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Slowly Oscillating Wave Solutions of a Single Species Reaction-Diffusion Equation with Delay

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Slowly oscillating wave solutions of a single species reaction-diffusion equation with delay

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Dedicated to Professor Anatoliy Samoilenko on the occasion of his 70th birthday

Abstract. We study positive bounded wave solutions $u(t,x) = \phi(\nu \cdot x + ct)$, $\phi(-\infty) = 0$, of equation $u_t(t,x) = \Delta u(t,x) - u(t,x) + g(u(t-h,x)),\ x \in \mathbb{R}^m\ (*)$. It is supposed that Eq. $(*)$ has exactly two non-negative equilibria: $u_1 \equiv 0$ and $u_2 \equiv \kappa > 0$. The birth function $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies a few mild conditions: it is unimodal and differentiable at $0, \kappa$. Some results also require the positive feedback of $g : [g(\max g), \max g] \to \mathbb{R}_+$ with respect to κ. If additionally $\phi(+\infty) = \kappa$, the above wave solution $u(t,x)$ is called a travelling front. We prove that every wave $\phi(\nu \cdot x + ct)$ is eventually monotone or slowly oscillating about κ. Furthermore, we indicate $c^* \in \mathbb{R}_+ \cup \{+\infty\}$ such that $(*)$ does not have any travelling front (neither monotone nor non-monotone) propagating at velocity $c > c^*$. Our results are based on a detailed geometric description of the wave profile ϕ. In particular, the monotonicity of its leading edge is established. We also discuss the uniqueness problem indicating a subclass $\mathcal{G}$ of *'asymmetric'* tent maps such that given $g \in \mathcal{G}$, there exists exactly one travelling front for each fixed admissible speed.

Keyword: Time-delayed reaction-diffusion equation, slow oscillations, small solution, semi-wavefront, travelling front, single species population model.

2000 Mathematics Subject Classification: 34K12, 35K57, 92D25

1. **Introduction and main results.** We study travelling wave solutions of the delayed reaction-diffusion equation

$$u_t(t,x) = \Delta u(t,x) - u(t,x) + g(u(t-h,x)),\ u(t,x) \geq 0,\ x \in \mathbb{R}^m, \qquad (1)$$

which has exactly two non-negative equilibria $u_1 \equiv 0,\ u_2 \equiv \kappa > 0$. The nonlinearity g is called *the birth function*, therefore it is non-negative. Throughout all the paper we assume that g satisfies the following unimodality condition

(UM): $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and has only one positive local extremum point $x = x_M$ (global maximum point). Furthermore, $g(0) = 0,\ g(\kappa) = \kappa$ and there exist $g'(0) > 1,\ g'(\kappa)$.

It is clear that (**UM**) implies that g is strictly monotone on $[0, x_M]$ and on $[x_M, +\infty)$. For example, (**UM**) is satisfied for the diffusive Nicholson's blowflies equation. Since the biological interpretation of u is the size of an adult population, we consider only positive bounded wave solutions $u(x,t) = \phi(\nu \cdot x + ct), \|\nu\| = 1$.

Before going further, let us fix some terminology. We say that wave solution $u(x,t) = \phi(\nu \cdot x + ct),\ \|\nu\| = 1$, is a wavefront (or a travelling front), if the profile function ϕ satisfies $\phi(-\infty) = 0$ and $\phi(+\infty) = \kappa$. In fact, such a profile ϕ is a positive heteroclinic solution of the delay differential equation

$$\epsilon x''(t) - x'(t) - x(t) + g(x(t-h)) = 0, \quad t \in \mathbb{R}, \qquad (2)$$

where $\epsilon = c^{-2} > 0$. Notice that ϕ may not be monotone. Following [8], we call positive bounded solutions $\phi(\nu \cdot x + ct)$ of (1) satisfying $\phi(-\infty) = 0$ semi-wavefronts. Slightly abusing the terminology we will also call ϕ the wavefront (or semi-wavefront) for Eq. (2).

If we take $h = 0$ in (1), we get a *monostable* reaction-diffusion equations without delay. The problem of existence of travelling fronts for this equation is quite well understood. In particular, for each such equation we can indicate a positive real number c_* such that, for every $c \geq c_*$, it has exactly one travelling front $u(x,t) = \phi(\nu \cdot x + ct)$, see [8, Theorem 8.3 (ii) and Theorem 8.7]. To find c_* we can use one of the variational principles for the front speeds, e.g., see [4, 8, 10]. Furthermore, Eq. (1) does not have any travelling front propagating at the velocity $c < c_*$. The profile ϕ is necessarily strictly increasing function, e.g. see [8, Theorem 2.39].

However, the situation will change drastically if we take $h > 0$. In fact, at the present moment, it seems that we are far from proving similar results concerning the existence, uniqueness and geometric properties of wavefronts for delayed equation (1). This despite that fact that the existence of travelling fronts in (1) was recently intensively studied (e.g. see [6, 7, 12, 13, 18, 19, 20, 21, 22]) for some specific subclasses of birth functions. Certainly, so called monotone case (when g is monotone on $[0, \kappa]$) is that one for which the most information is available. But so far, even for equations with monotone birth functions nothing is known about the number of wavefronts (modulo translation) for an arbitrary fixed $c \geq c_*$. In fact, we don't know how to determine c_* for a monotone g which does not meet the sublinearity condition $g(x) \leq g'(0)x$. The situation when g is not monotone on $[0, \kappa]$ is much more complicated. For example, it is not clear whether exists Eq. (1) which does not have any travelling front.

The main results of this paper answer some questions raised above:

Theorem 1.1 (Monotonicity of the leading edge of semi-wavefronts)**.** *Consider semi-wavefront $u(x,t) = \phi(\nu \cdot x + ct)$, $\|\nu\| = 1$, to Eq. (1). Then we can indicate some $\tau_3 > \tau_2 > \tau_1 \in \mathbb{R} \cup \{+\infty\}$ such that $\phi'(s) > 0$ on $(-\infty, \tau_1) \cup (\tau_2, \tau_3)$ and $\phi'(s) < 0$ on (τ_1, τ_2). Furthermore, τ_1 is finite if and only if $\phi(\tau_1) > \kappa$. Similarly, τ_2 is finite if and only if $\phi(\tau_2) < \kappa$.*

It is worth to mention that $\liminf_{s \to +\infty} \phi(s) \geq d > 0$, where d does not depend on c, ϕ. See [21] where the uniform permanence of semi-wavefronts was proved.

To state other theorems, we will need the concept of slowly oscillating solutions to Eq. (2). In Definition 1, we follow closely [16].

Definition 1.1. *Assume that the restriction $g : [g(\max g), \max g] \to \mathbb{R}_+$ has the positive feedback with respect to the equilibrium κ (i.e. $(g(x) - \kappa)(x - \kappa) < 0$, $x \neq \kappa$). Set $\mathbb{K} = [-h, 0] \cup \{1\}$. For any $\phi \in C(\mathbb{K}) \setminus \{0\}$ we define the number of sign changes by* $\mathrm{sc}(\phi) =$

$$\sup\{k \geq 1 : \text{ there exist } t_0 < \cdots < t_k \text{ such that } \phi(t_{j-1})\phi(t_j) < 0 \text{ for } j \geq 1\}.$$

We set $\mathrm{sc}(\phi) = 0$ if $\phi(s) \geq 0$ or $\phi(s) \leq 0$ for $s \in \mathbb{K}$. Being $x : [a - h, +\infty) \to \mathbb{R}$ a non-monotone solution of Eq. (2), we set $(\bar{x}_t)(s) = x(t + s) - \kappa$ if $s \in [-h, 0]$, and $(\bar{x}_t)(1) = x'(t)$. We will say that $x(t)$ is slowly oscillating about κ if, for each $t \geq a$, we have either $sc(\bar{x}_t) = 1$ or $sc(\bar{x}_t) = 2$.

The critical speeds c_*, c^* are defined below:

Definition 1.2. (a) *$c^* \geq 0$ is the biggest real number such that equation*

$$(c^*)^{-2} z^2 - z - 1 + g'(\kappa) \exp(-zh) = 0$$

has only one root in the half plane $\{\Re z > 0\}$.

(b) *$c_* \geq 0$ is the smallest real number such that equation*

$$(c_*)^{-2} z^2 - z - 1 + g'(0) \exp(-zh) = 0$$

has at least one real root in the half plane $\{\Re z > 0\}$.

Theorem 1.2 (Semi-wavefronts are either monotone or slowly oscillating). *Assume that* $g : [g(\max g), \max g] \to \mathbb{R}_+$ *has the positive feedback with respect to* κ *and* $g'(\kappa) < 0,\ g'(0) > 1$. *If* $u(x,t) = \phi(\nu \cdot x + ct),\ \|\nu\| = 1$, *is a semi-wavefront to Eq. (1), then* ϕ *is eventually either monotone or slowly oscillating around* κ. *Furthermore, if* $c > c^*$ *then the profile* ϕ *has to develop non-decaying slow oscillations around* κ.

It follows from [17] that these non-decaying slow oscillations are asymptotically periodic if $g : [g(\max g), \max g] \to \mathbb{R}_+$ is decreasing.

Corollary 1.1 (Admissible wavefront speeds and non-existence of fronts). *If all the conditions of Theorem 1.2 are satisfied then Eq. (1) does not have any travelling front (neither monotone nor non-monotone) propagating at velocity* $c > c^*$ *or* $c < c_*$. *In consequence, if* c^* *is less than* c_*, *then Eq. (1) does not possess any travelling front.*

The above result gives a strong argument supporting the conjecture from [21] that if the Schwarz derivative of sublinear g is negative and $c_* \leq c^*$, then the set of all admissible wavefront speeds coincides with the interval $[c_*, c^*]$. Observe that c_* is the minimal speed of propagation of semi-wavefronts if $g(x) \leq g'(0)x$, $x \geq 0$, e.g. see [21].

Finally, we discuss the uniqueness (up to translations) of positive wavefront for a given admissible speed c. There exist a very few theoretical studies devoted to this problem. To the best of our knowledge, the uniqueness was established only in two limit cases: for small delays in [1] and for large speeds in [2]. Here, we indicate a family $\mathcal{G}$ of unimodal and piece-wise linear g for which the problem of the existence of travelling fronts can be solved in the closed form. The elements of $\mathcal{G}$ are defined as follows:

Let $d > 1,\ \theta > 0,\ a \in [-1, 1)$ be given and satisfy $a\theta + b = d\theta,\ a\kappa + b = \kappa$ for some b, κ. Then $b > 0,\ \kappa > \theta$ and the piece-wise linear function

$$g(x) = g(x, a, d, \theta) := \begin{cases} dx, & \text{for } x \in [0, \theta]; \\ ax + b, & \text{if } x \in [\theta, \max\{\kappa, d\theta\}], \end{cases}$$

is continuous and $g(0) = 0,\ g(\kappa) = \kappa$. Set $\mathcal{G} = \{g(x, a, d, \theta) : a \in [-1, 1), d > 1, \theta > 0\}$. It is clear that $\mathcal{G}$ is sufficiently representative since *'asymmetric'* tent maps mimic the main features of general unimodal birth functions. Thus we hope that Theorem 1.3 below can be extended for all unimodal smooth nonlinearity g, in this way the above mentioned uniqueness result from [8, Theorem 8.7] could be proved for equations with delay.

Theorem 1.3 (On the uniqueness of the travelling front). *For* $g \in \mathcal{G}$, *there exists exactly one wavefront for each fixed admissible speed.*

The structure of this paper is as follows: in the next section, we prove the monotonicity of the leading edge of semi-wavefronts. This monotonicity will imply that the initial segment of the semi-wavefronts considered within positive feedback invariant domain is monotone or slowly oscillating. In the third section, we study the dependence of roots to the characteristic equation in the positive steady state of (2) on the parameter $\epsilon = c^{-2}$. In Section 4, under the positive feedback condition, we establish that semi-wavefronts are (eventually) either monotone or slowly oscillating. This section also contains the hardest part of the proof of Theorem 1.2: *if $c > c^*$ then the profile ϕ has to develop non-decaying slow oscillations around κ.* Finally, in Section 5 we show how the problem of travelling wavefronts can be solved in the closed form for the birth functions in $\mathcal{G}$. This will imply that, given $g \in \mathcal{G}$, there exists exactly one wavefront for each fixed admissible speed.

2. **Monotonicity of the leading edge of semi-wavefronts.** For given $\epsilon > 0$ we will denote by $\lambda < 0 < \mu$ the roots of $\epsilon z^2 - z - 1 = 0$. Also, we set $\epsilon' := \epsilon(\mu - \lambda)$. In this section, always assuming (**UM**), we study the monotonicity properties of semi-wavefronts to the equation

$$\epsilon x''(t) - x'(t) - x(t) + g(x(t-h)) = 0, \quad t \in \mathbb{R}. \tag{3}$$

Lemma 2.1. *Let x be a semi-wavefront to Eq. (3). Then $x'(t) > 0$ on some maximal interval $(-\infty, \sigma)$.*

Proof. Looking for a contradiction, we admit that there exists a sequence $t_n \to -\infty$ such that $x'(t_n) = 0$ for every n. Set $\xi(t) = g(x(t-h))/x(t-h)$, $y_n(t) = x(t+t_n)/x(t_n)$. Since $x(-\infty) = 0$, without the loss of generality we can suppose that $x(t) \le x(t_n),\ \xi(s+t_n) < 2g'(0)$ for all $t \le t_n,\ s \le 0$. It is clear that $y_n(0) = 1 = \max_{t\le 0} y_n(t),\ y_n'(0) = 0$, and that $y_n(t) > 0$ satisfies

$$\epsilon y''(t) - y'(t) - y(t) + \xi(t+t_n)y(t-h) = 0. \tag{4}$$

A partial integration of (4) yields

$$y_n'(t) = \frac{1}{\epsilon}\int_0^t e^{(t-s)/\epsilon}(y_n(s) - \xi(s+t_n)y_n(s-h))ds, \tag{5}$$

from which we deduce the uniform boundedness of the sequence $\{y_n'(t)\}$:

$$|y_n'(t)| \le 1 + 2g'(0), \quad t \le 0,\ n \in \mathbb{N}. \tag{6}$$

Together with $0 < y_n(t) \le 1,\ t \le 0$, inequality (6) implies the pre-compactness of $\{y_n(t),\ n \in \mathbb{N}\}$ in the compact open topology of $C(\mathbb{R}_-, \mathbb{R}_+)$. Therefore, by the Arzelà-Ascoli theorem, there is a subsequence $y_{n_j}(t)$ converging uniformly on bounded subsets of $\mathbb{R}_-$ to a continuous function $y(t)$. Integrating (5) between

t and 0 and then taking the limit as $n_j \to \infty$ in the obtained expression, we establish that $y(t),\ t \le 0$, satisfies

$$\epsilon y''(t) - y'(t) - y(t) + g'(0)y(t-h) = 0. \tag{7}$$

Additionally, $y'(0) = 0$ and $0 \le y(t) \le 1 = y(0),\ t \le 0$. Since (3) possesses a semi-wavefront x, equation (7) has exactly two real positive eigenvalues (counting multiplicity) $0 < \lambda_2(\epsilon) \le \lambda_1(\epsilon)$ while other eigenvalues satisfy $\Re\lambda_j(\epsilon) < \lambda_2(\epsilon)$, see [20, 21]. Therefore, for every $b > \lambda_1(\epsilon)$, it holds that

$$y(t) = w(t) + \exp(bt)o(1),\ t \to -\infty,$$

where $w(t)$ is a finite sum of eigensolutions of (7) associated to the eigenvalues λ_j with $\Re\lambda_j \ge 0$. Moreover, the positivity of y implies that

$$y(t) = \begin{cases} A_2\exp(\lambda_2(\epsilon)t) + A_1\exp(\lambda_1(\epsilon)t) + \zeta(t), & \text{if } \lambda_2(\epsilon) < \lambda_1(\epsilon); \\ \exp(\lambda_1(\epsilon)t)(A_2 + A_1 t) + \zeta(t), & \text{if } \lambda_2(\epsilon) = \lambda_1(\epsilon), \end{cases}$$

where ζ is a *small* solution of (7) at $-\infty$ in the sense that $\lim_{t\to-\infty}\zeta(t)\exp(bt) = 0$ for every $b \in \mathbb{R}$.

We claim that $\zeta(t) = 0$ for all $t \le 0$. Indeed, suppose that $\zeta(q) \ne 0$ for some $q \le 0$ and consider another small solution $u(t) = \zeta(q+t), t \le 0, u(0) \ne 0$, of (7). Multiplying this equation by $\exp(-zt)$ and then integrating obtained expression on $(-\infty, 0]$, we get that

$$\hat{u}(z) = \Phi(z)/\Delta(z), \text{ where } \hat{u}(z) = \int_{-\infty}^{0} e^{-zs}u(s)ds,\ \Delta(z) = \epsilon z^2 - z - 1 + g'(0)e^{-zh},$$

$\Phi(z) = \epsilon(zu(0) + u'(0)) - u(0) - g'(0)\int_0^h e^{-zs}u(s-h)ds$.

Since u is a small solution, we find that $\hat{u}$ is an entire function. Furthermore, since $g'(0)u(0) \ne 0$ the entire functions $\Phi(z), \Delta(z)$ are of the same exponential type h (see [5, Theorem 2.1, p. 137]). On the other hand, $\Phi(z), \Delta(z)$ are polynomially bounded in the closed right half-plane. Thus, by [5, Corollary 2.3, p.138], we get that $\hat{u}(z)$ is an entire function of exponential type 0. It is easy to see that $z\hat{u}(z)$ is uniformly bounded in $\Re z \ge 0$. Hence, an application of the Paley-Wiener theorem (see [5, Theorem 2.1]) yields $\hat{u} = 0$. Therefore $u(t) = 0$ for all $t \le 0$ contradicting to $u(0) \ne 0$.

In consequence, $y_{n_j}(t)$ converges to

$$y(t) = \begin{cases} A_2\exp(\lambda_2(\epsilon)t) + A_1\exp(\lambda_1(\epsilon)t), & \text{if } \lambda_2(\epsilon) < \lambda_1(\epsilon); \\ \exp(\lambda_1(\epsilon)t)(A_2 + A_1 t), & \text{if } \lambda_2(\epsilon) = \lambda_1(\epsilon). \end{cases} \tag{8}$$

Next, observe that $y_{n_j}(t),\ y_{n_j}(0) = 1,\ y'_{n_j}(0) = 0$, satisfy, for all $t \in \mathbb{R}$,

$$y_{n_j}(t) = \frac{\mu e^{\lambda t} - \lambda e^{\mu t}}{\mu - \lambda} + \frac{1}{\epsilon'}\int_0^t \left(e^{\lambda(t-s)} - e^{\mu(t-s)}\right)\xi(s + t_{n_j})y_{n_j}(s-h)ds. \tag{9}$$

Taking limit, as $j \to \infty$, in (9) on $[0, h]$, we see that $y_{n_j}(t)$ converges to $y(t)$ uniformly on $[0, h]$. Repeating the above procedure consecutively on the intervals $[0, 2h]$, $[0, 3h], \dots$, we establish that, in fact, $y_{n_j}(t)$ converges to $y(t)$ uniformly on every bounded subset of $\mathbb{R}$. Therefore $y(t)$, $t \in \mathbb{R}$, given by (8) must take only the non-negative values. It is easy to see that this requirement is incompatible with $y(0) = 1, y'(0) = 0$. □

Fix some semi-wavefront x of (3) and set $\Gamma(t) := g(x(t-h))$. Applying the variation of constants formula to (3), we obtain that

$$x(t) = A'e^{\lambda t} + B'e^{\mu t} + \frac{1}{\epsilon'}\left\{\int_a^t e^{\lambda(t-s)}\Gamma(s)ds + \int_t^b e^{\mu(t-s)}\Gamma(s)ds\right\}. \qquad (10)$$

Suppose for a moment that Γ is of bounded variation on $[a, b]$. Differentiating (10) and then integrating by parts Riemann-Stiltjes integrals [3, Theorem 7.6], we find that, for some $A, B \in \mathbb{R}$, the derivative $z(t) = x'(t),\ t \in [a, b]$, satisfies

$$z(t) = Ae^{\lambda t} + Be^{\mu t} + \frac{1}{\epsilon'}\left\{\int_a^t e^{\lambda(t-s)}d\Gamma(s) + \int_t^b e^{\mu(t-s)}d\Gamma(s)\right\}. \qquad (11)$$

Lemma 2.2. *If z meets the boundary conditions $z(a) = z_0,\ z(0) = 0$, then*

$$\begin{aligned} z(t) &= \frac{e^{\lambda t} - e^{\mu t}}{e^{\lambda a} - e^{\mu a}}\left\{z_0 + \frac{1}{\epsilon'}\int_a^t (e^{\lambda(a-u)} - e^{\mu(a-u)})d\Gamma(u)\right\} \\ &+ \frac{e^{\mu(t-a)} - e^{\lambda(t-a)}}{\epsilon'}\int_t^0 \frac{e^{-\mu u} - e^{-\lambda u}}{e^{-\mu a} - e^{-\lambda a}}d\Gamma(u); \qquad (12) \\ z'(0) &= \frac{\lambda - \mu}{e^{\lambda a} - e^{\mu a}}\left\{z_0 + \frac{1}{\epsilon'}\int_a^0 (e^{\lambda(a-u)} - e^{\mu(a-u)})d\Gamma(u)\right\}; \\ z'(a) &= \frac{\lambda e^{\lambda a} - \mu e^{\mu a}}{e^{\lambda a} - e^{\mu a}}z_0 + \frac{\mu - \lambda}{\epsilon'}\int_a^0 \frac{e^{-\mu u} - e^{-\lambda u}}{e^{-\mu a} - e^{-\lambda a}}d\Gamma(u). \end{aligned}$$

Proof. Formula (12) follows from (11) after taking into consideration the boundary conditions. The representations for $z'(0), z'(a)$ can be obtained in the following way: first, we integrate by parts the both Riemann-Stiltjes integrals in (12). Then we find $z'(t)$ differentiating the obtained expression with respect to t. To get the above formulae for $z'(0), z'(a)$, we need once more to integrate by parts. Observe that, in general, we cannot differentiate Riemann-Stiltjes integrals in (12). □

Remark 2.1 (Critical points of $x(t)$ are isolated)**.** *Lemma 2.1 does not allow to have $\Gamma'(t) = 0$ on any interval (p, q) since otherwise $x'(t) = 0$ for all $t \in \cup_{j\ge 1}(p - jh, q - jh)$. Other consequence of Lemmas 2.1, 2.2 is that the closed set $K = \{s : x'(s) = 0\}$ does not have finite limit points. Indeed, let s_1 be the first limit point of K. Since function $g(x(t-h))$ is strictly monotone in both small* *one-sided neighborhoods $\mathcal{O}_l, \mathcal{O}_r$ of s_1, we see that $\Gamma(t)$ is of bounded variation on $\mathcal{O}_l \cup \mathcal{O}_r$. In consequence, Lemma 2.2 can be used near s_1 to find that $x''(s_1) \neq 0$. Therefore s_1 must be isolated in K. Notice that, under additional conditions of C^2-smoothness of g at 0 and the hyperbolicity of Eq. (7), Lemma 2.1 was proved in* [21, Remark 5.5].

Remark 2.2 implies that, for a semi-wavefront $x(t)$, function $\Gamma(t) = g(x(t-h))$ is piece-wise monotone, with finite number of local extrema on every compact subinterval of $\mathbb{R}$. In this way, Γ is locally of bounded variation, that is why we don't require this condition explicitly in Lemmas 2.2, 2.3.

Lemma 2.3. *If $z(t) = x'(t)$ satisfies $z(-\infty) = 0,\ z(0) = 0$ then*

$$\begin{aligned} z(t) &= \frac{1}{\epsilon'}\left\{(e^{\lambda t} - e^{\mu t})\int_{-\infty}^{t} e^{-\lambda s} d\Gamma(s) + e^{\mu t}\int_{t}^{0}(e^{-\mu s} - e^{-\lambda s}) d\Gamma(s)\right\} \\ &= \frac{1}{\epsilon'}\left\{(e^{\lambda t} - e^{\mu t})\int_{-\infty}^{0} e^{-\lambda s} d\Gamma(s) + \int_{0}^{t}(e^{\lambda(t-s)} - e^{\mu(t-s)}) d\Gamma(s)\right\}. \quad (13) \end{aligned}$$

Proof. Formula (13) follows from (11) after taking into consideration the boundary conditions. To justify the convergence of the improper Riemann-Stiltjes integrals, it suffices to integrate them by parts. □

Theorem 2.1. *Let x be a semi-wavefront to Eq. (3). If $\tau \in \mathbb{R}$ is the leftmost point where $x(\tau) = \kappa$ then $x'(t) > 0,\ t \in (-\infty, \tau]$.*

Proof. Take σ as in Lemma 2.1. Since $\sigma = +\infty$ implies that $x(+\infty) = \kappa$ and $x(t) < \kappa,\ t \in \mathbb{R}$, we may assume that $\sigma = 0$ and $z(0) = x'(0) = 0$. Thus $z(t) = x'(t) > 0$ for all $t < 0$. Next, arguing as in (5), (6), we find that $x'(t) \leq x(0)(1+2g'(0))$. Due to (3), this yields the uniform boundedness of $|x''(t)|$ on $\mathbb{R}_-$. Therefore $x'(t)$ is uniformly continuous on $\mathbb{R}_-$. An application of the Barbalat lemma (e.g. see [22, Lemma 2.3]) gives $x'(-\infty) = 0$.

First, we consider the case when $x(0) \leq x_M$. Then $\Gamma(t) = g(x(t-h))$ is strictly increasing on $(-\infty, h)$. Since $z(t) = x'(t)$ satisfies boundary conditions $z(-\infty) = 0$, $z(0) = 0$, we get from (13) that $z(t) < 0$ for all $t \in (0, h]$. Thus $z(t) < 0$ on some maximal interval $(0, \sigma_1)$. Notice that σ_1 must be a finite real number since otherwise $x'(t) < 0$ on $(0, +\infty)$ implying $x(+\infty) = 0$. However, this contradicts the uniform persistence of semi-wavefronts established in Lemma 4.3 of [21]. In consequence, $\sigma_1 > h$ is finite so that $x'(\sigma_1) = z(\sigma_1) = 0,\ x''(\sigma_1) \geq 0$ and $x(\sigma_1) < x(\sigma_1 - h)$. On the other hand, we see that (3) implies

$$\epsilon x''(\sigma_1) - x(\sigma_1) + g(x(\sigma_1 - h)) = 0,$$

from which we obtain $x(\sigma_1 - h) > x(\sigma_1) \geq g(x(\sigma_1 - h))$, a contradiction.

Let us suppose now that $x(0) \in (x_M, \kappa]$. Then $x(t_*) = x_M$ for a unique $t_* < 0$.

Case I. If $t_* + h > 0$, then we can again use Lemma 2.3 to find that $z(t) < 0$, $t \in (0, t_* + h]$. Moreover, $x''(0) = z'(0) < 0$ in view of Lemma 2.2. Therefore, if $x(0) \leq \kappa$ and if $\sigma_2 > 0$ denotes the leftmost positive point where $x'(\sigma_2) = 0$, then $\sigma_2 > t_* + h,\ x''(\sigma_2) \geq 0$ and $x(\sigma_2) < \kappa$.

Case II. Now, assume that $t_* + h \leq 0$. Then $x''(0) < 0$, since $x''(0) = 0$ implies $\kappa \geq x(0) = g(x(-h)) > \kappa$, what of course is not true. Suppose that $x'(a) = 0$ for some $a \in (0, h]$. Since $\Gamma(t) = g(x(t-h)),\ t \in [0, a]$ is strictly decreasing, an application of Lemma 2.2 yields $x'(t) = z(t) < 0,\ t \in (0, a)$ and $x''(a) = z'(a) > 0$. Hence, we can find at most one critical point $a \in (0, h]$. In any case, we see that if $t_* + h \leq 0$ then $x'(t) < 0$ on $(0, a)$.

The above considerations show that if $\sigma_2 > 0$ denotes the leftmost positive point where $x'(\sigma_2) = 0$, then $\sigma_2 > t_* + h$ and $x''(\sigma_2) \geq 0,\ x(\sigma_2) < \kappa$.

Finally, let us suppose for a moment that $x(\sigma_2) < x_M$. Then $\sigma_2 > t_* + h$ implies that $x(\sigma_2) < x(\sigma_2 - h) \leq \kappa$, a contradiction in view of $x(\sigma_2) \geq g(x(\sigma_2 - h)) > x(\sigma_2 - h)$. Therefore we have to suppose that $x(\sigma_2) \geq x_M$. But then $\sigma_2 > t_* + h$ implies that $\kappa \geq x(\sigma_2 - h) \geq x_M$ so that $x(\sigma_2) \geq g(x(\sigma_2 - h)) \geq \kappa$. This is again a contradiction.

The above said shows that $x(t)$ is strictly increasing with $x'(t) > 0$, at least until its first intersection with the positive equilibrium κ. □

Arguments used in the proof of Theorem 2.1 allows us to establish the strict monotonicity of *all* semi-wavefronts of Eq. (3) once g is monotone on $[0, \kappa]$:

Corollary 2.1. *Assume that continuous $g : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing on $[0, \kappa]$, there exists $g'(0) > 1$ and equation $g(x) - x = 0$ has only two roots: 0 and κ. Then every semi-wavefront x of Eq. (3) in fact is a travelling front. Moreover, $x'(t) > 0$ for all $t \in \mathbb{R}$.*

Proof. Since we can set formally $x_M = +\infty$, we find from the proof of Theorem 2.1 that $x'(t) > 0$ on some maximal semi-infinite interval $(-\infty, \sigma)$. If $\sigma = +\infty$, Corollary 2.1 is proved. If σ is finite, then $x(\sigma) > \kappa,\ x'(\sigma) = 0$ and therefore, by Lemma 2.2, $x''(\sigma) < 0$. This leads to the following contradiction: $x(\sigma) < g(x(\sigma - h)) \leq \max\{\kappa, x(\sigma - h)\}$. □

Lemma 2.4. *Let x be a non-monotone semi-wavefront to Eq. (3). Then there exist $\tau_1 > \tau$ such that $x'(t) > 0$ on $(-\infty, \tau)$, $x(\tau) > \kappa$, $x'(\tau_1) = 0$ and $x'(t) < 0$ on (τ, τ_1). If τ_1 is finite then $x(\tau_1) < \kappa,\ x''(\tau_1) \geq 0$. Finally, if $\tau_1 \in (\tau, \tau + h]$ then $x''(\tau_1) > 0$ and $x'(t) > 0$ on $(\tau_1, \tau + h]$.*

Observe that if $\tau_1 = +\infty$, then wavefront x can have only one extremum (global maximum) at τ. However, we don't know whether this can happen under our assumption **(UM)**.

Proof. Set $\tau = \sup\{t : x'(s) > 0,\ s \in (-\infty, t)\}$. This number is finite since x is not monotone and Theorem 2.1 implies that $x(\tau) > \kappa,\ x'(\tau) = 0$. Next, let t_* be the unique point on $(-\infty, \tau)$ where $x(t_*) = x_M$. We will consider two cases depending on possible mutual positions of the points $t_* + h$ and τ.

Case A First, suppose that $t_* + h \leq \tau$. ¿From *Case II* of the proof of Theorem 2.1, we find out immediately that either the inequality $x'(t) < 0$ on $(\tau, \tau+h]$ or the existence of a critical point $a \in (\tau, \tau + h]$ of x imply all conclusions of Lemma 2.4 (with $\tau_1 = a$) but the inequality $x(\tau_1) < \kappa$. Now, to see that the latter inequality holds, let us consider $b := \sup\{t : x'(s) > 0,\ s \in (\tau_1, t)\}$. If $x(\tau_1) \geq \kappa$ then b is finite, $b - \tau > h$ and $x'(b) = 0,\ x''(b) \leq 0$. This gives $\kappa < x(b) \leq g(x(b-h)) \leq \kappa$, a contradiction. Finally, if we consider the third possibility that $x'(t) > 0$ on $(\tau, \tau + h]$ and set $c := \sup\{t : x'(s) > 0,\ s \in (\tau, t)\}$, then we find that c is finite, $x'(c) = 0$ and $\Gamma(t) = g(x(t - h))$ strictly decreases on (τ, c). Applying Lemma 2.2, we get a contradiction: $x''(c) = z'(c) > 0$.

Case B So we have only to study the case when $t_* + h > \tau$. As we already have established in *Case I* of the proof of Theorem 2.1, $x'(t) < 0$ on $(\tau, t_* + h]$. Let us suppose for a moment that there exists $a \in (t_* + h, \tau + h]$ such that $x'(a) = 0$. Then applying Lemma 2.2 on $[t_* + h, a]$, we obtain that $x''(a) > 0$ (and thus we may set $\tau_1 = a$). This means that there is at most one critical point of x on $(t_* + h, \tau + h]$ (and, in consequence, on $[\tau, \tau + h]$). The proof of the inequality $x(a) = x(\tau_1) < \kappa$ is as above.

Now, if $t_* + h > \tau$ and $x'(t) < 0$ on $(\tau, \tau + h]$, we can set $\tau_1 = \sup\{t : x'(s) < 0,\ s \in (\tau, t)\} > \tau + h$. When τ_1 is finite, it holds $x'(\tau_1) = 0,\ x''(\tau_1) \geq 0$. Furthermore, we claim that $x'(t) > 0$ in some right neighborhood of $\tau_1 \in \mathbb{R}$. Indeed, otherwise $x''(\tau_1) = 0$ and therefore $x(\tau_1) = g(x(\tau_1 - h))$. This implies that $x(\tau_1) < \kappa < x(\tau_1 - h)$. In consequence, if $x'(t) < 0$ on (τ_1, τ_2) and $x'(\tau_2) = 0$ then $\tau_2 - \tau_1 < h$. Moreover, since $\kappa < x(\tau_2 - h)$ we find that $\Gamma(t) = g(x(t - h))$ strictly increases on (τ_1, τ_2). Applying Lemma 2.2, we get a contradiction: $x''(\tau_1) > 0$.

Finally, since $x'(t) > 0$ in some right neighborhood of $\tau_1 \in \mathbb{R}$, we finalize our studies of *Case B* proving the inequality $x(\tau_1) < \kappa$ as it was done above. □

Corollary 2.2. *Let x be a non-monotone semi-wavefront to Eq. (3). Let τ be the leftmost critical point of $x(t)$. Then $sc(\bar{x}_{\tau+h}) = 1$ or $sc(\bar{x}_{\tau+h}) = 2$ and $g(g(x_M)) \leq x(t) \leq g(x_M) = \max_{x>0} g(x)$ for all $t \geq \tau$.*

Proof. We need only to prove that $x(t) \geq g(g(x_M))$. Let τ_1 be as in Lemma 2.4. First, we see that $x(\tau_1) \geq g(x(\tau_1 - h)) \geq g(g(x_M))$ because of $x(\tau_1 - h) > \kappa$. Now, suppose that $\hat{\tau}$ is the first critical point where $x(\hat{\tau}) < g(g(x_M)) < \kappa$. We have $x'(\hat{\tau}) = 0,\ x''(\hat{\tau}) \geq 0,\ x(\hat{\tau} - h) > \kappa > x(\hat{\tau})$ and therefore $x(\hat{\tau}) \geq g(x(\hat{\tau} - h)) \geq g(g(x_M))$. □

Remark 2.2 (Monotonicity without assuming the unimodality)**.** *Some of the proofs given above don't use the full force of condition* (**UM**)*. For example, as it can be easily checked, Lemma 2.1 holds true for all positive (including unbounded) solutions* $x,\ x(-\infty)=0$, *if continuous* g *satisfies* $g'(0)>0$ *and* $g(x)>0, x\in\mathbb{R}$. *One can consider also the following assumption proposed in* [21]*:*

(B): $g:\mathbb{R}_+\to\mathbb{R}_+$ *is continuous and such that, for some* $0<\zeta_1<\zeta_2$,

1. $g([\zeta_1,\zeta_2])\subseteq[\zeta_1,\zeta_2]$ *and* $g([0,\zeta_1])\subseteq[0,\zeta_2]$*;*
2. $\min_{s\in[\zeta_1,\zeta_2]} g(s)=g(\zeta_1)$*;*
3. $g(x)>x$ *for* $x\in(0,\zeta_1]$ *and there exists* $g'(0)>1$*;*
4. *In* $[0,\zeta_2]$, *the equation* $g(x)=x$ *has exactly two fixed points* 0 *and* κ.

We can repeat the first part of the proof of Theorem 2.1 to establish

Proposition 2.1. *Assume* **(B)** *with* $\sup_{s\ge 0} g(s)\le\zeta_2$, *and suppose that* g *increases on* $[0,x_M],\ x_M\in[\zeta_1,\zeta_2]$. *Let* ϕ *be a positive semi-wavefront to Eq. (3). Then there exists a unique* τ *such that* $\phi(\tau)=x_M$ *and* $\phi'(s)>0$ *for all* $s\le\tau$.

3. **Variational equation at the positive equilibrium.** In this section, we study the zeros of the characteristic function

$$\psi(z,\epsilon):=\epsilon z^2-z-1+a\exp(-zh),\ a=g'(\kappa)<0,$$

associated with the variational equation

$$\epsilon x''(t)-x'(t)-x(t)+ax(t-h)=0 \tag{14}$$

along the equilibrium κ of Eq. (3). It is easy to check (e.g., see [21]) that all complex zeros of ψ are simple and that, for some $\epsilon_0>0$, equation $\psi(z,\epsilon_0)=0$ has a negative real root z_0 of the multiplicity 2. In fact, (z_0,ϵ_0) is a bifurcation point where two real roots merge and disappear as $\epsilon\to\epsilon_0+$.

Lemma 3.1. *Fix* $a<0,\ h>0,\ p\in[0,1]$, *and suppose that* $\epsilon>\max\{2,-2ae^h\}$. *Then function*

$$\psi_p(z,\epsilon):=\epsilon z^2-p(z+1)+a\exp(-zh)$$

1) *has exactly two roots* $\lambda_0,\ \lambda_1$, *in the half-plane* $\Re\lambda>-1$. *Furthermore, these roots are real and* $\lambda_1<0<\lambda_0$*;*
2) *does not have any root in the semi-infinite horizontal strips* $(-\infty,0]\times(\pi(1+2k)/h,\pi(2+2k)/h)$, $(-\infty,0]\times(-\pi(2+2k)/h,-\pi(1+2k)/h)$, $k\in\mathbb{N}\cup\{0\}$*;*
3) *has at most two roots (counting multiplicity) on the vertical line* $\Re z=\alpha$, *for every fixed* $\alpha\in\mathbb{R}$.

Proof. 1). Let $\mu=\mu(\epsilon,p)\le 0,\ \nu=\nu(\epsilon,p)\ge 0$, be the roots of $\epsilon z^2-pz-p=0$. Since $\mu(\epsilon,p)>-1/2$ for $\epsilon>2$, we have that

$$|\epsilon z^2-pz-p|=\epsilon|z-\mu||z-\nu|\ge\epsilon/2>|a|e^{-\Re zh}$$

for all z from the boundary of sufficiently large rectangles $[-1, A] \times [-B, B] \subset \mathbb{C}$. An application of the Rouché theorem ends the proof of 1).

2). Indeed, if we take $z = x + iy,\ x \leq 0,\ yh \in (\pi + 2\pi k, 2\pi + 2\pi k)$, then

$$\Im\psi_p(z, \epsilon) = 2\epsilon xy - py - ae^{-xh}\sin(yh) < 0.$$

3). Suppose that $z_1 = \alpha + iv \neq z_2 = \alpha + iu$, $|u| \neq |v|$ satisfy $\psi_p(z_j, \epsilon) = 0$. Then

$$|\epsilon z_1^2 - pz_1 - p|^2 = |a|^2\exp(-2\alpha h) = |\epsilon z_2^2 - pz_2 - p|^2,$$

that implies

$$\epsilon^2(u^2 + v^2) + 2(\epsilon\alpha - 0.5p)^2 + 2\epsilon p + p^2/2 = 0,$$

a contradiction. □

As it was observed in [21, Lemma 2.1 and Remark 2.2], if for fixed $a < 0, h$, $\epsilon_1 > 0$, the equation $\psi(\lambda, \epsilon_1) = 0$ has a unique root in the half plane $\{\Re z > 0\}$, then this property will be maintained for all $\epsilon > \epsilon_1$. In consequence, from Lemma 3.1 (with $p = 1$) we can deduce the following

Corollary 3.1. *Fix $a < 0, h > 0$. Then there is a unique $\epsilon^* \in [0, \max\{2, -2ae^h\}]$ such that $\psi(z, \epsilon)$ has only one zero in the half plane $\{\Re z > 0\}$ if and only if $\epsilon \geq \epsilon^*$.*

Remarks 3.1, 3.2 below are motivated by [14, Section 6].

Remark 3.1. *Fix $a < 0,\ h > 0,\ p \in [0, 1]$, and suppose that $\epsilon > \max\{2, -2ae^h\}$. Then Lemma 3.1 implies that each zero $\lambda,\ \Im\lambda \geq 0$, of $\psi_p(\lambda, \epsilon)$ belongs to the set $\{\lambda_0\} \cup_{k\geq 0} S_k$, where*

$$S_0 = (-\infty, 0] \times [0, \pi/h],\ \ S_k = (-\infty, 0] \times [\pi 2k/h, \pi(1 + 2k)/h],\ \ k \in \mathbb{N}.$$

Next, it is straightforward to see that $|a| \leq (\epsilon|z|^2 + |z| + 1)e^{\Re zh}$ for every zero z of $\psi_p(\lambda, \epsilon)$. In consequence, for each j we can indicate $x_j(|a|, \epsilon, h) < -1$ such that every root $z \in S_j$ satisfies $\Re z \in [x_j(|a|, \epsilon, h), -1]$. Hence, equations $\psi(z, \epsilon) = 0$ and

$$z^2 = \rho\exp(-zh),\ \ \rho = |a|/\epsilon > 0, \tag{15}$$

have the same number of roots in each S_j.

Remark 3.2. *Consider (15) for $\rho > 0$. All complex roots of (15) are simple, and the unique multiple (double) real root is $z = -2/h$, it appears when $\rho = \rho_\sharp := 4/(he)^2$. If ρ is sufficiently small then all roots $z,\ \Im z \geq 0$, (excepting one positive) of this equation belong to $\cup_{k\geq 0} S_k$. Now, fix some j and take $z = z_j(\rho) \in S_j$. If we let ρ increase, then $\exp(\Re z_j(\rho)h)|z_j(\rho)|^2 = \rho$ yields that $\Re z_j(\rho) > 0$ for sufficiently large ρ. If $z_j(\rho_j) = i\nu_j$, then we have*

$$\nu_j h = \pi(2j + 1),\ \ \rho_j = \nu^2 = (\pi/h)^2(2j + 1)^2,\ \ \rho_0 > \rho_\sharp.$$

In consequence, every strip $S_j, j > 0$ possesses a unique root $z_j(\rho)$ for all $\rho \le \rho_j$. When ρ increases trough ρ_j, this root crosses the imaginary axis from left to right. Hence, S_j does not contain any root of (15) for $\rho > \rho_j$. The same for the strip S_0, with the unique exception that S_0 contains two real roots $z_{01} \le z_{02} < 0$ for $\rho \le \rho_\sharp$. Furthermore, Lemma 3.1 (3) implies that $\Re z_j(\rho) < \Re z_i(\rho),\ \rho > 0$, if and only if $j > i$. If $\rho \le \rho_\sharp$, then

$$\dots < \Re z_2(\rho) < \Re z_1(\rho) < z_{01} \le z_{02}.$$

Remarks 3.1, 3.2 imply the following

Lemma 3.2. *Take $a < 0,\ h > 0$, and $\epsilon \ge \epsilon^*$. Then the set*

$$\Lambda = \{\lambda_j\}_{j>0} \cup \{\lambda_0, \lambda_{01}, \lambda_{02}\}$$

of all zeros $\lambda_j, \Im\lambda_j \ge 0, j > 0$, of ψ can be enumerated in such a way that either

$$\lambda_0 > 0 > \lambda_{01} \ge \lambda_{02} > \Re\lambda_1 > \Re\lambda_2 > \dots$$

or

$$\lambda_0 > 0 \ge \Re\lambda_{01} = \Re\lambda_{02} > \Re\lambda_1 > \Re\lambda_2 > \dots$$

Furthermore, $\lambda_j \in S_j$ and $\lambda_{0k} \in S_0$.

The next result is key to the proof of Theorem 1.2 :

Theorem 3.1. *If $\epsilon^* > 0$ is as in Corollary 3.1, then $\psi(z, \epsilon)$ does not have any zero in the strip $S_{00} := (-\infty, 0] \times [-2\pi/h, 2\pi/h]$ for every $\epsilon < \epsilon^*$.*

Proof. By Lemma 3.2, Theorem 3.1 holds if $\epsilon^* - \epsilon > 0$ is close to 0. Therefore, if S_{00} contains zero $\lambda_j(\hat{\epsilon})$ of ψ for some $\hat{\epsilon} < \epsilon^*$, it should enter the strip S_{00} crossing the interval $\mathcal{J} := [-2\pi i/h, 2\pi i/h]$ from right to left as ϵ is decreasing. This means that $\lambda_j(\epsilon)$ crosses $\mathcal{J}$ from left to right as ϵ increases from $\hat{\epsilon}$ to ϵ^*.

Now, the root $\lambda_j := \lambda_j(\hat{\epsilon}) \notin \mathbb{R}$ of $\psi(z, \hat{\epsilon}) = 0$ determines a unique smooth function $\lambda_j(\cdot) : (\alpha_j, \beta_j) \to \mathbb{C}$ defined on some maximal open interval $(\alpha_j, \beta_j) \subseteq [0, +\infty)$ containing $\hat{\epsilon}$ and such that $\lambda_j(\hat{\epsilon}) = \lambda_j, \psi(\lambda_j(\epsilon), \epsilon) = 0$. We claim that the path $\lambda_j(\cdot) : (\alpha_j, \beta_j) \to \mathbb{C}$ can not cross the imaginary axis from left to right. Indeed, we have that

$$\lambda_j'(\epsilon) = -\frac{z^2}{2\epsilon z - 1 + h(\epsilon z^2 - z - 1)},$$

so that, at the moment $\tilde{\epsilon}$ of the eventual intersection we have $\lambda_j(\tilde{\epsilon}) = i\omega$ and

$$\Re\lambda_j'(\tilde{\epsilon}) = -\omega^2(1 + h + \epsilon h\omega^2)/((1 + h(\epsilon\omega^2 + 1))^2 + \omega^2(2\epsilon - h)^2) < 0,$$

a contradiction. Theorem 3.1 is completely proved. □

4. **Proof of Theorem 1.2.** Let x be a non-monotone semi-wavefront solution of Eq. (3). By Remark 2.2, all critical points of x are isolated so that $x(t) \not\equiv const$ on every open subinterval of $\mathbb{R}$. Let τ be as in Lemma 2.4. Then Corollary 2.2 implies that $x(t) \in [g(\max g), \max g]$ for every $t > \tau$ and that $\mathrm{sc}(\bar{x}_{\tau+h}) = 1$ or $\mathrm{sc}(\bar{x}_{\tau+h}) = 2$. Applying [16, Theorem 2.1], we find that $\mathrm{sc}(\bar{x}_t) \in \{0, 1, 2\}$ for every $t > \tau + h$. It is immediate to check that $\mathrm{sc}(\bar{x}_s) > 0$ for all $s > \tau + h$. Now, $\mathrm{sc}(\bar{x}_t) = 1$ for all large t if and only if $x(t)$ is eventually monotone. If $\mathrm{sc}(\bar{x}_t) : (\tau+h, \infty) \to \{1, 2\}$ is not constant, then x is a slowly oscillating solution.

It is easy to see that x can not be monotone if $c > c^*$. Indeed, if x is monotone then necessarily $x(+\infty) = \kappa$. However, if $c > c^*$ then the characteristic function has not negative real roots (and therefore $g'(\kappa) < 0$). This means that x should oscillate around κ, see [7], [20, Remark 3.2], [21, Remark 5.2]. Hence, if $c > c^*$, then x is slowly oscillating around the positive steady state. In the remaining part of this section, we show that these oscillations are non-decaying.

Arguing by contradiction, assume that $x(+\infty) = \kappa$ for some $c > c^*$. Then $w(t) = x(t) - \kappa$, $w(+\infty) = 0$, solves

$$\epsilon w''(t) - w'(t) - w(t) + g_1(w(t-h)) = 0, \ t \in \mathbb{R}, \tag{16}$$

where $g_1(u) := g(u+\kappa) - \kappa, \ g_1(0) = 0, g_1'(0) = g'(\kappa)$, satisfies the positive feedback condition with respect to 0.

Since $w(+\infty) = 0$, there exists a sequence $t_n \to +\infty$ such that $|w(t_n)| = \max_{s \geq t_n} |w(s)|$. It is evident that $w(t_n) \neq 0$. Additionally, we can suppose that w attains its local extremum at t_n so that $w'(t_n) = 0$, $w''(t_n)w(t_n) \leq 0$. Due to the positive feedback condition, this implies immediately that $w(t_n)w(t_n - h) < 0$ and therefore $\mathrm{sc}(\bar{w}_{t_n}) = 1$ (observe that $\mathrm{sc}(\bar{w}_{t_n})$ must be an odd integer). In fact, there are a unique $z_n \in (t_n - h, t_n)$ and a finite set F_n such that $w(s) < 0$ for $s \in [t_n - h, z_n) \setminus F_n$ and $w(s) \geq 0$ for $s \in [z_n, t_n]$. Without restricting the generality, we can suppose that $|w(t_n)| = \max\{|w(s)| : s \in [z_n, t_n]\}$, and that $\{r_n\}, \ r_n := t_n - z_n \in (0, h)$, is monotonically converging to $r_* \in [0, h]$.

Now, $y_n(t) = w(t + z_n)/w(t_n), \ t \in \mathbb{R}$, satisfies

$$\epsilon y''(t) - y'(t) - y(t) + p_n(t-h)y(t-h) = 0, \tag{17}$$

where

$$p_n(t) = \begin{cases} g_1(w(t+z_n))/w(t+z_n), & \text{if } w(t+z_n) \neq 0; \\ g'(\kappa), & \text{if } w(t+z_n) = 0. \end{cases}$$

It is clear that $y_n(0) = 0$ and $|y_n(t)| \leq 1, \ t \geq 0$, and that $\lim_{n\to\infty} p_n(t) = g'(\kappa)$ uniformly in $t \in \mathbb{R}_+$. As a consequence, we may suppose that $p_n(t)/g'(\kappa) \in [0.9, 1.1]$ for all n and $t \geq 0$. We have also that $y_n(r_n) = 1, \ y_n(r_n - h) < 0$.

Next, we need to estimate $|y_n'(t)|$. Let $\{s_n\}, \ \lim(s_n - z_n) = +\infty$ be such that $w'(s_n) = 0$. Since $v_n(t) = y_n'(t)$ solves the initial value problem $v_n(s_n - z_n) = 0$

for

$$\epsilon v'(t) - v(t) - y_n(t) + p_n(t-h)y_n(t-h) = 0, \quad t \in \mathbb{R},$$

we obtain that

$$y_n'(t) = v_n(t) = \frac{1}{\epsilon}\int_{s_n - z_n}^{t} e^{(t-s)/\epsilon}(y_n(s) - p_n(s-h)y_n(s-h))ds.$$

For all $t \in [h, s_n - z_n]$, we have

$$|y_n'(t)| \le |\frac{1}{\epsilon}\int_{s_n-z_n}^{t} e^{(t-s)/\epsilon}(|y_n(s)| + \sup_{x>0}|\frac{g(x)-\kappa}{x-\kappa}||y_n(s-h)|)ds| \le$$

$$\le (\sup_{x>0}|\frac{g(x)-\kappa}{x-\kappa}| + 1)\frac{1}{\epsilon}\int_{t}^{s_n-z_n} e^{(t-s)/\epsilon}ds \le (\sup_{x>0}|\frac{g(x)-\kappa}{x-\kappa}| + 1) := \rho.$$

$$|y_n''(t)| \le \epsilon^{-1}[|y_n'(t)| + |y_n(t)| + |p_n(t-h)||y_n(t-h)|] \le 2\epsilon^{-1}\rho.$$

Hence, the sequences $y_n(t), y_n'(t)$ have subsequences which converges on $[h, +\infty)$, in the compact-open topology, to continuous function $y_*(t), y_*'(t)$. Recalling the properties of y_n, we find that $\max\{|y_*(s)|, s \ge h\} \le 1$. Next, for all $t \in [2h, +\infty)$, it holds that

$$g_n(t) := p_n(t-h)y_n(t-h) \to g_*(t) := g'(\kappa)y_*(t-h).$$

We have $0 \le |g_*(t)| \le |g'(\kappa)|$ for $t \ge 2h$.

In order to establish some further properties of $y_*(t)$, we find the family of all solutions to (17) which are bounded at $+\infty$:

$$y(t) = Ae^{\lambda t} + \frac{1}{\epsilon(\mu-\lambda)}\left\{\int_{2h}^{t} e^{\lambda(t-s)}g_n(s)ds + \int_{t}^{+\infty} e^{\mu(t-s)}g_n(s)ds\right\}, \ t \ge 2h. \quad (18)$$

Replacing $y(t)$ with $y_n(t)$ in (18) and taking limit as $n \to +\infty$ (through passing to a subsequence if necessary) we find that $y_*(t),\ t \ge 2h$, satisfies

$$y_*(t) = Ae^{\lambda t} + \frac{1}{\epsilon(\mu-\lambda)}\left\{\int_{2h}^{t} e^{\lambda(t-s)}g_*(s)ds + \int_{t}^{+\infty} e^{\mu(t-s)}g_*(s)ds\right\}, \quad (19)$$

with some finite A. Next, (19) implies that $y_*(t)$ satisfies the linear equation

$$\epsilon y''(t) - y'(t) - y(t) + g'(\kappa)y(t-h) = 0, \ t \ge 2h. \quad (20)$$

We claim that $y_*(t)$ is not a small solution, the proof of this claim, given below, is motivated by [14, Section 10].

Indeed, on the contrary, let us suppose that $y_*(t)$ has superexponential decay. Then [11, Theorem 3.1] assures that $y_*(t) = 0$ for all $t \ge 3h$. But then Eq. (20) implies that $y_*(t) = 0, y_*'(t) = 0$ for all $t \ge 2h$ and, in consequence, $y_*(t) = 0$, $y_*'(t) = 0$ all $t \ge h$.

Next, by the Banach-Alaoglu theorem, we can suppose that $y_{n,h} \rightharpoonup \phi$ in $*$-weak topology of $L^\infty[-h,0]$. Since $y_n(h) \to y_*(h) = 0$, then integrating (17) between h and $t \geq h$, we get

$$\epsilon y_n'(t) - \epsilon y_n'(h) - (y_n(t) - y_n(h)) - \int_h^t (y_n(s) + p_n(s-h)y_n(s-h)ds = 0. \quad (21)$$

After taking limit as $n \to \infty$, we find that $\int_h^t g'(\kappa)\phi(s-2h)ds = 0,\ t \in [h,2h]$. Hence, $\phi = 0$ and therefore

$$\lim_{n\to\infty} \inf_{t\in[a,b]} |y_n(t)| = 0 \quad (22)$$

for every subinterval $[a,b] \subseteq [0,h]$. (Indeed, otherwise there exists $\varepsilon_0 > 0$ and a subsequence $\{y_{n_k}\}$ such that either $y_{n_k}(t) \geq \varepsilon_0$ or $y_{n_k}(t) \leq -\varepsilon_0$ for all $t \in [a,b]$. This means that $\int_a^b y_{n_k}(s)ds \not\to 0$, contradicting to $y_{n,h} \rightharpoonup 0$).

We claim that there exists a sequence $\{s_{n_j}\}$, $s_{n_j} \in (r_{n_j}, h)$, such that $s_{n_j} \to r_*$ and $y'_{n_j}(s_{n_j}) < 2g'(\kappa) - 1$, $y''_{n_j}(s_{n_j}) = 0$. Below, we prove this statement considering three different situations $(i), (ii), (iii)$.

(i) If $r_* = h$, then we can define s_n by

$$y_n'(s_n) = \min_{s\in[r_n,h]} y_n'(s).$$

Recall that $y_n(h),\ y_n'(h) \to 0$, so that, in fact, $s_n \in (r_n, h)$. Thus $y''(s_n) = 0$.

(ii) Next, suppose that $r_* < h$, and that $\{r_n\}$ is increasing. For an arbitrary j satisfying $-(2^j - 2^{-j}) < 2g'(\kappa) - 1$, we will fix two intervals

$$I_1 = [r_*, r_* + 2^{-j-1}],\ I_2 = [r_* + 2^{-j}, r_* + 3\cdot 2^{-j-1}].$$

In view of (22), we can find $d_k \in I_k$ and integer n_j such that $r_* - r_{n_j} \leq 2^{-j-1}$; $|y_{n_j}(d_k)| \leq 4^{-j}$. But then $1 - 4^{-j} \leq y_{n_j}(r_{n_j}) - y_{n_j}(d_1) = y'_{n_j}(\theta_{n_j})(r_{n_j} - d_1)$ so that

$$y'_{n_j}(\theta_{n_j}) \leq \frac{1-4^{-j}}{r_{n_j} - d_1} \leq -(2^j - 2^{-j}) < 2g'(\kappa) - 1.$$

Similarly,

$$-4^{-j} - 4^{-j} \leq y_{n_j}(d_2) - y_{n_j}(d_1) = y'_{n_j}(\xi_{n_j})(d_2 - d_1),$$

so that

$$y'_{n_j}(\xi_{n_j}) \geq \frac{-2\cdot 4^{-j}}{d_2 - d_1} \geq \frac{-2\cdot 4^{-j}}{2^{-j-1}} = -2^{-j+2} \geq y'_{n_j}(\theta_{n_j}),\ \theta_{n_j} < \xi_{n_j}.$$

Accordingly, if we set

$$y'_{n_j}(s_{n_j}) = \min_{s\in[r_{n_j}, r_* + 3\cdot 2^{-j-1}]} y'_{n_j}(s),$$

then

$$y''_{n_j}(s_{n_j}) = 0,\ y'_{n_j}(s_{n_j}) < 2g'(\kappa) - 1,\ s_{n_j} - r_{n_j} \leq 2^{-j+1}.$$

(iii) Finally, if $r_* < h$, and $\{r_n\}$ is decreasing, a similar argument works, if we take $I_1 = [r_*+2^{-j-2}, r_*+2^{-j-1}]$, $I_2 = [r_*+2^{-j}, r_*+3\cdot 2^{-j-1}]$, $r_{n_j} \in [r_*, r_*+2^{-j-2})$.

Hence, the above claim and (17) imply that

$$y_{n_j}(s_{n_j} - h) = \frac{y'_{n_j}(s_{n_j}) + y_{n_j}(s_{n_j})}{p_{n_j}(s_{n_j} - h)} \geq \frac{2}{p_{n_j}(s_{n_j} - h)/g'(\kappa)} \geq 2/1.1 > 1,$$

a contradiction, since $-h < r_{n_j} - h < s_{n_j} - h < r_{n_j}$ and

$$y_{n_j}(s) \leq 0, \ s \in [r_{n_j} - h, 0), \quad 0 \leq y_{n_j}(s) < 1, \ s \in [0, r_{n_j}).$$

Therefore $y_*(t)$ is not a small solution.

Hence, by [15, Proposition 7.2], for every sufficiently large $\nu < 0$, we have that

$$y_*(t) = u(t) + O(\exp(\nu t)), \ t \to +\infty,$$

where u is a *non empty* finite sum of eigensolutions of (20) associated to the eigenvalues $\lambda_j \in F = \{\nu < \Re\lambda_j \leq 0\}$. Now, Theorem 3.1 says that, for every $\epsilon \in (0, \epsilon_0]$,

$$F \cap (-\infty, 0] \times [-2\pi/h, 2\pi/h] = \emptyset.$$

In consequence, there exist $A > 0$, $\beta > 2\pi/h$, $\alpha \geq 0$, $\varphi \in \mathbb{R}$, such that

$$y_*(t) = (A\cos(\beta t + \varphi) + o(1))e^{-\alpha t}, \ t \geq 0.$$

This implies the existence of an interval $(a, a+h)$, $a > 3h$, such that $y_*(t)$ changes its sign on $(a, a+h)$ exactly three times. Since $y_{n_j}(t), y'_{n_j}(t) \to y_*(t)$ uniformly on $[a, a+h]$, we can conclude that $\mathrm{sc}(\bar{y}_{n_j, a+h}) \geq 3$ for all large j, a contradiction since $y_{n_j}(t)$ is a slowly oscillating function. In consequence, the equality $x(+\infty) = \kappa$ can not hold for $c > c^*$.

5. **Uniqueness in the case of piece-wise linear birth functions.** Let $d > 1$, $\theta > 0$, κ, b, and $a \in [-1, 1)$ satisfy the relations $a\theta + b = d\theta$, $a\kappa + b = \kappa$. Then $\theta < \kappa$, $b > 0$, and the piece-wise linear function

$$g(x) = \begin{cases} dx, & \text{for } x \in [0, \theta]; \\ ax + b, & \text{if } x \in [\theta, \max\{\kappa, d\theta\}], \end{cases}$$

is continuous and it holds that $g(0) = 0$ and $g(\kappa) = \kappa$. Moreover, if $a \in [-1, 0)$ then $g : [g(\max g), \max g] \to \mathbb{R}_+$ is decreasing so that the positive feedback condition is satisfied automatically.

In this section, given $a \in [-1, 1)$, $\theta > 0$, $d > 1$, we show how all the heteroclinic solutions of the equation

$$\epsilon x''(t) - x'(t) - x(t) + g(x(t-h)) = 0 \tag{23}$$

can be found in the closed form. It should be noticed here that Eq. (23) has at least one heteroclinic solution (say ϕ) for every $\epsilon \in (0, (c_*(a, h))^{-2}]$ independently on the value of delay h, see [13, 21].

Now, Lemma 2.4 (or Corollary 2.1) assures the existence of t_0 such that $\phi'(t) > 0$, $t \leq t_0 - h$, and $\phi(t_0 - h) = \theta$. Set $t_0 = 0$. Then, for all $t \leq 0$, such ϕ is a positive solution of the linear equation

$$\epsilon x''(t) - x'(t) - x(t) + dx(t-h) = 0. \tag{24}$$

The characteristic equation for (24) is

$$\epsilon \lambda^2 - \lambda - 1 + de^{-h\lambda} = 0, \tag{25}$$

and it has two positive real roots $0 < \lambda_1 \leq \lambda_2$ which dominate each complex root λ_j of (25) in the sense that $\Re \lambda_j < \lambda_1$, e.g. see [21, Lemma 2.3].

Case of the simple positive roots. At first we assume that $\lambda_1 < \lambda_2$. Then we get, for some $p \geq 0,\ p + q > 0$, that

$$\phi(t) = pe^{\lambda_1(t+h)} + qe^{\lambda_2(t+h)},\ t \leq 0. \tag{26}$$

From (26) we get $\phi(-h) = p + q = \theta$, so that $p = \theta - q$.

By Corollary 2.2, we have that $\phi(t) \geq \theta$ for all $t \geq -h$. Hence, if $t > 0$, then

$$\epsilon \phi''(t) - \phi'(t) - \phi(t) + a\phi(t-h) + b = 0.$$

The change of variables $\phi = y + \kappa$ transforms this equation into

$$\epsilon y''(t) - y'(t) - y(t) + ay(t-h) = 0. \tag{27}$$

Set $\psi(s) = \phi(s) - \kappa,\ s \geq -h$. Then

$$\psi(s) = (\theta - q)e^{\lambda_1(h+s)} + qe^{\lambda_2(h+s)} - \kappa,\ s \in [-h, 0],$$

$$\psi(0) = (\theta - q)e^{\lambda_1 h} + qe^{\lambda_2 h} - \kappa,\ \psi'(0) = \lambda_1(\theta - q)e^{\lambda_1 h} + q\lambda_2 e^{\lambda_2 h}.$$

Applying the Laplace transform $(\mathcal{L}y)(z) = \int_0^\infty e^{-zs} y(s) ds$ to Eq. (27), we get

$$\chi(z)(\mathcal{L}y)(z) = \epsilon(\psi'(0) + z\psi(0)) - \psi(0) - ae^{-zh} \int_{-h}^{0} \psi(s) e^{-zs} ds. \tag{28}$$

Here $\chi(z) = \epsilon z^2 - z - 1 + ae^{-hz}$. Since $|a| \leq 1$, characteristic function χ has a unique positive root ν while other characteristic values have negative real parts, see [21]. Therefore $\lim y(t) = 0,\ t \to \infty$, only if $(\mathcal{L}y)(\nu) = 0$. The last equation has the form $P(\nu, \lambda_1, \lambda_2)q + Q(\nu, \lambda_1, \lambda_2) = 0$, where $P(\nu, \lambda_1, \lambda_2) =$

$$\epsilon \lambda_2 e^{\lambda_2 h} - \epsilon \lambda_1 e^{\lambda_1 h} + (e^{\lambda_2 h} - e^{\lambda_1 h})(\epsilon \nu - 1) - ae^{-h\nu} \int_{-h}^{0} e^{-\nu s}(e^{\lambda_2(h+s)} - e^{\lambda_1(h+s)}) ds,$$

$$Q(\nu, \lambda_1, \lambda_2) = \epsilon(\theta \lambda_1 e^{\lambda_1 h} + \nu \theta e^{\lambda_1 h} - \nu \kappa) + \kappa - \theta e^{\lambda_1 h} - ae^{-h\nu} \int_{-h}^{0} e^{-\nu s}(\theta e^{\lambda_1(h+s)} - \kappa) ds.$$

Next, we establish that $P(\nu,\lambda_1,\lambda_2) > 0$, proving that the partial derivative $P_{\lambda_2}(\nu,\lambda_1,\lambda_2) > 0$. Observe here that $\lambda_2 > \lambda_1$ and $P(\nu,\lambda_1,\lambda_1) = 0$. Since $a < 1, \lambda_2 > 0$, we have

$$P_{\lambda_2}(\nu,\lambda_1,\lambda_2) = e^{\lambda_2 h}\left(\epsilon + \epsilon\lambda_2 h + h(\epsilon\nu - 1)\right) - a\int_{-h}^{0} e^{-\nu(s+h)}e^{\lambda_2(h+s)}(h+s)ds \geq$$

$$\geq e^{\lambda_2 h}\left(\epsilon + \epsilon\lambda_2 h + h(\epsilon\nu - 1)\right) - he^{\lambda_2 h}\int_{-h}^{0} e^{-\nu(s+h)}ds =$$

$$= e^{\lambda_2 h}\left(\epsilon + \epsilon\lambda_2 h + h(\epsilon\nu - 1) + h\frac{e^{-\nu h}-1}{\nu}\right) \geq e^{\lambda_2 h}(\epsilon + \epsilon\lambda_2 h) > 0.$$

Notice that

$$h\frac{e^{-\nu h}-1}{\nu} \geq h\frac{-\epsilon\nu^2+\nu}{\nu} = (-\epsilon\nu + 1)h$$

due to relations $0 = \epsilon\nu^2 - \nu - 1 + ae^{-\nu h} \leq \epsilon\nu^2 - \nu - 1 + e^{-\nu h}$.

Hence $q = -Q(\nu)/P(\nu)$ is determined uniquely and we can find $y(t)$ from (28) using the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left[\frac{\epsilon(\psi'(0) + z\psi(0)) - \psi(0) - ae^{-zh}\int_{-h}^{0}\psi(s)e^{-zs}ds}{\chi(z)}\right](t). \tag{29}$$

We observe here that [21, Lemma 3.1] says that $p > 0$ and $q < 0$ in (26).

Case of the multiple positive roots. Now, let us consider the case when $\lambda_1 = \lambda_2$. Then, for some $p \geq 0,\ p + q > 0$, we have

$$\phi(t) = e^{\lambda_1(t+h)}(p + q(t+h)),\ t \leq 0. \tag{30}$$

From (30) we get $\phi(-h) = p = \theta$, so that $p = \theta$.

Therefore $\psi(s) = \phi(s) - \kappa,\ s \in [-h, 0]$, satisfies :

$$\begin{aligned} \psi(s) &= e^{\lambda_1(h+s)}(\theta + q(s+h)) - \kappa, \\ \psi(0) &= e^{\lambda_1 h}(\theta + qh) - \kappa,\ \psi'(0) = e^{\lambda_1 h}(\lambda_1\theta + \lambda_1 qh + q). \end{aligned} \tag{31}$$

We next apply Laplace transform to Eq. (27) considered together with initial conditions (31). Considering relation $(\mathcal{L}y)(\nu) = 0$ (which is necessary to have $y(+\infty) = 0$), we find that it can be written as $P(\nu,\lambda_1)q + Q(\nu,\lambda_1) = 0$, where $Q(\nu,\lambda_1)$ is as above (but with $\lambda_1 = \lambda_2$) and

$$P(\nu,\lambda_1) = e^{\lambda_1 h}\left(\epsilon\lambda_1 h + \epsilon + \epsilon\nu h - h\right) - ae^{-h\nu}\int_{-h}^{0} e^{-\nu s + \lambda_1(h+s)}(h+s)ds.$$

Since $a < 1, \lambda_1 > 0$, we have

$$P(\nu,\lambda_1) \geq e^{\lambda_1 h}\left(\epsilon\lambda_1 h + \epsilon + (\epsilon\nu - 1)h - h\int_{-h}^{0} e^{-\nu(h+s)}ds\right) =$$

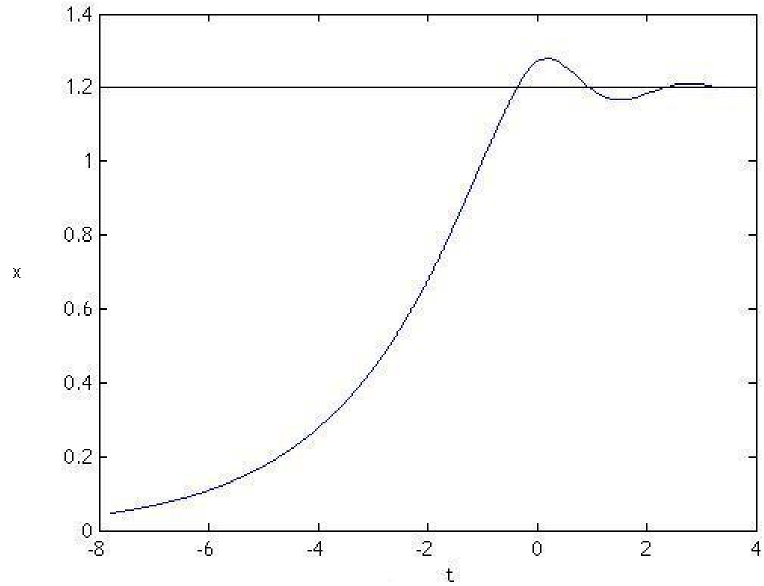

FIGURE 1. Non-monotone wavefront for Eq. (32).

$$= e^{\lambda_1 h}\left(\epsilon\lambda_1 h + \epsilon + (\epsilon\nu - 1)h + \frac{h}{\nu}(e^{-\nu h} - 1)\right) \geq e^{\lambda_1 h}\left(\epsilon\lambda_1 h + \epsilon\right) > 0.$$

Hence $q = -Q(\nu)/P(\nu)$ is determined uniquely and $y(t)$ is given by (29).

Example. Partially, the above technique works even when $a < -1$. Consider

$$x''(t) - x'(t) - x(t) + g(x(t-1)) = 0, \tag{32}$$

with continuous

$$g(x) = \begin{cases} 2x, & \text{for } x \in [0,1]; \\ -4x + 6, & \text{if } x \in [1, 1.4]; \\ \text{is positive decreasing} & \text{when } x > 1.4. \end{cases}$$

Eq. (32) has two non-negative equilibria $x_1 \equiv 0$ and $x_2 \equiv 1.2$.

It is easy to see that the characteristic equation $z^2 - z - 1 + g'(0)\exp(-z) = 0$ has two positive real roots, and that the roots $\lambda = (1-\sqrt{5})/2$ and $\mu = (1+\sqrt{5})/2$ of the equation $z^2 - z - 1 = 0$ satisfy the condition

$$\frac{\mu - \lambda}{\mu e^{-\lambda} - \lambda e^{-\mu}} = 0.715... > \frac{\Gamma^2 + \Gamma}{\Gamma^2 + 1} = 0.705..., \text{ where } \Gamma := g'(\kappa) = -4.$$

Therefore, the existence of non-monotone travelling front in (32) is guaranteed by [21, Theorem 1.1]. Then we use the Laplace transform to find and picture the non-monotone wavefront, see Fig. 1.

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