A)$ the rational span of A and put $V = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. We endow V with its natural topology and embed A into V canonically.*

(a) *The convex hull K of A in V is compact and does not contain 0.*

(b) *The pointed convex cone C spanned by A in V is closed and proper and spanned by its extremal rays. An extremal ray has the form $\mathbb{R}_+ \cdot \alpha$ for suitable $\alpha \in A$; in particular, A contains elements α such that $\mathbb{R}_+ \cdot \alpha$ is an extremal ray of C.*

Proof. (a) Recall that the convex hull of A consists of all real linear combinations $x = \sum_{\alpha \in A} r_\alpha \cdot \alpha$ where $r_\alpha > 0$ with the property that $\sum r_\alpha = 1$ [**7**, II, §2.3, Cor. 1 of Prop. 8]. Assume to the contrary that $x = 0$ has such a representation. Consider the finite-dimensional rational vector space $\mathbb{Q}^A$ with basis $(e_\alpha)_{\alpha \in A}$ and the canonical map $f\colon \mathbb{Q}^A \to V_{\mathbb{Q}}$ sending e_α to α. Let $f_{\mathbb{R}}\colon \mathbb{R}^A \to V$ be the $\mathbb{R}$-linear extension of f, let $W_{\mathbb{Q}} = \operatorname{Ker}(f) \subset \mathbb{Q}^A$ and $W = \operatorname{Ker}(f_{\mathbb{R}}) \subset \mathbb{R}^A$. Since the exact sequence

$$0 \longrightarrow W_{\mathbb{Q}} \longrightarrow \mathbb{Q}^A \xrightarrow{f} V_{\mathbb{Q}} \longrightarrow 0$$

remains exact upon tensoring with $\mathbb{R}$, we have $W_{\mathbb{Q}} \otimes \mathbb{R} \cong \operatorname{Ker}(f_{\mathbb{R}}) = W$, so $W_{\mathbb{Q}}$ is a rational form of W.

Let $U = W \cap \mathbb{R}^A_{++}$. Then U is open in W because $\mathbb{R}^A_{++}$ is open in $\mathbb{R}^A$, and by our assumption, $(r_\alpha)_{\alpha \in A} \in U$. By (1.2.1), $U \cap W_{\mathbb{Q}}$ is not empty as well. So there exists $u = (q_\alpha) \in \mathbb{Q}^A_{++}$ such that $0 = f(u) = \sum_{\alpha \in A} q_\alpha \cdot \alpha$. By multiplying this relation with the product of the denominators of the q_α we obtain a non-trivial relation $0 = \sum_{\alpha \in A} n_\alpha \cdot \alpha$ where $n_\alpha \in \mathbb{N}_+$. This contradicts the fact that A is $\mathbb{N}$-free. Finally, K is compact by [**7**, II, §2.6, Cor. 1 of Prop. 15].

(b) The pointed convex cone C (with vertex 0) spanned by A is

$$C = \mathbb{R}_+[A] = \Big\{ \sum_{\alpha \in A} r_\alpha \cdot \alpha : r_\alpha \in \mathbb{R}_+ \Big\},$$

and this is clearly the same as the smallest pointed cone which contains K. By [**7**, II, §7.3, Prop. 6], C is proper and closed in V. Hence by [**7**, II, §7.2, Prop. 5], C is the closed convex hull of the union of its extremal rays; in particular, such rays exist. One sees easily (cf. [**40**, B.1]) that an extremal ray of C has the form $\mathbb{R}_+ \cdot \alpha$ for some $\alpha \in A$, so A contains elements α such that $\mathbb{R}_+ \cdot \alpha$ is an extremal ray of C.

By a *height function* for a subset A of X we mean a homomorphism $h\colon \mathbb{Z}[A] \to \mathbb{Z}$ of abelian groups taking strictly positive values on A.

1.4. Proposition. *Any subset of X admitting a height function is $\mathbb{N}$-free. Conversely, a finite $\mathbb{N}$-free subset admits a height function.*

Proof. Suppose h is a height function for A, and let $(n_\alpha)_{\alpha \in A}$ in $\mathbb{N}^{(A)}$ with $\sum_{\alpha \in A} n_\alpha \cdot \alpha = 0$. Applying h yields $\sum_{\alpha \in A} n_\alpha \cdot h(\alpha) = 0$, and since all $h(\alpha)$ are positive, it follows that all $n_\alpha = 0$.

Conversely, suppose A is finite and $\mathbb{N}$-free. We use the notations of Lemma 1.3. By that lemma, K is compact and does not contain 0. By [**7**, II, §5.3, Prop. 4], there exists a hyperplane separating $\{0\}$ and K strictly. Thus there exists a linear form $g \in V^*$, the dual of V, and $c \in \mathbb{R}$ such that $g(0) - c < 0$ and $g(x) - c > 0$ for all $x \in K$; in particular, $g(\alpha) > 0$ for all $\alpha \in A$.

V^* has the rational form $V^*_{\mathbb{Q}} = \{f \in V^* : f(V_{\mathbb{Q}}) \subset \mathbb{Q}\}$, and for each $\alpha \in A$, the set $U_\alpha = \{f \in V^* : f(\alpha) > 0\}$ is open in V^*. Since A is finite, $U := \bigcap_{\alpha \in A} U_\alpha$ is open as well, and $g \in U$ by the above. By (1.2.1), $U \cap V^*_{\mathbb{Q}} \neq \emptyset$, so there exists $f \in V^*$ such that $f(\alpha) = q_\alpha \in \mathbb{Q}_{++}$, for all $\alpha \in A$. Multiplying f with the product of the denominators of the q_α yields the desired height function.

1.5. The category $\mathbf{SV}_{\Bbbk}$, closed and strictly positive sets. We introduce the category $\mathbf{SV}_{\Bbbk}$ of sets in vector spaces over $\Bbbk$ whose objects are pairs (R, X) consisting of a $\Bbbk$-vector space X and a subset $R \subset X$ which spans X and satisfies $0 \in R$. The morphisms $f\colon (R, X) \to (S, Y)$ of $\mathbf{SV}_{\Bbbk}$ are the $\Bbbk$-linear maps $f\colon X \to Y$ satisfying $f(R) \subset S$. If $\Bbbk$ is unimportant we will abbreviate $\mathbf{SV}_{\Bbbk}$ by $\mathbf{SV}$. The elements of

$$R^\times = R \setminus \{0\}$$

will often be referred to as *roots*. More generally, for any subset A of R, we put $A^\times = A \setminus \{0\}$. Note that the locally finite root systems of [**40**], see also 4.6, form a full subcategory of $\mathbf{SV}_{\mathbb{R}}$.

Let $(R, X) \in \mathbf{SV}_{\Bbbk} = \mathbf{SV}$. Generalizing a concept of [**40**, 10.2], a subset $C \subset R$ is called *additively closed in R* (or simply *closed* if there is no ambiguity) if $C = R \cap \mathbb{N}_+[C]$, i.e., if for all $\alpha_1, \ldots, \alpha_n \in C$ with $\beta := \alpha_1 + \cdots + \alpha_n \in R$, we have $\beta \in C$. The *additive closure* A^c of a subset A of R is the smallest additively closed subset containing A; it is given by

$$A^c = R \cap \mathbb{N}_+[A]. \tag{1}$$

In the special case $A = \{\alpha, \beta\}$, we write

$$\llbracket \alpha, \beta \rrbracket := \{\alpha, \beta\}^c = \{m\alpha + n\beta : m, n \in \mathbb{N},\ m + n > 0\} \tag{2}$$

and call it the *closed root interval* between α and β. If $f\colon (R, X) \to (R', X')$ is a morphism of $\mathbf{SV}$, then

$$f(A^c) \subset f(A)^c. \tag{3}$$

This is immediate from the definitions.

A subset A of R is called *strictly positive* if it is additively closed and $\mathbb{N}$-free. We remark that

$$A \text{ is strictly positive} \iff A \text{ is closed and } 0 \notin A. \tag{4}$$

Indeed, the implication from left to right is clear because an $\mathbb{N}$-free set does not contain 0. Conversely, let A be closed and $0 \notin A$. If $\sum n_\alpha \cdot \alpha = 0$ then $0 \in A$ since A is closed in R and $0 \in R$, contradiction. Let us also remark that a subset A is positive in the sense of [**40**, 10.5] if and only if $A^\times$ is strictly positive.

1.6. Commutator sets. Let $(R, X) \in \mathbf{SV}$. For arbitrary subsets A, B of R we define the *commutator set*

$$\big(A, B\big) := R \cap \big(\mathbb{N}_+[A] + \mathbb{N}_+[B]\big). \tag{1}$$

Thus $\gamma \in \big(A, B\big)$ if and only if γ belongs to R and has the form

$$\gamma = \alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_q \tag{2}$$

where $p, q \geqslant 1$, $\alpha_i \in A$, and $\beta_j \in B$.

If $A = \{\alpha\}$ consists of a single element, we simply write $\big(\alpha, B\big)$ instead of $\big(\{\alpha\}, B\big)$, and similarly

$$\big(\alpha, \beta\big) := \big(\{\alpha\}, \{\beta\}\big) = R \cap \big(\mathbb{N}_+\alpha + \mathbb{N}_+\beta\big), \tag{3}$$

called the *open root interval* from α to β. The following properties follow easily from the definition:

$$\big(A, \emptyset\big) = \emptyset, \qquad A \cup \big(A, A\big) = A^c = \big(A, 0\big), \tag{4}$$

$$A \quad \text{is closed} \quad \Longleftrightarrow \quad \big(A, A\big) \subset A, \tag{5}$$

$$0 \in B^c \quad \Longrightarrow \quad A^c \subset \big(A, B\big), \tag{6}$$

$$\big(A, B\big) = \big(B, A\big) = \big(A^c, B\big) = \big(A^c, B^c\big) = \big(A, B\big)^c, \tag{7}$$

$$A' \subset A, \quad B' \subset B \quad \Longrightarrow \quad \big(A', B'\big) \subset \big(A, B\big), \tag{8}$$

$$\big(A \cup B\big)^c = A^c \cup \big(A, B\big) \cup B^c, \tag{9}$$

$$\big(A, \big(A, B\big)\big) \subset \big(A, B\big). \tag{10}$$

If $f\colon (R, X) \to (R', X')$ is a morphism of $\mathbf{SV}$ then for $A, B \subset X$,

$$f\big(A, B\big) \subset \big(f(A), f(B)\big) \cap f(R). \tag{11}$$

Let $A \subset R$ be additively closed. A subset B of A is called *normal* (in A) if $\big(A, B\big) \subset B$. We remark that in [**55**, p. 24], the terminology "B is an ideal in A" is employed. By (5) and (8), a normal subset is in particular closed. Moreover, by (4) and (5), $\emptyset$ and A are always normal subsets of A, and by (6) any proper normal subset B of A has $0 \notin B^c$.

1.7. The lower central series. Let $(R, X) \in \mathbf{SV}$ and let $A \subset R$ be an arbitrary subset. The *lower central series of* A is defined inductively by

$$\mathscr{C}^1(A) = A^c, \qquad \mathscr{C}^{n+1}(A) = \big(A, \mathscr{C}^n(A)\big). \tag{1}$$

From (1.6.7) and (1.6.8) it follows by induction that

$$\mathscr{C}^n(A) = \mathscr{C}^n(A^c) = \mathscr{C}^n(A)^c, \tag{2}$$

$$\mathscr{C}^n(A) \supset \mathscr{C}^{n+1}(A), \tag{3}$$

and (1.6.6) and (1.6.4) yield

$$0 \in A^c \quad \Longrightarrow \quad \mathscr{C}^n(A) = A^c, \tag{4}$$

for all $n \geqslant 1$. Thus the lower central series is mainly of interest for closed subsets not containing 0, i.e., for strictly positive subsets, cf. (1.5.4). We note also that all $\mathscr{C}^n(A)$ are normal subsets of A if A is closed. The lower central series behaves well with respect to inclusions and morphisms:

$$B \subset A \quad \Longrightarrow \quad \mathscr{C}^n(B) \subset \mathscr{C}^n(A), \tag{5}$$

$$f\big(\mathscr{C}^n(A)\big) \ \subset \ \mathscr{C}^n\big(f(A)\big). \tag{6}$$

Indeed, (5) is a consequence of (1.6.8) while (6) follows from (1.5.3) and (1.6.11).

1.8. The upper central series. Let $(R, X) \in \mathbf{SV}$ and let $A \subset R$ be a closed subset. We define the *upper central series* of A inductively by

$$\mathscr{Z}_0(A) = \emptyset, \quad \mathscr{Z}_n(A) = \{\gamma \in A : \big(\gamma, A\big) \subset \mathscr{Z}_{n-1}(A)\}, \tag{1}$$

and the *centre* of A by

$$\mathscr{Z}(A) := \mathscr{Z}_1(A) = \{\gamma \in A : \big(\gamma, A\big) = \emptyset\}. \tag{2}$$

From the definition, it is clear that

$$\emptyset = \mathscr{Z}_0(A) \subset \mathscr{Z}_1(A) \subset \mathscr{Z}_2(A) \subset \cdots \subset A, \tag{3}$$

and that

$$\big(A, \mathscr{Z}_n(A)\big) \subset \mathscr{Z}_{n-1}(A), \tag{4}$$

in particular, the $\mathscr{Z}_n(A)$ are normal in A.

As for the lower central series, only the case $0 \notin A$ is of interest, because $0 \in A$ implies $\gamma = \gamma + 0 \in \big(\gamma, A\big)$ for all $\gamma \in A$, so $\mathscr{Z}(A)$ and therefore also all the other $\mathscr{Z}_n(A)$ are empty.

1.9. Prenilpotent and nilpotent subsets. Let $(R, X) \in \mathbf{SV}$. A subset A of R is said to be *prenilpotent* if $\mathscr{C}^n(A) = \emptyset$ for sufficiently large n, and it is called *nilpotent* if it is closed and prenilpotent. From 1.5 and 1.7 it is immediate that a prenilpotent set cannot contain 0 and that the following implications hold:

$$A \text{ prenilpotent} \quad \Longrightarrow \quad 0 \notin A^c, \tag{1}$$

$$B \subset A \text{ and } A \text{ prenilpotent} \quad \Longrightarrow \quad B \text{ prenilpotent}, \tag{2}$$

$$f(A) \text{ prenilpotent} \quad \Longrightarrow \quad A \text{ prenilpotent}, \tag{3}$$

$$A \text{ prenilpotent} \quad \Longrightarrow \quad A^c \text{ nilpotent}, \tag{4}$$

$$A \text{ nilpotent} \quad \Longrightarrow \quad A \text{ strictly positive}. \tag{5}$$

The *class* of a nilpotent A is the smallest k such that $\mathscr{C}^{k+1}(A) = \emptyset$. Thus

$$\begin{aligned} k \leqslant 1 &\iff A = \mathscr{Z}(A) &&\iff \big(A, A\big) = \emptyset, \\ k \leqslant 2 &\iff \big(A, A\big) \subset \mathscr{Z}(A) &&\iff \big(A, \big(A, A\big)\big) = \emptyset, \end{aligned}$$

and we will call an A of class $\leqslant 1$ resp. $\leqslant 2$ *abelian* resp. *2-step nilpotent*.

As in the case of groups, nilpotence can also be characterized by the upper central series. More generally, let A be a strictly positive subset of R. A chain of subsets $A \supset A_1 \supset A_2 \supset \cdots$ is called a *central chain* if $\big(A, A_n\big) \subset A_{n+1}$ for all $n \geqslant 1$. For example, the lower central series is a central chain, and so is $A_i := \mathscr{Z}_{m+1-i}$ for some fixed m, provided we define $\mathscr{Z}_j(A) = \emptyset$ for $j < 0$.

Clearly the terms A_n of a central chain are normal in A. From (1.7.1) and (1.8.1) it follows easily that

$$A_1 = A \implies A_i \supset \mathscr{C}^i(A), \tag{6}$$

$$A_{n+1} = \emptyset \implies A_i \subset \mathscr{Z}_{n+1-i}(A). \tag{7}$$

Now (6) shows

$$A \text{ is nilpotent of class } \leqslant n \iff \begin{array}{l}\text{there exists a central chain} \\ \text{with } A_1 = A \text{ and } A_{n+1} = \emptyset,\end{array} \tag{8}$$

and (7) implies

$$A \text{ is nilpotent of class } \leqslant n \iff \mathscr{Z}_n(A) = A. \tag{9}$$

Let us also note that the length of the upper central series of a nilpotent A of class k is exactly k. Indeed, $\mathscr{Z}_k(A) = A$ holds by (9). Assuming $\mathscr{Z}_{k-1}(A) = A$ would yield a central chain $A_i := \mathscr{Z}_{k-i}(A)$ with $A_1 = A$ and $A_k = \mathscr{Z}_0(A) = \emptyset$, so A would have class $\leqslant k - 1$, contradiction.

1.10. Lemma. *Let $(R, X) \in$ **SV** and let $A \subset R$ be a finite strictly positive subset of cardinality n.*

(a) *There exist total orders $\geqslant$ on A compatible with the partial order $\succcurlyeq_A$ defined by A, cf.* (1.1.1), *in the sense that $\alpha \succcurlyeq_A \beta$ implies $\alpha \geqslant \beta$.*

(b) *Let $\geqslant$ be as in* (a), *and enumerate $A = \{\alpha_1, \ldots, \alpha_n\}$ in such a way that $\alpha_1 < \cdots < \alpha_n$. Then $A_i := \{\alpha_i, \ldots, \alpha_n\}$ for $i = 1, \ldots, n$, and $A_i := \emptyset$ for $i > n$, is a central chain of A. In particular, A is nilpotent of class $\leqslant n$.*

Proof. (a) This follows from the Szpilrajn-Marczewski Lemma [**23**, Ch. 8, Section 8.6].

(b) We show $\big(A, A_i\big) \subset A_{i+1}$. By (1.6.2), an element $\gamma \in \big(A, A_i\big)$ has the form $\gamma = \alpha_{i_1} + \cdots + \alpha_{i_p} + \alpha_{j_1} + \cdots + \alpha_{j_q}$ where $p, q \geqslant 1$, $i_\lambda \in \{1, \ldots, n\}$ and $j_\mu \in \{i, \ldots, n\}$; in particular, $\gamma \succ_A \alpha_{j_1}$. On the other hand, $\gamma \in A$ because A is closed, say, $\gamma = \alpha_k$. Hence $k > j_1 \geqslant i$ so $\alpha_k \in A_k \subset A_{i+1}$.

The statement about the nilpotence of A now follows from (1.9.8).

1.11. Lemma. *Let F be a finite set and let $\mathbb{N}^F$, the set of functions $F \to \mathbb{N}$, be equipped with the partial order*

$$v \leqslant w \quad \iff \quad v(\alpha) \leqslant w(\alpha) \text{ for all } \alpha \in F.$$

Then every infinite subset S of $\mathbb{N}^F$ contains a strictly increasing sequence $v_1 < v_2 < \cdots$.

Proof. The proof is by induction on the cardinality of F, the case $F = \emptyset$ being trivial. If S has no maximal element then the assertion is clear. Otherwise, let m be a maximal element of S. Then $v > m$ holds for no $v \in S$, i.e., for every $v \in S$ there exists an element $\alpha \in F$ such that $v(\alpha) \leqslant m(\alpha)$. Letting $S_\alpha := \{v \in S : v(\alpha) \leqslant m(\alpha)\}$, we thus have $S = \bigcup_{\alpha \in F} S_\alpha$. Since S is infinite, there must be a $\beta \in F$ such that S_β is infinite. Consider the evaluation map $S_\beta \to \mathbb{N}$, $v \mapsto v(\beta)$, whose image is contained in the finite interval $I := \{0, 1, \ldots, m(\beta)\}$ of $\mathbb{N}$. Since S_β is infinite, there exists $i \in I$ such that the fibre $S_\beta^i := \{v \in S_\beta : v(\beta) = i\}$ is infinite. Let $F' := F \setminus \{\beta\}$, denote by res: $\mathbb{N}^F \to \mathbb{N}^{F'}$ the restriction map induced by the inclusion $F' \hookrightarrow F$, and put $S' := \mathrm{res}(S_\beta^i) \subset \mathbb{N}^{F'}$. Clearly, res: $S_\beta^i \to S'$ is bijective, with inverse ext: $S' \to S_\beta^i$ given by extending an element $v' \in S'$ (which after all is a map $F' \to \mathbb{N}$) to a map $F \to \mathbb{N}$ via $\beta \mapsto i$. By induction, there exists a strictly increasing sequence $v_1' < v_2' < \cdots$ in S'. Then $v_k := \mathrm{ext}(v_k')$ is the desired sequence in S.

1.12. Proposition. *Let $(R, X) \in \mathbf{SV}$. For a subset $F \subset R$ with closure $F^c = A$, the following conditions are equivalent:*

(i) *F is finite and prenilpotent,*
(ii) *A is finite and nilpotent,*
(iii) *A is finite and strictly positive,*
(iv) *A is finite and $0 \notin A$.*

Proof. (i) $\iff$ (ii): F is prenilpotent if and only if A is nilpotent by (1.9.4), so it remains to show that F finite implies A is finite. Assume, by way of contradiction, that A is infinite. Then by definition of the closure of a set in (1.5.1) we have $S := \kappa^{-1}(A) \subset \mathbb{N}^F$ infinite, where κ is defined in 1.1. Choose a sequence $(v_k)_{k \geqslant 1}$ in S as in Lemma 1.11 and put $\gamma_k = \kappa(v_k)$. We will show by induction that $\gamma_k \in \mathscr{C}^k(F)$ for all $k \geqslant 1$, contradicting the fact that $\mathscr{C}^k(F) = \emptyset$ for sufficiently big k, by nilpotence of A. Obviously, $\gamma_1 \in A = \mathscr{C}^1(F)$. Suppose we have $\gamma_k \in \mathscr{C}^k(F)$. Then $\gamma_{k+1} - \gamma_k = \sum_{\alpha \in F} n_\alpha \alpha$ where all $n_\alpha := v_{k+1}(\alpha) - v_k(\alpha) \in \mathbb{N}$, and at least one n_α is positive because $v_{k+1} > v_k$. Hence $\gamma_{k+1} \in \big(F, \gamma_k\big) \subset \big(F, \mathscr{C}^k(F)\big) = \mathscr{C}^{k+1}(F)$.

(ii) $\Longrightarrow$ (iii) is (1.9.5), and the implication (iii) $\Longrightarrow$ (ii) is a consequence of Lemma 1.10(b). The equivalence of (iii) and (iv) follows from (1.5.4).

1.13. Corollary. *A finite prenilpotent subset admits a height function.*

Proof. Let F be finite and prenilpotent with closure A. By Proposition 1.12(iii), A is finite and strictly positive, hence in particular $\mathbb{N}$-free, see 1.5. By Proposition 1.4, A admits a height function and hence so does F.

1.14. Corollary. *The following conditions on a subset A of R are equivalent:*

(i) *A is closed in R and every finite subset of A is prenilpotent,*
(ii) *A is strictly positive and every finite subset of A has finite closure.*

This follows easily from Proposition 1.12. A subset satisfying these conditions is called *locally nilpotent.* In particular, if R is a locally finite root system, cf. 4.6, then a subset of R is locally nilpotent if and only if it is strictly positive.

§2. Groups with commutator relations

2.1. Nilpotent pairs and division of roots. Let $(R, X) \in \mathbf{SV}_{\Bbbk} = \mathbf{SV}$ and $\alpha, \beta \in R$. Recall from (1.5.2) and (1.6.9) that the *closed root interval* from α to β is

$$\begin{aligned} \big[\alpha, \beta\big] &= \{\alpha, \beta\}^c = \{\alpha\}^c \cup \big(\alpha, \beta\big) \cup \{\beta\}^c \\ &= R \cap \{p\alpha + q\beta : \ p, q \in \mathbb{N}, \ p + q \geqslant 1\}. \end{aligned} \tag{1}$$

We will call (α, β) a *nilpotent pair* if $\big[\alpha, \beta\big]$ is a nilpotent subset of R, in other words, if the subset $\{\alpha, \beta\}$ of R is prenilpotent, see 1.9. Clearly, if (α, β) is a nilpotent pair then so is (β, α). By Proposition 1.12,

$$(\alpha, \beta) \text{ is a nilpotent pair} \quad \iff \quad \big[\alpha, \beta\big] \text{ is finite and } 0 \notin \big[\alpha, \beta\big]. \tag{2}$$

On the other hand, one shows easily that, for $\alpha, \beta \in R^{\times}$,

$$0 \in \big[\alpha, \beta\big] \iff 0 \in \big(\alpha, \beta\big) \iff \alpha \in \big(\alpha, \beta\big) \iff \beta \in \big(\alpha, \beta\big). \tag{3}$$

Let $\alpha, \beta \in R$. We say α *divides* β, written $\alpha \big| \beta$, if $\beta \in \mathbb{N} \cdot \alpha$, i.e., $\beta = n\alpha$ for some $n \in \mathbb{N}$. Note that

$$\alpha \big| 0 \text{ always holds, while } 0 \big| \beta \text{ implies } \beta = 0. \tag{4}$$

Divisibility is a partial order, in particular it is transitive:

$$\alpha \big| \beta \quad \text{and} \quad \beta \big| \gamma \quad \Longrightarrow \quad \alpha \big| \gamma. \tag{5}$$

2.2. Definition. Let G be a group. The commutator of $a, b \in G$ is $\big(a,\, b\big) = aba^{-1}b^{-1}$. If X and Y are subsets of G, we use the notation $\big(X, Y\big)$ for the subgroup of G generated by all $\big(a, b\big)$, $a \in X$, $b \in Y$. If $(X_i)_{i \in I}$ is a family of subsets of G, $\big\langle X_i : \ i \in I \big\rangle$ denotes the subgroup of G generated by the union of the X_i.

Let $(R, X) \in \mathbf{SV}_{\Bbbk}$ and let $(U_\alpha)_{\alpha \in R}$ be a family of subgroups of G. For a subset A of R we put

$$U_A = \Big\langle U_\alpha : \ \alpha \in A \Big\rangle;$$

thus in particular $U_\emptyset = \{1\}$. We say G *has R-commutator relations with root groups* $(U_\alpha)_{\alpha \in R}$ if the following conditions hold for all $\alpha, \beta \in R$:

$$U_0 \ = \ \{1\}, \tag{1}$$

$$\alpha \text{ divides } \beta \quad \Longrightarrow \quad U_\beta \subset U_\alpha, \tag{2}$$

$$(\alpha, \beta) \text{ nilpotent} \quad \Longrightarrow \quad \big(U_\alpha, U_\beta\big) \subset U_{(\alpha, \beta)}. \tag{3}$$

Note that by (2.1.4) and (1), the relation (2) holds automatically if α or β is zero. We will usually refer to the subgroups U_α as *root groups*. Because of (1) only the U_α, $\alpha \neq 0$, are of interest. In particular, when considering examples it is sufficient to specify U_α, $\alpha \in R^\times$.

Note also that we do not require G to be generated by its root subgroups. On the other hand, the subgroup

$$\mathfrak{r}(G) = U_R$$

generated by all root subgroups clearly has R-commutator relations as well. If $G = \mathfrak{r}(G)$ we say that G is *tight* or the $(U_\alpha)_{\alpha\in R}$ form a *generating family.*

2.3. Examples. (a) *The case* $R = \mathrm{A}_1$*.* The reader will find many examples of groups with R-commutator relations throughout this book. A very simple case is $R = \{0, 1, -1\} \subset \Bbbk$, the root system of type A_1. Then a group G has A_1-commutator relations if and only if $U_1 = U^+$ and $U_{-1} = U^-$ are two abelian subgroups. In particular, the projective elementary group $\mathrm{PE}(V)$ of a Jordan pair V as in 9.2 has A_1-commutator relations with root subgroups $U^\pm = \exp_\pm(V^\pm)$.

(b) Commutator relations are inherited by homomorphic images: if $\varphi\colon G \to H$ is a group homomorphism then H has R-commutator relations with root groups $\varphi(U_\alpha)$.

(c) *Reductive algebraic groups over fields.* Let G be a connected reductive algebraic group over an algebraically closed field in the sense of [**6**, **54**], and let $\Phi = \Phi(G, T)$ be the root system of G with respect to a maximal torus T. For $\alpha \in \Phi$ let U_α be the root group defined in [**6**, Thm. 13.18(4.d)]. Then $R = \Phi \cup \{0\}$ is a reduced finite root system and G has R-commutator relations with respect to the family $(U_\alpha)_{\alpha\in R}$ [**6**, 14.5 (*)], which is generating in case G is semisimple [**54**, Thm. 8.1.5].

More generally, let G be a connected reductive algebraic group defined over an arbitrary field k in the sense of [**6**, **54**], and let Φ' be the set of k-roots of G with respect to a maximal k-split torus T' of G. One knows [**6**, Thm. 21.6] that $R' = \Phi' \cup \{0\}$ is a finite but not necessarily reduced root system. Moreover, for every $\alpha' \in \Phi'$ there exists a unique closed connected unipotent k-subgroup $U'_{\alpha'}$ normalized by the centralizer of T' and with Lie algebra $\mathfrak{g}_{\alpha'} \oplus \mathfrak{g}_{2\alpha'}$, where $\mathfrak{g}_{2\alpha'} = 0$ if $2\alpha' \notin \Phi'$ [**6**, Prop. 21.9]. The construction of the $U'_{\alpha'}$ in [**6**] is a special case of the construction given in Proposition 3.3(a). We note that even if G is generated by the U_α, this will in general no longer hold for the $U'_{\alpha'}$.

The analogous statement also holds for the groups $G(k)$ and $U'_{\alpha'}(k)$ of k-rational points of G and the U'_α: the group $G(k)$ has R'-commutator relations. Note, however, that $G(k)$ is in general not generated by the $U'_{\alpha'}(k)$, even when G is semisimple and simply connected, see for example [**17**, Ch. 2.2.E] for a discussion of this question for groups of type A (the Tannaka-Artin problem).

(d) *Split reductive group schemes.* Let $\mathbf{G}$ be a split reductive group scheme over a scheme S [**13**, Exp. XXII, Déf. 1.13]. Recall [**13**, Exp. XXII, Prop. 1.14] that the root system R of $\mathbf{G}$ is reduced. Let $\mathbf{U}_\alpha$, $\alpha \in R^\times$, be the root subgroups of $\mathbf{G}$

(denoted $\mathbf{P}_\alpha$ in [**13**]). Let $S' \to S$ be a morphism of schemes. Then $G = \mathbf{G}(S')$ is a group with R-commutator relations with respect to the subgroups $U_\alpha = \mathbf{U}_\alpha(S')$ [**13**, Exp. XXII, Cor. 5.5.2]. If $\mathbf{G}$ is simply connected and S' is a local scheme the group G is generated by the root subgroups [**13**, Exp. XXII, Cor. 5.7.6].

Let in particular $S = \mathrm{Spec}(\mathbb{Z})$ and let $S' = \mathrm{Spec}(k)$ where k is any field. Then $U_R = \langle U_\alpha : \alpha \in R\rangle$ is a Chevalley group in the sense of [**55**, §3] and hence has R-commutator relations.

(e) *Groups associated to Moufang buildings.* Let $\mathscr{B}$ be a thick irreducible spherical Moufang building over $I = \{1, \dots, l\}$ with $l \geqslant 2$ and different from an octagon (we use the notation of [**56**, II, §5] and [**61**]). Let Φ be the set of roots of an apartment $\mathscr{A}$ of $\mathscr{B}$. It is known [**56**, p. 126] that $\Phi \cup \{0\}$ can be identified with a finite irreducible reduced root system. Moreover, there exist an irreducible finite root system $R \supset \Phi$ and a subgroup $G \subset \mathrm{Aut}(\mathscr{B})$ with root groups U_α, $\alpha \in R$, such that $G = U_R$ has R-commutator relations. We have $R = \Phi \cup \{0\}$ if Φ is not of type B or C, and $R \in \{\mathrm{B}_l, \mathrm{C}_l, \mathrm{BC}_l\}$ otherwise depending on $\mathscr{B}$.

The construction of these groups is for example given in [**56**, II, §5]. It is immediate from this construction that the relation (2.2.2) holds. Note that it only has to be verified in case α and $2\alpha \in R$. The relation (2.2.3) for the nilpotent pair (α, α) is also clear from the construction. To verify (2.2.3) in the remaining cases, we can therefore in view of 4.8 assume that α and β are $\mathbb{Q}$-linearly independent. For $l = 2$ the commutator relation (2.2.3) then follows by comparing the list in 4.8 with the one in [**60**] or [**62**, §5.4]. The case $l \geqslant 3$ can be reduced to the case $l = 2$, see e.g. [**56**, II, (5.7)]. For example, for a Moufang quadrangle of type F_4 in the sense of [**61**, (16.7)], the root system R is of type BC_2, see [**61**, (40.59)].

(f) The reader can find more examples of groups with commutator relations in 2.14 (nilpotent groups), 2.16 (groups with unique factorization, in particular elementary linear groups), 3.19 (Tits' Steinberg group), 6.7 (rank one groups), 10.3 (elementary groups of special Jordan pairs), 10.16 (Steinberg groups $\mathrm{St}_n(R)$ for R an associative k-algebra). In particular, we will see in 2.14 that nilpotent groups provide natural examples of groups with R-commutator relations where $\{\alpha\}^c$ can have any finite cardinality.

2.4. Remarks. Let G have R-commutator relations with root groups U_α.

(a) Suppose $(\alpha, \beta) \in R^\times \times R^\times$ is not nilpotent, so that either $0 \in [\![\alpha, \beta]\!]$ or $[\![\alpha, \beta]\!]$ is infinite. In the first case, (2.1.3) implies that even

$$\Big\langle U_\alpha \cup U_\beta \Big\rangle \subset U_{(\!(\alpha, \beta)\!)} \tag{1}$$

holds. In the second case we do not have nor do we require any relations.

(b) In many interesting examples, e.g. if $R = \mathrm{Re}(S)$ where R is the set of reflective roots of a partial root system S in the sense of [**42**], R satisfies the finiteness condition

$$\text{(F1):} \qquad \{\alpha\}^c \ \text{ is finite, for every } \alpha \in R. \tag{2}$$

Then a pair $(\alpha,\beta) \in R^\times \times R^\times$ is nilpotent if and only if the commutator set $\big(\alpha,\beta\big)$ is finite and does not contain zero.

(c) Suppose (R,X) satisfies the stronger finiteness condition

$$\text{(F2):} \qquad \big[\alpha,\beta\big] \text{ is finite, for all } \alpha,\beta \in R. \tag{3}$$

Then also (F1) holds, and for $\alpha,\beta \in R^\times$ either (α,β) is nilpotent or $0 \in \big[\alpha,\beta\big]$. Hence by (1), a group with R-commutator relations actually satisfies

$$\big(U_\alpha, U_\beta\big) \subset U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} \qquad \text{for all } \alpha,\beta \in R. \tag{4}$$

Note that (F2) is always fulfilled if (R,X) is a *locally finite root system* in the sense of [**40**], see 4.6.

(d) Let $R = \mathrm{Re}(S)$ where S is a partial root system, see [**42**, Def. 3.3]; in particular, $R = S$ could be a locally finite root system. Then $\big(\alpha,\alpha\big) \subset \{2\alpha\}$ for all $\alpha \in R$, so (2.2.3) implies

$$\big(U_\alpha, U_\alpha\big) \subset U_{2\alpha}, \qquad \big(U_\alpha, U_{2\alpha}\big) = 1. \tag{5}$$

Hence all root groups are 2-step nilpotent (the derived group is central), and they are even abelian if R is reduced.

2.5. Lemma. *Let G be a group with R-commutator relations and let (α,β) be a nilpotent pair.*

(a) *For all $\gamma,\delta \in \big[\alpha,\beta\big]$, the pair (γ,δ) is nilpotent and satisfies*

$$\big(\gamma,\delta\big) \subset \big(\alpha,\beta\big). \tag{1}$$

(b) *U_α and U_β normalize $U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$, and*

$$U_{\boldsymbol{[}\alpha,\beta\boldsymbol{]}} = U_\alpha \cdot U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} \cdot U_\beta. \tag{2}$$

Proof. (a) Clearly $\big[\gamma,\delta\big] \subset \big[\alpha,\beta\big]$, and $\big(\gamma,\delta\big) \subset \big(\alpha,\beta\big)$ holds by (1.6.8). Since subsets of prenilpotent sets are prenilpotent (cf. (1.9.2)), we have (a).

(b) Let $\gamma,\delta \in \big[\alpha,\beta\big]$. By (a) and the commutator relation (2.2.3), we have

$$\mathrm{Int}(U_\gamma) \cdot U_\delta \subset \big(U_\gamma, U_\delta\big) \cdot U_\delta \subset U_{\boldsymbol{(}\gamma,\delta\boldsymbol{)}} \cdot U_\delta. \tag{3}$$

In particular, for $\gamma \in \{\alpha,\beta\}$ and $\delta \in \big(\alpha,\beta\big)$, (3) and (1) imply $\mathrm{Int}(U_\gamma)\cdot U_\delta \subset U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$, from which it follows that U_α and U_β normalize $U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$. Hence $U_\alpha \cdot U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} = U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} \cdot U_\alpha$ and $K = U_\beta \cdot U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} = U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} \cdot U_\beta$ are subgroups of G.

The inclusion from right to left in (2) is clear from (2.1.1). For the reverse inclusion, let $H = U_\alpha \cdot K$ denote the right hand side of (2). Note first that, because of (2.1.1) and the relation (2.2.2), H contains all U_γ, $\gamma \in \big[\alpha,\beta\big]$. Hence it suffices to show that H is a subgroup of G. Now, for $(\gamma,\delta) = (\alpha,\beta)$, (3) shows that $\mathrm{Int}(U_\alpha) \cdot U_\beta \subset K$. Thus U_α normalizes the

2.6. Commutator formulas. Let G be a group. For elements a, b of G we write ${}^a b = aba^{-1}$, and denote by $\mathscr{Z}(G)$ the centre of G. Then the following formulas hold:

$$\begin{aligned}
\big(a,b\big)^{-1} &= \big(b,a\big), && (1)\\
\big(ab,c\big) &= {}^a\big(b,c\big)\cdot\big(a,c\big) = \big(a,\big(b,c\big)\big)\cdot\big(b,c\big)\cdot\big(a,c\big), && (2)\\
\big(a,bc\big) &= \big(a,b\big)\cdot{}^b\big(a,c\big) = \big(a,b\big)\cdot\big(a,c\big)\cdot\big(\big(c,a\big),b\big), && (3)\\
\big(a^{-1},b\big) &= a^{-1}\big(a,b\big)^{-1}a, && (4)\\
\big(a,b^{-1}\big) &= b^{-1}\big(a,b\big)^{-1}b. && (5)\\
\big(a,\ \big(b,c\big)\big) &= \big(ab,\ c\big)\cdot\big(c,a\big)\cdot\big(c,b\big), && (6)\\
\big(\big(a,b\big),\ c\big) &= \big(a,b\big)\cdot\big(c,b\big)\cdot\big(b,ca\big), && (7)\\
a\equiv a' \textit{ and } b\equiv b' \bmod \mathscr{Z}(G) \quad &\Longrightarrow \quad \big(a,b\big) = \big(a',b'\big). && (8)
\end{aligned}$$

We also have the following relations, see [**9**, I, §6.2]. Formula (11) is a group-theoretic analogue of the Jacobi identity in Lie algebras.

$$\begin{aligned}
\big(a,bc\big)\cdot\big(b,ca\big)\cdot\big(c,ab\big) &= 1, && (9)\\
\big(ab,c\big)\cdot\big(ca,b\big)\cdot\big(bc,a\big) &= 1, && (10)\\
\big({}^b a,\big(c,b\big)\big)\cdot\big({}^c b,\big(a,c\big)\big)\cdot\big({}^a c,\big(b,a\big)\big) &= 1. && (11)
\end{aligned}$$

The proofs are straightforward verifications.

2.7. Lemma. *Let G be a group, H a subgroup and let X_1, X_2 be subsets of G normalizing H and satisfying $\big(X_1, X_2\big) \subset H$. Then the subgroups G_i generated by X_i normalize H and $\big(G_1, G_2\big) \subset H$.*

Proof. Since the normalizer of any subset is a subgroup, it is clear that the G_i normalize H. For the proof of the second claim, we first note that $\big(x_1^{\pm1}, x_2^{\pm1}\big) \in H$ by (2.6.4) and (2.6.5). We may therefore assume $X_i = X_i^{-1}$, so that G_i is the submonoid generated by X_i. Then $\big(g_1, g_2\big) \in H$ for all $g_i \in G_i$ by a straightforward induction, using (2.6.2) and (2.6.3), and this in turn implies $\big(G_1, G_2\big) \subset H$.

2.8. Lemma. *Let $(R, X) \in \mathbf{SV}$ and let G be a group with a family of subgroups U_α indexed by $\alpha \in R$, which satisfy* (2.2.1) *and* (2.2.2). *For each $\alpha \in R$ let $X_\alpha = X_\alpha^{-1} \subset U_\alpha$ be a symmetric set of generators of U_α, and suppose that*

$$\big(X_\alpha, X_\beta\big) \subset U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} \tag{1}$$

holds for all nilpotent pairs (α, β). Then G has R-commutator relations with root groups U_α.

Proof. It remains to verify the commutator relation (2.2.3) for every nilpotent pair (α, β). To do so, we apply Lemma 2.7 to $X_1 = X_\alpha$, $X_2 = X_\beta$ and $H = U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$, so we must show that X_α and X_β normalize H. Now H is generated by all U_γ, $\gamma \in \big(\alpha, \beta\big)$, and (α, γ) is a nilpotent pair, by 2.5(a). Thus our hypothesis (1) yields $x_\alpha y_\gamma x_\alpha^{-1} \in U_{(\alpha,\gamma)} \cdot y_\gamma \subset H$ for all $x_\alpha \in X_\alpha$, $y_\gamma \in X_\gamma$. As U_γ is generated by X_γ, this implies $x_\alpha U_\gamma x_\alpha^{-1} \subset H$. By definition, H is generated by all U_γ, $\gamma \in \big(\alpha, \beta\big)$. Conjugation with x_α is an automorphism so $x_\alpha H x_\alpha^{-1} \subset H$. As also $x_\alpha^{-1} \in X_\alpha$ by symmetry of X_α, we have $x_\alpha H x_\alpha^{-1} = H$, so x_α does normalize H. In the same way, one shows that X_β normalizes H. Now Lemma 2.7 yields $\big(U_\alpha, U_\beta\big) \subset H$. This completes the proof.

Recall that the lower and upper central series of a group H are defined inductively by $\mathscr{C}^1(H) = H$, $\mathscr{C}^{n+1}(H) = \big(H, \mathscr{C}^n(H)\big)$ and $\mathscr{Z}_0(H) = \{1\}$ and $\mathscr{Z}_n(H) = \{a \in H : \big(a, H\big) \subset \mathscr{Z}_{n-1}(H)\}$, respectively.

2.9. Lemma. *Let G be a group with R-commutator relations.*

(a) *Let A and B be subsets of R with the property that, for all $\alpha \in A$, $\beta \in B$ and $\gamma \in \big(A, B\big)$, the pairs (α, β), (α, γ) and (β, γ) are nilpotent. Then the subgroups U_A and U_B normalize $U_{(A,B)}$, and the* generalized commutator relations

$$\big(U_A, U_B\big) \subset U_{(A,B)} \tag{1}$$

hold.

(b) *Let A be a strictly positive subset (cf.* 1.5*) of R with the property that (α, β) is a nilpotent pair, for all $\alpha, \beta \in A$. Then a central chain $A \supset A_1 \supset A_2 \supset \cdots$ in A gives rise to a central chain $U_A \supset U_{A_1} \supset U_{A_2} \supset \cdots$ in U_A. The lower and upper central series of A and U_A are related by*

$$\mathscr{C}^n(U_A) \subset U_{\mathscr{C}^n(A)}, \qquad U_{\mathscr{Z}_n(A)} \subset \mathscr{Z}_n(U_A). \tag{2}$$

If $D \subset A$ is a normal subset then U_D is a normal subgroup of U_A.

Proof. (a) Let $C := \big(A, B\big)$ and put $X_1 := \bigcup_{\alpha \in A} U_\alpha$, $X_2 := \bigcup_{\beta \in B} U_\beta$ and $H := U_C$. Since $U_A = \langle X_1 \rangle$ and $U_B = \langle X_2 \rangle$, our claim will follow from Lemma 2.7 once we verify the assumptions of that lemma. First, since the pair (α, β) is nilpotent for all $\alpha \in A$, $\beta \in B$ and satisfies $\big(\alpha, \beta\big) \subset C$ by (1.6.8), we obtain $\big(X_1, X_2\big) \subset U_C = H$. Thus, in order to apply 2.7, it remains to show that X_1 and X_2 normalize H. By symmetry, it is enough to do so for X_1. Let $\alpha \in A$ and $\gamma \in C$. Then (α, γ) is a nilpotent pair by assumption, and $\big(\alpha, \gamma\big) \subset \big(A, C\big) \subset C$ by (1.6.10). Hence, for all $x_\alpha \in U_\alpha$,

$$x_\alpha U_\gamma x_\alpha^{-1} \subset \big(U_\alpha, U_\gamma\big) \cdot U_\gamma \subset U_{(\alpha,\gamma)} \cdot U_\gamma \subset U_C = H.$$

This implies $x_\alpha H x_\alpha^{-1} \subset H$ because the U_γ generate H, and even $x_\alpha H x_\alpha^{-1} = H$ because U_α, being a subgroup, is closed under inversion. Thus X_1 normalizes H, as required.

(b) Note first that formula (1) applies to any subset B of A. Indeed, $\big(A,B\big) \subset A$ because A is closed, so our assumption on A shows that the property required in (a) holds. By 1.9, a central chain in A satisfies $\big(A, A_i\big) \subset A_{i+1}$. Hence $\big(U_A, U_{A_i}\big) \subset U_{\left(A,A_i\right)} \subset U_{A_{i+1}}$, showing the U_{A_i} form a central chain in U_A.

For $n = 1$ the first formula of (2) is clear. The induction step follows by putting $B = \mathscr{C}^n(A)$, whence $\mathscr{C}^{n+1}(U_A) = \big(U_A, \mathscr{C}^n(U_A)\big) \subset \big(U_A, U_B\big)$ (by induction) $\subset U_{\left(A,B\right)} = U_{\mathscr{C}^{n+1}(A)}$ (by (1.7.1)).

The second formula of (2) obviously holds for $n = 0$. Assume it is true for $n-1$ and let $\alpha \in \mathscr{Z}_n(A)$. Then $\big(U_\alpha, U_A\big) \subset U_{\left(\alpha,A\right)} \subset U_{\mathscr{Z}_{n-1}(A)}$ (by (1.8.1)) $\subset \mathscr{Z}_{n-1}(U_A)$ (by induction). A normal subset D of A satisfies $\big(A,D\big) \subset D$. Hence the last statement follows from $\mathrm{Int}(U_A) \cdot U_D \subset \big(U_A, U_D\big)U_D \subset U_{\left(A,D\right)}U_D \subset U_D$.

The following lemma is due to J. Tits [**59**, 4.7, Lemma 2], for (b) see also [**55**, Lemma 18] .

2.10. Lemma. *Let X be a group generated by subgroups $X_1, \dots, X_n$. Suppose that X has a central chain $X = Z_1 \supset Z_2 \supset \cdots \supset Z_h \supset Z_{h+1} = \{1\}$ such that, for all $i \in \{1, \dots, h\}$, there exists $j \in \{1, \dots, n\}$ for which the inclusion $Z_i \subset X_j \cdot Z_{i+1}$ holds. Then:*

(a) *For every permutation $\sigma \in \mathfrak{S}_n$, the product map $X_{\sigma(1)} \times \cdots \times X_{\sigma(n)} \to X$ is surjective.*

(b) *If that map is injective for one permutation σ, it is injective for all σ.*

2.11. Indivisibility. Let $A \subset R$. A root $\beta \in A$ is said to be *indivisible in A* if $\beta \neq 0$ and, for all $\alpha \in A$, the relation $\alpha \big| \beta$ implies $\alpha = \beta$. We denote the set of indivisible roots in A by A_{ind}. Note that indivisibility depends very much on A:

$$B \subset A \quad \Longrightarrow \quad B \cap A_{\mathrm{ind}} \subset B_{\mathrm{ind}}, \tag{1}$$

and this is in general a proper inclusion. For example, if α and 2α belong to $R^\times$ then 2α is indivisible in $B = \{2\alpha\}$ while it is divisible in $A = \{\alpha, 2\alpha\}$.

2.12. Proposition. *Let G be a group with R-commutator relations and root groups U_α.*

(a) *If $A \subset R$ is a nilpotent subset of class at most k then U_A is a nilpotent subgroup of G of class at most k.*

(b) *Let A be a finite nilpotent subset of R. Then U_A is nilpotent of class at most* $\mathrm{Card}(A)$*, and for any ordering $A_{\mathrm{ind}} = \{\beta_1, \dots, \beta_n\}$,*

$$U_A = U_{A_{\mathrm{ind}}} = U_{\beta_1} \cdots\ U_{\beta_n}. \tag{1}$$

If the product map $U_{\beta_1} \times \cdots \times U_{\beta_n} \to U_A$ is injective for one ordering of A_{ind} then it is so for all orderings.

Proof. (a) A is strictly positive by (1.9.5), and (α, β) is a nilpotent pair for all $\alpha, \beta \in A$. Thus Lemma 2.9(b) is applicable to A, and the assertion follows from the definition of nilpotence of a given class in (1.9.8) which is analogous to the definition for groups.

(b) Let $h = \mathrm{Card}(A)$, order $A = \{\alpha_1, \ldots, \alpha_h\}$ as in Lemma 1.10(a), and consider the central chain $A_i = \{\alpha_i, \ldots, \alpha_h\}$ of A as in Lemma 1.10(b). By 2.9(b), the $Z_i := U_{A_i}$ form a central chain in U_A. Since $Z_1 = U_A$ and $Z_{h+1} = \{1\}$, U_A is nilpotent of class at most h. Now let $A_{\mathrm{ind}} = \{\beta_1, \ldots, \beta_n\}$ and put $X_j := U_{\beta_j}$, for $j = 1, \ldots, n$. By finiteness of A, for every $\alpha \in A$ there exists a $\beta \in A_{\mathrm{ind}}$ dividing α. Hence for every $i \in \{1, \ldots, h\}$ there exists some $j = j(i) \in \{1, \ldots, n\}$ such that $\beta_j \big| \alpha_i$, and therefore $X_j = U_{\beta_j} \supset U_{\alpha_i}$ (by (2.2.2)). This shows that the X_j generate U_A. The members of a central chain are normal subgroups. Hence $U_{\alpha_i} \cdot Z_{i+1}$ is a subgroup of U_A, and therefore

$$Z_i = \Big\langle U_{\alpha_i} \cup \cdots \cup U_{\alpha_n} \Big\rangle = \Big\langle U_{\alpha_i} \cup Z_{i+1} \Big\rangle = U_{\alpha_i} \cdot Z_{i+1} \subset X_{j(i)} \cdot Z_{i+1}.$$

Now the assertion follows from Tits' Lemma 2.10.

2.13. Corollary. *Let G be a group having R-commutator relations with root groups U_α.*

(a) *If $\{\alpha\}^c$ is finite then U_α is nilpotent.*

(b) *If $A \subset R$ is locally nilpotent in the sense of* 1.14 *then U_A is a locally nilpotent group.*

(c) *Let R be a locally finite root system and let (α, β) be a nilpotent pair which does not fall under the cases* 7 *or* 8 *of the table in* 4.8. *Then $U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$ is abelian. In particular, this is so if* Card $\boldsymbol{(}\alpha, \beta\boldsymbol{)} \leqslant 2$.

Proof. (a) We may assume $\alpha \neq 0$. The set $A := \{\alpha\}^c$ is finite by assumption, and obviously strictly positive, hence nilpotent by Proposition 1.12. As $A_{\mathrm{ind}} = \{\alpha\}$, the assertion follows from Proposition 2.12.

(b) We must show that every finite subset E of U_A is contained in a nilpotent subgroup. Now $E \subset U_F$ where F is a suitable finite subset of A. By Corollary 1.14, $B := F^c$ is nilpotent, and therefore so is U_B by Proposition 2.12(a). Since $U_F \subset U_B$ we are done.

(c) It follows from 4.8 that in the cases $1 - 6$, the set $A = \boldsymbol{(}\alpha, \beta\boldsymbol{)}$ is abelian. Hence by Proposition 2.12(a), U_A is abelian as well.

We will now show how nilpotent groups fit into our framework.

2.14. Corollary. *For $k \in \mathbb{N}_+$ let $R = \{0, 1, \ldots, k\} \subset \Bbbk$. Then any group G with R-commutator relations and generated by its root groups $(U_i)_{i \in R}$ is nilpotent of class at most k. Conversely, if G is a nilpotent group of class at most k then G has R-commutator relations with root groups $U_i = \mathscr{C}^i(G)$ for $i = 1, \ldots, k$.*

Proof. Let $A = R^\times = \{1, \dots, k\}$. One shows easily by induction that $\mathscr{C}^i(A) = \{i, \dots, n\}$, so A is nilpotent of class k. By Proposition 2.12(a), $G = U_A$ is nilpotent of class $\leqslant k$.

Conversely, let G be nilpotent of class $\leqslant k$ and put $U_i = \mathscr{C}^i(G)$ for $i = 1, \dots, k$ (and of course $U_0 = \{1\}$). Then the R-commutator relations hold. Indeed, the condition (2.2.2) follows from the fact that $U_i \supset U_j$ for $i \leqslant j$. To verify (2.2.3) observe that the nilpotent pairs of R are the pairs (i, j) with $1 \leqslant i, j \leqslant k$. They satisfy $i+j \in \big(i,j\big) \subset \{l \in \mathbb{N} : i+j \leqslant l \leqslant k\}$ if $i+j \leqslant k$, while $\big(i,j\big) = \emptyset$ if $i+j > k$. It now follows that

$$\big(U_i, U_j\big) = \big(\mathscr{C}^i(G), \mathscr{C}^j(G)\big) \subset \mathscr{C}^{i+j}(G) = U_{i+j} \subset U_{\left(i,j\right)}$$

in either case.

Remark. There is a similar result for a group G containing two nilpotent subgroups, say U^+ of class at most k and U^- of class at most l, respectively. Indeed, put $R = \{-l, \dots, -1, 0, 1, \dots, k\} \subset \Bbbk$. Then

$$G \text{ has } R\text{-commutator relations with root groups } U_{\pm i} = \mathscr{C}^i(U^\pm). \tag{1}$$

Conversely, if G has R-commutator relations with root groups U_i then the subgroups U_1 and U_{-1} are nilpotent of class $\leqslant k$ and of class $\leqslant l$ respectively. Details are left to the reader.

2.15. Groups with unique factorization. Let G have R-commutator relations and let $A \subset R$ be a finite nilpotent subset. We say G *has unique factorization for* A if there exists an enumeration $A_{\mathrm{ind}} = \{\gamma_1, \dots, \gamma_n\}$ such that the product map

$$\mu \colon U_{\gamma_1} \times \cdots \times U_{\gamma_n} \to U_A$$

is injective. Recall from Proposition 2.12(b) that μ is surjective and that the injectivity of μ is independent of the choice of enumeration. A group having unique factorization for all finite nilpotent subsets is said to have (unqualified) *unique factorization*.

It is convenient to introduce the following weaker form, called *unique factorization for nilpotent pairs*: For all nilpotent pairs (α, β), unique factorization holds for the sets $\big[\alpha, \beta\big]$ and $\big(\alpha, \beta\big)$.

In general, unique factorization for $\big[\alpha, \beta\big]$ will not imply this property for $\big(\alpha, \beta\big)$. For example, let $\alpha \in R^\times$ and assume that $\{\alpha\}^c = \{\alpha, 2\alpha, 3\alpha\}$. Then $\big[\alpha, \alpha\big]_{\mathrm{ind}} = \{\alpha\}$ so unique factorization for $\big[\alpha, \alpha\big]$ holds trivially, while $\big(\alpha, \alpha\big) = \{2\alpha, 3\alpha\} = \big(\alpha, \alpha\big)_{\mathrm{ind}}$, and so unique factorization for $\big(\alpha, \alpha\big)$ means that $U_{2\alpha} \cap U_{3\alpha} = \{1\}$. On the other hand,

$$\alpha \text{ and } \beta \text{ linearly independent over } \mathbb{Q} \quad \Longrightarrow \quad \big(\alpha, \beta\big)_{\mathrm{ind}} \subset \big[\alpha, \beta\big]_{\mathrm{ind}},$$

and hence unique factorization for $\big[\alpha, \beta\big]$ implies that for $\big(\alpha, \beta\big)$.

For the proof, let $\gamma \in \big(\alpha, \beta\big)_{\rm ind}$ and assume that γ is divisible by some $\delta \neq \gamma$ in $\big[\alpha, \beta\big]$. Thus $\delta = p\alpha + q\beta$ where $p, q \in \mathbb{N}$, and $\gamma = np\alpha + nq\beta$ for some $n \geqslant 2$. Since α and β are linearly independent over $\mathbb{Q}$, $\big(\alpha, \beta\big)$ contains no multiple of α or β, so we have $np \geqslant 1$ and $nq \geqslant 1$. But then also $p \geqslant 1$ and $q \geqslant 1$, whence $\delta \in \big(\alpha, \beta\big)$, contradicting the fact that γ is indivisible in $\big(\alpha, \beta\big)$.

As a consequence, we see: if R has Card $\big(\alpha, \beta\big) \leqslant 1$ for all $\mathbb{Q}$-linearly dependent nilpotent pairs, which is for instance the case when R is a locally finite root system by 4.8, then unique factorization for nilpotent pairs follows from that for all $\big[\alpha, \beta\big]$.

2.16. Examples. (a) The condition that the map μ of 2.15 be injective is trivially fulfilled for any finite nilpotent A with $\mathrm{Card}(A_{\rm ind}) = 1$. For example, let $R = \mathrm{A}_1 = \{0, 1, -1\}$, let V be a Jordan pair, and let $G = \mathrm{PE}(V)$. As noted in 2.3(a), G then has R-commutator relations with root groups $U_{\pm 1} = U^{\pm}$. Since $\{1\}$ and $\{-1\}$ are the only nilpotent subsets of R, $\mathrm{PE}(V)$ has unique factorization.

(b) Here is an example which shows that not every group has unique factorization for nilpotent pairs and that unique factorization for nilpotent pairs is weaker than unqualified unique factorization. Let $X = \Bbbk^n$ with standard basis $B = \{\varepsilon_1, \ldots, \varepsilon_n\}$ and let $R = \{0\} \cup B$. Then B is an abelian subset of R in the sense of 1.9. A group G with R-commutator relations is a group with a family of abelian subgroups $U_i = U_{\varepsilon_i}$ which commute pairwise. We may replace G by the subgroup generated by the U_i. Then G is commutative, and in additive notation, we have $G = \sum_{i=1}^n U_i$. The nilpotent pairs (α, β) are the pairs $(\varepsilon_i, \varepsilon_j)$, and since $\big(\varepsilon_i, \varepsilon_j\big) = \emptyset$, we have $\big[\varepsilon_i, \varepsilon_j\big] = \{\varepsilon_i, \varepsilon_j\}$, so that $U_{[\varepsilon_j, \varepsilon_j]} = U_i + U_j$. Hence G has unique factorization for all nilpotent pairs if and only if $U_i \cap U_j = \{0\}$ for $i \neq j$. On the other hand, G has (unqualified) unique factorization if and only if $G = \bigoplus_{i=1}^n U_i$.

(c) *Elementary linear groups.* Let I be an index set, let A be a unital associative ring and $M = A^{(I)}$ the free right A-module with standard basis $(e_i)_{i \in I}$. Let $\mathrm{E}_I(A) \subset \mathrm{GL}(M)$ be the elementary linear group, that is, the subgroup of $\mathrm{GL}(M)$ generated by all transvections

$$e_{ij}(a) = \mathrm{Id} + E_{ij}(a) \quad (a \in A, \ i \neq j),$$

where the $E_{ij}(a)$ are the usual matrix units mapping e_k to $\delta_{jk} e_i a$. Let $R = \dot{\mathrm{A}}_I = \{\varepsilon_i - \varepsilon_j : i, j \in I\}$ be the locally finite root system as in [**40**, 8.1], see also (4.7.1). Then it is well-known and easy to see (cf. [**17**]) that $G = \mathrm{E}_I(A)$ has $\dot{\mathrm{A}}_I$-commutator relations and root groups $U_{\varepsilon_i - \varepsilon_j} = e_{ij}(A)$ for $i \neq j$. Moreover, it is well-known that G has unqualified unique factorization. But we will give an elementary proof for the special case of unique factorization for nilpotent pairs now.

Thus, let (α, β) be a nilpotent pair in R. In the present situation, this means $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_k - \varepsilon_l$ where $i \neq j$, $k \neq l$, and $\alpha + \beta \neq 0$. It suffices to treat the case $\alpha \neq \beta$. By 4.8, either $\alpha + \beta \notin R^\times$ and then $\big(\alpha, \beta\big) = \emptyset$, or $\alpha + \beta \in R^\times$ and then $\big(\alpha, \beta\big) = \{\alpha + \beta\}$ (all this holds for any simply laced locally finite root system).

Case 1: $\alpha+\beta \notin R$. This is equivalent to $j \neq k$ and $i \neq l$. We must show that the multiplication map $U_\alpha \times U_\beta \to G$ is injective, which is equivalent to $U_\alpha \cap U_\beta = \{1\}$. Thus assume $e_{ij}(a) = e_{kl}(b) \in U_\alpha \cap U_\beta$. Applying this to e_j and e_l yields

$$\begin{aligned} e_{ij}(a) \cdot e_j = e_j + e_i a = e_{kl}(b) \cdot e_j = e_j + \delta_{jl} e_k b, \\ e_{ij}(a) \cdot e_l = e_l + \delta_{jl} e_i a = e_{kl}(b) \cdot e_l = e_l + e_k b. \end{aligned}$$

If $i \neq k$ then these equations imply $a = b = 0$, as required. If $i = k$ then necessarily $j \neq l$, otherwise $\alpha = \beta$. Hence these equations again show $a = b = 0$.

Case 2: $\gamma = \alpha + \beta \in R^\times$. Possibly after exchanging α and β we may assume $j = k$ and have $i \neq l$, so $\gamma = \varepsilon_i - \varepsilon_l$. We show that the multiplication map $U_\alpha \times U_\beta \times U_\gamma \to G$ is injective. Let $g = e_{ij}(a)e_{jl}(b)e_{il}(c)$. Then a simple computation shows that

$$g \cdot e_j = e_j + e_i a, \quad g \cdot e_l = e_l + e_j b + e_i(c + ab).$$

This shows that a, b, c are uniquely determined by g and proves our claim.

(d) Any of the groups considered in Examples 2.3(c)–(f) has unique factorization. Indeed, if G is a connected reductive algebraic group defined over a field k and so has commutator relations with respect to some finite root system R, it follows from [**42**, Lemma 3.4] that any nilpotent subset $A \subset R$ lies in a positive system of R. Then [**6**, Prop. 14.5] shows that the product map $\prod_{\gamma \in A} U_\gamma \to U_A$ is bijective if k is algebraically closed. The case of an arbitrary base field k then follows from the algebraically closed case. For split reductive group schemes as in Example 2.3(d) unique factorization is a consequence of [**13**, Exp. XXII, Prop. 5.5.1] while for Chevalley groups this is proven in [**55**, p. 24, Lemma 17].

For the groups in Example 2.3(e) one can argue as above: any nilpotent subset lies in a positive system so that it suffices to know that the product map $\prod_{\gamma \in P} U_\gamma \to U_P$ is bijective. This is for example proven in [**61**, 8.10] for $l = 2$ or [**63**, Prop. 11.11] in general. Other examples are discussed in 3.18.

2.17. Lemma. *If a subset A of R is the disjoint union of two closed subsets B and C, then*

$$A_{\mathrm{ind}} = B_{\mathrm{ind}} \mathbin{\dot\cup} C_{\mathrm{ind}}. \tag{1}$$

Proof. Since B and C are disjoint so is the union on the right hand side of (1). By (2.11.1) we have $B \cap A_{\mathrm{ind}} \subset B_{\mathrm{ind}}$ and $C \cap A_{\mathrm{ind}} \subset C_{\mathrm{ind}}$. This proves the inclusion from left to right. Conversely, let $\beta \in B_{\mathrm{ind}}$ and assume $\beta \notin A_{\mathrm{ind}}$. Then $\beta = n\alpha$ for some $\alpha \in A$ and $n \geqslant 2$. We cannot have $\alpha \in B$ because β is indivisible in B. Hence $\alpha \in C$. But then also $n\alpha \in C$ because C is closed, whence $\beta \in B_{\mathrm{ind}} \cap C = \emptyset$, a contradiction which proves $\beta \in A_{\mathrm{ind}}$. By symmetry, we have $C_{\mathrm{ind}} \subset A_{\mathrm{ind}}$, so the inclusion from right to left in (1) holds as well.

2.18. Corollary. *Let α and β be linearly independent over $\mathbb{Q}$. Then*

$$\big[\alpha, \beta\big]_{\mathrm{ind}} = \{\alpha\} \cup \big(\alpha, \beta\big)_{\mathrm{ind}} \cup \{\beta\}.$$

If moreover (α, β) *is a nilpotent pair and* G *is a group with* R*-commutator relations and root groups* U_α *and unique factorization for nilpotent pairs, then* $U_\beta \cap U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} = \{1\}$.

Proof. From linear independence it follows that the union $A := \boldsymbol{[}\alpha, \beta\boldsymbol{]} = \{\alpha\}^c \cup \boldsymbol{(}\alpha, \beta\boldsymbol{)} \cup \{\beta\}^c$ is disjoint. Moreover, $B := \{\alpha\}^c$ is closed by definition, and $C := \boldsymbol{(}\alpha, \beta\boldsymbol{)} \cup \{\beta\}^c$ is easily seen to be closed. By Lemma 2.17, $A_{\mathrm{ind}} = B_{\mathrm{ind}} \cup C_{\mathrm{ind}}$, and obviously $B_{\mathrm{ind}} = \{\alpha\}$. Repeating this argument for the disjoint decomposition $C = \boldsymbol{(}\alpha, \beta\boldsymbol{)} \dot\cup \{\beta\}^c$ into two closed subsets yields the first assertion. The second is then immediate from the definitions.

2.19. Lemma. *Let* G *be a group with* R*-commutator relations and assume* R *and* G *have the following property: For every finite non-empty nilpotent subset* A *of* R *there exists* $\alpha_0 \in A$ *such that*

(i) $B := A \setminus \{\alpha_0\}^c$ *is closed, and*
(ii) $U_{\alpha_0} \cap U_B = \{1\}$.

Then G *has unique factorization.*

Proof. Let A be a finite nilpotent subset of R and assume $\alpha_0 \in A$ satisfies condition (i). Then

$$A_{\mathrm{ind}} = \{\alpha_0\} \dot\cup B_{\mathrm{ind}} \tag{1}$$

by Lemma 2.17. We show by induction on $n = \mathrm{Card}(A_{\mathrm{ind}})$: there exists an enumeration $A_{\mathrm{ind}} = \{\alpha_1, \ldots, \alpha_n\}$ such that the multiplication map $U_{\alpha_1} \times \cdots \times U_{\alpha_n} \to G$ is injective.

This is trivial for $\mathrm{Card}(A_{\mathrm{ind}}) = 1$. Let $\mathrm{Card}(A_{\mathrm{ind}}) = n + 1$ and let $\alpha_0 \in A$ and B be as in (i). Observe that B is nilpotent, being a closed subset of a nilpotent set, cf. (1.9.4) and (1.9.2). By (1), B_{ind} has cardinality n, so by induction hypothesis, there exists an enumeration $B_{\mathrm{ind}} = \{\alpha_1, \ldots, \alpha_n\}$ such that the multiplication map $U_{\alpha_1} \times \cdots \times U_{\alpha_n} \to G$ is injective. Let $g_i, h_i \in U_{\alpha_i}$ for $i = 0, \ldots, n$, and assume $g_0 g_1 \cdots g_n = h_0 h_1 \cdots h_n$. Then $h_0^{-1} g_0 = h_1 \cdots h_n g_n^{-1} \cdots g_1^{-1} \in U_{\alpha_0} \cap U_B = \{1\}$. This implies $g_0 = h_0$, hence also $g_1 \cdots g_n = h_1 \cdots h_n$, and therefore, by induction, $g_i = h_i$ for $i = 1, \ldots, n$. Hence the multiplication map $U_{\alpha_0} \times \cdots \times U_{\alpha_n} \to G$ is injective.

Remark. Condition (i) is always fulfilled if R is reduced in the sense that $R \cap \mathbb{N}_+\alpha = \{\alpha\}$ for every $\alpha \in R^\times$. Indeed, choose $\alpha_0 \in A$ of minimal height with respect to a height function h (Corollary 1.13). If $B = A \setminus \{\alpha_0\}$ were not closed there would exist $\beta_1, \ldots \beta_p \in B$ such that $\beta_1 + \cdots + \beta_p = \alpha_0$. Applying h to this equation yields $p = 1$ and $\alpha_0 = \beta_1 \in B$, contradiction.

2.20. Locally nilpotent endomorphisms and exponentials. Let V be a vector space over a field $\Bbbk$ of characteristic 0. Recall, see e. g. [**46**, 1.5], that an endomorphism f of V is called locally nilpotent if for every $v \in V$ there exists $m \in \mathbb{N}$, possibly depending on v, such that $f^m(v) = 0$. In this case the exponential of f,

$$\exp(f) = \sum_{n\in\mathbb{N}} \frac{f^n}{n!},$$

is a well-defined invertible endomorphism of V with inverse $\exp(f)^{-1} = \exp(-f)$. If d is a locally nilpotent derivation of a $\Bbbk$-algebra A, one knows that $\exp d \in \mathrm{Aut}(A)$. For example, if L is a Lie algebra over $\Bbbk$ and $x \in L$ is locally ad-nilpotent, i.e., $\operatorname{ad} x$ is locally nilpotent, then $\exp \operatorname{ad} x$ is a so-called elementary automorphism of L.

If f and g are locally nilpotent and commuting endomorphisms of V, then $\exp(f)\exp(g) = \exp(f+g)$ holds. In general, $\exp(f)\exp(g)$ need not be an exponential. The cases where this is still true are governed by the theory of the Hausdorff series, which we quickly review.

Let $\mathfrak{F}$ be the free Lie $\Bbbk$-algebra on two generators X, Y and denote by $\widehat{\mathfrak{F}}$ its standard completion [**8**, II, §6.2]. Then $\exp X \exp Y = \exp H(X,Y)$ where $H(X,Y) \in \widehat{\mathfrak{F}}$ is the Hausdorff series. One knows

$$H(X,Y) = \sum_{n\geqslant 1} H_n(X,Y) = X + Y + \frac{1}{2}[X,Y] + \cdots \tag{1}$$

where the $H_n(X,Y)$ are rational linear combinations of higher order commutators in X and Y, homogeneous of total degree n (thus $H_1(X,Y) = X+Y$ and $H_2(X,Y) = \frac{1}{2}[X,Y]$). Their precise form is for example given in [**8**, II, §6.4, Th. 2]. It follows from that theorem that $H_n(X,Y) \in \mathscr{C}^n(\mathfrak{F})$, the nth term of the central descending series of $\mathfrak{F}$.

If x, y are elements of a $\Bbbk$-Lie algebra L, we denote by $H_n(x,y)$ the element in $\mathscr{C}^n(L)$ obtained by the substitution $(X,Y) \mapsto (x,y)$. If L is nilpotent, only finitely many $H_n(x,y)$ are different from zero, thus $H(x,y)$ is a finite sum. The form of the $H_n(x,y)$ shows in particular that, for ideals L_i of L and $x_i \in L_i$,

$$H(x_1,x_2) \equiv x_1 + x_2 \mod [L_1, L_2]. \tag{2}$$

2.21. Lemma. *Let L be a nilpotent Lie algebra and let $\varrho\colon L \to \mathfrak{gl}(V)$ be a representation such that $\varrho(x)$ is locally nilpotent for all $x \in L$. Then for all $x, y \in L$,*

$$\exp \varrho(x) \exp \varrho(y) = \exp \varrho(H(x,y)) \tag{1}$$

in $\mathrm{GL}(V)$.

Proof. Since L and $\varrho(L)$ are nilpotent, it follows from 2.20 that $H(x,y)$ and $H(\varrho(x),\varrho(y))$ are well-defined elements of L and $\varrho(L) \subset \mathfrak{gl}(V)$, respectively. Because $\varrho(H_n(x,y)) = H_n(\varrho(x),\varrho(y))$ holds for all $n \in \mathbb{N}$, we get that $\varrho(H(x,y)) = H(\varrho(x),\varrho(y))$ is a locally nilpotent endomorphism of V, whence $\exp \varrho(H(x,y)) = \exp(H(\varrho(x),\varrho(y))$ is well-defined. The equality (1) can then be checked for each $v \in V$.

We can now get another example of a group with commutator relations.

2.22. Proposition. *Let Λ be a torsion-free abelian group, $\Bbbk$ a field of characteristic zero. In order to apply the results of section 1 we view Λ canonically embedded in the $\Bbbk$-vector space $\Lambda \otimes_{\mathbb{Z}} \Bbbk$. Assume further that $L = \bigoplus_{\lambda \in \Lambda} L_\lambda$ is a Λ-graded Lie algebra, that $\varrho \colon L \to \mathfrak{gl}(V)$ is a representation of L written as $\varrho(x)(v) = x \cdot v$ for $x \in L$ and $v \in V$, and that R is a subset of $S := \{0\} \cup \{\lambda \in \Lambda : L_\lambda \neq 0\}$ satisfying $0 \in R$ and the following conditions:*

- (i) *$\varrho(L_\alpha)$ consists of locally nilpotent endomorphisms of V, for all $\alpha \in R^\times$,*
- (ii) *for any nilpotent pair (α, β) in R we have*
  - (ii.1) *$S \cap \{p\alpha + q\beta : p, q \in \mathbb{N}, p + q \geqslant 1\} \subset R$, i.e., the root interval $[\![\alpha, \beta]\!]_S$ calculated in S is the same as the root interval $[\![\alpha, \beta]\!]_R$ calculated in R, and*
  - (ii.2) *for every $x \in \bigoplus_{\gamma \in (\!(\alpha, \beta)\!)} L_\gamma$, the endomorphism $\varrho(x)$ is locally nilpotent,*
- (iii) *$L_{n\alpha} = \{0\}$ for $n \geqslant 2$ and $\alpha \in R^\times$.*

(a) *Then $\mathrm{GL}(V)$ has R-commutator relations with respect to the family of abelian subgroups $\mathscr{U} = (U_\alpha)_{\alpha \in R}$ defined by $U_0 = \{1\}$ and $U_\alpha = \exp \varrho(L_\alpha)$ for $\alpha \neq 0$.*

(b) *Assume that $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$ is Λ-graded and that ϱ is a graded representation, i.e., $L_\alpha \cdot V_\lambda \subset V_{\alpha+\lambda}$ for all $\alpha, \lambda \in \Lambda$. Then $(\mathrm{GL}(V), \mathscr{U})$ has unique factorization in the sense of* 2.15.

Proof. (a) That the U_α are subgroups of $\mathrm{GL}(V)$ follows from the identities mentioned in 2.20: $(\exp f)^{-1} = \exp(-f)$ and $\exp f \exp g = \exp(f+g)$ for $f = \varrho(x)$, $g = \varrho(y)$, $x, y \in L_\alpha$. The latter identity holds since $[\varrho(x), \varrho(y)] \in \varrho(L_{2\alpha}) = 0$ in view of the assumption (iii).

If $\beta = n\alpha$, $n \geqslant 2$, then $L_\beta = \{0\}$ by (iii), so $U_\beta = \{1\} \subset U_\alpha$ shows (2.2.2). Thus it remains to show the relation (2.2.3) for a nilpotent pair (α, β) in R, i.e., $[\![\alpha, \beta]\!]$ is a nilpotent subset of R. For any subset $A \subset [\![\alpha, \beta]\!]$ we put $L_A = \bigoplus_{\gamma \in A} L_\gamma$. In order to apply Lemma 2.21, we first observe that

$$L_{[\![\alpha,\beta]\!]} \text{ is a nilpotent subalgebra of } L.$$

Indeed, for arbitrary subsets $A, B \subset [\![\alpha, \beta]\!]$ we have

$$[L_A, L_B] \subset L_{(\!(A,B)\!)}$$

using the assumption (ii.1) and that L is Λ-graded. In particular, L_A for $A = [\![\alpha, \beta]\!]$ is a subalgebra of L. Denoting by $\mathscr{C}^n$ the nth term of the central descending series, nilpotency of L_A will follow from $\mathscr{C}^n(L_A)) \subset L_{\mathscr{C}^n(A)}$ which is proved by induction: $\mathscr{C}^{n+1}(L_A) = [L_A, \mathscr{C}^n(L_A)] \subset [L_A, L_{\mathscr{C}^n(A)}] \subset L_{(\!(A, \mathscr{C}^n(A))\!)} = L_{\mathscr{C}^{n+1}(A)}$. Next we show

$$\exp \varrho(x) \in U_{(\!(\alpha,\beta)\!)} \text{ for } x \in L_{(\!(\alpha,\beta)\!)}. \tag{1}$$

We abbreviate $B = (\!(\alpha, \beta)\!)$, and enumerate $B = \{\beta_1, \ldots, \beta_n\}$ as in Lemma 1.10(b). Thus $B_i = \{\beta_i, \ldots, \beta_n\}$, $1 \leqslant i \leqslant n$ and $B_i = \emptyset$ for $i > n$ is a central chain for B. In particular, $\big(B, B_i\big) \subset B_{i+1}$ holds, which implies that $K_i = L_{B_i}$ is an ideal of the nilpotent subalgebra $K_1 = L_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$ with $[K, K_i] \subset K_{i+1}$. To prove (1), it suffices to show by downward induction on i that

$$\exp \varrho(x) \in U_{B_i} = \big\langle U_\gamma : \gamma \in B_i \big\rangle \text{ for } x \in K_i. \tag{2}$$

Write $x \in K_i$ in the form $x = x_i + x'$ with $x_i \in L_{\beta_i}$ and $x' \in K_{i+1}$. From (2.20.2) we know that $H(-x_i, x) \equiv x' \bmod [K_i, K_i] \subset K_{i+1}$, and therefore $H(-x_i, x) \in K_{i+1}$. Applying (2.21.1) now yields $\exp \varrho(-x_i) \exp \varrho(x) = \exp \varrho(H(-x_i, x)) \in U_{B_{i+1}}$ proving (2) and thus also (1).

Finally, we are ready to show the commutator relation (2.2.3). Let $x \in L_\alpha$ and $y \in L_\beta$. Then, by [**46**, Proposition 6.1.2] applied to the nilpotent Lie algebra $L_{\boldsymbol{[}\alpha,\beta\boldsymbol{]}}$, we obtain

$$\exp(\varrho(x))\, \varrho(y)\, (\exp \varrho(x))^{-1} = \varrho(z), \quad z = (\exp \operatorname{ad} x)(y) \in L_{\boldsymbol{[}\alpha,\beta\boldsymbol{]}},$$

whence also $\exp(\varrho(x))\, \varrho(y)^n\, (\exp \varrho(x))^{-1} = \varrho(z)^n$, and since $\varrho(y)$ and $\varrho(z)$ are locally nilpotent we get

$$\exp(\varrho(x))\, \exp(\varrho(y))\, (\exp \varrho(x))^{-1} = \exp \varrho(z).$$

Therefore, by (2.21.1),

$$\big(\exp \varrho(x), \exp \varrho(y)\big) = \exp \varrho(z)\, \exp \varrho(-y) = \exp \varrho(H(z, -y))$$

But $z = \sum_{n \in \mathbb{N}} \frac{1}{n!} (\operatorname{ad} x)^n (y) \equiv y \bmod L_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$, whence $H(z, -y) \in [L_{\boldsymbol{[}\alpha,\beta\boldsymbol{]}}, L_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}] \subset L_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$. Then (1) shows $\exp \varrho(H(z, -y)) \in U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$.

(b) Let $A \subset R$ be a finite nilpotent subset. By assumption (iii), there is no harm in assuming $A = A_{\mathrm{ind}}$. Let $A = \{\alpha_1, \ldots, \alpha_n\}$ be an enumeration of A as in Lemma 1.10(b). As explained in 2.15, it is sufficient to show that the product map μ: $U_{\alpha_1} \times \cdots \times U_{\alpha_n} \to U_A$ is injective. This will follow from the claim that the product map

$$U_{\alpha_i} \times U_{\alpha_{i+1}} \times \cdots \times U_{\alpha_n} \to U_{A_i}, \quad A_i = \{\alpha_i, \ldots, \alpha_n\}, \tag{3}$$

is injective for all i, $1 \leqslant i \leqslant n$. We will prove (3) by downward induction from $i = n$ to $i = 1$. Thus, suppose $\exp \varrho(x_i) u_{i+1} = \exp \varrho(x_i') u_{i+1}'$ for some $x_i, x_i' \in L_{\alpha_i}$ and $u_{i+1}, u_{i+1}' \in U_{A_{i+1}}$. Then $\exp \varrho(x_i - x_i') = u_{i+1}^{-1} u_{i+1}' \in U_{A_{i+1}}$ because all U_{A_i} are subgroups of U_A. Let $v_\lambda \in V_\lambda$. Then $\exp \varrho(x_i - x_i')(v_\lambda) = v_\lambda + \varrho(x_i - x_i')(v_\lambda) + \cdots \in \bigoplus_{n \in \mathbb{N}} V_{\lambda + n\alpha_i}$ while $u_{i+1}^{-1} u_{i+1}'(v_\lambda) \in \bigoplus_\mu V_\mu$, the sum being taken over all μ of the form $\mu = \lambda + m_{i+1}\alpha_{i+1} + \cdots + m_n \alpha_n$, $m = (m_{i+1}, \ldots, n_n) \in \mathbb{N}^{n-i}$. Hence, if $0 \neq \varrho(x_i - x_i')(v_\lambda) \in V_{\lambda + \alpha_i}$, then $\alpha_i = m_{i+1}\alpha_{i+1} + \cdots + m_n \alpha_n$ for some $m = (m_{i+1}, \ldots, m_n) \in \mathbb{N}^{n-i}$. At least two of the components m_j of m are non-zero, whence $\alpha_i \in \big(A, A_j\big) \subset A_{j+1}$ for some $j > i$. contradiction. Thus $\varrho(x_i - x_i') = 0$, $\exp \varrho(x_i) = \exp \varrho(x_i')$, and injectivity of the map in (3) follows.

We will consider two types of Lie algebras to which the proposition applies, Kac-Moody algebras in 2.23 and extended affine Lie algebras in 2.26.

2.23. Example: Category $\mathscr{O}$-representation of a Kac-Moody algebra. Let L be a Kac-Moody algebra over $\Bbbk$ [**26**, **46**] with standard Cartan subalgebra H, and let $\varrho\colon L \to \mathfrak{gl}(V)$ be an integrable representation of L in category $\mathscr{O}$. For $\sigma \in H^*$ let $L_\sigma = \{x \in L : [h,x] = \sigma(h)x \text{ for all } h \in H\}$ and let $S = \{\sigma \in H^* : L_\sigma \neq 0\}$. It is known that S is a reflection system, see Section 4, in fact even a partial root system in the sense of [**42**, §3]. In particular, $0 \in S$ since $L_0 = H \neq 0$. We will show that Proposition 2.22 can be applied with $R = \mathrm{Re}(S)$.

The root space decomposition $L = \bigoplus_{\alpha \in H^*} L_\alpha$ with respect to H is a grading with grading group $\Lambda = H^*$. It is well-known that assumption (iii) of Proposition 2.22 holds, see for instance [**26**, Prop. 5.1], while the condition (ii.1) follows from [**42**, Theorem 3.7]. By definition a representation $\varrho\colon L \to \mathfrak{gl}(V)$ is integrable if it satisfies the following two conditions:

(i) V has a weight space decomposition $V = \bigoplus_{\lambda \in H^*} V_\lambda$ with respect to the action of H, in particular the assumption 2.22(b) holds: $L_\alpha \cdot V_\lambda \subset V_{\alpha+\lambda}$.

(ii) Let $B = \{\alpha_1, \dots, \alpha_l\}$ be the standard root basis of S. Then $L_{\pm\alpha_i}$ acts by locally nilpotent endomorphisms. It is known, see e.g. [**46**, Proposition 6.1.3], that then also condition (i) of Proposition 2.22 holds.

To establish assumption (ii.2) of that proposition we use that ϱ is a representation in the category $\mathscr{O}$. Denoting by $\mathscr{P}(V) = \{\lambda \in H^* : V_\lambda \neq 0\}$ the set of weights of V and putting $\mathscr{Q}_+ = \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_l$, a requirement for ϱ to be in $\mathscr{O}$ is that there exist $\lambda_1, \dots, \lambda_s \in H^*$ such that

$$\mathscr{P}(V) \subset \bigcup_{i=1}^{s} (\lambda_i - \mathscr{Q}_+). \tag{1}$$

Let now (α, β) be a nilpotent pair in R. Assume first that α, β are positive roots, i.e., $\alpha, \beta \in R_+ = R \cap \mathscr{Q}_+$. It is then clear from (1) that $\varrho(x)$ for $x \in \bigoplus_{\gamma \in (\!(\alpha,\beta)\!)} L_\gamma$ is locally nilpotent. For an arbitrary nilpotent pair (α, β) there exists $w \in W(R)$ such that $w([\![\alpha, \beta]\!]) \subset R_+$ [**42**, Theorem 3.9]. It is known, see e.g. [**46**, Proposition 4.1.4], that there exists an elementary automorphism w_L of L such that $w_L(L_\sigma) = L_{w(\sigma)}$ for all $\sigma \in S$. Moreover, there also exist $w_V \in \mathrm{GL}(V)$ satisfying $\varrho(w_L y) = w_V\, \varrho(y) w_V^{-1}$ for all $y \in L$, see e.g. [**26**, 3.8]. Condition (ii.2) follows by combining these results.

We can now apply Proposition 2.22 and obtain that $\mathrm{GL}(V)$ *is a group with commutator relations with respect to the family* $\mathscr{U} = (U_\alpha)_{\alpha \in R}$ *of root groups* $U_\alpha = \exp \varrho(L_\alpha)$ *for* $\alpha \in R^\times$ *and* $U_0 = \{1\}$. *Moreover,* $(\mathrm{GL}(V), \mathscr{U})$ *has unique factorization.*

For our second example, we assume that the support set $\mathrm{supp}_\Lambda L = \{\lambda \in \Lambda : L_\lambda \neq 0\}$ is an affine reflection system. A structure theory of affine reflection systems is developed in [**42**, §5], to which the reader is referred for all unexplained notions used in the following lemma and its proof.

2.24. Lemma. *Let (R, X) be an affine reflection system and let $\alpha, \beta \in R^{\mathrm{re}}$. We denote by (S, Y) the quotient root system of (R, X) and by $f\colon (R, X) \to (S, Y)$ the canonical projection.*

Then (α, β) is a nilpotent pair of (R, X) if and only if $(f(\alpha), f(\beta))$ is a nilpotent pair in (S, Y). In this case $[\![\alpha, \beta]\!] \subset R^{\mathrm{re}}$.

Nilpotent pairs in root systems are characterized in 4.8.

Proof. We first take care of the easy direction, namely assume that $(f(\alpha), f(\beta))$ is a nilpotent pair of (S, Y), i.e., $f(\{\alpha, \beta\})$ is a prenilpotent subset of (S, Y). By (1.9.3), $\{\alpha, \beta\} \subset R$ is a prenilpotent subset, hence (α, β) is a nilpotent pair. We also know $f([\![\alpha, \beta]\!]) = f(\{\alpha, \beta\}^c) \subset \{f(\alpha), f(\beta)\}^c = [\![f(\alpha), f(\beta)]\!]$ by (1.5.3). Since $0 \notin [\![f(\alpha), f(\beta)]\!]$, we conclude that $[\![\alpha, \beta]\!] \subset R^{\mathrm{re}}$.

Let us now assume that (α, β) is a nilpotent pair in R. To show that then $(f(\alpha), f(\beta))$ is a nilpotent pair in S, we recall that (R, X) is isomorphic to the extension of the root system (S, Y) by an extension datum $(\Lambda_\xi)_{\xi \in S}$ of type (S, S_{ind}, Z). To simplify the notation we assume that (R, X) is in fact equal to this extension. Thus, $R = \bigcup_{\xi \in S} \xi \oplus \Lambda_\xi \subset X = Y \oplus Z$, $f\colon X \to Y$ is the projection along Z, and $\gamma \in R^{\mathrm{re}} \iff f(\gamma) \in S^\times = S^{\mathrm{re}}$. We write α, β as $\alpha = \xi + \lambda_\alpha$ and $\beta = \tau + \lambda_\beta$ with $\xi, \tau \in S^\times$, $\lambda_\alpha \in \Lambda_\xi$ and $\lambda_\beta \in \Lambda_\tau$.

The characterization of nilpotent pairs in root systems (4.8) says that (ξ, τ) is a nilpotent pair in (S, Y) if and only if $\{\xi, \tau\}$ is $\mathbb{N}$-free. To finish the proof of the lemma, it is therefore sufficient to show that if $\tau = -s\xi$ for $s \in \{1, 2\}$ then (α, β) is not nilpotent, i.e., $[\![\alpha, \beta]\!]$ is infinite or contains zero. For $n \in \mathbb{N}$ and $m = 1 + sn \in \mathbb{N}$ we claim that $m\alpha + n\beta = \xi \oplus ((1 + ns)\lambda_\alpha + n\lambda \in R^{\mathrm{re}}$, i.e.,

$$\lambda_n := (1 + ns)\lambda_\alpha + n\lambda_\beta \in \Lambda_\xi \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

For the proof of (1) we will use that Λ_ξ is a symmetric reflection subspace of $(Z, +)$, i.e., $2\Lambda_\xi - \Lambda_\xi \subset \Lambda_\xi = -\Lambda_\xi$, which contains $\Lambda_{s\xi} = \Lambda_{-s\xi}$, in particular $\lambda_\beta \in \Lambda_\xi$. We will prove (1) by induction. For $s = 1$ we have $\lambda_0 = \lambda_\alpha \in \Lambda_\xi$, $\lambda_1 = 2\lambda_\alpha + \lambda_\beta \in \Lambda_\xi$, and hence by induction $\lambda_{2p} = (2p+1)\lambda_\alpha + 2p\lambda_\beta = 2((p+1)\lambda_\alpha + p\lambda_\beta) - \lambda_\alpha = 2\lambda_p - \lambda_\alpha \in \Lambda_\xi$ and $\lambda_{2p+1} = (2p+2)\lambda_\alpha + (2p+1)\lambda_\beta = 2((p+1)\lambda_\alpha + p\lambda_\beta) + \lambda_\beta = 2\lambda_p + \lambda_\beta \in \Lambda_\xi$. The case $s = 2$ can be shown in the same way. If $\{\lambda_n : n \in \mathbb{N}\}$ is not infinite, there exist $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$ such that $\lambda_{n_1} = \lambda_{n_2}$, whence $(n_1 - n_2)s\lambda_\alpha = (n_2 - n_1)\lambda_\beta$ and therefore $\lambda_\beta = -s\lambda_\alpha$. But then $0 = (s\xi + \tau) + (s\lambda_\alpha + \lambda_\beta) = s\alpha + \beta \in [\![\alpha, \beta]\!]$.

2.25. Corollary. *Let Λ be a torsion-free abelian group, and let $L = \bigoplus_{\lambda \in \Lambda} L_\lambda$ be a Λ-graded Lie algebra defined over a field $\Bbbk$ of characteristic zero such that $R = \{\lambda \in \Lambda : L_\lambda \neq 0\}$ is an affine reflection system in $\Lambda \otimes_{\mathbb{Z}} \Bbbk$.*

(a) *Then $\operatorname{ad} x$ is nilpotent of class $\leqslant 4$ for all $x \in L_\alpha$, $\alpha \in R^{\mathrm{re}}$, and nilpotent of class $\leqslant 10$ for all $x \in \bigoplus_{\gamma \in [\![\alpha,\beta]\!]} L_\gamma$, (α, β) a nilpotent pair in R with $\alpha, \beta \in R^{\mathrm{re}}$.*

(b) *If R is reduced, then $(\mathrm{Aut}(L), \mathscr{U})$ is a group with $\mathrm{Re}(R)$-commutator relations with respect to the family $\mathscr{U} = (U_\alpha)_{\alpha \in \mathrm{Re}(R)}$ of subgroups defined by*

$U_\alpha = \{\exp \operatorname{ad} x : x \in L_\alpha\}$, $\alpha \in R^{\mathrm{re}}$. *Moreover,* $(\operatorname{Aut}(L), \mathscr{U})$ *has unique factorization.*

Proof. As in the proof of Lemma 2.24, it is no harm to assume that R is the extension of the root system S by the extension datum $(\Lambda_\xi)_{\xi \in S}$.

(a) Let $\alpha = \xi \oplus \lambda \in R^{\mathrm{re}}$ with $\xi \in S^{\mathrm{re}}$, $\lambda \in \Lambda_\xi$ and let $x \in L_\alpha$. For an arbitrary $\gamma = \zeta \oplus \nu \in R$ we get $(\operatorname{ad} x)^n(L_\gamma) \subset L_{(\zeta + n\xi)\oplus(\nu + n\lambda)}$. Hence $(\operatorname{ad} x)(L_\gamma) = 0$ as soon as $\zeta + n\xi \notin S$, which by [**10**, VI, §1.3, Cor. de la Prop. 9] or [**40**, A.5] is always the case for $n > 4$.

Let now (α, β) be a nilpotent pair in R with $\alpha, \beta \in R^{\mathrm{re}}$ and let $f\colon R \to S$ be the canonical projection. By Lemma 2.24, $(f(\alpha), f(\beta)) =: (\xi, \tau)$ is a nilpotent pair of S. We claim that $(\operatorname{ad} x)^{11}(y) = 0$ for any $y \in L$. To prove this, we may assume that y is homogeneous, say $y \in L_\gamma$. Let $S' = S \cap (\Bbbk\alpha + \Bbbk\beta + \Bbbk\gamma)$. Then S' is a closed subsystem of S, whence $L' = \bigoplus_{f(\delta) \in S'} L_\delta$ is a subalgebra of L containing y and any $x \in L_{[\![\alpha,\beta]\!]} = \bigoplus_{\delta \in [\![\alpha,\beta]\!]} L_\delta$. Hence, to show that $(\operatorname{ad} x)^{11}(y) = 0$, we may without loss of generality assume that S is finite of rank $\leqslant 3$.

By Proposition 1.12, $[\![\xi, \tau]\!]$ is a strictly positive subset of S. Hence by [**40**, Proposition 10.13], $[\![\xi, \tau]\!]$ is contained in a positive system, which by [**40**, Lemma 11.1] coincides with the set of non-negative roots with respect to some root basis B of R. Let $\operatorname{ht}\colon \mathbb{Z}[S] \to \mathbb{Z}$ be the corresponding height function, defined by $\operatorname{ht}(B) = 1$. For any $z = \sum_{\delta \in R} z_\delta \in L$ we define the level $l(z)$ of z by $l(z) = \min\{\operatorname{ht}\big(f(\delta)\big) : z_\delta \neq 0\}$. Then $x \in L_{[\![\alpha,\beta]\!]}$ has positive level and $l([x, z]) > l(z)$ holds for all $z \in L$, in particular for $y \in L_\gamma$. Since the height and hence the level is bounded, there exists $n \in \mathbb{N}$ such that $(\operatorname{ad} x)^n(y) = 0$. More precisely, it is immediate from the classification of root systems of rank $\leqslant 3$ that the maximal height in S is 5, so certainly $n = 11$ will do.

(b) Let R be reduced. To verify condition (iii) of Proposition 2.22, let $\alpha \in R^{\mathrm{re}}$ and $n \in \mathbb{N}_+$. We can assume $n\alpha \in R$ since otherwise $L_{n\alpha} = \{0\}$. But then $n\alpha \in R^{\mathrm{re}}$ follows, whence $n = 1$ by definition of a reduced reflection system in 4.2. Thus condition (iii) of 2.22 holds. For ϱ the adjoint representation the other conditions follow from (a) and Lemma 2.24.

2.26. Example: The automorphism group of extended affine Lie algebras. Let L be an affine reflection Lie algebra in the sense of [**48**, §6]. Hence L contains an ad-diagonalizable (= split toral) subalgebra $H \neq \{0\}$, inducing a decomposition $L = \bigoplus_{\alpha \in H^*} L_\alpha$ where $L_\alpha = \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}$ such that $R = \{\alpha \in H^* : L_\alpha \neq 0\}$ is an affine reflection system. Thus the assumptions of Corollary 2.25 are fulfilled in case R is reduced.

Any extended affine Lie algebra E [**48**, 6.11] is an example of an affine reflection Lie algebra with a reduced R. Hence $\operatorname{Aut}(E)$ *is a group with commutator relations and unique factorization.* One can also consider suitably defined classes of integrable representations of an extended affine Lie algebra to which Proposition 2.22 applies, in the spirit of 2.23. Details are left to the interested reader.

§3. Categories of groups with commutator relations

3.1. The categories $\mathbf{gc}_R$ and gc. Let $(R, X) \in \mathbf{SV}$. We define a category $\mathbf{gc}_R$ as follows: its objects are pairs $\big(G, (U_\alpha)_{\alpha\in R}\big)$ consisting of a group G having R-commutator relations with respect to the family $(U_\alpha)_{\alpha\in R}$ of subgroups, called root groups. Its morphisms

$$\varphi\colon (G, (U_\alpha)_{\alpha\in R}) \to (G', (U'_\alpha)_{\alpha\in R})$$

are group homomorphisms $\varphi\colon G \to G'$ preserving root groups: $\varphi(U_\alpha) \subset U'_\alpha$ for all $\alpha \in R$. By abuse of notation, we often do not indicate explicitly the root groups of an object of $\mathbf{gc}_R$ and thus simply write $G \in \mathbf{gc}_R$ etc. As in 2.2, we put $\mathfrak{r}(G) = U_R$ (which has again R-commutator relations with the same root groups) and say G is tight if $G = \mathfrak{r}(G)$. The tight groups in $\mathbf{gc}_R$ form a full coreflective subcategory. We leave it to the ardent reader to elaborate on this theme ...

Next, we define a category $\mathbf{gc}$ encompassing all $\mathbf{gc}_R$. The objects of $\mathbf{gc}$ are triples $\big((R, X), G, (U_\alpha)_{\alpha\in R}\big)$ where

(i) $(R, X) \in \mathbf{SV}$,
(ii) $\big(G, (U_\alpha)_{\alpha\in R}\big) \in \mathbf{gc}_R$.

A *morphism* from $\big((R, X), G, (U_\alpha)_{\alpha\in R}\big)$ to $\big((S, Y), H, (V_\xi)_{\xi\in S}\big)$ in $\mathbf{gc}$ is a pair (f, φ), where $f\colon (R, X) \to (S, Y)$ is a morphism of $\mathbf{SV}$ and $\varphi\colon G \to H$ is a group homomorphism, such that

$$\varphi(U_\alpha) \subset V_{f(\alpha)} \quad \text{for all } \alpha \in R. \tag{1}$$

By our convention that $U_0 = \{1\}$ (cf. (2.2.1)), this means in particular that

$$f(\alpha) = 0 \quad\Longrightarrow\quad U_\alpha \subset \operatorname{Ker}(\varphi). \tag{2}$$

It is easily verified that, together with the natural composition of morphisms, this defines indeed a category. To improve readability we will sometimes abbreviate an object $\big((R, X), G, (U_\alpha)_{\alpha\in R}\big)$ of $\mathbf{gc}$ by $(R, G, (U_\alpha))$ or even by (R, G). Likewise, we will sometimes refer to an object (R, X) of $\mathbf{SV}$ simply by R, omitting the vector space X. Thus $\mathbf{gc}$ is the category of all groups with R-commutator relations, for all possible $(R, X) \in \mathbf{SV}$. Strictly speaking, $\mathbf{gc}$ still depends on the choice of the base field $\Bbbk$, but this dependency will be suppressed in the notation.

The assignments $(R, G) \mapsto R$ and $(f, \varphi) \mapsto f$ define a covariant functor

$$\mathit{\Pi} : \mathbf{gc} \to \mathbf{SV}. \tag{3}$$

For fixed $R \in \mathbf{SV}$ we may identify $\mathbf{gc}_R$ with the fibre of $\mathit{\Pi}$ at R, i.e., with the subcategory of $\mathbf{gc}$ whose objects are of the form $\big(R, G, (U_\alpha)\big)$, and whose morphisms are the morphism (Id_R, φ) of $\mathbf{gc}$.

The category $\mathbf{gc}$ admits the following direct sum construction. However, even if the index set I is finite, this construction does not have the categorical properties of a direct product or coproduct.

3.2. Lemma. *Let I be an arbirary index set, and let G_i be groups with R_i-commutator relations and root groups $U^{(i)}_\alpha$, $\alpha \in R_i$. Let $G = \bigoplus_{i\in I} G_i$ be their restricted direct product, i.e., the subgroup of the full direct product $\prod_{i\in I} G_i$ whose elements have only finitely many components different from 1, and let $R = \bigcup_{i\in I} R_i$ be the direct sum of the R_i as in 4.3. Identify G_i with the subgroup of G obtained by injection into the ith factor. Then G has R-commutator relations with root groups*

$$U_\alpha = U^{(i)}_\alpha$$

for $\alpha \in R_i \subset R$. Any family of morphisms $\varphi_i\colon G_i \to G'_i$ in $\mathbf{gc}_{R_i}$ canonically yields a morphism $\bigoplus_{i\in I} \varphi_i\colon \bigoplus_i G_i \to \bigoplus_i G'_i$ of the category $\mathbf{gc}_R$.

Proof. If $\alpha\big|\beta$ and $\alpha \neq 0$, then α and β lie in the same component of R, say $\alpha, \beta \in R_i$, so that (2.2.2) holds by definition of the root groups. To prove (2.2.3), suppose (α, β) is a nilpotent pair, say, $\alpha \in R_i$ and $\beta \in R_j$. If $i = j$ then the commutator relation follows from the one in G_i. If $i \neq j$ then $\big(\alpha, \beta\big) = \emptyset$ and $\big(U_\alpha, U_\beta\big) = \{1\}$ by definition of the restricted direct product. The last statement is evident.

3.3. Proposition. *Let $(G, U_\alpha) \in \mathbf{gc}_R$ be a group with R-commutator relations and let $f\colon (R, X) \to (S, Y)$ be a morphism of $\mathbf{SV}$. For all $\xi \in S$ define subsets $R[\xi]$ of R by*

$$R[\xi] := R[\xi, f] := \{\alpha \in R : f(\alpha) \neq 0 \text{ and } \xi\big|f(\alpha)\}, \tag{1}$$

in particular $R[0] = \emptyset$, and define subgroups U'_ξ of G by

$$U'_\xi := U_{R[\xi]}. \tag{2}$$

(a) *Then G has S-commutator relations with root groups U'_ξ.*

(b) *Let $g\colon (S, Y) \to (T, Z)$ be a second morphism of $\mathbf{SV}$ and for $\tau \in T$, define subgroups of G by*

$$U''_\tau := \Big\langle U_\alpha :\ g(f(\alpha)) \neq 0,\ \ \tau\big|(g\circ f)(\alpha)\Big\rangle \ = \ U_{R[\tau, g\circ f]}. \tag{3}$$

Then

$$U''_\tau = \Big\langle U'_\xi :\ g(\xi) \neq 0,\ \ \tau\big|g(\xi)\Big\rangle \ = \ U'_{S[\tau,g]}. \tag{4}$$

(c) *The assignment $(G, U_\alpha) \mapsto (G, U'_\xi)$ on objects and $\varphi \mapsto \varphi$ on morphisms is a covariant functor $f_\bullet\colon \mathbf{gc}_R \to \mathbf{gc}_S$.*

(d) *$f_\bullet$ itself depends functorially on f; i.e., $(\mathrm{Id})_\bullet = \mathrm{Id}$ and $(g\circ f)_\bullet = g_\bullet \circ f_\bullet$ for all morphisms $g\colon (S, Y) \to (T, Z)$ of $\mathbf{SV}$.*

Remark. A special case of part (a) is the construction of root groups in reductive algebraic groups over arbitrary fields [**6**, Prop. 21.9], see Example 2.3(c).

Proof. (a) We have $R[0] = \emptyset$ by (2.1.4), so $U'_0 = U_\emptyset = \{1\}$. Next, $\eta\big|\xi$ implies $R[\eta] \supset R[\xi]$ by (2.1.5), from which the relation $U'_\eta \supset U'_\xi$ follows. It remains to verify the commutator relation (2.2.3). Let $X_\xi = \bigcup\{U_\alpha : \alpha \in R[\xi]\}$, and note $X_\xi = X_\xi^{-1}$ and $\langle X_\xi \rangle = U'_\xi$. Hence, by Lemma 2.8 applied to the family of subgroups $(U'_\xi)_{\xi\in S}$, it suffices to prove that

$$\big(U_\alpha, U_\beta\big) \subset U'_{\boldsymbol{(}\xi,\eta\boldsymbol{)}}, \tag{5}$$

whenever (ξ, η) is a nilpotent pair in S and $\alpha \in R[\xi]$ and $\beta \in R[\eta]$. By (1.9.3), $\{\alpha, \beta\}$ is prenilpotent, hence the R-commutator relations for G yield $\big(U_\alpha, U_\beta\big) \subset U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}}$, and $U_{\boldsymbol{(}\alpha,\beta\boldsymbol{)}} \subset U'_{\boldsymbol{(}\xi,\eta\boldsymbol{)}}$ follows from $f\big(\big(\alpha, \beta\big)\big) \subset \big(\xi, \eta\big)$ and $U_\gamma \subset U'_{f(\gamma)}$ for $f(\gamma) \neq 0$. This establishes (5) and completes the proof of (a).

(b) For the inclusion from left to right in (4), let $\alpha \in R[\tau, g\circ f]$ and put $\xi = f(\alpha)$. Then $g(\xi) = g(f(\alpha)) \neq 0$ and $\tau\big|g(\xi)$, so $\xi \in S[\tau, g]$. Obviously, $\alpha \in R[\xi, f]$, and hence $U_\alpha \subset U'_\xi$.

To prove the inclusion from right to left in (4), let $\xi \in S[\tau, g]$. We must show $U'_\xi \subset U''_\tau$. By (2), U'_ξ is generated by all U_α, $\alpha \in R[\xi, f]$, so it suffices to show that $\alpha \in R[\xi, f]$ and $\xi \in S[\tau, g]$ imply $\alpha \in R[\tau, g \circ f]$, because then $U_\alpha \subset U''_\tau$ will follow. Now g is linear, so $f(\alpha) = n\xi \neq 0$ and $g(\xi) = p\tau \neq 0$ for suitable $n, p \in \mathbb{N}_+$ imply $g(f(\alpha)) = ng(\xi) = np\tau \neq 0$, whence $\alpha \in R[\tau, g \circ f]$, as desired.

(c) By (a) we have $f_\bullet(G, U_\alpha) \in \mathbf{gc}_S$. Now let also $H = (H, (V_\alpha)_{\alpha\in R}) \in \mathbf{gc}_R$ and let φ: $G \to H$ be a morphism of $\mathbf{gc}_R$, so that $\varphi(U_\alpha) \subset V_\alpha$ for all $\alpha \in R$. We use the notations of (1) and (2) for H as well. Then $\varphi(U'_\xi) \subset V'_\xi$ is clear from the definition (2), so $f_\bullet(\varphi) = \varphi$: $G \to H$ is a morphism of $\mathbf{gc}_S$. It is immediately verified that $f_\bullet(\mathrm{Id}_G) = \mathrm{Id}_{f_\bullet(G)}$ and $f_\bullet(\psi \circ \varphi) = f_\bullet(\psi) \circ f_\bullet(\varphi)$, showing $f_\bullet$ is indeed a covariant functor.

(d) From (2) and the fact that $U_\beta \subset U_\alpha$ for $\alpha\big|\beta$ (by (2.2.2)) it follows that $\mathrm{Id}_\bullet(G) = G$ and $\mathrm{Id}_\bullet(\varphi) = \varphi$ for a morphism φ of $\mathbf{gc}_R$. Now let g: $(S, Y) \to (T, Z)$ be a morphism of $\mathbf{SV}$. We must show that the functors $(g\circ f)_\bullet$ and $g_\bullet\circ f_\bullet$ have the same effect on objects and morphisms of $\mathbf{gc}_R$. We have $(g\circ f)_\bullet(G) = (G, (U''_\tau)_{\tau\in T}) \in \mathbf{gc}_T$ where U''_τ is given by (3). Hence $g_\bullet(f_\bullet(G)) = (g \circ f)_\bullet(G)$ follows from (4). Thus $g_\bullet \circ f_\bullet$ and $(g \circ f)_\bullet$ agree on objects of $\mathbf{gc}_R$. That also $g_\bullet(f_\bullet(\varphi)) = (g \circ f)_\bullet(\varphi)$ for morphisms of $\mathbf{gc}_R$, is then an easy consequence.

Remark. Let $\mathfrak{CAT}$ be the meta-category of all categories, whose objects are categories, and whose morphisms are functors between categories. Then Proposition 3.3 can be expressed by saying that there is a functor $\mathbb{G}$: $\mathbf{SV} \to \mathfrak{CAT}$, given by $\mathbb{G}(R) = \mathbf{gc}_R$ and $\mathbb{G}(f) = f_\bullet$. In the terminology of [**25**, B1.3], $\mathbb{G}$ is an $\mathbf{SV}^{\mathrm{op}}$-indexed category. The Grothendieck construction [**25**, B1.3] then yields a fibred category over $\mathbf{SV}^{\mathrm{op}}$ whose objects (but not morphisms!) are those of $\mathbf{gc}$. On the other hand, we will see that $\mathit{\Pi}$: $\mathbf{gc} \to \mathbf{SV}$ is a cofibration in a natural way. We first recall this concept.

3.4. Cofibrations. Let $\Pi\colon \mathscr{C} \to \mathscr{S}$ be a covariant functor between categories $\mathscr{C}$ and $\mathscr{S}$. For $R \in \mathscr{S}$, denote by $\mathscr{C}_R$ the subcategory of $\mathscr{C}$ with objects all $G \in \mathscr{C}$ such that $\Pi(G) = R$, and morphisms those morphisms $\vartheta\colon G \to G'$ of $\mathscr{C}$ satisfying $\Pi(\vartheta) = \mathrm{Id}_R$. Objects of $\mathscr{C}_R$ will also be called objects *over* R.

A morphism $\varphi\colon G \to H$ of $\mathscr{C}$, say with $\Pi(\varphi) = f\colon R = \Pi(G) \to S = \Pi(H)$, will be called a *morphism over* $f\colon R \to S$. We say φ is *cocartesian* if for every morphism $\chi\colon G \to K$ of $\mathscr{C}$ over $h\colon R \to T = \Pi(K)$, every factorization $h = g \circ f$ in $\mathscr{S}$ can be lifted uniquely to a factorization of χ, i.e., there exists a unique morphism $\psi\colon H \to K$ of $\mathscr{C}$ such that $\chi = \psi \circ \varphi$:

$$\begin{tikzcd}
G \arrow[rr, "\chi"] \arrow[dr, "\varphi"'] \arrow[dd, dash] & & K \arrow[dd, dash] \\
& H \arrow[ur, dotted, "\exists!", "\psi"'] \arrow[dd, dash] & \\
R \arrow[rr, "h"] \arrow[dr, "f"'] & & T \\
& S \arrow[ur, "g"'] &
\end{tikzcd} \tag{1}$$

Here a vertical line like $\begin{array}{c} G \\ | \\ R \end{array}$ indicates that $\Pi(G) = R$, i.e., that G is an object over R.

The functor Π is called a *cofibration*, or $\mathscr{C}$ is said to be *cofibred over* $\mathscr{S}$, if for every morphism $f\colon R \to S$ of $\mathscr{S}$ and every object $G \in \mathscr{C}$ over R, there exists $H \in \mathscr{C}_S$ and a cocartesian morphism $\varphi\colon G \to H$ over f. Such φ and H are unique up to unique isomorphism.

Let $\mathscr{C}$ be cofibred over $\mathscr{S}$. For every $G \in \mathscr{C}$ and every morphism $f\colon R \to S$ of $\mathscr{S}$ choose a cocartesian morphism $\varphi\colon G \to H$, and put $f_*(G) := H$ and $\omega_f(G) := \varphi$. Then for every morphism $\vartheta\colon G \to G'$ in the category $\mathscr{C}_R$ there exists a unique morphism $f_*(\vartheta)\colon f_*(G) \to f_*(G')$ in $\mathscr{C}_S$ such that the diagram

$$\begin{tikzcd}
G \arrow[r, "\vartheta"] \arrow[d, "\omega_f(G)"'] & G' \arrow[d, "\omega_f(G')"] \\
f_*(G) \arrow[r, "f_*(\vartheta)"'] & f_*(G')
\end{tikzcd} \tag{2}$$

commutes. This follows by applying the diagram (1) to the case where $\chi = \omega_f(G') \circ \vartheta$, $K = f_*(G')$ and $g = \mathrm{Id}_S$. In this way, we obtain a functor $f_*\colon \mathscr{C}_R \to \mathscr{C}_S$. The choice of f_* and $\omega_f(G)$ for all f and G is called a *cleavage* of $\mathscr{C}$ over $\mathscr{S}$. There are unique isomorphisms $(\mathrm{Id}_R)_* \cong \mathrm{Id}_{\mathscr{C}_R}$ and $g_* \circ f_* \cong (g \circ f)_*$, but these are in general not equalities, so the assignment $R \mapsto \mathscr{C}_R$, $f \mapsto f_*$ is merely a pseudofunctor (or lax 2-functor) from $\mathscr{S}$ to $\mathfrak{CAT}$.

3.5. Proposition. *The functor* Π: $\mathbf{gc} \to \mathbf{SV}$ *of* (3.1.3) *is a cofibration. A cleavage of* Π *is given as follows. For a morphism* $f\colon (R,X) \to (S,Y)$ *of* $\mathbf{SV}$ *and an object* $G = (R, G, (U_\alpha)_{\alpha\in R}) \in \mathbf{gc}$*, let* N_f *be the normal subgroup of* G *generated by* $\{U_\alpha : f(\alpha) = 0\}$*. Let* $H = G/N_f$ *and denote* $\mathrm{can}\colon G \to H$ *the canonical map. For* $\xi \in S$ *define* $V_\xi := \mathrm{can}(U'_\xi)$ *where the* U'_ξ *are as in* (3.3.2)*. Then* $f_*(G) := \bigl(S, H, (V_\xi)_{\xi\in S}\bigr) \in \mathbf{gc}$*, and* $\omega_f(G) := (f, \mathrm{can})\colon G \to f_*(G)$ *is cocartesian.*

Proof. By Proposition 3.3(a), G has S-commutator relations with root groups U'_ξ, so as remarked in 2.3, the homomorphic image H has S-commutator relations with root groups V_ξ, i.e., $f_*(G)$ is an object of $\mathbf{gc}$ over S. It remains to show that $\omega_f(G) = (f, \mathrm{can})\colon G \to f_*(G)$ is cocartesian. Explicitly, this means:

$(*)$ Let $K = \bigl(T, K, (W_\tau)_{\tau\in T}\bigr)$ be an object of $\mathbf{gc}$ over T, let $(h,\chi)\colon G \to K$ be a morphism of $\mathbf{gc}$ over $h\colon R \to T$, and let $h = g \circ f$ be factored via a morphism $g\colon S \to T$ of $\mathbf{SV}$. Then there exists a unique morphism $(g,\psi)\colon f_*(G) \to K$ in $\mathbf{gc}$ over g such that $\chi = \psi \circ \mathrm{can}$.

First note that $\omega_f(G)$ is a morphism of $\mathbf{gc}$ lying over f. Indeed, $\mathrm{can}\colon G \to G/N_f$ is a group homomorphism, and $U_\alpha \subset U'_{f(\alpha)}$ obviously holds by the definition (3.3.2), whence $\mathrm{can}(U_\alpha) \subset \mathrm{can}(U'_{f(\alpha)}) = V_{f(\alpha)}$ for all $\alpha \in R$. Now let $K = \bigl(T, K, (W_\tau)_{\tau\in T}\bigr)$, let $(h,\chi)\colon G \to K$ and $h = g \circ f$ as in $(*)$. For $\alpha \in R$ with $f(\alpha) = 0$ we have $h(\alpha) = g(f(\alpha)) = 0$ and therefore $U_\alpha \subset \mathrm{Ker}(\chi)$ by (3.1.2), which implies that also $N_f \subset \mathrm{Ker}(\chi)$. Hence there exists a unique group homomorphism $\psi\colon H \to K$ such that $\chi = \psi \circ \mathrm{can}$, and it remains to show that ψ is a homomorphism of $\mathbf{gc}$ over g, i.e., that $\psi(V_\xi) \subset W_{g(\xi)}$ for all $\xi \in S$. Now $V_\xi = \mathrm{can}(U'_\xi)$ is generated by all $\mathrm{can}(U_\alpha)$ where $f(\alpha) \neq 0$ and $\xi \big| f(\alpha)$. For such α, we have $\psi(\mathrm{can}(U_\alpha)) = \chi(U_\alpha) \subset W_{h(\alpha)}$, since $\chi\colon G \to K$ is a morphism over h. But $\xi \big| f(\alpha)$ implies $g(\xi) \big| h(\alpha)$ (because $h = g \circ f$), and hence $W_{g(\xi)} \supset W_{h(\alpha)}$ by (2.2.2). This shows $\psi(V_\xi) \subset W_{g(\xi)}$, as desired.

3.6. Remarks. (a) Let $\vartheta\colon G \to G'$ be a morphism of groups in $\mathbf{gc}_R$, and let N'_f be the normal subgroup of G' defined analogously to N_f. Then it easy to see that $\vartheta(N_f) \subset N'_f$, and that the homomorphism $f_*(\vartheta)\colon G/N_f \to G'/N'_f$ of (3.4.2) is the one induced from ϑ by passing to the quotient groups.

(b) Let $g\colon S \to T$ be a morphism of $\mathbf{SV}$. The isomorphisms of functors $\mathrm{Id}_{\mathbf{gc}_R} \cong (\mathrm{Id}_R)_*$ and $(g \circ f)_* \cong g_* \circ f_*$ mentioned in 3.4 are given on objects as follows. For $f = \mathrm{Id}_R$, we have $N_f = \{1\}$, and hence $(\mathrm{Id}_R)_*(G) = G/\{1\} \cong G$ in the obvious way. Let $N_{g\circ f}$ be the normal subgroup of G generated by all U_α with $g(f(\alpha)) = 0$ and let $\bar N_g$ the normal subgroup of $\bar G$ generated by all $\bar U_\xi$ with $g(\xi) = 0$. Then $N_{g\circ f}/N_f = \bar N_g$, and we have an isomorphism of groups $(g \circ f)_*(G) = G/N_{g\circ f} \cong \bar G/\bar N_g = g_*(f_*(G))$ by the first isomorphism theorem. One checks easily that this is compatible with the respective root subgroups.

(c) If $f\colon R \to S$ is an *immersion* in the sense that $R \cap \mathrm{Ker}(f) = \{0\}$ then $N_f = \{1\}$ and therefore the underlying group of $f_*(G)$ is the same as that of G, although of course the root groups differ.

3.7. Definition. A morphism $\varphi\colon (G,(U_\alpha)_{\alpha\in R}) \to (G',(U'_\alpha)_{\alpha\in R})$ of the category $\mathbf{gc}_R$ is called *injective (surjective, bijective) on root groups* if, for all $\alpha \in R$, the restriction $\varphi\colon U_\alpha \to U'_\alpha$ has the respective property. Morphisms of this type are stable under composition, and hence define (non-full) subcategories of $\mathbf{gc}_R$.

If G' is tight (generated by its root subgroups) then a morphism which is surjective on root groups is actually surjective. An analogous result does not hold for morphisms which are injective on root groups. However, we have a positive result under suitable assumptions on unique factorization.

3.8. Lemma. *Let $\varphi\colon (G,(U_\alpha)_{\alpha\in R}) \to (G',(U'_\alpha)_{\alpha\in R})$ be a morphism of $\mathbf{gc}_R$.*

(a) *If φ is surjective on root groups then $\varphi(U_A) = U'_A$ for any subset A of R.*

(b) *Suppose φ is injective on root groups and G' has unique factorization for a finite nilpotent subset A of R, cf.* 2.15. *Then G has unique factorization for A as well, and $\varphi\colon U_A \to U'_A$ is injective.*

(c) *Let again φ be injective on root groups. If G' has unique factorization for all finite nilpotent subsets then so does G, and $\varphi\colon U_A \to U'_A$ is injective, for all (possibly infinite) nilpotent subsets A of R.*

(d) *If φ is bijective on root groups and G' has unique factorization then G has unique factorization, and $\varphi\colon U_A \to U'_A$ is an isomorphism, for all nilpotent $A \subset R$.*

Proof. (a) This is evident from the definitions.

(b) Enumerate $A_{\mathrm{ind}} = \{\gamma_1,\dots,\gamma_n\}$. Then the diagram

$$\begin{tikzcd}
U_{\gamma_1}\times\cdots\times U_{\gamma_n} \arrow[r, "\mu"] \arrow[d] & U_A \arrow[d] \\
U'_{\gamma_1}\times\cdots\times U'_{\gamma_n} \arrow[r, "\mu'"'] & U'_A
\end{tikzcd}$$

commutes, where the horizontal maps are the product maps of G and G', respectively, and the vertical maps are induced by φ. By (2.12.1) the horizontal maps are surjective. Since φ is injective on root groups, the left hand map is injective, and since G' has unique factorization for A, the bottom map μ' is injective. Hence μ and the right hand map are injective as well. In particular, G has unique factorization for A.

(c) By (b), G has unique factorization for all finite nilpotent subsets. Now let A be an arbitrary nilpotent subset, and let $u \in U_A$ with $\varphi(u) = 1$. Then there exists a finite subset $F \subset A$ such that $u \in U_F$. By Proposition 1.12, the closure F^c is finite and nilpotent, so after replacing F by F^c we may assume F nilpotent. By what we proved in the finite case, $u = 1$, as desired.

(d) This follows immediately from (a) – (c).

3.9. Steinberg categories. Fix $(R, X) \in \mathbf{SV}$. As in 3.1, we will frequently not make explicit the root subgroups U_α of an object $(G, (U_\alpha)_{\alpha\in R})$ of $\mathbf{gc}_R$ and thus write $G = (G, (U_\alpha)_{\alpha\in R})$ if there is no danger of confusion. Let us also fix an object $\bar{G} = (\bar{G}, (\bar{U}_\alpha)_{\alpha\in R}) \in \mathbf{gc}_R$. We introduce a category $\mathbf{st}(\bar{G})$, called the *Steinberg category of* $\bar{G}$, as follows.

An *object* of $\mathbf{st}(\bar{G})$ is a morphism $\pi\colon G \to \bar{G}$ of $\mathbf{gc}_R$ with two additional properties:

(i) π is bijective on root groups, and
(ii) $\pi\colon U_{\boldsymbol{[}\alpha,\beta\boldsymbol{]}} \to \bar{U}_{\boldsymbol{[}\alpha,\beta\boldsymbol{]}}$ is bijective, for all nilpotent pairs (α, β).

A *morphism* from $G \xrightarrow{\pi} \bar{G}$ to $G' \xrightarrow{\pi'} \bar{G}$ of $\mathbf{st}(\bar{G})$ is a morphism $\varphi\colon G \to G'$ of $\mathbf{gc}_R$ making the diagram

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & G' \arrow[dl, "\pi'"] \\ & \bar{G} & \end{tikzcd} \tag{1}$$

commutative. Thus $\mathbf{st}(\bar{G})$ is a (in general non-full) subcategory of the comma-category $(\mathbf{gc}_R \downarrow \bar{G})$, cf. [**43**, II.6].

Let $\pi\colon G \to \bar{G}$ be in $\mathbf{st}(\bar{G})$ and let $\mathfrak{r}(G)$ and $\mathfrak{r}(\bar{G})$ be the subgroups generated by the respective root subgroups. Then it is clear that $\pi\colon \mathfrak{r}(G) \to \bar{G}$ belongs to $\mathbf{st}(\bar{G})$, and that $\pi\colon \mathfrak{r}(G) \to \mathfrak{r}(\bar{G})$ is in $\mathbf{st}(\mathfrak{r}(\bar{G}))$.

3.10. Remark. (a) The reader may wonder about the relation between (i) and (ii). The following examples show that these conditions are in general independent.

(i) *does not imply* (ii): Let $\varepsilon_1, \varepsilon_2$ be the standard basis of $\Bbbk^2$ and let $R = \{0, \varepsilon_1, \varepsilon_2\} \subset \Bbbk^2$. Let e_1, e_2 be the standard basis of $\mathbb{Z}^2$ and put $G = \mathbb{Z}^2$ with $U_{\varepsilon_i} = \mathbb{Z}\cdot e_i$. Define $\bar{G} = \mathbb{Z}$ with $\bar{U}_{\varepsilon_i} = \mathbb{Z}$ and let $\pi\colon G \to \bar{G}$ be defined by $\pi(e_i) = 1$. All pairs $\{\varepsilon_i, \varepsilon_j\}$ for $i, j \in \{1, 2\}$ are nilpotent pairs. Then π is bijective on all U_α, but $\pi\big|U_{\boldsymbol{[}\varepsilon_1,\varepsilon_2\boldsymbol{]}}$ is not.

(ii) *does not imply* (i): Let $R = \mathbb{N} \subset \Bbbk$, let $G = \mathbb{Z}$ (additive group), and put $U_1 = G$ and $U_n = \{0\}$ for $n \neq 1$. Let $\bar{G} = \{1\}$ and let $\pi\colon G \to \bar{G}$ be the only possible map. There are no nilpotent pairs, because $\boldsymbol{[}\alpha, \beta\boldsymbol{]}$ either contains 0 (in case $\alpha = 0$ or $\beta = 0$), or it is infinite. Hence (ii) holds trivially. On the other hand, $\pi\colon U_1 \to \bar{U}_1 = \bar{G} = \{1\}$ is not injective.

(b) However, if R satisfies the finiteness condition (F1) of (2.4.2), that is, if $\{\alpha\}^c$ is finite for all α, then (ii) implies (i). Indeed, then any pair (α, α) (for $\alpha \in R^\times$) is nilpotent, and $\boldsymbol{[}\alpha, \alpha\boldsymbol{]} = \{\alpha\}^c$ as well as $U_{\boldsymbol{[}\alpha,\alpha\boldsymbol{]}} = U_\alpha$ because of the relation (2.2.2). In particular, this is so in the important case where R is a locally finite root system.

3.11. We note some elementary properties of the Steinberg category.

(a) *Assume $\bar{G}$ has unique factorization for all nilpotent sets of the form $\boldsymbol{[}\alpha, \beta\boldsymbol{]}$ where (α, β) is a nilpotent pair. Then a morphism $\pi\colon G \to \bar{G}$ of $\mathbf{gc}_R$ is an object of $\mathbf{st}(\bar{G})$ if and only if it is bijective on root groups.*

This follows immediately from Lemma 3.8.

(b) Suppose φ is a morphism of $\mathbf{st}(\bar{G})$ as in (3.9.1). Then:

(b0) *φ: $U_\alpha \to U'_\alpha$ is an isomorphism, for all $\alpha \in R$,*
(b1) *if G is tight then φ is uniquely determined,*
(b2) *if G' is tight then φ is surjective.*

Indeed, from (3.9.1) we obtain the commutative diagrams

$$\begin{tikzcd} U_\alpha \arrow[rr, "\varphi|_{U_\alpha}"] \arrow[dr, "\cong", "\pi"'] & & U'_\alpha \arrow[dl, "\cong"', "\pi'"] \\ & \bar{U}_\alpha & \end{tikzcd} \tag{1}$$

which show that the restriction of φ to each U_α is uniquely determined and an isomorphism onto U'_α. Hence φ is uniquely determined, provided the U_α generate G. This also shows that φ is surjective if G' is generated by the U'_α.

(c) *A morphism φ: $G \to G'$ of $\mathbf{st}(\bar{G})$ can be considered as an object of $\mathbf{st}(G')$.*

Indeed, φ is bijective on root groups by (1), and φ: $U_{[\alpha,\beta]} \to U'_{[\alpha,\beta]}$ is bijective, for all nilpotent pairs, as follows from the commutative triangles

$$\begin{tikzcd} U_{[\alpha,\beta]} \arrow[rr, "\varphi"] \arrow[dr, "\cong", "\pi"'] & & U'_{[\alpha,\beta]} \arrow[dl, "\cong"', "\pi'"] \\ & \bar{U}_{[\alpha,\beta]} & \end{tikzcd} \tag{2}$$

(d) Let π: $G \to \bar{G}$ be in $\mathbf{st}(\bar{G})$ and let $K \subset G$ be a normal subgroup with the property that $K \subset \mathrm{Ker}(\pi)$. Let can: $G \to \dot{G} := G/K$ be the canonical map, and equip $\dot{G}$ with the root groups $\dot{U}_\alpha = \mathrm{can}(U_\alpha)$ is in 2.3(b). Then π induces a homomorphism $\dot{\pi}$: $\dot{G} \to \bar{G}$ such that $\pi = \dot{\pi} \circ \mathrm{can}$, and $\dot{\pi} \in \mathbf{st}(\bar{G})$. Indeed, it follows immediately from the commutative diagrams

$$\begin{tikzcd} U_\alpha \arrow[rr, "\mathrm{can}"] \arrow[dr, "\cong", "\pi"'] & & \dot{U}_\alpha \arrow[dl, "\dot{\pi}"] \\ & \bar{U}_\alpha & \end{tikzcd} \qquad \begin{tikzcd} U_{[\alpha,\beta]} \arrow[rr, "\mathrm{can}"] \arrow[dr, "\cong", "\pi"'] & & \dot{U}_{[\alpha,\beta]} \arrow[dl, "\dot{\pi}"] \\ & \bar{U}_{[\alpha,\beta]} & \end{tikzcd}$$

that $\dot{\pi}$ satisfies conditions (i) and (ii) of 3.9.

Finally, it is clear that

(e) Id: *$\bar{G} \to \bar{G}$ is a terminal object of $\mathbf{st}(\bar{G})$.*

Our next aim is to show that $\mathbf{st}(\bar{G})$ has an initial object. The proof is based on a direct limit construction of groups with commutator relations.

3.12. Lemma. *Let $(R, X) \in \mathbf{SV}$. For all $\alpha \in R^\times$ and all nilpotent pairs $(\alpha, \beta) \in R \times R$, let L_α and $L_{(\alpha,\beta)}$ be groups. For simpler notation, we write $L_{\alpha\beta} = L_{(\alpha,\beta)}$. Let*

$$\begin{aligned} i_\alpha^\beta &\colon L_\beta \to L_\alpha && \textit{whenever } \alpha \big| \beta, \textit{ and} && (1)\\ i_{\alpha\beta}^\gamma &\colon L_\gamma \to L_{\alpha\beta} && \textit{for all } \gamma \in \boldsymbol{[}\alpha, \beta\boldsymbol{]}, && (2) \end{aligned}$$

be group homomorphisms satisfying

$$L_{\alpha\beta} = \Big\langle i_{\alpha\beta}^\gamma(L_\gamma) : \gamma \in \boldsymbol{[}\alpha, \beta\boldsymbol{]} \Big\rangle, \tag{3}$$

$$\boldsymbol{(} i_{\alpha\beta}^\alpha(L_\alpha), i_{\alpha\beta}^\beta(L_\beta) \boldsymbol{)} \subset \Big\langle i_{\alpha\beta}^\gamma(L_\gamma) : \gamma \in \boldsymbol{(}\alpha, \beta\boldsymbol{)} \Big\rangle. \tag{4}$$

Let L be the inductive limit of the groups L_α and $L_{\alpha\beta}$ with respect to the maps i_α^β and $i_{\alpha\beta}^\gamma$. Denote by $j_\alpha\colon L_\alpha \to L$ and $j_{\alpha\beta}\colon L_{\alpha\beta} \to L$ the canonical maps into the inductive limit. Then L has R-commutator relations with root groups

$$Y_0 = \{1\}, \qquad Y_\alpha = j_\alpha(L_\alpha) \quad (\alpha \in R^\times),$$

and is generated by the Y_α; i.e., $\big(L, (Y_\alpha)_{\alpha \in R}\big) \in \mathbf{gc}_R$.

Proof. By standard facts [**43**, **51**, **52**], the inductive limit exists and has the following properties which characterize it uniquely up to unique isomorphism: the homomorphisms j_α and $j_{\alpha\beta}$, for all $\alpha \in R^\times$ and all nilpotent pairs (α, β), make the inner left hand triangles of the diagrams

$$\begin{tikzcd} L_\beta \arrow[dd, "i_\alpha^\beta"'] \arrow[dr, "j_\beta"] \arrow[drr, bend left, "\varphi_\beta"] & & \\ & L \arrow[r, dotted, "\exists!\,\varphi"] & H \\ L_\alpha \arrow[ur, "j_\alpha"'] \arrow[urr, bend right, "\varphi_\alpha"'] & & \end{tikzcd} \qquad \begin{tikzcd} L_\gamma \arrow[dd, "i_{\alpha\beta}^\gamma"'] \arrow[dr, "j_\gamma"] \arrow[drr, bend left, "\varphi_\gamma"] & & \\ & L \arrow[r, dotted, "\exists!\,\varphi"] & H \\ L_{\alpha\beta} \arrow[ur, "j_{\alpha\beta}"'] \arrow[urr, bend right, "\varphi_{\alpha\beta}"'] & & \end{tikzcd} \tag{5}$$

commutative, for all $\alpha \big| \beta$ and all $\gamma \in \boldsymbol{[}\alpha, \beta\boldsymbol{]}$, (α, β) nilpotent. Furthermore, given any group H and given homomorphisms

$$\varphi_\alpha\colon L_\alpha \to H, \qquad \varphi_{\alpha\beta}\colon L_{\alpha\beta} \to H$$

making the outer triangles of (5) commute, there exists a unique homomorphism $\varphi\colon L \to H$ making the entire diagrams commute.

We first observe that

$$Y_{\boldsymbol{[}\alpha,\beta\boldsymbol{]}} = j_{\alpha\beta}(L_{\alpha\beta}), \tag{6}$$

for all nilpotent pairs (α, β). (Recall that, by the definition given in 2.2, for a subset A of R, Y_A is the subgroup of L generated by all Y_γ, $\gamma \in A$. This is applied here to the subsets $\boldsymbol{[}\alpha, \beta\boldsymbol{]}$ of R.) Indeed,

$$
\begin{aligned}
Y_{[\alpha,\beta]} &= \Big\langle Y_\gamma : \gamma \in \big[\alpha,\beta\big] \Big\rangle = \Big\langle j_\gamma(L_\gamma) : \gamma \in \big[\alpha,\beta\big] \Big\rangle \\
&= \Big\langle j_{\alpha\beta} \circ i^\gamma_{\alpha\beta}(L_\gamma) : \gamma \in \big[\alpha,\beta\big] \Big\rangle && \text{(by (5))} \\
&= j_{\alpha\beta} \Big\langle i^\gamma_{\alpha\beta}(L_\gamma) : \gamma \in \big[\alpha,\beta\big] \Big\rangle && \text{(since } j_{\alpha\beta} \text{ is a homomorphism)} \\
&= j_{\alpha\beta}(L_{\alpha\beta}) && \text{(by (3)).}
\end{aligned}
$$

By uniqueness of φ in (5), see also [**52**, Chapter I, 1.1], L is generated by the subgroups $j_\alpha(L_\alpha) = Y_\alpha$, $\alpha \in R$, and $j_{\alpha\beta}(L_{\alpha\beta})$, (α, β) nilpotent. Now (6) shows that L is already generated by the Y_α.

Next, the relation (2.2.2) for L, i.e., $Y_\beta \subset Y_\alpha$ if $\alpha \big| \beta$, follows immediately from the first diagram of (5) and the definition of the root groups, so it remains to verify that L satisfies the commutator relations (2.2.3). Let (α, β) be a nilpotent pair. Then, since α and β belong to $\big[\alpha,\beta\big]$, by (5) and by (4),

$$
\begin{aligned}
\big(Y_\alpha, Y_\beta\big) &= \big(j_\alpha(L_\alpha), j_\beta(L_\beta)\big) = \big((j_{\alpha\beta} \circ i^\alpha_{\alpha\beta})(L_\alpha), (j_{\alpha\beta} \circ i^\beta_{\alpha\beta})(L_\beta)\big) \\
&= j_{\alpha\beta}\Big(\big(i^\alpha_{\alpha\beta}(L_\alpha), i^\beta_{\alpha\beta}(L_\beta)\big)\Big) \subset j_{\alpha\beta} \Big\langle i^\gamma_{\alpha\beta}(L_\gamma) : \gamma \in \big(\alpha,\beta\big) \Big\rangle \\
&= \Big\langle (j_{\alpha\beta} \circ i^\gamma_{\alpha\beta})(L_\gamma) : \gamma \in \big(\alpha,\beta\big) \Big\rangle = \Big\langle j_\gamma(L_\gamma) : \gamma \in \big(\alpha,\beta\big) \Big\rangle \\
&= \Big\langle Y_\gamma : \gamma \in \big(\alpha,\beta\big) \Big\rangle = Y_{(\alpha,\beta)}.
\end{aligned}
$$

This completes the proof.

Remark. For this computation to work it is essential that $i^\alpha_{\alpha\beta}\colon L_\alpha \to L_{\alpha\beta}$ and $i^\beta_{\alpha\beta}\colon L_\beta \to L_{\alpha\beta}$ be defined. This explains why we have to allow $\gamma \in \big[\alpha,\beta\big]$ in (2); it would not be sufficient to require (2) only for $\gamma \in \big(\alpha,\beta\big)$.

The following result is inspired by [**59**, 3.6].

3.13. Theorem. *Let $\big(\bar G, (\bar U_\alpha)_{\alpha\in R}\big) \in \mathbf{gc}_R$. The Steinberg category $\mathbf{st}(\bar G)$ has an initial object $\hat\pi\colon \hat G \to \bar G$, and the group $\hat G$ is generated by its root subgroups.*

By standard facts, this initial object is uniquely determined up to unique isomorphism. We call it the *Steinberg group of $\bar G$* and denote it by $\mathrm{St}(\bar G)$. By abuse of terminology, the group $\hat G$ will also be referred to as the Steinberg group of $\bar G$.

Proof. We apply Lemma 3.12 to the situation where $L_\alpha := \bar U_\alpha$, $L_{\alpha\beta} := \bar U_{[\alpha,\beta]}$, and $i^\beta_\alpha\colon L_\beta \to L_\alpha$ and $i^\gamma_{\alpha\beta}\colon L_\gamma \to L_{\alpha\beta}$ are the inclusion maps. Then (3.12.3) holds by definition of $\bar U_{[\alpha,\beta]}$, and (3.12.4) follows from (2.2.3). Let $L = \hat G$ be the inductive limit of the L_α and $L_{\alpha\beta}$ as in Lemma 3.12.

We claim that there exists a unique homomorphism $\varphi = \hat\pi\colon \hat G \to \bar G$ such that $\hat\pi \circ j_\alpha = \mathrm{inc}\colon \bar U_\alpha \hookrightarrow \bar G$ for all $\alpha \in R$, and $\hat\pi$ is an object of the category $\mathbf{st}(\bar G)$. For the proof, we use the universal property of $\hat G$ (cf. (3.12.5)) in the case where $H = \bar G$ and the $\varphi_\alpha\colon L_\alpha \to \bar G$ and $\varphi_{\alpha\beta}\colon L_{\alpha\beta} \to \bar G$ are the inclusion maps. Then the outer triangles of the diagrams (3.12.5) obviously commute, proving the existence of $\hat\pi$.

From the first diagram of (3.12.5) we see that $\hat{\pi} \circ j_\alpha = \mathrm{Id}_{\bar{U}_\alpha}$. By definition, the root groups of $\hat{G}$ are $Y_\alpha = j_\alpha(L_\alpha) = j_\alpha(\bar{U}_\alpha)$. Hence $j_\alpha\colon \bar{U}_\alpha \to Y_\alpha$ is an isomorphism, so $\hat{\pi}$ is bijective on root groups. In the same way, the second diagram of (3.12.5) shows that $\hat{\pi} \circ j_{\alpha\beta}$ is the identity on $\bar{U}_{[\alpha,\beta]}$. Hence $\hat{\pi}$ satisfies condition (ii) of 3.9 as well, so $\hat{\pi}$ is an object of $\mathbf{st}(\bar{G})$.

Next we show that $\hat{\pi}$ is in fact an initial object of $\mathbf{st}(\bar{G})$. Thus let $G = (G, (U_\alpha)_{\alpha \in R}) \in \mathbf{gc}_R$ and let $\pi\colon G \to \bar{G}$ be a morphism satisfying (i) and (ii) of 3.9, i.e., an object of $\mathbf{st}(\bar{G})$. We must show that there exists a unique homomorphism $\varphi\colon \hat{G} \to G$ preserving root groups and making the diagram

$$\begin{tikzcd}
\hat{G} \arrow[rr, "\varphi"] \arrow[dr, "\hat{\pi}"'] & & G \arrow[dl, "\pi"] \\
& \bar{G} &
\end{tikzcd}$$

commute. We use again the universal property of the inductive limit. Define $\varphi_\alpha\colon \bar{U}_\alpha \to G$ and $\varphi_{\alpha\beta}\colon \bar{U}_{[\alpha,\beta]} \to G$ to be the inverses of $\pi\big|U_\alpha$ and $\pi\big|U_{[\alpha,\beta]}$. Since π satisfies (i) and (ii) of 3.9, this makes sense. Then the outer triangles of (3.12.5) are clearly commutative, so we have a unique $\varphi\colon \hat{G} \to G$ making the diagrams (3.12.5) commute. From these diagrams one sees that φ preserves root groups, and uniqueness of φ follows from 3.9(c1). Hence $\hat{\pi}\colon \hat{G} \to \bar{G}$ is an initial object of $\mathbf{st}(\bar{G})$.

3.14. Remarks. The Steinberg group of a group $\bar{G}$ does not change when replacing $\bar{G}$ by some $G \in \mathbf{st}(\bar{G})$ or by a suitable quotient of $\bar{G}$. More precisely:

(a) Let $\bar{G} \in \mathbf{gc}_R$ and let $\hat{\pi}\colon \hat{G} \to \bar{G}$ be its Steinberg group. Let $\pi\colon G \to \bar{G}$ be an object of $\mathbf{st}(\bar{G})$ and $\varphi\colon \hat{G} \to G$ the unique morphism such that $\hat{\pi} = \pi \circ \varphi$. By 3.11(c), φ is an object of $\mathbf{st}(G)$, and it is in fact an initial object. Indeed, let $\varrho\colon H \to G$ be in $\mathbf{st}(G)$. Then $\pi \circ \varrho\colon H \to G \to \bar{G}$ belongs to $\mathbf{st}(\bar{G})$, so by the universal property of $\hat{G}$ there exists a unique $\varphi'\colon \hat{G} \to H$ making the diagram

$$\begin{tikzcd}
& H \arrow[dr, "\varrho"] & \\
\hat{G} \arrow[ur, dotted, "\varphi'"] \arrow[rr, "\varphi"] \arrow[dr, "\hat{\pi}"'] & & G \arrow[dl, "\pi"] \\
& \bar{G} &
\end{tikzcd}$$

commutative. Now φ and $\varrho \circ \varphi'$ are morphisms from $\hat{G}$ to G in $\mathbf{st}(\bar{G})$, so $\varphi = \varrho \circ \varphi'$ by 3.11(b1).

(b) Let N be a normal subgroup of $\bar{G}$ such that $N \cap \bar{U}_\alpha = N \cap \bar{U}_{[\alpha,\beta]} = \{1\}$ for all α and all nilpotent pairs (α, β) in R. Let $\kappa\colon \bar{G} \to \bar{G}/N$ be the canonical homomorphism. Then $\bar{G}/N$ has commutator relations with root groups $\kappa(\bar{U}_\alpha)$ by

2.4(e), and we have $\mathrm{St}(\bar G) = \mathrm{St}(\bar G/N)$, more precisely, $\kappa \circ \hat\pi \colon \hat G \to \bar G \to \bar G/N$ is the Steinberg group of $\bar G/N$. Indeed, the assumptions on N imply that κ satisfies (i) and (ii) of 3.9, so $\kappa \in \mathbf{st}(\bar G/N)$. Now the claim follows from (a) (with G and $\bar G$ replaced by $\bar G$ and $\bar G/N$, respectively), and the essential uniqueness of the Steinberg group.

The Steinberg group commutes with the direct sum construction of Lemma 3.2:

3.15. Lemma. *Let $\bar G_i$ be a family of groups in $\mathbf{gc}_{R_i}$ and let $\bar G = \bigoplus \bar G_i \in \mathbf{gc}_R$ as in Lemma 3.2. Let $\hat\pi_i \colon \hat G_i \to \bar G_i$ be the Steinberg groups of the $\bar G_i$. Then the Steinberg group of $\bar G$ is $\bigoplus \hat\pi_i \colon \bigoplus \hat G_i \to \bar G$.*

Proof. Put $\hat G = \bigoplus \hat G_i$ and $\hat\pi = \bigoplus \hat\pi_i$. We show first that $\hat\pi$ belongs to the Steinberg category $\mathbf{st}(\bar G)$, so we must verify the conditions (i) and (ii) of 3.9. But this follows easily from the definition of the respective root groups in Lemma 3.2 and the fact that for $\alpha, \beta \in R = \bigcup R_i$, the root interval $\big[\alpha, \beta\big]$ is empty unless α and β belong to the same component of R.

Next we show that $\hat\pi$ has the required universal property. Thus let $\pi \colon G \to \bar G$ be in $\mathbf{st}(\bar G)$ and let $H_i = U_{R_i}$ be the subgroup of G generated by all U_α, $\alpha \in R_i$. From the R-commutator relations in G it follows that

$$\big(H_i, H_j\big) = \{1\} \quad \text{for } i \neq j \tag{1}$$

but note that $H_i \cap H_j$ may be non-trivial. We identify $\bar G_i$ with a subgroup of $\bar G$ by means of the injection into the ith factor. Then $\pi(H_i) \subset \bar G_i$, and $\pi_i := \pi\big|H_i \colon H_i \to \bar G_i$ belongs to the Steinberg category $\mathbf{st}(\bar G_i)$. By the universal property of $\hat\pi_i$, there exist unique $\varphi_i \colon \hat G_i \to H_i$ making the diagrams

$$\begin{tikzcd} \hat G_i \arrow[rr, "\varphi_i"] \arrow[dr, "\hat\pi_i"'] & & H_i \arrow[dl, "\pi_i"] \\ & \bar G_i & \end{tikzcd}$$

commutative. Now define $\varphi \colon \hat G \to G$ by

$$\varphi\big((g_i)_{i\in I}\big) = \prod_i \varphi_i(g_i). \tag{2}$$

Since only finitely many factors g_i are different from 1 and the subgroups H_i of G commute pairwise by (1), this is a well-defined group homomorphism, and it makes the diagram

$$\begin{tikzcd} \hat G \arrow[rr, "\varphi"] \arrow[dr, "\hat\pi"'] & & G \arrow[dl, "\pi"] \\ & \bar G & \end{tikzcd}$$

commutative.

It remains to show uniqueness of φ. Suppose also $\pi \circ \psi = \hat\pi$. Then $\psi_i := \psi\big|H_i$ satisfies $\pi_i \circ \psi_i = \hat\pi_i$, so we have $\psi_i = \varphi_i$ by the universal property of $\hat\pi_i$, and therefore $\psi = \varphi$ by (2).

3.16. Another construction of the Steinberg group. Generalizing again [**59**, 3.6], we now give a more concrete (but less canonical) description of $\mathrm{St}(\bar{G})$ in case $\bar{G}$ has unique factorization for nilpotent pairs, as defined in 2.15.

For every nilpotent pair (α, β) choose an ordering $\big(\alpha, \beta\big)_{\mathrm{ind}} = \{\gamma_1, \dots, \gamma_n\}$ (where $n = n_{\alpha\beta}$ will of course depend on α, β). Then there are well-defined functions $f^i_{\alpha\beta}\colon \bar{U}_\alpha \times \bar{U}_\beta \to \bar{U}_{\gamma_i}$ such that

$$\big(\bar{a}, \bar{b}\big) = \prod_{i=1}^{n} f^i_{\alpha\beta}(\bar{a}, \bar{b}), \tag{1}$$

for all $\bar{a} \in \bar{U}_\alpha$, $\bar{b} \in \bar{U}_\beta$. (The $f^i_{\alpha\beta}$ will in general depend on the chosen ordering).

Let F be the free product of the groups $\bar{U}_\alpha$, $\alpha \in R$, and let $h_\alpha\colon \bar{U}_\alpha \to F$ be the canonical injections. Let N be the normal subgroup of F generated by all

$$h_\beta(\bar{b})^{-1} \cdot h_\alpha(\bar{b}), \tag{2}$$

$$\big(h_\alpha(\bar{a}), h_\beta(\bar{b})\big)^{-1} \cdot \prod_{i=1}^{n} h_{\gamma_i}\big(f^i_{\alpha\beta}(\bar{a}, \bar{b})\big), \tag{3}$$

where $\alpha\big|\beta$ and $\bar{b} \in \bar{U}_\beta$ in the first formula, and (α, β) is nilpotent and $\bar{a} \in \bar{U}_\alpha$, $\bar{b} \in \bar{U}_\beta$ in the second. Let $L := F/N$ and denote by can: $F \to L$ the canonical map. We define $k_\alpha := \mathrm{can} \circ h_\alpha\colon \bar{U}_\alpha \to L$ and put $Y_\alpha = k_\alpha(\bar{U}_\alpha) \subset L$.

3.17. Theorem. *Suppose $(\bar{G}, (\bar{U}_\alpha)_{\alpha \in R}) \in \mathbf{gc}_R$ has unique factorization for nilpotent pairs. Then, with the notations of* 3.16*, $(L, (Y_\alpha)_{\alpha \in R})$ belongs to $\mathbf{gc}_R$, and there exists a unique homomorphism $\hat{\pi}\colon L \to \bar{G}$ making L an initial object of $\widetilde{\mathbf{st}}(\bar{G})$ (and hence L "is" the Steinberg group $\mathrm{St}(\bar{G})$).*

Proof. We note first that L has R-commutator relations with root groups Y_α and is generated by the Y_α, i.e., $(L, (Y_\alpha))$ is an object in $\mathbf{gc}_R$. Indeed, since F is generated by the $h_\alpha(\bar{U}_\alpha)$, it follows that L is generated by the Y_α. Next, if $\alpha\big|\beta$, we have $Y_\beta \subset Y_\alpha$ by applying can to the relations (3.16.2), and the commutator relations (2.2.3) follow in the same way by applying can to (3.16.3).

Now let $\pi\colon G \to \bar{G}$ be an object of $\mathbf{st}(\bar{G})$; in particular, $G = (G, (U_\alpha)) \in \mathbf{gc}_R$. Since F is the free product of the $\bar{U}_\alpha$ and π is bijective on root groups, there is a homomorphism $\kappa\colon F \to G$ making all diagrams

$$\begin{tikzcd} \bar{U}_\alpha \arrow[r, "h_\alpha"] & F \arrow[d, "\kappa"] \\ U_\alpha \arrow[u, "\pi", "\cong"'] \arrow[r, "\mathrm{inc}"'] & G \end{tikzcd} \qquad (\alpha \in R) \tag{1}$$

commutative. We show that κ factors via F/N, i.e., that all elements of type (3.16.2) and (3.16.3) belong to the kernel of κ. Let us first remark that

$$\pi\big(\kappa(h_\alpha(\bar{a}))\big) = \bar{a}, \quad \text{for all } \bar{a} \in \bar{U}_\alpha,\ \alpha \in R. \tag{2}$$

Indeed, by (1) we have $a = \kappa\big(h_\alpha(\pi(a))\big)$ for all $a \in U_\alpha$. Applying π yields $\pi(a) = \pi\big(\kappa(h_\alpha(\pi(a)))\big)$, and since $\pi\colon U_\alpha \to \bar{U}_\alpha$ is in particular surjective, (2) follows.

Let $\alpha\big|\beta$ and $\bar{b} = \pi(b) \in \bar{U}_\beta$. Then $\bar{U}_\beta \subset \bar{U}_\alpha$ and (2) imply

$$\pi\big(\kappa(h_\beta(\bar{b}))\big) = \bar{b} = \pi\big(\kappa(h_\alpha(\bar{b}))\big).$$

Since $\pi\big|U_\alpha$ is injective, we conclude $\kappa(h_\beta(\bar{b})) = \kappa(h_\alpha(\bar{b}))$, so κ vanishes on elements of type (3.16.2).

Next, let (α, β) be a nilpotent pair, and let $\bar{a} \in \bar{U}_\alpha$, $\bar{b} \in \bar{U}_\beta$. We must show that

$$\kappa\big(h_\alpha(\bar{a}),\, h_\beta(\bar{b})\big) = \kappa\Big(\prod_{i=1}^{n} h_{\gamma_i}\big(f^i_{\alpha\beta}(\bar{a}, \bar{b})\big)\Big). \tag{3}$$

By (1), $\kappa(h_\alpha(\bar{a})) \in U_\alpha$. Hence the left hand side is, since κ is a homomorphism and G has R-commutator relations,

$$\kappa\big(h_\alpha(\bar{a}),\, h_\beta(\bar{b})\big) = \big(\kappa(h_\alpha(\bar{a})),\, \kappa(h_\alpha(\bar{b}))\big) \in \big(U_\alpha, U_\beta\big) \subset U_{(\alpha,\beta)}.$$

In the same way, one sees that the right hand side belongs to $U_{\gamma_1} \cdots U_{\gamma_n} = U_{(\alpha,\beta)}$, cf. Proposition 2.12(b). Now π satisfies (ii) of 3.9, in particular $\pi\colon U_{(\alpha,\beta)} \to \bar{U}_{(\alpha,\beta)}$ is injective. Hence it suffices that (3) hold after applying π. This follows now immediately from (2) and (3.16.1): The left hand side is

$$\pi\big(\kappa(\big(h_\alpha(\bar{a}),\, h_\beta(\bar{b})\big))\big) = \big(\bar{a}, \bar{b}\big) = \prod_{i=1}^{n} f^i_{\alpha\beta}(\bar{a}\bar{b}),$$

while the right hand side is

$$\prod_{i=1}^{n} \pi\Big(\kappa\big(h_{\gamma_i}(f^i_{\alpha\beta}(\bar{a}, \bar{b}))\big)\Big) = \prod_{i=1}^{n} f^i_{\alpha\beta}(\bar{a}, \bar{b}).$$

Hence also the generators of type (3.16.3) belong to the kernel of κ.

We now have a unique homomorphism $\psi\colon F/N = L \to G$ satisfying $\kappa = \psi \circ \mathrm{can}\colon F \to L \to G$, and from (2) it follows that

$$\pi\big(\psi(k_\alpha(\bar{a}))\big) = \bar{a}, \quad \text{for all } \bar{a} \in \bar{U}_\alpha,\ \alpha \in R.$$

Since $k_\alpha(\bar{U}_\alpha) = Y_\alpha$ by definition of the root groups in L, we see that $\psi\colon Y_\alpha \to U_\alpha$ is an isomorphism. Since $\bar{G}$ has unique factorization for nilpotent pairs, 3.9(b) shows that ψ satisfies condition (ii) of 3.9. Thus $\psi\colon L \to G$ is an object of $\mathbf{st}(G)$.

By specializing $\pi\colon G \to \bar{G}$ to $\mathrm{Id}\colon \bar{G} \to \bar{G}$ we obtain a morphism $\hat{\pi}\colon L \to \bar{G}$ which satisfies (i) and (ii) of 3.9, so that L, equipped with the subgroups Y_α and with $\hat{\pi}$, is an object of $\mathbf{st}(\bar{G})$. Now the above proof shows that it is in fact an initial object of $\mathbf{st}(\bar{G})$. This completes the proof.

3.18. Examples. (a) Let $R = \{0, 1, -1\}$ be the root system A_1 and let $\bar{G} \in \mathbf{gc}_R$. Since R contains only the nilpotent pairs $(1,1)$ and $(-1,-1)$, $\bar{G}$ is simply a group generated by two abelian subgroups $\bar{U}_1$ and $\bar{U}_{-1}$. It is immediately seen that the free product of $\bar{U}_1$ and $\bar{U}_{-1}$ is an initial object of $\mathbf{st}(\bar{G})$. This applies in particular to the projective elementary group of a Jordan pair, cf. §9.

(b) We take up Example (b) of 2.16 and let $\bar{G}$ be a group with R-commutator relations and root groups $U_i = \bar{U}_{\varepsilon_i}$. Suppose that $\bar{G}$ has unique factorization for all nilpotent pairs, i.e., that $U_i \cap U_j = \{0\}$ for $i \neq j$. Then it follows easily from Theorem 3.17 that $L = \bigoplus_{i=1}^n U_i$, with $\hat{\pi}(x_1 \oplus \cdots \oplus x_n) = x_1 + \cdots + x_n$, is the Steinberg group of $\bar{G}$. As shown in 2.16, $\bar{G}$ has unique factorization if and only if $\bar{G} = \bigoplus_{i=1}^n U_i$, i.e., if and only if $L = \bar{G}$ is its own Steinberg group.

(c) The linear elementary group $\mathrm{E}_I(A)$ has unique factorization for nilpotent pairs by Example (c) of 2.16. Hence it follows easily from Theorem 3.17 that $\mathrm{St}(\mathrm{E}_I(A))$ is the usual Steinberg group, at least when I is finite or countably infinite:

$$\mathrm{St}(\mathrm{E}_n(A)) = \mathrm{St}_n(A), \quad \mathrm{St}(\mathrm{E}_{\mathbb{N}}(A)) = \mathrm{St}(A)$$

in the notation of [**17**, 1.4], see also 10.16 where we will relate $\mathrm{St}_n(A)$ to the Steinberg group of the Jordan pair $V = (\mathrm{Mat}_{pq}(A), \mathrm{Mat}_{qp}(A))$.

(d) Similarly, the usual elementary unitary group $\mathrm{EU}_{2n}(A, \Lambda)$ ($n \geqslant 3$) of a form ring (A, Λ) in the sense of [**17**] has C_n-commutator relations, and

$$\mathrm{St}(\mathrm{EU}_{2n}(A, \Lambda)) = \mathrm{StU}_{2n}(A, \Lambda)$$

is the usual unitary Steinberg group.

3.19. Example: Tits' Steinberg group. Let $R = \Delta^{\mathrm{re}} \cup \{0\}$, where Δ^{re} is the set of real roots of the partial root system associated to a generalized Cartan matrix, see [**42**, Example 3.1(c)]. In [**59**, Prop. 1], Tits proves the existence of $\mathbb{Z}$-group schemes $\mathbf{U}_\alpha$, $\alpha \in \Delta^{\mathrm{re}}$, and $\mathbf{U}_{[\alpha,\beta]}$, (α, β) nilpotent, as well as monomorphisms $\mathfrak{i}^\gamma_{\alpha\beta}\colon \mathbf{U}_\gamma \to \mathbf{U}_{[\alpha,\beta]}$ for all $\gamma \in [\alpha, \beta]$ such that for any order on $[\alpha, \beta]$ the associated multiplication map

$$\mathfrak{m}\colon \prod_{\gamma \in [\alpha,\beta]} \mathbf{U}_\gamma \to \mathbf{U}_{[\alpha,\beta]}, \quad (u_\gamma) \mapsto \prod_\gamma \mathfrak{i}^\gamma_{\alpha\beta}(u_\gamma) \tag{1}$$

is an isomorphism of schemes. We note that Tits' result actually holds for all subsets A that are T-nilpotent [**42**] with respect to the standard positive system Δ^{re}_+. It is however only the special case $A = [\alpha, \beta]$ for (α, β) a T-nilpotent pair that will be needed below. Because of [**42**, Cor. 3.8] the concepts of a T-nilpotent pair with respect to Δ^{re}_+ and of a nilpotent pair in our sense are the same.

Let now k be a commutative ring. For $\alpha \in R^\times$, nilpotent pairs $(\alpha, \beta) \subset R$ and all $\gamma \in [\alpha, \beta]$ define groups U_α, $U_{[\alpha,\beta]}$ and maps $i^\gamma_{\alpha\beta}$ by evaluating the corresponding functors at k:

$$U_\alpha = \mathbf{U}_\alpha(k), \quad U_{[\alpha,\beta]} = \mathbf{U}_{[\alpha,\beta]}(k) \quad \text{and} \quad i^\gamma_{\alpha\beta} = \mathfrak{i}^\gamma_{\alpha\beta}(k).$$

Then the assumptions of Lemma 3.12 are fulfilled. Indeed, since R is reduced we have $\beta \big| \alpha \iff \beta = \alpha$, so that there are no conditions arising from divisible roots. The assumption (3.12.3) is immediate from (1), while (3.12.4) is shown in the proof of Prop. 1 on [**59**, p. 560/1]. Thus, by Theorem 3.13, the inductive limit $G = \mathfrak{St}(k)$ of the U_α and $U_{[\alpha,\beta]}$ is a group in $\mathbf{gc}_R$ which is its own Steinberg group. We note that this group enters into the definition of Tits' Kac-Moody group functor.

§4. Reflection systems

4.1. Reflection systems. We will introduce a subcategory of the category **SV** of 1.5 by requiring the existence of reflections for certain roots.

Let X be a vector space over a field $\Bbbk$ of characteristic zero. Recall that a (hyperplane) reflection is an element σ of $\mathrm{GL}(X)$ with $\sigma^2 = \mathrm{Id}$ and fixed point set a hyperplane. We denote by $\mathrm{Ref}(X)$ the union of $\{\mathrm{Id}_X\}$ and all hyperplane reflections of X, thus considering Id_X as an improper reflection.

Now let $(R, X) \in \mathbf{SV}$, and let $s\colon R \to \mathrm{Ref}(X)$ be a map, written $\alpha \mapsto s_\alpha$. We denote by

$$R^{\mathrm{re}} := \{\alpha \in R : s_\alpha \neq \mathrm{Id}\}$$

the set of *reflective roots*. The triple (R, X, s) is called a *reflection system* if the following axioms hold for all $\alpha \in R$:

(ReS1) $\alpha \in R^{\mathrm{re}}$ implies $\alpha \neq s_\alpha(\alpha) = -\alpha \in R^{\mathrm{re}}$;

(ReS2) $s_\alpha(R) = R$ and $s_\alpha(R^{\mathrm{re}}) = R^{\mathrm{re}}$.

(ReS3) $s_{c\alpha} = s_\alpha$ whenever $c \in \Bbbk^\times$ and both α and $c\alpha$ belong to R^{re}, and

(ReS4) $s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha$ for all $\alpha, \beta \in R$.

By abuse of notation we will often refer to a reflection system simply by R or (R, X) instead of (R, X, s).

Let (S, Y, s) be a second reflection system. Unless this might lead to confusion, we will use the same letter s for the maps $R \to \mathrm{Ref}(X)$ and $S \to \mathrm{Ref}(Y)$. A *morphism* $f\colon (R, X, s) \to (S, Y, s)$ is a linear map $f\colon X \to Y$ such that $f(R) \subset S$ and

$$f(s_\alpha(\beta)) = s_{f(\alpha)}(f(\beta)), \tag{1}$$

for all $\alpha, \beta \in R$. We denote by **ReS** the category of reflection systems, which is thus a subcategory of the category **SV**.

Note that (ReS1) and (ReS2) imply $0 \notin R^{\mathrm{re}}$ and $R^{\mathrm{re}} = -R^{\mathrm{re}}$. The automorphism group of (R, X, s) is denoted by $\mathrm{Aut}(R, X, s)$ or simply by $\mathrm{Aut}(R)$. The condition (ReS4) is equivalent to the condition $s_\alpha \in \mathrm{Aut}(R)$ for all $\alpha \in R^{\mathrm{re}}$. The subgroup of $\mathrm{Aut}(R)$ generated by all s_α, $\alpha \in R$, is called the *Weyl group* of R and denoted $W(R)$. It is a normal subgroup of $\mathrm{Aut}(R)$.

Let (R, X, s) be a reflection system. For every reflective root α, there exists a unique linear form $\alpha^\vee$ on X such that s_α is given by the familiar formula

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha. \tag{2}$$

In particular, $s_\alpha(\alpha) = -\alpha \iff \langle \alpha, \alpha^\vee \rangle = 2$. For $\alpha \in R \setminus R^{\mathrm{re}}$ we put $\alpha^\vee = 0$. Then ${}^\vee\colon R \to X^*$ is a well-defined map and (2) holds for all $\alpha \in R$ and $x \in X$. Conversely, given $(R, X) \in \mathbf{SV}$ with a map ${}^\vee\colon R \to X^*$, taking (2) as the definition of s_α and putting $R^{\mathrm{re}} = \{\alpha \in R : \alpha^\vee \neq 0\}$, the axioms of a reflection system can also be phrased in terms of $(R, X, {}^\vee)$, see [**42**, 2.3] for details.

4.2. Elementary properties of reflection systems. Let (R, X, s) be a reflection system. We will say R is

(i) *reduced* if $\alpha \in R^{\mathrm{re}}$, $c \in \Bbbk^\times$ and $c\alpha \in R^{\mathrm{re}}$ imply $c = \pm 1$;
(ii) *integral* if $\langle R, R^\vee \rangle \subset \mathbb{Z}$;
(iii) *symmetric* if $R = -R$.

For a reflection system (R, X, s) and $\alpha, \beta \in R^{\mathrm{re}}$ we have ([**42**, (2.11)])

$$s_\alpha = s_\beta \quad \Longleftrightarrow \quad \beta \in \Bbbk^\times \alpha. \tag{1}$$

Moreover, if R is integral then by [**42**, (2.10)],

$$\alpha \in R^{\mathrm{re}} \text{ and } c\alpha \in R^{\mathrm{re}} \text{ for some } c \in \Bbbk^\times \quad \Longrightarrow \quad c \in \{\pm\tfrac{1}{2}, \pm 1, \pm 2\}, \tag{2}$$

and $\alpha/2$ and 2α cannot both be in R^{re}.

4.3. Direct sums. A family $(R_i, X_i)_{i \in I}$ in **SV** has the coproduct

$$(R, X) = \coprod_{i \in I} (R_i, X_i) = \big(\bigcup_{i \in I} R_i, \bigoplus_{i \in I} X_i\big)$$

cf. [**40**, 1.2]. Following tradition, we also write $R = \bigoplus_{i \in I} R_i$ and call R the *direct sum* of the R_i.

If each R_i is a reflection system so is R. Indeed, we extend each s_{α_i}, $\alpha_i \in R_i$, to a reflection on X by $s_{\alpha_i}\big|X_j = \mathrm{Id}$ for $i \neq j$, and in this way obtain a map $s\colon R \to \mathrm{Ref}(X)$ which is easily seen to satisfy (ReS1) – (ReS4). The linear form on X corresponding to $\alpha_i \in R_i$ is just the extension by zero of $\alpha_i^\vee$. It is immediate that $R^{\mathrm{re}} = \bigcup_{i \in I} R_i^{\mathrm{re}}$, and $W(R) \cong \bigoplus_{i \in I} W(R_i)$, the restricted direct product of the $W(R_i)$.

4.4. Subsystems. Let $R = (R, X, s)$ be a reflection system. A *subsystem* of R is a subset R' of R satisfying $0 \in R'$ and $s_\alpha(\beta) \in R'$ for all $\alpha, \beta \in R'$. Equivalently, the inclusion $(R', \mathrm{span}(R')) \hookrightarrow (R, X)$ is a morphism of reflection systems. It follows from (ReS2) that

$$\mathrm{Re}(R) := R^{\mathrm{re}} \cup \{0\}$$

is always a subsystem of any reflection system R. Similarly,

$$R_{\mathrm{ind}} := \{\alpha \in R^{\mathrm{re}} : \frac{\alpha}{2} \notin R^{\mathrm{re}}\} \cup \{0\}, \tag{1}$$

is a subsystem. We call its elements the *indivisible roots*. By (4.2.2), R_{ind} is a reduced subsystem of any integral reflection system R.

4.5. Examples of reflection systems. The most important examples of reflection systems for this book are the finite or locally finite root systems. Indeed, the usual finite root systems in the sense of [**10**] (augmented by 0) are precisely the reflection systems which are finite, integral, and satisfy $R^{\mathrm{re}} = R \setminus \{0\}$, and the locally finite root systems have a similar description, see below.

Other important examples of reflection systems are: the roots of a Kac-Moody Lie algebra with R^{re} being the real roots, the extended affine root systems occurring in extended affine Lie algebras, the roots of classical Lie superalgebras, the non-crystallographic finite root systems, or the root systems associated to the geometric representation of Coxeter groups. The reader can find many more examples in [**42**, 2.10, 2.12, 3.1, and 4.3]. Moreover, this paper also introduces partial root systems which form an important subcategory of **ReS**.

4.6. Locally finite root systems [40]. For the convenience of the reader we now give a more explicit description of locally finite root systems.

A *locally finite root system* is a pair (R, X) consisting of a vector space X over $\Bbbk$ and a subset R such that the following conditions hold:

(i) $0 \in R$, and R spans X as a vector space,
(ii) for every $\alpha \in R^\times = R \setminus \{0\}$ there exists $\alpha^\vee$ in the dual X^* of X such that $\langle \alpha, \alpha^\vee \rangle = 2$, $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\beta \in R$, and $s_\alpha(R) = R$ for $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$,
(iii) R is locally finite in the sense that $R \cap Y$ is finite for every finite-dimensional subspace Y of X.

As in [**40**] the term *root system* will be an abbreviation for "locally finite root system", and a *finite root system* will be a root system (R, X) with $\mathrm{Card}(R) < \infty$, equivalently, $\dim X < \infty$. The *rank* of a root system (R, X) is by definition the dimension of X.

We will consider root systems as a subcategory of **SV** or of **ReS**, as required by the context. But we note that an isomorphism between root systems in **SV** satisfies (4.1.1), so that isomorphisms between root systems are the same in the categories **SV** and **ReS** ([**40**, 3.6]).

A direct sum of root systems is again a root system. A nonzero root system is called *irreducible* if it is not isomorphic to a direct sum of two nonzero root systems. Any root system decomposes uniquely into a direct sum of irreducible root subsystems, called its *irreducible components* [**42**, 3.13] which we will describe now.

4.7. Classification of root systems. Let I be a non-empty set, let $X = \mathrm{span}_\Bbbk(I) = \bigoplus_{i \in I} \mathbb{R}\varepsilon_i$ be the free vector space on the set I, and let $\dot{X} = \mathrm{Ker}(t) \subset X$ be the kernel of the *trace form* t, defined as the linear form on X taking the value 1 on each ε_i. We define

$$\dot{\mathrm{A}}_I = \{\varepsilon_i - \varepsilon_j : i, j \in I\}, \tag{1}$$
$$\mathrm{D}_I = \dot{\mathrm{A}}_I \cup \{\pm(\varepsilon_i + \varepsilon_j) : i \neq j\}, \tag{2}$$
$$\mathrm{B}_I = \mathrm{D}_I \cup \{\pm\varepsilon_i : i \in I\}, \tag{3}$$
$$\mathrm{C}_I = \mathrm{D}_I \cup \{\pm 2\varepsilon_i : i \in I\} = \{\pm\varepsilon_i \pm \varepsilon_j : i, j \in I\}, \tag{4}$$
$$\mathrm{BC}_I = \mathrm{B}_I \cup \mathrm{C}_I = \{\pm\varepsilon_i : i \in I\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : i, j \in I\}. \tag{5}$$

Then $\dot{\mathrm{A}}_I$ is a root system in $\dot{X}$ and the others are root systems in X, with the exception of D_I for $|I| = 1$ where $\mathrm{D}_I = \{0\}$ does not span X. The rank of $\dot{\mathrm{A}}_I$ is $\mathrm{Card}(I) - 1$ while the rank in the other cases is $\mathrm{Card}(I)$. The notation $\dot{\mathrm{A}}$ (instead of the traditional A) serves to indicate this fact. For a finite I, say $|I| = n$, we will use the standard notation $\mathrm{B}_n = \mathrm{B}_I$, $\mathrm{C}_n = \mathrm{C}_I$, $\mathrm{D}_n = \mathrm{D}_I$ and $\mathrm{BC}_n = \mathrm{BC}_I$, while the usual notation A_n is linked to our notation by

$$\mathrm{A}_n = \dot{\mathrm{A}}_{\{0,1,\dots,n\}} = \dot{\mathrm{A}}_{n+1}.$$

A root system R is called *classical* if it is isomorphic to one of the root systems (1) – (5) for a suitable, possibly infinite, set I.

The *exceptional root systems* are the well-known finite irreducible root systems of type $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4$ and G_2, see for example [**10**]. An irreducible root system is either classical or isomorphic to an exceptional root system [**40**, Th. 8.4].

Recall the notion of prenilpotent subset introduced in 1.9. The following lemma gives a detailed description of the prenilpotent two-element subsets of locally finite root systems.

4.8. Lemma. *Let (R, X) be a locally finite root system, and let $\{\alpha, \beta\} \subset R^\times$. Then*

$$\{\alpha, \beta\} \textit{ is prenilpotent} \iff \beta \notin \{-\alpha, -2\alpha, -\frac{\alpha}{2}\}. \tag{1}$$

Assume this to be the case and put $R_{\alpha\beta} := R \cap \big(\mathbb{Z}\alpha + \mathbb{Z}\beta\big)$ and $C := \big(\alpha, \beta\big)$. Then

$$\big[\alpha, \beta\big] = \{\alpha\}^c \cup C \cup \{\beta\}^c \tag{2}$$

is nilpotent of class $k \leqslant 5$ and of cardinality $\leqslant 6$. Moreover, $\mathrm{Card}\, C \leqslant 4$, $\mathrm{Card}\, \big(C, C\big) \leqslant 1$, *and $C \neq \emptyset$ if and only if $\alpha + \beta \in R$.*

Proof. By (1.5.2) and 1.12, $\{\alpha, \beta\}$ is not prenilpotent if and only if there exist $m, n \in \mathbb{N}$, $m + n > 0$, such that $m\alpha + n\beta = 0$. Since $\alpha, \beta \neq 0$, this is equivalent to $m\alpha + n\beta = 0$ for some $m, n \in \mathbb{N}_+$, i.e., $-\beta$ is a positive rational multiple of α. Since R is locally finite we have $R^\times \cap \mathbb{Q}\alpha \subset \{\pm\alpha, \pm 2\alpha, \pm\alpha/2\}$ which proves (1). The formula (2) is a consequence of (1.6.9). The remaining assertions follow easily from the classification of root systems of rank $\leqslant 2$ in [**10**]. The details are left to the reader.

Note that (2) easily implies

$$\mathscr{C}^2\big(\big[\alpha, \beta\big]\big) = \big(\{\alpha\}^c \setminus \{\alpha\}\big) \cup C \cup \big(\{\beta\}^c \setminus \{\beta\}\big).$$

Also, $\{\alpha\}^c = \{\alpha, 2\alpha\}$ or $\{\alpha\}^c = \{\alpha\}$ depending on whether 2α does or does not belong to R.

We now list the cases where $C \neq \emptyset$ in more detail. It is no restriction to assume that $\|\alpha\| \leqslant \|\beta\|$ with respect to some invariant inner product.

Case	$\langle\alpha,\beta^\vee\rangle$	$\langle\beta,\alpha^\vee\rangle$	$C = \boldsymbol{(}\alpha,\beta\boldsymbol{)}$	k	$\lvert\boldsymbol{[}\alpha,\beta\boldsymbol{]}\rvert$	$R_{\alpha\beta}$
1	2	2	2α	2	2	BC_1
2	1	1	$\alpha+\beta$	2	3	G_2
3	0	0	$\alpha+\beta$	2	3 or 5	B_2 or BC_2
4	-1	-1	$\alpha+\beta$	2	3	A_2
5	-1	-1	$\alpha+\beta,\ 2\alpha+\beta,\ \alpha+2\beta$	3	5	G_2
6	-1	-2	$\alpha+\beta,\ 2\alpha+\beta$	3	4	B_2
7	-1	-2	$\alpha+\beta,\ 2\alpha+\beta,\ 2\alpha+2\beta$	4	6	BC_2
8	-1	-3	$\alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,$ $3\alpha+2\beta$	5	6	G_2

Remarks. We put $B := \{\alpha,\beta\}$.

Case 1: Here $\alpha=\beta$.

Case 2: α and β are two short roots of G_2 whose sum is a long root.

Case 3: α and β are weakly orthogonal short roots.

Case 4: B is a root basis of A_2.

Case 5: B is a root basis for the subsystem of short roots of G_2.

Case 6: $R_{\alpha\beta}=\mathrm{B}_2$ and B is a root basis of B_2.

Case 7: B is a root basis of BC_2.

Case 8: B is a root basis of G_2.

§5. Weyl elements I

5.1. Weyl elements and Weyl triples. Let $(R, X, s) \in \mathbf{ReS}$ be a reflection system, see 4.1, and let $(G, \mathscr{U}) \in \mathbf{gc}_R$ be a group with R-commutator relations and root groups U_α as in 2.2. Let $\alpha \in \mathrm{Re}(R) = R^{\mathrm{re}} \cup \{0\}$ and let s_α be the reflection associated to α. An element $w \in G$ is called a *generalized Weyl element for* α if conjugation by w realizes the reflection s_α on the root groups in the sense that

$$w\, U_\beta\, w^{-1} = U_{s_\alpha(\beta)} \quad \text{for all } \beta \in R. \tag{1}$$

We say w is a *Weyl element for* α if it is a generalized Weyl element for α and $w \in U_\alpha\, U_{-\alpha}\, U_\alpha$. This follows the terminology of Faulkner [**15**]. Thus a Weyl element for α has a representation $w = u_1 u_2 u_3$ where $u_1, u_3 \in U_\alpha$ and $u_2 \in U_{-\alpha}$. In general, w does not determine the triple (u_1, u_2, u_3) uniquely. Therefore, we define: a *Weyl triple for* α is a triple $t = (u_1, u_2, u_3) \in U_\alpha \times U_{-\alpha} \times U_\alpha$ such that $\mu(t) = u_1 u_2 u_3$ is a Weyl element for α. Here μ denotes the multiplication map.

We denote by $M_\alpha = M_\alpha(G)$ the (possibly empty) set of generalized Weyl elements for the root α, by $W_\alpha = W_\alpha(G)$ the set of Weyl elements for α, and by $\mathfrak{T}_\alpha = \mathfrak{T}_\alpha(G)$ the set of Weyl triples for α. Strictly speaking, we should write $M_\alpha(G, (U_\beta)_{\beta \in R})$ or $M_\alpha(G, \mathscr{U})$ etc. since the notions of (generalized) Weyl element and Weyl triple depend of course on the family of root groups $(U_\beta)_{\beta \in R}$. But we will use the simplified notation, hoping that the reader will keep this dependence in mind. Thus

$$W_\alpha = M_\alpha \cap \big(U_\alpha U_{-\alpha} U_\alpha\big) \subset M_\alpha \quad \text{and} \quad \mu\colon \mathfrak{T}_\alpha \to W_\alpha \quad \text{is surjective.} \tag{2}$$

Note in particular that

$$M_0 = \bigcap_{\beta \in R} \mathrm{Norm}_G(U_\beta), \quad W_0 = \{1\}, \quad \mathfrak{T}_0 = \{(1,1,1)\}, \tag{3}$$

since by our conventions, $0 \in \mathrm{Re}(R)$, $s_0 = \mathrm{Id}$ and $U_0 = \{1\}$.

It is convenient to consider also the following sets:

$$\Theta_\alpha(G) = U_\alpha \times U_{-\alpha} \times U_\alpha, \qquad \Theta(G) = \coprod_{\alpha \in R} \Theta_\alpha(G) \tag{4}$$

as well as

$$\mathfrak{T}(G) = \coprod_{\alpha \in \mathrm{Re}(R)} \mathfrak{T}_\alpha(G). \tag{5}$$

Clearly, $\Theta_\alpha(G)$ and $\Theta(G)$ depend functorially on G: if $\varphi\colon G \to H$ is a morphism of $\mathbf{gc}_R$ then $\Theta_\alpha(\varphi)\colon \Theta_\alpha(G) \to \Theta_\alpha(H)$ is defined component-wise by $(x_1, x_2, x_3) \mapsto (\varphi(x_1), \varphi(x_2), \varphi(x_3))$. The sets $M_\alpha(G)$, $W_\alpha(G)$ and $\mathfrak{T}_\alpha(G)$ in general do not depend functorially on G. However, if φ is surjective on root groups, then $\varphi\big(M_\alpha(G)\big) \subset M_\alpha(H)$ which is seen by applying φ to (1). This easily implies $\varphi\big(W_\alpha(G)\big) \subset W_\alpha(H)$ and $\mathfrak{T}_\alpha(\varphi)\big(\mathfrak{T}_\alpha(G)\big) \subset \mathfrak{T}_\alpha(H)$ as well, where we define $\mathfrak{T}_\alpha(\varphi) = \Theta_\alpha(\varphi)\big|\mathfrak{T}_\alpha(G)$.

5.2. Example. Let $G = \mathrm{GL}_2(A)$ where A is a (unital associative) ring. We view G as a group with commutator relations with root system $R = \mathrm{A}_1 = \{0, \pm 1\}$ and root groups $U_{\pm 1} = U^{\pm} = \mathrm{e}_{\pm}(A)$, where $\mathrm{e}_{\pm}\colon A \to G$ is defined by

$$\mathrm{e}_+(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{e}_-(y) = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}.$$

For $u, v \in A^\times$, the set of units of A, we define

$$\mathrm{w}_{u,v} = \begin{pmatrix} 0 & u \\ -v^{-1} & 0 \end{pmatrix} \in G, \quad \text{with inverse} \quad \mathrm{w}_{u,v}^{-1} = \begin{pmatrix} 0 & -v \\ u^{-1} & 0 \end{pmatrix} = \mathrm{w}_{-v,-u}.$$

Straightforward matrix calculations show that

$$M_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in A^\times \right\}, \quad M_1 = \{\mathrm{w}_{u,v} : u, v \in A^\times\} = M_{-1}. \tag{1}$$

The action of $\mathrm{w}_{u,v}$ on the root groups is given by

$$\mathrm{w}_{u,v}\, \mathrm{e}_+(x)\, \mathrm{w}_{u,v}^{-1} = \mathrm{e}_-(v^{-1} x u^{-1}), \quad \mathrm{w}_{u,v}\, \mathrm{e}_-(y)\, \mathrm{w}_{u,v}^{-1} = \mathrm{e}_+(uyv). \tag{2}$$

We now determine the Weyl elements for $\alpha = 1$ and claim that

$$\mathrm{w}_{u,v} \in W_1 \quad \iff \quad u = v.$$

Indeed, if $u = v$ then by straightforward calculation,

$$\mathrm{w}_{u,u} = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix} = \mathrm{e}_+(u)\, \mathrm{e}_-(u^{-1})\, \mathrm{e}_+(u) \in U^+ U^- U^+.$$

Conversely, suppose that

$$\mathrm{w}_{u,v} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in U^+ U^- U^+.$$

By working out the product on the right, we obtain

$$\mathrm{w}_{u,v} = \begin{pmatrix} 0 & u \\ -v^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 - xy & x + z - xyz \\ -y & 1 - yz \end{pmatrix}.$$

This shows that $y \in A^\times$ with inverse $y^{-1} = x = z$, so $x = u$ and $y = v = x^{-1}$. Hence the Weyl elements for $\alpha = 1$ are precisely the elements

$$\mathrm{w}_u = \mathrm{w}_{u,u} = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}. \tag{3}$$

By interchanging the roles of 1 and -1, one sees that the Weyl elements for the root -1 are the elements

$$\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \begin{pmatrix} 1 & u^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} = \begin{pmatrix} 0 & u^{-1} \\ -u & 0 \end{pmatrix} = \mathrm{w}_{u^{-1}}, \quad u \in A^\times. \tag{4}$$

Hence, the Weyl triples for $\alpha = \sigma 1$ are the elements

$$t_\sigma(u) = \big(\mathrm{e}_\sigma(u),\, \mathrm{e}_{-\sigma}(u^{-1}),\, \mathrm{e}_\sigma(u)\big), \quad u \in A^\times. \tag{5}$$

In particular, this shows that $W_\alpha \subsetneq M_\alpha$ as soon as $A^\times$ has at least two elements. We also see that here the multiplication maps $\mu\colon \mathfrak{T}_\alpha \to W_\alpha$ are bijective.

All this can be generalized to the group $\mathrm{GL}_n(A)$, viewed as group with A_{n-1}-commutator relations, see for example [**17**, 1.4E].

More examples of groups with (generalized) Weyl elements will be given later in 6.7, 6.9 and 6.10. We will see that apart from the obvious inclusion $W_\alpha \subset M_\alpha$, no other general relation is true.

5.3. Proposition. *Let $(R, X, s) \in$ **ReS** be a reflection system and let G be a group with R-commutator relations.*

(a) *The sets M_α and W_α, $\alpha \in \mathrm{Re}(R)$, satisfy the following relations:*

$$M_\alpha = M_\alpha^{-1}, \qquad W_\alpha = W_\alpha^{-1}, \tag{1}$$

$$M_\alpha = M_{r\alpha} \qquad \text{if } r \in \mathbb{k}^\times \text{ and } r\alpha \in \mathrm{Re}(R), \tag{2}$$

$$W_\alpha = W_{-\alpha}, \qquad W_{n\alpha} \subset W_\alpha \quad \text{if } n \in \mathbb{N}_+ \text{ and } n\alpha \in R, \tag{3}$$

$$M_\alpha = M_0 M_\alpha M_0, \qquad W_\alpha = (M_0 \cap U_\alpha) W_\alpha (M_0 \cap U_\alpha). \tag{4}$$

In particular, if w is a (generalized) Weyl element for α then so is w^{-1}, and both w and w^{-1} are (generalized) Weyl elements for $-\alpha$.

(b) *If M_α is not empty and $m_\alpha \in M_\alpha$ then $M_\alpha = m_\alpha M_0 = M_0 m_\alpha$ is a coset of M_0, and*

$$M_\alpha M_\alpha = M_0, \tag{5}$$

$$m_\alpha M_\beta m_\alpha^{-1} = M_\alpha M_\beta M_\alpha = M_{s_\alpha \beta}, \tag{6}$$

$$m_\alpha W_\beta m_\alpha^{-1} = W_{s_\alpha \beta}, \tag{7}$$

$$m_\alpha \mathfrak{T}_\beta m_\alpha^{-1} = \mathfrak{T}_{s_\alpha \beta}, \tag{8}$$

where $\beta \in \mathrm{Re}(R)$ and conjugation by m_α in (8) is understood componentwise.

Proof. (a) Let $m \in M_\alpha$. Since $s_\alpha^2 = \mathrm{Id}$ we have $m^{-1} U_\beta m = m^{-1} U_{s_\alpha^2 \beta} m = m^{-1} m^2 U_\beta m^{-2} m = m U_\beta m^{-1} = U_{s_\alpha \beta}$ for all $\beta \in R$, whence $m^{-1} \in M_\alpha$. This proves $M_\alpha = M_\alpha^{-1}$. Furthermore, $W_\alpha^{-1} = \big(M_\alpha \cap (U_\alpha U_{-\alpha} U_\alpha)\big)^{-1} = M_\alpha^{-1} \cap (U_\alpha U_{-\alpha} U_\alpha)^{-1} = M_\alpha \cap (U_\alpha U_{-\alpha} U_\alpha) = W_\alpha$. Next, (2) follows immediately from the definition of M_α and the fact that $s_\alpha = s_\beta$ whenever $\beta = r\alpha \in \mathrm{Re}(R)$ for some $r \in \mathbb{k}^\times$, by (ReS3) in 4.1. Now let $w \in W_\alpha$. Then $s_\alpha(\pm\alpha) = \mp\alpha$ implies

$$\begin{aligned} w = w w w^{-1} &\in w \cdot U_\alpha U_{-\alpha} U_\alpha \cdot w^{-1} \\ &= w U_\alpha w^{-1} \cdot w U_{-\alpha} w^{-1} \cdot w U_\alpha w^{-1} = U_{-\alpha} U_\alpha U_{-\alpha}, \end{aligned}$$

and therefore $w \in W_{-\alpha}$. Thus $W_\alpha \subset W_{-\alpha}$ and then also $W_{-\alpha} \subset W_{-(-\alpha)} = W_\alpha$. Furthermore, if $n\alpha \in R$ then by (5.1.2) and (2.2.2), $W_{n\alpha} = M_{n\alpha} \cap (U_{n\alpha} U_{-n\alpha} U_{n\alpha}) \subset M_\alpha \cap (U_\alpha U_{-\alpha} U_\alpha) = W_\alpha$. Finally, (4) follows immediately from the definitions of M_α and (5.1.3).

(b) If $m, m' \in M_\alpha$ then $mm'U_\beta m'^{-1}m^{-1} = mU_{s_\alpha\beta}m^{-1} = U_{s_\alpha^2\beta} = U_\beta$ for all $\beta \in R$, so $mm' \in M_0$. Furthermore, since $m^{-1} \in M_\alpha$ by (1), we have $m' = m(m^{-1}m') = (m'm^{-1})m \in mM_0 \cap M_0m$, so $M_\alpha = mM_0 = M_0m$ is a coset of M_0, and we also see $M_\alpha^2 = M_0m \cdot mM_0 = M_0$. If $m_\alpha \in M_\alpha$ and $m_\beta \in M_\beta$ (where now $\alpha, \beta \in \mathrm{Re}(R)$) then, for all $\gamma \in R$,

$$m_\alpha m_\beta m_\alpha^{-1} U_\gamma m_\alpha m_\beta^{-1} m_\alpha^{-1} = U_{s_\alpha s_\beta s_\alpha \gamma} = U_{s_{s_\alpha\beta}\gamma}$$

by (ReS4) in 4.1. This shows $m_\alpha M_\beta m_\alpha^{-1} \subset M_{s_\alpha\beta}$, and in fact we have equality because from $s_\alpha^2 = \mathrm{Id}$ and $m_\alpha^{-1} \in M_\alpha$ we see $m_\alpha^{-1} M_{s_\alpha\beta} m_\alpha \subset M_{s_\alpha^2\beta} = M_\beta$. The second equation of (6) now follows from $M_\alpha M_\beta M_\alpha = M_0 m_\alpha M_\beta m_\alpha^{-1} M_0 = M_0 M_{s_\alpha\beta} M_0 = M_{s_\alpha\beta}$. Finally,

$$\begin{aligned} m_\alpha W_\beta m_\alpha^{-1} &= m_\alpha(M_\beta \cap (U_\beta U_{-\beta} U_\beta)) m_\alpha^{-1} \\ &= m_\alpha M_\beta m_\alpha^{-1} \cap m_\alpha(U_\beta U_{-\beta} U_\beta) m_\alpha^{-1} \\ &= M_{s_\alpha\beta} \cap (U_{s_\alpha\beta} U_{-s_\alpha\beta} U_{s_\alpha\beta}) = W_{s_\alpha\beta}, \end{aligned}$$

and (8) is an immediate consequence of (7) and the definitions.

5.4. An algebraic structure on the set of Weyl triples. Let G be a group with R-commutator relations and root groups U_α and let $\mathfrak{T}_\alpha = \mathfrak{T}_\alpha(G)$ be the set of Weyl triples for $\alpha \in \mathrm{Re}(R)$. Let $\mathfrak{T} = \mathfrak{T}(G)$ be as in (5.1.5). We define the following operations on $\mathfrak{T}$. First, let $x \in \mathfrak{T}_\alpha$, say, $x = (x_1, x_2, x_3) \in U_\alpha \times U_{-\alpha} \times U_\alpha$, and let $w = \mu(x) = x_1x_2x_3$ be the corresponding Weyl element. By (5.3.1), $w^{-1} = x_3^{-1}x_2^{-1}x_1^{-1}$ is again a Weyl element for α. Hence, the triple

$$x^{-1} := (x_3^{-1}, x_2^{-1}, x_1^{-1}) \in \mathfrak{T}_\alpha. \tag{1}$$

This yields a unary operation $(\ )^{-1}\colon \mathfrak{T} \to \mathfrak{T}$ which maps each $\mathfrak{T}_\alpha$ to itself and is obviously involutive:

$$(x^{-1})^{-1} = x. \tag{2}$$

Next, consider the triples

$$x^\sharp := (wx_3w^{-1}, x_1, x_2), \qquad x^\flat := (x_2, x_3, w^{-1}x_1w). \tag{3}$$

Since $w\,U_\alpha\,w^{-1} = U_{-\alpha}$, we see that $x^\sharp$ and $x^\flat$ are in $U_{-\alpha} \times U_\alpha \times U_{-\alpha}$. Moreover,

$$\mu(x^\sharp) = wx_3x_3^{-1}x_2^{-1}x_1^{-1} \cdot x_1x_2 = w \tag{4}$$

and similarly $\mu(x^\flat) = w$, so $x^\sharp$ and $x^\flat$ are Weyl triples for the root $-\alpha$. This yields two more unary operations on $\mathfrak{T}$.

Let x and w be as before, and let $y = (y_1, y_2, y_3) \in \mathfrak{T}_\beta$. Then

$$x \bullet y := \mathrm{Int}\bigl(\mu(x)\bigr) \cdot y = (wy_1w^{-1}, wy_2w^{-1}, wy_3w^{-1}) \in \mathfrak{T}_{s_\alpha(\beta)} \tag{5}$$

by (5.3.8). This defines a binary operation $\bullet$ on $\mathfrak{T}$. Finally, we let $1 = (1,1,1) \in \mathfrak{T}_0$ and define a projection $p\colon \mathfrak{T} \to \mathrm{Re}(R)$ by mapping the elements of $\mathfrak{T}_\alpha$ to α.

Example. Let $G = \mathrm{GL}_2(A)$ as in 5.2. Then the unary operations are

$$t_\sigma(u)^{-1} = t_\sigma(-u), \qquad t_\sigma(u)^\sharp = t_\sigma(u)^\flat = t_{-\sigma}(u^{-1}). \tag{6}$$

This follows easily from (5.2.2). For the products $\bullet$ one computes

$$t_+(u) \bullet t_-(v) = t_+(uvu). \tag{7}$$

From this, one obtains formulas of type $t_\sigma(u) \bullet t_\tau(v)$ by using (5.5.2), for example $t_+(u) \bullet t_+(v) = t_+(u) \bullet t_-(v^{-1})^\sharp = (t_+(u) \bullet t_-(v^{-1}))^\sharp = t_+(uv^{-1}u)^\sharp = t_-(u^{-1}vu^{-1})$.

5.5. Lemma. (a) *The operations just introduced satisfy the following rules.*

$$\begin{aligned}
x \bullet (x^{-1} \bullet y) = x^{-1} \bullet (x \bullet y) &= y && (1)\\
(x \bullet y)^{-1} = x \bullet y^{-1}, \quad (x \bullet y)^\sharp &= x \bullet y^\sharp, \quad (x \bullet y)^\flat = x \bullet y^\flat, && (2)\\
x \bullet (y \bullet z) &= (x \bullet y) \bullet (x \bullet z), && (3)\\
1 \bullet x &= x, \quad x \bullet 1 = 1, && (4)\\
p(x \bullet y) &= s_{p(x)} p(y), \quad p(x^{-1}) = p(x), \quad p(x^\sharp) = p(x^\flat) = -p(x), && (5)\\
(x^\sharp)^\flat &= (x^\flat)^\sharp = x, && (6)\\
(x^\sharp)^{-1} &= (x^{-1})^\flat, \quad (x^\flat)^{-1} = (x^{-1})^\sharp, && (7)\\
x^{\sharp\sharp\sharp} &= x \bullet x, \quad x^{\flat\flat\flat} = x^{-1} \bullet x. && (8)
\end{aligned}$$

(b) *Let $\varphi\colon G \to H$ be a morphism of $\mathbf{gc}_R$ which is surjective on root groups and define $\mathfrak{T}(\varphi)\colon \mathfrak{T}(G) \to \mathfrak{T}(H)$ by $\mathfrak{T}(\varphi)\big|\mathfrak{T}_\alpha = \mathfrak{T}_\alpha(\varphi)$ as in 5.1. Then $\mathfrak{T}(\varphi)$ preserves the algebraic operations.*

Proof. (a) (1) is immediate from the definition, since $\mu(x^{-1}) = x_3^{-1}x_2^{-1}x_1^{-1} = \mu(x)^{-1}$. To prove the first formula of (2), put $w = \mu(x)$ and observe that

$$\begin{aligned}
(x \bullet y)^{-1} &= \Big(\mathrm{Int}\, w \cdot (y_1, y_2, y_3)\Big)^{-1} = (wy_1w^{-1}, wy_2w^{-1}, wy_3w^{-1})^{-1}\\
&= (wy_3^{-1}w^{-1}, wy_2^{-1}w^{-1}, wy_1^{-1}w^{-1}) = \mathrm{Int}(w) \cdot y^{-1} = x \bullet y^{-1}.
\end{aligned}$$

The proof of the second and third formula is similar. For (3), we compute

$$\begin{aligned}
x \bullet (y \bullet z) &= \mathrm{Int}\, \mu(x) \cdot \big(\mathrm{Int}\, \mu(y) \cdot z\big) = \mathrm{Int}(\mu(x)\mu(y)) \cdot z\\
&= \mathrm{Int}(\mu(x)\mu(y)\mu(x)^{-1})\, \mathrm{Int}(\mu(x) \cdot z).
\end{aligned}$$

On the other hand,

$$\mu(x)\mu(y)\mu(x)^{-1} = \mu\big(\mathrm{Int}(\mu(x)) \cdot y\big) = \mu(x \bullet y),$$

from which the assertion follows. It is obvious that (4) and (5) hold.

For (6), observe $(x^\sharp)^\flat = (wx_3w^{-1}, x_1, x_2)^\flat = (x_1, x_2, w^{-1}wx_3w^{-1}w) = x$, and similarly for the second formula. The first formula of (7) follows from $(x^\sharp)^{-1} = (wx_3w^{-1}, x_1, x_2)^{-1} = (x_2^{-1}, x_1^{-1}, wx_3^{-1}w^{-1}) = (x^{-1})^\flat$, and the second formula is proved similarly. Finally, since $\mu(x^\sharp) = \mu(x^\flat) = \mu(x) = w$, we have

$$\begin{aligned} x^{\sharp\sharp\sharp} &= (wx_3w^{-1}, x_1, x_2)^{\sharp\sharp} = (wx_2w^{-1}, wx_3w^{-1}, x_1)^\sharp \\ &= (wx_1w^{-1}, wx_2w^{-1}, wx_3w^{-1}) = x \bullet x, \end{aligned}$$

and similarly for the second formula.

(b) This follows immediately from the definitions.

Note that the left multiplications $L_x \colon y \mapsto x \bullet y$ are bijective, with $L_x^{-1} = L_{x^{-1}}$, by (1). Formulas (2) and (3) say that L_x is an automorphism of $\mathfrak{T}$, equipped with the algebraic structures of multiplication, inversion, $\sharp$ and $\flat$. By (6), $\flat$ is just the inverse map of $\sharp$, and by (7), $\flat$ can also be defined in terms of $\sharp$ and inversion as

$$x^\flat = ((x^{-1})^\sharp)^{-1} \quad \text{and} \quad x^\sharp = ((x^{-1})^\flat)^{-1}. \tag{9}$$

For example, this can be used to establish the formulas for $\flat$ in (2) and (8), once the corresponding formula for $\sharp$ has been established.

5.6. Subsystems. We say a subset $\mathfrak{S}$ of $\mathfrak{T}$ is *closed* or a *subsystem* if it contains the element $1 \in \mathfrak{T}_0$ and is closed under the operations of multiplication, inversion, and $\sharp$ or, equivalently, $\flat$. If $\mathfrak{X} \subset \mathfrak{T}$ is an arbitrary subset, the closure of $\mathfrak{X}$ (or the subsystem generated by $\mathfrak{X}$), denoted $\langle \mathfrak{X} \rangle$, is defined as the smallest subsystem of $\mathfrak{T}$ containing $\mathfrak{X}$. Its existence and uniqueness is clear: just take the intersection of all subsystems containing $\mathfrak{X}$, this set being non-empty because $\mathfrak{T}$ belongs to it. We now give a more explicit description of the closure of a subset $\mathfrak{X}$.

Example. Let $G = \mathrm{GL}_2(A)$ as in 5.2 and let $\mathfrak{X} = \mathfrak{T}_1 = \{t_+(u) : u \in A^\times\}$ as in (5.2.5). Then $\langle \mathfrak{X} \rangle = \mathfrak{T}$ follows from (5.4.6).

5.7. Lemma. *Let $\mathfrak{X} \subset \mathfrak{T}$ be an arbitrary subset.*

(a) *The set $\langle \mathfrak{X} \rangle$ is obtained as follows. Let $\mathfrak{Y} = \mathfrak{X} \cup \mathfrak{X}^{-1} \cup \{1\}$ and put*

$$\tilde{\mathfrak{Y}} := \bigcup_{n \in \mathbb{Z}} \mathfrak{Y}^{n\sharp},$$

where $x^{n\sharp} = x^{\sharp\cdots\sharp}$ (n times) for $n \geqslant 0$ and $x^{n\sharp} = x^{\flat\cdots\flat}$ (n times) for $n < 0$. Then $\langle \mathfrak{X} \rangle$ is the set of all finite products, with arbitrary parentheses, of elements taken from $\tilde{\mathfrak{Y}}$.

(b) *If $\varphi \colon G \to H$ is a morphism of $\mathbf{gc}_R$ which is surjective on root groups, then $\mathfrak{T}(\varphi)(\langle \mathfrak{X} \rangle) = \langle \mathfrak{T}(\varphi)(\mathfrak{X}) \rangle$.*

Proof. (a) Clearly, $\tilde{\mathfrak{Y}} \subset \langle \mathfrak{X} \rangle$. It follows from (5.5.6) and (5.5.7) that $\tilde{\mathfrak{Y}}$ is stable under the unary operations $(\ )^{-1}$, $\sharp$ and $\flat$. Let $\mathfrak{P}$ be the set of all products of elements from $\tilde{\mathfrak{Y}}$. Then clearly $\mathfrak{P} \subset \langle \mathfrak{X} \rangle$ so it suffices to show that $\mathfrak{P}$ is a subsystem. Evidently, $\mathfrak{P}$ contains 1 and is closed under products. To show it is closed under the unary operations, we use induction on the length of a product. Products of length 1 are just the elements of $\tilde{\mathfrak{Y}}$. A product of length $n > 1$ is of the form $a \bullet b$ where a and b are products of length $< n$. Then $(a \bullet b)^{-1} = a \bullet b^{-1}$ by (5.5.2), and by induction $b^{-1} \in \mathfrak{P}$. Hence $(a \bullet b)^{-1} \in \mathfrak{P}$ as well. Similarly, one shows that $\mathfrak{P}$ is stable under $\sharp$ and $\flat$, using the second and third formula of (5.5.2).

(b) This follows from Lemma 5.5(b).

5.8. Steinberg categories defined by sets of Weyl triples. Let $\bar{G} = (\bar{G}, \bar{U}_\alpha) \in \mathbf{gc}_R$ and let $\mathbf{st}(\bar{G})$ be the corresponding Steinberg category as in 3.9. We define full subcategories of $\mathbf{st}(\bar{G})$ depending on a set $\bar{\mathfrak{X}}$ of Weyl triples of $\bar{G}$ as follows.

Let π: $(G, U_\alpha) \to (\bar{G}, \bar{U}_\alpha)$ be an object of $\mathbf{st}(\bar{G})$ as in 3.9 and define $\Theta(G)$ and $\Theta(\bar{G})$ as in (5.1.4). Since π is in particular bijective on root subgroups, the induced maps $\Theta(\pi)$: $\Theta(G) \to \Theta(\bar{G})$ sending a triple $x = (x_1, x_2, x_3) \in U_\alpha \times U_{-\alpha} \times U_\alpha$ to $\pi(x) = \big(\pi(x_1), \pi(x_2), \pi(x_3)\big)$, are bijective as well. By abuse of notation, we will often simply write π instead of $\Theta(\pi)$ or $\mathfrak{T}(\pi)$. For an element $t = (t_1, t_2, t_3) \in \Theta(\bar{G})$ we call $\pi^{-1}(t) \in \Theta(G)$ the *lift* of t to G. The lift of a Weyl triple for $\bar{G}$ will in general no longer be a Weyl triple for G. Therefore, we define the full subcategory $\mathbf{st}(\bar{G}, \bar{\mathfrak{X}})$ of $\mathbf{st}(\bar{G})$ by

$$\big(\pi\colon G \to \bar{G}\big) \in \mathbf{st}(\bar{G}, \bar{\mathfrak{X}}) \quad \iff \quad \pi^{-1}(\bar{\mathfrak{X}}) \subset \mathfrak{T}(G).$$

This subcategory has the following property: if $\pi \in \mathbf{st}(\bar{G}, \bar{\mathfrak{X}})$ and φ: $\pi \to \eta$ is a morphism of $\mathbf{st}(\bar{G})$ then also $\eta \in \mathbf{st}(\bar{G}, \bar{\mathfrak{X}})$. Indeed, the morphism φ is a commutative triangle

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & H \arrow[dl, "\eta"] \\ & \bar{G} & \end{tikzcd} \tag{1}$$

of morphisms of $\mathbf{gc}_R$, so φ is in particular bijective on root groups. Hence it induces a commutative triangle of bijections

$$\begin{tikzcd} \Theta(G) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \Theta(H) \arrow[dl, "\eta"] \\ & \bar{\Theta}(G) & \end{tikzcd}$$

which implies

$$\varphi(\pi^{-1}(t)) = \eta^{-1}(t) \tag{2}$$

for all $t \in \Theta(\bar{G})$. Since φ is surjective on root groups, the image of a Weyl triple of G under φ is a Weyl triple of H, as noted in 5.1. Hence $\eta^{-1}(\bar{\mathfrak{X}}) = \varphi(\pi^{-1}(\bar{\mathfrak{X}})) \subset \varphi(\mathfrak{T}(G)) \subset \mathfrak{T}(H)$.

5.9. Lemma. *The subcategories* $\mathbf{st}(\bar{G}, \bar{\mathfrak{X}})$ *have the following properties.*

$$\mathbf{st}(\bar{G}, \emptyset) = \mathbf{st}(\bar{G}, \{1\}) = \mathbf{st}(\bar{G}), \tag{1}$$

$$\bar{\mathfrak{X}} \subset \bar{\mathfrak{Y}} \quad \Longrightarrow \quad \mathbf{st}(\bar{G}, \bar{\mathfrak{X}}) \supset \mathbf{st}(\bar{G}, \bar{\mathfrak{Y}}), \tag{2}$$

$$\mathbf{st}(\bar{G}, \bar{\mathfrak{X}}) = \mathbf{st}(\bar{G}, \langle \bar{\mathfrak{X}} \rangle). \tag{3}$$

Proof. (1) and (2) are evident from the definition. We prove (3). Since $\bar{\mathfrak{X}} \subset \langle \bar{\mathfrak{X}} \rangle$, we have $\mathbf{st}(\bar{G}, \bar{\mathfrak{X}}) \supset \mathbf{st}(\bar{G}, \langle \bar{\mathfrak{X}} \rangle)$ by (2). Conversely, let $\pi\colon G \to \bar{G}$ belong to $\mathbf{st}(\bar{G}, \bar{\mathfrak{X}})$, so $\mathfrak{X} := \pi^{-1}(\bar{\mathfrak{X}}) \subset \mathfrak{T}(G)$. We must show that $\pi^{-1}(\langle \bar{\mathfrak{X}} \rangle) \subset \mathfrak{T}(G)$ as well. By 5.4, $\langle \mathfrak{X} \rangle \subset \mathfrak{T}(G)$, so it suffices to show that $\pi^{-1}(\langle \bar{\mathfrak{X}} \rangle) = \langle \mathfrak{X} \rangle$. But this follows from Lemma 5.7 and bijectivity of π on $\Theta(G)$: $\pi\big(\langle \mathfrak{X} \rangle\big) = \big\langle \pi(\mathfrak{X}) \big\rangle = \langle \bar{\mathfrak{X}} \rangle$.

5.10. Theorem. *Let R be a reflection system, let $\bar{G} \in \mathbf{gc}_R$ be a group with R-commutator relations and let $\bar{\mathfrak{X}}$ be a set of Weyl triples for $\bar{G}$. Then the category $\mathbf{st}(\bar{G}, \bar{\mathfrak{X}})$ is a reflective subcategory of the Steinberg category $\mathbf{st}(\bar{G})$: the inclusion functor $i\colon \mathbf{st}(\bar{G}, \bar{\mathfrak{X}}) \to \mathbf{st}(\bar{G})$ has a left adjoint $\ell : \mathbf{st}(\bar{G}) \to \mathbf{st}(\bar{G}, \bar{\mathfrak{X}})$.*

Proof. We put $\bar{\mathfrak{X}}_\alpha = \mathfrak{X} \cap \mathfrak{T}_\alpha(\bar{G})$, so that $\bar{\mathfrak{X}} = \coprod_{\alpha \in \mathrm{Re}(R)} \bar{\mathfrak{X}}_\alpha$. Let $\pi\colon G \to \bar{G}$ be an object of $\mathbf{st}(\bar{G})$, let $t \in \bar{\mathfrak{X}}_\alpha$ be a Weyl triple, and let $\bar{w}_t = \mu(t)$ be the corresponding Weyl element. Also let $x = \pi^{-1}(t) \in \Theta(G)$ be the lift of t to G. Since π is bijective on root groups and $\bar{w}_t$ is a Weyl element for α in $\bar{G}$, there exists, for every $\beta \in R$, a unique isomorphism $f^G_{\alpha\beta}(t)\colon U_\beta \to U_{s_\alpha(\beta)}$ making the diagram

$$\begin{tikzcd} U_\beta \arrow[r, "f^G_{\alpha\beta}(t)"] \arrow[d, "\pi"', "\cong"] & U_{s_\alpha(\beta)} \arrow[d, "\cong"', "\pi"] \\ \bar{U}_\beta \arrow[r, "\mathrm{Int}(\bar{w}_t)"'] & \bar{U}_{s_\alpha(\beta)} \end{tikzcd} \tag{1}$$

commutative. It is clear that $w^G_t := \mu(x)$ is a Weyl element for α in G if and only if

$$\mathrm{Int}(w^G_t) \cdot u = f^G_{\alpha\beta}(t) \cdot u, \tag{2}$$

for all $u \in U_\beta$ and all $\beta \in R$. We now pass to the largest quotient of G for which the relations (2) hold. In more detail, let $K(G)$ be normal subgroup of G generated by all elements

$$Z^G(t, u, \alpha, \beta) := \big(\mathrm{Int}(w^G_t) \cdot u\big)^{-1} \big(f^G_{\alpha\beta}(t) \cdot u\big)$$

where $\alpha \in R^{\mathrm{re}}$, $\beta \in R$, $t \in \bar{\mathfrak{X}}_\alpha$ and $u \in U_\beta$. Let $\dot{G} = G/K(G)$ and let can: $G \to \dot{G}$ be the canonical map. By 3.11(d), π factors $\pi = \dot{\pi} \circ \mathrm{can}$, and $\dot{G}$ with root groups $\dot{U}_\alpha = \mathrm{can}(U_\alpha)$ and projection $\dot{\pi}$ belongs to $\mathbf{st}(\bar{G})$.

It follows from the definition of $K(G)$ that $\mathrm{can}(w^G_t) = w^{\dot{G}}_t$ is a Weyl element (and hence $\mathrm{can}(\pi^{-1}(t)) = \dot{\pi}^{-1}(t)$ is a Weyl triple) for α in $\dot{G}$, for all $t \in \bar{\mathfrak{X}}_\alpha$ and all $\alpha \in R^{\mathrm{re}}$. Hence $\dot{G}$ (more precisely, $\dot{\pi}$) belongs to $\mathbf{st}(\bar{G}, \bar{\mathfrak{X}})$.

We define the functor ℓ on objects by $\ell(\pi) = \dot\pi$, and on morphisms as follows. Let $\varphi\colon \pi \to \eta$ be a morphism of $\mathbf{st}(\bar G)$, cf. (5.8.1). To have an induced homomorphism $\dot\varphi\colon \dot G \to \dot H$ it suffices to show that $\varphi(K(G)) \subset K(H)$. By 3.11(b0), $\varphi\colon U_\beta \to V_\beta$ is an isomorphism, for all $\beta \in R$. Since φ is a group homomorphism we have, using (5.8.2),

$$\varphi(w_t^G) = \varphi(\mu(\pi^{-1}(t))) = \mu(\varphi(\pi^{-1}(t))) = \mu(\eta^{-1}(t)) = w_t^H. \tag{3}$$

From (1) and the analogous diagram for H and the fact that η is bijective on root groups it follows that

$$\varphi\big(f^G_{\alpha\beta}(t)\cdot u\big) = f^H_{\alpha\beta}(t)\cdot\varphi(u), \tag{4}$$

for all $u \in U_\alpha$. Now (3) and (4) imply

$$\varphi\big(Z^G(t,u,\alpha,\beta)\big) = Z^H(t,\varphi(u),\alpha,\beta) \in K(H).$$

Hence $\varphi(K(G)) \subset K(H)$, so we have an induced homomorphism $\dot\varphi\colon \dot G \to \dot H$ making the diagram

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\pi"] \arrow[dd, "\mathrm{can}"'] & & H \arrow[dl, "\eta"'] \arrow[dd, "\mathrm{can}"] \\
& \bar G & \\
\dot G \arrow[ur, "\dot\pi"] \arrow[rr, "\dot\varphi"'] & & \dot H \arrow[ul, "\dot\eta"']
\end{tikzcd}$$

commutative. Now a straightforward verification shows that $\dot\varphi$ is a morphism in $\mathbf{st}(\bar G)$ and that the assignments $\pi \mapsto \dot\pi$ on objects and $\varphi \mapsto \dot\varphi$ on morphisms define a functor $\ell\colon \mathbf{st}(\bar G) \to \mathbf{st}(\bar G, \bar{\mathfrak{X}})$.

It remains to show that ℓ is left adjoint to the inclusion functor i, that is, to find natural bijections

$$\mathrm{Mor}_{\mathbf{st}(\bar G,\bar{\mathfrak{X}})}\big(\ell(\pi),\eta\big) \cong \mathrm{Mor}_{\mathbf{st}(\bar G)}\big(\pi, i(\eta)\big), \tag{5}$$

for all $\pi\colon G \to \bar G$ in $\mathbf{st}(\bar G)$ and $\eta\colon H \to \bar G$ in $\mathbf{st}(\bar G, \bar{\mathfrak{X}})$. Thus let $\psi\colon \ell(\pi) \to \eta$ be a morphism of $\mathbf{st}(\bar G, \bar{\mathfrak{X}})$ and let $\mathrm{can}\colon G \to \dot G = \ell(G)$ be the canonical map. Then $\psi \circ \mathrm{can}\colon \pi \to i(\eta)$ is a morphism of $\mathbf{st}(\bar G)$. Conversely, let $\varphi\colon \pi \to i(\eta)$ be a morphism of $\mathbf{st}(\bar G)$. Since $H \in \mathbf{st}(\bar G, \bar{\mathfrak{X}})$, the elements $Z^H(t,v,\alpha,\beta)$ generating the normal subgroup $K(H)$ of H (in the notation used earlier) are all trivial, so $\dot H$ is canonically identified with H and therefore $\dot\varphi = \ell(\varphi)\colon \ell(\pi) \to \eta$ is a morphism in $\mathbf{st}(\bar G, \bar{\mathfrak{X}})$. It is easily verified that these constructions are natural and inverses of each other.

5.11. Corollary. *Let $\bar G \in \mathbf{gc}_R$ be a group with R-commutator relations and let $\bar{\mathfrak{X}}$ be a set of Weyl triples for $\bar G$. Let $\mathrm{St}(\bar G) = \hat\pi\colon \hat G \to \bar G$ be its Steinberg group in $\mathbf{st}(\bar G)$ as in Theorem* 3.13. *Then $\ell\big(\mathrm{St}(\bar G)\big)$ is an initial object in $\mathbf{st}(\bar G, \bar{\mathfrak{X}})$, called* *the* Steinberg group of $(\bar{G}, \bar{\mathfrak{X}})$ *and denoted* $\mathrm{St}(\bar{G}, \bar{\mathfrak{X}})$. *Moreover, the Steinberg group of* $(\bar{G}, \bar{\mathfrak{X}})$ *does not change when replacing* $\bar{\mathfrak{X}}$ *by its closure:*

$$\mathrm{St}(\bar{G}, \bar{\mathfrak{X}}) = \mathrm{St}(\bar{G}, \langle \bar{\mathfrak{X}} \rangle). \tag{1}$$

Proof. This is immediate from (5.10.5) and the fact that $\mathrm{St}(\bar{G})$ is an initial object in $\mathbf{st}(\bar{G})$. The last statement follows from (5.9.3).

More explicitly, let $\hat{\pi}\colon \hat{G} \to \bar{G}$ be the Steinberg group of $\bar{G}$. Then the Steinberg group of $(\bar{G}, \bar{\mathfrak{X}})$ is $\ell(\hat{\pi})\colon \hat{G}/K(\hat{G}) \to \bar{G}$. By abuse of notation, we will also speak of the group $\hat{G}/K(\hat{G})$ as of the *Steinberg group of* $(\bar{G}, \bar{\mathfrak{X}})$.

5.12. Example: $\mathrm{St}_2(A)$ **of a ring.** Let A be a unital associative ring. Recall that the *linear Steinberg group* $\mathrm{St}_2(A)$ is the group presented by generators $\mathrm{x}_\sigma(a)$, $a \in A$, $\sigma \in \{+, -\}$ and relations

$$\mathrm{x}_\sigma(a + b) = \mathrm{x}_\sigma(a)\mathrm{x}_\sigma(b), \tag{1}$$

$$\mathrm{w}_\sigma(u)\mathrm{x}_{-\sigma}(a)\mathrm{w}_\sigma(u)^{-1} = \mathrm{x}_\sigma(uau), \tag{2}$$

where $a, b \in A$, $u \in A^\times$ and $\mathrm{w}_\sigma(u) := \mathrm{x}_\sigma(u)\mathrm{x}_{-\sigma}(u^{-1})\mathrm{x}_\sigma(u)$. The reader will easily verify that this definition agrees with the one in [**45**, Def. 10.4], see also [**17**, p. 57], by setting $\mathrm{x}_+(a) = x_{12}(a)$ and $\mathrm{x}_-(a) = x_{21}(-a)$. We show that this group is the Steinberg group of an appropriately defined $(\bar{G}, \bar{\mathfrak{X}})$ in the sense of 5.11. First observe that (1) and (2) imply

$$\mathrm{w}_\sigma(u)\mathrm{x}_\sigma(a)\mathrm{w}_\sigma(u)^{-1} = \mathrm{x}_{-\sigma}(u^{-1}au^{-1}) \tag{3}$$

for $u \in A^\times$ and $a \in A$. Indeed, $\mathrm{w}_\sigma(u)^{-1} = \mathrm{w}_\sigma(-u)$ by (1), whence $\mathrm{x}_{-\sigma}(a) = \mathrm{w}_\sigma(-u)\mathrm{x}_\sigma(uau)\mathrm{w}_\sigma(-u)^{-1}$ by (2), so that (3) follows by replacing u by $-u$ and a by $u^{-1}au^{-1}$.

Let $R = \mathrm{A}_1 = \{0, \pm 1\}$. We have already noted in 2.3(a) that the objects of the category $\mathbf{gc}_R$ are the groups G with abelian subgroups $U^\pm = U_{\pm 1}$. In particular, this is so for the group $\bar{G} = \mathrm{GL}_2(A)$ of Example 5.2, with subgroups $\bar{U}^\pm = \mathrm{e}_\pm(A)$. It is immediately seen that the Steinberg group $\hat{G} = \mathrm{St}(\bar{G})$, in the sense of Theorem 3.13 can be identified with the free product $\bar{U}^+ * \bar{U}^-$, i.e., $\hat{G}$ is the group presented by generators $\mathrm{x}_\sigma(a)$, $a \in A$, and the relations (1).

For an object $\pi\colon G \to \bar{G}$ of $\mathbf{st}(\bar{G})$, let $\mathrm{x}_\sigma\colon A \to U^\sigma$ be the unique isomorphism satisfying $\pi(\mathrm{x}_\sigma(a)) = \mathrm{e}_\sigma(a)$, for all $a \in A$. Let $t = t_\sigma(u)$ be a Weyl triple of $\bar{G}$ as in (5.2.5). Then

$$\pi^{-1}(t) = \big(\mathrm{x}_\sigma(u),\, \mathrm{x}_{-\sigma}(u^{-1}),\, \mathrm{x}_\sigma(u)\big),$$

so $\mu(\pi^{-1}(t)) = \mathrm{w}_\sigma(u)$. By specializing the relations (5.2.2) to the case where $u = v$ one finds that $\pi^{-1}(t)$ is a Weyl triple for G if and only if the relations (2) and (3) hold. It follows easily that $\mathrm{St}_2(A)$ is canonically isomorphic to the Steinberg group $\mathrm{St}(\bar{G}, \bar{\mathfrak{X}})$ where $\bar{\mathfrak{X}}$ is the set of all Weyl triples of $\bar{G}$. In view of (5.11.1) and the example in 5.6, one obtains the same group by taking for $\bar{\mathfrak{X}}$ only the set $\mathfrak{T}_1(\bar{G})$ of all Weyl triples for the root $\alpha = 1$.

5.13. Balanced Weyl triples. We return to a group G with R-commutator relations and root groups U_α. Recall the unary operations $\sharp$ and $\flat$ on the set $\mathfrak{T}$ of Weyl triples from (5.4.3). We say that a Weyl triple x is *balanced* if $x^\sharp = x^\flat$.

We now derive some properties of balanced Weyl triples. First, the following conditions for a Weyl triple x with $w = \mu(x)$ are equivalent:

(i) x is balanced,
(ii) $x_1 = x_3$ and $wx_1w^{-1} = x_2$,
(iii) $x_1 = x_3$ and $wx_2w^{-1} = x_1$,
(iv) $x_1 = x_3$ and $x_1x_2x_1 = x_2x_1x_2$,

In this case, $x^\sharp = x^\flat = (x_2, x_1, x_2)$.

Indeed, let $x^\sharp = x^\flat$. By (5.4.3) this means

$$(wx_3w^{-1}, x_1, x_2) = (x_2, x_3, w^{-1}x_1w),$$

equivalently, $x_1 = x_3$ and $x_2 = wx_1w^{-1} = w^{-1}x_1w$. This proves (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii). Now suppose (ii) holds. Since $w = x_1x_2x_1$, it follows that $x_1x_2x_1x_1 = wx_1 = x_2w = x_2x_1x_2x_1$ whence $x_2x_1x_2 = x_1x_2x_1$, proving (iv). In the same way, one shows (iii) $\Longrightarrow$ (iv). Finally, suppose (iv) holds. Then $w = x_1x_2x_1 = x_2x_1x_2$, which implies $wx_3w^{-1} = wx_1w^{-1} = x_2x_1x_2 \cdot x_1 \cdot x_1^{-1}x_2^{-1}x_1^{-1} = x_2$, and $w^{-1}x_1w = x_1^{-1}x_2^{-1}x_1^{-1} \cdot x_1 \cdot x_2x_1x_2 = x_2$. But this says $x^\sharp = x^\flat = (x_2, x_1, x_2)$ by (5.4.3).

As a consequence of these characterizations, we note:

$$\text{\textit{Let x and y be balanced, having one component in common and satisfying $w = \mu(x) = \mu(y)$. Then $x = y$.}} \tag{1}$$

Indeed, if $x_1 = y_1$ then $x_2 = wx_1w^{-1} = y_2$ by (ii), and if $x_2 = y_2$ then $x_1 = y_1$ follows from (iii). See also Proposition 6.5 for a similar result.

From Lemma 5.5(a), one sees immediately that the set of balanced Weyl triples is stable under the unary operations $(\ )^{-1}$, $\sharp$ and $\flat$, as well as under all left multiplications by elements of $\mathfrak{T}$. In particular,

$$\text{\textit{the set of balanced Weyl triples is a subsystem of $\mathfrak{T}$,}} \tag{2}$$

$$\text{\textit{if $\mathfrak{X} \subset \mathfrak{T}$ is balanced, then so is $\langle \mathfrak{X} \rangle$.}} \tag{3}$$

Example. Let $G = \mathrm{GL}_2(A)$. Then by (5.4.6), all Weyl triples are balanced, so a Weyl triple is uniquely determined by any one of its components (which is also evident from the explicit formulas (5.2.3) and (5.2.5)). Moreover,

$$\mathrm{w}_u = \mu(t_+(u)) = \mu(t_-(u)) = \mathrm{e}_-(u^{-1})\mathrm{e}_+(u)\mathrm{e}_-(u^{-1}).$$

5.14. Lemma. *If $\varphi\colon G \to H$ is a morphism in $\mathbf{gc}_R$ which is surjective on root groups, then $\mathfrak{T}(\varphi)$ preserves balanced Weyl triples, and if φ is bijective on root groups, a Weyl triple $x \in \mathfrak{T}(G)$ is balanced if and only if $\mathfrak{T}(\varphi)(x)$ is balanced.*

Proof. The first statement follows immediately from Lemma 5.5(b). For the second, it suffices to remark that $\mathfrak{T}(\varphi)$ is injective if φ is bijective on root groups.

§6. Weyl elements II

6.1. Proposition. *Let $(R,X,s)\in\mathbf{ReS}$ and let G be a group with R-commutator relations.*

(a) *The sets*

$$S=\{\alpha\in\mathrm{Re}(R): M_\alpha\neq\emptyset\},\quad S'=\{\alpha\in\mathrm{Re}(R): W_\alpha\neq\emptyset\}\subset S$$

are subsystems of $\mathrm{Re}(R)$, *in particular, they are stable under the group $W(S)$ generated by all $\{s_\alpha:\alpha\in S\}$. The group $W(S')$ is normal in $W(S)$.*

(b) *Let M, M' and M'_0 be the subgroups of G generated by*

$$\bigcup_{\alpha\in S} M_\alpha,\qquad \bigcup_{\alpha\in S'} W_\alpha,\qquad \bigcup_{\alpha\in S'} W_\alpha^2,$$

respectively. Then M', M_0 and M'_0 are normal subgroups of M, and $M'_0\subset M'\cap M_0$.

Proof. (a) From (5.3.6) it follows that $\alpha,\beta\in S$ implies $s_\alpha\beta\in S$. Also, by (5.1.3), $0\in S'\subset S$, so S is a subsystem of $\mathrm{Re}(R)$. Moreover, (5.3.7) shows that $s_\alpha\beta\in S'$ for $\alpha\in S$ and $\beta\in S'$, whence S' is stable under $W(S)$. Since $W(S)$ and $W(S')$ are generated by the reflections in the roots of S and S', respectively, formula (ReS4) of 4.1 implies that $W(S')$ is normal in $W(S)$.

(b) By (5.3.7) we have M' normal in M, and normality of M_0 in M follows from 5.3(b): $m_\alpha M_0 m_\alpha^{-1}=M_\alpha m_\alpha^{-1}=M_0 m_\alpha m_\alpha^{-1}=M_0$, for any $m_\alpha\in M_\alpha$. Furthermore, $W_\alpha^2\subset M_\alpha^2=M_0$ whence $M'_0\subset M'\cap M_0$. That M'_0 is normal in M follows from (5.3.7).

6.2. Proposition. *Let (R,X) be a locally finite root system and let G be a group with R-commutator relations. We use the notations introduced in* 6.1.

(a) *There are surjective homomorphisms $\psi\colon W(S)\to M/M_0$ and $\psi'\colon W(S')\to M'/M'_0$ such that $\psi(s_\alpha)=M_\alpha$ $(\alpha\in S)$ and $\psi'(\beta)=W_\beta M'_0$ $(\beta\in S')$, respectively. If u denotes the map $\alpha\mapsto U_\alpha$ from R to the set $\mathbf{U}=u(R)$ of root subgroups of G, then the diagram*

$$\begin{tikzcd} W(S)\times R \arrow[r] \arrow[d, "\psi\times u"'] & R \arrow[d, "u"] \\ (M/M_0)\times\mathbf{U} \arrow[r] & \mathbf{U} \end{tikzcd} \tag{1}$$

is commutative, where the top map is given by the natural action of $W(S)$ on R, and the bottom map is induced from conjugation. There is a unique surjective homomorphism $\psi''\colon W(S)/W(S')\to M/M'M_0$ making the following diagram commutative with exact rows:

$$\begin{tikzcd}
& 1 \arrow[r] \arrow[d] & W(S') \arrow[r] \arrow[d, "\psi'"] & W(S) \arrow[r] \arrow[d, "\psi"] & W(S)/W(S') \arrow[r] \arrow[d, "\psi''"] & 1 \\
1 \arrow[r] & M_0 \cap M'/M'_0 \arrow[r] & M'/M'_0 \arrow[r] & M/M_0 \arrow[r] & M/M'M_0 \arrow[r] & 1
\end{tikzcd} \tag{2}$$

(b) *If ψ is injective (hence an isomorphism) then so are ψ' and ψ'', and $M'_0 = M_0 \cap M'$.*

(c) *If $u\big|S$ is injective then ψ is injective. If $u\big|S'$ is injective then ψ' is injective, and $M'_0 = M_0 \cap M'$.*

Remark. The reader of the proof below will notice that it is not crucial, though convenient, to assume that (R, X) is a locally finite root system. Rather, all that is needed is a reflection system (R, X) for which the subsystems (S, Y) and (S', Y') have the property that the groups $W(S)|Y = \{w|Y : w \in W(S)\}$ and $W(S')|Y' = \{w|Y' : w \in W(S')\}$ have a so-called presentation by conjugation. For $W(S)|Y$ this means that it is presented by generators g_α, $\alpha \in S$ and relations $g_\alpha^2 = 1$ for $\alpha \in S^\times$, $g_\alpha = g_\beta$ for α, β linearly dependent and $g_\alpha g_\beta g_\alpha = g_{s_\alpha(\beta)}$ for $\alpha, \beta \in S^\times$. This type of presentation exists not only in the case of a locally finite root system (R, S), but also if, for example, $W(S)|Y$ is a Coxeter group [**18**, Prop. 4.2].

Proof. (a) By (4.2.1), $s_\alpha = s_\beta$ for $\alpha, \beta \in R^\times$ if and only if α and β are linearly dependent. Let Y be the span of S. By [**40**, 5.8], the restriction map $W(S) \to \mathrm{GL}(Y)$ is injective. Hence (5.3.2) and (5.3.6) show that the cosets $M_\alpha \in M/M_0$ ($\alpha \in S^\times$) satisfy the relations of the presentation of $W(S)$ given in [**40**, Th. 5.12]. This proves the statement concerning ψ. Next, let $[W_\alpha] = W_\alpha M'_0$ be the coset in M'/M'_0 determined by W_α. For linearly dependent roots $\alpha = c\beta$ the factor c must be in $\{\pm 1, \pm 2, \pm(1/2)\}$. Hence (5.3.3) shows that $[W_\alpha] = [W_\beta]$ for all linearly dependent $\alpha, \beta \in S'$. By definition of M'_0 we have $[W_\alpha]^2 = 1$. Hence (5.3.7), specialized to the case where $m_\alpha \in W_\alpha$, shows that $[W_\alpha][W_\beta][W_\alpha] = [W_{s_\alpha\beta}]$ for all $\alpha, \beta \in S'$. Now the existence of ψ' follows again from [**40**, Th. 5.12].

The group M acts on $\mathbf{U}$ by conjugation, and clearly M_0 acts trivially, so that M/M_0 acts on $\mathbf{U}$. For $\alpha \in S$ we have $\psi(s_\alpha) = M_\alpha$, and by (5.3.6), this acts on $\mathbf{U}$ via $U_\beta \mapsto U_{s_\alpha\beta}$. Hence we have commutativity of (1) on the generators s_α of $W(S)$, which is sufficient. The rest is straightforward.

(b) Suppose ψ is injective and thus (by (a)) an isomorphism. From (2) it is then immediate that the maps ψ' and $M'/M'_0 \to M/M_0$ are injective. Hence ψ' is an isomorphism, and $M_0 \cap M' = M'_0$. Moreover, by chasing the diagram, one sees easily that ψ'' is injective as well, and therefore an isomorphism.

(c) Let u be injective on S. As we saw in the proof of (a), the restriction map $W(S) \to \mathrm{GL}(Y)$ is injective. Hence the group $W(S)$ acts faithfully on S by permutations. From (1) we thus deduce the commutative diagram

$$\begin{tikzcd}
W(S) \arrow[r] \arrow[d, "\psi"'] & \mathrm{Sym}(S) \arrow[d] \\
M/M_0 \arrow[r] & \mathrm{Sym}(u(S))
\end{tikzcd}$$

where Sym( ) denotes the symmetric group. In this diagram the top and the right arrow are injective and bijective, respectively. Therefore ψ is injective.

Finally, let u be injective on S'. Similarly as before, we have the commutative diagram

$$\begin{tikzcd}
W(S') \arrow[rr] \arrow[d, "\psi'"'] & & \mathrm{Sym}(S') \arrow[d] \\
M'/M'_0 \arrow[rr] \arrow[dr] & & \mathrm{Sym}(u(S')) \\
& M'/M_0 \cap M' \arrow[ur] &
\end{tikzcd}$$

from which the assertion readily follows.

6.3. Proposition. *Let (R,X) and (S,Y) be reflection systems, $f\colon (R,X) \to (S,Y)$ a morphism in* **SV** *and $(G,(U_\alpha)_{\alpha\in R}) \in \mathbf{gc}_R$ a group with R-commutator relations. Recall from Proposition* 3.3(a) *that G then also has S-commutator relations with root groups $U'_\xi = U_{R[\xi]}$, $\xi \in S$, for $R[\xi] = \{\alpha \in R : f(\alpha) \neq 0,\ \xi \big| f(\alpha)\}$.*

Let $\xi \in S$ and suppose we have $\alpha_1, \dots, \alpha_n \in R[\xi]$ satisfying the following conditions:

(i) $\big(\pm\alpha_i, \pm\alpha_j\big) = \emptyset$ *for $i \neq j$,*
(ii) *for all $\beta \in R$,*

$$\langle f(\beta), \xi^\vee\rangle = \sum_{i=1}^{n} \langle \beta, \alpha_i^\vee\rangle. \tag{1}$$

Then $W_{\alpha_1} \cdots W_{\alpha_n} \subset W_\xi$. In particular, if all $W_{\alpha_i} \neq \emptyset$ then G has a Weyl element for the root ξ.

Proof. Assumption (i) and the commutator relations imply $\big(U_{\pm\alpha_i}, U_{\pm\alpha_j}\big) = \{1\}$ for $i \neq j$. Hence $W_{\alpha_1} \cdots W_{\alpha_n} \subset U'_\xi U'_{-\xi} U'_\xi$. Let now $w_i \in W_{\alpha_i}$, let $\eta \in S^\times$ and pick an element $\beta \in R[\eta]$. Then

$$(w_1 \cdots w_n) U_\beta (w_1 \cdots w_n)^{-1} = U_{s_{\alpha_1} \cdots s_{\alpha_n}(\beta)},$$

because the w_i are Weyl elements for the roots α_i. It therefore suffices to show that

$$s_{\alpha_1} \cdots s_{\alpha_n}(\beta) \in R[s_\xi(\eta)]. \tag{2}$$

For the proof of (2), let us first observe that

$$\langle \alpha_i, \alpha_j^\vee\rangle = 0 \quad \text{for } i \neq j \tag{3}$$

in view of (i) and integrality of R. We also have

$$\alpha_i \in R \cap f^{-1}(\xi). \tag{4}$$

Indeed, we know $f(\alpha_i) = m_i\xi$ for some $m_i \in \mathbb{N}_+$, whence by (ii) and (3), $2m_i = \langle f(\alpha_i), \xi^\vee\rangle = \sum_{j=1}^n \langle \alpha_i, \alpha_j^\vee\rangle = 2$, so $m_i = 1$. Now we prove (2):

$$\begin{aligned} f\big(s_{\alpha_1}\cdots s_{\alpha_n}(\beta)\big) &= f\big(\beta - \sum_{i=1}^n \langle\beta, \alpha_i^\vee\rangle\alpha_i\big) && \text{(by (3))}\\ &= f(\beta) - \Big(\sum_{i=1}^n \langle\beta,\alpha_i^\vee\rangle\Big)\xi && \text{(by (4))}\\ &= f(\beta) - \langle f(\beta), \xi^\vee\rangle\xi && \text{(by (1))}\\ &= s_\xi\big(f(\beta)\big). \end{aligned}$$

Since $f(\beta) \in \mathbb{N}_+\eta$, this proves $f\big(s_{\alpha_1}\cdots s_{\alpha_n}(\beta)\big) \in \mathbb{N}_+ s_\xi(\eta)$, whence (2).

Remark. If we replace assumption (i) by (3), the proof above also shows $M_{\alpha_1}\cdots M_{\alpha_n} \subset M_\xi$.

Example. An example of a morphism $f\colon R \to S$ satisfying the conditions (i) and (ii) can be constructed as follows. Let R be the root system $R = \mathrm{C}_I \subset X = \bigoplus_{i\in I}\mathbb{R}\varepsilon_i$ and suppose $\sim$ is an equivalence relation on the index set I. Denote by $I' = I/{\sim}$ the set of equivalence classes $[i]$ of elements $i \in I$, put $Y = \bigoplus_{J\in I'}\mathbb{R}\varepsilon_J$ and define $f\colon X \to Y$ by $f(\varepsilon_i) = \varepsilon_{[i]}$, cf. [**40**, 12.14]. Then f is a morphism from $R = \mathrm{C}_I$ to $S = \mathrm{C}_{I'}$. Let $J \in I'$ be a finite equivalence class and let $\xi = 2\varepsilon_J$. The reader will easily verify that (i) and (ii) hold for the roots $\alpha_i = 2\varepsilon_i$, $i \in J$.

We have seen in 5.13 that a balanced Weyl triple is uniquely determined by any one of its components. In Proposition 6.5 we investigate this property for not necessarily balanced Weyl triples, and begin with a lemma.

6.4. Lemma. *Let R be an integral symmetric reflection system, see* 4.2*, and let $\big(G, (U_\alpha)_{\alpha\in R}\big)$ be a group with R-commutator relations and unique factorization for nilpotent pairs. Let $\alpha \in R$ and suppose that there exists a reflective root $\beta \in R$ with the following properties:*

(i) *α and β are $\mathbb{Q}$-linearly independent, and both (α, β) and $(-\alpha, \beta)$ are nilpotent pairs,*
(ii) *$\big(\alpha, \beta\big) = \emptyset$,*
(iii) *$s_\beta(\alpha) \neq \alpha$,*
(iv) *$W_\beta \neq \emptyset$.*

Let M_0 be as in (5.1.3) *and let $z \in M_0 \cap (U_\alpha U_{-\alpha} U_\alpha)$, say, $z = z_1z_2z_3$ with $z_1, z_3 \in U_\alpha$ and $z_2 \in U_{-\alpha}$. Then $z_2 = z = 1$.*

Proof. Since (α, β) is a nilpotent pair by (i), it follows from (ii) that

$$\big(z_1, U_\beta\big) \subset \big(U_\alpha, U_\beta\big) \subset U_{\left(\alpha,\beta\right)} = \{1\}, \tag{1}$$

and in the same way $\big(z_3, U_\beta\big) = \{1\}$. By definition of M_0, z normalizes all root subgroups. It follows that, for all $b \in U_\beta$,

$$U_\beta \ni zbz^{-1} = z_1 z_2 (z_3 b z_3^{-1}) z_2^{-1} z_1^{-1} = z_1 \cdot z_2 b z_2^{-1} \cdot z_1^{-1}.$$

Conjugating this relation with z_1^{-1} and using (1) shows

$$z_2 b z_2^{-1} = zbz^{-1} \in U_\beta.$$

Since the pair $(-\alpha, \beta)$ is nilpotent we have

$$\big(z_2, b\big) \in \big(U_{-\alpha}, U_\beta\big) \subset U_{\boldsymbol{(}-\alpha,\beta\boldsymbol{)}}.$$

Hence $\big(z_2, b\big) \in U_{\boldsymbol{(}-\alpha,\beta\boldsymbol{)}} \cap U_\beta$. By unique factorization and $\mathbb{Q}$-linear independence of α and β, it follows from Corollary 2.18 that this intersection is trivial, so

$$\big(z_2, U_\beta\big) = 1. \tag{2}$$

The reflection s_β is given by

$$s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta,$$

where $0 \neq \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ by (iii) and integrality of R. Moreover, (ii) implies $\langle \alpha, \beta^\vee \rangle > 0$, else $s_\beta(\alpha) \in \big(\alpha, \beta\big)$. Hence we have

$$s_\beta(\alpha) \in \big(\alpha, -\beta\big). \tag{3}$$

Now choose a Weyl element $w_\beta \in U_\beta U_{-\beta} U_\beta$. Since $\big(z_2, U_\beta\big) = \{1\}$ and $\big(z_2, U_{-\beta}\big) \subset \big(U_{-\alpha}, U_{-\beta}\big) \subset U_{\boldsymbol{(}-\alpha,-\beta\boldsymbol{)}} = U_{-\boldsymbol{(}\alpha,\beta\boldsymbol{)}} = \{1\}$ by symmetry of R and (ii), it follows that z_2 is fixed under conjugation with w_β. On the other hand, $w_\beta z_2 w_\beta^{-1} \in U_{s_\beta(-\alpha)}$, so by (3) and unique factorization,

$$z_2 = w_\beta \cdot z_2 \cdot w_\beta^{-1} \in U_{-\alpha} \cap U_{s_\beta(-\alpha)} \subset U_{-\alpha} \cap U_{\boldsymbol{(}-\alpha,\beta\boldsymbol{)}} = \{1\}.$$

It follows that $z = z_1 z_3 \in U_\alpha$. Since z normalizes all root subgroups, we have $\big(z, U_{-\beta}\big) \subset U_{-\beta}$. But also

$$\big(z, U_{-\beta}\big) \subset U_{-\beta} \cap \big(U_\alpha, U_{-\beta}\big) \subset U_{-\beta} \cap U_{\boldsymbol{(}\alpha,-\beta\boldsymbol{)}} = \{1\},$$

by unique factorization for the nilpotent pair $(\alpha, -\beta)$. We have shown above that z_1 and z_3 commute with U_β, hence so does z. It follows that z commutes with w_β, so we have

$$z = w_\beta z w_\beta^{-1} \in U_\alpha \cap U_{s_\beta(\alpha)} \subset U_\alpha \cap U_{\boldsymbol{(}\alpha,-\beta\boldsymbol{)}} = \{1\},$$

by unique factorization for the nilpotent pair $(\alpha, -\beta)$.

Remarks. (a) If β has properties (i) – (iv) for α then $-\beta$ has these properties for $-\alpha$. This follows from symmetry of R and $W_\beta = W_{-\beta}$ (by (5.3.3)).

(b) Suppose R satisfies the condition (F2) of 2.4. Then the $\mathbb{Q}$-linear independence of α and β implies already that (α, β) and $(\alpha, -\beta)$ are nilpotent pairs.

(c) Let R be locally finite root system without irreducible components of rank 1. Then for every $\alpha \in R^\times$ there exists $\beta \in R^\times$ satisfying (i) – (iii).

Indeed, since R has no irreducible components of rank 1, there exists a $\mathbb{Q}$-linearly independent β not orthogonal to α. Possibly after replacing β by its negative, we may assume $\langle \alpha, \beta^\vee \rangle > 0$. Since $s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta$, it is clear that (iii) holds for β. Now we distinguish two cases. First suppose that $\alpha + \beta \notin R$. From the structure of the commutator set of two roots given in 4.8, it follows that $\big(\alpha, \beta\big) = \emptyset$. Hence β has the required property (ii) as well, and condition (i) holds by (b).

Now suppose that $\gamma := \alpha + \beta \in R$. Then we modify β as follows. First, note that $\langle \gamma, \beta^\vee \rangle = 3$. Indeed, by standard facts [**40**, A.2], $\langle \alpha, \beta^\vee \rangle \in \{1, 2, 3\}$. Hence $\langle \gamma, \beta^\vee \rangle = \langle \alpha + \beta, \beta^\vee \rangle = \langle \alpha, \beta^\vee \rangle + 2 \in \{3, 4, 5\}$. Assuming $\langle \gamma, \beta^\vee \rangle = 4$ yields, by loc. cit., $\gamma = 2\beta$ which implies $\alpha = \beta$, contradicting linear independence of α and β. The case $\langle \gamma, \beta^\vee \rangle = 5$ is impossible, again by loc. cit. Now it follows from 4.8, case 8, that $B = \{-\beta, \gamma\}$ is a root basis of a subsystem of type G_2, with $-\beta$ the short root. From the well-known structure of such root systems, one sees easily that $\beta' = -\gamma = -\beta - \alpha$ has the required properties.

6.5. Proposition. *Let G and R be as in Lemma* 6.4, *and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be Weyl triples for the root α. Suppose there exists β satisfying the conditions* (i) – (iv). *If x and y have one component in common then $x = y$.*

Remark. We have seen in Remark 6.4(c) that the assumptions on R are in particular fulfilled when R is a locally finite root system without irreducible components of rank 1.

Proof. (a) Let $\mu(x) = w = x_1 x_2 x_3$ and $\mu(y) = \tilde{w} = y_1 y_2 y_3$ and suppose $x_1 = y_1$. Then $z := w^{-1}\tilde{w} \in M_0$ by (5.3.5). On the other hand,

$$z = x_3^{-1} \cdot (x_2^{-1} y_2) \cdot y_3 = z_1 z_2 z_3 \in U_\alpha \cdot U_{-\alpha} \cdot U_\alpha.$$

Let β have the properties (i)–(iv) of Lemma 6.4. Then $z = 1 = x_2^{-1} y_2$, so $x_2 = y_2$ and then $x_3 = y_3$ follows from $z = 1$.

(b) Now let $x_2 = y_2$. Then $x^\flat = (x_2, x_3, w^{-1} x_1 w)$ and $y^\flat = (y_2, y_3, \tilde{w}^{-1} y_1 \tilde{w})$ are two Weyl triples for $-\alpha$ with the same first component. By Remark (a) of 6.4, $-\beta$ satisfies the assumptions of Lemma 6.4 for $-\alpha$. Hence $x^\flat = y^\flat$ by (a), and therefore $x = y$ by (5.5.6).

(c) Finally, suppose $x_3 = y_3$. Then x^{-1} and y^{-1} are two Weyl triples for α with the same first component $x_3^{-1} = y_3^{-1}$, so $x^{-1} = y^{-1}$ by (a) and therefore $x = y$ by (5.4.2).

6.6. Definition. A *rank one group* is a group G together with two distinct non-trivial nilpotent subgroups U^+ and U^- which generate G and satisfy the following conditions: for each $x \in U^{+*} := U^+ \setminus \{1\}$ there exists an element $y \in U^-$ such that

$$xyU^+ = U^-xy, \tag{1}$$

and for each $y \in U^{-*} := U^- \setminus \{1\}$ there exists $z \in U^+$ such that

$$yzU^- = U^+yz. \tag{2}$$

This is an easy reformulation of the definition in [**56**, I, §1]. Note that the element y in (1) is different from 1, otherwise we would have $U^- = xU^+x^{-1} = U^+$, contradicting $U^+ \neq U^-$. In the same way, $z \neq 1$ in (2). Here are some standard properties of rank one groups, see [**56**, §1, §2].

(a) *The normalizer of U^+ in U^- is trivial.* Indeed, assume $n \in U^{-*}$ normalizes U^+. Then by (2), there exists $z \in U^+$ such that $nzU^- = U^+nz$. This implies $U^- = z^{-1}n^{-1}U^+nz = z^{-1}U^+z = U^+$, contradicting $U^+ \neq U^-$. In the same way, the normalizer of U^- in U^+ is trivial. In particular, $U^+ \cap U^- = \{1\}$.

(b) *For a given $x \in U^{+*}$ resp. $y \in U^{-*}$, the elements y of* (1) *and z of* (2) *are uniquely determined.* Indeed, assume $xyU^+ = U^-xy$ as well as $xy'U^+ = U^-xy'$. Then $x^{-1}U^-x = yU^+y^{-1} = y'U^+(y')^{-1}$, so $n = y^{-1}y' \in U^-$ normalizes U^+ whence $n = 1$. The second case follows by symmetry. Therefore, there is a well-defined map $x \mapsto x^\vee$ from U^{+*} to U^{-*} such that $x^\vee = y$ whenever x and y are as in (1). Similarly, one defines $y^\vee = z$ in the situation of (2).

(c) *The maps $x \mapsto x^\vee$ and $y \mapsto y^\vee$ are bijective.* Indeed, assume $x^\vee = u^\vee$ for $x, u \in U^{+*}$. Then $uyU^+ = U^-uy$ and (1) imply $u^{-1}U^-u = yU^+y^{-1} = x^{-1}U^-x$, so $xu^{-1} \in U^+$ normalizes U^- and therefore $u = x$. To show surjectivity, let $y \in U^{-*}$. Then also $y^{-1} \in U^{-*}$, so by (2) there exists $z \in U^+$ such that $y^{-1}zU^- = U^+y^{-1}z$. Inverting this relation yields $U^-z^{-1}y = z^{-1}yU^+$ and shows $y = (z^{-1})^\vee$. Bijectivity of the map $y \mapsto y^\vee$ follows by symmetry.

(d) Let

$$\Omega = \{U^+\} \cup \{xU^-x^{-1} : x \in U^+\}, \tag{3}$$

so Ω is a subset of the set Σ of all subgroups of G. It follows easily from (1) and (2) that also

$$\Omega = \{U^-\} \cup \{yU^+y^{-1} : y \in U^-\}, \tag{4}$$

and the unions are disjoint because $U^+ \neq U^-$. The group G acts on Σ by conjugation. From (3) it is clear that Ω is stable under the action of U^+, and (4) shows that it is invariant under the action of U^- as well. Since G is generated by U^+ and U^-, it follows that Ω is precisely the orbit of U^+ (or U^-) under the action of G on Σ. Moreover, *G acts doubly transitively on Ω.* Indeed, by transitivity of G on Ω, it suffices to show that, for some $\omega \in \Omega$, the isotropy group G_ω of ω in G acts transitively on $\Omega \setminus \{\omega\}$. Let $\omega = U^+$. Then $U^+ \subset G_\omega$ and (3) shows that U^+ is transitive on $\Omega \setminus \{\omega\}$.

(e) *If $\omega, \eta \in \Omega$ with $\omega \cap \eta \neq \{1\}$ then $\omega = \eta$.* Indeed, assume to the contrary that $\omega \neq \eta$. By the double transitivity of G on Ω, there exists $g \in G$ such that $\omega = gU^+g^{-1}$ and $\eta = gU^-g^{-1}$. Hence $\omega \cap \eta \neq \{1\}$ implies $U^+ \cap U^- \neq \{1\}$, contradicting what we proved in (a).

We now characterize rank one groups in terms of Weyl triples.

6.7. Proposition. *Let G be a group generated by two nilpotent subgroups U^+ and U^- of class $\leqslant k$ and $\leqslant l$, respectively. Let $R = \{-l, \ldots, -1, 0, 1, \ldots, k\} \subset X = \Bbbk$ be the reflection system defined by $s_i(j) = -j$ for $i \in R^\times$ and $j \in R$. Recall from (2.14.1) that G has R-commutator relations with root groups $U_{\pm i} = \mathscr{C}^i(U^\pm)$; in particular, $U_{\pm 1} = U^\pm$, and denote by $\mathfrak{T}_i$ the set of Weyl triples for the root i. Then the following conditions are equivalent:*

(i) *G is a rank one group,*
(ii) *the first projections $\mathrm{pr}_1\colon \mathfrak{T}_{\pm 1} \to U^\pm$ are bijections $\mathfrak{T}_{\pm 1} \to U^{\pm *}$.*

In this case, the inverse of the first projection is given by $x \mapsto t_x := (x, x^\vee, x^{\vee\vee})$.

Remark. Since the maps $x \mapsto x^\vee$ are bijective, it follows that the second and third projections $\mathrm{pr}_2\colon \mathfrak{T}_{\pm 1} \to U^{\mp *}$ and $\mathrm{pr}_3\colon \mathfrak{T}_{\pm 1} \to U^{\pm *}$ are bijective as well.

Proof. (i) $\Longrightarrow$ (ii): Suppose $t = (x, y, z) \in \mathfrak{T}_1$ and put $w = xyz \in W_1$. Then $x \neq 1$, else $wU^+ = yzU^+ = yU^+ = U^-w = U^-yz = U^-z$ which implies $U^+ = U^+z^{-1} = y^{-1}U^- = U^-$, contradiction. We show that $y = x^\vee$. Indeed, $wU^+ = xyzU^+ = xyU^+ = U^-w = U^-xyz$ implies $xyU^+ = xyU^+z^{-1} = U^-xy$, so $y = x^\vee$ by (6.6.1). Similarly, $xyzU^- = wU^- = U^+w = U^+xyz = U^+yz$ implies $yzU^- = x^{-1}U^+yz = U^+yz$, so $z = y^\vee$ by (6.6.2). Thus the first projection $\mathrm{pr}_1\colon \mathfrak{T}_1 \to U^{+*}$ is injective. To see that it also surjective, let $x \in U^{+*}$. We claim that $(x, x^\vee, x^{\vee\vee}) \in \mathfrak{T}_1$, i.e., that $w = xx^\vee x^{\vee\vee}$ is a Weyl element for the root $\alpha = 1$. Indeed, write $y = x^\vee$ and $z = y^\vee$. Then

$$
\begin{aligned}
&wU^+w^{-1} = (xy)zU^+z^{-1}(xy)^{-1} = xyU^+(xy)^{-1} = U^- \text{ (by (6.6.1))}, \quad \text{and} \\
&wU^-w^{-1} = x(yz)U^-(yz)^{-1}x^{-1} = xU^+x^{-1} \text{ (by (6.6.2))} = U^+.
\end{aligned}
$$

Thus, the relation (5.1.1) holds for $\alpha = 1$ and $\beta = \pm 1$. Since $U_{\pm i} = \mathscr{C}^i(U^\pm)$ for $i \geqslant 1$, (5.1.1) follows for all $\beta \in R$, so we have $w \in W_1$. The statement concerning $\mathfrak{T}_{-1}$ follows by symmetry.

(ii) $\Longrightarrow$ (i): We first show that U^+ and U^- are distinct and non-trivial subgroups of G. Assume $U^+ = U^- = \{1\}$. Then $U^{+*} = \emptyset$ but $\mathfrak{T}_1$ contains the element $(1, 1, 1)$, contradiction. Next, assume $U^+ \neq \{1\} = U^-$. Since $U^{+*} \neq \emptyset$ there exists a Weyl triple $t = (x, y, z) \in \mathfrak{T}_1$ with $x \in U^{+*}$. Let $w = \mu(t)$ be the corresponding Weyl element. Then $U^- = wU^+w^{-1} \neq \{1\}$, contradiction. In the same way, one shows that $U^- \neq \{1\} = U^+$ is impossible. Now both U^+ and U^- are non-trivial, and it remains to show that they are different. Assume to the contrary that $U^+ = U^-$ and let $x \in U^{+*}$. Then $t = (x, 1, 1)$ and $t' = (x, x, 1)$ are Weyl triples in $\mathfrak{T}_1$ having the same first component, contradicting the fact that $\mathrm{pr}_1\colon \mathfrak{T}_1 \to U^{+*}$ is injective.

It remains to verify (6.6.1) and (6.6.2). Let $x \in U^{+*}$. Since $\mathrm{pr}_1\colon \mathfrak{T}_1 \to U^{+*}$ is surjective, there exists a Weyl triple $t_x = (x, y, z) \in \mathfrak{T}_1$. Let $w = xyz$ be the corresponding Weyl element. Then

$$wU^+ = xyzU^+ = U^- w = U^- xyz.$$

Since $z \in U^+$, this implies $xyU^+ = U^- xy$, so condition (6.6.1) holds. Similarly, (6.6.2) follows from surjectivity of $\mathrm{pr}_1\colon \mathfrak{T}_{-1} \to U^{-*}$.

Example. Consider $\mathrm{GL}_2(A)$ as in 5.2 and let G be the subgroup generated by U^+ and U^-. Then G is a rank one group if and only if A is a division ring. In this case $\exp_\sigma(u)^\vee = \exp_{-\sigma}(u^{-1})$ for all $0 \neq u \in A$.

Indeed, if G is a rank one group then, by the proposition, every $1 \neq \exp_\sigma(u)$, i.e., $u \neq 0$, is the first component of a Weyl triple. By (5.2.5), u is invertible and $\exp_\sigma(u)^\vee = \exp_{-\sigma}(u^{-1})$. Conversely, if A is a division ring then, again by (5.2.5), every $1 \neq \exp_\sigma(u)$ is part of a Weyl triple, so that G is a rank one group.

6.8. Proposition. *Let G be a rank one group with subgroups $U^\pm$. For an element $x \in U^{+*}$ with associated Weyl triple $t_x = (x, x^\vee, x^{\vee\vee})$ as in Proposition 6.7, the following conditions are equivalent:*

(i) *t_x is balanced,*
(ii) *$x^{\vee\vee} = x$,*
(iii) *$x^\vee x U^- = U^+ x^\vee x$,*
(iv) *$(x^\vee)^{-1} = (x^{-1})^\vee$,*
(v) *$x x^\vee x = x^\vee x x^\vee$.*

We leave it to the reader to formulate the analogous result for an element of U^{-*}.

Proof. By 5.13, a balanced Weyl triple is symmetric in the outer components, so we have (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii): Let $y = x^\vee$ for simpler notation. Since $w = xyx$ is a Weyl element for the root $\alpha = 1$, we have $wU^- w^{-1} = U^+$, whence $xyxU^- = U^+ xyx$. This implies $yxU^- = (x^{-1}U^+ x)yx = U^+ yx$.

(iii) $\Longrightarrow$ (i): We verify condition (iv) of 5.13 and claim first that $w' = xyx$ is a Weyl element for $\alpha = 1$. Indeed,

$$\begin{aligned} w'U^+(w')^{-1} &= (xy)xU^+x^{-1}(xy)^{-1} = (xy)U^+(xy)^{-1} = U^- \text{ (by (6.6.1))}, \\ w'U^-(w')^{-1} &= x(yx)U^-(yx)^{-1}x^{-1} = xU^+x^{-1} \text{ (by (iii))} = U^+. \end{aligned}$$

Now t_x and (x, y, x) are Weyl triples having the same first component, so $x^{\vee\vee} = x$ follows from Proposition 6.7. It remains to show that $xyx = yxy$. From (iii) it follows in particular that there exists $v \in U^-$ such that $yxv = xyx$, so it suffices to have $v = y$. Now

$$xvU^+ = y^{-1}xy(xU^+) = y^{-1}xyU^+ = y^{-1}U^-xy \text{ (by (6.6.1))} = U^-xy = xyU^+,$$

whence $vU^+ = yU^+$. This implies $v^{-1}y \in U^+ \cap U^- = \{1\}$.

(iii) $\iff$ (iv): The defining property of $(x^{-1})^\vee$ is

$$x^{-1}(x^{-1})^\vee U^+ = U^- x^{-1}(x^{-1})^\vee.$$

Thus $y^{-1} = (x^{-1})^\vee$ is equivalent to the equation $x^{-1}y^{-1}U^+ = U^-x^{-1}y^{-1}$, which by inversion is equivalent to (iii).

(i) $\implies$ (v): This is a consequence of 5.13(iv).

(v) $\implies$ (iii): From (v) we get $y^{-1}xy = xyx^{-1}$ and this element is $\neq 1$ because $x \neq 1$. Hence the groups xU^-x^{-1} and $y^{-1}U^+y$ have non-trivial intersection, so they are equal by 6.6(e). This implies (iii).

Remark. The equivalence of (iv) and (v) is [**56**, Lemma (2.2)].

Following [**56**, I, Definition (1.1)], a rank one group is called *special* if every $x \in U^{+*}$ satisfies the equivalent conditions of Proposition 6.8. Not all rank one groups are special. For example, the 1-dimensional affine group group over a field with more than four elements is not special by [**56**, Example (1.7)].

6.9. Examples. The following three examples show that, except for the inclusion $W_\alpha \subset M_\alpha$, there is no general relation between between the set of Weyl elements and generalized Weyl elements.

(a) *Example* $\emptyset = W_\alpha \subsetneq M_\alpha$: The group $G = \mathrm{GL}_2(\mathbb{C})$ of 5.2 also has A_1-commutator relations with respect to the root groups $U_{\pm 1} = \exp_\pm(2\mathbb{Z})$. One easily sees that the set $M_{\pm 1}$ of generalized Weyl elements with respect to these root groups coincides with the set $M_1 = \{\mathrm{w}_{u,v} : u, v \in \mathbb{C}^\times\}$ of (5.2.1). Moreover, it follows as in 5.2 that for $\mathrm{w}_{u,v}$ to lie in $U_1U_{-1}U_1$ one would need $v = u^{-1}$ for $u, v \in 2\mathbb{Z}$, which is impossible. Thus the set W_1 of Weyl elements is empty.

(b) *Example* $\emptyset \neq W_\alpha = M_\alpha$: For a commutative ring k we define $\mathrm{SL}_2(k) = \{g \in \mathrm{GL}_2(k) : \det(g) = 1\}$. Then $\mathrm{SL}_2(k)$ has A_1-commutator relations with respect to the root groups $U_{\pm 1}$ of 5.2. Moreover, it is immediate from 5.2 that $\emptyset \neq W_{\pm 1} = M_{\pm 1}$ for this group.

(c) *Example* $\emptyset = W_\alpha = M_\alpha$: Let k be a commutative ring and put $G = \mathrm{SL}_3(k)$. This group has R-commutator relations for $R = \mathrm{A}_2 = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i, j \leqslant 3\}$ with root groups

$$U_\alpha = \mathrm{Id} + kE_{ij} = U_{ij}, \quad \alpha = \varepsilon_i - \varepsilon_j \neq 0.$$

Let $S = \mathrm{A}_1 = \{0, \pm(\delta_1 - \delta_2)\}$ and define a morphism $f : R \to S$ in **SV** by $f(\varepsilon_1) = \delta_1$ and $f(\varepsilon_2) = \delta_2 = f(\varepsilon_3)$. By Proposition 3.3, $\mathrm{SL}_3(k)$ has A_1-commutator relations with root groups

$$U'_{\delta_1-\delta_2} = U_{12}U_{13} \quad \text{and} \quad U'_{\delta_2-\delta_1} = U_{21}U_{31}.$$

But G does not have generalized Weyl elements for the root $\pm(\delta_1 - \delta_2)$, as can be seen by a straightforward matrix calculation.

6.10. Example: $\mathrm{PGL}_2(A)$ **of a ring.** The centre of the group $G = \mathrm{GL}_2(A)$ is $\mathscr{Z}(G) = \{z \cdot 1_2 : z \in Z(A)^\times\}$ where $Z(A)^\times$ is the set of invertible elements of the centre $Z(A)$ of A and where 1_2 denotes the 2×2 identity matrix. Let

$$\pi\colon G \to \bar{G} := \mathrm{PGL}_2(A) := G/\mathscr{Z}(G)$$

be the canonical map. Then $\bar{G}$ has A_1-commutator relations with respect to the root groups $\bar{U}_{\pm 1} = \pi(U_{\pm 1})$. Since π is surjective on root groups, it sends (generalized) Weyl elements of G to (generalized) Weyl elements of $\bar{G}$. In fact, standard matrix calculations show

$$\pi\big(M_{\pm 1}(G)\big) = M_{\pm 1}(\bar{G}) \quad \text{and} \quad \pi\big(W_{\pm 1}(G)\big) = W_{\pm 1}(\bar{G}). \tag{1}$$

The induced map $\mathfrak{T}(\pi)$ on the Weyl triples is well-defined by 5.1, injective since $\mathscr{Z}(\mathrm{GL}_2(A)) \cap U^\pm = \emptyset$ and surjective by (1), whence

$$\mathfrak{T}(\pi)\colon \mathfrak{T}(G) \to \mathfrak{T}(\bar{G}) \quad \text{is a bijection.}$$

In particular it follows from 5.14 and the example in 5.13 that *all Weyl triples in* $\mathrm{PGL}_2(A)$ *are balanced.*

6.11. Example: Reductive algebraic groups. Let G be a connected reductive algebraic group defined over a field k as in Example (c) of 2.3. We have seen there that G has commutator relations with respect to some finite root system R. One knows that G has generalized Weyl elements for all $\alpha \in R$ [**6**, 21.2]. If k is algebraically closed, then G also has Weyl elements [**54**, Lemma 8.1.4], and all Weyl triples are balanced.

For the proof, let R be the root system of G with respect to a maximal torus T, and let $(U_\alpha)_{\alpha \in R}$ be the corresponding family of root groups in G. For $\alpha \in R^\times$ let G_α be the centralizer of the subtorus $(\operatorname{Ker} \alpha)^0 \subset T$ in G. By [**6**, Thm. 13.18], G_α is a connected reductive algebraic group of semisimple rank 1. Hence [**6**, Prop. 13.13] there exists an epimorphism φ from G_α onto $\mathrm{PGL}_2(k)$. One knows that $\operatorname{Ker}(\varphi) = \mathscr{Z}(G_\alpha) \subset T$. The group G_α has A_1-commutator relations with root groups $U_{\pm\alpha} \subset G_\alpha$, which are mapped isomorphically onto root groups of $\mathrm{PGL}_2(k)$. By conjugacy of maximal tori and hence of the associated root groups, we can assume that $\varphi(U_{\pm\alpha})$ are the root groups of Example 6.10.

Let now $x \in \mathfrak{T}_\alpha(G)$ be a Weyl triple. Then x is in particular a Weyl triple of G_α and so, by 5.1, $\mathfrak{T}(\varphi)(x)$ is a Weyl triple of $\mathrm{PGL}_2(k)$, hence balanced by 6.10. But then x is balanced by Lemma 5.14.

Similarly, if G is a Chevalley group as in Example 2.3(d), say with root system R, then again G has balanced Weyl triples for all $\alpha \in R$ by [**55**, Chapter 3, Lemma 19].

6.12. Example: Moufang polygons. Let G be the group associated in 2.3(e) to a Moufang building. There we have seen that G has R-commutator relations with respect to root groups U_α, $\alpha \in R$, where R is a finite irreducible root system of rank $l \geqslant 2$. A theorem of Tits (see for example [**63**, Prop. 11.22]) says that for any $\alpha \in R^\times$ and $1 \neq u_{-\alpha} \in U_{-\alpha}$ there exist u'_α and $u''_\alpha \in U_\alpha$ such that $u'_\alpha u_{-\alpha} u''_\alpha \in W_\alpha$. In particular, $W_\alpha \neq \emptyset$.

6.13. Example: Lie algebras. (a) Let L be a Kac-Moody Lie algebra and $\varrho\colon L \to \mathfrak{gl}(V)$ be an integrable representation in category $\mathscr{O}$. Let R be the real (= reflective) roots of L augmented by 0. We have seen in 2.23 that $\mathrm{GL}(V)$ is a group with R-commutator relations for a suitable definition of root groups.

Using [**46**, Prop. 6.1.8] it is not hard to show that $\mathrm{GL}(V)$ has Weyl elements for every root $\alpha \in R^\times$. Moreover, since $U_\alpha U_{-\alpha} U_\alpha \subset \mathrm{GL}(V)$ is a homomorphic image of $\mathrm{SL}_2(\Bbbk)$ by [**46**, Prop. 6.1.7], it follows from 6.11 that all Weyl triples for $(\mathrm{GL}(V), \mathscr{U})$ are balanced.

(b) Let $\mathrm{Aut}(E)$ be the automorphism group of an extended affine Lie algebra E. We have seen in 2.26 that $\mathrm{Aut}(E)$ is a group with commutator relations for $R = \{0\} \cup R^{\mathrm{an}}$ where R^{an} is the set of anisotropic (= reflective) roots. It is a standard result in the theory of extended affine Lie algebras, see e.g. [**1**, Proposition 1.27], that for every $\alpha \in R^{\mathrm{an}}$ there exists an elementary automorphism w_α satisfying $w_\alpha(L_\beta) = L_{s_\alpha(\beta)}$ for all $\beta \in R$, the set of all roots of E. This easily implies that w_α is a Weyl element for $\alpha \in R^{\mathrm{an}}$. By construction in loc. cit., $w_\alpha = \mu(t_\alpha)$ for t_α a balanced Weyl triple.

CHAPTER II

GROUPS ASSOCIATED TO JORDAN PAIRS

§7. Introduction to Jordan pairs

7.1. The elementary group of a Morita context. Let us start with something very simple, namely 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with coefficients in a ring R. Recall that the elementary group $\mathrm{E}_2(R)$ is the subgroup of $\mathrm{GL}_2(R)$ generated by the elementary matrices

$$\mathrm{e}_+(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{e}_-(y) = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} \qquad (x \in R).$$

More generally, one considers the elementary group $\mathrm{E}_n(R) \subset \mathrm{GL}_n(R)$, generated by all $1_n + xE_{ij}$, $i \neq j$, $x \in R$. This can also be done with (formal) 2×2 matrices by subdividing an $n \times n$ matrix in 4 blocks, say of size $p \times p$, $p \times q$, $q \times p$, $q \times q$, with $p + q = n$. It is easy to see that $\mathrm{E}_n(R)$ is already generated by the matrices

$$\begin{pmatrix} 1_p & x \\ 0 & 1_q \end{pmatrix}, \qquad \begin{pmatrix} 1_p & 0 \\ -y & 1_q \end{pmatrix} \qquad \bigl(x \in \mathrm{Mat}_{pq}(R),\ y \in \mathrm{Mat}_{qp}(R)\bigr).$$

This suggests to consider right away the following situation: Replace $\mathrm{Mat}_n(R)$ by a (unital associative) ring $\mathfrak{A}$ with a formal block matrix decomposition, namely the Peirce decomposition of $\mathfrak{A}$ with respect to an idempotent $e \in \mathfrak{A}$. Putting $f = 1 - e$, we have

$$\mathfrak{A} = \begin{pmatrix} e\mathfrak{A}e & e\mathfrak{A}f \\ f\mathfrak{A}e & f\mathfrak{A}f \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix};$$

in other words: $\mathfrak{M} = (A, B, C, D)$ is a Morita context. Then one defines the *elementary group of* $\mathfrak{M}$ by

$$\mathrm{E}(\mathfrak{M}) = \left\langle \begin{pmatrix} 1_A & B \\ 0 & 1_D \end{pmatrix} \cup \begin{pmatrix} 1_A & 0 \\ C & 1_D \end{pmatrix} \right\rangle \subset \mathfrak{A}^\times.$$

Let us now always work over an arbitrary commutative base ring k. All objects for which this makes sense are modules over k, rings are k-algebras, and so on. If $\mathfrak{A}$ and $\mathfrak{M}$ are as above, then $M^+ := B$ and $M^- := C$ are in particular k-modules (and M^+ is an (A, D)-bimodule etc.).

The associative algebra $\mathfrak{A}$ gives rise to a Lie algebra $\mathfrak{A}^-$ having the same underlying k-module and the Lie bracket $[a,b] = ab - ba$. Note that $\mathfrak{A}^- = \bigoplus_{i\in\mathbb{Z}} \mathfrak{A}_i$ is a $\mathbb{Z}$-graded Lie algebra with the definitions

$$\mathfrak{A}_{-1} = \begin{pmatrix} 0 & 0 \\ M^- & 0 \end{pmatrix}, \quad \mathfrak{A}_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathfrak{A}_1 = \begin{pmatrix} 0 & M^+ \\ 0 & 0 \end{pmatrix},$$
$$\mathfrak{A}_i = 0 \quad \text{for } i \notin \{-1,0,1\}.$$

A $\mathbb{Z}$-graded Lie algebra concentrated in degrees $-1,0,1$ is also called 3-*graded.*

7.2. Generalized elementary groups. This will now be generalized as follows. Let $V^\pm \subset M^\pm$ be k-submodules and let V be the pair (V^+, V^-). We consider the subgroup

$$\mathrm{E}(\mathfrak{M}, V) = \left\langle \begin{pmatrix} 1 & V^+ \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 \\ V^- & 1 \end{pmatrix} \right\rangle$$

of $\mathrm{E}(\mathfrak{M})$, called the *elementary group of* V. Since the $V^\pm$ are in particular additive subgroups of $M^\pm$, it is clear that $\begin{pmatrix} 1 & V^+ \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ V^- & 1 \end{pmatrix}$ are multiplicative subgroups of $\mathfrak{A}^\times$, isomorphic to the additive groups $V^\pm$ under the maps $x \mapsto \mathrm{e}_+(x)$ and $y \mapsto \mathrm{e}_-(y)$. For example, let $\mathfrak{M}$ be the Morita context where $\mathfrak{A}$ is $\mathrm{Mat}_{2n}(k)$, subdivided into 4 blocks of size $n \times n$. Then $\mathrm{E}(\mathfrak{M}) = \mathrm{E}_{2n}(k)$ is the elementary group as in 7.1. Choosing $V^\pm = \mathrm{H}_n(k)$, the $n \times n$ symmetric matrices, yields for $\mathrm{E}(\mathfrak{M}, V)$ the elementary symplectic group $\mathrm{ESp}_{2n}(k)$, and choosing $V^\pm = \mathrm{Alt}_n(k)$, the alternating $n \times n$ matrices, i.e., skew-symmetric with zeros on the diagonal, $\mathrm{E}(\mathfrak{M}, V)$ is the elementary orthogonal group $\mathrm{EO}_{2n}(k)$, see [**17**, 5.3A, 5.3B].

Returning to the general situation, we define k-submodules $\mathfrak{e}_i$ of $\mathfrak{A}_i$ by

$$\mathfrak{e}_{-1} = \begin{pmatrix} 0 & 0 \\ V^- & 0 \end{pmatrix}, \qquad \mathfrak{e}_1 = \begin{pmatrix} 0 & V^+ \\ 0 & 0 \end{pmatrix},$$
$$\mathfrak{e}_0 = k \cdot \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} + k \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_D \end{pmatrix} + [\mathfrak{e}_1, \mathfrak{e}_{-1}],$$
$$\mathfrak{e}_i = \{0\} \qquad \text{for } i \notin \{-1,0,1\},$$

and put

$$\mathfrak{e}(\mathfrak{M}, V) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{e}_i = \mathfrak{e}_{-1} \oplus \mathfrak{e}_0 \oplus \mathfrak{e}_1.$$

Let us consider the following closure conditions for V:

$$x, z \in V^\sigma,\ y \in V^{-\sigma} \quad \Longrightarrow \quad xyz + zyx \in V^\sigma, \tag{1}$$
$$x \in V^\sigma,\ y \in V^{-\sigma} \quad \Longrightarrow \quad xyx \in V^\sigma. \tag{2}$$

Here and in the sequel the index σ always takes values in $\{+,-\}$ and $-\sigma$ has the obvious meaning. Note that (2) implies (1) by linearization, since $V^\pm$ is in particular an abelian subgroup of $M^\pm$:

$$xyz + zyx = (x+z)y(x+z) - xyx - zyz.$$

Note also that in the examples treated so far, the conditions (1) and (2) are satisfied. Their significance is shown by the following lemma.

7.3. Lemma. *Let $V = (V^+, V^-)$ be a pair of submodules of (M^+, M^-).*

(a) *V satisfies* (7.2.1) $\iff$ *$\mathfrak{e}(\mathfrak{M}, V)$ is a graded subalgebra of the Lie algebra $\mathfrak{A}^-$.*

(b) *V satisfies* (7.2.2) $\iff$ *$\mathfrak{e}(\mathfrak{M}, V)$ is stable under conjugation by elements of* $\mathrm{E}(\mathfrak{M}, V)$.

Proof. (a) "$\Longrightarrow$": This is shown by direct computation. For example, the rule $[\mathfrak{e}_0, \mathfrak{e}_1] \subset \mathfrak{e}_1$ follows from the relations

$$\Big[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\Big] = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = -\Big[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\Big], \tag{1}$$

$$\Big[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -y & 0 \end{pmatrix}\Big] = \begin{pmatrix} -xy & 0 \\ 0 & yx \end{pmatrix}, \tag{2}$$

$$\Big[\begin{pmatrix} xy & 0 \\ 0 & -yx \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}\Big] = \begin{pmatrix} 0 & xyz + zyx \\ 0 & 0 \end{pmatrix}. \tag{3}$$

Similarly, the fact that $\mathfrak{e}_0$ is a subalgebra of $\mathfrak{A}^-$ follows from the formula

$$[xy, uv] = xyuv - uvxy = (xyu + uyx)v - u(yxv + vxy).$$

The details are left to the reader.

"$\Longleftarrow$": We know $[\mathfrak{e}_1, \mathfrak{e}_{-1}] \subset \mathfrak{e}_0$ and $[\mathfrak{e}_0, \mathfrak{e}_1] \subset \mathfrak{e}_1$, so (2) and (3) show that (7.2.1) holds for $\sigma = +$, and the case $\sigma = -$ is proved similarly.

(b) "$\Longrightarrow$": Since (7.2.2) implies (7.2.1), $\mathfrak{e}(\mathfrak{M}, V)$ is a 3-graded Lie algebra by (a). It follows easily from the formula

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -y & 0 \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -xy & xyx \\ -y & yx \end{pmatrix} \tag{4}$$

that $\mathfrak{e}(\mathfrak{M}, V)$ is stable under conjugation with $\begin{pmatrix} 1 & V^+ \\ 0 & 1 \end{pmatrix}$, and a similar computation shows stability under the remaining generators of $\mathrm{E}(\mathfrak{M}, V)$.

"$\Longleftarrow$": From (4) we see that $xyx \in V^+$ and similarly one has $yxy \in V^-$, for all $x \in V^+$, $y \in V^-$. Hence V satisfies (7.2.2).

For any pair V satsifying (7.2.2) we call $\mathfrak{e}(\mathfrak{M}, V)$ the *elementary Lie algebra* of $(\mathfrak{M}, V)$.

7.4. Definition of Jordan pairs. We will first define "concrete" Jordan pairs as pairs of off-diagonal submodules of a Morita context satifying (7.2.2), and then, by abstracting from their properties, Jordan pairs in general. This follows a well-established procedure in algebra. For example, concrete Lie algebras are submodules of associative algebras closed under the commutator product $[a, b] = ab - ba$, abstract Lie algebras are modules equipped with an alternating product $[a, b]$ satisfying "the same" identities (in this case, the Jacobi identity) as concrete Lie algebras.

Let $\mathfrak{M}$ be a Morita context as above. A *Jordan subpair of* $\mathfrak{M}$ is a pair of submodules $V = (V^+, V^-)$ of (M^+, M^-) satisfying condition (7.2.2). Thus V comes equipped with the following somewhat unusual algebraic structure: a pair of maps $Q^+\colon V^+ \times V^- \to V^+$ and $Q^-\colon V^- \times V^+ \to V^-$, given by

$$Q^+(x; y) = xyx \quad \text{and} \quad Q^-(y; x) = yxy. \tag{1}$$

Clearly, these maps are quadratic in the first and linear in the second variable. They can also be considered as quadratic maps $Q_+\colon V^+ \to \mathrm{Hom}_k(V^-, V^+)$ and $Q_-\colon V^- \to \mathrm{Hom}_k(V^+, V^-)$, by defining

$$Q_+(x) \cdot y = Q^+(x; y), \quad Q_-(y) \cdot x = Q^-(y; x).$$

We now come to the definition of abstract Jordan pairs. Thus we have to find the relevant identities holding for the quadratic-linear compositions of a Jordan subpair V as above. This turns out to be fairly complicated. To avoid a proliferation of parentheses and indices $\pm$, we introduce the following conventions: for $x \in V^\sigma$ and $y \in V^{-\sigma}$ (where $\sigma \in \{+, -\}$), we simply write

$$Q_\sigma(x)y = Q_x y \quad (= xyx \text{ in the concrete situation}).$$

This notation does not lead to confusion as long as care is taken to ensure that in an expression $Q_x y$, the elements x and y are taken in different spaces: $x \in V^+$ and $y \in V^-$, or vice versa.

We will also need efficient notation for the linearizations of the quadratic-linear expression $Q_x y$. First, we denote the linearization of $Q_x y$ with respect to x in the direction of z by

$$Q_{x,z} y = Q(x, z)y = Q_{x+z} y - Q_x y - Q_z y.$$

Next, define trilinear compositions $\{-, -, -\}\colon V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ by

$$\{xyz\} = \{zyx\} = D_{x,y} z = Q_{x,z} y. \tag{2}$$

Again, the entries in the trilinear product $\{xyz\}$ have to be taken alternatingly in V^+ and V^-. In the concrete situation, we have

$$\{xyz\} = xyz + zyx.$$

Among all the identities satisfied by the compositions of a Jordan subpair V as above, the following three have turned out to be the essential ones:

$$\{x, y, Q_x v\} = Q_x\{y, x, v\}, \tag{JP1}$$
$$\{Q_x y, y, z\} = \{x, Q_y x, z\}, \tag{JP2}$$
$$Q_{Q_x y} v = Q_x Q_y Q_x v, \tag{JP3}$$

for all $x, z \in V^\sigma$, $y, v \in V^{-\sigma}$, and $\sigma \in \{+, -\}$. Here and in the sequel, the enumeration of the identities (JPx) follows the one in [**34**]. In the concrete situation of a Jordan subpair of a Morita context $\mathfrak{M}$, (JP1) amounts to the following computation, valid because of the associativity of $\mathfrak{A}$:

$$\{x, y, Q_x v\} = xy(xvx) + (xvx)yx = x(yxv + vxy)x = Q_x\{y, v, x\}.$$

Similarly, (JP2) and (JP3) say concretely

$$\begin{aligned} \{Q_x y, y, z\} &= (xyx)yz + zy(xyx) = x(yxy)z + z(yxy)x = \{x, Q_y x, z\}, \\ Q_{Q_x y} v &= (xyx)v(xyx) = x(y(xvx)y)x = Q_x Q_y Q_x v. \end{aligned}$$

Thus, the identities (JP1) – (JP3) should not be regarded as saying that V is a non-associative algebraic system but rather as an expression of the essential associativity of the non-linear composition xyx.

Inspection shows that (JP1) is of degree 3 in x and (JP3) is of degree 4 in x. In turns out that one needs all (formal) linearizations of these identities to hold as well. (For (JP2) this is automatically the case because it is only of degree 2 in x and y). A more concise way of expressing this fact is as follows. Suppose $R \in k$-**alg** is an arbitrary commutative associative unital k-algebra, and let $V_R = (V_R^+, V_R^-)$ be the corresponding base ring extension. Since the maps $Q^\sigma\colon V^\sigma \times V^{-\sigma} \to V^\sigma$ of (1) are of bi-degree $(2, 1)$, they have natural extensions to maps $Q_R^\sigma\colon V_R^\sigma \otimes V_R^{-\sigma} \to V_R^\sigma$ of R-modules, again of bi-degree $(2, 1)$.

The formal definition of an (abstract) Jordan pair is now as follows: a *Jordan pair over the commutative ring* k is a pair of k-modules $V = (V^+, V^-)$ equipped with a pair $Q_\sigma\colon V^\sigma \times V^{-\sigma} \to V^\sigma$ of maps, bi-homogeneous of bi-degree $(2, 1)$, such that, using the notations introduced before, the identities (JP1) – (JP3) hold in all base ring extensions.

From the definition, it is evident that Jordan pairs admit arbitrary base change: if V is a Jordan pair over k then V_R is a Jordan pair over R, for all $R \in k$-**alg**.

As expected, a *homomorphism* $h\colon V \to W$ *of Jordan pairs* is a pair (h_+, h_-) of k-linear maps $h_\sigma\colon V^\sigma \to W^\sigma$ satisfying $h_\sigma(Q_x y) = Q_{h_\sigma(x)} h_{-\sigma}(y)$ for all $x \in V^\sigma$, $y \in V^{-\sigma}$. Jordan pairs then form a category admitting arbitrary base ring extensions. The definition of isomorphisms and automorphisms is clear. For example, any unit $\mu \in k^\times$ gives rise to an automorphism $(\mu \mathrm{Id}_{V^+}, \mu^{-1}\mathrm{Id}_{V^-})$ of V.

Unlike the case of rings, it makes no sense to define the opposite of a Jordan pair by reversing the order of the factors in a product. However, it is possible to interchange the roles of V^+ and V^-, so we define: the *opposite* of V is the Jordan pair $V^{\mathrm{op}} = (V^-, V^+)$ with quadratic maps $Q_x^{\mathrm{op}} y = Q_x y$ for $x \in (V^{\mathrm{op}})^\sigma = V^{-\sigma}$ and $y \in (V^{\mathrm{op}})^{-\sigma} = V^\sigma$. If $V = (B, C)$ is the Jordan pair of a Morita context with $\mathfrak{A}$ and idempotents e and f as in 7.1, then V^{op} is the Jordan pair associated with the opposite algebra $\mathfrak{A}^{\mathrm{op}}$ since $e\mathfrak{A}^{\mathrm{op}} f = f\mathfrak{A}e = C = V^-$ and $f\mathfrak{A}^{\mathrm{op}} e = e\mathfrak{A}f = V^+$.

The reader will not be surprised to learn that a *subpair* of a Jordan pair V is a pair $S = (S^+, S^-)$ of submodules of $V = (V^+, V^-)$ satisfying $Q(S^\sigma)S^{-\sigma} \subset S^\sigma$ for $\sigma = \pm$, while an *ideal* of V is a pair $I = (I^+, I^-)$ of submodules such that $Q(I^\sigma)V^{-\sigma} + Q(V^\sigma)I^{-\sigma} + \{V^\sigma, V^{-\sigma}, I^\sigma\} \subset I^\sigma$ holds for $\sigma = \pm$. If I is an ideal of V, then $V/I = (V^+/I^+, V^-/I^-)$ is a Jordan pair with the obvious operations. The role of one-sided ideals in ring theory is played in Jordan theory by the *inner ideals*: these are the k-submodules $M \subset V^\sigma$ satisfying $Q_M V^{-\sigma} \subset M$.

7.5. Examples, special and exceptional Jordan pairs. A natural question arises here: is every abstract Jordan pair a subpair of some Morita context? For Lie algebras, a positive answer to the analogous question is, at least over fields, furnished by the Poincaré-Birkhoff-Witt theorem. For Jordan pairs, the answer is no: there are Jordan pairs even over the complex numbers, called *exceptional*, which cannot be embedded into any Morita context. This leads to the following definition. An (abstract) Jordan pair is called *special* if it can be embedded into some Moritat context $\mathfrak{M}$ (this may be possible in many different ways). The most important examples of special Jordan pairs are the following.

(a) *Rectangular matrices* $\mathbb{M}_{pq}(A)$. Let A be an arbitrary associative (not necessarily commutative) k-algebra and put $V^+ = \mathrm{Mat}_{pq}(A)$, $V^- = \mathrm{Mat}_{qp}(A)$.

(b) *Alternating matrices* $\mathbb{A}_n(k)$. Here $V^+ = V^- = \mathrm{Alt}_n(k)$, alternating $n \times n$ matrices over k.

(c) *Symmetric matrices* $\mathbb{H}_n(k)$. Here $V^+ = V^- = \mathrm{H}_n(k)$, symmetric $n \times n$ matrices over k.

In all three cases, we obtain special Jordan pairs with composition $Q_x y = xyx$ (matrix product).

Examples (b) and (c) are special cases of the following more general example.

(d) *Hermitian matrices over a form ring.* Let $(A, J, \varepsilon, \Lambda)$ be a form ring in the sense of [**17**, 5.1C]. Thus A is an associative unital k-algebra, J is an anti-automorphism of A, $\varepsilon \in A^\times$ is a unit of A with the property that $\varepsilon^J = \varepsilon^{-1}$ and $a^{JJ} = \varepsilon a \varepsilon^{-1}$ for all $a \in A$, and Λ is a form parameter; i.e., a k-submodule of A with the property that

$$\{a - a^J \varepsilon : a \in A\} \subset \Lambda \subset \{a \in A : a = -a^J \varepsilon\}$$

and $a^J \lambda a \in \Lambda$ for all $a \in A$ and $\lambda \in \Lambda$. We extend J to an anti-automorphism of $\mathrm{Mat}_n(A)$ by defining $x^J = (x_{ji}^J)$ for an $n \times n$-matrix $x = (x_{ij})$ with entries from A. Now put $\Lambda^+ = \varepsilon^{-1}\Lambda$, $\Lambda^- = \Lambda$ and define

$$V^+ = \{x \in \mathrm{Mat}_n(A) : x^J = -\varepsilon x \text{ and } x_{ii} \in \Lambda^+ \text{ for all } i\},$$
$$V^- = \{y \in \mathrm{Mat}_n(A) : y^J = -y\varepsilon^{-1} \text{ and } y_{ii} \in \Lambda^- \text{ for all } i\}.$$

Then $V = (V^+, V^-)$ is a Jordan subpair of the Jordan pair $(\mathrm{Mat}_n(A), \mathrm{Mat}_n(A))$ of Example (a), hence special. Observe that $V^- = \Lambda_n$ and $V^+ = \Lambda_n^J$ in the notation of [**17**, 5.1C].

For example, by letting $A = K$ be a commutative associative unital k-algebra and taking $J = \mathrm{Id}$, $\varepsilon = +1$ and $\Lambda = 0$, we obtain the Jordan pair $\mathbb{A}_n(K) = (\mathrm{Alt}_n(K), \mathrm{Alt}_n(K))$ of alternating matrices over K, generalizing example (b) above.

Let $\varepsilon = -1$. Then J is an involution of A and a form parameter Λ is a k-submodule satisfying $\{a + a^J : a \in A\} \subset \Lambda \subset \{a \in A : a^J = a\}$ and $a^J \Lambda a \subset \Lambda$ for all $a \in A$. In this case, $\Lambda^+ = \Lambda = \Lambda^-$ and $V^+ = V^-$ consists of hermitian $n \times n$ matrices over A with diagonal entries in Λ. The corresponding Jordan pair is denoted $\mathbb{H}_n(A, J, \Lambda)$. For $A = k$, $J = \mathrm{Id}$ and $\Lambda = k$ we get the example (c) of symmetric matrices above. Important examples of form parameters Λ are the ones with $1 \in \Lambda$, traditionally called ample subspaces. For $A = \mathbb{Z}$, $J = \mathrm{Id}$, $\Lambda = 2\mathbb{Z}$ is a form parameter which is not ample.

(e) Examples of *exceptional Jordan pairs* are obtained by taking in (a) and (d) an alternative instead of an associative coordinate algebra, but only for small sizes of the respective matrices. Thus let A be an alternative k-algebra. Then $\mathbb{M}_{12}(A) = (\mathrm{Mat}_{12}(A), \mathrm{Mat}_{21}(A))$ is still a Jordan pair with quadratic operators

$$Q_x y = x(yx), \qquad Q_y x = (yx)y,$$

for $x \in \mathrm{Mat}_{12}(A)$, $y \in \mathrm{Mat}_{21}(A)$. If A is an octonion algebra this Jordan pair is exceptional. Similarly, there is a natural (but not so easily described) Jordan pair structure on the 3×3-hermitian matrices over an octonion algebra which is exceptional.

(f) Let $q\colon X \to k$ be a quadratic form on a k-module X and denote by $b(x, y) = q(x+y) - q(x) - q(y)$ the polar form of q. Then $V = (X, X)$ is a Jordan pair, called the *Jordan pair of* q, with quadratic operators $Q_x y = b(x, y)x - q(x)y$.

(g) In some of the examples above, the Jordan pairs $V = (V^+, V^-)$ had the property that $V^+ = V^-$ and $Q^+ = Q^-$. These types of Jordan pairs are essentially the same as *Jordan triple systems*, see [**34**, 1.13] for details.

7.6. Identities. Jordan theory requires a large amount of sometimes non-trivial identities, all of which are consequences of the defining identities (JP1) – (JP3). We derive some of them here and refer to [**34**] for a more complete list.

Let us define bilinear maps $D_\sigma\colon V^\sigma \times V^{-\sigma} \to V^\sigma$ by

$$D_\sigma(x, y) \cdot z = \{x, y, z\} \quad (= Q_{x,z} y).$$

We follow the same convention as for Q and drop the index σ at D. To save parentheses, we will often write $D_{x,y}$ instead of $D(x, y)$.

Since the right hand side of (JP1) is symmetric in y and v so must be the left hand side. This yields $\{x, y, Q_x v\} = \{x, v, Q_x y\} = Q_x\{yxv\}$, or in operator form:

$$D_{x,y} Q_x = Q_x D_{y,x} = Q(x, Q_x y). \tag{JP4}$$

Linearizing (JP2) with respect to x in the direction of u resp. with respect to y in the direction of v yields

$$\begin{aligned}\{\{xyu\}, y, z\} &= \{x, Q_y u, z\} + \{u, Q_y x, z\},\\ \{x, \{yxv\}, z\} &= \{Q_x y, v, z\} + \{Q_x v, y, z\}.\end{aligned}$$

Written in operator form, this becomes

$$D(\{xyu\}, y) = D(x, Q_y u) + D(u, Q_y x), \tag{JP7}$$
$$D(x, \{yxv\}) = D(Q_x y, v) + D(Q_x v, y), \tag{JP8}$$
$$D_{z,y} D_{x,y} = Q_{x,z} Q_y + D(z, Q_y x), \tag{JP9}$$
$$Q_{x,z} D_{y,x} = Q(Q_x y, z) + D_{z,y} Q_x. \tag{JP10}$$

Similarly, linearize (JP1) with respect to x in direction z:

$$\{x, y, \{xvz\}\} + \{z, y, Q_x v\} = Q_x\{\{yzv\} + Q_{x,z}\{yxv\}.$$

Reading this as a function of y yields

$$Q(x, \{xvz\}) + Q(z, Q_x v) = Q_x D_{v,z} + Q_{x,z} D_{v,x}.$$

Replace here v by y and add the result to (JP10). After switching x and z, we obtain

$$D_{x,y} Q_z + Q_z D_{y,x} = Q(z, \{xyz\}). \tag{JP12}$$

Applying this to v and reading the result as a function of x yields

$$D(Q_z v, y) + Q_z Q_{y,v} = D_{z,v} D_{z,y}. \tag{JP13}$$

Linearizing (JP13) with respect to z in the direction u and applying the result to an element x shows

$$\{\{zvu\}, y, x\} + \{z, \{yxv\}, u\} = \{z, v, \{uyx\}\} + \{u, v, \{zyx\}\}.$$

By reading this as a function of z, we see

$$[D_{x,y}, D_{u,v}] = D(\{xyu\}, v) - D(u, \{yxv\}). \tag{JP15}$$

The identities derived so far are all consequences of (JP1) and (JP2). The following two identities require (JP3). For the proof, we refer to [**34**, 2.10].

$$Q(\{xyz\}) + Q(Q_x y, Q_z y) = Q_x Q_y Q_z + Q_z Q_y Q_x + Q_{x,z} Q_y Q_{x,z}, \tag{JP20}$$
$$Q(\{xyz\}) + Q(Q_x Q_y z, z) = Q_x Q_y Q_z + Q_z Q_y Q_x + D_{x,y} Q_z D_{y,x}. \tag{JP21}$$

For some applications it is useful to know that under suitable conditions on V the identity (JP15) implies (JP1)–(JP3). For example, let 2 be a unit in k, let $V = (V^+, V^-)$ be a pair of k-modules without 3-torsion and suppose $V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$, $(x, y, z) \to \{xyz\} =: D(x, y)z$ are trilinear maps which are symmetric in the outer variables and satisfy (JP15). Then V becomes a Jordan pair with respect to $Q(x)y = \frac{1}{2}\{xyx\}$. This can be used in the following situation. Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a 3-graded Lie algebra over a ring k in which 6 is a unit. Then $V = (L_1, L_{-1})$ becomes a Jordan pair with respect to $\{xyz\} = -[[x, y], z]$.

7.7. Derivations and inner derivations. Derivations of Jordan pairs are defined by the usual mechanism: a pair $\varDelta = (\varDelta_+, \varDelta_-)$ of linear maps $\varDelta_\sigma \in \mathrm{End}(V^\sigma)$ is called a *derivation* if $\mathrm{Id} + \varepsilon\varDelta$ is an automorphism of the base ring extension $V \otimes k(\varepsilon)$ where $k(\varepsilon)$ is the algebra of dual numbers. A simple computation shows that this is equivalent to the conditions

$$\varDelta_\sigma(Q_z v) = \{\varDelta_\sigma(z), v, z\} + Q_z \varDelta_{-\sigma}(v), \tag{1}$$

for all $z \in V^\sigma$, $v \in V^{-\sigma}$. With component-wise operations, the derivations of V form a Lie subalgebra $\mathrm{Der}(V)$ of $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$.

Identity (JP12) says precisely that, for any pair $(x, y) \in V^+ \times V^-$, the pair

$$\delta(x, y) := (D_{x,y}, -D_{y,x}) \tag{2}$$

is a derivation of V. We call this the *inner derivation determined by* (x, y). From (JP15) it follows that the k-linear span

$$\mathrm{Inder}(V) = \mathrm{span}\{\delta(x, y) : (x, y) \in V\} \tag{3}$$

is a subalgebra of $\mathrm{Der}(V)$, called the *inner derivation algebra of* V. In fact, by linearizing (1), one sees easily that $\mathrm{Inder}(V)$ is an ideal in $\mathrm{Der}(V)$.

In any Jordan pair, we have the derivation

$$\zeta_V = (\mathrm{Id}_{V^+}, -\mathrm{Id}_{V^-}) \tag{4}$$

which obviously belongs to the centre of $\mathrm{Der}(V)$.

7.8. The Bergmann operators. For a pair $(x, y) \in V^\sigma \times V^{-\sigma}$ we define $B(x, y) = B_{x,y} \in \mathrm{End}\, V^\sigma$ by

$$B_{x,y} = \mathrm{Id}_{V^\sigma} - D_{x,y} + Q_x Q_y.$$

The name "Bergmann" comes from the fact that Jordan pairs over the complex numbers equipped with positive hermitian involutions are in correspondence with bounded symmetric domains. Then the determinant of $B_{x,y}$ is related to the Bergmann kernel of the domain, see [**35**].

The Bergmann operators play a fundamental role in the theory of Jordan pairs. Of the many identities satisfied by them, we list only the following two and refer to [**34**, 2.11] for more.

$$B(x, y)^2 = B(2x - Q_x y, y) = B(x, 2y - Q_y x), \tag{JP25}$$

$$Q(B_{x,y} z) = B_{x,y} Q_z B_{y,x}. \tag{JP26}$$

In our standard example of the Jordan pair (B, C) of a Morita context $\mathfrak{M} = (A, B, C, D)$ we have

$$B(x, y)z = (1 - xy)z(1 - yx). \tag{1}$$

7.9. The quasi-inverse. A pair $(x, y) \in V$ is called *quasi-invertible* if the Bergmann operator $B(x, y)$ is invertible (as an endomorphism of V^+). Then, the *quasi-inverse of* (x, y) is defined as the element

$$x^y := B(x, y)^{-1}(x - Q_x y) \tag{1}$$

of V^+. Quasi-invertibility and the quasi-inverse of $(y, x) \in V^{\mathrm{op}}$ are then well-defined, too.

Example. Let $V = (B, C)$ be the Jordan pair of a Morita context $\mathfrak{M} = (A, B, C, D)$ as in 7.1. Then the following conditions are equivalent:

(a) (x, y) is quasi-invertible in V,
(b) $1_A - xy \in A$ is a unit in A,
(c) $1_D - yx$ is a unit in D,
(d) (y, x) is quasi-invertible in V^{op}.

In this case, the quasi-inverses are given by

$$x^y = (1 - xy)^{-1}x = x(1 - yx)^{-1}, \qquad y^x = (1 - yx)^{-1}y = y(1 - xy)^{-1}.$$

A proof is given in [**38**, 4.5]. In particular, if $\mathfrak{M}$ is the Morita context of 2×2 matrices over a ring R, then the pair $(x, 1_R)$ is quasi-invertible if and only if x is quasi-invertible in the sense of ring theory [**50**, p. 180]. The group-theoretic significance of quasi-invertibility will be seen in (9.2.5) and (9.2.6).

We return to an arbitrary Jordan pair. Proofs of the following facts can be found in [**34**, §3] and [**38**, Theorem 4.10], keeping in mind that $B(x, y) = F(x, y, 0, 0)$ in the notation of [**38**]. The following conditions on a pair $(x, y) \in V$ are equivalent.

(i) (x, y) is quasi-invertible,
(ii) $B(x, y)$ is surjective,
(iii) $2x - Q_x y$ belongs to the image of $B(x, y)$,
(iv) x belongs to the image of $B(x, y)$.

In ring theory, invertibility in a ring and the opposite ring are equivalent. The analogue for Jordan pairs is the "symmetry principle":

$$(x, y) \text{ is quasi-invertible in } V \iff (y, x) \text{ is quasi-invertible in } V^{\mathrm{op}}, \tag{2}$$

and in this case, the quasi-inverses are related by the formula

$$x^y = x + Q_x y^x. \tag{3}$$

Let $h\colon V \to W$ be a homomorphism of Jordan pairs. Condition (iii) (or (iv)) together with (1) immediately imply: If (x, y) is quasi-invertible in V then $(h_+(x), h_-(y))$ is quasi-invertible in W, and then

$$h_+\big(x^y\big) = h_+(x)^{h_-(y)}, \quad h_-(y^x) = h_-(y)^{h_+(x)}. \tag{4}$$

An important property of the quasi-inverse, and the reason for the exponential notation x^y is the following. Suppose (x, y) is quasi-invertible and let $v \in V^-$. Then $(x, y+v)$ is quasi-invertible if and only if (x^y, v) is quasi-invertible, in which case

$$(x^y)^v = x^{y+v}, \quad (y+v)^x = y^x + B(y,x)^{-1} \cdot v^{(x^y)}. \tag{5}$$

We refer to [**34**, 3.7] for the proof.

Of the numerous identities relating the quasi-inverse and the Bergmann operators, we list the following and refer to [**34**, 3.6] for more. Here it is always assumed that (x, y) is quasi-invertible, while $z \in V^+$ and $v \in V^-$ can be arbitrary.

$$B(x,y)B(x^y,v) = B(x,y+v), \tag{JP33}$$

$$B(z,y^x)B(x,y) = B(x+z,y), \tag{JP34}$$

$$B(x,y)^{-1} = B(x^y,-y) = B(-x,y^x). \tag{JP35}$$

7.10. Structural transformations. Besides homomorphisms, the following types of maps between Jordan pairs play an important role. Let $V = (V^+, V^-)$ and $W = (W^+, W^-)$ be Jordan pairs. A *structural transformation from V to W* is a pair of k-linear maps $f\colon V^+ \to W^+$ and $g\colon W^- \to V^-$ (note the change of direction!) such that

$$Q_{f(x)} = f\,Q_x\,g \quad \text{and} \quad Q_{g(y)} = g\,Q_y\,f,$$

for all $x \in V^+$ and $y \in W^-$. We write this as

$$(f,g)\colon V \rightleftharpoons W$$

and note that

$$(f,g)\colon V \rightleftharpoons W \quad \iff \quad (g,f)\colon W^{\mathrm{op}} \to V^{\mathrm{op}}.$$

The basic examples are given by the quadratic operators and the Bergmann operators: for all $x \in V^+$, $y \in V^-$, we have the structural transformations

$$(Q_x,Q_x)\colon V^{\mathrm{op}} \rightleftharpoons V, \quad (Q_y,Q_y)\colon V \rightleftharpoons V^{\mathrm{op}}, \quad (B_{x,y},B_{y,x})\colon V \rightleftharpoons V.$$

This is just another way of expressing the identities (JP3) and (JP26). We also note that for any scalar $\lambda \in k$, the homotheties $f(x) = \lambda x$ and $g(y) = \lambda y$ define a structural transformation from V to itself.

An invertible structural transformation is essentially an isomorphism; more precisely, the following conditions are equivalent:

(i) $(f,g)\colon V \rightleftharpoons W$ is a structural transformation with f and g invertible,
(ii) $(f,g^{-1})\colon V \to W$ is an isomorphism.

The proof is immediate from the definitions.

In particular, let (x,y) be quasi-invertible. Then both $B_{x,y}$ and $B_{y,x}$ are invertible, so

$$\beta(x, y) := (B_{x,y}, B_{y,x}^{-1})$$

is an automorphism of V, called the *inner automorphism determined by* (x, y). The *inner automorphism group* $\mathrm{Inn}(V)$ is the subgroup of $\mathrm{Aut}(V)$ generated by all $\beta(x, y)$, $(x, y) \in V$ quasi-invertible.

Structural transformations relate well to the quasi-inverse. Let $(f, g)\colon V \rightleftharpoons W$ be structural and let $x \in V^+$ and $y \in W^-$. Then $(f(x), y)$ is quasi-invertible in W if and only if $(x, g(y))$ is quasi-invertible in V, in which case the formula

$$f(x)^y = f\bigl(x^{g(y)}\bigr) \tag{1}$$

holds, known as the "shifting principle".

Indeed, by linearizing the defining equations of a structural transformation, one obtains the formulas

$$D(f(x), y) \circ f = f \circ D(x, g(y)), \quad D(g(y), x) = g \circ D(y, f(x))$$

and then also

$$B(f(x), y) \circ f = f \circ B(x, g(y)), \quad B(g(y), x) = g \circ B(y, f(x)),$$

for all $x \in V^+$, $y \in V^-$. Now suppose $(x, g(y))$ quasi-invertible. Then there exists $z \in V^+$ such that $x = B(x, g(y))z$. Applying f to this equation and using the above formula yields $f(x) = B(f(x), y)f(z)$, so $(f(x), y)$ is quasi-invertible.

Next, let $(f(x), y)$ be quasi-invertible. By the symmetry principle, $(y, f(x))$ is quasi-invertible, and since (g, f) is structural, it follows that $(g(y), x)$ is quasi-invertible, which implies $(x, g(y))$ quasi-invertible, again by symmetry. Finally, we have $B(x, g(y))x^{g(y)} = x - Q_x g(y)$. Applying f to this (1) and using the above formulas shows

$$f\bigl(B(x, g(y))x^{g(y)}\bigr) = B(f(x), y)f\bigl(x^{g(y)}\bigr) = f(x) - fQ_x g(y) = f(x) - Q_{f(x)}y.$$

By applying $B(f(x), y)^{-1}$ to this we obtain (1).

7.11. Inverses and Jordan algebras. An element $u \in V^\sigma$ is called *invertible* if $Q_u\colon V^{-\sigma} \to V^\sigma$ is invertible (as a linear map). In general, a Jordan pair will not contain any invertible elements. For example, the Jordan pair of $p \times q$ and $q\times$ matrices over a commutative ring (as in Example (a) of 7.5) contains invertible elements if and only if $p = q$. If $u \in V^\sigma$ is invertible then the inverse of u is defined by

$$u^{-1} = Q_u^{-1}u \in V^{-\sigma}$$

Recall here that Q_u maps $V^{-\sigma}$ to V^σ, so $Q_u^{-1}\colon V^\sigma \to V^{-\sigma}$. It follows easily from (JP3) that

$$Q_u^{-1} = Q_{u^{-1}} \quad \text{and} \quad (u^{-1})^{-1} = u \tag{1}$$

for an invertible $u \in V^\sigma$. For the example $\mathbb{M}_{11}(A) = (A, A)$ of an associative unital k-algebra A, an element is invertible in $\mathbb{M}_{11}(A)$ if and only if it is invertible in

A. Also, care has to be taken to distinguish between x^{-1_A}, the quasi-inverse of $(x, -1_A)$, from the inverse x^{-1}.

We say V is a *Jordan division pair* if $V \neq (0,0)$, and if every non-zero element of V^σ is invertible. For example, the Jordan pair $\mathbb{M}_{11}(A) = (A, A)$ of an associative k-algebra is a Jordan division pair if and only if A is a division algebra.

Jordan pairs containing invertible elements are in the following relation with (unital quadratic) Jordan algebras. First, we recall from [**24**] that a unital quadratic Jordan algebra over k is a k-module J equipped with a distinguished element $1_J \in J$ and a quadratic-linear map $U\colon J \to \mathrm{End}_k(J)$ such that the identities

$$U_{1_J} = \mathrm{Id}_J, \tag{QJ1}$$
$$U_{U_x y} = U_x U_y U_x, \tag{QJ2}$$
$$U_x V_{y,x} = V_{x,y} U_x \tag{QJ3}$$

hold in all scalar extensions. Here $V_{x,y} \in \mathrm{End}_k(J)$ is defined (similarly to $D_{x,y}$ for Jordan pairs) by

$$V_{x,y} z = U_{x+z} y - U_x y - U_z y.$$

Now any Jordan algebra J determines a Jordan pair $(V^+, V^-) = (J, J)$ with quadratic operators $Q_x = U_x$ ($x \in V^\pm$). Indeed, (JP1) is (QJ3), (JP3) is (QJ2), and (JP2) is the identity QJ21 of [**24**, p. 3.10]. Conversely, let V be a Jordan pair containing an invertible element $v \in V^-$. Then the k-module V^+ becomes a unital quadratic Jordan algebra J by defining $1_J = v^{-1}$ and $U_x = Q_x Q_v$ for all $x \in V^+$, and the Jordan pair (J, J) is isomorphic to V under the pair of maps $(\mathrm{Id}_{V^+}, Q_v)\colon (J, J) \to (V^+, V^-)$. The Jordan algebras arising from choosing a different invertible element in V^- are not necessarily isomorphic, but they are isotopic. For details, we refer to [**34**, §1].

There is the following relation between inverse and quasi-inverse. Suppose $u \in V^+$ (resp. $v \in V^-$) is an invertible element of the Jordan pair V, and let $y \in V^-$ (resp. $x \in V^+$) be arbitrary. Then

$$B(u, y) = Q_u Q(u^{-1} - y), \qquad B(x, v) = Q(x - v^{-1}) Q_v. \tag{2}$$

Moreover, (u, y) is quasi-invertible if and only if $u^{-1} - y$ is invertible in V^-, and then the formula

$$u^y = (u^{-1} - y)^{-1} \tag{3}$$

holds, see [**34**, 2.12, 3.13] for a proof.

7.12. Idempotents and Peirce decomposition. Let V be a Jordan pair. A pair $e = (e_+, e_-) \in V^+ \times V^-$ is called an *idempotent* if

$$Q_{e_+} e_- = e_+ \quad \text{and} \quad Q_{e_-} e_+ = e_-.$$

Clearly, idempotents are mapped to idempotents under Jordan pair homomorphisms.

If V is a special Jordan pair embedded in a Morita context as in 7.4 then this means $e_+ = e_+e_-e_+$ and $e_- = e_-e_+e_-$, so e_+ and e_- are in particular von Neumann regular. Also, e_+e_- and e_-e_+ are ring idempotents in A and D, respectively.

An idempotent e of V gives rise to the important *Peirce decomposition* as follows. For $\sigma \in \{+,-\}$ and $i \in \{0,1,2\}$ define endomorphisms E_i^σ of V^σ by

$$E_2^\sigma = Q_{e_\sigma}Q_{e_{-\sigma}}, \quad E_1^\sigma = D(e_\sigma, e_{-\sigma}) - 2E_2^\sigma, \quad E_0^\sigma = B(e_\sigma, e_{-\sigma}).$$

Then the E_i^σ are orthogonal projections whose sum is the identity on V^σ, so that

$$V^\sigma = V_2^\sigma \oplus V_1^\sigma \oplus V_0^\sigma \quad \text{where} \quad V_i^\sigma = \operatorname{Im}(E_i^\sigma).$$

This is the Peirce decomposition of V with respect to e, the $V_i = V_i^\sigma(e)$ are called the *Peirce spaces* of V with respect to e. We often put $V_i = (V_i^+, V_i^-)$ and write the Peirce decomposition as $V = V_2 \oplus V_1 \oplus V_0$, to be read component-wise. The Peirce spaces can also be described by

$$\begin{aligned} V_2^\sigma &= \{x \in V^\sigma : Q_{e_\sigma}Q_{e_{-\sigma}}x = x\} = \operatorname{Im} Q_{e_\sigma}, \\ V_1^\sigma &= \{x \in V^\sigma : \{e_\sigma, e_{-\sigma}, x\} = x\}, \\ V_0^\sigma &= \{x \in V^\sigma : Q_{e_{-\sigma}}x = \{e_\sigma, e_{-\sigma}, x\} = 0\}. \end{aligned}$$

In particular, for $x \in V_2^\sigma$ we have $x = E_2^\sigma(x)$ and $0 = E_1^\sigma(x) = \{e_\sigma, e_{-\sigma}, x\} - E_2^\sigma(x)$, whence $\{e_\sigma, e_{-\sigma}, x\} = 2x$.

Idempotents and Peirce decompositions behave well with respect to homomorphisms: suppose $h\colon V \to W$ is a homomorphism of Jordan pairs. Then the image $h(e) = (h_+(e_+), h_-(e_-))$ of an idempotent of V is an idempotent of W, and it follows from the definition of the Peirce spaces that $h\bigl(V_i^\sigma(e)\bigr) \subset W_i^\sigma\bigl(h(e)\bigr)$. For $h \in \operatorname{Aut}(V)$ we obviously have $h(\mathfrak{P}_e) = \mathfrak{P}_{h(e)}$.

The Peirce spaces satisfy the following multiplication rules, where we put $V_j^\sigma = 0$ for $j \notin \{0,1,2\}$ [**34**, Theorem 5.4]:

$$\begin{aligned} Q(V_i^\sigma)V_j^{-\sigma} &\subset V_{2i-j}^\sigma, \\ \{V_i^\sigma, V_j^{-\sigma}, V_l^\sigma\} &\subset V_{i-j+l}^\sigma, \\ \{V_2^\sigma, V_0^{-\sigma}, V^\sigma\} &= \{V_0^\sigma, V_2^{-\sigma}, V^\sigma\} = 0. \end{aligned}$$

Let in particular $u \in V^+$ be invertible with inverse $u^{-1} \in V^-$. Then $e = (u, u^{-1})$ is an idempotent with the property that $V = V_2(e)$. Conversely, if e is an idempotent with $V = V_2(e)$ then e_+ is invertible with inverse e_-. In general, the $+$-component e_+ of an idempotent e is always invertible in the subpair $V_2(e)$, and its inverse is e_-.

Two idempotents c and d are called *orthogonal* if $c \in V_0(d)$. This is equivalent to $d \in V_0(c)$, and then $c + d$ (defined component-wise) is again an idempotent.

Suppose $e_1, \dots, e_n$ is a finite family of pairwise orthogonal idempotents, and define $V_{ij}^\sigma \subset V^\sigma$ for $i, j \in \{0, 1, \dots, n\}$ by

$$V_{ii} = V_2(e_i), \quad V_{ij} = V_{ji} = V_1(e_i) \cap V_1(e_j) \quad (i \neq j),$$
$$V_{00} = \bigcap_{i=1}^{n} V_0(e_i), \quad V_{i0} = V_{0i} = V_1(e_i) \cap \bigcap_{j \neq i} V_0(e_j).$$

Then V decomposes as

$$V = \bigoplus_{0 \leqslant i \leqslant j \leqslant n} V_{ij}. \tag{1}$$

To formulate the multiplication rules which these spaces satisfy, we consider triples of unordered pairs of indices (ij, lm, pq) taken from $\{0, 1, \dots, n\}$, and furthermore identify this with (pq, lm, ij). We call such a triple connected if it is of the form (ij, jm, mp). Then the following composition rules hold:

$$\{V_{ij}^{\sigma}, V_{jm}^{-\sigma}, V_{mp}^{\sigma}\} \subset V_{ip}^{\sigma}.$$

If (ij, jl, ij) is connected and $ij = lm$ then

$$Q(V_{ij}^{\sigma})V_{jl}^{-\sigma} \subset V_{im}^{\sigma}.$$

If (ij, lm, pq) resp. (ij, lm, ij) is not connected then

$$\{V_{ij}^{\sigma}, V_{lm}^{-\sigma}, V_{pq}^{\sigma}\} = Q(V_{ij}^{\sigma})V_{lm}^{-\sigma} = 0.$$

Proofs can be found in [**34**, Theorem 5.14].

§8. Peirce gradings

8.1. Peirce gradings. In this section we let $V = (V^+, V^-)$ be a Jordan pair over a unital associative and commutative ring k, see Section 7. A *$\mathbb{Z}$-grading* of V [**36**] consists of decompositions $V^\sigma = \bigoplus_{i\in\mathbb{Z}} V_i^\sigma$ ($\sigma \in \{+,-\}$) into direct sums of k-submodules satisfying the multiplication rules

$$\{V_i^\sigma V_j^{-\sigma} V_l^\sigma\} \subset V_{i-j+l}^\sigma, \qquad Q(V_i^\sigma)V_j^{-\sigma} \subset V_{2i-j}^\sigma. \tag{1}$$

The convention for numbering the V_i^- differs from that of [**36**] by a sign. A *homomorphism* of $\mathbb{Z}$-graded Jordan pairs is a Jordan pair homomorphism $h\colon V \to V'$ satisfying $h(V_i) \subset V_i'$. From (1) it follows immediately that $V_i = (V_i^+, V_i^-)$ is a subpair of V, for all $i \in \mathbb{Z}$.

A *Peirce grading* $\mathfrak{P}$ of V is a $\mathbb{Z}$-grading with $V_i^\sigma = 0$ for $i \notin \{0,1,2\}$ and the additional *orthogonality relations*

$$D(V_2^\sigma, V_0^{-\sigma}) = D(V_0^\sigma, V_2^{-\sigma}) = 0. \tag{2}$$

To simplify notation, we will usually write $\mathfrak{P} : V = V_2 \oplus V_1 \oplus V_0$ or simply $V = V_2 \oplus V_1 \oplus V_0$ to specify a Peirce grading of V. If $V = V_2 \oplus V_1 \oplus V_0$ is already a $\mathbb{Z}$-grading, formula (2) is equivalent to

$$\{V_2^\sigma V_0^{-\sigma} V_0^\sigma\} = \{V_0^\sigma V_2^{-\sigma} V_2^\sigma\} = 0. \tag{3}$$

Also, (1) implies that $\{V_i^\sigma V_j^{-\sigma} V_l^\sigma\} = 0$ if $i - j + l \notin \{0,1,2\}$. The following properties are immediate from the definition.

$$V_0^\sigma \text{ and } V_2^\sigma \text{ are inner ideals,} \tag{4}$$

$$V_i^* = V_{2-i} \text{ is again a Peirce grading,} \tag{5}$$

$$B(V_i^\sigma, V_j^{-\sigma}) = \mathrm{Id} \text{ for } |i-j| = 2. \tag{6}$$

We call the Peirce grading of (5) the *reverse* of $\mathfrak{P}$.

The automorphism group $\mathrm{Aut}(V)$ acts on the set of Peirce gradings of V in the obvious way: if $f \in \mathrm{Aut}(V)$ and $\mathfrak{P} : V = V_2 \oplus V_1 \oplus V_0$ is a Peirce grading, then the Peirce grading $f(\mathfrak{P}) : V = \tilde V_2 \oplus \tilde V_1 \oplus \tilde V_0$ is given by $\tilde V_i = f(V_i)$.

8.2. Examples. For any Jordan pair V and a fixed $i \in \{0,1,2\}$ there is always the *trivial Peirce grading* $V_i = V$ and $V_j = 0$ for $j \neq i$. A Peirce grading with $V_1 = 0$ is the same as a direct sum decomposition $V = V_0 \oplus V_2$ of V into ideals. If $\mathfrak{P}_U : U = U_2 \oplus U_1 \oplus U_0$ and $\mathfrak{P}_W : W = W_2 \oplus W_1 \oplus W_0$ are Peirce graded Jordan pairs then the direct sum $V = U \oplus W$ has a Peirce grading $\mathfrak{P}_U \oplus \mathfrak{P}_W$ given by $V_i = U_i \oplus W_i$.

The main examples of Peirce gradings are the ones defined by an idempotent e of V. Indeed, from 7.12 it is clear that the the Peirce spaces $V_i = V_i(e)$ define a Peirce grading of V, denoted by $\mathfrak{P}_e$.

However, there are important examples of Peirce gradings which are not induced by an idempotent, for instance the decomposition of the Jordan pair $\mathbb{M}_{p,q}(R)$ of $p \times q$ and $q \times p$ matrices over a ring R given by the following block decomposition:

$$V^+ = \begin{array}{c} \\ r\left\{\right. \\ p-r\left\{\right. \end{array}\!\!\begin{array}{cc} \overbrace{\quad}^{s} & \overbrace{\quad}^{q-s} \\ \hline \multicolumn{1}{|c|}{V_2^+} & \multicolumn{1}{c|}{V_1^+} \\ \hline \multicolumn{1}{|c|}{V_1^+} & \multicolumn{1}{c|}{V_0^+} \\ \hline \end{array}, \qquad V^- = \begin{array}{c} \\ s\left\{\right. \\ q-s\left\{\right. \end{array}\!\!\begin{array}{cc} \overbrace{\quad}^{r} & \overbrace{\quad}^{p-r} \\ \hline \multicolumn{1}{|c|}{V_2^-} & \multicolumn{1}{c|}{V_1^-} \\ \hline \multicolumn{1}{|c|}{V_1^-} & \multicolumn{1}{c|}{V_0^-} \\ \hline \end{array}. \tag{1}$$

Suppose R is unital. If $r = s$ then this Peirce grading is idempotent. Conversely, if it comes from an idempotent, say $e = (e_+, e_-)$, then $e_+ \colon R^s \to R^r$ is an R-module isomorphism with inverse e_-. Thus $r = s$ if R is a ring with invariant basis number, for instance, if R is commutative, local, or has stable rank 1, see e.g. [**33**, §1.5].

In 8.5 we will establish some multiplication rules for Jordan pairs with a Peirce grading. These formulas hold in fact in a more general setting, which we will review now.

8.3. Kernels and annihilators. We recall from [**39**] and [**34**, 10.3] the definition of the kernel $\operatorname{Ker} X$ and annihilator $\operatorname{Ann} X$ of a subset $X \subset V^\sigma$:

$$\begin{aligned} \operatorname{Ker} X &= \{v \in V^{-\sigma} : Q_X v = Q_X Q_v X = 0\}, \\ \operatorname{Ann} X &= \{v \in V^{-\sigma} : D(v, X) = D(X, v) = Q_v X = Q_X v \\ &\qquad\qquad = Q_v Q_X = Q_X Q_v = 0\}. \end{aligned}$$

Despite the nonlinear character of the defining conditions, kernel and annihilator are in fact k-submodules, and clearly $\operatorname{Ann} X \subset \operatorname{Ker} X$. If $X = \{x\}$ consists of a single element, we simply write $\operatorname{Ker} x$ and $\operatorname{Ann} x$. Note the symmetry in the definition of the annihilator:

$$v \in \operatorname{Ann} x \iff x \in \operatorname{Ann} v. \tag{1}$$

For example, if $\mathfrak{P}$ is a Peirce grading of V then it follows easily from the multiplication rules that

$$V_{2-i}^{-\sigma} \oplus V_1^{-\sigma} \subset \operatorname{Ker} V_i^\sigma, \qquad V_{2-i}^{-\sigma} \subset \operatorname{Ann} V_i^\sigma \tag{2}$$

for $i \in \{0, 2\}$. If $\mathfrak{P} = \mathfrak{P}_e$ is idempotent then, by [**34**, 10.3], $V_0^- = \operatorname{Ann} V_2^+ = \operatorname{Ann} e^+$. Also, by definition of the extreme radical $\operatorname{Extr}(V)$ in 9.8,

$$\operatorname{Ann}(V^-) = \{z \in \operatorname{Extr}(V^+) : Q_{V^-} z = 0\}. \tag{3}$$

8.4. Proposition. (a) *Let $(x,v) \in V^\sigma \times V^{-\sigma}$ and $v \in \operatorname{Ker} x$, i.e., $Q_x v = Q_x Q_v x = 0$. Then (x,v) is quasi-invertible with quasi-inverses $x^v = x$, $v^x = v + Q_v x$. For all $y \in V^-$ we have $\{yxv\} \in \operatorname{Ker} x$ and $v \in \operatorname{Ker} Q_x y$, and the following "shift formulas" hold:*

$$
\begin{aligned}
D(Q_x y, v) &= D(x, \{yxv\}), & D(v, Q_x y) &= D(\{yxv\}, x), && (1)\\
Q(Q_x y)Q_v &= Q_x Q_{\{yxv\}}, & Q_v Q(Q_x y) &= Q_{\{yxv\}} Q_x, && (2)\\
B(Q_x y, v) &= B(x, \{yxv\}), & B(v, Q_x y) &= B(\{yxv\}, x). && (3)
\end{aligned}
$$

(b) *Let $(x,v) \in V^\sigma \times V^{-\sigma}$ and $v \in \operatorname{Ann} x$. Then (x,v) is quasi-invertible with quasi-inverses $x^v = x$ and $v^x = v$, and $B(x,v)$ and $B(v,x)$ is the identity. For all $(z,y) \in V^\sigma \times V^{-\sigma}$ we have $\{yzv\} \in \operatorname{Ker} x$ and $\{xyz\} \in \operatorname{Ker} v$, and the following shift formulas hold:*

$$
\begin{aligned}
D(x, \{yzv\}) &= D(\{xyz\}, v), \qquad D(\{yzv\}, x) = D(v, \{xyz\}), && (4)\\
Q_x Q_{\{yzv\}} &= Q_x Q_y Q_z Q_v = Q_{\{xyz\}} Q_v, && (5)\\
Q_v Q_{\{xyz\}} &= Q_v Q_z Q_y Q_x = Q_{\{yzv\}} Q_x, && (6)\\
B(x, \{yzv\}) &= B(\{xyz\}, v), \qquad B(\{yzv\}, x) = B(v, \{xyz\}), && (7)\\
Q_{\{xyz\}} v &= Q_x Q_y Q_z v. && (8)
\end{aligned}
$$

Proof. (a) We have $B_{x,v} x = x - 2Q_x v + Q_x Q_v x = x$, so (x,v) is quasi-invertible by (iv) of 7.9 with quasi-inverse $x^v = x$. By the symmetry principle (7.9.2) and formula (7.9.3), we have (v,x) quasi-invertible, with $v^x = v + Q_v x^v = v + Q_v x$.

From the fundamental formula (JP3) it follows easily that $v \in \operatorname{Ker} Q_x y$. Before showing $\{yxv\} \in \operatorname{Ker} x$, we establish the shift formulas (1)–(3). By (JP8) we have $D(Q_x y, v) = -D(Q_x v, y) + D(x, \{yxv\}) = D(x, \{yxv\})$, proving the first formula of (1), and the second one follows similarly from (JP7). Furthermore, the identities (JP20), (JP3), (JP13) and (JP2) yield

$$
\begin{aligned}
Q_x Q_{\{yxv\}} &= Q_x Q_y Q_x Q_v + Q_x Q_v Q_x Q_y + Q_x Q_{y,v} Q_x Q_{y,v} - Q_x Q(Q_v x, Q_y x)\\
&= Q(Q_x y)Q_v + Q(Q_x v)Q_y + Q(Q_x y, Q_x v)Q_{y,v} - Q_x Q(Q_v x, Q_y x)\\
&= Q(Q_x y)Q_v - Q_x\big[D(x, Q_v x)D(x, Q_y x) + D(Q_x Q_v x, Q_y x)\big]\\
&= Q(Q_x y)Q_v - Q_x D(Q_x v, v)D(x, Q_y x) = Q(Q_x y)Q_v.
\end{aligned}
$$

This establishes the first formula of (2). The second formula is proved by a similar computation. Finally, (3) is immediate from (1) and (2) and the definition of the B-operators.

We can now show $\{yxv\} \in \operatorname{Ker} x$. Indeed, $Q_x\{yxv\} = D(x,y)Q_x v = 0$ by (JP1), and $Q_x Q_{\{yxv\}} = Q_x Q_y \big(Q_x Q_v x\big) = 0$ by (2).

(b) From the definition of the annihilator it is clear that $B(x,v)$ and $B(v,x)$ are the identity and that $x^v = x$ and $v^x = v$. As before, we first prove the shift formulas. Since $D(x,v) = 0$, (JP15) yields $0 = \big[D_{z,y}, D_{x,v}\big] = D(\{zyx\}, v) - D(x, \{yzv\})$, and and similarly one shows the second formula of (4). Next, by (JP20),

$$Q_xQ_{\{yzv\}} = Q_xQ_yQ_zQ_v + Q_xQ_vQ_zQ_y + Q_xQ_{y,v}Q_zQ_{y,v} - Q_xQ(Q_yz, Q_vz).$$

The second term on the right vanishes by definition of the annihilator. Furthermore, $Q_xQ_{y,v} = D(x,v)D(x,y) - D(Q_xv, y) = 0$ by (JP13), and again by (JP13) and (JP7), for any $t \in V^{-\sigma}$,

$$\begin{aligned} Q_xQ(t, Q_vz) &= D(x, Q_vz)D(x,t) - D(Q_xQ_vz, t) \\ &= \big(- D(z, Q_vx) + D(\{xvz\}, v)\big)D(x,t) = 0. \end{aligned}$$

This establishes the first formula of (5). The second one is proved similarly, and (6) follows from annihilator symmetry (8.3.1). As before, (7) is a consequence of the definition of the B-operators. For (8) we obtain from ((JP20)) that $Q_xQ_yQ_zv = Q_{\{xyz\}}v + \{Q_xy, v, Q_zy\}$. But, by ((JP8)), $D(Q_xy, v) = -D(Q_v, y) + D(x, \{vxy\}) = 0$. Finally we show $\{yzv\} \in \operatorname{Ker} x$ which, again by annihilator symmetry, also establishes $\{xyz\} \in \operatorname{Ker} v$ by switching the roles of x, z and v, y. We have $Q_x\{yzv\} = -D(z,y)Q_xv + \{xv\{zyx\}\} = 0$ by (JP12), and $Q_xQ_{\{yzv\}}x = Q_xQ_yQ_zQ_vx = 0$ by (5).

8.5. Corollary. *Let $V = V_0 \oplus V_1 \oplus V_2$ be a Peirce grading of V and let subscripts indicate membership in the corresponding Peirce space.*

(a) *If $i \neq j$ then (x_i, y_j) is quasi-invertible with quasi-inverses*

$$x_i^{y_j} = x_i + Q(x_i)y_j, \qquad y_j^{x_i} = y_j + Q(y_j)x_i, \tag{1}$$

where either $Q(x_i)y_j = 0$ or $Q(y_j)x_i = 0$, and

$$\beta(x_i, y_j)^{-1} = \beta(x_i, -y_j). \tag{2}$$

(b) *For $i \in \{0, 2\}$ we have the formulas*

$$D(y, x_i)D(v_1, x_i) = Q(y, v_1)Q(x_i), \tag{3}$$
$$D(x_i, v_1)D(x_i, z) = Q(x_i)Q(v_1, z), \tag{4}$$
$$D(Q_{x_i}y, v_1) = D(x_i, \{yx_iv_1\}), \quad D(v_1, Q_{x_i}y) = D(\{v_1x_iy\}, x_i), \tag{5}$$
$$D(x_i, \{yzv_{2-i}\}) = D(\{x_iyz\}, v_{2-i}), \tag{6}$$
$$D(\{v_{2-i}zy\}, x_i) = D(v_{2-i}, \{zyx_i\}), \tag{7}$$
$$Q(Q_{x_i}y)Q_{v_1} = Q_{x_i}Q_{\{yx_iv_1\}}, \quad Q_{v_1}Q(Q_{x_i}y) = Q_{\{v_1x_iy\}}Q_{x_i}, \tag{8}$$
$$Q_{x_i}Q_{\{yzv_{2-i}\}} = Q_{x_i}Q_yQ_zQ_{v_{2-i}} = Q_{\{x_iyz\}}Q_{v_{2-i}}, \tag{9}$$
$$Q_{v_{2-i}}Q_{\{zyx_i\}} = Q_{v_{2-i}}Q_zQ_yQ_{x_i} = Q_{\{v_{2-i}yz\}}Q_{x_i}, \tag{10}$$
$$Q_{\{x_iyz\}}v_{2-i} = Q_{x_i}Q_yQ_zv_{2-i} \tag{11}$$
$$B(Q_{x_i}y, v_1) = B(x_i, \{yx_iv_1\}), \quad B(v_1, Q_{x_i}y) = B(\{v_1x_iy\}, x_i), \tag{12}$$
$$B(x_i, \{yzv_{2-i}\}) = B(\{x_iyz\}, v_{2-i}), \tag{13}$$
$$B(\{v_{2-i}yz\}, x_i) = B(v_{2-i}, \{zyx_i\}), \tag{14}$$
$$\beta(x_i, v_{2-i}) = \mathrm{Id}, \tag{15}$$
$$\beta(Q(x_i)y, v_1) = \beta(x_i, \{yx_iv_1\}), \tag{16}$$
$$\beta(\{x_iyz\}, v_{2-i}) = \beta(x_i, \{yzv_{2-i}\}). \tag{17}$$

Proof. (a) We do the case $i = 2$ and $j = 1$ and leave the other cases, which follow a similar pattern, to the reader. By (8.3.2), $y_1 \in \mathrm{Ker}(x_2)$, so by Proposition 8.4(a), we have $x_2^{y_1} = x_2$ and $y_1^{x_2} = y_1 + Q(y_1)x_2$. By (JP35), $\beta(x_2, y_1)^{-1} = \beta(x_2^{y_1}, -y_1) = \beta(x_2, -y_1)$ which proves (2).

(b) Since $Q(x_i)v_1 = 0$, (3) follows from (JP9). Similarly, (4) follows from (JP13) while (15) is immediate from $v_{2-i} \in \mathrm{Ann}\, x_i$. The remaining formulas are all special cases of 8.4 since, by (8.3.2), $V_{2-i}^- \oplus V_1^- \subset \mathrm{Ker}\, V_i^+ \subset \mathrm{Ker}\, x_i$ and $V_{2-i}^- \subset \mathrm{Ann}\, V_i^+ \subset \mathrm{Ann}\, x_i$.

8.6. Corollary. *Let $\mathfrak{P} = \mathfrak{P}_e$ be the Peirce grading determined by an idempotent e of V. Then in addition to the formulas of* 8.5(b)*, we have the following relations, where $j = 0, 1$ and again subscripts indicate membership in the corresponding Peirce space:*

$$\begin{aligned}
D(e^\sigma, y_2) &= D(Q_{e^\sigma} y_2, e^{-\sigma}), & D(x_2, e^{-\sigma}) &= D(e^\sigma, Q_{e^{-\sigma}} x_2), && (1)\\
D(x_{j+1}, y_j) &= D(e^\sigma, \{e^{-\sigma} x_{j+1} y_j\}), & D(u_j, v_{j+1}) &= D(\{u_j v_{j+1} e^\sigma\}, e^{-\sigma}), && (2)\\
Q_{x_{j+1}} Q_{y_j} &= Q_{e^\sigma} Q_{\{e^{-\sigma} x_{j+1} y_j\}}, & Q_{u_j} Q_{v_{j+1}} &= Q_{\{u_j v_{j+1} e^\sigma\}} Q_{e^{-\sigma}}, && (3)\\
B(x_{j+1}, y_j) &= B(e^\sigma, \{e^{-\sigma} x_{j+1} y_j\}), & B(u_j, v_{j+1}) &= B(\{u_j v_{j+1} e^\sigma\}, e^{-\sigma}), && (4)\\
\beta(x_{j+1}, y_j) &= \beta(e^\sigma, \{e^{-\sigma} x_{j+1} y_j\}), & \beta(u_j, v_{j+1}) &= \beta(\{u_j v_{j+1} e^\sigma\}, e^{-\sigma}). && (5)
\end{aligned}$$

Proof. For (1) we use the identity (JP8) and get

$$\begin{aligned}
D(Q_{e^\sigma} y_2, e^{-\sigma}) &= -D(Q_{e^\sigma} e^{-\sigma}, y_2) + D(e^\sigma, \{e^{-\sigma} e^\sigma y_2\})\\
&= -D(e^\sigma, y_2) + 2D(e^\sigma, y_2) = D(e^\sigma, y_2).
\end{aligned}$$

The second formula can be proved similarly using (JP7). The remaining formulas now all follow from 8.5(b). We will prove the first formula in (2) and leave the proof of the rest, which follows a similar pattern, as an exercise. For $j = 1$ we have, using (8.5.5) and (1), $D(x_2, y_1) = D(Q_{e^\sigma} Q_{e^{-\sigma}} x_2, y_1) = D(e^\sigma, \{Q_{e^{-\sigma}} x_2, e^\sigma, y_1\}) = D(e^\sigma, \{e^{-\sigma}, x_2, y_1\}$. For $j = 1$ we use (8.5.6) and get $D(x_1, y_0) = D(\{e^\sigma e^{-\sigma} x_1\}, y_0)$ $= D(e^\sigma, \{e^{-\sigma} x_1 y_0\})$.

In the following lemma we use the abbreviations $\{V_i V_j V_k\} = \big(\{V_i^+ V_j^- V_k^+\}, \{V_i^- V_j^+ V_k^-\}\big)$ and $Q_{V_i} V_j = \big(Q(V_i^+) V_j^-, Q(V_i^-) V_j^+\big)$.

8.7. Lemma. *Let $\mathfrak{P} : V = V_2 \oplus V_1 \oplus V_0$ be a Peirce grading of a Jordan pair V.*

(a) *The ideal of V generated by $V_2 \oplus V_1$ is*

$$\big\langle V_2 \oplus V_1 \big\rangle = V_2 \,\oplus\, V_1 \,\oplus\, \Big(\{V_1 V_1 V_0\} + Q_{V_1} V_2 + Q_{V_0} Q_{V_1} V_2\Big). \tag{1}$$

(b) *If $\mathfrak{P} = \mathfrak{P}_e$ is an idempotent Peirce grading then the ideals generated by V_2, $V_1 \oplus V_0$ and V_1 are*

$$\begin{aligned}
\langle V_2 \rangle &= V_2 \oplus V_1 \oplus Q_{V_1} V_2, && (2)\\
\langle V_1 \oplus V_0 \rangle &= \Big(\{V_2\, V_1\, V_1\} + Q_{V_1} V_0 + Q_{V_2} Q_{V_1} V_0 \Big) \oplus V_1 \oplus V_0 && (3)\\
&= \Big(\{e\, V_1\, V_1\} + Q_{V_1} V_0 + Q_e Q_{V_1} V_0 \Big) \oplus V_1 \oplus V_0, && (4)\\
\langle V_1 \rangle &= \Big(\{V_2\, V_1\, V_1\} + Q_{V_1} V_0 + Q_{V_2} Q_{V_1} V_0 \Big) \oplus V_1 \oplus Q_{V_1} V_2 && (5)\\
&= \Big(\{e\, V_1\, V_1\} + Q_{V_1} V_0 + Q_e Q_{V_1} V_0 \Big) \oplus V_1 \oplus Q_{V_1} V_2. && (6)
\end{aligned}$$

In particular, if $2 \in k^\times$ *then*

$$\langle V_1 \rangle = \Big(\{V_2\, V_1\, V_1\} + \{V_1\, V_0\, V_1\} \Big) \oplus V_1 \oplus \{V_1\, V_2\, V_1\}. \tag{7}$$

Remark. (2) and (6) are due to K. McCrimmon [**44**, 2.13], with a different proof. He also shows:

$$\text{if } V \text{ is simple then so are } V_2(e) \text{ and } V_0(e). \tag{8}$$

This is still true for arbitrary Peirce gradings by [**3**]. On the other hand, $V_1(e)$ need not be simple if V is, see (8.2.1) where V_1 is the direct product $M_{r,q-s}(R) \times M_{p-r,s}(R)$. For other properties of V that are inherited by V_2 and V_0 see [**39**, 4.1] and [**3**, **4**].

Proof. (a) Let I denote the right hand side of (1). Since obviously $I \subset \langle V_2 \oplus V_1 \rangle$, it remains to show

(i) $\{V V I\} \subset I$, (ii) $Q(V) I \subset I$, (iii) $Q(I) V \subset I$.

Let I_i be the Peirce components of I. Obviously only the 0-component in (i) – (iii) is of interest.

(i) It suffices to check that $\{V_i\, V_j\, I_l\} \subset I_0$ for $i - j + l = 0$. Because of (8.1.2), this leads to the condition $\{V_i\, V_{i+1}\, V_1\} \subset I_0$, which holds by definition of I_0, and to $\{V_i\, V_i\, I_0\} \subset I_0$. Concerning this last condition, we have $\{V_i\, V_i\, \{V_1\, V_1\, V_0\}\} \subset \{V_1\, V_1\, V_0\}$ by (JP15) and $\{V_i\, V_i\, V_j\} \subset V_j$. Moreover, $\{V_i, V_i, Q_{V_1} V_2\} \subset Q_{V_1} V_2$ by (JP12). A second application of (JP12) then shows

$$\begin{aligned}
\{V_i, V_i, Q_{V_0} Q_{V_1} V_2\} &\subset Q_{V_0} \{V_i, V_i, Q_{V_1} V_2\} + \{\{V_i\, V_i\, V_0\}, Q_{V_1} V_2, V_0\}\\
&\subset Q_{V_0} Q_{V_1} V_2.
\end{aligned}$$

(ii) It is immediate that $\{V_i\, I_j\, V_l\} \subset I_0$ for $i - j + l = 0$, $i \neq l$. Hence it is enough to prove $Q_{V_i} I_j \subset I_0$ for $2i = j$. The case $Q_{V_1} V_2 \subset I_0$ holds by definition, so only $Q_{V_0} I_0 \subset I_0$ has to be checked, and this follows from

$$\begin{aligned}
Q_{V_0} \{V_1\, V_1\, V_0\} &= Q_{V_0} Q(V_1, V_0) \subset D(V_0, V_1) D(V_0, V_0) V_1\\
&\subset \{V_0\, V_1\, V_1\} \quad \text{(by (8.5.3))},\\
Q_{V_0} Q_{V_0} Q_{V_1} V_2 &= Q(\{V_0\, V_0\, V_1\}) V_2 \subset Q_{V_1} V_2 \quad \text{(by (8.4.8))}.
\end{aligned}$$

(iii) In view of (i) it suffices to verify $Q_{I_i}V_j \subset I_0$ for $2i = j$, and since $Q_{V_1}V_2 \subset I_0$ by definition, only the case $i = 0 = j$ is of interest. Again by (i) it is sufficient to check the following cases:

$$\begin{aligned} Q(Q_{V_1}V_2)V_0 &\subset Q_{V_1}Q_{V_2}Q_{V_1}V_0 \subset Q_{V_1}V_2, \\ Q(Q_{V_0}Q_{V_1}V_2)V_0 &\subset Q_{V_0}Q_{V_1}Q_{V_2}V \subset Q_{V_0}Q_{V_1}V_2, \end{aligned}$$

which follow from the fundamental formula (JP3), and

$$\begin{aligned} Q(\{V_0\,V_1\,V_1\})V_0 \subset{} & Q\big(Q_{V_0}Q_{V_1}V_1, V_1\big)V_0 + Q_{V_1}Q_{V_1}Q_{V_0}V_0 \\ & \qquad + Q_{V_0}Q_{V_1}Q_{V_1}V_0 + \{V_0, V_1, Q_{V_1}\{V_1\,V_0\,V_0\}\} \\ \subset{} & 0 + Q_{V_1}V_2 + Q_{V_0}Q_{V_1}V_2 + \{V_0\,V_1\,V_1\} \end{aligned}$$

which follows from (JP21) and $Q_{V_0}V_1 = 0$.

(b) Now assume $V_i = V_i(e)$. Then $\{e_\sigma e_{-\sigma} x_1\} = x_1$ for all $x_1 \in V_1^\sigma$, whence the ideals generated by V_2 and $V_2 \oplus V_1$ agree. Also,

$$\begin{aligned} &\{V_1\,V_1\,V_0\} = \{V_0\,V_1\,V_1\} = \{\{V_0\,V_1\,e\}\,e\,V_1\} \subset Q_{V_1}V_2 \quad \text{(by (8.6.2))}, \\ &Q_{V_0}Q_{V_1}V_2 = Q(\{V_0\,V_1\,e\})Q_eV_2 \subset Q_{V_1}V_2 \quad \text{(by (8.6.3))} \end{aligned}$$

which proves (2). For the proof of (3) we apply (1) to the reverse Peirce grading $V_i^* = V_{2-i}(e)$ and obtain that

$$\Big(\{V_1V_1V_2\} + Q_{V_1}V_0 + Q_{V_2}Q_{V_1}V_0\Big) \,\oplus\, V_1 \,\oplus\, V_0$$

is an ideal, whence the ideal generated by $V_1 \oplus V_0$. However, $\{V_1\,V_1\,V_2\} = \{e\,V_1\,V_1\}$ by (8.6.2) and $Q_{V_2}Q_{V_1}V_0 = Q_eQ(\{e\,V_2\,V_1\})V_0 \subset Q_eQ_{V_1}V_0$ by (8.6.3). This proves (4). The ideal (5) is the intersection of the ideals (2) and (3). Formula (6) follows in the same way.

Finally, suppose $2 \in k^\times$. Then $Q_{V_1}V_0 = \{V_1V_0V_1\}$ and

$$Q_{V_2}Q_{V_1}V_0 = Q_{V_2}Q_{V_1,V_1}V_0 \subset \{V_2V_1\{V_2V_1V_0\}\}(\text{ by (8.5.4)}) \subset \{V_2V_1V_1\}$$

by the Peirce rules. This proves (7).

8.8. Definition. Let $\mathfrak{P}$ be a Peirce grading of V. The *group of $\mathfrak{P}$-elementary automorphisms* of V is the subgroup

$$\mathrm{EA}(V, \mathfrak{P}) = \big\langle \beta(V_i^+, V_j^-) : i \neq j \big\rangle$$

of the inner automorphism group $\mathrm{Inn}(V)$. If $\mathfrak{P} = \mathfrak{P}_e$ is an idempotent Peirce grading, we write $\mathrm{EA}(V, e) := \mathrm{EA}(V, \mathfrak{P}_e)$. It follows from (8.6.5) and (8.5.15) that

$$\mathrm{EA}(V, e) = \big\langle \beta(e_+, V_1^-) \cup \beta(V_1^+, e_-) \big\rangle.$$

§9. The projective elementary group of a Jordan pair

9.1. The Tits-Kantor-Koecher algebra. Let V be a Jordan pair over k. We use the notation introduced in 7.7 and note that $\mathfrak{L}_0(V) = k\cdot\zeta_V + \mathrm{Inder}(V)$ is a subalgebra of $\mathrm{Der}(V)$ and ζ_V is central in $\mathfrak{L}_0(V)$ (indeed, in all of $\mathrm{Der}(V)$).

In this book, the *Tits-Kantor-Koecher algebra of* V (or *TKK-algebra* for short) is the Lie algebra

$$\mathfrak{L}(V) = V^+ \oplus \mathfrak{L}_0(V) \oplus V^- \tag{1}$$

with multiplication

$$[V^\sigma, V^\sigma] = 0, \quad [D, z] = D_\sigma(z), \quad [x, y] = -\delta(x, y)$$

for $D = (D_+, D_-) \in \mathfrak{L}_0(V)$, $z \in V^\pm$ and $(x, y) \in V$. This definition of a TKK-algebra is different from the one used elsewhere, e.g. in [**15**, **47**] (see 9.11 for a comparison), but it is the most appropriate for our purposes.

We put $\mathfrak{g}_0 = \mathfrak{L}_0(V)$, $\mathfrak{g}_1 = V^+$, $\mathfrak{g}_{-1} = V^-$ and $\mathfrak{g}_i = \{0\}$ for $i \neq 0, \pm 1$. Then

$$\mathfrak{L}(V) = \mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

is a $\mathbb{Z}$-graded Lie algebra: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}$ which is 3-graded because $\mathfrak{g}_i = 0$ if $i \notin \{\pm 1, 0\}$. Moreover, $\mathrm{ad}\,\zeta_V$ is the grading derivation, i.e., $[\zeta_V, X] = iX$ for $X \in \mathfrak{g}_i$. From the definition, it follows easily that the derived algebra of $\mathfrak{L}(V)$ is

$$\mathrm{D}\mathfrak{L}(V) = [\mathfrak{L}(V), \mathfrak{L}(V)] = V^+ \oplus \mathrm{Inder}(V) \oplus V^-. \tag{2}$$

We also note that $\mathfrak{L}(V)$ has trivial centre. Indeed, if $Z = x \oplus \varDelta \oplus y \in V^+ \oplus \mathfrak{g}_0 \oplus V^-$ is central in $\mathfrak{g}$ then $0 = [\zeta, Z] = x \oplus 0 \oplus (-y)$ shows $Z = \varDelta \in \mathfrak{g}_0$, and since the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ is faithful, it follows that $Z = 0$.

The TKK-algebra depends functorially on V but only with respect to surjective homomorphisms. In more detail, let $f = (f_+, f_-)\colon V \to W$ be a surjective homomorphism of Jordan pairs. Then it follows easily from the definitions that f induces a surjective homomorphism of Lie algebras

$$f_0\colon \mathfrak{L}_0(V) \to \mathfrak{L}_0(W), \quad \delta(x, y) \mapsto \delta(f_+(x), f_-(y)), \quad \zeta_V \mapsto \zeta_W$$

mapping $\mathrm{Inder}(V)$ onto $\mathrm{Inder}(W)$, and a surjective homomorphism of the TKK-algebras

$$\mathfrak{L}(f)\colon \mathfrak{L}(V) \to \mathfrak{L}(W), \quad x \oplus D \oplus y \mapsto f_+(x) \oplus f_0(D) \oplus f_-(y),$$

which maps $\mathrm{D}\mathfrak{L}(V)$ onto $\mathrm{D}\mathfrak{L}(W)$.

9.2. The projective elementary group. For any $x \in V^\sigma$ ($\sigma \in \{\pm\}$) one defines an endomorphism $\exp_\sigma(x)$ of $\mathfrak{g} = \mathfrak{L}(V)$ (as a k-module) by

$$\exp_\sigma(x)z = z, \quad \exp_\sigma(x)\varDelta = \varDelta + [x, \varDelta], \quad \exp_\sigma(x)y = y + [x, y] + Q_x y, \tag{1}$$

where $z \in V^\sigma$, $\varDelta \in \mathfrak{g}_0$ and $y \in V^{-\sigma}$. With respect to the decomposition $\mathfrak{g} = V^+ \oplus \mathfrak{g}_0 \oplus V^-$, the maps $\exp_\sigma$ are given by formal 3×3 matrices of homomorphisms. In this identification we have, for $(x, y) \in V$,

$$\exp_+(x) = \begin{pmatrix} 1 & \operatorname{ad} x & Q_x \\ 0 & 1 & \operatorname{ad} x \\ 0 & 0 & 1 \end{pmatrix}, \qquad \exp_-(y) = \begin{pmatrix} 1 & 0 & 0 \\ \operatorname{ad} y & 1 & 0 \\ Q_y & \operatorname{ad} y & 1 \end{pmatrix}. \tag{1$'$}$$

It is known [**38**, 1.2] that $\exp_\sigma(z)$ is an automorphism of $\mathfrak{g}$ and in fact

$$\exp_\sigma \colon V^\sigma \to \operatorname{Aut}(\mathfrak{g}) \quad \text{is an injective homomorphism.} \tag{2}$$

We put $U^\pm := \operatorname{Im}(\exp_\pm)$ and define the *projective elementary group of* V as the subgroup

$$G = \mathrm{PE}(V)$$

of $\operatorname{Aut}(\mathfrak{g})$ generated by $U^+ \cup U^-$. Since the groups $U^\pm$ are abelian, it is clear that $\mathrm{PE}(V)$ has A_1-commutator relations with root groups $U_{\pm 1} = U^\pm$.

Example. Let $V = \mathbb{M}_{pq}(k) = (\mathrm{Mat}_{pq}(k), \mathrm{Mat}_{qp}(k))$ be the rectangular matrix pair of 7.5 and let $\mathfrak{M} = (\mathrm{Mat}_{pp}(k), \mathrm{Mat}_{pq}(k), \mathrm{Mat}_{qp}(k), \mathrm{Mat}_{qq}(k))$ be the associated Morita context. It is immediate (see the calculation in 10.16) that the elementary group $\mathrm{E}(\mathfrak{M}, V) = \mathrm{E}(\mathfrak{M})$ of 7.1 and 7.2 coincides with the elementary matrix group $\mathrm{E}_n(k)$, $n = p + q$. For simplicity assume from now on that k is a field. Then $\mathrm{E}_n(k) = \mathrm{SL}_n(k)$ by [**17**, 2.2.6]. Since $\mathrm{PE}(V) = \mathrm{E}(\mathfrak{M})/\mathscr{Z}(\mathrm{E}(\mathfrak{M}))$ by [**38**, Cor. 2.12], we have

$$\mathrm{PE}(\mathbb{M}_{pq}(k)) \cong \mathrm{SL}_n(k)/\mathscr{Z}(\mathrm{SL}_n(k)), \quad (n = p + q,\ k \text{ a field}).$$

We note that $\mathrm{PE}(\mathbb{M}_{pq}(k)) \cong \mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/\mathscr{Z}(\mathrm{GL}_n(k))$ if k contains all nth roots of unity, e.g. if k is algebraically closed.

From the functoriality of $\mathfrak{L}(V)$ it follows that $\mathrm{PE}(V)$ depends functorially on V with respect to surjective homomorphisms as well, cf. [**38**, 1.6]. The *diagonal subgroup* of G is the group

$$\mathrm{PE}_0(V) = G \cap \operatorname{Aut}(V)$$

where $\operatorname{Aut}(V)$ is diagonally embedded in $\operatorname{Aut}(\mathfrak{g})$, operating on $\mathfrak{g}_0$ by conjugation and on $V^\pm$ in the natural way. We often write

$$H = \mathrm{PE}_0(V)$$

for short. The subgroups $U^{\pm}$ are normalized by H; more precisely, for $h = (h_+, h_-) \in H$ and $x \in V^\sigma$,

$$h \exp_\sigma(x) h^{-1} = \exp_\sigma(h_\sigma(x)). \tag{3}$$

The *big cell* of G is defined as

$$\Omega = U^- H U^+.$$

By [**38**, 1.5],

$$\text{the map } V^- \times H \times V^+ \to \Omega,\ (y, h, x) \mapsto \exp_-(y) h \exp_+(x), \quad \text{is bijective.} \tag{4}$$

By [**38**, 1.4],

$$\begin{aligned} (x, y) \quad \text{is quasi-invertible} \quad &\iff \quad \exp_+(x)\exp_-(y) \in \Omega \\ &\iff \quad \exp_-(y)\exp_+(x) \in \Omega^{-1}, \end{aligned} \tag{5}$$

and then

$$\exp_+(x)\exp_-(y) = \exp_-(y^x)\beta(x, y)\exp_+(x^y). \tag{6}$$

In particular, this shows that the inner automorphism group $\mathrm{Inn}(V)$ defined in 7.10 is contained in H.

9.3. Higher order quasi-inverses. Faulkner [**16**] has extended (9.2.6) to arbitrary products. For $\mathbf{x} = (x_1, \ldots, x_{2n}) \in V^n = (V^+ \times V^-)^n$ define

$$\exp(\mathbf{x}) = \exp_+(x_1) \cdots \exp_-(x_{2n}) \in G.$$

Observe that every element of G is of the form $\exp(\mathbf{x})$ for a suitable $\mathbf{x}$ and n, since one can always add trivial factors $\exp_\pm(0)$. We put $\mathbf{x}^{\mathrm{op}} = (x_{2n}, \ldots, x_1)$ and $\exp(\mathbf{x}^{\mathrm{op}}) = \exp_-(x_{2n}) \cdots \exp_+(x_1) \in G$. Now define generalized Bergmann operators $B(\mathbf{x}) \in \mathrm{End}(V^+)$ and $B(\mathbf{x}^{\mathrm{op}}) \in \mathrm{End}(V^-)$ by

$$B(\mathbf{x})u \equiv \exp(\mathbf{x})(u) \bmod \mathfrak{g}_0 \oplus V^-, \quad B(\mathbf{x}^{\mathrm{op}})v \equiv \exp(\mathbf{x}^{\mathrm{op}})(v) \bmod \mathfrak{g}_0 \oplus V^+, \tag{1}$$

for $(u, v) \in V$. A recursive definition of these operators is given in [**16**]. The equivalence with our definition follows from [**16**, Lemma 2]. The recursive definition implies in particular

$$B(\mathbf{x}) = B(-\mathbf{x}), \qquad B(\mathbf{x}^{\mathrm{op}}) = B(-\mathbf{x}^{\mathrm{op}}). \tag{2}$$

For example,

$$\begin{aligned} B(x, y, z, v) = B(x, y)B(z, v) - D(x, v) + Q_x Q_v + Q_x Q(y, v) \\ + Q(x, z)Q_v - Q_x D(y, z) Q_v. \end{aligned} \tag{3}$$

Generalizing from the case $n = 1$, we call a $2n$-tuple $\mathbf{x}$ *quasi-invertible* if both $B(\mathbf{x})$ and $B(\mathbf{x}^{\mathrm{op}})$ are invertible, and then define $\beta(\mathbf{x}) = (B(\mathbf{x}), B(\mathbf{x}^{\mathrm{op}})^{-1})$. Note that $(B(\mathbf{x}), B(\mathbf{x}^{\mathrm{op}}))\colon V \rightleftharpoons V$ is a structural transformation for all $\mathbf{x}$ [**16**, Cor. 4] and that therefore $\beta(\mathbf{x}) \in \mathrm{Aut}(V)$ for a quasi-invertible $\mathbf{x}$. By [**16**, Theorem 1], $\mathbf{x}$ is quasi-invertible if and only if $\exp(\mathbf{x}) \in \Omega$, and in this case

$$\exp(\mathbf{x}) = \exp_-\big(q(\mathbf{x}^{\mathrm{op}})\big)\beta(\mathbf{x})\exp_+\big(q(\mathbf{x})\big). \tag{4}$$

The unique elements $q(\mathbf{x}) \in V^+$ and $q(\mathbf{x}^{\mathrm{op}}) \in V^-$ appearing in this formula are called the *quasi-inverses* of $\mathbf{x}$ and $\mathbf{x}^{\mathrm{op}}$.

As an application of (4) we show that the diagonal subgroup H is in fact the subgroup of G consisting of "diagonal" maps with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$:

$$\{\beta(\mathbf{x}) : \mathbf{x} \text{ quasi-invertible}\} = H = \{f \in G : f(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for } i = \pm 1, 0\}. \tag{5}$$

Indeed, by (4) we have $\{\beta(\mathbf{x}) : \mathbf{x} \text{ quasi-invertible}\} \subset H$, while $H \subset \{f \in G : f(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for } i = \pm 1, 0\}$ is immediate from the definition of H. Conversely, let $g = \mathrm{diag}(a_1, a_2, a_3) \in G$ be diagonal and write $g = \exp(\mathbf{x})$ for a suitable $\mathbf{x} \in V^n$. It follows from (1) and (2) that $B(\mathbf{x}) = a_1$ and $B(\mathbf{x}^{\mathrm{op}}) = B(-\mathbf{x}^{\mathrm{op}}) = a_3^{-1}$ are invertible. Thus $g = \exp_-(y)\, h \exp_+(x) \in \Omega$ and therefore $h = \beta(\mathbf{x})$ by (4). We have

$$\begin{aligned} g(\zeta) = a_2\zeta &= \exp_-(y)h(-x \oplus \zeta) = \exp_-(y)(-h_+(x) \oplus \zeta) \\ &= -h_+(x) \oplus (\zeta + [h_+(x), y]) \oplus (y - Q_y h_+(x)) \in \mathfrak{g}_0. \end{aligned}$$

This shows $h_+(x) = 0 = y$ and hence also $x = 0$, proving $g = h = \beta(\mathbf{x})$.

9.4. The Weyl element defined by an idempotent. Let $e = (e_+, e_-)$ be an idempotent of V. We introduce the notations

$$\begin{aligned} \theta_e &= \big(\exp_+(e_+), \exp_-(e_-), \exp_+(e_+)\big) \in U^+ \times U^- \times U^+, \\ \omega_e &= \mu(\theta_e) = \exp_+(e_+)\exp_-(e_-)\exp_+(e_+) \in G = \mathrm{PE}(V). \end{aligned}$$

As noted in 9.2, G has A_1-commutator relations with root groups $U_1 = U^+$ and $U_{-1} = U^-$, so $\theta_e \in \Theta_1(G)$ in the notation of (5.1.4). If e is an invertible idempotent, cf. 7.12, then we will see in 9.7 that ω_e a Weyl element and hence θ_e is a Weyl triple for G. In general, this is not the case. However, it will be shown later that the Peirce decomposition of V with respect to e gives rise to C_2-commutator relations for G (for suitably defined root groups) and then ω_e is indeed a Weyl element for one of the long roots. By abuse of language, we will therefore often refer to ω_e (resp. θ_e) as the Weyl element (resp. Weyl triple) defined by e.

Let $V^{\mathrm{op}} = (V^-, V^+)$ be the opposite Jordan pair, see 7.4. We define $e^{\mathrm{op}} = (e_-, e_+)$ and correspondingly

$$\begin{aligned} \theta_{e^{\mathrm{op}}} &= \big(\exp_-(e_-), \exp_+(e_+), \exp_-(e_-)\big) \in U^- \times U^+ \times U^-, \\ \omega_{e^{\mathrm{op}}} &= \exp_-(e_-)\exp_+(e_+)\exp_-(e_-). \end{aligned}$$

Note that $-e$ is an idempotent having the same Peirce spaces as e. Since $\exp_\sigma$ is a group homomorphism it is evident that

$$\omega_e^{-1} = \omega_{-e}.$$

9.5. Lemma. *Let e be an idempotent of V and let $V = V_2 \oplus V_1 \oplus V_0$ be the associated Peirce decomposition as in* 7.12. *Then the action of ω_e on the generators of $\mathfrak{L}(V)$ is given by*

$$\omega_e \cdot x = \left\{ \begin{array}{ll} x & \text{if } x \in V_0^\sigma \\ [e_\sigma, x] & \text{if } x \in V_1^\sigma \\ Q_{e_{-\sigma}} \cdot x & \text{if } x \in V_2^\sigma \\ x + [e_+, e_-] & \text{if } x = \zeta \end{array} \right\}. \tag{1}$$

Proof. For the first formula, let $x = x_0^\sigma \in V_0^\sigma$. Then $\exp_\sigma(e_\sigma) \cdot x = x$ holds by definition of exp in (9.2.1), while

$$\exp_{-\sigma}(e_{-\sigma}) \cdot x_0^\sigma = x_0 + [e_{-\sigma}, x_0] + Q_{e_{-\sigma}} \cdot x_0 = x_0$$

follows from the multiplication rules for the Peirce spaces, see 7.12. Next, let $x_1 \in V_1^+$. Then $\exp_-(e_-) \cdot x_1 = x_1 + [e_-, x_1]$ since $Q_{e_-} x_1 = 0$ by the Peirce relations, hence

$$\begin{aligned} \omega_e \cdot x_1 &= \exp_+(e_+) \exp_-(e_-) \cdot x_1 = \exp_+(e_+) \cdot \big(x_1 + [e_-, x_1]\big) \\ &= x_1 + [e_-, x_1] + [e_+, [e_-, x_1]]. \end{aligned}$$

Here $[e_+, [e_-, x_1]] = -\{e_+, e_-, x_1\} = -x_1$, so we have shown the case $\sigma = +$ of the second formula of (1). For the case $\sigma = -$, let $y_1 \in V_1^-$. Then similarly

$$\begin{aligned} \omega_e \cdot y_1 &= \exp_+(e_+) \exp_-(e_-) \cdot \big(y_1 + [e_+, y_1]\big) \\ &= \exp_+(e_+) \cdot \big(y_1 + [e_+, y_1] + [e_-, [e_+, y_1]]\big) = \exp_+(e_+) \cdot [e_+, y_1] \\ &= [e_+, y_1] + [e_+, [e_+, y_1]] = [e_+, y_1], \end{aligned}$$

since $[e_+, [e_+, y_1]] = \{e_+, y_1, e_+\} = 2Q_{e_+} y_1 = 0$.

Now let $x_2 \in V_2^+$. Then

$$\begin{aligned} \omega_e \cdot x_2 &= \exp_+(e_+) \exp_-(e_-) \cdot x_2 = \exp_+(e_+) \cdot \big(x_2 + [e_-, x_2] + Q_{e_-} x_2\big) \\ &= x_2 + \big([e_-, x_2] + [e_+, [e_-, x_2]]\big) + \big(Q_{e_-} x_2 + [e_+, Q_{e_-} x_2] + Q_{e_+} Q_{e_-} x_2\big). \end{aligned}$$

The map $Q_{e_-} \colon V_2^+ \to V_2^-$ is bijective with inverse $Q_{e_+} \colon V_2^- \to V_2^+$. Moreover, $[e_+, [e_-, x_2]] = -\{e_+, e_-, x_2\} = -2x_2$ and $[Q_{e_-} x_2, e_+] = \delta(e_+, Q_{e_-} x_2) = \delta(x_2, e_-)$ (by (8.6.1)). Hence $\omega_e \cdot x_2 = Q_{e_-} x_2$, proving the case $\sigma = +$ of the third formula of (1). The case $\sigma = -$ can be done by a similar computation, or by arguing as follows: recall that $\omega_{-e} = \omega_e^{-1}$ and that e and $-e$ have the same Peirce spaces, while Q_x is a quadratic function of x. Hence $\omega_{-e} \cdot x_2 = \omega_e^{-1} \cdot x_2 = Q_{e_-} x_2$ or $x_2 = \omega_e \cdot Q_{e_-} x_2$ for all $x_2 \in V_2^+$. By putting $y_2 = Q_{e_-} x_2$ we see that $\omega_e \cdot y_2 = Q_{e_+} y_2$ for all $y_2 \in V_2^-$.

It remains to compute the action of ω_e on ζ. Since $\operatorname{ad} \zeta$ is the grading derivation, we have $[e_+, \zeta] = -e_+$ and $[e_-, \zeta] = e_-$. Hence

$$\begin{aligned} \omega_e \cdot \zeta &= \exp_+(e_+) \exp_-(e_-) \cdot \big(\zeta + [e_+, \zeta]\big) \\ &= \exp_+(e_+) \cdot \big((\zeta + [e_-, \zeta]) - e_+ + [e_-, -e_+] + Q_{e_-}(-e_+)\big) \\ &= \exp_+(e_+) \cdot \big(-e_+ + (\zeta + [e_+, e_-])\big) \\ &= -e_+ + \zeta - e_+ + [e_+, e_-] + [e_+, [e_+, e_-]] = \zeta + [e_+, e_-], \end{aligned}$$

because $Q_{e_-} e_+ = e_-$ and $[e_+, [e_+, e_-]] = 2Q_{e_+} e_- = 2e_+$.

9.6. Proposition. *Let $e \in V$ be an idempotent. Then for all $u_i \in V_i^+(e)$, $v_i \in V_i^-(e)$ and $z_0 \in V_0^\sigma(e)$, the element ω_e satisfies the conjugation formulas*

$$\omega_e \, \exp_+(u_2)\, \omega_e^{-1} = \exp_-\big(Q_{e_-} u_2\big), \tag{1}$$

$$\omega_e \, \exp_-(v_2)\, \omega_e^{-1} = \exp_+\big(Q_{e_+} v_2\big), \tag{2}$$

$$\omega_e \, \exp_+(u_1)\, \omega_e^{-1} = \beta(u_1, -e_-), \tag{3}$$

$$\omega_e \, \exp_-(v_1)\, \omega_e^{-1} = \beta(e_+, v_1), \tag{4}$$

$$\omega_e \, \exp_\sigma(z_0)\, \omega_e \ = \exp_\sigma(z_0). \tag{5}$$

Moreover, θ_e is balanced and hence ω_e satisfies

$$\omega_e = \omega_{e^{\mathrm{op}}}. \tag{6}$$

Proof. (Cf. [**15**, Theorem 7]) It is easier to prove (1) in the form

$$\omega_e \, \exp_+(u_2) = \exp_-(Q_{e_-} u_2)\, \omega_e, \tag{7}$$

for all $u_2 \in V_2^+$. Since both sides of (7) are automorphisms of the TKK-algebra $\mathfrak{g} = \mathfrak{L}(V)$ which is generated by $V^+ = \mathfrak{g}_1$, $V^- = \mathfrak{g}_{-1}$ and ζ, it suffices to show that both sides agree when applied to the generators $x = x_i^\sigma \in V_i^\sigma(e)$ ($\sigma \in \{+,-\}$, $i \in \{0,1,2\}$) and $x = \zeta$. This amounts to seven cases. We do the case $x = x_2^- = y_2 \in V_2^-$, using (9.5.1), and leave the others to the reader:

$$\begin{aligned} \big(\omega_e \, \exp_+(u_2)\big) \cdot y_2 &= \omega_e \cdot \big(y_2 + [u_2, y_2] + Q_{u_2} y_2\big) \\ &= Q_{e_+} y_2 + [\omega_e(u_2), \omega_e(y_2)] + Q_{e_-} Q_{u_2} y_2. \end{aligned}$$

In the second step, we have used the fact that ω_e is an automorphism of $\mathfrak{g}$. On the other hand,

$$\begin{aligned} \big(\exp_-(Q_{e_-} u_2)\, \omega_e\big) \cdot y_2 &= \exp_-(Q_{e_-} u_2) \cdot Q_{e_+} y_2 \\ &= Q_{e_+} y_2 + [Q_{e_-} u_2, Q_{e_+} y_2] + Q(Q_{e_-} u_2) Q_{e_+} y_2. \end{aligned}$$

The second terms agree by (9.5.1), and so do the third terms by the Jordan identity (JP3) and the fact that $Q_{e_-} Q_{e_+}$ is the identity on $V_2^-(e)$: $Q(Q_{e_-} u_2) Q_{e_+} y_2 = Q_{e_-} Q_{u_2} Q_{e_-} Q_{e_+} y_2 = Q_{e_-} Q_{u_2} y_2$.

The relation (2) is now a consequence of the one just proved and the following observation. Since (1) holds for all idempotents, it does so in particular for $-e$. Now $\omega_{-e} = \omega_e^{-1}$ and Q is a quadratic map, so putting $u_2 = Q_{e_+} v_2$, we have $\omega_e^{-1} \exp_+(Q_{e_+} v_2) \omega_e = \exp_-(Q_{-e_-} Q_{e_+} v_2) = \exp_-(v_2)$ which is (2).

The remaining formulas (3) – (5) can be proved in the same way.

Finally, put $v_2 = e_-$ in (2) and use $Q_{e_+} e_- = e_+$. Then $\omega_e \exp_-(e_-) \omega_e^{-1} = \exp_+(e_+)$, so θ_e is balanced by 5.13(ii), which implies (6) by 5.13(iv).

9.7. Corollary. *Let V be a Jordan pair with invertible elements, and let e be an invertible idempotent of V, i.e., $e_+ \in V^+$ is invertible with inverse e_-. Consider $G = \mathrm{PE}(V)$ as a group with A_1-commutator relations as in 9.2. Then θ_e is a balanced Weyl triple and ω_e is a Weyl element for the root $\alpha = 1$ in the sense of 5.1.*

Proof. We have $V = V_2(e)$, so $Q_{e_-}\colon V^+ \to V^-$ is an isomorphism with inverse $Q_{e_+}\colon V^- \to V^+$. Now the corollary follows from (9.6.1) and (9.6.2).

Note, however, that in general not all Weyl elements of $\mathrm{PE}(V)$ are of this type; see Proposition 10.11 for details.

9.8. The extreme radical. Recall from [**34**, 4.21] that the *extreme radical* $\mathrm{Extr}(V) = (E^+, E^-)$ of a Jordan pair V is

$$E^\sigma = \{z \in V^\sigma : Q_z = D(z, V^{-\sigma}) = D(V^{-\sigma}, z) = 0\} \quad (\sigma \in \{+,-\}). \tag{1}$$

The extreme radical is a characteristic ideal. From the formulas for the Bergmann operators and the quasi-inverse in 7.8 and 7.9 it is easy to see that E^+ can also be characterized by

$$z \in E^+ \quad \iff \quad z^y = z \text{ and } \beta(z, y) = \mathrm{Id} \text{ for all } y \in V^-, \tag{2}$$

and similarly for E^-. We note

$$\text{if } V \text{ has invertible elements then } 2z = 0 \text{ for any } z \in E^\sigma. \tag{3}$$

Indeed, suppose $u \in V^+$ is invertible. By 7.12, $e = (u, u^{-1})$ is an idempotent with $V = V_2(e)$, so that $D(z, V^{-\sigma}) = 0$ implies $0 = \{z e_{-\sigma} e_\sigma\} = 2z$.

We now describe the normalizer of $U^\pm$ and the centre $\mathscr{Z}(G)$ of the projective elementary group $G = \mathrm{PE}(V)$ in terms of the extreme radical.

9.9. Theorem. (a) *Let N be the intersection of the normalizers of U^+ and U^-. Then N is given by*

$$N = \exp_-(E^-) \cdot H \cdot \exp_+(E^+), \tag{1}$$

where $E = (E^+, E^-)$ is the extreme radical. In particular, the normalizer of $U^{-\sigma}$ in U^σ is $\exp_\sigma(E^\sigma)$.

(b) *An element g belongs to $\mathscr{Z}(G)$ if and only if $g = \exp_-(v) h \exp_+(z)$ where $(z, v) \in \mathrm{Extr}(V)$ and $h = (h_+, h_-) \in H$ is determined by v and z by means of the formulas*

$$h_+(x) = x + Q_x v, \quad h_-(y) = y + Q_y z \quad (x \in V^+,\ y \in V^-). \tag{2}$$

In particular,

$$\mathscr{Z}(G) \cap H = \{1\}, \tag{3}$$
$$\operatorname{Extr}(V) = 0 \quad \Longrightarrow \quad \mathscr{Z}(G) = 1. \tag{4}$$

Proof. (a) We first show that $N \subset \Omega$. Let $g \in N$ and $t \in V^-$. Since g normalizes U^-, there exists $t' \in V^-$ such that $g \exp_-(t) = \exp_-(t')g$. With respect to the decomposition $\mathfrak{g} = V^+ \oplus \mathfrak{g}_0 \oplus V^-$, the automorphism g is given by a formal 3×3-matrix of homomorphisms, say $g = (a_{ij})$. With this identification, we obtain

$$\begin{aligned} g\ \exp_-(t)(\zeta) &= g(\zeta + t) = (a_{12}\zeta + a_{13}t) \oplus \cdots \\ &= \exp_-(t')g(\zeta) = \exp_-(t')(a_{12}\zeta \oplus a_{22}\zeta \oplus a_{32}\zeta) = a_{12}\zeta \oplus \cdots. \end{aligned}$$

Hence $a_{13} = 0$. An analogous computation yields, for $s \in V^+$,

$$\begin{aligned} g\ \exp_-(t)(s) &= g(s \oplus [t,s] \oplus Q_t s) = (a_{11}s + a_{12}[t,s]) \oplus \cdots \\ &= \exp_-(t')g(s) = \exp_-(t')(a_{11}s \oplus \cdots) = a_{11}s \oplus \cdots. \end{aligned}$$

Since $(s,t) \in V$ is arbitrary, this implies $a_{12}(\operatorname{Inder}(V)) = 0$. Note that every automorphism of $\mathfrak{g}$ leaves the derived algebra $\mathfrak{L}'(V)$ (cf. (9.1.2)) invariant. For $g^{-1} = (b_{ij})$, the $\mathfrak{g}_0$-component of $g^{-1}(s)$ is $b_{21}s$ which, by the remark just made, belongs to $\operatorname{Inder}(V)$. It follows that $a_{12}b_{21} = 0$. The relation $gg^{-1} = 1$ now yields $\operatorname{Id}_{V^+} = \sum_i a_{1i}b_{i1} = a_{11}b_{11}$. By switching the roles of g and g^{-1} we also have $\operatorname{Id}_{V^+} = b_{11}a_{11}$. Hence, a_{11} is invertible with inverse b_{11}. Similarly, one shows that b_{33} is invertible with inverse a_{33} by using the fact that g normalizes U^+. Letting $g = \exp(\mathbf{x})$ for a suitable $\mathbf{x} \in V^n$ we have $a_{11} = B(\mathbf{x})$, $b_{33} = B(-\mathbf{x}^{\mathrm{op}}) = B(\mathbf{x}^{\mathrm{op}})$ in view of (9.3.1) and (9.3.2). Thus, $\mathbf{x}$ is quasi-invertible so (9.3.4) shows $g \in \Omega = U^-HU^+$.

By (9.2.3), H normalizes U^σ and also $\exp_\sigma(E^\sigma)$ since the extreme radical is stable under all automorphisms of V. Hence it remains to show that an element $g = \exp_-(v)\exp_+(z)$ belongs to N if and only if $(z,v) \in \operatorname{Extr}(V)$. Now for any $y \in V^-$,

$$g \exp_-(y)\, g^{-1} = \exp_-(v)\exp_+(z)\exp_-(y)\exp_+(-z)\exp_-(-v) \in U^-$$

if and only if $\exp_+(z)\exp_-(y) \in U^- \exp_+(z)$, which, by (9.2.5) and (9.2.6), is equivalent to (z,y) being quasi-invertible, $z = z^y$ and $\beta(z,y) = 1$. By (9.8.2), this is equivalent to $z \in E^+$. Similarly, g normalizes U^+ if and only if $v \in E^-$.

(b) By (a), any $g \in \mathscr{Z}(G)$ has the form $g = \exp_-(v)h\exp_+(z)$ with $(z,v) \in \operatorname{Extr}(V)$. Furthermore, for all $x \in V^+$,

$$\begin{aligned} g\exp_+(x)\, g^{-1} &= \exp_-(v)h\ \exp_+(x)\, h^{-1}\ \exp_-(-v) \\ &= \exp_-(v)\exp_+(h_+(x))\exp_-(-v) = \exp_+(x) \end{aligned}$$

if and only if $\exp_+(x)\exp_-(v) = \exp_-(v)\exp_+(h_+(x))$. By (9.2.6) and $v \in E^-$ this is equivalent to $h_+(x) = x^v = x - Q_xv = x + Q_xv$, since $2Q_xv = \{xvx\} = 0$. By symmetry, $z \in E^+$ and $h_-(y) = y + Q_yz$ for all $y \in V^-$. This proves that g belongs to the centre if and only if it has the stated form, and (3) and (4) are then immediate consequences.

9.10. Remarks. (i) Let $g = \exp_-(v)h\exp_+(z) \in \mathscr{Z}(G)$. In view of (9.9.2), the maps $h_\pm - \mathrm{Id}_{V^\pm}$ are both linear and quadratic over k. Therefore, they must be zero (and hence $h = 1$) provided V satisfies the condition

$$(\lambda - \lambda^2)u = 0 \quad \text{for all } \lambda \in k \quad \Longrightarrow \quad u = 0, \tag{1}$$

for any $u \in V^\sigma$. In particular, this is the case if V has no 2-torsion or if there exists an element $\lambda \in k^\times$ with $1 - \lambda \in k^\times$, for example, when k is a field with at least 3 elements. On the other hand, we will give an example where $h \neq 1$ in 9.17.

(ii) The converse of (9.9.4) is not true, as the following example shows. Let k be a field of characteristic 2 and let $V^+ = V^- = J$ where J is the quadratic (non-special) three-dimensional Jordan algebra $k1 \oplus ka \oplus ka^3$ [**22**, I.5, Example (3)]. Then $E^\pm = ka^3$ but $\mathscr{Z}(G) = 1$. Indeed, since $Q_1 = \mathrm{Id}$, the centre of G will be trivial as soon as we know that the H-component of an arbitrary $g \in \mathscr{Z}(G)$ is trivial. If k has more than two elements this is clear by (1) above. If k has two elements we argue as follows.

In an arbitrary Jordan algebra, the powers of an element b satisfy $\{b^m b^n b^p\} = 2b^{m+n+p}$ [**22**, p. I.23]. Hence $\{V, V, V\} = \mathrm{Inder}(V) = 0$ because of $2k = 0$. Since V is finite-dimensional it is in particular stable [**38**], so that H is generated by all $\beta(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, x_3, x_4) \in V \times V$ is quasi-invertible. Now a straightforward computation shows that $B(\mathbf{x})$ has the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix}$ with respect to the basis $1, a, a^3$. On the other hand, if $g = \exp_-(v)h\ \exp_+(z) \in \mathscr{Z}(G)$ and $v \neq 0$ then $v = a^3$ and therefore, by (9.9.1) and a lengthy calculation, h_+ has the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. This contradiction shows that $v = 0$, and in the same way $z = 0$, proving $g = 1$. We will see another example in 9.17.

9.11. Faulkner's projective elementary group. J. Faulkner [**15**, Sect. 3] introduced a group which is closely related to the projective elementary group $\mathrm{PE}(V)$ as defined in 9.2. We describe here the precise relation between these two groups.

Faulkner's Tits-Kantor-Koecher algebra is $\mathfrak{L}^{\mathrm{Fau}}(V) = V^+ \oplus \mathrm{Inder}(V) \oplus V^-$ with the multiplication

$$[V^\sigma, V^\sigma] = 0, \quad [D, z] = D_\sigma(z), \quad [x, y] = \delta(x, y),$$

for $D = (D_+, D_-) \in \mathrm{Inder}(V)$, $z \in V^\sigma$ and $(x, y) \in V$. Observe that $\mathfrak{L}^{\mathrm{Fau}}(V)$ differs from our Tits-Kantor-Koecher-algebra $\mathfrak{L}(V)$ of 9.1 in two respects: the 0-part of $\mathfrak{L}(V)$ is enlarged by the degree derivation ζ_V, and for $(x, y) \in V$ we have $[x, y] = -\delta(x, y)$ in $\mathfrak{g}$. One easily verifies that

$$f : \mathfrak{L}^{\mathrm{Fau}}(V) \to \mathrm{D}\mathfrak{L}(V), \qquad x_+ \oplus D \oplus y_- \mapsto x_+ \oplus D \oplus (-y_-)$$

is an isomorphism.

Faulkner's group, which we denote $\mathrm{FPE}(V)$, is the subgroup of $\mathrm{Aut}(\mathfrak{L}^{\mathrm{Fau}}(V))$ generated by automorphisms $\mathrm{x}_F(V^\sigma)$ as defined in [**15**]. For $(u, v) \in V$ one checks that

$$f \circ \mathrm{x}_F(u) \circ f^{-1} = \exp_+(u)\big|\mathrm{D}\mathfrak{L}(V), \quad f \circ \mathrm{x}_F(v) \circ f^{-1} = \exp_-(-v)\big|\mathrm{D}\mathfrak{L}(V).$$

Hence, if we let ϱ: $\mathrm{PE}(V) \to \mathrm{PE}(V)\big|\mathrm{D}\mathfrak{L}(V)$ be the restriction map, we have

$$\mathrm{FPE}(V) \cong \varrho\big(\mathrm{PE}(V)\big) \cong \mathrm{PE}(V)/\operatorname{Ker}(\varrho). \tag{1}$$

9.12. Proposition. *Let* $\mathrm{Extr}(V) = (E^+, E^-)$ *be the extreme radical of* V*. The map*

$$\varphi : E^+ \times E^- \to \operatorname{Ker}(\varrho), \quad (u, v) \mapsto \exp_-(v)\exp_+(u)$$

is a group isomorphism. In particular, $\operatorname{Ker}(\varrho)$ *is abelian, and if* $\mathrm{Extr}(V) = 0$ *then* $\mathrm{PE}(V) \cong \mathrm{FPE}(V)$*.*

Proof. It is an easy consequence of the definition of the extreme radical and the properties of $\exp_\pm$ that the map φ is an injective group homomorphism into $\operatorname{Ker}(\varrho)$, namely, (9.2.6), (9.8.2) and (9.2.4). Any $g \in \mathrm{PE}(V)$ is of the form $g = \exp(\mathbf{x})$ for a suitable $\mathbf{x} \in V^n$. If $g \in \operatorname{Ker}(\varrho)$ we have $B(\mathbf{x}) = \mathrm{Id}$ where B is the generalized Bergmann operator introduced in 9.3. Similarly, $g^{-1} \in \operatorname{Ker}(\varrho)$ implies $B(\mathbf{x}^{\mathrm{op}}) = \mathrm{Id}$. Hence, by (9.3.4), $\mathbf{x}$ is quasi-invertible and $g \in \Omega$, so that $g = \exp_-(v)h\,\exp_+(u)$ for $(u, v) \in V$ and $h \in H$. Now for all $x \in V^+$,

$$x = g(x) = \exp_-(v)h_+(x) = h_+(x) \oplus [v, h_+(x)] \oplus Q_v h_+(x)$$

whence $h_+ = \mathrm{Id}$ and $Q_v = D(v, V^+) = D(V^+, v) = 0$, i.e., $v \in E^-$. Similarly, $h_- = \mathrm{Id}$ and $u \in E^+$, proving that φ is surjective.

9.13. The projective elementary group of a subpair. Consider a subpair $V' = (V'^+, V'^-)$ of V, let $\mathfrak{g} = \mathfrak{L}(V)$ and define $\mathfrak{g}' \subset \mathfrak{g}$ by

$$\mathfrak{g}' = V'^+ \oplus \big(k \cdot \zeta_V + [V'^+, V'^-]\big) \oplus V'^- = \mathfrak{g}'_1 \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}'_{-1}.$$

Since V' is a subpair, it follows from the definition of the multiplication in $\mathfrak{g}$ that $[[V'^+, V'^-], V'^\sigma] \subset V'^\sigma$ and this implies that $\mathfrak{g}'$ is a (graded) subalgebra of $\mathfrak{g}$. Let $\mathfrak{L}(V') = V'^+ \oplus \big(k \cdot \zeta_{V'} + \mathrm{Inder}(V')\big) \oplus V'^-$ be the TKK-algebra of V'. It is easily verified that there is a surjective homomorphism φ: $\mathfrak{g}' \to \mathfrak{L}(V')$ of graded Lie algebras given by $\varphi(x') = x'$ for $x' \in \mathfrak{g}'_{\pm 1}$, and on $\mathfrak{g}'_0$ by restricting an element $D' \in \mathfrak{g}'_0$ to $V'^\pm$. Thus we have an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g}' \xrightarrow{\varphi} \mathfrak{L}(V') \longrightarrow 0$$

with $\mathrm{Ker}(\varphi) = \mathfrak{k} \subset \mathfrak{g}'_0$, and $X \in \mathfrak{k}$ if and only if $[X, V'^{\pm}] = 0$. This implies $[\mathfrak{k}, \mathfrak{g}'] = 0$, so $\mathfrak{k}$ is central in $\mathfrak{g}'$.

Now consider the projective elementary groups $\mathrm{PE}(V)$ of V and $\mathrm{PE}(V')$ of V' with exponential maps $\exp_\pm$ and $\exp'_\pm$, respectively. Let $G' \subset \mathrm{PE}(V)$ be the subgroup generated by $\exp_+(V'^+) \cup \exp_-(V'^-)$. We claim that there is a unique surjective group homomorphism

$$\psi \colon G' \to \mathrm{PE}(V') \tag{1}$$

with the property that

$$\psi\big(\exp_\sigma(x')\big) = \exp'_\sigma(x'), \tag{2}$$

for all $x' \in V'^\sigma$ and $\sigma \in \{+, -\}$.

Indeed, it follows from (9.2.1) and the fact that V' is a subpair of V that $\exp_\sigma(x')$ stabilizes $\mathfrak{g}'$. Moreover, for $X \in \mathfrak{k}$ we have $\exp_\sigma(x') \cdot X = X + [x', X]$ (by (9.2.1)) $= X$, so the generators of G' fix the elements of $\mathfrak{k}$. Hence G' stabilizes $\mathfrak{g}'$ and $\mathfrak{k}$, showing there is a well-defined homomorphism $\psi \colon G' \to \mathrm{GL}\big(\mathfrak{L}(V')\big)$. By applying φ to (9.2.1) (with x and y now belonging to V') we see that the diagrams

$$\begin{tikzcd}
\mathfrak{g}' \arrow[r, "\exp_\sigma(x')"] \arrow[d, "\varphi"'] & \mathfrak{g}' \arrow[d, "\varphi"] \\
\mathfrak{L}(V') \arrow[r, "\exp'_\sigma(x')"'] & \mathfrak{L}(V')
\end{tikzcd}$$

are commutative. Hence (2) holds so ψ maps G' onto $\mathrm{PE}(V')$, as asserted.

9.14. The projective elementary group of a direct sum. Let $V = \bigoplus_{i \in I} V_i$ be a direct sum of ideals. Our aim is to show that there is a natural isomorphism

$$\mathrm{PE}(V) \cong \bigoplus_{i \in I} \mathrm{PE}(V_i) \tag{1}$$

where the direct sum symbol on the right denotes the restricted direct product of groups, i.e., the subgroup of the full direct product whose elements have only finitely many components different from 1.

Clearly, the inner derivation algebra of V commutes with direct sums:

$$\mathrm{Inder}(V) \cong \bigoplus_{i} \mathrm{Inder}(V_i). \tag{2}$$

Recall from 9.1 that $\mathrm{D}\mathfrak{L}(V) = V^- \oplus \mathrm{Inder}(V) \oplus V^+$ is the derived algebra of the TKK-algebra $\mathfrak{L}(V)$ of V. Then (2) immediately implies that the functor $\mathrm{D}\mathfrak{L}$ also commutes with direct sums:

$$\mathrm{D}\mathfrak{L}(V) \cong \bigoplus_{i \in I} \mathrm{D}\mathfrak{L}(V_i). \tag{3}$$

The relation between the full TKK-algebra $\mathfrak{L}(V) = k \cdot \zeta_V + \mathrm{D}\mathfrak{L}(V)$ of V and that of the V_i is more complicated.

Let $p_i\colon V \to V_i$ be the projection onto the i-th factor. Since $\mathfrak{L}$ is functorial with respect to surjective homomorphisms by 9.1, there are induced homomorphisms $f_i = \mathfrak{L}(p_i)\colon \mathfrak{L}(V) \to \mathfrak{L}(V_i)$, so we have a homomorphism

$$f\colon \mathfrak{L}(V) \to \prod_{i \in I} \mathfrak{L}(V_i)$$

with components f_i. Explicitly, f_i and f are given as follows. Let

$$X = x \oplus (\lambda \cdot \zeta_V + d) \oplus y \in \mathfrak{L}(V)$$

where $x = \sum x_i \in V^+ = \bigoplus V_i^+$, $y = \sum y_i \in \bigoplus V_i^-$, $\lambda \in k$, and $d = \sum d_i \in \bigoplus \mathrm{Inder}(V_i)$. Then

$$f_i(X) = x_i \oplus (\lambda \zeta_{V_i} + d_i) \oplus y_i, \quad f(X) = (f_i(X))_{i \in I} = x \oplus (\lambda \xi + d) \oplus y, \qquad (4)$$

where we identify $\mathrm{D}\mathfrak{L}(V)$ with the subalgebra of $\prod_i \mathfrak{L}(V_i)$ determined by (3) and the embeddings $\mathrm{D}\mathfrak{L}(V_i) \subset \mathfrak{L}(V_i)$, and $\xi := (\zeta_{V_i})_{i \in I} \in \prod \mathfrak{L}(V_i)$. From (4) it is clear that f is an isomorphism of $\mathfrak{L}(V)$ onto the subalgebra $\mathfrak{g} := k \cdot \xi + \mathrm{D}\mathfrak{L}(V)$ of $\hat{\mathfrak{g}} := \prod \mathfrak{L}(V_i)$. We will therefore *identify* the TKK-algebra of V with the subalgebra $\mathfrak{g}$ of $\hat{\mathfrak{g}}$.

Let $G_i := \mathrm{PE}(V_i)$ and $G' = \bigoplus_i G_i \subset \prod_i G_i$. The latter group acts on $\hat{\mathfrak{g}}$ diagonally (i.e., componentwise) by automorphisms. Then the assertion (1) is a consequence of the following result:

9.15. Proposition. *With the notations introduced above, G' stabilizes the subalgebra $\mathfrak{g} \cong \mathfrak{L}(V)$, and the induced homomorphism $\psi\colon G' \to \mathrm{Aut}(\mathfrak{g})$ is an isomorphism of G' onto $\mathrm{PE}(V)$.*

Proof. We start with the following remark. Let W be an arbitrary Jordan pair with TKK-algebra $\mathfrak{L}(W) = k \cdot \zeta_W + \mathrm{D}\mathfrak{L}(W)$ and let $h \in \mathrm{PE}(W)$. Then h stabilizes $\mathrm{D}\mathfrak{L}(W)$ (since h is an automorphism of $\mathfrak{L}(W)$ and the derived algebra is a characteristic ideal). Moreover, $h(\zeta_W) \equiv \zeta_W \pmod{\mathrm{D}\mathfrak{L}(W)}$. Indeed, it suffices to check this for the generators $\exp_\pm(x)$ of $\mathrm{PE}(W)$, where $\exp_\pm(x) \cdot \zeta_W = \zeta_W + [x, \zeta_W] = \zeta_W \mp x$.

Now let $g = (g_i)_{i \in I} \in G'$. Then clearly g stabilizes $\mathrm{D}\mathfrak{L}(V)$. Moreover, applying the remark above to $W = V_i$ and $h = g_i$, we have $g_i(\zeta_{V_i}) = \zeta_{V_i} + X_i$ where $X_i \in \mathrm{D}\mathfrak{L}(V_i)$, and since $g_i \neq 1$ for only finitely many i, only finitely many X_i are different from 0. Hence $g(\xi) = \xi + (X_i)_{i \in I} \in \xi + \mathrm{D}\mathfrak{L}(V) \subset \mathfrak{g}$. This shows that $\mathfrak{g}$ is indeed stable under G' and proves the existence of a group homomorphism ψ.

Next, we show that ψ is injective. If $g = (g_i)_{i \in I} \in \mathrm{Ker}(\psi)$ then g_i acts like the identity on $\mathrm{D}\mathfrak{L}(V_i)$, and $g(\xi) = \xi$ implies $g_i(\zeta_{V_i}) = \zeta_{V_i}$, for all $i \in I$, whence $g = 1$.

Finally, we show $\psi(G') = \mathrm{PE}(V)$. Denote the exponential maps of $\mathfrak{L}(V_i)$ $(i \in I)$ and of $\mathfrak{L}(V)$ by $\exp_\sigma^{(i)}\colon V_i^\sigma \to G_i$ and $\exp_\sigma\colon V^\sigma \to \mathrm{Aut}(\mathfrak{g})$, respectively. Then one easily verifies that the diagram

$$\begin{tikzcd} V_i^\sigma \arrow[r, "\mathrm{inc}"] \arrow[d, "\exp_\sigma^{(i)}"'] & V^\sigma \arrow[d, "\exp_\sigma"] \\ G_i \arrow[r, "\psi"'] & \mathrm{Aut}(\mathfrak{g}) \end{tikzcd}$$

commutes. Since $\mathrm{PE}(V)$ is generated by $\exp_\sigma(V^\sigma)$ ($\sigma \in \{+,-\}$), and $\exp_\sigma(x) = \exp_\sigma(\sum x_i) = \prod_i \exp_\sigma(x_i) = \prod_i \psi\big(\exp_\sigma^{(i)}(x_i)\big)$, this shows that $\psi(G') = \mathrm{PE}(V)$.

9.16. Example. In the rest of this section we will elucidate some of the concepts introduced above by considering the example $V = (J, J)$ where $J = J_n$ is the (Jordan algebra determined by the) unital commutative associative algebra over $k = \mathbb{F}_2$, the field with two elements, generated by a nilpotent element a of index $n \geqslant 0$, i.e.,

$$J = k[t]/(t^{n+1}) = k1 \oplus ka \oplus \cdots \oplus ka^n, \qquad a^n \neq 0.$$

Clearly, J is local (see [**34**, 1.10]) with Jacobson radical $\mathrm{Rad}(J) = ka \oplus \cdots \oplus ka^n$. We let $a^0 = 1$. The Jordan product and the Bergmann operators are then given by

$$\begin{aligned} &Q_{a^i}(a^j) = a^{2i+j} \quad (0 \leqslant i, j \leqslant n), \qquad Q_{x,y} = 0, \\ &B(x,y)z = (1 - xy)z(1 - yx) = z + x^2y^2z \end{aligned} \tag{1}$$

for $x, y, z \in J$. In particular $\mathrm{Inder}(V) = 0$, and hence $\mathfrak{g}(V) = J \oplus k\zeta \oplus J$ is a 2-step solvable Lie algebra. The operator $A = Q_a$ is nilpotent of index $m = [\frac{n}{2}]$. The algebra $\mathfrak{A} \subset \mathrm{End}(J)$ generated by A is therefore

$$\mathfrak{A} = k\mathrm{Id} \oplus kA \oplus \cdots \oplus kA^m,$$

and the set $\mathfrak{A}^\times$ of invertible elements of $\mathfrak{A}$ has the description

$$\mathfrak{A}^\times = \mathrm{Id} \oplus k\,A \oplus \cdots \oplus kA^m. \tag{2}$$

For any $x \in J$ we let (x_i) be the coordinates of x with respect to the basis $(1, a, a^2, \ldots, a^n)$. Then

$$Q_x = \sum_{i=0}^{m} x_i A^i \in \mathfrak{A}. \tag{3}$$

Hence, (1) and (3) together with (9.8.1) imply

$$\mathrm{Extr}(V) = (E, E) \quad \text{where} \quad E = k\,a^{m+1} \oplus \cdots \oplus ka^n. \tag{4}$$

Thus $\mathrm{Extr}(V) \neq 0$ if $n \geqslant 1$. Also, (3) shows

$$B(x,y) = \mathrm{Id} + Q_xQ_y = B(y,x) \equiv (1 + x_0y_0) \bmod \mathrm{Rad}(\mathfrak{A}) \tag{5}$$

for $x, y \in J$. Therefore, $(x, y) \in V$ is quasi-invertible if and only if $x_0y_0 = 0$.

We claim that the diagonal subgroup H of $G = \mathrm{PE}(V)$ is

$$H = \{(b, b^{-1}) : b \in \mathfrak{A}^\times\}. \tag{6}$$

Indeed, for any $y \in J$ we have $B(a, y) = \mathrm{Id} + \sum_{i=1}^m y_{i-1}A^i = B(y, a)$ in view of (3) and (5). Since $\beta(a, y) = (B(a, y), B(y, a)^{-1}) \in H$ this proves that $(b, b^{-1}) \in H$ for any $b \in \mathfrak{A}^\times$. On the other hand, by [**38**, 5.3], stability of V implies that H is generated by the generalized Bergman operators

$$\beta(x, y, z, u) = (B(x, y, z, u), B(u, z, y, x)^{-1})$$

for quasi-invertible $(x, y, z, u) \in V \times V$. In our setting, formula (9.3.3) shows, for any $(x, y, z, v) \in V \times V$,

$$B(x, y, z, v)s = \big((1 - xy)(1 - zv) - xv\big)^2 s$$

and therefore any element of H has the form (b, b^{-1}) for some $b \in \mathfrak{A}^\times$.

We note in particular that $H = \{1\}$ if and only if $n = 0$ or 1. The centre of G has the following description:

9.17. Lemma. *$\mathscr{Z}(G) = \{1\}$ if and only if $n = 0$ or n is odd; $|\mathscr{Z}(G)| = 2$ if and only if $n \geqslant 2$ is even. In the second case, the non-trivial element of $\mathscr{Z}(G)$ is*

$$\exp_-(a^n)\,(b, b)\,\exp_+(a^n), \qquad \textit{where} \quad b = \mathrm{Id} + A^{n/2}.$$

Proof. By 9.9 and 9.16.6, any $g \in \mathscr{Z}(G)$ has the form

$$g = \exp_-(v)\,(b, b^{-1})\,\exp_+(z)$$

where $v, z \in E$ and $b \in \mathfrak{A}^\times$ satisfies

$$(b - \mathrm{Id})x = Q_x v, \quad (b^{-1} - \mathrm{Id})y = Q_y z \tag{1}$$

for all $x, y \in J$. It is immediate (by specializing $x = y = 1$) that $g = 1$ if $b = \mathrm{Id}$; in particular this is so when $H = \{1\}$. We can therefore assume $n \geqslant 2$. By (9.16.2) there exist $\mu_i \in k$ such that $b - \mathrm{Id} = \sum_{i=1}^m \mu_i A^i$. Hence, condition (1) for $x = 1$ yields $v = \sum_{i=1}^m \mu_i a^{2i} \in E$. Then choosing $x = a$ gives the relation

$$\sum_{i=1}^m \mu_i a^{2i+1} = \sum_{i=1}^m \mu_i a^{2i+2}.$$

For $i < m$ we have $2i + 1 < n + 1$. Since then $a^{2i+1} \neq 0$ we must have $\mu_i = 0$. For $i = m$ and n odd we obtain $2m + 1 = n$ and hence also $\mu_m = 0$. Therefore in this case $v = 0$, $b = \mathrm{Id} = b^{-1}$ and so $g = 1$. For n even we showed $v = \mu_m a^n$ and $b - \mathrm{Id} = \mu_m A^m$. For these choices, $(b - \mathrm{Id})x = \mu_m x_0 a^n = Q_x v$ holds for all $x \in J$. Since $b = \mathrm{Id} + \mu_m A^m = b^{-1}$ we get $z = \mu_m a^n = v$. This finishes the proof of the lemma.

Let us point out that for odd n we have $\mathrm{Extr}(V) \neq 0$ but $\mathscr{Z}(G) = 1$, yielding yet another example that the converse of the implication (9.9.4) is not true.

Next, we identify the groups $G_n := \mathrm{PE}(J_n, J_n)$. Denote by Sym_n the symmetric group on n letters.

9.18. Lemma. $G_0 = \mathrm{Sym}_3$ *and* $G_1 = \mathrm{Sym}_4$.

Proof. We will use the well-known fact that the group Sym_{n+1} is presented by generators $s_1, \dots, s_n$ and relations $s_i^2 = 1 = (s_i s_{i+1})^3 = (s_i s_j)^2$ for $i < j - 1$. For $n = 0$ we have $J = k$ and hence G_0 is generated by the two elements $g_1 = \exp_+(1)$ and $g_2 = \exp_-(1)$ which, by characteristic 2, satisfy the relation $g_i^2 = 1$. Let $e = (1, 1)$. Since by (9.6.6) the Weyl element ω_e is symmetric,

$$\omega_e = \exp_+(1) \exp_-(1) \exp_+(1) = \exp_-(1) \exp_+(1) \exp_-(1),$$

we have $g_1 g_2 g_1 = g_2 g_1 g_2$ or, equivalently, $(g_1 g_2)^3 = 1$. Hence there is a surjective homomorphism $\varphi\colon \mathrm{Sym}_3 \to G_0$. But $|G_0| \geqslant |\Omega| \geqslant |J|^2 = 4$ where Ω is the big cell defined in 9.2, so that φ is an isomorphism.

We will proceed in the same manner for $n = 1$ where $J = k1 \oplus ka$, $a^2 = 0$, is the algebra of dual numbers. Let us first observe that $\mathfrak{A} = k\mathrm{Id}$ and so, by (9.16.5), (x, y) is quasi-invertible if and only if $x_0 y_0 = 0$, in which case $B(x, y) = \mathrm{Id}$ and $x^y = x - Q_x y$. In particular $H = \{1\}$ by (9.16.6). Thus $1^a = 1 + a, (1 + a)^a = 1$ and $a^y = a$ for all $y \in J$. Hence, by (9.2.6), we have the following relations in G_1 for $\sigma = \pm$:

$$\begin{aligned} \exp_\sigma(a) \exp_{-\sigma}(1) &= \exp_{-\sigma}(1 + a) \exp_\sigma(a), \\ \exp_\sigma(a) \exp_{-\sigma}(a) &= \exp_{-\sigma}(a) \exp_\sigma(a), \\ \exp_\sigma(a) \exp_{-\sigma}(1 + a) &= \exp_{-\sigma}(1) \exp_\sigma(a). \end{aligned} \tag{1}$$

It follows from (1) that G_1 is generated by

$$g_1 = \exp_+(1), \quad g_2 = \exp_-(1), \quad g_3 = \exp_+(1 + a).$$

As in case $n = 0$ we have $g_i^2 = 1 = (g_1 g_2)^3$. Since U^+ is abelian also $(g_1 g_3)^2 = 1$ holds. We will show

$$(g_2 g_3)^3 = 1, \tag{2}$$

equivalently, $g_2 g_3 g_2 = g_3 g_2 g_3$. Indeed, by (1) we have

$$\begin{aligned} g_2 g_3 &= \exp_-(1) \exp_+(1 + a) = \exp_-(1) \exp_+(a) \exp_+(1) \\ &= \exp_+(a) \exp_-(a) \exp_-(1) \exp_+(1) \end{aligned}$$

and hence $g_2 g_3 g_2 = \exp_+(a) \exp_-(a) \omega_e$. On the other hand,

$$g_3 g_2 g_3 = \exp_+(a) \omega_e \exp_+(a) = \exp_+(a) \exp_-(a) \omega_e$$

by the relation (9.6.1). This finishes the proof of (2). We have now shown that the generators g_1, g_2, g_3 of G_1 satisfy the defining relations for Sym_4. Hence there exists a surjective homomorphism $\mathrm{Sym}_4 \to G_1$, and because $|G_1| \geqslant |J_1|^2 = 16$ this is an isomorphism.

9.19. Proposition. *G_n is a solvable group of order*

$$|G_n| = 3 \cdot 2^{2n+1+[\frac{n}{2}]}.$$

Proof. We denote by $X_n = X(V_n)$ the projective space of the Jordan pair $V_n = (J_n, J_n)$ [**37**]. Since J_n is local, Proposition 9.20 below implies

$$|X_n| = |J_n| + |ka \oplus \cdots \oplus ka^n| = 2^{n+1} + 2^n = 3 \cdot 2^n.$$

Because V_n is stable, $X_n = G_n/P^-$ where $P^- = U^- H$ has order $|U^-| \cdot |H| = 2^{n+1+m}$ with $m = [\frac{n}{2}]$, and hence $|G_n| = 3 \cdot 2^{2n+1+m}$. By Burnside's $p^a q^b$-Theorem G_n is solvable. This can also be seen directly by induction on n. The cases $n = 0$ and $n = 1$ hold by 9.18. For $n \geqslant 1$, the Jordan algebra J_n contains the proper ideal ka^n and $V_n/(ka^n, ka^n) \cong V_{n-1}$. Hence by [**38**, 1.6], $G_{n-1} \cong G_n/K$ for some normal subgroup K of G_n. Since K is a 2-group (namely of order 4 if n is odd and of order 8 if n is even) it is solvable [**2**, 11, Cor. 5]. Thus, solvability of G_{n-1} implies solvability of G_n.

9.20. Proposition. *Let J be a local Jordan algebra and let $X = X(J, J)$ be the projective space determined by J. We identify J with $(J\colon 0) \subset X$. Then the map $a \mapsto (1\colon 1 - a)$ is a bijection from the set of non-units of J onto $X \setminus J$.*

Proof. Since a is not invertible, $(1, 1 - a)$ is not quasi-invertible [**34**, 3.13]. Let

$$w = \omega_e = \exp_-(1) \exp_+(1) \exp_-(1)$$

be the Weyl element of the idempotent $(1, 1)$. As a local Jordan algebra, J is stable and hence w leaves X invariant and $wU^+ = U^- w$ [**38**, 6.6, 6.4, 6.3]. Hence injectivity follows from $(1\colon 1 - a) = \exp_-(-a)(1\colon 1) = \exp_-(-a)w(0\colon 0) = w \exp_+(-a)(0\colon 0) = w(-a\colon 0)$. To show surjectivity, let $(x\colon y) \in X \setminus J$. Then (x, y) is not quasi-invertible. We claim that x is invertible. Indeed, otherwise x would belong to the radical since J is local and so (x, y) would be quasi-invertible. Moreover, because (x, y) is not quasi-invertible, $a = x^{-1} - y$ is not invertible. One now easily sees that $(x\colon y) = (1\colon 1 - x^{-1} + y) = (1\colon 1 - a)$.

§10. Groups over Jordan pairs

10.1. Groups over a Jordan pair. Let $R = \{0, 1, -1\}$ be the root system of type A_1. As in Example 2.3(a), we denote the root groups of a group with A_1-commutator relations by $U^{\pm} = U_{\pm 1}$.

Let us fix a Jordan pair $V = (V^+, V^-)$. We will modify the notations of 9.2 for the projective elementary group and its subgroups as follows:

$$\bar{G} = \mathrm{PE}(V), \quad \bar{U}^{\pm} = \exp_{\pm}(V^{\pm}), \quad \bar{G}_0 = \mathrm{PE}_0(V).$$

We specialize 3.9 to the present situation, cf. also Example (a) of 3.18. The Steinberg category $\mathbf{st}(\bar{G}, \bar{U}^+, \bar{U}^-)$ will simply be denoted by $\mathbf{st}(V)$ and its objects will be called *groups over* V. Thus an object of $\mathbf{st}(V)$ can be considered as a quadruple $(G, U^{\pm}, \pi)$ where G is a group and $\pi\colon G \to \bar{G}$ is a homomorphism such that $\pi\big|U^{\sigma}\colon U^{\sigma} \to \bar{U}^{\sigma}$ is an isomorphism, for $\sigma \in \{+, -\}$; in particular, U^{σ} is abelian. Specializing the definition of tightness in 2.2, G is tight if and only if U^+ and U^- generate G. For any group $(G, U^{\pm}, \pi)$ over V, the subgroup $\mathfrak{r}(G)$ generated by U^+ and U^- is a tight group over V.

We define isomorphisms $\mathrm{x}_{\pm}\colon V^{\pm} \xrightarrow{\cong} U^{\pm}$ by the commutative diagrams

$$\begin{tikzcd}
V^{\pm} \arrow[rr, "\mathrm{x}_{\pm}", "\cong"'] \arrow[dr, "\cong", "\exp_{\pm}"'] & & U^{\pm} \arrow[dl, "\cong"', "\pi"] \\
& \bar{U}^{\pm} &
\end{tikzcd} \tag{1}$$

Then an object of $\mathbf{st}(V)$ can also be identified with a quadruple $(G, \mathrm{x}_{\pm}, \pi)$ consisting of a group G and homomorphisms $\mathrm{x}_{\pm}\colon V^{\pm} \to G$ and $\pi\colon G \to \bar{G}$ satisfying (1). It is tight if and only if $\mathrm{x}_+(V^+) \cup \mathrm{x}_-(V^-)$ generates G. A *morphism* $\varphi\colon (G, \mathrm{x}_+, \mathrm{x}_-, \pi) \to (G', \mathrm{x}'_+, \mathrm{x}'_-, \pi')$ of groups over V is then the same as a group homomorphism $\varphi\colon G \to G'$ making the diagrams

$$\begin{tikzcd}
& V^{\pm} \arrow[dl, "\mathrm{x}_{\pm}"'] \arrow[dr, "\mathrm{x}'_{\pm}"] & \\
G \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & G' \arrow[dl, "\pi'"] \\
& \bar{G} &
\end{tikzcd} \tag{2}$$

commutative. To simplify notation, we will often denote an object of $\mathbf{st}(V)$ simply by G, and also use the same letters $\mathrm{x}_{\pm}$ and π for different groups G in $\mathbf{st}(V)$.

Example. We note that in general groups over V are not tight. For example, let V be the rectangular matrix pair $\mathbb{M}_{pq}(k)$ for k an algebraically closed field and put $n = p+q$. We have seen in the example in 9.2 that the elementary group of this special Jordan pair is $\mathrm{SL}_n(k)$ and that $\mathrm{PE}(\mathbb{M}_{pq}(k)) \cong \mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/k^\times \cdot 1_n$. It is immediate that $\mathrm{GL}_n(k)$ is a non-tight group over V with respect to the canonical map $\pi\colon \mathrm{GL}_n(k) \to \mathrm{PGL}_n(k)$ and the natural subgroups $U^\pm \subset \mathrm{SL}_n(k)$.

10.2. Lemma. *Let $G \in \mathbf{st}(V)$ with root groups $U^\sigma = \mathrm{x}_\sigma(V^\sigma)$. Define subgroups G_0 and N of G by*

$$G_0 := \pi^{-1}(\bar{G}_0), \qquad N := \mathrm{Norm}_G(U^+) \cap \mathrm{Norm}_G(U^-).$$

Let $\bar{\Omega} = \bar{U}^- \cdot \bar{G}_0 \cdot \bar{U}^+$ be the big cell of $\bar{G} = \mathrm{PE}(V)$ as in 9.2 and define

$$\Omega := \pi^{-1}(\bar{\Omega}).$$

(a) *The map $\Phi\colon V^- \times G_0 \times V^+ \to G$, $(y,h,x) \mapsto \mathrm{x}_-(y) \cdot h \cdot \mathrm{x}_+(x)$, is injective (equivalently, the map $\mu\colon U^- \times G_0 \times U^+ \to G$ given by multiplication is injective) with image Ω.*

(b) *Let $h \in G_0$ and let $\pi(h) = \bar{h} = (h_+, h_-) \in \mathrm{PE}_0(V)$. Then*

$$h \in N \iff h\,\mathrm{x}_\sigma(v)h^{-1} = \mathrm{x}_\sigma(h_\sigma(v)) \text{ for all } v \in V^\sigma,\ \sigma = \pm. \tag{1}$$

(c) *$\mathrm{Ker}(\pi) \subset G_0$ and $N \cap \mathrm{Ker}(\pi)$ centralizes $U^\pm$. If G is tight then $N \cap \mathrm{Ker}(\pi)$ is central in G.*

(d) *Let $\varphi\colon G \to G'$ be a morphism of groups over V, with subgroups $U^\sigma, G_0, N \subset G$ and $U'^\sigma, G'_0, N' \subset G'$ as above. Then*

- (i) *$\varphi\colon U^\sigma \to U'^\sigma$ is an isomorphism, and $\varphi^{-1}(U'^\sigma) = U^\sigma \cdot \mathrm{Ker}(\varphi) = \mathrm{Ker}(\varphi) \cdot U^\sigma$.*
- (ii) *$\varphi^{-1}(G'_0) = G_0$ and $\varphi(G_0) \subset G'_0$. If φ is surjective (for example, if G' is tight) then $\varphi(G_0) = G'_0$.*
- (iii) *$\varphi(N) \subset N'$.*

Proof. (a) If $\mathrm{x}_-(y)h\mathrm{x}_+(x) = \mathrm{x}_-(y')h'\mathrm{x}_+(x')$ then by applying π and (9.2.4) we obtain $y = y'$ and $x = x'$ whence also $h = h'$. Clearly Φ has range contained in $\pi^{-1}(\bar{\Omega})$. Conversely, if $g \in \pi^{-1}(\bar{\Omega})$ and, say, $\pi(g) = \exp_-(y)\bar{h}\exp_+(x)$ then $\mathrm{x}_-(-y)g\mathrm{x}_+(-x) \in \pi^{-1}(\mathrm{PE}_0(V)) = G_0$ so that g is in the range of Φ.

(b) By (9.2.3) we always have $\pi(h\,\mathrm{x}_\sigma(v)h^{-1}) = \bar{h}\exp_\sigma(v)\bar{h}^{-1} = \exp_\sigma(h_\sigma(v)) = \pi(\mathrm{x}_\sigma(h_\sigma(v)))$. The asserted equivalence then follows from injectivity of π on U^σ.

(c) Clearly $\mathrm{Ker}(\pi) = \pi^{-1}(\{1\}) \subset \pi^{-1}(\bar{G}_0) = G_0$. Let $h \in N \cap \mathrm{Ker}(\pi) \subset N \cap G_0$. Then $\pi(h) = \bar{h} = 1$ and therefore (1) shows that h centralizes the subgroups $U^\pm$.

(d) For (i), we have $\varphi(U^\sigma) = U'^\sigma$ by (10.1.2). Since $\pi\big|U^\sigma = (\pi' \circ \varphi)\big|U^\sigma$ is an isomorphism, it follows that $\varphi\big|U^\sigma$ is injective. Next, let $g \in \varphi^{-1}(U'^\sigma)$, say,

$\varphi(g) = \mathrm{x}'_\sigma(v)$. Then $\varphi(g\mathrm{x}_\sigma(-v)) = \mathrm{x}'_\sigma(v)\mathrm{x}'_\sigma(-v) = 1_{G'}$, so $g \in \mathrm{Ker}(\varphi) \cdot U^\sigma$. Thus $\varphi^{-1}(U'^\sigma) \subset \mathrm{Ker}(\varphi) \cdot U^\sigma$, and the reverse inclusion is clear. The second formula is proved similarly.

(ii) We have

$$\varphi^{-1}(G'_0) = \varphi^{-1}(\pi'^{-1}(\mathrm{PE}_0(V))) = (\pi' \circ \varphi)^{-1}(\mathrm{PE}_0(V)) = \pi^{-1}(\mathrm{PE}_0(V)) = G_0.$$

For the second statement, let $h \in G_0$. Then $\pi'(\varphi(h)) = \pi(h) \in G_0$ whence $\varphi(h) \in G'_0$. Now assume φ surjective, and let $h' \in G'_0$, say, $h' = \varphi(g)$, $g \in G$. Then $\pi(g) = \pi'(\varphi(g)) = \pi'(h') \in \bar{G}_0$, whence $g \in G_0$.

(iii) is immediate from the definitions.

10.3. Example: Elementary groups of special Jordan pairs. Let $\mathfrak{M} = (A, B, C, D)$ be a Morita context, let $V \subset (B, C)$ be a special Jordan pair and let $G = \mathrm{E}(\mathfrak{M}, V)$ be the elementary group of $(\mathfrak{M}, V)$ as in 7.2. By [**38**, Th. 2.8], there is a well-defined surjective homomorphism π: $G \to \bar{G} = \mathrm{PE}(V)$ satisfying

$$\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \exp_+(x), \qquad \pi \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} = \exp_-(y), \tag{1}$$

for all $x \in V^+$, $y \in V^-$. Thus G is a group over V provided we define the homomorphisms x_σ: $V^\sigma \to G$ by

$$\mathrm{x}_+(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{x}_-(y) = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}. \tag{2}$$

By [**38**, Th. 2.8], $\pi^{-1}(\mathrm{PE}_0(V)) = G_0$ and $\pi^{-1}(\bar{\Omega}) = \Omega \subset G$ are given by

$$G_0 = \{g \in G : ge_1 = e_1 g\} = G \cap \begin{pmatrix} A^\times & 0 \\ 0 & D^\times \end{pmatrix}, \quad \Omega = U^- G_0 U^+, \tag{3}$$

and the kernel of π is central in G, more precisely,

$$\mathrm{Ker}(\pi) = \mathscr{Z}(G) \cap G_0.$$

10.4. Lifting Jordan pair homomorphisms. Let V and V' be Jordan pairs, and let G and G' be groups over V and V', respectively. Consider a homomorphism f: $V \to V'$ of Jordan pairs. We say a group homomorphism φ_f: $G \to G'$ *is a lift of* f if the diagrams

$$\begin{tikzcd} G \arrow[r, "\varphi_f"] & G' \\ V^\pm \arrow[u, "\mathrm{x}_\pm"] \arrow[r, "f_\pm"'] & V'^\pm \arrow[u, "\mathrm{x}'_\pm"'] \end{tikzcd} \tag{1}$$

commute. This condition determines φ_f uniquely on the subgroup $\langle U^+ \cup U^- \rangle$. If $f'\colon V' \to V''$ is a second Jordan pair homomorphism with lift $\varphi_{f'}\colon G' \to G''$ then $\varphi_{f'} \circ \varphi_f$ is a lift of $f' \circ f$. Note that the lifts of the identity Id: $V \to V$ are just the morphisms of the category $\mathbf{st}(V)$.

Suppose $G \in \mathbf{st}(V)$. An automorphism a of V is said to *normalize* G if there exist lifts $\varphi_a\colon G \to G$ of a and $\varphi_{a^{-1}}\colon G \to G$ of a^{-1}, necessarily unique if G is tight. Let $A = \mathrm{Aut}(V)$ be the automorphism group of V and denote by $\mathrm{Norm}_A(G)$ the set of $a \in A$ normalizing G. One shows easily that $\mathrm{Norm}_A(G)$ is a subgroup of A, and, if G is tight then the map $\varphi\colon \mathrm{Norm}_A(G) \to \mathrm{Aut}(G)$, $a \mapsto \varphi_a$, is an injective group homomorphism.

With the notations of 10.2 let $h \in N \cap G_0$. By (10.2.1),

$$a = \pi(h) \in \mathrm{Norm}_A(G) \cap \bar{G}_0, \qquad \varphi_a = \mathrm{Int}\, h, \tag{2}$$

the inner automorphism of G determined by h, is a lift of a. Thus the diagram

$$\begin{tikzcd} G_0 \cap N \arrow[rr, "\mathrm{Int}"] \arrow[dr, "\pi"'] & & \mathrm{Aut}(G) \\ & \bar{G}_0 \cap \mathrm{Norm}_A(G) \arrow[ur, "\varphi"'] & \end{tikzcd} \tag{3}$$

is commutative.

10.5. Groups induced by subpairs. Let G be a group over V and let $V' \subset V$ be a subpair of V. Define $U'^{\pm} = \mathrm{x}_\pm(V'^{\pm}) \subset U^\pm$ and let $G' \subset G$ be the subgroup generated by $U'^+ \cup U'^-$. Then $\mathrm{x}'_\pm := \mathrm{x}_\pm \big| V'^\pm\colon V'^\pm \to U'^\pm$ is an isomorphism. We claim that, with the following definition of the projection $\pi'\colon G' \to \mathrm{PE}(V')$, the quadruple $(G', \mathrm{x}'_\pm, \pi')$ is a group over V' (in fact, G' is tight), and that the inclusion $G' \subset G$ is a lift (in the sense of 10.4) of the inclusion $V' \subset V$.

Let $\mathrm{PE}(V')$ be the projective elementary group of V' with exponential maps $\exp'_\pm$, and let $\bar{G}' \subset \bar{G} = \mathrm{PE}(V)$ be the subgroup generated by $\exp_+(V'^+) \cup \exp_-(V'^-)$ as in 9.13. From (10.1.1), it follows that $\pi(G') = \bar{G}'$. By (9.13.1) we have a surjective homomorphism $\psi\colon \bar{G}' \to \mathrm{PE}(V')$ satisfying (9.13.2). Now define $\pi'\colon G' \to \mathrm{PE}(V')$ by $\pi' = \psi \circ (\pi \big| G')$.

To show that $(G', \mathrm{x}'_\pm, \pi') \in \mathbf{st}(V')$, we need to verify (10.1.1). By the definition of π', we must show that the diagram

$$\begin{tikzcd} V'^{\pm} \arrow[r, "\mathrm{x}'_\pm"] \arrow[d, "\exp'_\pm"'] & U'^{\pm} \arrow[d, "\pi"] \arrow[dl, "\pi'"'] \\ \exp'_\pm(V'^{\pm}) & \exp_\pm(V'^{\pm}) \arrow[l, "\psi"] \end{tikzcd}$$

is commutative and its maps are isomorphisms. But this follows readily from 9.13. Finally, the fact that the inclusion $G' \subset G$ is a lift of the inclusion $V' \subset V$ follows immediately from the definition of $\mathrm{x}'_\pm$.

10.6. The elements $\mathrm{b}(x,y)$**.** Let G be a group over V. For a quasi-invertible pair $(x,y)\in V$ we define the element $\mathrm{b}(x,y)\in G$ by the formula

$$\mathrm{x}_+(x)\cdot\mathrm{x}_-(y)=\mathrm{x}_-(y^x)\cdot\mathrm{b}(x,y)\cdot\mathrm{x}_+(x^y), \tag{1}$$

equivalently,

$$\big(\mathrm{x}_-(-y),\,\mathrm{x}_+(x)\big)=\mathrm{x}_-(y^x-y)\cdot\mathrm{b}(x,y)\cdot\mathrm{x}_+(x^y-x). \tag{2}$$

Then (9.2.6) shows that

$$\pi(\mathrm{b}(x,y))=\beta(x,y);\quad\text{hence}\quad \mathrm{b}(x,y)\in G_0=\pi^{-1}(\bar G_0). \tag{3}$$

One also sees immediately from the definition that

$$\mathrm{b}(x,0)=\mathrm{b}(0,y)=1.$$

If there is no risk of confusion, we will use the same letter b for different groups over V. If $f\colon V\to V'$ is a homomorphism of Jordan pairs and $\varphi_f\colon G\to G'$ is a lift of f as in 10.4 then

$$\varphi_f(\mathrm{b}(x,y))=\mathrm{b}(f_+(x),f_-(y)). \tag{4}$$

Indeed, if (x,y) is quasi-invertible then $(f_+(x),f_-(y))$ is quasi-invertible in V' and $f_+(x^y)=f_+(x)^{f_-(y)}$ as well as $f_-(y^x)=f_-(y)^{f_+(x)}$, by (7.9.4). Hence

$$\begin{aligned}\varphi_f\big(\mathrm{x}_+(x)\cdot\mathrm{x}_-(y)\big)&=\mathrm{x}_+(f_+(x))\cdot\mathrm{x}_-(f_-(y))\\&=\mathrm{x}_-\big((f_-(y))^{f_+(x)}\big)\cdot\mathrm{b}(f_+(x),f_-(y))\cdot\mathrm{x}_+\big((f_+(x))^{f_-(y)}\big)\\&=\mathrm{x}_-(f_-(y^x))\cdot\mathrm{b}(f_+(x),f_-(y))\cdot\mathrm{x}_+(f_+(x^y))\end{aligned}$$

and also

$$\begin{aligned}\varphi_f\big(\mathrm{x}_+(x)\cdot\mathrm{x}_-(y)\big)&=\varphi_f\big(\mathrm{x}_-(y^x)\cdot\mathrm{b}(x,y)\cdot\mathrm{x}_+(x^y)\big)\\&=\mathrm{x}_-(f_-(y^x))\cdot\varphi_f\big(\mathrm{b}(x,y)\big)\cdot\mathrm{x}_+(f_+(x^y)),\end{aligned}$$

so that (4) follows by comparison. In particular, this applies to $f=a=\pi(h)$ where $h\in N\cap G_0$, and then (10.4.2) yields the formula

$$h\cdot\mathrm{b}(x,y)\cdot h^{-1}=\mathrm{b}(a_+(x),a_-(y)). \tag{5}$$

Example. Let $G=\mathrm{E}(\mathfrak{M},V)$ as in 10.3 and let $(x,y)\in V$. Then

$$(x,y)\text{ is quasi-invertible}\iff 1_A-xy\in A^\times\iff 1_D-yx\in D^\times, \tag{6}$$

and in this case

$$x^y = (1-xy)^{-1}x = x(1-yx)^{-1}, \quad y^x = (1-yx)^{-1}y = y(1-xy)^{-1}, \quad (7)$$

$$\mathrm{b}(x,y) = \mathrm{x}_-(-y^x)\mathrm{x}_+(x)\mathrm{x}_-(y)\mathrm{x}_+(-x^y) = \begin{pmatrix} 1-xy & 0 \\ 0 & (1-yx)^{-1} \end{pmatrix}. \quad (8)$$

Indeed, (6) and (7) holds for (B, C) by 7.9. Since the inclusion $V \subset (B, C)$ is a Jordan pair homomorphism and quasi-invertibility and the quasi-inverse behave well with respect to Jordan pair homomorphisms by (7.9.4), the same is true for V. Matrix computation shows $\mathrm{x}_+(x)\mathrm{x}_-(y) = \begin{pmatrix} 1-xy & x \\ -y & 1 \end{pmatrix}$. Now let (x, y) be quasi-invertible. Then

$$\begin{aligned} \mathrm{b}(x,y) &= \mathrm{x}_-(-y^x)\mathrm{x}_+(x)\mathrm{x}_-(y)\mathrm{x}_+(-x^y) = \begin{pmatrix} 1 & 0 \\ y^x & 1 \end{pmatrix} \begin{pmatrix} 1-xy & x \\ -y & 1 \end{pmatrix} \begin{pmatrix} 1 & -x^y \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-xy & x \\ 0 & 1+y^x x \end{pmatrix} \begin{pmatrix} 1 & -x^y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-xy & 0 \\ 0 & 1+y^x x \end{pmatrix} \\ &= \begin{pmatrix} 1-xy & 0 \\ 0 & (1-yx)^{-1} \end{pmatrix}, \end{aligned}$$

where we used (7) and in the last step

$$\begin{aligned} 1 + y^x x &= 1 + y(1-xy)^{-1}x = 1 + yx^y = 1 + yx(1-yx)^{-1} \\ &= (1 - yx + yx)(1-yx)^{-1} = (1-yx)^{-1}. \end{aligned}$$

10.7. The relations $\mathfrak{B}(x, y)$. Let G be a group over V and let $(x, y) \in V$ be a quasi-invertible pair. We say G *satisfies the relations* $\mathfrak{B}(x, y)$ if the formulas

$$\mathrm{b}(x,y) \cdot \mathrm{x}_+(z) \cdot \mathrm{b}(x,y)^{-1} = \mathrm{x}_+(B(x,y)z), \quad (1)$$

$$\mathrm{b}(x,y)^{-1} \cdot \mathrm{x}_-(v) \cdot \mathrm{b}(x,y) = \mathrm{x}_-(B(y,x)v) \quad (2)$$

hold for all $(z, v) \in V^+ \times V^-$. Since $B(x,y)z = z - \{xyz\} + Q_x Q_y z$, an equivalent formulation is

$$\big(\mathrm{b}(x,y),\, \mathrm{x}_+(z)\big) = \mathrm{x}_+\big(-\{xyz\} + Q_x Q_y z\big), \quad (3)$$

$$\big(\mathrm{b}(x,y)^{-1},\, \mathrm{x}_-(v)\big) = \mathrm{x}_-\big(-\{yxv\} + Q_y Q_x v\big), \quad (4)$$

for all $(z, v) \in V$. From (10.6.3) and (10.2.1) we see that

$$G \text{ satisfies } \mathfrak{B}(x,y) \quad \Longleftrightarrow \quad \mathrm{b}(x,y) \in N. \quad (5)$$

In particular, it follows from (9.2.6) that $\bar{G}$ satisfies $\mathfrak{B}(x, y)$ for all quasi-invertible pairs (x, y). The same is true for elementary groups of special Jordan pairs:

$$\mathrm{E}(\mathfrak{M}, V) \text{ satisfies the relations } \mathfrak{B}(x,y) \text{ for all quasi-invertible } (x,y) \in V. \quad (6)$$

Indeed, this follows by a simple matrix computation from (10.6.8).

Suppose G' is a group over a Jordan pair V' as well and $f\colon V \to V'$ is a surjective homomorphism of Jordan pairs which lifts to a group homomorphism $\varphi_f\colon G \to G'$ as in 10.4. Then by (10.6.4) and (10.4.1),

$$\textit{if } G \textit{ satisfies } \mathfrak{B}(x,y) \textit{ then } G' \textit{ satisfies } \mathfrak{B}(f_+(x), f_-(y)). \quad (7)$$

In particular, if G satisfies $\mathfrak{B}(x, y)$ and $a = (a_+, a_-) \in \mathrm{Norm}_A(G)$ then G also satisfies $\mathfrak{B}(a_+(x), a_-(y))$.

10.8. Lemma. *Let (x, y) and (u, v) be quasi-invertible and let $(s, t) \in V$ with the property that $(s+x, y)$ and $(x, y+t)$ are quasi-invertible. If G satisfies $\mathfrak{B}(x, y)$ then the following formulas hold:*

$$\mathrm{b}(s+x, y) = \mathrm{b}(s, y^x) \cdot \mathrm{b}(x, y), \tag{1}$$
$$\mathrm{b}(x, y+t) = \mathrm{b}(x, y) \cdot \mathrm{b}(x^y, t), \tag{2}$$
$$\mathrm{b}(x, y)^{-1} = \mathrm{b}(-x,\, y^x) = \mathrm{b}(x^y,\, -y), \tag{3}$$
$$\mathrm{b}(x, y) = \mathrm{b}\big(Q_x y - x,\, -y^x\big) = \mathrm{b}\big(-x^y,\, Q_y x - y\big), \tag{4}$$
$$\mathrm{b}(x, y) \cdot \mathrm{b}(u, v) \cdot \mathrm{b}(x, y)^{-1} = \mathrm{b}\big(B(x, y)u,\, B(y, x)^{-1}v\big). \tag{5}$$

Proof. By (7.9.5) applied to V^{op}, the formulas

$$(s+x)^y = x^y + B(x, y)^{-1}\big(s^{(y^x)}\big), \qquad y^{(x+s)} = (y^x)^s \tag{6}$$

hold. From (10.6.1) we therefore obtain

$$\begin{aligned} \mathrm{x}_+(s+x) \cdot \mathrm{x}_-(y) &= \mathrm{x}_-(y^{s+x}) \cdot \mathrm{b}(s+x, y) \cdot \mathrm{x}_+((s+x)^y) \\ &= \mathrm{x}_-(y^{s+x}) \cdot \mathrm{b}(s+x, y) \cdot \mathrm{x}_+\big(x^y + B(x, y)^{-1} s^{(y^x)}\big). \end{aligned}$$

On the other hand, (10.7.1) shows

$$\begin{aligned} \mathrm{x}_+(s+x) \cdot \mathrm{x}_-(y) &= \mathrm{x}_+(s) \cdot \mathrm{x}_-(y^x) \cdot \mathrm{b}(x, y) \cdot \mathrm{x}_+(x^y) \\ &= \mathrm{x}_-((y^x)^s) \cdot \mathrm{b}(s, y^x) \cdot \mathrm{x}_+(s^{(y^x)}) \cdot \mathrm{b}(x, y) \cdot \mathrm{x}_+(x^y) \\ &= \mathrm{x}_-(y^{x+s}) \cdot \mathrm{b}(s, y^x) \cdot \mathrm{b}(x, y) \cdot \mathrm{x}_+\big(B((x, y)^{-1} s^{(y^x)} + x^y\big) \end{aligned}$$

so that (1) follows by comparison. Formula (2) follows similarly from (10.7.2). We obtain (3) by setting $s = -x$ and $t = -y$ in (1) and (2).

To prove (4) first note that (3) shows G satisfies the relations $\mathfrak{B}(-x, y^x)$ and $\mathfrak{B}(x^y, -y)$ since $\mathrm{b}(x, y)^{-1} \in N$. Hence by (3),

$$\begin{aligned} \mathrm{b}(x, y) &= \big(\mathrm{b}(x, y)^{-1}\big)^{-1} = \mathrm{b}(-x, y^x)^{-1} \\ &= \mathrm{b}\big((-x)^{(y^x)}, -y^x\big) = \mathrm{b}(Q_x y - x, -y^x), \end{aligned}$$

because (6) yields, for $s = -x$, the Hua-type relation

$$(-x)^{(y^x)} = -B(x, y)x^y = -(x - Q_x y). \tag{7}$$

The second formula of (4) is proved similarly. Finally, replace (x, y) in (10.6.5) by (u, v) and put $h = \mathrm{b}(x, y)$ and $a = \beta(x, y)$. Then (5) follows from (10.6.5) and (10.7.5).

10.9. Groups over V^{op}. Let $V^{\mathrm{op}} = (V^-, V^+)$ be the opposite of a Jordan pair V and let G be a group over V. Then G can and will be considered as a group over V^{op} by switching the roles of U^+ and U^-, i.e., by setting

$$\mathrm{x}_\sigma^{\mathrm{op}}(v) = \mathrm{x}_{-\sigma}(v) \quad (v \in V^{-\sigma},\ \sigma \in \{+,-\}); \tag{1}$$

more precisely,

$$(G, \mathrm{x}_+, \mathrm{x}_-, \pi)^{\mathrm{op}} = (G, \mathrm{x}_+^{\mathrm{op}}, \mathrm{x}_-^{\mathrm{op}}, \pi^{\mathrm{op}}) = (G, \mathrm{x}_-, \mathrm{x}_+, \pi)$$

in the notation of 10.1. In particular, this applies to the projective elementary group of V and provides an identification of $\mathrm{PE}(V)$ and $\mathrm{PE}(V^{\mathrm{op}})$, cf. [**38**, 1.3]. The assignment $(G, \mathrm{x}_+, \mathrm{x}_-, \pi) \mapsto (G, \mathrm{x}_-, \mathrm{x}_+, \pi)$ from $\mathbf{st}(V)$ to $\mathbf{st}(V^{\mathrm{op}})$ is then an isomorphism of categories.

Let (x, y) be quasi-invertible in V. By the symmetry principle (7.9.2), this is equivalent to (y, x) being quasi-invertible in V^{op}. Invert (10.6.1), replace (x, y) by $(-x, -y)$ and use the fact that $(-\mathrm{Id}_{V^+}, -\mathrm{Id}_{V^-})$ is an automorphism of V. The result is

$$\mathrm{x}_-(y) \cdot \mathrm{x}_+(x) = \mathrm{x}_+(x^y) \cdot \mathrm{b}(-x, -y)^{-1} \cdot \mathrm{x}_-(y^x) \tag{2}$$

which, when read in $(G, \mathrm{x}_+, \mathrm{x}_-, \pi)^{\mathrm{op}}$, says

$$\mathrm{b}^{\mathrm{op}}(y, x) = \mathrm{b}(-x, -y)^{-1}. \tag{3}$$

With the aim of achieving greater symmetry in formulas, we will often use the notation

$$\mathrm{b}_+(x, y) = \mathrm{b}(x, y), \qquad \mathrm{b}_-(y, x) = \mathrm{b}^{\mathrm{op}}(y, x). \tag{4}$$

Then (3) implies

$$\mathrm{b}_\sigma(x, y)^{-1} = \mathrm{b}_{-\sigma}(-y, -x), \tag{5}$$

and (10.6.1) and (2) can be subsumed into the single formula

$$\mathrm{x}_\sigma(x) \cdot \mathrm{x}_{-\sigma}(y) = \mathrm{x}_{-\sigma}(y^x) \cdot \mathrm{b}_\sigma(x, y) \cdot \mathrm{x}_\sigma(x^y). \tag{6}$$

or the equivalent commutator formula

$$\big(\mathrm{x}_{-\sigma}(-y),\, \mathrm{x}_\sigma(x)\big) = \mathrm{x}_{-\sigma}(y^x - y) \cdot \mathrm{b}_\sigma(x, y) \cdot \mathrm{x}_\sigma(x^y - x). \tag{7}$$

10.10. The elements t_e and w_e. Let $e = (e_+, e_-) \in V$ be an idempotent and consider a group G over V. In analogy to 9.4, we introduce the notations

$$\mathrm{t}_e = \big(\mathrm{x}_+(e_+),\, \mathrm{x}_-(e_-),\, \mathrm{x}_+(e_+)\big) \in U^+ \times U^- \times U^+, \tag{1}$$

$$\mathrm{w}_e = \mathrm{x}_+(e_+) \cdot \mathrm{x}_-(e_-) \cdot \mathrm{x}_+(e_+) \in G. \tag{2}$$

Again as in 9.4, we put

$$\mathrm{t}_{e^{\mathrm{op}}} = \big(\mathrm{x}_-(e_-),\, \mathrm{x}_+(e_+),\, \mathrm{x}_-(e_-)\big), \tag{3}$$
$$\mathrm{w}_{e^{\mathrm{op}}} = \mathrm{x}_-(e_-)\cdot \mathrm{x}_+(e_+)\cdot \mathrm{x}_-(e_-). \tag{4}$$

Since $-e = (-e_+, -e_-)$ is an idempotent as well, we have the following formula for the inverse:

$$\mathrm{w}_e^{-1} = \mathrm{w}_{-e}. \tag{5}$$

Clearly, $\pi(\mathrm{t}_e) = \theta_e$ and $\pi(\mathrm{w}_e) = \omega_e$ are the elements defined in 9.4. However, even when V has invertible elements and hence θ_e is a Weyl triple for $\bar{G}$ considered as a group with A_1-commutator relations (see 9.7), this is in general no longer the case for G. We now discuss this question in more detail.

10.11. Proposition. *Let V be a Jordan pair and let G be a group over V, considered as a group with* A_1*-commutator relations and root subgroups $U_{\pm 1} = U^{\pm}$.*

(a) *Suppose $t \in U^+ \times U^- \times U^+$ is a Weyl triple for the root $\alpha = 1$. Then t has the form*

$$t = \big(\mathrm{x}_+(z+u),\, \mathrm{x}_-(u^{-1}),\, \mathrm{x}_+(u+z')\big) \tag{1}$$

where $u \in V^+$ is invertible and $z, z' \in \mathrm{Extr}(V^+)$.

(b) *Conversely, if $G = \mathrm{PE}(V)$ then any triple*

$$\big(\exp_+(z+u),\, \exp_-(u^{-1}),\, \exp_+(u+z')\big),$$

where $u \in V^+$ is invertible and $z, z' \in \mathrm{Extr}(V^+)$, is a Weyl triple for $\alpha = 1$.

Proof. (a) Let $w = \mu(t)$ be the Weyl element determined by t. Since π is surjective on root groups, $\omega := \pi(w)$ is a Weyl element for $\alpha = 1$ in $\mathrm{PE}(V)$. Hence

$$\omega \cdot \exp_+(x) \cdot \omega^{-1} = \exp_-\big(f(x)\big) \tag{2}$$

for all $x \in V^+$, where $f\colon V^+ \to V^-$ is an isomorphism of additive groups. Write $t = \big(\mathrm{x}_+(a),\, \mathrm{x}_-(v),\, \mathrm{x}_+(a')\big)$ where $a, a' \in V^+$ and $v \in V^-$. Then

$$\omega = \exp_+(a)\, \exp_-(v)\, \exp_+(a'), \tag{3}$$

and equation (2) is equivalent to

$$\exp_-(v)\cdot \exp_+(x)\cdot \exp_-(-v)\cdot \exp_+(-a) = \exp_+(-a)\cdot \exp_-(f(x)). \tag{4}$$

By applying both sides of (4) to the element $\zeta \in \mathfrak{L}_0(V)$ of the Tits-Kantor-Koecher algebra and comparing the terms in V^- we obtain, by a lengthy but straightforward computation using (JP4), the formula

$$f(x) = Q_v\big(x - Q_x(v - Q_v a)\big). \tag{5}$$

Since f is surjective, this shows that $Q_v\colon V^+ \to V^-$ is surjective. In particular, there exists $u \in V^+$ such that $v = Q_v u$. But Q_v is injective as well: indeed, $Q_v x = 0$ and (5) imply

$$f(x) = -Q_vQ_xv + Q_vQ_xQ_va = Q_vQ_xQ_v(-u+a) = Q(Q_vx)(a-u) = 0,$$

and therefore $x = 0$, because f is a group isomorphism. Thus Q_v is invertible. It follows that v is an invertible element with inverse $u := v^{-1} = Q(v)^{-1}v$. We put $z := a - u$ and $z' := a' - u$ and show that z and z' belong to the extreme radical.

By 7.12, $e = (u, u^{-1})$ is an idempotent with $V = V_2(e)$, and from (3) and 9.4 we see that $\omega = \exp_+(z)\,\omega_e\,\exp_+(z')$. By 9.7, ω_e is a Weyl element for $\alpha = 1$ as well. Put $n = \exp_+(z)$ and $n' = \exp_+(z')$. Then $n = \omega\,(n')^{-1}\,\omega_e^{-1}$ which implies

$$\begin{aligned} n\bar{U}^-n^{-1} &= \omega(n')^{-1}\omega_e^{-1}\bar{U}^-\omega_e n'\omega^{-1} \\ &= \omega(n')^{-1}\bar{U}^+n'\omega^{-1} = \omega\bar{U}^+\omega^{-1} = \bar{U}^-, \end{aligned}$$

so n normalizes $\bar{U}^-$. Similarly, one shows that n' normalizes $\bar{U}^-$. Thus $z, z' \in \mathrm{Extr}(V^+)$ by 9.9(a).

(b) This follows easily from the fact that ω_e is a Weyl element in $\mathrm{PE}(V)$ by 9.7, and that $\exp_+(\mathrm{Extr}(V^+))$ normalizes $\bar{U}^-$, by Theorem 9.9(a).

10.12. Corollary. *The following conditions on a Jordan pair V are equivalent.*

(i) *V is a Jordan division pair,*
(ii) *$\mathrm{PE}(V)$ is a rank one group.*

In this case, $\mathrm{PE}(V)$ is a special rank one group.

Proof. We write $G = \mathrm{PE}(V)$ and $U^\pm$ instead of $\bar{U}^\pm$ for simpler notation.

(i) $\Longrightarrow$ (ii): First observe that a Jordan division pair has trivial extreme radical. Indeed, if $0 \neq z \in \mathrm{Extr}(V^+)$ then $Q_z = 0$ and $Q_z\colon V^- \to V^+$ is an isomorphism of k-modules whence $V^\pm = 0$, contradiction. Now Proposition 10.11 shows that the set $\mathfrak{T}_1$ of Weyl triples for $\alpha = 1$ is precisely the set of all $\big(\exp_+(u), \exp_-(u^{-1}), \exp_+(u)\big)$ where $0 \neq u \in V^+$. Hence $\mathrm{pr}_1\colon \mathfrak{T}_1 \to U^{+*} = U^+ \setminus \{1\}$ is bijective. By passing to V^{op}, one sees that $\mathrm{pr}_1\colon \mathfrak{T}_{-1} \to U^{-*}$ is bijective as well, so G is a rank one group by Proposition 6.7. Since the Weyl triples θ_e are balanced by 9.7, G is special by Proposition 6.8.

(ii) $\Longrightarrow$ (i): By Proposition 6.7, $\mathrm{pr}_1\colon \mathfrak{T}_1 \to U^{+*}$ is bijective. Hence, every $1 \neq x \in U^+$ is the first component of a Weyl triple. By Proposition 10.11 this shows that every element $0 \neq a \in V^+$ has the form $a = z + u$ where $z \in \mathrm{Extr}(V^+)$ and $u \in V^+$ is invertible. From the definition of the extreme radical in 9.8 we have $\{z, V^-, V^+\} = 0$, whence $Q(z, V^+) = 0$. Hence $Q_a = Q_z + Q_{z,u} + Q_u = Q_u$ is invertible. Also, $U^+ \neq \{1\}$ is part of the definition of a rank one group. Hence $V^\pm \neq 0$, so V is a Jordan division pair.

If G is the elementary group of a special Jordan pair, we have the following more precise description of the Weyl elements and Weyl triples. Note that this generalizes 5.2.

10.13. Proposition. *Let V be a special Jordan pair, embedded in a Morita context $\mathfrak{M} = (A, B, C, D)$, and let $G = \mathrm{E}(\mathfrak{M}, V)$ be the corresponding elementary group as in* 10.3. *We consider G as a group with A_1-commutator relations and root groups $U_{\pm 1} = U^{\pm}$. Then G has Weyl elements for $\alpha = 1$ if and only if V has invertible elements. In this case, the Weyl triples and Weyl elements are given by*

$$t_u = \big(\mathrm{x}_+(u),\, \mathrm{x}_-(u^{-1}),\, \mathrm{x}_+(u)\big), \tag{1}$$

$$w_u = \mu(t_u) = \begin{pmatrix} 1_A - uu^{-1} & u \\ -u^{-1} & 1_D - u^{-1}u \end{pmatrix}, \tag{2}$$

where $\mathrm{x}_\pm$ *is defined in* (10.3.2) *and* $u \in V^+$ *is invertible in* V *with inverse* $u^{-1} \in V^-$. *In particular, every Weyl triple is balanced and the multiplication map $\mu\colon \mathfrak{T}_1 \to W_1$ is bijective.*

We emphasize that u^{-1} in (1) and (2) is the inverse in the Jordan pair V. It is in general not true that $uu^{-1} = 1_A$ or $u^{-1}u = 1_D$.

Proof. Let $u, u' \in V^+$ and $v \in V^-$ and consider the element

$$w = w(u, v, u') = \mathrm{x}_+(u)\mathrm{x}_-(v)\mathrm{x}_+(u') = \begin{pmatrix} 1 - uv & u + u' - uvu' \\ -v & 1 - vu' \end{pmatrix} \tag{3}$$

of G. A matrix computation shows that

$$w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w^{-1} = \begin{pmatrix} 1 + (1 - uv)xv & (1 - uv)x(1 - vu) \\ -vxv & 1 - vx(1 - vu) \end{pmatrix}, \tag{4}$$

for all $x \in V^+$.

First suppose that v is invertible in V with inverse $v^{-1} = u = u'$ and abbreviate $w_u := w(u, u^{-1}, u)$. Then $u = uvu$ so (3) shows that w_u has the form claimed in (2). Also, since $Q_u\colon V^- \to V^+$ is bijective, we have $x = Q_u y = uyu$ for a unique $y \in V^-$. Hence

$$(1 - uv)x = x - uv(uyu) = x - (uvu)yu = x - uyu = 0, \tag{5}$$

and similarly

$$x(1 - vu) = 0 \tag{6}$$

for all $x \in V^+$, so (4) yields

$$w_u \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w_u^{-1} = \begin{pmatrix} 1 & 0 \\ -vxv & 1 \end{pmatrix}$$

whence $w_u U^+ w_u^{-1} = U^-$. By a similar computation as before we have

$$w_u \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} w_u^{-1} = \begin{pmatrix} 1 & uyu \\ 0 & 1 \end{pmatrix}.$$

This proves that w_u is indeed a Weyl element and hence t_u is a Weyl triple for $\alpha = 1$. It is balanced since one easily sees by direct computation that also $w_u = \mathrm{x}_-(u^{-1})\mathrm{x}_+(u)\mathrm{x}_-(u^{-1})$.

Conversely, suppose that w as in (3) is a Weyl element for $\alpha = 1$. Then in particular $wU^+w^{-1} = U^-$. Hence (4) shows that $Q_v\colon V^+ \to V^-$, $x \mapsto vxv$, is bijective (i.e., v is invertible in V) and, for all $x \in V^+$,

$$(1 - uv)xv = 0 = vx(1 - vu), \tag{7}$$

$$(1 - uv)x(1 - vu) = 0. \tag{8}$$

From (7) we obtain, for $x = u$, that $uv = uvuv$ and hence $Q_v u = vuv = vuvuv = Q_v Q_u v$. By injectivity of Q_v this implies $u = Q_u v$. From (8) and the fact that $B(u, v)x = (1 - uv)x(1 - vu)$ (cf. (7.8.1)) we conclude with (7.11.2) that $0 = B(u, v) = Q(u - v^{-1})Q_v$ and therefore $Q(u - v^{-1}) = 0$. This implies $0 = Q(u - v^{-1})v = Q_u v - \{uvv^{-1}\} + Q(v^{-1})v = u - 2u + v^{-1}$ whence $u = v^{-1}$. It remains to show that $u' = u$. Let $z = u' - u$ and $n = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = w_u^{-1} \cdot w$. Then (!!) both w and w_u are Weyl elements for $\alpha = 1$ so $n \in W_1^{-1} \cdot W_1 \subset M_1 M_1 \subset M_0$ (by (5.3.1) and (5.3.5)) normalizes U^-. It now follows from (3) that

$$n \cdot \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \cdot n^{-1} = w(z, v, -z) = \begin{pmatrix} 1 - zv & zvz \\ -v & 1 + vz \end{pmatrix},$$

and this belongs to U^- if and only if $zv = vz = zvz = 0$. Hence $vzv = Q_v z = 0$ and therefore $z = 0$ because Q_v is invertible.

10.14. Lemma. *Let $e \in V$ be an idempotent with Peirce spaces $V_i^\sigma = V_i^\sigma(e)$. Let G be a group over V and define $U_i^\sigma = \mathrm{x}_\sigma\big(V_i^\sigma(e)\big)$, for $i = 0, 1, 2$ and $\sigma \in \{+, -\}$. Consider the following conditions:*

$$\mathrm{w}_e\, \mathrm{x}_\sigma(z_2)\, \mathrm{w}_e^{-1} = \mathrm{x}_{-\sigma}\big(Q_{e_{-\sigma}} z_2\big) \quad (z \in V_2^\sigma,\ \sigma \in \{+, -\}), \tag{1}$$

$$\mathrm{w}_{e^{\mathrm{op}}}\, \mathrm{x}_\sigma(z_2)\, (\mathrm{w}_{e^{\mathrm{op}}})^{-1} = \mathrm{x}_{-\sigma}\big(Q_{e_{-\sigma}} z_2\big) \quad (z \in V_2^\sigma,\ \sigma \in \{+, -\}), \tag{2}$$

$$\mathrm{w}_e\, U_2^\sigma\, \mathrm{w}_e^{-1} = U_2^{-\sigma} \quad (\sigma \in \{+, -\}), \tag{3}$$

$$\mathrm{w}_{e^{\mathrm{op}}}\, U_2^\sigma\, (\mathrm{w}_{e^{\mathrm{op}}})^{-1} = U_2^{-\sigma} \quad (\sigma \in \{+, -\}), \tag{4}$$

$$\mathrm{w}_e = \mathrm{w}_{e^{\mathrm{op}}}. \tag{5}$$

Then (1) – (4) *are all equivalent and imply* (5).

Proof. In the presence of (5), it is clear that (1) $\iff$ (2). Hence the equivalence of (1) and (2) will follow once we have shown (1) $\Longrightarrow$ (5) $\Longleftarrow$ (2). By putting $z_2 = e_\sigma$ in (1) and using $Q_{e_{-\sigma}} e_\sigma = e_{-\sigma}$ we see

$$\mathrm{w}_e\, \mathrm{x}_\sigma(e_\sigma)\, \mathrm{w}_e^{-1} = \mathrm{x}_{-\sigma}(e_{-\sigma}).$$

This implies $\mathrm{w}_e\, \mathrm{w}_{e^{\mathrm{op}}}\, \mathrm{w}_e^{-1} = \mathrm{w}_e$ and therefore (5), as required. A similar argument shows (2) $\Longrightarrow$ (5).

Evidently, (1) implies (3). Conversely, if (3) holds and $z_2 \in V_2^\sigma(e)$ then $\mathrm{w}_e\, \mathrm{x}_\sigma(z_2)\, \mathrm{w}_e^{-1} = \mathrm{x}_{-\sigma}(v_2)$ for some $v_2 \in V_2^{-\sigma}(e)$. Applying π to this relation and comparing with (9.6.1) and (9.6.2) yields $v_2 = Q(e_{-\sigma})z_2$, so we have (1). In the same way, one proves the equivalence of (2) and (4).

10.15. The Weyl relations. Let G be a group over V and let $e \in V$ be an idempotent. We say G *satisfies the Weyl relations* $\mathfrak{W}(e)$ if the equivalent conditions (10.14.1) – (10.14.4) of Lemma 10.14 hold. Clearly, by that lemma, $\mathfrak{W}(e)$ and $\mathfrak{W}(e^{\mathrm{op}})$ are equivalent and imply $\mathrm{w}_e = \mathrm{w}_{e^{\mathrm{op}}}$.

By Proposition 9.6, the projective elementary group satisfies these relations. Also, the elementary group $G = \mathrm{E}(\mathfrak{M}, V)$ of a special Jordan pair satisfies the Weyl relations. Indeed, since $\mathrm{w}_e = w(e_+, e_-, e_+)$ as in (10.13.3) one obtains

$$\mathrm{w}_e = \begin{pmatrix} 1_A - e_+e_- & e_+ \\ -e_- & 1_D - e_-e_+ \end{pmatrix}.$$

Now let $x_2 \in V_2^+(e)$, so $x_2 = Q_{e_+}Q_{e_-}x_2 = e_+e_-x_2e_-e_+$. Since e_+e_- is an idempotent in A, this implies

$$(1 - e_+e_-)x_2 = (1 - e_+e_-)e_+e_-x_2e_-e_+ = 0.$$

We also have $e_-x_2 = e_-(e_+e_-x_2e_-e_+) = e_-x_2e_-e_+$. Now (10.13.4) shows that (10.14.1) holds for $\sigma = +$, and the case $\sigma = -$ follows similarly. On the other hand, the Weyl relations do not hold in all groups over V. For example, the free product $F(V) = V^+ * V^-$ of the additive groups V^+ and V^- is a group over V, namely the Steinberg group of $\bar{G} = \mathrm{PE}(V)$, cf. 3.18(a). The relation (10.14.5) is not satisfied in $F(V)$.

For an automorphism $h \in \mathrm{Norm}_A(G)$ we have:

$$\textit{if } G \textit{ satisfies } \mathfrak{W}(e) \textit{ then } G \textit{ also satisfies } \mathfrak{W}(h(e)). \tag{1}$$

Indeed, let $e' = h(e)$. Then for $u \in V_2^\sigma(e')$,

$$\begin{aligned} \mathrm{w}_{e'}\mathrm{x}_\sigma(u)\mathrm{w}_{e'}^{-1} &= \varphi_h(\mathrm{w}_e\mathrm{x}_\sigma(h_\sigma^{-1}(u))\mathrm{w}_e^{-1}) = \varphi_h\big(\mathrm{x}_{-\sigma}(Q(e_{-\sigma})h_\sigma^{-1}(u))\big) \\ &= \mathrm{x}_{-\sigma}\big(Q(e'_{-\sigma})\,u\big), \end{aligned}$$

i.e., $\mathfrak{W}(h(e))$ holds in G.

10.16. The Steinberg groups $\mathrm{St}_n(R)$**.** Let R be an associative unital k-algebra, let $n = p + q \geqslant 3$ where $p \geqslant 1$, $q \geqslant 1$, and consider the Morita context $\mathfrak{M} = (A, B, C, D)$ of matrices of size $p \times p$, $p \times q$, $q \times p$ and $q \times q$ over R as in 7.1, with associated algebra $\mathfrak{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Mat}_n(R)$. Then $V = (B, C) = (\mathrm{Mat}_{pq}(R), \mathrm{Mat}_{qp}(R)) = \mathbb{M}_{pq}(R)$ is the Jordan pair of $p \times q$ matrices over R. It is easy to see that the group $\mathrm{E}(\mathfrak{M}, V)$ is indeed the usual elementary group $\mathrm{E}_n(R)$ in the sense of [**17**, 1.2C], cf. Example 2.16(c).

Recall [**45**, **17**] that the Steinberg group $\mathrm{St}_n(R)$ is the group presented by generators $\mathrm{x}_{ij}(r)$ ($r \in R$, $i \neq j$, $i, j \in \{1, \dots, n\}$) and relations

$$\begin{aligned} \mathrm{x}_{ij}(r)\mathrm{x}_{ij}(s) &= \mathrm{x}_{ij}(r+s), & (1)\\ \big(\mathrm{x}_{ij}(r),\, \mathrm{x}_{jl}(s)\big) &= \mathrm{x}_{il}(rs) \quad \text{for } i \neq l, & (2)\\ \big(\mathrm{x}_{ij}(r),\, \mathrm{x}_{kl}(s)\big) &= 1 \quad \text{for } j \neq k,\ i \neq l, & (3) \end{aligned}$$

where $r, s \in R$. To see that $\mathrm{St}_n(R)$ is indeed a group over V in the sense of 10.1, define $\mathrm{x}_\sigma\colon V^\sigma \to \mathrm{St}_n(R)$ by

$$\mathrm{x}_+(u) = \prod_{\substack{1\leqslant i\leqslant p\\ 1\leqslant j\leqslant q}} \mathrm{x}_{i,p+j}(u_{ij}), \qquad \mathrm{x}_-(v) = \prod_{\substack{1\leqslant i\leqslant p\\ 1\leqslant j\leqslant q}} \mathrm{x}_{p+j,i}(-v_{ji}), \tag{4}$$

for $u = (u_{ij}) \in V^+$ und $v = (v_{ji}) \in V^-$. From the defining relations of $\mathrm{St}_n(R)$ it follows easily that the order of the factors in (4) is immaterial, and that x_σ is in fact a homomorphism of the additive group V^σ into $\mathrm{St}_n(R)$.

It is well known [**17**, 1.4C] that there is a homomorphism $\psi\colon \mathrm{St}_n(R) \to \mathrm{E}_n(R) = \mathrm{E}(\mathfrak{M}, V)$ satisfying $\psi(\mathrm{x}_{ij}(r)) = 1_n + rE_{ij}$. Combining this with the map $\pi\colon \mathrm{E}(\mathfrak{M}, V) \to \mathrm{PE}(V)$ of 10.3, we have a homomorphism $\tilde{\pi} = \pi \circ \psi\colon \mathrm{St}_n(R) \to \mathrm{PE}(V)$ satisfying $\tilde{\pi} \circ \mathrm{x}_\sigma = \exp_\sigma$. Hence $\mathrm{St}_n(R)$ is a group over V. Moreover, $\mathrm{St}_n(R)$ is already generated by $\mathrm{x}_+(V^+) \cup \mathrm{x}_-(V^-)$. Indeed, the generators of $\mathrm{St}_n(R)$ not contained in this set are the $\mathrm{x}_{ij}(r)$ where i, j belongs to $\{1, \ldots, p\}$ or to $\{p+1, \ldots, p+q\}$. In the first case we have, by (2),

$$\mathrm{x}_{ij}(r) = \big(\mathrm{x}_{in}(r), \mathrm{x}_{nj}(1)\big) = \big(\mathrm{x}_+(rE_{iq}), \mathrm{x}_-(-E_{qj})\big).$$

The missing generators of the second type are recovered similarly.

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