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On the directionally Newton-non-degenerate singularities of complex hypersurfaces

Dmitry Kerner

Abstract. We introduce a minimal generalization of Newton-non-degenerate singularities of hypersurfaces. Roughly speaking, an isolated hypersurface singularity is called directionally Newton-non-degenerate if the local embedded topological singularity type can be restored from a collection of Newton diagrams. A singularity that is not directionally Newton-non-degenerate is called essentially Newton-degenerate .

For plane curves we give an explicit and simple characterization of directionally Newton-non-degenerate singularities, for hypersurfaces we give some examples.

Then we treat the question: is Newton-non-degenerate or directionally Newton-non-degenerate a property of singular types or of particular representatives. Namely, is the non-degeneracy preserved in an equi-singular family? This is proved for curves. For hypersurfaces we give an example of a Newton-non-degenerate hypersurface whose equi-singular deformation consists of essentially Newton-degenerate hypersurfaces.

Finally, the classical formulas for the Milnor number (Kouchnirenko) and the zeta function (Varchenko) of the Newton-non-degenerate singularity are generalized to some classes of directionally Newton-non-degenerate singularities.

Contents

1. Introduction

We work with germs of complex algebraic (or locally analytic) hypersurfaces in $\mathbb{C}^n$, mostly with isolated singularities. For the standard notions from singularity theory cf. [**AGLV-book**] and [**GLS-book**].

1.1. To every germ of singular hypersurface (with fixed local analytic coordinates) the Newton diagram is associated. A germ $V_f = \{f = 0\} \subset (\mathbb{C}^n, 0)$ is called Newton-non-degenerate (or non-degenerate with respect to its Newton diagram Γ_f) if for each face $\sigma \in \Gamma_f$ the truncation f_σ of f to σ is non-degenerate (i.e. the corresponding hypersurface has no singular points in the torus $(\mathbb{C}^*)^n$). A germ is called generalized Newton-non-degenerate if it is Newton-non-degenerate for some choice of coordinates. Otherwise it is called generalized Newton-degenerate .

The Newton diagram of a Newton-non-degenerate germ is a *complete* invariant of the local embedded topological singularity type of the germ. Namely, if $(V_f, 0)$ and $(V_g, 0)$ are two Newton-non-degenerate germs, such that $\Gamma_f = \Gamma_g$ then they have the same singularity type. This distinguishes the generalized Newton-non-degenerate germs as especially simple to deal with. For them many topological invariants of the singularity can be expressed via the geometry of the Newton diagram in a relatively simple manner. For example:

- the Milnor number [**Kouchnirenko76**] (cf. also [**GLS-book**, I.2.1])
- the modality (with respect to right equivalence) for functions of two variables (conjectured in [**Arnol'd74**, 9.9], proved in [**Kouchnirenko76**, proposition 7.2])
- the zeta function of monodromy [**Varchenko76**] (cf. also [**AGLV-book**, II.3.12])
- the spectrum [**Steenbrink76, Khovanski-Varchenko85**](cf. also [**Kulikov98**, II.8.5])

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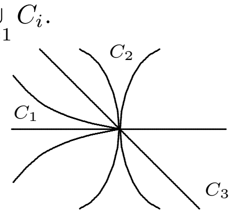
• the Hodge numbers $h^{p,q}$ [**Danilov-Khovanski87**]
• the Bernstein-Saito polynomial [**BGMM89**]

Unfortunately, the condition to be generalized Newton-non-degenerate is very restrictive, even in the case of plane curves.

EXAMPLE 1.1. For the germ $(C,0)\subset(\mathbb{C}^2,0)$ consider the *tangential decomposition*: $C=\bigcup\limits_{i=1}^{k}C_i$. Here each C_i has unique tangent line l_i (but may contain several branches). So, the tangent cone is $T_C=(l_1^{p_1}..l_k^{p_k})$, where p_i =the multiplicity of $(C_i,0)$ and $\sum p_i=p=$ the multiplicity of $(C,0)$. For example, for ordinary multiple point: $p_1=..=p_k=1$.

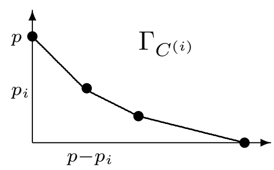


Note that $p_i=1$ iff C_i is a smooth branch, not tangent to any other. If $(C,0)$ is a generalized Newton-non-degenerate germ then $p_i>1$ for *at most* two i's.
Indeed, if $p_i>1$ and C_i is non-degenerate with respect to its diagram then a coordinate axis must be tangent to C_i (to reflect the fact that some monomials are absent). So, in general there are "not enough coordinate axes" to encode the singularity. And many singularities with small Milnor numbers and quite simple defining equations are not generalized Newton-non-degenerate.

On the other hand, among the locally irreducible curves (i.e. branches) the first examples of not generalized Newton-non-degenerate singularities are: $(x^2+y^3)^3+y^{10}$ with $\mu=44$ and $(x^3+y^4)^2+y^9$ with $\mu=38$.

1.2. The present work has originated from the observation that many germs of curves are "almost" generalized Newton-non-degenerate. Namely, many of their topological singularity type (and thus many of of their properties) are reflected on the Newton diagram, one just has to take *several choices* of coordinates.

EXAMPLE 1.2. Continue the previous example. Given the tangential decomposition $C=\bigcup\limits_{i=1}^{k}C_i$. For each $1\le i\le k$ let $C^{(i)}$ be a germ of curve with the tangential decomposition: $C^{(i)}=\Big(\bigcup\limits_{j=1}^{p-p_i}L_j\Big)\cup C_i$. Here $\{L_j\}$ are some lines, such that any two are distinct and none is tangent to C_i (but arbitrary otherwise). Call such a germ: *the directional approximation* of $(C,0)$. (The germ is non-unique, but its topological singularity type is unique and any two such approximations are connected by a $\mu=const$ family.)
If each C_i is generalized Newton-non-degenerate then so is each $C^{(i)}$ and its type can be restored from its Newton diagram (cf. the picture). Therefore, the local embedded topological singularity type of the original germ $(C,0)$ is completely determined from the *collection* of Newton diagrams (corresponding to all the directional approximations). The precise statement is in §2.4.1.



Note that if at least one branch of the curve is not generalized Newton-non-degenerate then no choice of coordinates can help recognize the topological type.
The Newton-non-degenerate directional approximations in higher dimensions are introduced in §2.5.

We generalize this observation by introducing the class of *directionally Newton-non-degenerate* singularities of hypersurface-germs (this in particular contains the generalized Newton-non-degenerate germs). It is not clear currently, how broad this class is or how to classify such germs (except for the case of curves, $n=2$). Thus we start from the end: the directionally Newton-non-degenerate singularities are defined as those germs whose local embedded topological singularity type can be completely determined from the collection of Newton diagrams (the precise definition is in §2). And then we describe some classes of directionally Newton-non-degenerate germs.
For plane curves ($n=2$) we give the complete classification in §2.4.1: a germ is directionally Newton-non-degenerate iff each branch of it is generalized Newton-non-degenerate and the union of any two branches is generalized Newton-non-degenerate.
For hypersurfaces ($n>2$) the situation is much more complicated. Some examples of directionally Newton-non-degenerate are some of the absolutely isolated singularities [**Melle00**] (i.e. those that can be resolved by blowups of points only), cf. §2.4.2.
Germs that are not directionally Newton-non-degenerate are called *essentially Newton-degenerate*.

1.3. A natural question arises: is directionally Newton-non-degenerate (or Newton-non-degenerate) a property of the topological type or of the germ? Namely, suppose a singular type has a directionally Newton-non-degenerate (or Newton-non-degenerate) representative. Is the generic representative of the type directionally Newton-non-degenerate (or generalized Newton-non-degenerate)? Or, is this notion preserved in a $\mu=const$ deformation?

This can be considered as a weakening of the constancy of Newton diagram along the $\mu = const$ stratum (which fails by the example of [**Briançon-Speder75**]).

The answers are yes for the case of curves (corollary 2.11) and no for higher dimensions. We give examples in §2.3 of a Newton-non-degenerate hypersurface germ whose $\mu = const$ deformation is essentially Newton-degenerate (i.e. not directionally Newton-non-degenerate). In fact for most singularity types this is the situation.

So, in general the ND-topological strata of hypersurfaces (i.e. those that can be brought to the given diagram by a locally analytic change of coordinates) are of positive codimension in the classical equisingular strata.

1.4. Once the germ is proven to be directionally Newton-non-degenerate, its singularity type is determined by the associated collection of Newton diagrams. Therefore, every topological singularity invariant can be expressed (at least theoretically) via the geometry of the diagrams. A natural task is to generalize the formulas known for the Newton-non-degenerate case.

In particular in §3.1 the formula for the Milnor number and in §3.2 the formula for the zeta function of monodromy are generalized.

In general we work in the space of all the locally analytic hypersurface germs in $\mathbb{C}^n$. Sometimes we pass to the space of germs of (high) bounded degrees (to have a finite dimensional space, to use algebraicity and Zariski topology). As the singularities are isolated this is always possible by finite determinacy.

Similarly, local changes of coordinates (in general diffeomorphisms) can be always assumed locally analytic.

The Newton diagrams are assumed to be commode (i.e. intersect all the coordinate axes), unless explicitly stated. Denote by f_σ the restriction of the function f to the face $\sigma \in \Gamma_f$.

1.5. Acknowledgements. This work would be impossible without numerous important discussions with E.Shustin and G.M.Greuel. Many thanks are also to V.Goryunov for important advices.

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2. Directionally Newton-non-degenerate hypersurfaces

2.1. Preparations for the definition. Start from the following observation. Let $(V_f, 0) = \{f = 0\} \subset (\mathbb{C}^n, 0)$ be a generalized Newton-non-degenerate isolated singularity. Let $\phi \circlearrowright (\mathbb{C}^n, 0)$ be a local diffeomorphism, such that $\phi^*(f)$ is non-degenerate with respect to its diagram $\Gamma_{\phi^*(f)}$. (By finite determinacy can assume ϕ to be a locally analytic change of coordinates.) In the space of all the hypersurface germs at the origin consider the stratum:

$$\Sigma_{(\phi,\Gamma_{\phi^*(f)})} := \overline{\{(V_g, 0) = (g = 0) \subset (\mathbb{C}^n, 0) |\ \Gamma_{\phi^*(g)} = \Gamma_{\phi^*(f)}\}} \tag{1}$$

Here the closure is taken in the classical topology (for the coefficients of the defining polynomial). Then for the generic point $(V_g, 0) \in \Sigma_{(\phi,\Gamma_{\phi^*(f)})}$ the *local embedded topological types* of $(V_f, 0)$ and $(V_g, 0)$ coincide [**Kouchnirenko76**].

Recall the notion of Newton weight function [**AGLV-book**, I.3.8] associated to every commode Newton diagram. Namely, $\lambda_\Gamma : \mathbb{R}^n_+ \to \mathbb{R}_+$ is defined uniquely by the conditions: $\lambda_\Gamma(\alpha\vec{x}) = \alpha\lambda_\Gamma(\vec{x})$ and $\lambda_\Gamma(\Gamma) = 1$.

Given two diagrams we say $\Gamma_1 \geq \Gamma_2$ if $\lambda_{\Gamma_1}(\Gamma_2) \leq 1$ (or $\lambda_{\Gamma_2}(\Gamma_1) \geq 1$).

Suppose a collection of pairs $\{(\phi_i, \Gamma_i)_i\}$ is given (with $\phi_i \circlearrowright (\mathbb{C}^n, 0)$ local diffeomorphisms and Γ_i some Newton diagrams).

Definition 2.1. *The stratum of hypersurfaces germs, associated to the collection $\{(\phi_i, \Gamma_i)\}$ is the closure of the set of all the germs giving the prescribed diagrams in the prescribed coordinates, i.e.*

$$\Sigma_{\{(\phi_i,\Gamma_i)\}} := \{(g = 0) \subset (\mathbb{C}^n, 0) |\ \forall i :\ \Gamma_{\phi_i^*(g)} \geq \Gamma_i\} \tag{2}$$

Proposition 2.2. *For any collection $\{(\phi_i, \Gamma_i)_i\}$ as above the associated stratum $\Sigma_{\{(\phi_i,\Gamma_i)\}}$ is a (non-trivial) linear subspace of the space of all the germs. (In particular it is closed, irreducible and the notion of the general point is well defined.)*

Proof: The condition $\Gamma_{\phi_i^*(g)} \geq \Gamma_i$ means the absence of some monomials in the Taylor expansion of $\phi_i^*(g)$. This says that some directional derivatives vanish: $\sum a_{i_1..i_n}\partial^{i_1}_{x_1}..\partial^{i_n}_{x_n}(g \circ \phi) = 0$. And these conditions are linear in g. ■

2.2. The main definition.

Definition 2.3. *The germ $(V_f,0)=\{f=0\}\subset(\mathbb{C}^n,0)$ is called directionally Newton-non-degenerate if there exist a finite number of coordinate choices (i.e. the diffeomorphisms $\phi_i \circlearrowright (\mathbb{C}^n,0)$) such that the transformed functions $\phi_i^*(f)$ give the collection of the diagrams Γ_i, such that the general point of the stratum $\Sigma_{\{(\phi_i,\Gamma_i)\}}$ corresponds to a hypersurface germ, whose locally embedded topological type is that of $(V_f,0)$.*

General here means: lying in the complement of a proper analytic subset.

Example 2.4. • Every generalized Newton-non-degenerate germ is directionally Newton-non-degenerate. In this case, by definition, just one pair (ϕ,Γ) suffices.
• Let $C=\overset{k}{\underset{i=1}{\cup}} C_i$ be the tangential decomposition of a plane curve singularity (cf. example 1.1). If each of C_i is generalized Newton-non-degenerate then C is directionally Newton-non-degenerate. Indeed, make k choices of coordinates with $\hat{y}$ axis generic and $\hat{x}$ axis chosen such that the germ C_i is Newton-non-degenerate. Then get the collection of Newton diagrams similar to those of example 1.2. Obviously, this collection specifies the topological type uniquely (the diagram Γ_i specifies the type of C_i and the fact that no other branch has the tangent common with C_i).
• The curve germ $(x^2-y^3)(x^2-y^3+x^3)=0$ is the union of two branches, each being Newton-non-degenerate, but the union is not directionally Newton-non-degenerate. It is easy to see that the germ $(x^2-y^3)(x^2+y^3)=0$ has the same Newton diagram as the original germ for any choice of coordinates.

Proposition 2.5. *(Consistency of the definition.) Let $(V_f,0)\subset(\mathbb{C}^n,0)$ be a directionally Newton-non-degenerate germ and $\{(\phi_i,\Gamma_i)_{i=1..k}\}$ a collection of pairs fulfilling the condition of the definition (i.e. specifying the topological type of V_f uniquely). Then for any additional pair $(\phi_{k+1},\Gamma_{k+1})$ the collection $\{(\phi_i,\Gamma_i)_{i=1..k+1}\}$ also fulfills the condition of the definition.*

Proof: As f is directionally Newton-non-degenerate all the point of $\Sigma_{\{(\phi_i,\Gamma_i)\}_{i=1..k}}$ (except for a proper analytic subset) have the same singularity type as $(V_f,0)$. In particular any small deformation of V_f inside $\Sigma_{\{(\phi_i,\Gamma_i)\}_{i=1..k}}$ is equi-singular. Thus all the small deformations of V_f in $\Sigma_{\{(\phi_i,\Gamma_i)\}_{i=1..k+1}}$ are equi-singular too. ■

Note that the minimal number of the coordinate choices (and the associated diagrams) needed to recover the singularity type of a directionally Newton-non-degenerate singularity can be arbitrarily big (even in the case of plane curves).

2.3. Germs vs types. (Continuation of §1.3.) For $n\geq 3$ being directionally Newton-non-degenerate/generalized Newton-non-degenerate is a property of germs (or of analytic singularity types) but not of the topological types. The examples below are based on two ideas: playing with several low singularities of the tangent cone (for the case of generalized Newton-non-degenerate) or playing with one high singularity of the tangent cone (for the case of directionally Newton-non-degenerate).

Example 2.6. Consider the super-isolated singularity $V_0=\{f_p+f_{p+1}=0\}\subset(\mathbb{C}^3,0)$ where f_{p+1} is generic and the projective curve $\{f_p=0\}\subset\mathbb{P}^2$ has three cusps (assume p is big enough). Arrange the coordinates such that the cusps are at $\hat{x}=(1,0,0)$, $\hat{y}=(0,1,0)$ and $\hat{z}=(0,0,1)$. Note that by now all the $GL(3)$ freedom is exhausted (up to permutations). To make V_0 Newton-non-degenerate assume that the tangents to the cusps are oriented along the coordinate axes, e.g. $z^{p-3}(zx^2+y^3)+x^{p-3}(xy^2+z^3)+y^{p-3}(yx^2+z^3)$.

Let V_t be the equi-singular family, with the cusps staying at their points $\hat{x},\hat{y},\hat{z}$, but their tangents changing freely. For example: $f_t(x,y,z)=z^{p-3}(z(x-ty)^2+y^3)+x^{p-3}(x(y+tz)^2+z^3)+y^{p-3}(y(x-tz)^2+z^3)$.

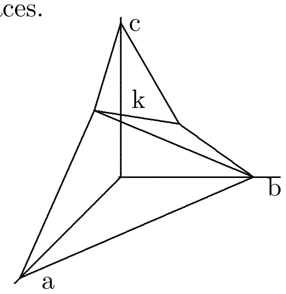
Then $V_{t\neq 0}$ is directionally Newton-non-degenerate but not generalized Newton-non-degenerate. Indeed, to bring $V_{t\neq 0}$ to a Newton-non-degenerate form one should keep the cusps at the points $\hat{x},\hat{y},\hat{z}$ and at the same time keep their tangents along the axes. And this is impossible as only $GL(3)$ transformations are relevant.

Example 2.7. Consider the family of surfaces $f_t=f_5+f_6=x^5+z(zx+ty^2)^2+y^5+z^6$. The projectivized tangent cone of these surfaces is the plane quintic $\{f_5=0\}\subset\mathbb{P}^2$ with one A_4 point at $(0,0,1)$. Note that $Sing(f_5=0)\cap(f_6=0)=\varnothing$. So, this is a super-isolated singularity (cf. [**AB-L-MH06**]). Thus $\mu=68=(5-1)^3+4$ (see §3.1 for the general formula). The family is equisingular in t, e.g. because each surface V_t is resolved by one blowup of the origin and the type of exceptional divisor is independent of t.

The singularity $V_{t=0}$ is Newton-non-degenerate (by direct check). For $t\neq 0$ the singularity is not generalized Newton-non-degenerate. To show this, we prove that the restriction of f_t to the face $Span(x^5,z^3x^2,zy^4,y^5)\subset\Gamma$ is degenerate for any choice of coordinates. Let $\phi\circlearrowright(\mathbb{C}^3,0)$ be a local diffeomorphism. As we are interested in the face whose monomials correspond to the tangent cone, the non-linear part of ϕ is irrelevant. So, assume $\phi\in GL(\mathbb{C}^3)$

and acts on $\mathbb{P}T_S = \{f_5 = 0\} \subset \mathbb{P}^2$. Thus the goal is to bring the singularity of this quintic to the Newton-non-degenerate form. But this is impossible for $t \neq 0$, since the non-linear (quadratic) transformation in local coordinates is needed. In other words, this is because the type A_4 is so-called "non-linear", cf. [**Kerner06**].

2.3.1. *Deformation to essentially Newton-degenerate.* In the last example all the fibres are directionally Newton-non-degenerate (by corollary 2.13). The deformation whose generic fibre is essentially Newton-degenerate (i.e. not directionally Newton-non-degenerate) is a simple modification: one changes the inclinations of the face on which the degeneration occurs ($Span(x^5, z^3x^2, zy^4, y^5)$ in the last example) and adds some other faces.



Consider the hypersurface $f = x^a + y^b + z^c + z^k(zx + y^2)^2$. (For $(a, b, c, k) = (5, 5, 6, 1)$ one has the previous example.) Suppose (a, b, c, k) are such that the Newton diagram consists of the three faces (cf. the picture): $Conv(x^a, x^2z^{k+2}, y^b)$, $Conv(y^b, x^2z^{k+2}, y^4z^k)$ and $Conv(x^2z^{k+2}, y^4z^k, z^c)$. This can be fulfilled by some convexity conditions.
Assume further that $a < b < k + 4 < c$ and also: if $\phi \circlearrowleft (\mathbb{C}^2, 0)$ is any locally analytic transformation whose linear part is identity (i.e. $(x, y, z) \to (x + \phi_x, y + \phi_y, z + \phi_z)$ with $\phi_i \in m^2$) then $\Gamma_f = \Gamma_{\phi^*(f)}$. This can be achieved e.g. if for each face all the slopes (with all the coordinate hyperplanes) are bounded $\frac{1}{2} < \tan(\alpha) < 2$.
In total, all the restrictions above are implied by the following inequalities:

$$b < \frac{k+2}{1-\frac{2}{a}} < \frac{k}{1-\frac{4}{b}} < \min(c, 2b),\ b < a < \min(2b, k+4) < c < k+6 \tag{3}$$

This implies $c = k + 5$ and $k > 10$. We consider (possibly the simplest case): $f_t = x^{14} + y^{13} + z^{16} + z^{11}(zx + y^2)^2$. By direct check this family is equisingular (e.g. $\mu_t = 2220 = const$, can be calculated using [**GPS-Singular**]).

The generic fibre $f_t^{-1}(0)$ is essentially Newton-degenerate by the following proposition.

PROPOSITION 2.8. *Let $f_{t\neq 0}$ as above and $g = x^{14} + y^{13} + z^{16} + z^{11}(zx)^2$. Then $f_{t\neq 0}$ and g have the same Newton diagram in any coordinate system.*

Proof: Let $\phi \circlearrowleft (\mathbb{C}^2, 0)$ be a locally analytic change of coordinates whose linear part is identity. By the construction it preserves the Newton diagram. Therefore it's enough to consider only *linear* coordinate changes. But then only the monomials x^a, y^b are relevant and their coefficients are the same in both cases. ■

In fact, as one sees, every non-linear generalized Newton-non-degenerate or directionally Newton-non-degenerate singularity of curves leads to a similar example. And since most of the singularity types of curves are non-linear, we see that this example is typical.

2.4. Criteria for directional Newton-non-degeneracy.

2.4.1. *Criteria for curves.* For curves it is possible to give a very explicit equivalent definition of a directionally Newton-non-degenerate germ.

PROPOSITION 2.9. *Let $C = \cup_i C_i$ be the tangential decomposition. C is directionally Newton-non-degenerate iff each C_i is directionally Newton-non-degenerate. And C_i is directionally Newton-non-degenerate iff the two conditions are satisfied:*
- *Each branch of C_i is generalized Newton-non-degenerate (so, locally it is of the type $x^p + y^q$ with $(p, q) = 1$).*
- *The union of any two branches is a generalized Newton-non-degenerate singularity. Namely, there does not exist a pair of singular branches in C_i with local equations (in some coordinates): $(x^p + y^q + ..)(x^p + y^q + ..)$. Here the dots mean higher order terms (i.e. monomials lying over the Newton diagram).*

Proof: The first equivalence is obvious. Regarding the second:
$\Rrightarrow$ If C_i contains a branch which is not generalized Newton-non-degenerate then for any choice of coordinates the Newton diagram cannot record the type of this branch. Indeed, among the smooth arcs $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ let γ_m be such that the local degree of intersection $< C_i, \gamma_m >$ is maximal. Choose γ_m as one coordinate axis, choose the second axis generically. Let Γ_{C_i} be the corresponding Newton diagram, it consists of one segment $(ar, 0)(0, br)$ for $gcd(a, b) = 1$, $1 < a < b$ (so, $C_i = \{x^{ar} + .. + y^{br} + .. = 0\}$). Let C' be the generic curve with such a diagram (in particular C' is non-degenerate). Then in any coordinates C, C' have equal Newton diagrams. But, of course, they have distinct singularity types.

The same applies to the case of $(x^p + y^q + ..)(x^p + y^q + ..)$: no choice of coordinates can distinguish this from the pair of branches $(x^p + y^q + ..)(x^p - y^q + ..)$ (which is certainly of different type).
$\Lleftarrow$ We should prove that by choosing different coordinates the topological type of C_i is determined by the collection of Newton diagrams. This amounts to the type of each branch and their intersection numbers.

Given any smooth branch in C_i, rectify it. Namely, choose the coordinates in which it is a line $y=0$, so $f_i = y(..)$ and the Newton diagram is non-commode. This fixes uniquely the intersection numbers of this branch with all the other branches.

So, it remains to consider only the singular branches and their intersections. Given a singular branch (generalized Newton-non-degenerate by assumption) bring to a Newton-non-degenerate form. Then its topological singularity type is fixed from the diagram. Finally, let C_1, C_2 be two singular branches, bring one of them to a Newton-non-degenerate form $x^{p_1} + y^{q_1} + ... = 0$, with $1 < p_1 < q_1$ (dots mean higher order terms). Suppose for such choice of coordinates the equation of the second branch is $(x + \sum_{j>1} \alpha_j y^j)^{p_2} + y^{q_2} + ... = 0$. Then the intersection $< C_1, C_2 >$ is completely determined by the types of branches (i.e. the numbers p_1, q_1, p_2, q_2) and the least j for which $\alpha_j \neq 0$. But this degree is also reflected on the diagram (e.g. as the inclination of the edge). ■

Example 2.10. The proposition immediately provides a simple example of a singularity which is not directionally Newton-non-degenerate: $f = (x^2 + y^3)(x^2 + y^3 + y^4)$. The simplest example of a branch which is not directionally Newton-non-degenerate i.e. a branch which is not generalized Newton-non-degenerate was given in the introduction.

The proposition allows also to answer positively the question from the introduction for curves: being generalized Newton-non-degenerate or directionally Newton-non-degenerate are properties of topological types and not only of their representatives.

Corollary 2.11. Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be a generalized Newton-non-degenerate (or directionally Newton-non-degenerate) germ of curve. Let $(C', 0) \subset (\mathbb{C}^2, 0)$ be a germ of the same (local embedded topological) singularity type as $(C, 0)$. Then $(C', 0)$ is also generalized Newton-non-degenerate (or directionally Newton-non-degenerate).

Proof: • For the directionally Newton-non-degenerate case the statement follows immediately from the proposition 2.9 (as the conditions are on the topological types of the branches).
• As was noticed in the introduction, if $T_C = (l_1^{p_1}..l_k^{p_k})$ is the tangent cone of a generalized Newton-non-degenerate singularity, then $p_i > 1$ for at most two cases. As the tangent cone is topological invariant it suffices, therefore, to consider the case of just one line in the tangent cone. So, let $(C, 0)$ be such a generalized Newton-non-degenerate germ, in particular each branch of C is generalized Newton-non-degenerate (and therefore of type $x^p + y^q$, $(p, q) = 1$).

Let $(C', 0) \subset (\mathbb{C}^2, 0)$ be any other representative of the type of $(C, 0)$. Then there is a 1:1 correspondence between the branches of C, C', preserving the types of the branches and their intersections numbers. Order the branches of C according to the edges of Γ_C (note that the inclinations of the edges are topological invariants). Let the edge corresponding to C_1 intersect the $\hat{y}$ axis at $(0, p)$. Choose coordinates for C', such that the edge of C'_1 does the same. Then the two edges coincide and can continue by induction.

Finally we have chosen the coordinates for C' such that $\Gamma_C = \Gamma_{C'}$. Then C' is Newton-non-degenerate (e.g. by the equality of their Milnor numbers). ■

2.4.2. *The case of hypersurfaces ($n > 2$).* Here the situation is much more complicated. First we show that the tangent cone cone of a hypersurface germ is completely fixed by the collection of Newton diagrams.

Proposition 2.12. *Let $V_f = f^{-1}(0)$ and $V_g = g^{-1}(0)$ be two hypersurface germs in $(\mathbb{C}^n, 0)$. Suppose for any choice of coordinates $\Gamma_f = \Gamma_g$. Let $f = f_p + f_{p+1} + ..$ and $g = g_p + g_{p+1} + ..$ be the Taylor expansions. Then $f_p = g_p$ (up to scaling).*

Proof: To prove $V_{f_p} = V_{g_p} \subset \mathbb{P}^{n-1}$ we start from the set-theoretic argument. Let $x \in V_{f_p}$ and $\phi \in GL(n)$ such that $\phi : x \to (1, 0...0) \in \mathbb{P}^{n-1}$. Then, in the new coordinates, the expansion of $\phi(f_p)$ contains no monomial x_1^p and this reflects on the Newton diagram $\Gamma_{\phi(f_p)}$. From the equality of Newton diagrams $(\Gamma_{\phi(f_p)} = \Gamma_{\phi(g_p)})$ one has: $x \in V_{g_p}$. By considering all points of V_{f_p}, V_{g_p} we get the equality $V_{f_p} = V_{g_p}$ as sets.

If the tangent cone is reduced, this gives $f_p = g_p$ up to scaling. Otherwise, let $f_p = \prod_{i=1}^{k} f_i^{n_i}$ and $g_p = \prod_{i=1}^{k} g_i^{m_i}$ be the prime decompositions. Let $x \in V_{f_i}$ be the generic point, so that x is a smooth point of the reduced cone $V_{\prod f_i}$. Apply linear transformation to $(\mathbb{C}^n, 0)$ to put $x = (1, 0..0)$. Then the monomial $x_1^{deg(f_i)}$ does not appear in f_i, while for any $j \neq i$ the monomial $x_1^{deg(f_j)}$ does appear in f_j. Thus the number $p - deg(f_i)n_i$ can be restored from the Newton diagram $\Gamma_{\phi(f_p)}$ by checking the monomial containing the highest power of x_1. And by equality of the Newton diagrams one gets $n_i = m_i$.

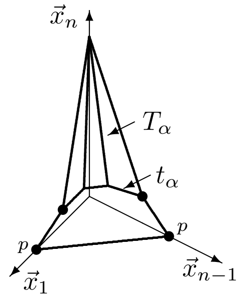
So the scheme structure of the projectivized tangent cone is also restored from the collection of the Newton diagrams. ■

Corollary 2.13. Let $f=f_p+f_k$ where $k>p$ and f_k is generic with respect to f_p (in particular $(V_{f_k},0)\subset(\mathbb{C}^n,0)$ is smooth). Then $(V_f,0)\subset(\mathbb{C}^n,0)$ is directionally Newton-non-degenerate.

In particular any super-isolated singularity is directionally Newton-non-degenerate.

2.5. The directional approximations. The directional approximations for curves were introduced in example 1.2, here we generalize this to the hypersurfaces. Let $(V_f,0)\subset(\mathbb{C}^n,0)$ be an isolated hypersurface germ of multiplicity p.

Suppose its projectivized tangent cone $\mathbb{P}T_{(V_f,0)}\subset\mathbb{P}(\mathbb{C}^n)$ has only isolated singularities, each one being of generalized Newton-non-degenerate type and can be brought locally to its (commode) Newton-non-degenerate form by a *linear* transformation of $\mathbb{P}^{n-1}$ (or of $(\mathbb{C}^n,0)$). Let $z_i\in Sing(\mathbb{P}T_{(V_f,0)})$, choose coordinates in $\mathbb{P}^{n-1}$ (i.e. local coordinates in $(\mathbb{C}^n,0)$) such that $z_i=(0,...,0,1)$ (i.e. corresponds to the $\hat{x}_n$ axis) and $(\mathbb{P}T_{(V_f,0)},z_i)$ is Newton-non-degenerate. Assume that for this basis the defining function f has monomials $x_1^p...x_{n-1}^p$, so the Newton diagram is as in the picture.



Let $\{t_\alpha\}$ be the top-dimensional faces of the Newton diagram of $(\mathbb{P}T_{(V_f,0)},z_i)$. They naturally correspond to some $(n-2)$ dimensional faces on $\Gamma_{(V_f,0)}$ (cf. the picture). Let T_α be those top-dimensional faces of the diagram $\Gamma_{(V_f,0)}$ which intersect the hyperplane $Span(x_1^p..x_n^p)$ along the faces $\{t_\alpha\}$.

Suppose the remaining axes $\hat{x}_1,..,\hat{x}_{n-1}$ can be oriented (preserving the Newton diagram) such that $\{f_{T_\alpha}\}$ and $\{f_{t_\alpha}\}$ are non-degenerate. (This is the analogue of the non-degeneracy of C_i in the case of curves.)

In this case consider a germ $(V_i,0)\subset(\mathbb{C}^n,0)$ defined by $f_{\cup T_\alpha}+\sum\limits_{j=1}^{n-1}b_jx_j^p$. (Here $f_{\cup T_\alpha}$ is the truncation of f to the corresponding faces, b_j are some non-zero numbers, such that the truncation $f_{\cup t_\alpha}+\sum\limits_{j=1}^{n-1}b_jx_j^p$ is Newton-non-degenerate.)

Definition 2.14. *For each $z_i\in Sing(\mathbb{P}T_{(V_f,0)})$ the so defined germ $(V_i,0)$ is called the Newton-non-degenerate directional approximation.*

As in the case of curves, the germ $(V_i,0)$ is not defined uniquely, but any two representatives have the same singularity type, the same Newton diagram and can be joined by a $\mu=const$ family.

Proposition 2.15. *Suppose for $(V_f,0)\subset(\mathbb{C}^n,0)$ the hypersurface $\mathbb{P}T_{(V_f,0)}\subset\mathbb{P}(\mathbb{C}^n)$ has isolated singularities only and for each singular point of it there exists a Newton-non-degenerate directional approximation. Then $(V_f,0)$ is directionally Newton-non-degenerate.*

Proof: The direct application of the following criterion. Let $(V_f,0)$, $(V_g,0)$ be two isolated hypersurface germs and $(\tilde{V}_f,0)$, $(\tilde{V}_g,0)$ be their strict transforms under the blowup $(Bl_0\mathbb{C}^n,E)\to(\mathbb{C}^n,0)$. Suppose the singular points of $E\cap\tilde{V}_f$ correspond bijectively to those of $E\cap\tilde{V}_g$ and for each singular point $z_i(f)\in Sing(E\cap\tilde{V}_f)$ there exists a neighborhood $z_i(f)\in U_f\subset Bl_0\mathbb{C}^n$ and the embedded homeomorphism $(U_f,z_i(f))\xrightarrow{\phi}(U_g,z_i(g))$ such that $(E,E\cap\tilde{V}_f,z_i(f))\xrightarrow{\phi}(E,E\cap\tilde{V}_g,z_i(g))$. Then $(V_f,0)$, $(V_g,0)$ are of the same topological type. ■

3. Some singularity invariants

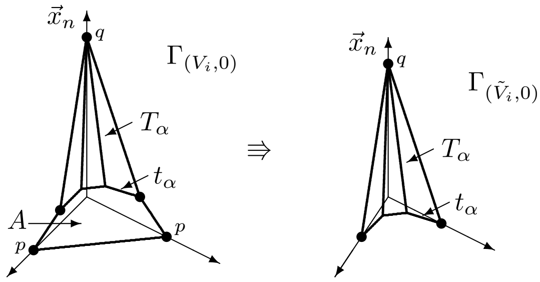
3.1. Kouchnirenko's formula for the Milnor number. The derivation of the formula is based on the following result [**Melle00**, Theorem 1]. For the hypersurface germ $(V,0)\subset(\mathbb{C}^n,0)$, let $\mathbb{P}T_{(V,0)}\subset\mathbb{P}^{n-1}$ be the projectivization of its tangent cone and $\tilde{V}\to V$ the strict transform under the blow-up of the origin. Assume, both $\tilde{V}$ and $\mathbb{P}T_{(V,0)}$ have isolated singularities only. Let $p=mult(V,0)$ then:

$$\mu(V,0)=(p-1)^n+\mu(\mathbb{P}^{n-1},\mathbb{P}T_{(V,0)})+\mu(Bl_0\mathbb{C}^n,\tilde{V}) \tag{4}$$

Theorem 3.1. *Let $(V,0)\subset(\mathbb{C}^n,0)$ be an isolated directionally Newton-non-degenerate hypersurface singularity. Suppose it has the Newton-non-degenerate directional approximations $(V_1,0)..(V_k,0)$ corresponding to $Sing(\mathbb{P}T_{(V,0)})=\{z_1..z_k\}$ (cf.§2.5). Then Kouchnirenko's formula holds in the following form:*

(5)
$$\mu(V,0)=\sum_{i=1}^{k}\mu(V_i,0)-(k-1)(p-1)^n,\ \ where\ \ \mu(V_i,0)=(n-1)!Vol_{n-1}(\Gamma_{(V_i,0)})-(n-2)!Vol_{n-2}(\Gamma_{(V_i,0)})+..+(-1)^n.$$

Proof: As a preparation consider the change of the Newton diagram of V_i under the blowup. As in §2.5, the top-dimensional faces of the diagram diagram are: a part of $Span(x_1^p..x_n^p)$ and $\{T_\alpha\}$. The intersections $T_\alpha \cap A = t_\alpha$ are faces of dimension $(n-2)$. Consider the strict transform of V_i under the blowup $\{(x_1..x_n) = (\sigma_1 : .. : \sigma_n)\} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$. (The two Newton diagrams are on the right.) The relevant chart is $\sigma_n \neq 0$, with the coordinates $(\frac{\sigma_1}{\sigma_n}..\frac{\sigma_{n-1}}{\sigma_n}, x_n)$. The total transform of the function is:
$f(x_1..x_n) \to x_n^p\Big(f_p(\frac{\sigma_1}{\sigma_n}..\frac{\sigma_{n-1}}{\sigma_n},1) + x_n f_{p+1}(\frac{\sigma_1}{\sigma_n}..\frac{\sigma_{n-1}}{\sigma_n},1) + .. + x_n^{q-p}\Big) + ...$
The polyhedron under $\Gamma_{(V_i,0)}$ is naturally subdivided into two parts, one being the pyramid under $Span(x_1^p..x_n^p)$. Denote the other polyhedron (under T_α) by Δ_{V_i}. Let $\tilde{\Delta}_{V_i}$ be the polyhedron under $\Gamma_{(\tilde{V}_i,0)}$.



The blowup induces the map $\Delta_{V_i} \to \tilde{\Delta}_{V_i}$ by $(a_1..a_n) \to (a_1..a_{n-1}, a_n - a_1 - .. - a_{n-1})$. So, there is the natural correspondence between the faces of Δ_{V_i} and $\tilde{\Delta}_{V_i}$.
While, Δ_{V_i} and $\tilde{\Delta}_{V_i}$ are not equal as polyhedra, their corresponding faces have equal volumes. Therefore $\mu(\Delta_{V_i}) = \mu(\tilde{\Delta}_{V_i})$. Here $\mu(\Delta_{V_i})$ is the standard expression: the main volume $(n-1)!Vol_{n-1}(\Delta_{V_i})$ minus the volume of top-dimensional faces $(n-2)!Vol_{n-2}(\Delta_{V_i})$, plus the sum of volumes of faces of codimension 2, etc.

Similarly, let $\Delta^b_{V_i} = \Delta \cap Span(x_1^p..x_n^p)$ and $\tilde{\Delta}^b_{V_i} = \tilde{\Delta} \cap \{x_n = 0\}$. Then the corresponding faces of $\Delta^b_{V_i}$ and $\tilde{\Delta}^b_{V_i}$ have equal volume.

Therefore the classical Kouchnirenko's formula gives (recall that $\tilde{V}_i$ and $\mathbb{P}T_{(V_i,0)}$ have Newton-non-degenerate singularities):

$$\mu(\Delta_{V_i}) + \mu(\Delta^b_{V_i}) = \mu(\tilde{\Delta}_{V_i}) + \mu(\tilde{\Delta}^b_{V_i}) = \mu(\tilde{V}_i) + \mu(\mathbb{P}T_{(V_i,0)}) \tag{6}$$

Finally,
• note that $\mu(\tilde{V}_i) = \mu(\tilde{V}, z_i)$ and $\mu(\mathbb{P}T_{(V_i,0)}) = \mu(\mathbb{P}T_{(V,0)}, z_i)$
• sum over all the singular point of the projectivized tangent cone $\mathbb{P}T_{(V,0)}$ (i.e. sum over all the directional approximations V_i) and apply equation (4)
• note that each $\mu(V_i,0)$ contains a contribution from the basic pyramid $(x_1^p..x_n^p)$, for which $\mu = (p-1)^n$. ■

Remark 3.2. For curves an especially simple proof can be given. It is based on the formulas for the δ invariant:
• $\mu = 2\delta - r + 1$, here r is the number of branches
• $\delta = \sum\limits_i \delta(C_i) + \sum\limits_{i<j} < C_i, C_j >$ (for the tangential decomposition $C = \bigcup\limits_{i=1}^k C_i$).
Using these formulas one gets:

$$\mu(C) = 2\Big(\frac{\sum\limits_i \mu(C_i) + r_i - 1}{2} + \sum_{i<j} < C_i, C_j > \Big) - \sum_i r_i + 1 = \sum_i \mu(C_i) + 1 - k + \sum_{i\neq j} p_i p_j \tag{7}$$

(for $p_i = mult(C_i)$). Assume that the curve-germ has a Newton-non-degenerate directional approximation $\{C^{(i)}\}$ (i.e. each C_i is generalized Newton-non-degenerate). Then the result follows from the observation: $\mu(C^{(i)}) = \mu(C_i) + p^2 - p_i^2 + 2p_i - 2p$.

3.2. Zeta function of monodromy. Recall the basic result of [**A'Campo75**] (cf. also [**AGLV-book**, II.3.12]). Given an isolated hypersurface singularity, construct its good resolution (cf. the diagram): $\tilde{V}$ is smooth, E consists of smooth components and $\tilde{V} \cup E$ is a normal crossing divisor. Let $\pi^{-1}(0) = \sum m_i E_i$, i.e. E_i is an irreducible component of E, of multiplicity m_i. Denote $S_m := \{x \in E : \ mult(E,x) = m\}$.
Then

$$\begin{array}{cc} (\tilde{V}, \tilde{V}\cap E) & \subset (Y,E) \\ \downarrow & \downarrow \pi \\ (V,0) & \subset (\mathbb{C}^n, 0) \end{array}$$

$$\zeta_{(V,0)}(z) = \prod_{m\geq 1} (1 - z^m)^{\chi(S_m)} \tag{8}$$

where χ is the Euler characteristic.

The product structure of this formula is the basic reason for the possibility to determine the zeta function by the geometry of the Newton diagram.

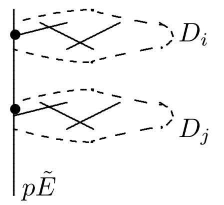
Proposition 3.3. *Let $(V,0) \subset (\mathbb{C}^n,0)$ be a directionally Newton-non-degenerate germ, whose projectivized tangent space has isolated singularities: $Sing(\mathbb{P}T_V) = \{y_1..y_k\}$ and the corresponding directional approximations $V_1..V_k$ are*

Newton-non-degenerate (cf.§2.5). Then A'Campo's formula can be written in the form

$$\zeta_{(V,0)}(z)=\frac{\prod\limits_{i=1}^{k}\zeta_{(V_i,0)}(z)}{(1-z^p)^{(k-1)(n-\chi_{p,n-1})}}$$

where $\zeta_{(V_i,0)}(z)$ *is the classical zeta-function of the (Newton-non-degenerate) hypersurface-germ,* $p=mult(V,0)$ *and* $\chi_{p,n-1}=\chi(V_{p,n-1})$ *for an arbitrary smooth hypersurface* $V_{p,n-1}\subset\mathbb{P}^{n-1}$ *of degree* p*.*

Proof: Blowup $\mathbb{C}^n$ at the origin. By the assumption the strict transform $\tilde{V}\to V$ has isolated singularities only and the exceptional divisor is pE for $E\approx\mathbb{P}^{n-1}\subset Bl_0(\mathbb{C}^n)$. Now, resolve the singularities of $\tilde{V}$.



Write the total preimage of the origin in the form: $\pi^{-1}(0)=p\tilde{E}+\sum\limits_{i=1}^{k}D_i$. Here each D_i corresponds to the resolution of $(\tilde{V},y_i)$ (cf. the picture). In particular: $D_i\cap D_j=\varnothing$ for $i\neq j$.
Thus the product $\prod_{m\geq 1}$ in the original formula (8) can be replaced by k copies (for each directional approximation X_i). Each such copy contributes the unnecessary factor $(1-z^p)^{\chi(\tilde{E}\setminus(D_i\cup\tilde{V}_i))}$ and no copy contains the needed factor $(1-z^p)^{\chi(\tilde{E}\setminus(\bigcup\limits_i D_i\cup\tilde{V}))}$.
So, the formula can be written in the form

$$\zeta_{(V,0)}(z)=(1-z^p)^{\chi(\tilde{E}\setminus(\bigcup\limits_i D_i\cup\tilde{V}))}\prod_{i=1}^{k}\frac{\zeta_{(V_i,0)}(z)}{(1-z^p)^{\chi(\tilde{E}\setminus(D_i\cup\tilde{V}_i))}}\tag{9}$$

Note that $\tilde{E}\setminus(\bigcup\limits_i D_i\cup\tilde{V})=E\setminus\mathbb{P}T_V$ and $\tilde{E}\setminus(D_i\cup\tilde{V}_i)=E\setminus\mathbb{P}T_{V_i}$, so the correction factor is:

$$\frac{(1-z^p)^{\chi(E\setminus\mathbb{P}T_V)}}{\prod_i(1-z^p)^{\chi(E\setminus\mathbb{P}T_{V_i})}}=\frac{(1-z^p)^{(1-k)\chi(\mathbb{P}^{n-1})}}{(1-z^p)^{\chi(\mathbb{P}T_V)-\sum\chi(\mathbb{P}T_{V_i})}}\tag{10}$$

Finally, for any isolated singularity: $\chi(\mathbb{P}T_{V_i})=\chi(V_{p,n-1})+N_i$, where $V_{p,n-1}\subset\mathbb{P}^{n-1}$ is a smooth hypersurface of degree p (in particular its Euler characteristic is independent of the hypersurface) and N_i is a number completely determined by the topological singularity type. Thus $\chi(\mathbb{P}T_V)-\sum\chi(\mathbb{P}T_{V_i})=(1-k)\chi(V_{p,n-1})$, proving the statement.
■

Using the last proposition it is immediate to generalize Varchenko's formula for the zeta function in terms of the Newton diagram. Recall ([**Varchenko76**],[**AGLV-book**, II.3.12]) that for an isolated Newton-non-degenerate hypersurface singularity $(V,0)\subset(\mathbb{C}^n,0)$ the zeta function of the monodromy can be written in the form:

$$\zeta_{(V,0)}(z)=\prod_{l=1}^{n}(\zeta^l(z))^{(-1)^{l-1}}\tag{11}$$

where $\{\zeta^l(z)\}_l$ are some polynomials completely determined by the geometry of $l-$dimensional faces of the Newton diagram.

Corollary 3.4. Under the assumptions of the proposition 3.3 Varchenko's formula is valid in the following form:

$$\zeta_{(V,0)}(z)=\frac{1}{(1-z^p)^{(k-1)(n-\chi_{p,n-1})}}\prod_{x_i\in Sing(\mathbb{P}T_{(V,0)})}\prod_{l=1}^{n}(\zeta^l_{(V_i,0)}(z))^{(-1)^{l-1}}$$

where, $\chi_{p,n-1}=\chi(V_{p,n-1})$ for an arbitrary smooth hypersurface $V_{p,n-1}\subset\mathbb{P}^{n-1}$ of degree p, and $\zeta^l_{(V_i,0)}(z)$ are the standard Varchenko polynomials (for the Newton-non-degenerate singularities V_i).

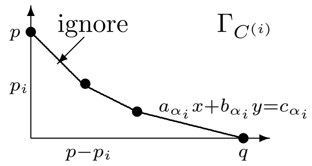
3.2.1. *Example: zeta function for some directionally Newton-non-degenerate singularities of curves.* In the case of curves all the objects are very explicit.

Let $C=\bigcup\limits_{i=1}^{k}C_i$ be the tangential decomposition, then the resolution tree consists of pE and k chains corresponding to $\{C_i\}$. Let $\{C^{(i)}\}$ be the directional approximations (cf. example 1.2). Then the zeta function of the monodromy is (cf. proposition 3.3):

$$\zeta_C(z)=\frac{\prod_i\zeta_{C^{(i)}}(z)}{(1-z^p)^{(k-1)(2-p)}}\tag{12}$$

Note that in the case of curves this result is valid without assumption that each C_i is generalized Newton-non-degenerate.

Assume now each C_i is generalized Newton-non-degenerate. To write Varchenko's formula introduce the parameters of the Newton diagram of $\Gamma_{C^{(i)}}$. For each edge l_α of the diagram (except for $\overline{(0,p),(p-p_i,p_i)}$) let $a_\alpha x + b_\alpha y = c_\alpha$ be the equation of the line it spans. Here $a_\alpha, b_\alpha, c_\alpha \in \mathbb{N}$ and $(a_\alpha, b_\alpha) = 1$ (so the coefficient are fixed uniquely). Let $|l_\alpha|$ be the number of integral points minus one on the edge α. Then the formula of corollary 3.4 reads:



$$\zeta_C(z) = \frac{1}{(1-z^p)^{k-2}} \prod_{i=1}^{k} \frac{1-z^{q_i}}{\prod_{\alpha_i}(1-z^{c_{\alpha_i}})^{|l_{\alpha_i}|}} \qquad (13)$$

Here α_i runs over the edges of $\Gamma_{C^{(i)}}$, each time omitting the edge $\overline{(0,p),(p-p_i,p_i)}$.

3.3. Some further invariants.

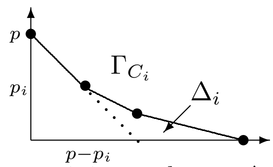
3.3.1. *Order of determinacy.* Suppose an isolated hypersurface germ $\{f=0\} = (V,0) \subset (\mathbb{C}^n, 0)$ has a Newton-non-degenerate directional approximation $(V_1,0)$... $(V_k,0)$. For each $(V_i,0)$ let $o.d.(V_i,0)$ be the (contact, topological) order of determinacy [**GLS-book**, I.2.2]. It is easily read from the diagram of $(V_i,0)$.

PROPOSITION 3.5. *The order of determinacy of* $(V,0)$ *is* $\max_i(o.d.(V_i,0))$

Proof: Let $q = \max_i(o.d.(V_i,0))$. Suppose $jet_q(f) = jet_q(g)$, then f,g have coinciding collections of Newton diagrams, isomorphic tangent cones and g is directionally Newton-non-degenerate with respect to its collection of the diagrams. So, the singular types of f,g coincide.

On the other hand the order of determinacy of $(V,0)$ is certainly at least $\max_i(o.d.(V_i,0))$. ■

3.3.2. *Right modality for functions of two variables.*
For the Newton-non-degenerate singularities the (right) modality can be calculated as the number of integral points (x,y) under the Newton diagram, satisfying $x,y \geq 2$ (cf. [**Kouchnirenko76**]). We can only propose a natural generalization. Let $C = \bigcup_i C_i$ be the tangential decomposition, assume each C_i is generalized Newton-non-degenerate. Let Γ_{C_i} be the corresponding Newton diagram.



It is naturally decomposed into the triangle $x+y \leq p$ for $p = mult(C)$ and the remaining polygon Δ_i. Let $\Delta_i \cap (2,2)$=number of integral points $(x,y) \in \Delta_i$, $x,y \leq 2$ strictly below the Newton diagram.
Conjecture: (right) modality=(modality of x^p+y^p)+$\sum_i(\Delta_i \cap (2,2))$.

3.3.3. τ^{es} *for linear types of curves.* Let $(C,0) \subset (\mathbb{C}^2,0)$ be a germ of curve whose directional approximations $\{C^i\}$ are Newton-non-degenerate with respect to their diagrams Γ_i. Assume also that the directional approximations are of *linear type* [**Kerner06**], i.e. any representative of the type of C^i can be brought to Γ_i by linear transformations only. Assume that every Γ_i is minimal.

As on the previous picture, let Δ_i be the distinguished polygon on Γ_i. Let $\sharp\Delta_i$ be the number of integral points in the closure $\bar{\Delta}_i$, which lie (strictly) under the Newton diagram Γ_i.

PROPOSITION 3.6. *Under the assumptions above:* $\tau^{es} = \sum \sharp\Delta_i + \binom{p+1}{2} - 2 - k$

Proof: Given $T = (l_1^{p_1}..l_k^{p_k})$, let $\Sigma_{l_1..l_k} = \{C | \forall 1 \leq i \leq k : \Gamma_C \geq \Gamma_{C_i}\ for\ coordinates\ (l_i,y)\}$. Thus $\tau^{es} = 2 + k + codim \Sigma_{l_1..l_k}$ where 2 is for the choice of the point in the plane and k for the dimension of space of k lines through the point. ■

COROLLARY 3.7. For linear singularities the right modality=$\mu + 2 + k - \sum \sharp\Delta_i - \binom{p+1}{2}$

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