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On the *δ=const* Collisions of Singularities of Complex Plane Curves

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On the $\delta = const$ collisions of singularities of complex plane curves.

Dmitry Kerner

ABSTRACT. We study a specific class of deformations of curve singularities: the case when the singular point splits to several ones, such that the total δ invariant is preserved. These are also known as equi-normalizable or equi-generic deformations. We restrict primarily to the deformations of singularities with smooth branches.

A new invariant of the singular type is introduced: the dual graph. It imposes severe restrictions on the possible collisions/deformations. And allows to prove some bounds on the variation of classical invariants in collisions.

We consider in details the $\delta = const$ deformations of ordinary multiple point, the deformation of a singularity into the collection of ordinary multiple points and the deformation of the type $x^p + y^{pk}$ into a collection of A_k's.

1. Introduction

In this note we continue the study of collisions of singular points (started in [**Kerner07**]). For the relevant notions from singularity theory cf. [**AGLV-book, GLS-book**]. Some of the notions are also recalled in §2. The singularity types are named according to Arnol'd's tables [**AGLV-book**, §I.2.1].

1.1. Setup. Consider plane complex affine singular reduced algebraic curves. Let $\{C_t\}_{t \in T} \subset \mathbb{C}^2 \times T$ be a (flat) family of such curves, with T a small neighborhood (in classical topology) of $0 \in \mathbb{C}^1$. Let $\sigma_1..\sigma_k : T \to \{C_t\}$ be the sections of the family, i.e. the fibre C_t is singular at the points $\sigma_1(t)...\sigma_k(t)$ (for some t the sections might intersect). We assume that one of the singular points stays at the origin, e.g. $\sigma_1(t) = (0,0) \in \mathbb{C}^2$. Such a family is called a *degeneration* (or degenerating family) if it is not equisingular over T (for local embedded topological equivalence).

DEFINITION 1.1. *A degenerating family is called collision (or split deformation) if the following is satisfied:*
- *The family is equisingular over $T \setminus \{0\}$ (for local embedded topological equivalence).*
- *The generic fibre $C_{t \neq 0}$ has at least two singular points.*
- *The central fibre $C_{t=0}$ has only one singular point.*

$C_t \qquad C_0$
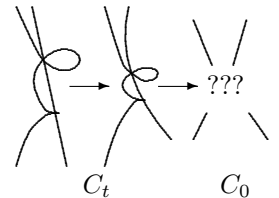
In this paper we always assume the degree of curves to be high enough for the given singularity types (of the generic and central fibres).

We work with topological singularity types and denote the collisions appropriately. So $(\mathbb{S}_1..\mathbb{S}_k) \to \mathbb{S}$ means that there exists a family of curves $\{C_t\}_{t \in T}$ with the generic fibre $C_{t \neq 0}$ having the singularities of the topological types $(\mathbb{S}_1..\mathbb{S}_k)$ and the central fibre having only one singular point of the type $\mathbb{S}$.

EXAMPLE 1.2. The simplest cases are $\bigcup_{i=1}^{k} A_{n_i} \to A_{\sum n_i + k - 1}$ and $\bigcup_{i=1}^{k} A_{n_i} \bigcup 2A_1 \to D_{\sum n_i + k + 2}$ Here the collisions are possible since the corresponding Dynkin diagrams decompose (see below the general result of [**Lyashko83**] for ADE).

The collision $(\mathbb{S}_1..\mathbb{S}_k) \to \mathbb{S}$ is the converse of the deformation $\mathbb{S} \to (\mathbb{S}_1..\mathbb{S}_k)$. So, the collision is possible if in the versal deformation of some singular germ of type $\mathbb{S}$ the singularities of types $(\mathbb{S}_1..\mathbb{S}_k)$ appear.

Note that in our case $(\mathbb{S}_1..\mathbb{S}_k) \to \mathbb{S}$ means only that *there exists a curve* with the given types which degenerates to the given type. In general the possibility of deformation $\mathbb{S} \to (\mathbb{S}_1..\mathbb{S}_k)$ depends not only on the (local embedded) topological singularity types but on the analytical types (i.e. on the moduli of the types) [**Pham70**] [**DamonGalligo93**] [**Jaworski94**] and [**duPlessis-Wall-book**, pg.209]. So the collision $(\mathbb{S}_1..\mathbb{S}_k) \to \mathbb{S}$ does not imply that every or generic representative can be degenerated in the prescribed manner.

DEFINITION 1.3. *A collision/deformation* $(\mathbb{S}_1..\mathbb{S}_k) \to \mathbb{S}$ *is called prime if it cannot be factorized non-trivially as* $(\mathbb{S}_1..\mathbb{S}_k) \to \mathbb{S}' \to \mathbb{S}$ *(the second arrow being a degeneration of the singularity type).*

The natural questions to ask are:
Given the singularity types $\mathbb{S}_1..\mathbb{S}_r$ *and* $\mathbb{S}_f$*. Is the collision* $(\mathbb{S}_1..\mathbb{S}_r) \to \mathbb{S}_f$ *possible? What are the possible results of prime collisions of* $\mathbb{S}_1..\mathbb{S}_r$*? What are the possible prime deformations of* $\mathbb{S}_f$*?*

These questions are highly important in various branches of singularity theory and algebraic geometry, but there is no hope to get any complete answer at such generality.

1.2. History. Some classical (and recent) results are:
• Every singular point can be deformed to a collection of nodes (A_1) in a $\delta = const$ way. (This was claimed in [**Albanese28**] and proved rigorously in [**Nobile84**, thm 1.4].)
• every singular point can be deformed to a collection of nodes (A_1) and cusps (A_2) in a $\delta = const$, $\kappa = const$ way [**DiazHarris88**, thm 1.1].
• Versal deformations of ADE singularities were studied classically. Let $\mathbb{S}$ be an ADE singularity. It deforms to a collection $\mathbb{S}_1..\mathbb{S}_k$ of (ADE) singularities iff the collection of Dynkin diagrams of $\mathbb{S}_1..\mathbb{S}_k$ can be obtained from that of $\mathbb{S}$ by removing some vertices [**Lyashko83**] (cf. also [**AGLV-book**, §5.9]). (All the diagrams are taken in the canonical bases of ADE.)
• Similar result (via Dynkin diagrams in distinguished bases) was obtained in [**Jaworski88**] (cf. also [**AGLV-book**, §5.11]) for parabolic singularity types ($P_8 = \tilde{E}_6$, $X_9 = \tilde{E}_7$, $J_{10} = \tilde{E}_8$).
• A thorough study of versal deformations of uni-modal types was done in [**Brieskorn79**], [**Brieskorn81**] and [**Brieskorn-book**] (by studying the properties of Milnor lattices).
• For many uni-modal and bi-modal types the necessary and sufficient conditions for the decomposition to ADE's are known. They are formulated in terms of some specific transformations of the canonical Dynkin diagrams. (cf. [**Urabe84**] for $J_{3,0}, Z_{1,0}, W_{1,0}, E_{k \le 12}, Z_{k \le 13}, W_{k \le 13}$ and [**Looijenga81**] for T_{pqr}).
• The dependence of some deformations of types $J_{k,0}$ on moduli was studied in [**Jaworski94**, thm 1-thm 4] by checking explicit equations for the singular germs.
• The deformations of T_{2pq} series (the singularity type of $x^p + x^2y^2 + y^q$) has been studied partially. (Here $X_9 = \tilde{E}_7 = T_{244}$.) The deformation $T_{244} \to A_7$ was known classically, the deformation $T_{2pp} \to A_{2p-1}$ was given in [**duPlessis-Wall-book**, pg.204]. The deformation $T_{245} \to A_8$ was constructed in [**Stevens04**] by brute force computation. The general case $T_{2pq} \overset{?}{\to} A_n$ is open.
• Some deformations of E,Z,Q series have been studied in [**duPlessis-Wall04**].
• The question of adjacency of just two singularities (i.e. a singular point deforms to just one singular point) seems to be more tractable. For the recent advances cf. [**Alberich-Roe05**].
• The case of surfaces in $\mathbb{C}^3$ is infinitely more complicated (e.g. the whole [**Urabe-book**] is a summary of the series of works studying the deformations of just 5 particular singularity types into ADE's).

As it seems, currently no other general results are known. Even worse, it is not clear how to answer (effectively) such questions in each particular case (except for the case of simple or uni-modal types). A kind of brute force computation was given in [**Kerner07**] for a specific class of singularity types (the so-called linear types). It does not seem to generalize easily to the arbitrary types.

Usually it is very difficult to prove that a deformation exists (e.g. to provide an example). Rather one seeks for various obstructions. The main classical obstructions are provided by the semi-continuity of various invariants (cf. §2).

1.3. The results. We restrict the consideration to the $\delta = const$ deformations, i.e. $\sum \delta(\mathbb{S}_i) = \delta(\mathbb{S})$, such that the initial type (i.e. the type of the central fibre) has all the branches smooth (§3). It is easy to see (proposition 3.1) that in such case all the resulting types have smooth branches too and $\kappa = const$ in the deformation.

To any type $\mathbb{S}$ with smooth branches we associate the *dual graph* $\Gamma_{\mathbb{S}}$. It is a complete invariant of the local embedded topological type (e.g. is equivalent to the resolution tree). The importance of the dual graph is due to the classical fact: a $\delta = const$ family is equi-normalizable (theorem 3.2). Thus a deformation $\mathbb{S} \to (\mathbb{S}_1..\mathbb{S}_k)$ corresponds to the decomposition $\Gamma_{\mathbb{S}} \to \oplus \Gamma_{\mathbb{S}_i}$ (the precise formulation is the theorem 5.1). So, the dual graph imposes various restrictions on the possible deformations. This is our main result.

These restrictions are stronger than some others (e.g. the restriction arising from the Milnor number). In general they are not weaker than others, e.g they are not implied by the semi-continuity of the spectrum (cf. example 2.7).

So we use them all together: those imposed by the dual graph, by the signature of the intersection form on the middle homology, by the local Bezout theorem, by the spectrum and the Hirzebruch inequality for ordinary multiple point(cf. equation (1) below).

As a result we get in many cases necessary conditions not known previously (to the best of author's knowledge). Below are some consequences of the method (proofs are in §5 §6).

PROPOSITION 1.4. *Let* $\mathbb{S} \to \bigcup \mathbb{S}_i$ *be a* $\delta = const$ *deformation (the types can have singular branches). Then, the number of branches is bounded:* $\binom{r}{2} \le \sum \binom{r_i}{2}$ *(the bound is sharp).*

For types with smooth branches the multiplicity is the number of branches, so this bounds the change of multiplicity.

The proposition gives an upper bound for the variation of Milnor numbers, as they satisfy $\mu - \sum \mu_i = \sum (r_i - 1) - (r - 1)$. The lower bound is directly obtained from the classical formulas (3), and is: $\frac{\mu}{\sum \mu_i} \le 1 + \frac{\sum (r_i - 1) - (r - 1)}{\sum (p_i - 1)^2}$ (the equality is realized for ordinary multiple points).

Unfortunately the conditions imposed by the dual graph are not sufficient (cf. remark 5.4). We do not know whether they can be strengthened/improved in any simple way to become sufficient.

1.3.1. *Results for ordinary multiple points.* Let K_p denote the topological type of the ordinary multiple point of multiplicity p. (So $K_2 = A_1$, $K_3 = D_4$, $K_4 = X_9$).

Question: Given the initial type K_p and the final collection $\mathbb{S}_1..\mathbb{S}_k$ with $\delta(K_p) = \sum \delta(\mathbb{S}_i)$. Is the $\delta = const$ deformation $K_p \to (\mathbb{S}_1..\mathbb{S}_k)$ possible? It's immediate (proposition 6.1) that each $\mathbb{S}_i$ must be an ordinary multiple point, so the deformation is $K_p \to (K_{p_1}..K_{p_k})$.

Even for such a particular case a satisfactory classification of possible deformations seems to be an open question. As the deformation is equi-normalizable we can trace each branch separately. Since all the branches of K_p are transversal and we consider only the germ of curve, can assume they are lines.

If one restricts the question further: assumes the deformed curve consists of lines only, then one has a local arrangement of lines. So, the classification of the possible splittings $K_p \to (K_{p_1}...K_{p_k})$ implies in particular the answer to the question:

Given p distinct lines in the plane, which patterns of intersection $(n_2 K_2, n_3 K_3, ...)$ can appear?

For small p this can be answered by direct classification. In general one can obtain some combinatorial restrictions (not sufficient). However there are restrictions of *non-combinatorial* origin. In [**Hirzebruch83**] the Miyaoka-Yau inequality for Chern numbers of surfaces was used to prove:

$$(1) \qquad n_2 + \frac{3}{4} n_3 \ge p + \sum_{i \ge 5} (2i - 9) n_i, \text{ provided } n_p = 0 = n_{p-1} = n_{p-2}$$

We do not know any generalizations or additional non-combinatorial restrictions. An elementary application of our method leads to the criterion (proved in §6.1):

PROPOSITION 1.5. *The* $(\delta = const)$ *deformation* $K_p \to \overset{k}{\underset{i=1}{\cup}} K_{p_i} + \left(\binom{p}{2} - \sum \binom{p_i}{2} \right) K_2$ *with* $\{p_i \ge k - 1\}$ *is possible iff* $p + \binom{k}{2} \ge \sum p_i$

So, for $k \ge 5$ this strengthens Hirzebruch's inequality. Note that for the codimension of the corresponding equisingular strata one has:

$$(2) \qquad \tau^{es}(K_p) - \tau^{es}\left(K_{p_1}..K_{p_k}, \left(\binom{p}{2} - \sum \binom{p_i}{2} \right) K_2 \right) = p - \sum p_i + 2(k - 1)$$

which can be non-positive. In this case the deformation does not exist for the *generic* representative of K_p, but only for a very special choice of moduli.

Another question is: *To which collections of ordinary multiple points can a given type be deformed in a $\delta = const$ way?* (Generalizing the classical deformation to δ nodes.)

The number of possible scenarios is quite big. One might hope to find some specific prime deformation, such that all other deformations factorize through this one. Unfortunately this is not the case. In §6.2 we propose a partial result: there always exists a *canonical* (prime) deformation into a bunch of ordinary multiple points. It is minimal in some sense but it does not factorize all others.

1.3.2. *Decompositions into ADE's.* An important question is: To which collections of ADE types deforms a given singularity?

In §6.3 we study a particular case: $\delta = const$ deformations of the type $x^p + y^{pk}$ (i.e. p smooth branches with equal tangency), denoted by $K_{p,k}$, into collections of A_j's. As was noticed above, only A_j's with smooth branches may appear. We apply the obstructions imposed by the dual graph, the signature of the middle homology lattice and the spectrum to get:

PROPOSITION 1.6. *Let* $K_{p,k>1} \overset{\delta=const}{\to} \bigcup n_i A_{2i-1}$. *Then* $n_{i>k} = 0$, $\sum_{i=1}^{k} i n_i = \binom{p}{2}k$ *and*

$$\sum n_i \geq \frac{(p-1)^2 k}{4} + (p-1) - \begin{cases} \frac{pk}{4} & p \text{ even} \\ 0 & p \text{ odd, } k \text{ even} \\ \frac{p-1}{2} & p \text{ odd, } k \text{ odd} \end{cases}$$

.

In particular: $n_k \leq \frac{(p-1)^2}{4}\frac{k}{k-1} + -\frac{p-1}{k-1} + \begin{cases} \frac{pk}{4(k-1)} \\ 0 \\ \frac{p-1}{2(k-1)} \end{cases}$ *and* $2n_1 + n_2 \geq \frac{(p-1)(p-3)k}{12} + (p-1) - \begin{cases} \frac{pk}{4} \\ 0 \\ \frac{p-1}{2} \end{cases}$

1.3.3. *Singular branches.* It is not clear how to approach the $\delta = const$ deformations when singular branches are present. An unpleasant fact is: a collection of types with smooth branches can collide ($\delta = const$) to a type with singular branches, such that the collision is prime (i.e. cannot be factorized). The simplest example is: $A_1 + A_3 \to D_5$.

When some of the initial branches are singular the dual graph can be still useful. Smoothen the branches (in a $\delta = const$ way): $\mathbb{S}_i \to \mathbb{S}_i^{def}$. Note that in general there are several distinct $\delta = const$ branch-smoothings. So, there are several collections $\{\mathbb{S}_i^{def}\}$. Then collide $\{\mathbb{S}_i^{def}\}$ (in all the possible ways). This produces some restrictions on the possible results of collision of $\{\mathbb{S}_i\} \to \mathbb{S}$ (for $\mathbb{S}$ with smooth branches).

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2. The classical semi-continuous invariants

Given a singular germ $(C, 0) \subset (\mathbb{C}^2, 0)$ and its normalization $\tilde{C} \overset{\nu}{\to} C$ the following are some simplest topological invariants: μ the Milnor number, $mult$ the multiplicity, the $\delta := \dim \nu_* \mathcal{O}_{\tilde{C}}/\mathcal{O}_C$ invariant (aka the genus discrepancy, virtual number of nodes etc.), the κ invariant (the multiplicity of intersection of the curve with its generic polar), $C = \bigcup C_i$ the branch decomposition, r the number of branches, Γ the resolution tree with multiplicities $\{m_i\}$. For the definitions and properties cf. [**GLS-book**, I.3.4]. Some classical formulas are:

$$(3) \qquad \delta = \frac{\mu + r - 1}{2}, \quad \delta = \sum_{p \in \Gamma} \frac{m_p(m_p - 1)}{2}, \quad \delta(C) = \sum \delta(C_i) + \sum_{i<j} <C_i C_j>, \quad \kappa = \mu + mult - 1$$

For the collection of types $(\mathbb{S}_1..\mathbb{S}_k)$ let $\tau^{es}(\mathbb{S}_1..\mathbb{S}_k) = \sum_i \tau^{es}(\mathbb{S}_i)$ be the codimension of the corresponding equsingular stratum in the space of its versal deformation. Let $Sp(\mathbb{S})$ be the spectrum [**Steenbrink76**] (cf. also [**Kulikov98**, II.8.5]).

EXAMPLE 2.1. $\star$ Let K_p be the ordinary multiple point of multiplicity p. Then:

$$(4) \quad \mu(K_p) = (p-1)^2, \quad \frac{\kappa}{2} = \delta(K_p) = \binom{p}{2}, \quad \tau^{es}(K_p) = \binom{p+1}{2}, \quad Sp(K_p) = t^{-1+\frac{2}{p}} + 2t^{-1+\frac{3}{p}} + 3t^{-1+\frac{4}{p}} + ... + 2t^{1-\frac{3}{p}} + t^{1-\frac{2}{p}}$$

⋆ Let $\mathbb{S}$ be the (topological) type of $x^p + y^q$ (with $p \leq q$). Then:

(5)
$$\mu(\mathbb{S}) = (p-1)(q-1), \ \kappa(\mathbb{S}) = (p-1)q, \ \delta(\mathbb{S}) = \frac{(p-1)(q-1)+gcd(p,q)-1}{2},$$
$$\tau^{es}(\mathbb{S}) = \sum_{i=2}^{\lfloor p(1-\frac{2}{q}) \rfloor} \lfloor q(1-\frac{i}{p})-1 \rfloor, \ Sp(\mathbb{S}) = t^{-1} \sum_{i=0}^{p-2} t^{\frac{i+1}{p}} \sum_{j=0}^{q-2} t^{\frac{j+1}{q}}$$

Some classical obstructions are:
• μ, κ, δ, $\mu - \delta$, *mult* do not increase under small deformations (e.g. [**Buchweitz-Greuel80**, theorem 6.1.7]). In particular, for $\delta = const$ deformation, $(r-1)$ is non-decreasing.
• If the *generic* representative of $\mathbb{S}$ can be deformed to a curve with $(\mathbb{S}_1..\mathbb{S}_k)$ then $\tau^{es}(\mathbb{S}) > \tau^{es}(\mathbb{S}_1..\mathbb{S}_k)$.
• The spectrum is semi-continuous in the following sense [**Steenbrink85**]. Let $Sp(\mathbb{S})$ be the spectrum of the central fibre and $Sp(\cup \mathbb{S}_i)$ be the joint spectrum of the generic fibre. Here $Sp(\cup \mathbb{S}_i) = \sum Sp(\mathbb{S}_i)$ so that the multiplicities sum up. For any half-open interval $B_\alpha = (\alpha, \alpha+1]$ let Sp_B be the number of spectral values in the interval (counting the multiplicities). Then, for every such half-open interval: $Sp_B(\mathbb{S}) \geq Sp_B(\cup \mathbb{S}_i)$
• The local Bezout theorem can be used as follows. It can be often proved that if a deformation $\mathbb{S} \to (\mathbb{S}_1..\mathbb{S}_k)$ exists then it must be realizable by a curve of small degree. Here one uses the strong criterion of [**Shustin87, GLS96**]:
Let C_d be a curve of degree d, with the singularity type $\mathbb{S}$, such that $\tau(\mathbb{S}) < 4d - 4$. Then the mini-versal deformation of $\mathbb{S}$ is induced from the parameter space of curves of degree d (i.e. $|\mathcal{O}_{\mathbb{P}^2}(d)|$).
When this criterion is applicable, one can try to show that a curve (of the small degree) with singularities $\mathbb{S}_1..\mathbb{S}_k$ must be reducible and non-reduced, thus forbidding the deformation.

EXAMPLE 2.2. Can an ordinary multiple point of multiplicity 4 (named X_9) be deformed to two ordinary multiple points of multiplicity 3 (named D_4)?
Note that $\tau^{es}(X_9) = 8 = \tau^{es}(2D_4)$ so the generic representative of X_9 does not deform to $2D_4$. The simplest classical invariants do not give any restriction: $\mu(X_9) = \tau(X_9) = 9 > 8 = \mu(2D_4) = \tau(2D_4)$, $\delta(X_9) = 6 = 2\delta(2D_4)$. However, by the previous statement, if the deformation exists it must be representable by curves of degree 4. But by Bezout theorem a curve of degree 4 with $2D_4$ must have a double line as a component. So, the deformation $X_9 \to 2D_4$ is impossible.
The deformation is also prohibited by the semi-continuity of the spectrum (cf. example 2.1), Hirzebruch's inequality (1) and the dual graph (proposition 1.5).

Heavy restrictions arise from the integer cohomology of the Milnor fibre. The cohomology ring is encoded by the lattice (product in the middle cohomology) [**AGLV-book**, I.1.6]. So the obstruction is:
If a representative of $\mathbb{S}$ deforms to a curve with singularities $(\mathbb{S}_1..\mathbb{S}_k)$ then the direct sum of the cohomology lattices of $(\mathbb{S}_1..\mathbb{S}_k)$ embeds into the lattice of $\mathbb{S}$. Correspondingly, there are some bases of vanishing cycles for $(\mathbb{S}_1..\mathbb{S}_k)$, $\mathbb{S}$ such that the collection of Dynkin diagrams of $(\mathbb{S}_1..\mathbb{S}_k)$ is obtained from that of $\mathbb{S}$ by removing some vertices.
This restriction is difficult to apply, since it is very difficult to check that one lattice cannot be embedded into another. Alternatively, one should check that the Dynkin diagram of $\mathbb{S}$ in all the possible bases cannot be decomposed. A very painful task even for the ADE types. (Various results on the behavior of Dynkin diagrams under the base change and specific criteria can be found e.g. in [**Brieskorn-book, Urabe-book**].)
In this work we use this obstruction only partially by the signature of the middle homology. Namely, consider the stabilization of the curve singularities to surfaces, i.e. instead of the curve $f(x,y) = 0$ one has the suspension $f(x,y) + z^2 = 0$, the surface singularity in $\mathbb{C}^3$. Then the intersection form in the middle (co)homology is symmetric. And if the signature forbids a collision for surfaces then so is for curves.
Given a lattice $(L, <, >)$ form the corresponding vector space $V = L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ with the induced quadratic form $Q : V \to \mathbb{R}$ (by $v \to < v, v >$). Thus if $L_1 \subset L_2$ one has $V_1 \subset V_2$ and $Q_1 = Q_2|_{V_1}$. This brings a restriction on the signature:

PROPOSITION 2.3. • *Let $(\mu_+(Q_i), \mu_0(Q_i), \mu_-(Q_i))$ be the number of positive/zero/negative eigenvalues. Then $\mu_+(Q_1) \leq \mu_+(Q_2)$, $\mu_-(Q_1) \leq \mu_-(Q_2)$ and $\mu_0(Q_1) \leq \mu_0(Q_2) + dim(V_2) - dim(V_1)$.*
• *Let $Q_i^{(N)} = Q_i \mod N$ be the quadratic form reduced modulo N. Then: $\mu_0(Q_1^{(N)}) \leq \mu_0(Q_2^{(N)}) + dim(V_2) - dim(V_1)$.*

Proof: Here only the bound on μ_0 possibly needs an explanation. Fix any base for L_1 and extend it to a base of L_2. Let Q_1, Q_2 be the matrices of the intersection forms in this basis. So, Q_1 is a submatrix of Q_2: $Q_2 = \begin{pmatrix} Q_1 & A \\ B & C \end{pmatrix}$. Apply the conjugation $Q_2 \to U_1 Q_2 U_2$ with $U_i \in GL(V_2)$ preserving $V_1 \subset V_2$ (so that the block structure is preserved). By such conjugation can "diagonalize" the blocks A, B:

$$
(6) \qquad Q_2 = \begin{pmatrix} Q_1 & \begin{pmatrix} * & 0 \\ 0 & * \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \end{pmatrix} & ** \end{pmatrix}
$$

Hence $\dim Ker(Q_1) - \dim Ker(Q_1) \leq dim(V_2) - dim(V_1)$ ∎

The numbers $\mu_\pm, \mu_0$ can be computed from the spectrum: $\mu_- = \sharp(Sp \cap (-\frac{1}{2}, \frac{1}{2}))$ and $\mu_+ = 2 \times \sharp(Sp \cap (-1, -\frac{1}{2}))$.
For the quasi-homogeneous case they can be calculated as follows [**Steenbrink77**].

Let $\{f(x_1, x_2, x_3) = 0\} \subset \mathbb{C}^3$ be a quasi-homogeneous surface singularity, with x_i of weight w_i and f of weight 1. Take the monomial basis for its Milnor algebra $\mathbb{C}[x_1, x_2, x_3]/j_f$: $e_1...e_\mu$. For each such monomial define the weight function $l(x_1^{n_1}..x_3^{n_3}) := \sum w_i(n_i + 1)$.

PROPOSITION 2.4. [**Steenbrink77**, theorem 2]*Given the germ of a quasi-homogeneous surface $\{f = 0\} \subset (\mathbb{C}^3, 0)$. Let $M = M_+ \oplus M_0 \oplus M_-$ be the decomposition of the (co)homology lattice according to the eigenvalues of the intersection product. Then:*
$$
(7)
$$
$$
M_0 = \{e_i \frac{dxdydz}{df} | l(e_i) \in \mathbb{Z}\}, \ M_+ = \{e_i \frac{dxdydz}{df} | l(e_i) \notin \mathbb{Z}, \ \lfloor l(e_i) \rfloor \ is \ even\}, \ M_- = \{e_i \frac{dxdydz}{df} | l(e_i) \notin \mathbb{Z}, \ \lfloor l(e_i) \rfloor \ is \ odd\}
$$

In particular $\mu_+ = |M_+|$, $\mu_0 = |M_0|$, $\mu_- = |M_-|$.

EXAMPLE 2.5. Let $\mathbb{S}$ be the topological type of $x^p + y^{pk} + z^2$ (for $k \geq 1$). Direct calculation gives:
• p even: $\mu_0 = p - 2$, $\mu_+ = \frac{p-2}{2}(\frac{pk}{2} - 2)$ and $\mu_- = \frac{3p-2}{4}kp - (p-1)$
• p odd, k even: $\mu_0 = p - 1$, $\mu_+ = (\frac{p-1}{2})^2 k - (p-1)$ and $\mu_- = \frac{(p-1)(3pk+k-4)}{4}$
• p odd, k odd: $\mu_0 = 0$, $\mu_+ = (\frac{p-1}{2})^2 k - \frac{p-1}{2}$ and $\mu_- = \frac{(p-1)(3pk+k-2)}{4}$

EXAMPLE 2.6. Can the singularity of type J_{10} (locally $x^3 + \alpha xy^4 + y^6$, α is the modulus) be deformed to three tacnodes ($3A_3$)? Since $\tau^{es}(J_{10}) = 9 = \tau^{es}(3A_3)$ the generic representative cannot be deformed. Other invariants are: $\mu(J_{10}) = \tau(J_{10}) = 10 > \mu(3A_4) = \tau(3A_3)$, $\delta(J_{10}) = 6 = \delta(3A_3)$, $\kappa(J_{10}) = 12 = \kappa(3A_3)$.

The local Bezout theorem does not give any restriction since the minimal degree of curve possessing J_{10} is 5.

The deformation is forbidden by the signature of the middle homology. In fact for J_{10} have $(\mu_+\mu_0, \mu_-) = (0, 2, 8)$, for $3A_3$ have $(\mu_+\mu_0, \mu_-) = (0, 0, 9)$. (Recall that the signatures are calculated for the stabilizations: surfaces in $\mathbb{C}^3$.)

The deformation is also forbidden by the spectrum.

EXAMPLE 2.7. • Does the deformation $K_5 \to 3K_3 + A_1$ exist? (Can a representative of the ordinary multiple point of multiplicity 5 be deformed to 3 points of types D_4 and one node?) The invariants as above do not forbid it:

$$
(8) \qquad \begin{array}{l} \delta(K_5) = 10 = 3\delta(K_3) + \delta(A_1), \ \mu_+(K_5) = 2, \ \mu_0(K_5) = 0, \ \mu_-(K_5) = 14 > 13 = 3\mu(K_3) + \mu(A_1), \\ Sp(K_5) = t^{-\frac{3}{5}} + 2t^{-\frac{2}{5}} + 3t^{-\frac{1}{5}} + 4t^0 + 3t^{\frac{1}{5}} + 2t^{\frac{2}{5}} + t^{\frac{3}{5}}, \ Sp(3K_3 \cup A_1) = 3t^{-\frac{1}{3}} + 7t^0 + 3t^{\frac{1}{3}} \end{array}
$$

Hirzebruch's inequality (1) is irrelevant (since $n_{p-2} \neq 0$).
• Let $K_{p,k}$ be the singularity of the type $x^p + y^{pk}$. Does the deformation $K_{4,3} \to 7A_3 + 4A_1$ exist? It is not forbidden by the classical invariants above.

However, in both cases the deformations are forbidden by the dual graph (cf. propositions 6.1 and 6.5).

3. The $\delta = const$ families.

In this note rather than restrict the singularity types, we restrict to the $\delta = const$ deformations. In fact we restrict further:

PROPOSITION 3.1. *Suppose the deformation $\mathbb{S} \stackrel{\delta=const}{\to} (\mathbb{S}_1..\mathbb{S}_k)$ is possible. If all the branches of $\mathbb{S}$ are smooth then all the branches of $(\mathbb{S}_1..\mathbb{S}_k)$ are smooth and the deformation is $\kappa = const$. If at least one branch of $\mathbb{S}$ is non-smooth and the deformation is $\kappa = const$, then at least one branch of $\cup\mathbb{S}_i$ is non-smooth.*

Proof: The $\delta = const$ means, cf. (3): $\mu + r - 1 = \sum(\mu_i + r_i - 1)$. The semi-continuity of κ means: $\mu + p - 1 \geq \sum(\mu_i + p_i - 1)$ (for $p_i = mult(\mathbb{S}_i)$). Thus $0 = p - r \geq \sum(p_i - r)$ implies $p_i = r_i$ i.e. the branches are smooth. ∎

The $\delta = const$ deformations/degenerations are called equi-generic, the $\delta = const$, $\kappa = const$ are called equi-classical [**DiazHarris88**].

The $\delta = const$ deformations/degenerations are very special: they are equinormalizable.

THEOREM 3.2. [**Teissier76**], *cf. also* [**GLS-book**, *Theorem I.2.54*]. *Let $S = \{C_t\}$ be a family of plane curves (considered as a fibred surface). Let $\tilde{S} \to S$ be the normalization of the surface and $\tilde{C}_0 \to C_0$ the corresponding map of the central fibres. Then $\delta(C_0) - \delta(C_{t \neq 0}) = \delta(\tilde{C}_0)$. In particular the family is equinormalizable (i.e. $\tilde{C}_0$ is smooth) iff it is $\delta = const$.*

This key property enables to associate a new obstruction: the dual graph.

4. The dual graph

EXAMPLE 4.1. As a motivation consider the ($\delta = const$) collision $A_{2k-1} + A_{2l-1} \to A_{2k+2l-1}$. Let $S = \{C_t\}_t \to T$ be the fibred surface formed by the degenerating family. S has non-isolated singularities. Consider the normalized surface $\tilde{S} = \{\tilde{C}_t\}_t \to T$. Since the collision is $\delta = const$, the surface is smooth and each fibre is the normalization of the corresponding curve. Let p_1, p_2 and q_1, q_2 be the pre-images of A_{2k-1}, A_{2l-1} (in the generic fibre). As p_1, p_2 are glued by the normalization connect them by a dotted line (with multiplicity k, the local intersection of the branches). Similarly for q_1, q_2. Then the collision can be traced on the normalized surface as the addition of graphs: $q_i \to p_i$, the edges merge (and their weights are added). Pictorially: $\bullet \overset{k}{-} \bullet + \bullet \overset{l}{-} \bullet = \bullet \overset{k+l}{--} \bullet$
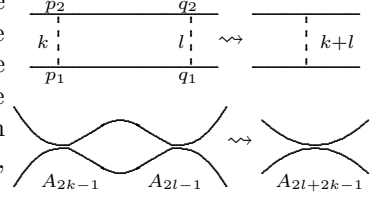
DEFINITION 4.2. *For the singularity $\mathbb{S}$ with smooth branches the dual graph $\Gamma_{\mathbb{S}}$ is a complete graph with weighted edges, such that:*
• *The vertices $\{v_i\}$ of $\Gamma_{\mathbb{S}}$ correspond bijectively to the branches $\{C_i\}$ of $\mathbb{S}$*
• *Let the vertices $v_i, v_j \in \Gamma_{\mathbb{S}}$ correspond to the branches $C_i C_j$. The weight of the edge $\overline{v_i v_j}$ equals $< C_i, C_j >$.*
If $C_i \pitchfork C_j$ (i.e. $< C_i C_j > = 1$) then the weight is omitted.

EXAMPLE 4.3. Below are some graphs for the ordinary multiple point of multiplicity p (denoted by K_p), for the type of $x^p + y^{pk}$ (denoted by $K_p(k)$) etc.
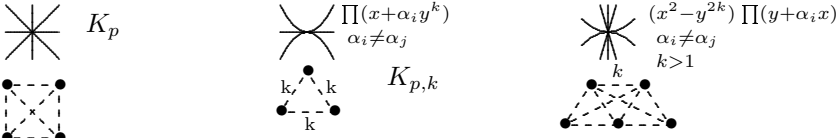
Though defined for a germ, the dual graph depends on the (local embedded topological) singularity type only.

PROPOSITION 4.4. *Let $(C, 0)$ be a germ with smooth branches. The dual graph $\Gamma_{\mathbb{S}(C)}$ is well defined (independent of the representative of $\mathbb{S}$) and is a complete invariant of the local embedded topological type $\mathbb{S}(C, 0)$. In particular:*
• *the number of branches=the multiplicity=the number of vertices of the graph*
• $\frac{\kappa(\mathbb{S})}{2} = \delta(\mathbb{S}) = \sum w(\overline{v_i v_j})$ *(the sum is over all the edges).*

Proof: •$\Gamma_{\mathbb{S}}$ is well defined because the number of branches and the multiplicities of their pairwise intersections are topological invariants. Conversely, from this data the resolution tree is immediately restored.
• Use $\delta(\mathbb{S}) = \sum_{i<j} < C_i, C_j >$, cf. (3) ∎
The weights of the dual graph satisfy some consistency conditions:

LEMMA 4.5. *For any path $< v_1...v_n >$ in the graph one has: $w(\overline{v_1 v_n}) \geq \min\left(w(\overline{v_1 v_2})w(\overline{v_2 v_3})..w(\overline{v_{n-1} v_n})\right)$. In particular, let $w = w(\overline{v_i v_j})$ be the minimal among the weights of edges in the graph. Then for any vertex v_n either $w(\overline{v_i v_n}) = w(\overline{v_i v_j})$ or $w(\overline{v_j v_n}) = w(\overline{v_i v_j})$. Moreover, let Γ' be a graph obtained from a dual graph by removing all the edges of weight $\leq k$. Then Γ' is the (disjoint) union of complete graphs (possibly isolated points).*

This follows from the observation $\min(< C_i C_j >, < C_j C_n >) \leq < C_i C_n >$ (immediate from the resolution tree) and consideration of all the paths $< v_i v_j v_n >$.

The converse is true: this condition is sufficient for a graph to be the dual graph.

LEMMA 4.6. *Let Γ be a complete graph with weighted edges, such that for each triple of vertices $v_i v_j v_k \in \Gamma$ the weights satisfy (possibly after a permutation): $w(\overline{v_i v_j}) = w(\overline{v_i v_k}) \leq w(\overline{v_k v_j})$. Then Γ is the dual graph for some topological type, i.e. $\Gamma = \Gamma_\mathbb{S}$.*

Proof: We construct an explicit representative of $\mathbb{S}$. To each vertex v_i associate an abstract smooth branch-germ $(C_i, 0) \approx (\mathbb{C}^1, 0)$ and embed them into $\mathbb{C}^2$ inductively. Start from $i_1 : (C_1, 0) \hookrightarrow (\mathbb{C}^2, 0)$. Let $i_2 : (C_2, 0) \hookrightarrow (\mathbb{C}^2, 0)$ be an embedding such that $< i_1(C_1) i_2(C_2) >= w(\overline{v_1 v_2})$, and is generic otherwise. Suppose the branches $C_1..C_k$ are embedded, such that $< i_j(C_j) i_l(C_l) >= w(\overline{v_j v_l})$ (for $1 \leq j < l \leq k$).

For v_{k+1} and C_{k+1} consider the integers $\{w(\overline{v_{k+1} v_j})\}_{j=1..k}$. Suppose the maximum is obtained for $w(\overline{v_{k+1} v_l})$ (if such l is non-unique, choose any of them). Embed $i_{k+1} : (C_{k+1}, 0) \hookrightarrow (\mathbb{C}^2, 0)$ such that $< i_{k+1}(C_{k+1}) i_l(C_l) >= w(\overline{v_{k+1} v_l})$ but is generic otherwise.

Then for any $j \leq k$: $< i_{k+1}(C_{k+1}) i_j(C_j) >\leq< i_{k+1}(C_{k+1}) i_l(C_l) >$. So, by the assumption of the proposition: $< i_{k+1}(C_{k+1}) i_j(C_j) >=< i_l(C_l) i_j(C_j) >\leq< i_{k+1}(C_{k+1}) i_l(C_l) >$. But $< i_l(C_l) i_j(C_j) >= w(\overline{v_l v_j}) \leq w(\overline{v_l v_{k+1}})$, thus $< i_{k+1}(C_{k+1}) i_j(C_j) >= w(\overline{v_j v_{k+1}})$. And this proves the criterion. ∎

The dual graphs can be often added and decomposed.

DEFINITION 4.7. *The (complete weighted) graph $\Gamma_\mathbb{S}$ decomposes into the union $\sqcup_i \Gamma_{\mathbb{S}_i}$ (write $\Gamma_\mathbb{S} = \oplus \Gamma_{\mathbb{S}_i}$) if there is a map $\phi : \sqcup_i \Gamma_{\mathbb{S}_i} \to \Gamma_\mathbb{S}$ (vertices-to-vertices, edges-to-edges) surjective on vertices and edges of $\Gamma_\mathbb{S}$ such that:*
- *For any two distinct vertices $v_1, v_2 \in \Gamma_{\mathbb{S}_i}$: $\phi(v_1) \neq \phi(v_2) \in \Gamma_\mathbb{S}$ and $\overline{\phi(v_1), \phi(v_2)} = \phi(\overline{v_1 v_2})$*
- *The weights of edges add up, for $v_1, v_2 \in \Gamma_\mathbb{S}$: $w(\overline{v_1, v_2}) = \sum\limits_{l \in \phi^{-1}(\overline{v_1, v_2})} w(l)$*
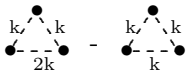
EXAMPLE 4.8. • Every dual graph can be decomposed to the union of K_2's (two vertices and edge of weight 1). This corresponds to the standard deformation of the the singularity to δ nodes.
• Let K_p be the complete graph on p vertices (all weights are one). Direct check shows that K_4 cannot be decomposed to the union of two K_3's. This corresponds to the impossibility of deformation $X_9 \to 2 D_4$.

Sometimes the dual graphs can be also subtracted.

DEFINITION 4.9. *Suppose there exists an embedding $\Gamma_1 \overset{i}{\hookrightarrow} \Gamma_2$ of vertices and edges with non-decreasing weights, i.e. for any two vertices $v_k v_j \in \Gamma_1$ have: $w(\overline{v_k v_j}) \leq w(\overline{i(v_k) i(v_j)})$. Denote by $\Gamma_2 - i(\Gamma_1)$ the weighted graph obtained from Γ_2 by subtracting the weights: $w(\overline{i(v_k) i(v_j)}) - w(\overline{v_k v_j})$ (for edges of Γ_2 in the image of Γ_1). If $w(\overline{i(v_k) i(v_j)}) = w(\overline{v_k v_j})$ then the edge $\overline{i(v_k) i(v_j)}$ is erased. All the isolated vertices are erased too.*

In general $\Gamma_2 - i(\Gamma_1)$ can have several connected components.

EXAMPLE 4.10.   This can be written also as:

(9)
$$\begin{array}{c}\text{k} \overset{\bullet}{\diagup} \diagdown \text{k} \\ \bullet \text{-}\text{-}\text{-} \bullet \\ \text{2k}\end{array} = \begin{array}{c}\text{k} \overset{\bullet}{\diagup} \diagdown \text{k} \\ \bullet \text{-}\text{-}\text{-} \bullet \\ \text{k}\end{array} + \bullet \overset{k}{\cdots} \bullet = \overset{k}{\underset{i=1}{\oplus}} \begin{array}{c}\overset{\bullet}{} \\ \bullet\text{-}\text{-}\text{-}\bullet\end{array} + \overset{k}{\underset{i=1}{\oplus}} \bullet \cdots \bullet$$

This is a particular case of an important operation: *the canonical decomposition* into ordinary multiple points.

Let w be the minimal among the weights of edges of $\Gamma_\mathbb{S}$. Let K_p be the complete graph on $|\Gamma_\mathbb{S}|$ vertices, with each edge of weight 1. Consider the graph $\Gamma_\mathbb{S} - w K_p$. It is immediate from the proposition 4.6 that each connected component of $\Gamma_\mathbb{S} - w K_p$ is the dual graph of some singularity type, i.e. $\Gamma_\mathbb{S} - w K_p = \oplus \Gamma_{\mathbb{S}_i}$. Apply the same procedure to each of $\Gamma_{\mathbb{S}_i}$, till one gets the collection of graphs with edges of weight 1.

DEFINITION 4.11. *The so defined decomposition $\Gamma_\mathbb{S} = \oplus w_i K_{p_i}$ is called the canonical decomposition for the type $\mathbb{S}$.*

An example of canonical decomposition is the last equality in equation (9). Note that the canonical decomposition is well defined and depends on the initial singularity type only.

REMARK 4.12. It is not clear how to define the dual graphs for types with singular branches. A trivial choice is, to ignore the singularities of branches and trace the contacts of branches only (i.e. the minimal number of blowups needed to separate them). Eventhough one looses lots of information, this leads to a new semi-continuous invariant (cf. corollary 5.6).

5. The new obstruction for the $\delta = const$ deformations

Consider $\delta = const$ family of plane curves: $\{C(t)\}_{t \in I}$. Let the central fibre possess only one singular point (at $0 \in \mathbb{C}^2$), with smooth branches. Let the generic fiber possess the singularities at $\{x_i \in \mathbb{C}^2\}$ (by the proposition 3.1 the all the branches are smooth).

Denote the result of the deformation as the map of germs $\bigsqcup (C_i, x_i) \to (C, 0)$. Let $\{\Gamma_{\mathbb{S}_i} := \Gamma(C_i, x_i)\}$ and $\Gamma_{\mathbb{S}} := \Gamma(C, 0)$ be the dual graphs.

THEOREM 5.1. *The $\delta = const$ deformation $(C, 0) \to \bigsqcup (C_i, x_i)$ (all the branches are smooth) induces the decomposition $\Gamma_{\mathbb{S}} = \oplus \Gamma_{\mathbb{S}_i}$.*

Proof: We should construct the needed map (cf. definition 4.7). Use Teissier's theorem 3.2.
Construction of the map. Let $\{\tilde{C}(t)\}_{t \in I} \to \{C(t)\}_{t \in I}$ be the normalization of the family. Namely, for every t, the curve $\tilde{C}(t)$ is smooth and the map $\tilde{C}(t) \xrightarrow{\nu_t} C(t)$ is the normalization. Let $(C, 0) = \bigcup (C^\alpha, 0)$ be the branch decomposition and $\bigsqcup (\tilde{C}^\alpha, 0^\alpha) \xrightarrow{\bigcup \nu_\alpha} \bigcup (C^\alpha, 0)$ the corresponding preimages.

As the singular points collide, the preimages of each of them in the normalized family $\{\tilde{C}_t\}_{t \in I}$ converge to one of the points 0^α. This defines the map π_i of the vertices of each graph $\Gamma(C_i, x_i)$ to the graph of the central fibre $\Gamma(C, 0)$.

Properties of the map. Let $(C^\alpha, 0)$ be a (smooth) branch of the central fibre, let $(\tilde{C}^\alpha, 0^\alpha)$ be the corresponding branch on the normalized surface. Let $\bigsqcup \tilde{C}_i^\alpha(t)$ be the normalization of all the branches that merge to $\tilde{C}^\alpha$. Correspondingly the (smooth) branches $\bigcup C_i^\alpha$ specialize to C^α. But C^α is smooth (and reduced), thus the branches $\{\nu(\tilde{C}_i^\alpha)\}_i$ cannot intersect.

For the corresponding graph this means: let v_i^α, v_j^α be two vertices of $\bigsqcup_k \Gamma_k$, sent by $\bigcup \pi_k$ to the same vertex $v^\alpha \in \Gamma_{\mathbb{S}}$. Then they are not connected (i.e. belong to distinct complete subgraphs). Therefore no edge of $\bigsqcup_k \Gamma_k$ is contracted by $\bigcup \pi_k$.

Similarly, note that $< C^\alpha C^\beta > \geq < \left(\bigsqcup \nu(\tilde{C}_i^\alpha(t)) \right) \left(\bigsqcup \nu(\tilde{C}_j^\beta(t)) \right) >$. But for the singular point with smooth branches $\delta(\bigcup C^\alpha) = \sum < C^\alpha C^\beta >$ (with the sum over all pairs of branches). Correspondingly (for $v^\alpha, v^\beta \in \Gamma_{\mathbb{S}}$):

$$(10) \qquad w(\overline{v^\alpha v^\beta}) = < C^\alpha C^\beta > = < \left(\bigsqcup \nu(\tilde{C}_i^\alpha(t)) \right) \left(\bigsqcup \nu(\tilde{C}_j^\beta(t)) \right) > = \sum_{\substack{\pi(v_i^\alpha) = v^\alpha \\ \pi(v_j^\beta) = v^\beta}} w(\overline{v_i^\alpha v_j^\beta})$$

Finally the map is surjective as no new branches are created in $\delta = const$ collisions. ∎

EXAMPLE 5.2. To illustrate the use of the last proposition, consider the collision of δ nodes. So we have δ graphs (each being just an edge with two vertices). From these building blocks we should glue a complete graph with weighted edges (such that the weights are added). Below are some examples for low δ.

(11)

Note that in the case of 5 nodes we discard the possibility as this graph is not a dual graph of any singularity (cf. proposition 4.5). Similarly for 6 nodes we discard the case , while for 7 nodes we discard and .

The obstruction imposed by the dual graph is stronger than some others and in particular provides a bound on the jump of the Milnor number.

PROPOSITION 5.3. *Let $(\mathbb{S}_1..\mathbb{S}_k), \mathbb{S}$ be the types with smooth branches such that the dual graph decomposes: $\Gamma_{\mathbb{S}} = \oplus \Gamma_{\mathbb{S}_i}$. Then $\delta(\mathbb{S}) = \sum \delta(\mathbb{S}_i)$, $\kappa(\mathbb{S}) = \sum \kappa(\mathbb{S}_i)$, $mult(\mathbb{S}) \geq \max \left(mult(\mathbb{S}_i) \right)$, $\mu_{\mathbb{S}} > \sum_i \mu_{\mathbb{S}_i}$. Moreover $\mu_{\mathbb{S}} - \sum_i \mu_{\mathbb{S}_i} = \sum (r_i - 1) - (r - 1)$, where $\sum \binom{r_i}{2} \geq \binom{r}{2}$.*

Proof: Here only the bound on the Milnor number should be explained. The equality $\mu_{\mathbb{S}} - \sum_i \mu_{\mathbb{S}_i} = \sum(r_i - 1) - (r - 1)$ arises from $\delta(\mathbb{S}) = \sum \delta(\mathbb{S}_i)$. The bound $\sum \binom{r_i}{2} \geq \binom{r}{2}$ arises from the surjectivity of the map $\sqcup \Gamma_{\mathbb{S}_i} \to \Gamma_{\mathbb{S}}$. ∎
It seems that the obstruction imposed by the dual graph is not implied by any known obstructions, in particular it is not weaker than the spectrum (cf. example 2.7).

REMARK 5.4. Unfortunately the necessary condition from the dual graph is far from being sufficient. For example, the deformation $J_{10} \to 3A_3$ is impossible (cf. example 2.6), but the corresponding graph certainly decomposes:

(12)

The conditions imposed by the dual graph are not sufficient even for deformations of an ordinary multiple point into ordinary multiple points. Indeed, there are classical examples of the decompositions of complete graphs: $K_p \to \oplus K_{p_i}$ with each $p_i > 3$. Contrary to Hirzebruch's inequality (1) for arrangement of lines.

5.1. The case of non-smooth branches in the initial types. For the case of non-smooth branches the $\delta = const$ deformation is still equi-normalizable (theorem 3.2). So, one can consider the dual graph (cf. 4.12) and the idea of the proof of the proposition 5.1 gives a weaker statement:

PROPOSITION 5.5. *Any $\delta = const$ deformation $\mathbb{S} \to \oplus \mathbb{S}_i$ induces the surjective map (on edges and vertices) $\oplus \Gamma_{\mathbb{S}_i} \to \Gamma_{\mathbb{S}}$.*

The map contracts some edges of the graphs $\Gamma_{\mathbb{S}_i}$ (so it is not weight additive) and does not give a significant restriction. However one has an immediate

COROLLARY 5.6. *Let $\mathbb{S} \to \oplus \mathbb{S}_i$ be any $\delta = const$ deformation, with $r, \{r_i\}$ the corresponding numbers of branches. Then $\binom{r}{2} \leq \sum \binom{r_i}{2}$.*

This follows just from counting the number of edges in $\oplus \Gamma_{\mathbb{S}_i}$ and $\Gamma_{\mathbb{S}}$.
Note that this strengthens the criterion from [**Buchweitz-Greuel80**, theorem 6.1.7]: $r - 1 \leq \sum(r_i - 1)$.

To get some restrictions on the possible results of deformations, start from smoothing the branches:

PROPOSITION 5.7. *For every singularity type $\mathbb{S}$ there exists a canonically defined type $\mathbb{S}^{def}$ with smooth branches such that $\mathbb{S}$ can be deformed to $\mathbb{S}^{def}$ and $\delta(\mathbb{S}^{def}) = \delta(\mathbb{S})$. This deformation preserves the multiplicity.*

Proof: Choose a branch, apply the minimal number of blowups till the branch becomes smooth (but tangent to the exceptional divisor). Once the branch is smooth, deform it to intersect the divisor transversally (while the intersection numbers with other branches are preserved). Note that this leaves all other branches intact. Do the same with all the branches. So, we get a point with smooth branches and by construction the deformation is canonical. ∎

So, can apply the following procedure.
• Smoothen the branches of each singular point (canonically) in the $\delta = const$ way. In this way from each graph $\Gamma(C)$ we get $\Gamma(C^{def})$ with a prescribed contraction map $\Gamma(C^{def}) \to \Gamma(C)$ defined as follows.
Each vertex of $\Gamma(C^{def})$ corresponds to a smooth branch C_α^{def} of C^{def}. Under the specialization $C^{def} \to C$ this branch is transformed to a branch C_α of C. So a vertex corresponding to C_α^{def} is sent to the vertex of C_α.

• We have a collection of singular points with smooth branches, whose graphs have marked subgraphs. Perform all the possible smooth-to-smooth collisions. Preserve the markings of the subgraphs.

• To each resulting singularity type apply the degeneration corresponding to the contraction of the marked subgraphs (if possible). If such a degeneration is possible and the Milnor number of the resulting type is bigger than the sum of Milnor numbers of the initial types, such a type is a potential candidate.

6. Applications

6.1. Deformations of ordinary multiple points.

PROPOSITION 6.1. • *Let $K_p \to (\mathbb{S}_1 .. \mathbb{S}_k)$ be a $\delta = const$ deformation. Then each $\mathbb{S}_i$ is an ordinary multiple point.*
• *The ($\delta = const$) deformation $K_p \to \bigcup_{i=1}^{k} K_{p_i} + \left(\binom{p}{2} - \sum \binom{p_i}{2} \right) K_2$ with $\{p_i \geq k - 1\}$ and $\{p_i \geq 3\}$ is possible iff $p + \binom{k}{2} \geq \sum p_i$*

Proof: • By proposition 3.1 all the $\mathbb{S}_i$ have smooth branches. Thus the dual graph forces each $\mathbb{S}_i$ to be an ordinary multiple point.

• Note that δ of the both sides is equal. Thus to prove sufficiency it's enough to construct the arrangement. Let K_k be a (non-embedded) complete graph and $\pi : K_k \to \mathbb{P}^2$ is its projection, such that π is injective on vertices. Let $\pi(K_k)$ be the arrangement of lines in the plane, generated by the image of K_k. For a vertex $v_i \in K_k$ let $\pi(v_i) \in \pi(K_k)$ be the corresponding ordinary multiple point. For each such point add $p_i - (k-1)$ lines through this point (but generic otherwise). So one has an arrangement of $\sum (p_i - (k-1)) + \binom{k}{2} = \sum p_i - \binom{k}{2}$ lines with ordinary multiple points of multiplicities $p_1..p_k$ and some nodes. Finally, add $p - \sum p_i + \binom{k}{2}$ generic lines. So one has an array of p lines with $\oplus K_{p_i}$ and the needed number of lines.

To prove the necessity of the condition consider the decomposition of the dual graph. We deal with ordinary multiple points, so all the dual graphs are just the complete graphs K_i (the weights of edges are 1). Order $p_1 \geq p_2 \geq ....$ Take a K_{p_1} inside K_p, now should construct the best packing of the remaining $K_{p_2} K_{p_3}....$ Any two of the embedded subgraphs cannot intersect in more than 1 vertex. Suppose the graphs $K_{p_1}...K_{p_{i-1}}$ are embedded. So the embedded K_{p_i} can have at most i vertices common with them, so $p_i - i + 1$ vertices are to be added (by the assumption $p_i \geq k - 1$). Altogether this gives at least $p_1 + (p_2 - 1) + .. = \sum p_i - \binom{k}{2}$ vertices needed to embed $K_{p_1}...K_{p_k}$, hence $p \geq \sum p_i - \binom{k}{2}$.
■
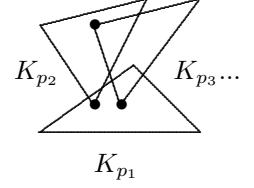
## 6.2. The canonical decomposition into ordinary multiple points.

PROPOSITION 6.2. *Given a type $\mathbb{S}$ with smooth branches, let $\Gamma_{\mathbb{S}} = \oplus n_i K_{p_i}$ be the canonical decomposition of its dual graph (cf. definition 4.11).*

There exists a $\delta = const$ deformation to the collection of ordinary multiple points: $\mathbb{S} \to \bigcup n_i K_{p_i}$ (called the canonical decomposition). The minimal number of ordinary multiple points, to which the type $\mathbb{S}$ can be $\delta = const$ deformed is $\sum n_i$.

Proof: As the canonical decomposition of the dual graph is done in steps it is enough to prove that for each step (i.e. subtraction $\Gamma_{\mathbb{S}} - K_p$) the corresponding deformation exists.

Let $(C,0)$ be a representative of the type $\mathbb{S}$ with the (reduced) tangent cone $T_C = (l_1..l_k)$. Decompose the germ accordingly: $C = \bigcup C_i$, such that $T_{C_i} = l_i$. Note that each C_i can be further locally reducible.

It is enough to prove that each germ $(C_i, 0)$ can be deformed into two singular points: an ordinary multiple point at the origin and the prescribed singularity at some other generic point (cf. the picture).
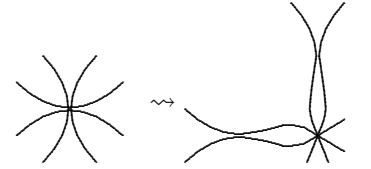
So, consider one germ $C_i = \{f(x,y) = 0\}$. Orient the tangent line along the y axis, so that $f = \prod_j (y(1 + f_j(x,y)) + x^2 g_j(x))$. Here $f_j \in m_{xy} \subset \mathbb{C}[[x,y]]$. Thus $(1 + f_j(x,y))$ is invertible and the defining series of the germ can be written as $\prod_j (y + \frac{x^2 g_j(x)}{1 + f_j(x,y)})$. Expand in powers of f, then the germ can be represented as $\prod_j (y(1 + x^2 f_j(x,y)) + x^2 g_j(x))$ (for some new f, g, such that $f_j \in m_{xy}$). Iterating this procedure one arrives at the expression $\prod_j (y(1 + x^{2N_j} f_j(x,y)) + x^2 g_j(x))$ for arbitrary large numbers $\{N_j\}$. And then, by finite determinacy, the term $x^{2N_j} f_j(x,y)$ is irrelevant. So, can assume the germ is given in the form: $\prod_j (y + \sum_k a_{jk} x^k)$.

Consider the deformation: $f_\epsilon = \prod_j (y + \sum_k a_{jk}(x - \epsilon)^{k-1} x)$. Then at the origin $f_{\epsilon \neq 0}$ defines an ordinary multiple point. And the germ $(f_\epsilon, 0)$ is precisely of the type whose dual graph is $\Gamma_f - K_p$. ■

REMARK 6.3. A natural question is: whether any other $\delta = const$ deformation of a singularity into ordinary multiple points factorizes through the canonical one? Or, at least, whether any other deformation corresponds to the further decomposition of the dual graphs: $\bigcup n_i K_{p_i} \to ...$? The following is a counterexample.

The canonical decomposition for the type $\mathbb{S} = (x^4 + y^{4p})$ is: $\mathbb{S} \to \overset{p}{\oplus} X_9$. Suppose the deformation $\mathbb{S} \to nD_4 + (6p - 3n)A_1$ exists. For $n \leq p$ it can be factorized as $\mathbb{S} \to pX_9 \to nD_4 + (6p - 3n)A_1$. But the case $n > p$ is the negative answer for both questions above, since the dual graph K_4 (of X_9) does not decompose into $2K_3$ (for D_4).

It remains to show that the deformation $\mathbb{S} \to nD_4 + (6p - 3n)A_1$ exists, e.g. for $n = p + 1$. The following construction for $n = 3$, $p = 2$ was given by E.Shustin. Let germ of curve be a line l and 3 conics $C_1..C_3$ such that the conics intersect the line at three points (so three triple points appear). The conics intersect also outside the line, adding $3A_1$.
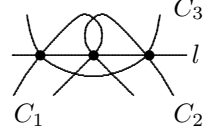
Note that of the four curves any pair intersects locally at two points. Thus as the three triple points merge (and the three nodal points also join them) the family degenerates to 4 (simply) tangent curves, i.e. the type of $x^4 + y^8$.

REMARK 6.4. The proposition does not generalize to the case of the initial type $\mathbb{S}$ with singular branches. Indeed, usually there are many non-equivalent ways to smoothen the branches in a $\delta = const$ way, resulting in different dual graphs.

An interesting question is: whether each deformation of a singularity to the ordinary multiple points factorizes through the smoothing of branches.

6.3. The $\delta = const$ deformations of the type $x^p + y^{pk}$ to A_k's. Denote the type $x^p + y^{pk}$ (i.e. p smooth branches, every two of them being $k-$tangent) by $K_{p,k}$. The corresponding dual graph is the complete graph on p vertices, with weights of all the edges: k.

PROPOSITION 6.5. Let $K_{p,k}$ deform into a bunch of A_i's. Then only A_{2i-1}'s appear. Let $K_{p,k>1} \overset{\delta=const}{\to} \bigcup n_i A_{2i-1}$. Then:

• $n_{i>k} = 0$

• For each i there exists a partition of the set $\{n_i A_{2i-1}\}$ into $\binom{p}{2}$ subsets $\{n_i^{(j)} A_{2i-1}\}_{1 \leq j \leq \binom{p}{2}}$ such that $\sum_{j=1}^{\binom{p}{2}} n_i^{(j)} = n_i$ and $\forall j: \sum_{i=1}^{k} i n_i^{(j)} = k$ (in particular $\sum_{i=1}^{k} i n_i = \binom{p}{2} k$)

•
$$\sum n_i \geq \frac{(p-1)^2 k}{4} + (p-1) - \begin{cases} \frac{pk}{4} : & p \text{ even} \\ 0 : & p \text{ odd, } k \text{ even} \\ \frac{p-1}{2} : & p \text{ odd, } k \text{ odd} \end{cases}$$

.

In particular: $n_k \leq \frac{(p-1)^2}{4} \frac{k}{k-1} - \frac{p-1}{k-1} + \begin{cases} \frac{pk}{4(k-1)} \\ 0 \\ \frac{p-1}{2(k-1)} \end{cases}$ and $2n_1 + n_2 \geq \frac{(p-1)(p-3)k}{12} + (p-1) - \begin{cases} \frac{pk}{4} \\ 0 \\ \frac{p-1}{2} \end{cases}$

Proof: Consider the corresponding dual graph decomposition. Comparison of weights of edges gives $n_{i>k} = 0$. the equality $\sum_{i=1}^{k} i n_i = \binom{p}{2} k$ is just the δ of both sides. The third inequality is obtained from the signatures of the middle homology lattice for $K_{p,k}$ and $\bigcup n_i A_{2i-1}$. In fact $\mu_0(\bigcup n_i A_{2i-1}) = 0 = \mu_+(\bigcup n_i A_{2i-1})$, so the only condition is from comparison of μ_-.

The bounds for n_k, n_1, n_2 are immediate consequence of these 3 conditions. ■

REMARK 6.6. For specific types of the deformations above, the bounds can be slightly refined. For example:

if $K_{p,k} \to nA_3 + (\binom{p}{2}k - 2n)A_1$ then $n \leq \min\left(\binom{p}{2}\lfloor \frac{k}{2} \rfloor, \frac{p^2 k}{4} - (p-1) + \begin{cases} 0 \\ -\frac{k}{4} \\ \frac{p-1}{2} \end{cases}\right)$

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