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DOMINANCE AND TRANSMISSIONS IN SUPERTROPICAL VALUATION THEORY

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

Abstract. This paper is a sequel of [IKR1], where we defined **supervaluations** on a commutative ring R and studied a **dominance relation** $\varphi \geqslant \psi$ between supervaluations φ and ψ on R, aiming at an enrichment of the algebraic tool box for use in tropical geometry.

A supervaluation $\varphi : R \to U$ is a multiplicative map from R to a supertropical semiring U, cf. [IR1], [IR2], [IKR1], with further properties, which mean that φ is a sort of refinement, or covering, of an m-valuation (= monoid valuation) $v : R \to M$. In the most important case, that R is a ring, m-valuations constitute a mild generalization of valuations in the sense of Bourbaki [B], while $\varphi \geqslant \psi$ means that $\psi : R \to V$ is a sort of coarsening of the supervaluation φ. If $\varphi(R)$ generates the semiring U, then $\varphi \geqslant \psi$ iff there exists a "**transmission**" $\alpha : U \to V$ with $\psi = \alpha \circ \varphi$.

Transmissions are multiplicative maps with further properties, cf. [IKR1, §5]. Every semiring homomorphism $\alpha : U \to V$ is a transmission, but there are others which lack additivity, and this causes a major difficulty. In the main body of the paper we study surjective transmissions via equivalence relations on supertropical semirings, often much more complicated than congruences by ideals in usual commutative algebra.

Contents

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Introduction

We set forth a study in supertropical valuation theory begun in [IKR1]. Generalizing Bourbaki's notion of a valuation on a commutative ring [B], we there introduced *m-valuations* (= monoid valuations) and then *supervaluations* on a commutative semiring. These are certain maps from a semiring R to a "*bipotent semiring*" M and a "*supertropical semiring*", respectively.

To repeat, if M is a **bipotent semiring**, here always commutative, then the set M is a totally ordered monoid under multiplication with smallest element 0, and the addition is given by $x + y = \max(x, y)$. Then an **m-valuation** on R is a multiplicative map $v : R \to M$, which sends 0 to 0, 1 to 1, and obeys the rule $v(a + b) \leqslant v(a) + v(b)$. We call v a **valuation**, if moreover the semiring M is cancellative. {In the classical case of a Krull valuation v, R is a field and $M = \mathcal{G} \cup \{0\}$, with $\mathcal{G}$ the valuation group of v in multiplicative notation.}

A **supertropical semiring** U is a – here always commutative – semiring such that $e := 1 + 1$ is an idempotent of U and some axioms hold ([IKR1, §3]), which imply in particular that the ideal $M := eU$ is a bipotent semiring. The elements of $M\backslash\{0\}$ are called **ghost** and those of $\mathcal{T}(U) := U\backslash M$ are called **tangible**. The zero element of U is regarded both ghost and tangible. For $x \in U$ we call ex the **ghost companion** of x. For $x, y \in U$ we have the rule

$$x + y = \begin{cases} y & \text{if} \quad ex < ey, \\ x & \text{if} \quad ex > ey, \\ ex & \text{if} \quad ex = ey. \end{cases}$$

Thus the addition in U is uniquely defined by the multiplication and the element e. We also mention that $ex = 0$ implies $x = 0$. We refer to [IKR1, §3] for all details.

Finally, a **supervaluation** on R is a multiplicative map $\varphi : R \to U$ to a supertropical semiring U sending 0 to 0 and 1 to 1, such that the map $e\varphi : R \to eU$, $a \mapsto e\varphi(a)$, is an m-valuation. We then say that φ **covers** the m-valuation $v := e\varphi$.

If $\varphi : R \to U$ is a supervaluation then $U' := \varphi(R) \cup e\varphi(R)$ is a sub-semiring of U and is again supertropical. In practice we nearly always may replace U by U' and then have a supervaluation at hands which we call **surjective**.

Given a surjective supervaluation $\varphi : R \to U$ and a map $\alpha : U \to V$ to a supertropical semiring V, the map $\alpha \circ \varphi$ is again a supervaluation iff α is multiplicative, sends 0 to 0, 1 to 1, e to e, and restricts to a semiring homomorphism from eU to eV. {We denote the elements $1+1$ in U and V both by "e".} We call such a map $\alpha : U \to V$ a **transmission**. Any semiring homomorphism from U to V is a transmission, but usually there exist also many transmissions which are not additive.

The study of transmissions is the central topic of the present paper. Transmissions are tied up with the relation of **dominance** defined in [IKR1, §5]. If $\varphi : R \to U$ and $\psi : R \to V$ are supervaluations and φ is surjective, then φ **dominates** ψ, which we denote by $\varphi \geqslant \psi$, iff there exists a transmission $\alpha : U \to V$ with $\psi = \alpha \circ \varphi$.

Already in [IKR1] we studied dominance for supervaluations which cover a fixed, say, surjective m-valuation $v : R \to M$. We called two such supervaluations φ, ψ **equivalent** if $\varphi \geqslant \psi$ and $\psi \geqslant \varphi$. The set $\mathrm{Cov}(v)$ of equivalent classes $[\varphi]$ of supervaluations $\varphi : R \to U$ covering v (having varying target U with $eU = M$) turns out to be a complete lattice under the dominance relation [IKR1, §7].

The bottom element of $\mathrm{Cov}(v)$ is the class $[v]$, with v viewed as a supervaluation. The top element is given by a surjective supervaluation $\varphi_v : R \to U(v)$, which we could describe explicitly in the case that v is valuation, i.e., M is cancellative [IKR1, Example 4.5 and Corollary 5.14].

We come to the contents of the present paper. If $v : R \to M$ is an m-valuation and $\gamma : M \to N$ is a homomorphism from M to a bipotent semiring N, then $\gamma \circ v$ clearly again is an m-valuation, called a **coarsening** of v. This generalizes the usual notion of coarsening for Krull valuations. It is of interest to look for relations between the lattices $\mathrm{Cov}(v)$ and $\mathrm{Cov}(\gamma \circ v)$. §1 gives a first step in this direction. Given $\gamma : M \to N$ and a supertropical semiring U with ghost ideal M we look for transmissions $\alpha : U \to V$ which **cover** γ, i.e., V has the ghost ideal N and $\alpha(x) = \gamma(x)$ for $x \in M$. Assuming that γ is surjective, we prove that there exists an **initial** such transmission $\alpha = \alpha_{U,\gamma} : U \to U_\gamma$. This means that any other transmission $\alpha' : U \to V'$ covering γ is obtained from α by composition with a transmission $\beta : U_\gamma \to V'$ covering the identity of N. This allows us to define an order preserving map

$$\gamma_* : \mathrm{Cov}(v) \to \mathrm{Cov}(\gamma \circ v),$$

sending a supervaluation $\varphi : R \to U$ to $\gamma_*(\varphi) := \alpha_{U,\gamma} \circ \varphi$. In good cases $\alpha_{U,\gamma}$ has a "pushout property" (cf. Definition 1.2), that is even stronger than to be initial, and $\alpha_{U,\gamma}$ can be described explicitly (cf. Theorem 1.11).

We defined in [IKR1, §2] **strong valuations** and in [IKR1, §9] **strong supervaluations**, which by definition are covers of strong valuations. Tangible strong supervaluations seems to be the most suitable supervaluations for applications in tropical geometry, hence our interest in them. Given a strong strong supervaluation $v : R \to M$ we proved that the set $\mathrm{Cov}_{\mathrm{t,s}}(v)$ of tangible strong supervaluations is a complete sublattice of $\mathrm{Cov}(v)$ [IKR1, §10]. In particular this set is not empty. In §2 of the present paper we study the behavior of such supervaluations covering v under the map γ_* from above. It turns out that $\gamma_*(\mathrm{Cov}_{\mathrm{t,s}}(v)) \subset \mathrm{Cov}_{\mathrm{t,s}}(\gamma \circ v)$.

Denoting a representative of the top element of $\mathrm{Cov}_{\mathrm{t,s}}(v)$ by $\overline{\varphi}_v$, we observe that $\gamma_*([\overline{\varphi}_v])$ is most often different from $[\overline{\varphi}_{\gamma\circ v}]$. On the other hand, $\gamma_*([\varphi_v]) = [\varphi_{\gamma\circ v}]$. This indicates that it is not advisable to restrict supervaluation theory from start to strong supervaluations, even if we are only interested in these.

The rest of the paper is devoted to an analysis and examples of surjective transmissions. After a preparatory §3, in which the construction of a large class of supertropical semirings is displayed, we study in §4 *"transmissive" equivalence relations.*

We call an equivalence relation E on a supertropical semiring U **transmissive**, if E is multiplicative (= compatible with multiplication), and the set of E-equivalence classes U/E admits the structure of a supertropical semiring such that the natural map $\pi_E : U \to U/E$ is a transmission. (There can be at most one such semiring structure on the set U/E.) Every surjective transmission $\alpha : U \to V$ has the form $\alpha \circ \pi_E$ with a (unique) transmissive equivalence relation E and an isomorphism $\rho : U/E \tilde{\to} V$. Thus having a hold on the transmissive equivalence relations means understanding transmissions in general.

In all following U denotes a supertropical semiring. The main result of §4 is an axiomatic description of those transmissive equivalence relations E on U, for which the ghost ideal of U/E is a cancellative semiring (Theorem 4.7, Definition 4.5). We also give a criterion that the transmission π_E is pushout, as defined in §1 (Theorem 4.13), and we analyse, which "orbital" equivalence relations, defined in [IKR1, §8], are transmissive. These exhaust *all* transmissive equivalence relations on U, if U is a *supertropical semifield*, i.e., all tangibles $\neq 0$ are invertible in U, and all ghosts $\neq 0$ are invertible in eU.

We call a transmissive equivalence relation on U **homomorphic** if the map $\pi_E : U \to U/E$ is a semiring homomorphism. In §5 we discuss a very special and easy, but important class of such equivalence relations. Then in the final section §6 we look at homomorphic equivalence relations in general.

Given a homomorphic equivalence relations Φ on $M := eU$ we classify all homomorphic equivalence relations E on U which extend Φ. Here **additivity** of E, i.e., compatibility with addition, causes the main difficulty. Thus, to ease understanding, we first perform the classification program for additive equivalence relations (Theorem 6.6′), and then add considerations on multiplicativity to find the homomorphic equivalence relations (Theorem 6.11).

We close the paper with examples of homomorphic equivalence relations using the classification, and also indicate consequences for other transmissive equivalence relations.

Notations. *Given sets X, Y we mean by $Y \subset X$ that Y is a subset of X, with $Y = X$ allowed. If E is an equivalence relation on X then X/E denotes the set of E-equivalence classes in X, and $\pi_E : X \to X/E$ is the map which sends an element x of X to its E-equivalence class, which we denote by $[x]_E$. If $Y \subset X$, we put $Y/E := \{[x]_E \mid x \in Y\}$.*

*If U is a supertropical semiring, we denote the sum $1 + 1$ in U by e, more precisely by e_U if necessary. If $x \in U$ the **ghost companion** ex is also denoted by $\nu(x)$ or x^ν, and the **ghost map** $U \to eU$, $x \mapsto \nu(x)$, is denoted by ν_U. If $\alpha : U \to V$ is a transmission, then the semiring homomorphism $eU \to eV$ obtained from α by restriction is denoted by α^ν and is called the **ghost part** of α. Thus $\alpha^\nu \circ \nu_U = \nu_V \circ \alpha$.*

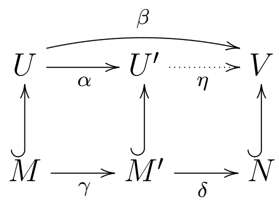
*If $v : R \to M$ is an m-valuation we call the ideal $v^{-1}(0)$ of R the **support** of v, and denote it by* $\operatorname{supp}(v)$. *If $\varphi : R \to U$ is a supervaluation covering v, we most often denote the equivalence class $[\varphi] \in \operatorname{Cov}(v)$ abusively again by φ*

1. Initial transmissions and a pushout property

We state the main problem which we address in this section.

Problem 1.1. *Assume that U is a supertropical semiring with ghost ideal $eU = M$, and $\gamma : M \to M'$ is a semiring homomorphism from M to a bipotent semiring M'. Find a supertropical semiring U' with ghost ideal $eU' = M'$ and a transmission $\alpha : U \to U'$ covering γ, i.e., $\alpha^\nu = \gamma$ (cf.* [IKR1, Definition 5.3]*), with the following universal property. Given a transmission $\beta : U \to V$ into a supertropical semiring V, with ghost ideal $N := eV$, and a semiring homomorphism $\delta : M' \to N$, such that $\beta^\nu = \delta\gamma$, there exists a unique transmission $\eta : U' \to V$ such that $\beta = \eta \circ \alpha$ and $\eta^\nu = \delta$.*

We indicate this problem by the following commuting diagram



where the vertical arrows are inclusion mappings.

We call such a map $\alpha : U \to U'$ a *pushout transmission covering* γ. This terminology alludes to the fact that our universal property means that the left square in the diagram above is a pushout (=cocartesian) square in the category STROP, whose objects are the supertropical semirings, and whose morphisms are the transmissions. To see this, just observe that a map $\rho : L \to W$ from a bipotent semiring L to a supertropical semiring W is transmissive iff ρ is a semiring homomorphism from L to eW followed by the inclusion $eW \hookrightarrow W$.

It is now obvious that, for a given homomorphism $\gamma : M \to M'$, Problem 1.1 has at most one solution up to isomorphism over M' and U. More precisely, if both $\alpha : U \to U'$ and $\alpha_1 : U \to U_1$ are solutions, there exists a unique isomorphism $\rho : U' \to U_1$ of semirings over M' with $\alpha_1 = \alpha' \circ \rho$.

We may cast the universal property above in terms of α alone and then arrive at the following formal definition.

Definition 1.2. *We call a map $\alpha : U \to V$ between supertropical semirings a* ***pushout transmission*** *if the following holds:*

1) *α is a transmission.*
2) *If $\beta : U \to W$ is a transmission from U to a supertropical semiring W and $\delta : eV \to eW$ is a semiring homomorphism with $\beta^\nu = \delta \circ \alpha^\nu$, then there exists a unique transmission $\eta : U \to W$ with $\eta^\nu = \delta$ and $\beta = \eta \circ \alpha$.*

We then also say that V is "the" ***pushout of*** *U* ***along*** *γ.*

The notion of a pushout transmission can be weakened by demanding the universal property in Definition 1.2 only for $W = V$ and δ the identity of eV. This is still interesting.

Definition 1.3. *We call a transmission $\alpha : U \to V$ between supertropical semirings an* ***initial transmission****, if, for any transmission $\beta : U \to W$ with $eW = eV$ and $\beta^\nu = \alpha^\nu$, there exists a unique semiring homomorphism[1] $\eta : V \to W$ over $eV = eW$ with $\beta = \eta \circ \alpha$.*

Given a supertropical semiring U and a semiring homomorphism $\gamma : eU \to N$ with N bipotent, it is again clear that there exists at most one initial transmission $\alpha : U \to V$ covering γ (in particular, $eV = N$) up to isomorphism over U and N.

We turn to the problem of existence, first for initial transmissions and then for pushout transmissions. In the first case we can apply results on supervaluations from [IKR1, §4 and §7], due to the following easy but important observation.

Proposition 1.4. *Let $\alpha : U \to V$ be a map between supertropical semirings and $\gamma : eU \to eV$ a semiring homomorphism. The following are equivalent:*

a) *α is a transmission covering γ.*
b) *α is a supervaluation on the semiring U with $\alpha(e_U) = e_V$ covering the strict m-valuation $v := \gamma \circ \nu_U : U \to eV$.*

We then have the commuting diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \alpha\ } & V \\ {\scriptstyle \nu_U}\downarrow & \searrow{\scriptstyle v} & \downarrow{\scriptstyle \nu_V} \\ eU & \xrightarrow{\ \gamma\ } & eV \end{array} .$$

[1]Every transmission η with η^ν injective is a homomorphism [IKR1, Proposition 5.10.iii].

Proof. We have to compare the axioms SV1–SV4 in [IKR1, §4] plus the condition $\alpha(e) = e$ with the axioms TM1–TM5 in [IKR1, §5]. The axioms SV1–SV3 say literally the same as TM1–TM3, and the condition $\alpha(e) = e$ is TM4.

We now assume that α fulfills TM1–TM4. For every $x \in U$ we have $\alpha(ex) = \alpha(e)\alpha(x) = e\alpha(x)$. That α is a transmission covering γ means that $\alpha(z) = \gamma(z)$ for all $z \in eU$. This is equivalent to $\alpha(ex) = \gamma(ex)$ for all $x \in U$; hence to the condition $e\alpha(x) = \gamma \circ \nu_U(x)$ for all $x \in U$. But this means that α is a supervaluation covering $\gamma \circ \nu_U$. □

Theorem 1.5. *Given a supertropical semiring U with ghost ideal $M := eU$ and a surjective homomorphism $\gamma : M \to M'$ to a bipotent semiring M', there exists an initial transmission $\alpha : U \to U'$ covering γ.*

Proof. We introduce the strict surjective valuation

$$v = \gamma \circ \nu_U : U \twoheadrightarrow M'.$$

By [IKR1, §7] there exists an initial surjective supervaluation $\varphi_v : U \to U(v)$ covering v. (In particular, $eU(v) = M'$.) The other surjective supervaluations $\psi : U \to V$ covering γ are the maps $\pi_T \circ \varphi_v$ with T running through the set of all MFCE-relations on $U(v)$, as explained in [IKR1, §7].

Let $f := \varphi_v(e_U)$ and $e := e_{U(v)} = 1_M$. Proposition 1.4 tells us that $\pi_T \circ \varphi_v$ is the initial transmission covering γ iff $f \sim_T e$ and moreover T is finer than any other MFCE-relation on $U(v)$ with this property. Now we invoke the following easy lemma, to be proved below.

Lemma 1.6. *If W is a supertropical semiring and X is a subset of W, there exists a unique finest MFCE-relation E on W with $x \sim_E e_W x$ for every $x \in X$.*

We apply the lemma to $W = U(v)$ and $X = \{f\}$, and obtain a finest equivalence relation T on $U(v)$ with $f \sim_T ef$. But

$$ef = \nu_{U(v)} \circ \varphi_v(e_U) = v(e_U) = e.$$

Thus, T is the unique finest MFCE-relation on $U(v)$ with $f \sim_T e$, and T gives us the wanted initial transmission $\alpha = \pi_T \circ \varphi_v$. □

Proof of Lemma 1.6. The set $\mathcal{M}$ of all MFCE-relations F on W with $x \sim_F ex$ for all $x \in X$ is not empty, since it contains the relation $E(\nu_W)$. The relation $E := \bigwedge \mathcal{M}$, i.e., the intersection of all $F \in \mathcal{M}$, has the desired property. □

Notation 1.7. *We denote "the" initial transmission in Theorem 1.5 by $\alpha_{U,\gamma}$, the semiring U' by U_γ, and the equivalence relation $E(\alpha_{U,\gamma})$ by $E(U, \gamma)$.*

This notation is sloppy, since $\alpha_{U,\gamma}$ is determined by U and γ only up to isomorphism. But $E(U, \gamma)$ truly depends only on U and γ. The ambiguity for $\alpha_{U,\gamma}$ can be avoided if γ is surjective, due to the following lemma.

Lemma 1.8. *If $\alpha : U \to V$ is an initial transmission covering a surjective homomorphism $\gamma : M \to M'$, then α itself is a surjective map.*

Proof. $V_1 := \alpha(V)$ is a subsemiring of V and thus a supertropical semiring itself. Replacing V by V_1 we obtain from α a surjective transmission $\alpha_1 : U \to V_1$. Since α is initial there exists a unique transmission $\eta : V \to V_1$ over M' with $\alpha_1 = \eta\alpha$. Also $\alpha = j\alpha_1$ with j the inclusion from V_1 to V. By the universal property of α we conclude from $\alpha = j\eta\alpha$ that $j\eta$ is the identity on V. This forces $V = V_1$. □

Thus, if γ is surjective, we have a canonical choice for U_γ and $\alpha_{U,\gamma}$, namely, $U_\gamma = U/E(U,\gamma)$ and $\alpha_{U,\gamma} = \pi_{E(U,\gamma)}$. Usually we will understand by U_γ and $\alpha_{U,\gamma}$ this semiring and transmission.

In light of Theorem 1.5 our main Problem 1.1 can be posed as follows: Given U and γ, is $\alpha_{U,\gamma} : U \to U_\gamma$ a pushout transmission?

We assume in the following that $\gamma : M \to M'$ *is surjective and* M' *is a cancellative bipotent domain*; hence $v = \gamma \circ \nu_U$ is a strict surjective valuation. In this case we will obtain a positive solution of the problem. The point here is that we can give an explicit description of U_γ and $\alpha_{U,\gamma}$, which allows us to check the pushout property.

We already have an explicit description of $\varphi_v : U \to U(v)$, given in [IKR1, §4]. Thus all we need is an explicit description of the finest MFCE-relation T on $U(v)$ with $f \sim_T e$. We develop such a description in a more general setting.

Assume that U is a supertropical semiring, $e := e_U$, and f is an idempotent of U. The ideal $L := fU$ of U is again a supertropical semiring with unit element f (under the addition and multiplication of U), since L is a homomorphic image of U. We have $e_L = f + f = ef$.

If F is an equivalence relation on the set L, there is a unique finest equivalence relation E on U extending F. It can be described as follows. Let $x_1, x_2 \in U$. Then $x_1 \sim_E x_2$ iff either $x_1 = x_2$ or $x_1 \in L$, $x_2 \in L$ and $x_1 \sim_F x_2$. We call E the **minimal extension** of the equivalence relation F to U.

Lemma 1.9. *Let F be an equivalence relation on fU, and let E denote the minimal extension of F to U.*

a) *If F is multiplicative, then E is multiplicative.*
b) *If F is fiber conserving, so is E.*

Proof. Assume that x_1, x_2 are elements of U with $x_1 \sim_E x_2$. Assume (without loss of generality) that also $x_1 \neq x_2$. Then $x_1, x_2 \in fU$ and $x_1 \sim_F x_2$.

If F is multiplicative then, for any $z \in U$,

$$x_1 z = x_1(fz) \sim_F x_2(fz) = x_2 z;$$

hence $x_1 z \sim_E x_2 z$. Thus E is multiplicative.

If F is fiber conserving, then

$$ex_1 = (ef)x_1 = (ef)x_2 = ex_2.$$

Thus E is fiber conserving. □

Proposition 1.10. *Assume that U is a supertropical semiring, $e := e_U$, and f is an idempotent of U. We define a binary relation E on U by decreeing ($x_1, x_2 \in U$)*

$$x_1 \sim_E x_2 \quad \text{iff either} \quad x_1 = x_2 \quad \text{or} \quad x_1, x_2 \in fU \quad \text{and} \quad ex_1 = ex_2.$$

a) *E is an MFCE-relation on U.*
b) *If $ef = e$, then $e \sim_E f$, and E is finer than any other multiplicative equivalence relation E' on U with $e \sim_{E'} f$.*

Proof. a) We apply the preceding lemma with F the relation $E(\nu_L)$ (cf. [IKR1, Example 6.4] on the supertropical semiring $L := fU$. The minimal extension of F to U is the relation E defined in the proposition. Indeed, for $x_1, x_2 \in L$ we have $x_1 \sim_F x_2$ if

$efx_1 = efx_2$. Since $fx_i = x_i$ $(i = 1, 2)$, this means that $ex_1 = ex_2$. By Lemma 1.9 the relation E is MFCE.

b) Assume now that $ef = e$, i.e., $e \in L$. Then $e \sim_E f$ by definition of E. Let E' be any multiplicative equivalence relation on U with $e \sim_{E'} f$. If $x_1, x_2 \in U$ and $x_1 \sim_E x_2$ we want to conclude that $x_1 \sim_{E'} x_2$. We may assume that $x_1 \neq x_2$. Then $x_1, x_2 \in fU$ and $ex_1 = ex_2$. Now $x_i \sim_{E'} ex_i$ $(i = 1, 2)$; hence $x_1 \sim_{E'} x_2$, as desired. $\square$

We are ready for a solution of Problem 1.1 in the case that $\gamma : M \to M'$ is surjective and M' is a cancellative bipotent semidomain; hence $v = \gamma \circ \nu_U$ is a strict surjective valuation. As before, let T denote the finest MFCE-relation on $U(v)$ with $f \sim_T e$ for $e := e_{U(v)}$ and $f := \varphi_v(e_U)$. Recall from the proof of Theorem 1.5 that $ef = e$. Thus Proposition 1.10 applies. We spell out what the proposition says in the present case.

For that we write the semiring $U(v)$ and the map φ_v in a way different from [IKR1, §4]. Let $\widehat{U}$ denote a copy of U disjoint from U with copying isomorphism $x \mapsto \hat{x}$. We use this to distinguish an element $x \in U \backslash \mathfrak{q}$, with $\mathfrak{q} := \operatorname{supp} v$, from the corresponding element in $\mathcal{T}(U(v))$. Thus we write

$$U(v) = (\widehat{U} \backslash \hat{\mathfrak{q}}) \ \dot{\cup}\ M'$$

with $\hat{\mathfrak{q}} := \{\hat{x} \mid x \in U,\ \gamma(e_U x) = 0\}$, and $\varphi_v(x) = \hat{x}$ for $x \in U \backslash \mathfrak{q}$, $\varphi_v(x) = 0$ for $x \in \mathfrak{q}$. Notice that $fU(v) = (\widehat{M} \backslash \hat{\mathfrak{q}}) \cup M'$ with $\widehat{M} := \{\hat{x} \mid x \in M\}$.

According to Proposition 1.10 the equivalence relation T has the following description. Let $y_1, y_2 \in U(v)$ be given with $y_1 \neq y_2$. Then $y_1 \sim_T y_2$ iff $y_1 = \hat{x}_1$, $y_2 = \hat{x}_2$, with either $x_1, x_2 \in M$ and $\gamma(e_U x_1) = \gamma(e_U x_2)$ or $x_1, x_2 \in U$ and $\gamma(e_U x_1) = \gamma(e_U x_2) = 0$. We may choose $U_\gamma = U(v)/T$ and $\alpha_{U,\gamma} = \pi_T \circ \varphi_v$. The transmission $\alpha := \alpha_{U,\gamma}$ is a surjective map from U to U_γ, and the equivalence relation $E(\alpha)$ is the relation $E(U, \gamma)$ defined in Notation 1.7. Thus $E := E(U, \gamma)$ has the following description: If $x_1, x_2 \in U$ and $x_1 \neq x_2$ then

$$x_1 \sim_E x_2 \ \Leftrightarrow\ \gamma(e_U x_1) = \gamma(e_U x_2), \text{ and if } x_1 \in \mathcal{T}(U) \text{ or } x_2 \in \mathcal{T}(U),\ \gamma(e_U x_1) = 0.$$

Having found $E(U, \gamma)$ we now redefine

$$U_\gamma := U/E(U, \gamma), \qquad \alpha_{U,\gamma} := \pi_{E(U,\gamma)}.$$

We arrive at the following theorem.

Theorem 1.11. *Let U be a supertropical semiring, $e := e_U$, $M := eU$, and assume that $\gamma : M \to M'$ is a surjective homomorphism from M to a cancellative bipotent semidomain M'. Then $E := E(U, \gamma)$ can be described as follows $(x_1, x_2 \in U)$:*

$$\begin{aligned} x_1 \sim_E x_2 \quad \textit{iff} \quad & x_1 = x_2, \\ \textit{or} \quad & \gamma(ex_1) = \gamma(ex_2),\ ex_1 = x_1,\ ex_2 = x_2, \\ \textit{or} \quad & \gamma(ex_1) = \gamma(ex_2) = 0. \end{aligned}$$

Scholium 1.12. *Thus this binary relation E on U is a multiplicative equivalence relation, and the multiplicative monoid U/E can be turned into a supertropical semiring in a unique way such that $\pi_E : U \to U/E$ is a transmission. It is the initial transmission covering γ.*

Most often π_E is not a homomorphism, cf. §6 below.

Theorem 1.13. *If γ is surjective and M' is a cancellative bipotent semidomain, then $\alpha_{U,\gamma}$ is a pushout transmission.*

Proof. Let $\alpha := \alpha_{U,\gamma} = \pi_E : U \to U/E$ with $E := E(U,\gamma)$. Assume that $\delta : M' \to N$ is a homomorphism from M' to a bipotent semiring N and $\beta : U \to V$ is a transmission covering $\delta\gamma : M \to N$, i.e., with $e_V \beta = \delta\gamma e_U$. (In particular $eV = N$.)

We want to verify that β respects the equivalence relation E, i.e., given $x_1, x_2 \in U$, that

$$x_1 \sim_E x_2 \quad \text{implies} \quad \beta(x_1) = \beta(x_2).$$

We may assume that $x_1 \neq x_2$. If x_1 or x_2 is tangible then $\gamma(ex_1) = \gamma(ex_2) = 0$; hence $e_V\beta(x_i) = \delta\gamma(ex_i) = 0$ for $i = 1, 2$. This implies $\beta(x_1) = \beta(x_2) = 0$. Assume now that both x_1 and x_2 are ghost. Then $\gamma(ex_1) = \gamma(ex_2)$; hence $\delta\gamma(ex_1) = \delta\gamma(ex_2)$, i.e., $e_V\beta(x_1) = e_V\beta(x_2)$. But both $\beta(x_1)$ and $\beta(x_2)$ are ghost or zero. Thus $\beta(x_1) = \beta(x_2)$ again.

Since α is surjective, it follows that we have a well-defined map $\rho : U/E \to V$ with $\beta = \rho\alpha$. Now [IKR1, Proposition 6.1.ii] tells us that ρ is a transmission, since both α and β are transmissions and α is surjective. We have

$$\nu_V \rho\alpha = \nu_V \beta = \delta\gamma\nu_U = \delta\nu_{U/E}\alpha.$$

Since α is surjective, this implies that $\nu_V\rho = \delta\nu_{U/E}$, i.e., ρ covers δ. The pushout property of α is verified. ☐

Remark 1.14. *If γ is surjective, but M' is not assumed to be cancellative, we have a description of $E(U,\gamma)$ in* [IKR3, §4]*, which is nearly as explicit as the description above in Theorem 1.11, but then often $\alpha_{U,\gamma}$ is not a pushout transmission.*

Assume now that U is **any** supertropical semiring, $M := eU$, and $\gamma : M \to M'$ is an **injective** semiring homomorphism from M to a bipotent semiring M'. Then Problem 1.1 can be solved affirmatively in an easy direct way, as we explicate now.

We may assume, without loss of generality, that M is a subsemiring of M' and γ is the inclusion from M to M'. We define a semiring U' as follows. As a set, U' is the disjoint union of the sets U and $M' \backslash M$. We have $U \subset U'$, $M' \subset U'$, $U \cup M' = U'$, $U \cap M' = M$. Let ν denote the ghost map from U to M, $\nu = \nu_U$. We define addition and multiplication on U by taking the given addition and multiplication on U and on M', and putting

$$x \cdot z = z \cdot x = \nu(x) \cdot z$$

$$x + z = z + x = \begin{cases} x & \text{if} \quad \nu(x) > z \\ z & \text{if} \quad \nu(x) \leqslant z \end{cases}$$

for $x \in U$, $z \in M'$. In the cases that $x \in M$ and $z \in M'$, or $x \in U$ and $z \in M$, these new products are the same as the ones in M' or U, respectively. Thus we have well-defined operations $\cdot$ and $+$ on U'. One checks in any easy and straightforward way that they obey all of the semiring axioms. Thus U' is now a commutative semiring with $1_{U'} = 1_U$. It clearly obeys the axioms (3.3′), (3.3″), (3.3) in [IKR1]. Thus U' is supertropical. We have $e_{U'} = e_U$, $eU' = M'$, $\mathcal{T}(U') = \mathcal{T}(U)$.

Definition 1.15. *We call U' the supertropical semiring* ***obtained from*** U ***by extension of the ghost ideal*** M ***to*** M'. *We also say, more briefly, that U' is a* ***ghost extension of*** U.

Let α denote the inclusion $U \hookrightarrow U'$. It is obvious that α is a transmission covering the inclusion $\gamma : M \hookrightarrow M'$. We verify that α is a pushout transmission.

Let $\delta : M' \to N$ be a homomorphism from M' to a bipotent semiring N and $\beta : U \to V$ a transmission covering $\delta\gamma$. This means that $eV = N$, and

(1) $\beta(x) = \delta(x)$ for $x \in M$.

Clearly, we have a unique well-defined map $\rho : U' \to V$ with $\rho|U = \beta$ and

(2) $\rho(x) = \delta(x)$ for $x \in M'$.

We have $\rho(0) = 0$, $\rho(1) = 1$, $\rho(e_{U'}) = e_V$. One checks easily that ρ is multiplicative.

We now know that ρ is a transmission covering δ. We have proved the following theorem.

Theorem 1.16. *Assume that M' is a bipotent semiring and M is a subring of M'. Assume further that U is a supertropical semiring with ghost ideal M, and U' is the supertropical semiring obtained from U by extension of the ghost ideal M to M'. Then the inclusion mapping $U \to U'$ is a pushout tranmission covering the inclusion mapping $M \hookrightarrow M'$.*

Combining Theorems 1.13 and 1.16, we obtain the most comprehensive solution of Problem 1.1 that we can offer in this section.

Theorem 1.17. [2] *Let $\gamma : M \to M'$ be a homomorphism between bipotent semirings, and assume that the bipotent semiring $\gamma(M)$ is cancellative. {N.B. This holds if M' is cancellative.} Let U be a supertropical semiring with $eU = M$. Then $\alpha_{U,\gamma} : U \to U_\gamma$ is a pushout transmission.*

Proof. We have a factorization $\gamma = i \circ \bar{\gamma}$, with $\bar{\gamma}$ the map $x \mapsto \gamma(x)$ from M to the subsemiring $\gamma(M)$ of M', and i the inclusion from $\gamma(M)$ to M'. By Theorems 1.13 and 1.16 there exist pushout transmissions $\alpha : U \to \overline{U}$ and $\beta : \overline{U} \to U'$ covering $\bar{\gamma}$ and i, respectively. Now look at the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\ \alpha\ } & \overline{U} & \xrightarrow{\ \beta\ } & U' \\ \uparrow & & \uparrow & & \uparrow \\ M & \underset{\bar{\gamma}}{\twoheadrightarrow} & \gamma(M) & \underset{i}{\hookrightarrow} & M' \end{array}$$

where the vertical arrows denote inclusions. Here the left and the right square are pushout diagrams in the category STROP of supertropical semirings and transmissions. Thus also the outer rectangle is a pushout in this category (cf., e.g., [ML, p.72, Execr.8]), i.e., $\beta\alpha$ is a pushout transmission. If $\alpha_{U,\gamma} : U \to U_\gamma$ is any prechosen initial covering of γ, there exists an isomorphism $\rho : U' \to U_\gamma$ over M' with $\rho\beta\alpha = \alpha_{U,\gamma}$. Thus also $\alpha_{U,\gamma}$ is a pushout transmission. □

2. Pushouts of tangible supervaluations

If $\varphi : R \to U$ and $\psi : R \to V$ are supervaluation on a semiring R, and φ dominates ψ, then we also say that ψ is a **coarsening** of φ . Recall that this happens iff there is a transmission $\alpha : U \to V$ with $\psi = \alpha \circ \varphi$. If in addition φ is surjective, i.e., $U = \varphi(R) \cup e\varphi(R)$, which is no essential loss of generality, then α is uniquely determined by φ and ψ, and we write $\alpha = \alpha_{\psi,\varphi}$ (cf. [IKR1, §5]).

Assume now that $v : R \to M$ is a surjective m-valuation and $\varphi : R \to U$ is a surjective supervaluation covering v (in particular $M = eU$). Moreover, let $\gamma : M \to N$ be a surjective homomorphism to another (bipotent) semiring N.

[2] In §5 and [IKR3, §1] we will meet pushout transmissions which are not covered by this theorem.

Definition 2.1. *We say that a surjective supervaluation $\psi : R \to V$ is the* ***initial coarsening of*** φ ***along*** γ*, if ψ is a coarsening of φ and $\alpha_{\psi,\varphi}$ is the initial transmission covering γ (cf. Definition 1.3). In the notation 1.7; which we will obey in the following, this means that $V = U_\gamma$ and $\alpha_{\psi,\varphi} = \alpha_{U,\gamma}$. We then write $\psi = \gamma_*(\varphi)$.*

In this way we obtain a map

$$\gamma_* : \mathrm{Cov}(v) \to \mathrm{Cov}(\gamma v)$$

between complete lattices.

[We could define such a map γ_* also if $\gamma : M \to N$ is not necessarily surjective. But in the present section this will give no additional insight.]

In the following, we will tacitly assume that all occurring supervaluations are surjective.

We write down a functional property of the initial transmissions $\alpha_{U,\gamma}$, which will give us simple properties of the maps γ_*. The map $\gamma : M \to N$ is always assumed to be a surjective homomorphism between bipotent semirings (as before).

Proposition 2.2. *Let U and V be supertropical semirings with $eU = eV = M$ and let $\lambda : U \to V$ be a transmission over M, hence a homomorphism*[3]*.*

(a) *Then there exists a unique transmission from U_γ to V_γ over N, denoted by λ_γ, such that*

$$\lambda_\gamma \circ \alpha_{U,\gamma} \ = \ \alpha_{V,\gamma} \circ \lambda.$$

We thus have a commuting diagram

$$\begin{array}{ccc} V & \xrightarrow{\alpha_{V,\gamma}} & V_\gamma \\ \uparrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle\lambda_\gamma} \\ U & \xrightarrow{\alpha_{U,\gamma}} & U_\gamma \\ \uparrow & & \uparrow \\ M & \xrightarrow{\gamma} & N \end{array}$$

with inclusion mappings $M \hookrightarrow U$ and $N \hookrightarrow U_\gamma$.

(b) *If $\xi : V \to W$ is a second homomorphism over M then*

$$\xi_\gamma \lambda_\gamma \ = \ (\xi\lambda)_\gamma.$$

Proof. a): $\alpha_{V,\gamma}\lambda : U \to V_\gamma$ is a transmission covering γ. Now use the universal property of the initial transmission $\alpha_{U,\gamma}$.

b): $\xi_\gamma\lambda_\gamma : U_\gamma \to W_\gamma$ is a transmission over N such that

$$\xi_\gamma\lambda_\gamma\alpha_{U,\gamma} = \xi_\gamma\alpha_{V,\gamma}\lambda = \alpha_{W,\gamma}\xi\lambda.$$

By the uniqueness part in a) we conclude that $\xi_\gamma\lambda_\gamma = (\xi\lambda)_\gamma$. □

As an immediate consequence of part b) we have

Corollary 2.3. *The map $\gamma_* : \mathrm{Cov}(v) \to \mathrm{Cov}(\gamma v)$ is order preserving in the weak sense, i.e., $\varphi \geqslant \psi$ implies $\gamma_*(\varphi) \geqslant \gamma_*(\psi)$.* □

[3] Any transmission $U \to W$, which is injective on eU, is a homomorphism, cf. [IKR1, Proposition 5.10.iii].

Corollary 2.4. *If $\varphi : R \to U$ and $\psi : R \to V$ are supervaluations covering v (in particular $eU = eV = M$) with $\varphi \geqslant \psi$ then*

$$\alpha_{\gamma_*(\psi),\gamma_*(\varphi)} \ = \ (\alpha_{\psi,\varphi})_\gamma.$$

Proof. We have $\psi = \lambda\varphi$ with $\lambda := \alpha_{\psi,\varphi}$. From this we conclude that

$$\gamma_*(\psi) \ = \ \alpha_{V,\gamma}\lambda\varphi \ = \ \lambda_\gamma\alpha_{U,\gamma}\varphi \ = \ \lambda_\gamma\gamma_*(\varphi).$$

Thus λ_γ is the transmission from $\gamma_*(\varphi)$ to $\gamma_*(\psi)$. □

Starting from now *we assume that the bipotent semirings M and N are cancellative*; hence $v : R \to M$ and $\gamma v : R \to N$ are valuations. We define

$$\mathfrak{p} := \gamma^{-1}(0), \qquad \mathfrak{q} := v^{-1}(0) = \operatorname{supp}(v), \qquad \mathfrak{q}' := v^{-1}(\mathfrak{p}) = \operatorname{supp}(\gamma v).$$

Notice that $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{q}'$ are prime ideals of M and R, respectively.

Given any supertropical semiring U with $eU = M$, we now know that $\alpha_{U,\gamma} : U \to U_\gamma$ is a pushout transmission (Theorem 1.13). Consequently, if $\varphi \in \operatorname{Cov}(v)$, we now call $\gamma_*(\varphi)$ the **pushout of φ along** γ (instead of "initial coarsening of φ along γ").

The good thing is that we now have an explicit descriptions of U_γ and $\alpha_{U,\gamma}$ which we recall from Theorem 1.11.

We start with a multiplicative equivalence relation $E(U,\gamma)$ on U established in Theorem 1.11. To repeat, for x, y in U

$$x \sim_{E(U,\gamma)} y \quad \Longleftrightarrow \quad \begin{array}{l} \text{either } x = y, \\ \text{or both } x, y \in M \text{ and } \gamma(x) = \gamma(y), \\ \text{or } ex \in \mathfrak{p},\ ey \in \mathfrak{p}. \end{array}$$

The restriction $E(U,\gamma)|M$ is the equivalence relation $E(\gamma)$ given by $\gamma : M \twoheadrightarrow N$. We identify every class $[x]_{E(U,\gamma)}$, $x \in M$, with the image $\gamma(x) \in N$ and then have

$$M/E(U,\gamma) = N.$$

As proved in §1, we may choose[4] $U_\gamma = U/E(U,\gamma)$ and then have

$$\alpha_{U,\gamma} = \pi_{E(U,\gamma)} : x \longmapsto [x]_{E(U,\gamma)}.$$

Let $x \in \mathcal{T}(U)$. If $ex \notin \mathfrak{p}$, then $[x]_{E(U,\gamma)} = \{x\}$, but if $ex \in \mathfrak{p}$, then $[x]_{E(U,\gamma)} = 0 \in N$. Thus we see that $\mathcal{T}(U_\gamma) = U_\gamma \backslash N$ is the bijective image of $\{x \in \mathcal{T}(U) \mid ex \notin \mathfrak{p}\}$. We identify $[x]_{E(U,\gamma)}$ with x, if x lies in this set, and then have

$$\mathcal{T}(U_\gamma) = \{x \in \mathcal{T}(U) \mid ex \notin \mathfrak{p}\}, \qquad U_\gamma = \{x \in \mathcal{T}(U) \mid ex \notin \mathfrak{p}\} \ \dot\cup \ N.$$

Notice that the multiplicative monoid $\mathcal{T}(U_\gamma)$ has become a submonoid of $\mathcal{T}(U)$, since $E(U,\gamma)$ is multiplicative, but the sum of two elements of $\mathcal{T}(U_\gamma)$, computed in the semiring U_γ, can be very different from their sum in U.

After all these identifications we have

Lemma 2.5. *For any $x \in U$,*

$$\alpha_{U,\gamma}(x) \ = \ \begin{cases} x & \text{if } x \in \mathcal{T}(U), ex \notin \mathfrak{p}, \\ 0 & \text{if } x \in \mathcal{T}(U), ex \in \mathfrak{p}, \\ \gamma(x) & \text{if } x \in M. \end{cases}$$

□

[4]Recall that $\alpha_{U,\gamma} : U \to U_\gamma$ is the solution of a universal problem.

Given a surjective valuation $v : R \to M$, as before, we denote by $\mathrm{Cov_t}(v)$ the set of all (equivalence classes of) tangible supervaluations covering v. It is an upper set, and hence a complete sublattice of the lattice $\mathrm{Cov}(v)$ with the same top element $\varphi_v : R \to U(v)$ as $\mathrm{Cov}(v)$ (cf. [IKR1, §10]).

Let $D(M)$ denote the unique supertropical semiring U such that $eU = M$ and ν_U maps $\mathcal{T}(U)$ bijectively onto $M\backslash\{0\}$. The bottom element of $\mathrm{Cov_t}(v)$ is given by the unique tangible supervaluation $\hat{v} : R \to D(M)$ covering v (cf. [IKR1, Example 9.16]).

Returning to an arbitrary covering $\varphi : R \to U$ of v, we read off from Lemma 2.5 the $\gamma_*(\varphi)$ is tangible if φ is tangible. This implies

Proposition 2.6. $\gamma_*(\mathrm{Cov_t}(v)) \subset \mathrm{Cov_t}(\gamma v)$.

We further have the following important fact.

Theorem 2.7. *The pushout of the initial covering $\varphi_v : R \to U(v)$ of v is the initial covering $\varphi_{\gamma v} : R \to U(\gamma v)$ of γv. In particular $U(\gamma v) = U(v)_\gamma$.*

Proof. Recall that $\mathcal{T}(U(v)) = R\backslash\mathfrak{q}$ and $\mathcal{T}(U(\gamma v)) = R\backslash\mathfrak{q}'$ with $\mathfrak{q} = \mathrm{supp}(v)$ and $\mathfrak{q}' = \mathrm{supp}(\gamma v) = v^{-1}(\mathfrak{p})$. Thus it is fairly obvious that $U(\gamma v) = U(v)_\gamma$. If $a \in R$, we have

$$\gamma_*(\varphi_v)(a) \ = \ \alpha_{U,\gamma}(\varphi_v(a));$$

hence, by Lemma 2.5, $\gamma_*(\varphi_v)(a) = \varphi(a)$ if $v(a) = e\varphi_v(a) \notin \mathfrak{p}$, while $\gamma_*(\varphi_v)(a) = 0$ if $v(a) \in \mathfrak{p}$. These are precisely the values attained by $\varphi_{\gamma v}$. □

We focus on the restriction

$$\gamma_{*,\mathrm{t}} : \mathrm{Cov_t}(v) \to \mathrm{Cov_t}(\gamma v)$$

of γ_* to tangible supervaluations. It maps the top element φ_v of $\mathrm{Cov_t}(\gamma)$ to the top element $\varphi_{\gamma v}$ of $\mathrm{Cov_t}(\gamma v)$. But it almost never maps the bottom element $\hat{v}$ of $\mathrm{Cov_t}(v)$ to the bottom element $\widehat{\gamma v}$ of $\mathrm{Cov_t}(\gamma v)$, as we will see below.

Our goal now is to exhibit a sublattice of $\mathrm{Cov_t}(v)$ which maps bijectively onto $\gamma_{*,\mathrm{t}}(\mathrm{Cov_t}(v))$ under the pushout map $\gamma_{*,\mathrm{t}}$. For that we need a construction of general interest.

In the following *we always assume that $eU = M$ and $\mathcal{T}(U)$ is closed under multiplication.*

Given an ideal $\mathfrak{a}$ of M we introduce the equivalence relation

$$E_\mathrm{t}(\mathfrak{a}) \ := \ E_{t,U}(\mathfrak{a}) \ := \ E_\mathrm{t} \cap E(M\backslash\mathfrak{a}),$$

with E_t and $E(M\backslash\mathfrak{a})$ the MFCE-relations defined in [IKR1, Examples 6.4.v and 6.12]. Clearly $E_\mathrm{t}(\mathfrak{a})$ is a ghost separating equivalence relation.

$E := E_\mathrm{t}(\mathfrak{a})$ has the following explicit description: Let $x, y \in U$ be given. If $x \in M$, or if $x \in \mathcal{T}(U)$, but $ex \notin \mathfrak{a}$, then $x \sim_E y$ iff $x = y$. If $x \in \mathcal{T}(U)$ and $ex \in \mathfrak{a}$, then $x \sim_E y$ iff $y \in \mathcal{T}(U)$ and $ex = ey$.

Definition 2.8.

(a) *We call the supertropical semiring $U/E_\mathrm{t}(\mathfrak{a})$ consisting of the $E_\mathrm{t}(\mathfrak{a})$-equivalence classes the* ***t-collapse (= tangible collapse)*** *of U over $\mathfrak{a}$ and we denote this semiring by $c_{\mathrm{t},\mathfrak{a}}(U)$.*

(b) *We call the natural semiring homomorphism*

$$\pi_{E_{\mathrm{t}}(\mathfrak{a})} : U \to c_{\mathrm{t},\mathfrak{a}}(U)$$

the ***t-collapsing map of*** U ***over*** $\mathfrak{a}$*, and we denote this map by* $\pi_{\mathrm{t},\mathfrak{a}}$*, or* $\pi_{\mathrm{t},\mathfrak{a},U}$ *if necessary.*

(c) *If* $\varphi : R \to U$ *is a tangible supervaluation covering* v*, we call the supervaluation*

$$\varphi/E_{\mathrm{t}}(\mathfrak{a}) = \pi_{\mathrm{t},\mathfrak{a}} \circ \varphi$$

the ***t-collapse of*** φ ***over*** $\mathfrak{a}$*, and we denote this supervaluation by* $c_{\mathrm{t},\mathfrak{a}}(\varphi)$*.*

(d) *Finally, we say that* U *is* ***t-collapsed over*** $\mathfrak{a}$*, if* $\pi_{\mathrm{t},\mathfrak{a}}$ *is an isomorphism, for which we abusively write* $c_{\mathrm{t},\mathfrak{a}}(U) = U$*, and we say that* φ *is* ***t-collapsed over*** $\mathfrak{a}$ *if* $c_{\mathrm{t},\mathfrak{a}}(\varphi)) = \varphi$ *(which happens iff* $c_{\mathrm{t},\mathfrak{a}}(U) = U$*, since our supervalutions are assumed to be surjective).*

We describe the semiring $c_{\mathrm{t},\mathfrak{a}}(U)$ more explicitly. Without essential loss of generality we assume that $e\mathcal{T}(U)_0 = M$.

If Z is any subset of M, let $U|_Z$ denote the preimage of Z under the ghost map ν_U,

$$U|_Z \ := \ \{x \in U \mid ex \in Z\}.$$

Now, if U is t-collapsed over $\mathfrak{a}$, every $z \in U$ has a unique tangible preimage under ν_U. We denote this preimage by $\check{z}$, and then have

$$U|_{\mathfrak{a}} \ = \ \mathfrak{a} \ \dot{\cup} \ \check{\mathfrak{a}}$$

with $\check{\mathfrak{a}} = \{\check{z} \mid z \in \mathfrak{a}\}$.

In general we identify

$$c_{\mathrm{t},\mathfrak{a}}(U)|_{M\setminus\mathfrak{a}} \ = \ U|_{M\setminus\mathfrak{a}}.$$

This makes sense since $[x]_{E_{\mathrm{t}}(\mathfrak{a})} = \{x\}$ for any $x \in U|_{M\setminus\mathfrak{a}}$. We then have

$$c_{\mathrm{t},\mathfrak{a}}(U) \ = \ (U|_{M\setminus\mathfrak{a}} \ \cup \ M) \ \dot{\cup} \ \check{\mathfrak{a}}$$

and

$$U|_{M\setminus\mathfrak{a}} \ \cap \ M \ = \ M\setminus\mathfrak{a}.$$

After these identifications the following is obvious.

Lemma 2.9.

(i) *If* $x \in U$ *then*

$$\pi_{\mathrm{t},\mathfrak{a}}(x) \ = \ \begin{cases} x & \text{if } x \in M \text{ or } ex \notin \mathfrak{a}, \\ \widetilde{(ex)} & \text{if } ex \in \mathfrak{a} \end{cases}$$

(ii) *If* $\varphi \in \mathrm{Cov}_{\mathrm{t}}(U)$ *and* $a \in R$*, then*

$$c_{\mathrm{t},\mathfrak{a}}(\varphi)(a) \ = \ \begin{cases} \varphi(a) & \text{if } v(a) \notin \mathfrak{a}, \\ \widetilde{v(a)} & \text{if } v(a) \in \mathfrak{a}. \end{cases}$$

□

We now look at the map

$$c_{\mathrm{t},\mathfrak{a}} : \mathrm{Cov}_{\mathrm{t}}(v) \to \mathrm{Cov}_{\mathrm{t}}(v)$$

which sends each $\varphi \in \mathrm{Cov}_{\mathrm{t}}(v)$ to its t-collapse $c_{\mathrm{t},\mathfrak{a}}(\varphi)$ over $\mathfrak{a}$. It is clearly order preserving, and is idempotent, i.e., $(c_{\mathrm{t},\mathfrak{a}})^2 = c_{\mathrm{t},\mathfrak{a}}$. We denote its image by $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v)$. Its elements are the t-collapsed tangible supervaluations over $\mathfrak{a}$ that cover v.

Using the description of suprema and infima in the complete lattice $\mathrm{Cov}(v)$ in [IKR1, §7], it is an easy matter to verify the following

Proposition 2.10. $\mathrm{Cov}_{\mathrm{t}}(v)$ *is a complete sublattice of* $\mathrm{Cov}(v)$*, and*

$$c_{\mathrm{t},\mathfrak{a}} : \mathrm{Cov}_{\mathrm{t}}(v) \to \mathrm{Cov}_{\mathrm{t}}(v)$$

respects suprema and infima in $\mathrm{Cov}_{\mathrm{t}}(v)$*. Thus, also* $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v)$ *is a complete sublattice of* $\mathrm{Cov}(v)$*.*

Remark 2.11. *Independently of this proposition it is clear that* $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v)$ *is a lower set in* $\mathrm{Cov}_{\mathrm{t}}(v)$ *with top element* $c_{\mathrm{t},\mathfrak{a}}(\varphi_v)$*. It follows that*

$$\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v) \;=\; \{\psi \in \mathrm{Cov}(v) \mid c_{\mathrm{t},\mathfrak{a}}(\varphi_v) \geqslant \psi \geqslant \breve{v}\ \}.$$

This proves again that $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v)$ *is a complete sublattice of* $\mathrm{Cov}(v)$*.*

We return to the surjective homomorphism $\gamma : M \to N$ and now choose for $\mathfrak{a}$ the prime ideal $\mathfrak{p} = \gamma^{-1}(0)$ of M.

Proposition 2.12. *Let* $V := c_{\mathrm{t},\mathfrak{p}}(U)$*.*

- (i) *The homomorphism* $\pi_{\mathrm{t},\mathfrak{p}} : U \to V$ *induces an isomorphism* $(\pi_{\mathrm{t},\mathfrak{p}})_\gamma : U_\gamma \tilde{\to} V_\gamma$ *over* N*. More precisely, using the identifications from above we have* $U_\gamma = V_\gamma$*, and then* $(\pi_{\mathrm{t},\mathfrak{p}})_\gamma$ *is the identity of* U_γ*.*
- (ii) $\alpha_{U,\gamma} = \alpha_{V,\gamma} \circ \pi_{\mathrm{t},\mathfrak{p}}$*.*
- (iii) *If* $\varphi \in \mathrm{Cov}_{\mathrm{t}}(v)$ *then* $\gamma_*(\varphi) = \gamma_*(c_{\mathrm{t},\mathfrak{p}}(\varphi))$*.*

Proof. We have the identification

$$\mathcal{T}(U|_{M\setminus\mathfrak{p}}) \;=\; \mathcal{T}(V|_{M\setminus\mathfrak{p}})$$

(see above). On the other hand, $\alpha_{U,\gamma}$ maps $U|_{\mathfrak{p}}$ to $\{0_N\}$, and $\alpha_{V,\gamma}$ maps $V|_{\mathfrak{p}}$ to $\{0_N\}$. Finally

$$\alpha_{U,\gamma}|_M \;=\; \alpha_{V,\gamma}|_M \;= \gamma.$$

Thus it is evident that, under our identifications, $U_\gamma = V_\gamma$ and then $\alpha_{U,\gamma} = \alpha_{V,\gamma} \circ \pi_{\mathrm{t},\mathfrak{p}}$. Reading this equality as

$$\mathrm{id}_{U_\gamma} \circ \alpha_{U,\gamma} \;=\; \alpha_{V,\gamma} \circ\; \pi_{\mathrm{t},\mathfrak{p}}$$

we conclude by Proposition 2.2.a that $(\pi_{\mathrm{t},\mathfrak{p}})_\gamma = \mathrm{id}_{U_\gamma}$. Finally, if $\varphi \in \mathrm{Cov}_{\mathrm{t}}(v)$, then

$$\gamma_*(c_{\mathrm{t},\mathfrak{p}}(\varphi)) \;=\; \alpha_{V,\gamma} \circ c_{\mathrm{t},\mathfrak{p}}(\varphi) \;=\; \alpha_{V,\gamma}(\pi_{\mathrm{t},\mathfrak{p}}(\varphi)) \;=\; \alpha_{U,\gamma}(\varphi) \;=\; \gamma_*(\varphi).$$

□

Lemma 2.13. *Let* U*,* V *be supertropical semirings with* $eU = eV = M$*, and* $\lambda : U \to V$ *a transmission over* M *with* $\lambda(\mathcal{T}(U)) \subset \mathcal{T}(V)$*. Assume further that* U *is t-collapsed over* $\mathfrak{p}$*. Finally assume that* $\lambda_\gamma : U_\gamma \to V_\gamma$ *is injective. Then* $\lambda : U \to V$ *is injective.*

Proof. The upper square of the of the diagram in Proposition 2.2.a restricts to a commuting square

$$\begin{array}{ccc} \mathcal{T}(U|_{M\setminus\mathfrak{p}}) & \xrightarrow[\text{id}]{\sim} & \mathcal{T}(U_\gamma) \\ {\scriptstyle ``\lambda"}\downarrow & & \downarrow{\scriptstyle ``\lambda_\gamma"} \\ \mathcal{T}(V|_{M\setminus\mathfrak{p}}) & \xrightarrow[\text{id}]{\sim} & \mathcal{T}(V_\gamma) \end{array}$$

Here the vertical arrows are restrictions of the maps λ and λ_γ. The vertical arrow on the right is an injective map by assumption. Thus, also the left vertical arrow is an injective map. The restriction $\lambda|\mathcal{T}(U|_\mathfrak{p})$ is a priori forced to be injective, since U is t-collapsed over $\mathfrak{p}$. Finally λ restricts to the identity on M. Thus, λ is injective. □

We now are ready for the main result of this section

Theorem 2.14. *As before assume that $\mathcal{T}(U)$ is closed under multiplication.*

(a) *The pushout map*
$$\gamma_{*,\mathrm{t}} : \mathrm{Cov}_\mathrm{t}(v) \ \to \ \mathrm{Cov}_\mathrm{t}(\gamma v)$$
restricts to a bijection from $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{p}}(v)$ *to* $\gamma_*(\mathrm{Cov}_\mathrm{t}(\gamma v))$. *Consequently* $\gamma_*(\mathrm{Cov}_\mathrm{t}(\gamma v))$ *is a sublattice of* $\mathrm{Cov}_\mathrm{t}(\gamma v)$ *isomorphic to* $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{p}}(v)$.

(b) *If* $\varphi,\psi \in \mathrm{Cov}_\mathrm{t}(v)$ *then* $\gamma_*(\varphi) = \gamma_*(\psi)$ *iff* φ *and* ψ *have the same t-collapse over* $\mathfrak{p}$.

Proof. a): Since we know already that $\gamma_*|\,\mathrm{Cov}_{\mathrm{t},c,\mathfrak{p}}(v)$ is a lattice homomorphism (Proposition 2.10), it suffices to verify the following: If $\varphi,\psi \in \mathrm{Cov}_\mathrm{t}(v)$ are t-collapsed over $\mathfrak{p}$ and $\varphi \geqslant \psi$, but $\varphi \neq \psi$, then $\gamma_*(\varphi) \neq \gamma_*(\psi)$.

We have a unique surjective transmission $\lambda := \alpha_{\psi,\varphi} : U \to V$ with $\psi = \lambda\varphi$. This implies $\gamma_*(\psi) = \lambda_\gamma\gamma_*(\varphi)$ by Corollary 2.4. If λ_γ were an isomorphism then also λ would be an isomorphism by Lemma 2.13 above. But this is not true. Thus λ_γ is not an isomorphism, and this means that $\gamma_*(\psi) \neq \gamma_*(\varphi)$.

b): We know by Proposition 2.12 that $\gamma_*(\varphi) = \gamma_*(c_{\mathrm{t},\mathfrak{p}}(\psi))$. Thus $\gamma_*(\varphi) = \gamma_*(\psi)$ iff $\gamma_*(c_{\mathrm{t},\mathfrak{p}}(\varphi)) = \gamma_*(c_{\mathrm{t},\mathfrak{p}}(\psi))$. By part a) this happens iff $c_{\mathrm{t},\mathfrak{p}}(\varphi) = c_{\mathrm{t},\mathfrak{p}}(\psi)$. □

We turn to the image of the map $\gamma_{*,\mathrm{t}} : \mathrm{Cov}_\mathrm{t}(v) \to \mathrm{Cov}_\mathrm{t}(\gamma v)$. Here we will put emphasis on strong supervaluations. Thus we now assume in addition that the surjective valuation $v : R \to M$ is strong.

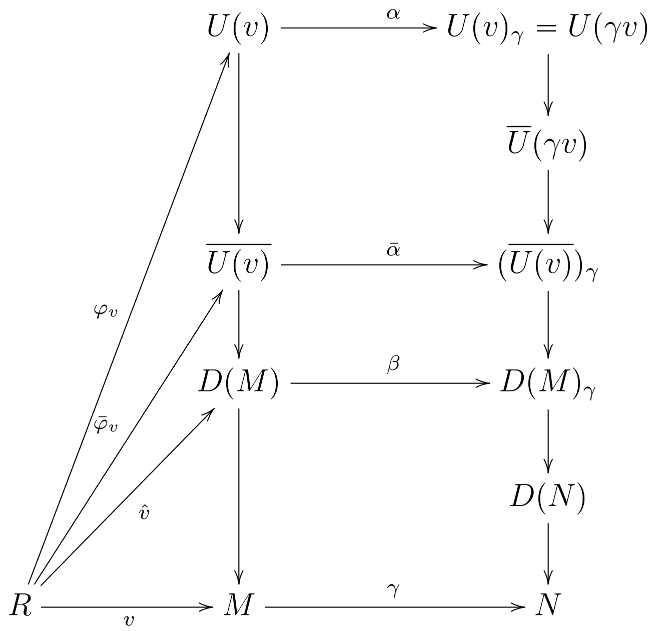
If $\varphi : R \to U$ is a strong supervaluation covering v, then $\gamma_*(\varphi) = \alpha_{U,\gamma} \circ \varphi$ is again a strong supervaluation, as follows from [IKR1, Lemma 10.1.ii] and the definition of "strong" [IKR1, Definition 9.9]. Thus

$$\gamma_*(\mathrm{Cov}_{\mathrm{t,s}}(\varphi)) \ \subset \mathrm{Cov}_{\mathrm{t,s}}(\gamma v).$$

We have seen that $\gamma_*(\varphi_v) = \varphi_{\gamma v}$, but we can only state that the pushout $\gamma_*(\overline{\varphi}_v)$ of the initial strong supervaluation $\overline{\varphi} : R \to \overline{U(v)}$ is dominated by $\overline{\varphi}_{\gamma v} : R \to \overline{U(\gamma v)}$. On the other side, the pushout $\gamma_*(\hat{v})$ of the bottom element $\hat{v} : R \to D(M)$ of both $\mathrm{Cov}_{\mathrm{t,s}}(\varphi)$ and $\mathrm{Cov}_\mathrm{t}(v)$ dominates $\widehat{\gamma v} : R \to D(N)$. Using the abbreviations

$$\alpha := \alpha_{U(v),\gamma}, \qquad \bar{\alpha} := \alpha_{\overline{U(v)},\gamma}, \qquad \beta := \alpha_{D(M),\gamma},$$

we thus have a commuting diagram



with surjective transmissions over M and N respectively as vertical arrows.

The following questions immediately come to mind.

Questions 2.15.

(1) *Can we expect that* $\overline{\varphi}_{\gamma v} = \gamma_*(\overline{\varphi}_v)$*?*
(2) *Can we expect that* $\breve{\gamma v} = \gamma_*(\breve{v})$*?*
(3) *Is* $\gamma_*(\mathrm{Cov_t}(v))$ *convex*[5] *in* $\mathrm{Cov_t}(\gamma v)$*?*
(4) *Is* $\gamma_*(\mathrm{Cov_{t,s}}(v))$ *convex in* $\mathrm{Cov_{t,s}}(\gamma v)$*?*

Recall that $\mathrm{Cov_{t,s}}(\gamma v)$ is convex in $\mathrm{Cov_t}(\gamma v)$, and $\mathrm{Cov_t}(\gamma v)$ is convex in $\mathrm{Cov}(\gamma v)$, as we have seen in [IKR1, §10].

Question (2) has a negative answer: If $z \in N\backslash\{0\}$, then the tangible fiber of $\{x \in D(M)_\gamma \mid ex = z\}$ is the union of the tangible fibers of $D(M)$ over the points of $\gamma^{-1}(z)$, and thus will quite often contain more than one point. The other questions will be answered here completely only in a special case to which we turn now.

Assume that $R\backslash\mathfrak{q}$ is a group under multiplication. Then we can give a very explicit description of the map $\gamma_{*,\mathrm{t}}$, and even γ_*.

Now $M\backslash\{0\} = v(R\backslash\mathfrak{q})$ and $N\backslash\{0\} = \gamma(M\backslash\{0\})$ are groups, i.e., M and N are bipotent semifields. This forces $\mathfrak{p} = 0$ and $\mathfrak{q} = \mathfrak{q}'$.

Since $\mathfrak{p} = 0$ we conclude from Theorem 2.14 and Proposition 2.12 that γ_* is an isomorphism of the lattice $\mathrm{Cov_t}(v)$ onto its image $\gamma_*(\mathrm{Cov_t}(v))$, By [IKR1, §8] the MFCE-relations on $U(v)$ except $E(\nu_U)$ are orbital, hence do not identify any tangibles with ghosts. Thus $\mathrm{Cov}(v) = \mathrm{Cov_t}(v) \cup \{v\}$ (as essentially observed in [IKR1, §8]). We have $\gamma_*(v) = \gamma v$, and we conclude that γ_* *is an isomorphism from* $\mathrm{Cov}(v)$ *onto its image.*

We have $M = \Gamma \cup \{0\}$ with Γ an ordered abelian group. Let

$$\Delta := \gamma^{-1}(1_N).$$

[5]A subset Y of a poset X is called convex in X if $y \leqslant x \leqslant z$ for $y, z \in Y$, $x \in X$ implies that $x \in Y$.

This is a convex subgroup of Γ, since $\gamma : M \to N$ is an order preserving monoid homomorphism. The map γ induces an isomorphism from $M/\Delta = \Gamma/\Delta \cup \{0\}$ onto N. In the following we assume without loss of generality that $N = M/\Delta$ and γ is the map $x \mapsto \Delta x$ from M to N. Excluding a trivial case we assume that $\Delta \neq 1$.

Returning to the notation from the end of [IKR1, §10] we have

$$\mathfrak{o}_v^* = \{a \in R \mid v(a) \in 1_M\} \quad \text{and} \quad \mathfrak{o}_{\gamma v}^* = \{a \in R \mid v(a) \in \Delta\},$$

further $\mathfrak{m}_v = \{a \in R \mid v(a) < 1_M\}$ and $\mathfrak{m}_{\gamma v} = \{a \in R \mid v(a) < \Delta\}$. $\{v(a) < \Delta$ means $v(a) < \delta$ for every $\delta \in \Delta.\}$

If H is a subgroup of $\mathfrak{o}_v^*$ then H is also a subgroup of $\mathfrak{o}_{\gamma v}^*$, since $\mathfrak{o}_v^*$ is a subgroup of $\mathfrak{o}_{\gamma v}^*$. Thus H gives us a transmission

$$\pi_{H,U(v)} : U(v) \;\to\; U(v)/E(H)$$

over M and a transmission

$$\pi_{H,U(\gamma v)} : U(\gamma v) \;\to\; U(\gamma v)/E(H)$$

over N. {Previously both maps sloppily had been denoted by π_H.}

Theorem 2.16. *If H is any subgroup of $\mathfrak{o}_v^*$, then*

(a) $(\pi_{H,U(v)})_\gamma = \pi_{H,U(\gamma v)}$,
(b) $\gamma_*(\varphi_v/H) = \varphi_{\gamma v}/H$.

Proof. a): Let $V := U(v)/E(H)$. We are done by Proposition 2.2.a if we verify that

$$\pi_{H,U(\gamma v)} \circ \alpha_{U(v),\gamma} \;=\; \alpha_{V,\gamma} \circ \pi_{H,U(v)}.$$

This is easily verified by use of Lemma 2.5.

b): We know (Theorem 2.7) that

$$\varphi_{\gamma v} \;=\; \gamma_*(\varphi_v) \;=\; \alpha_{U(v),\gamma} \circ \varphi_v$$

Thus

$$\varphi_{\gamma v}/H \;=\; \pi_{H,U(\gamma v)} \circ \alpha_{U(v),\gamma} \circ \varphi_v.$$

On the other hand

$$\gamma_*(\varphi_v/H) \;=\; \alpha_{V,\gamma}(\varphi_v/H) \;=\; \alpha_{V,\gamma} \circ \pi_{H,U(v)} \circ \varphi_v.$$

By step a) we conclude that indeed

$$\gamma_*(\varphi_v/H) = \varphi_{\gamma v}/H.$$

□

We learned before ([IKR1, §8]) that the elements φ of $\mathrm{Cov_t}$ correspond uniquely with the subgroups H of $\mathfrak{o}_v^*$ via $\varphi = \varphi_v/H$, and now conclude by Theorem 2.16 that

$$\gamma_*(\mathrm{Cov_t}(v)) \;=\; \{\varphi_{\gamma v}/H \mid H \leqslant \mathfrak{o}_v^*\}.$$

(" $\leqslant$ " means subgroup). On the other hand

$$\mathrm{Cov_t}(\gamma v) \;=\; \{\varphi_{\gamma v}/H \mid H \leqslant \mathfrak{o}_{\gamma v}^*\}.$$

Thus, $\gamma_*(\mathrm{Cov_t}(v))$ is an upper set of the complete lattice $\mathrm{Cov_t}(\gamma v)$ with bottom element

$$\gamma_*(\check{v}) \;=\; \varphi_{\gamma v}/\mathfrak{o}_v^*.$$

This element is definitely different from

$$\widetilde{\gamma v} \;=\; \varphi_{\gamma v}/\mathfrak{o}^*_{\gamma v},$$

since $\mathfrak{o}^*_{\gamma v}/\mathfrak{o}^*_v \cong \Delta$. Thus question 2.15.(2) has a negative answer (which we know already), while question 2.15.(3) has a positive answer.

How about question 2.15.(1)? The top element of $\mathrm{Cov}_{\mathrm{t,s}}(v)$ is $\overline{\varphi}_v$. We saw in [IKR1, §10] that $\overline{\varphi}_v = \varphi_v/1 + \mathfrak{m}_v$, and now conclude by Theorem 2.16 that

$$\gamma_*(\overline{\varphi}_v) \;=\; \varphi_{\gamma v}/(1 + \mathfrak{m}_v).$$

But

$$\overline{\varphi}_{\gamma v} \;=\; \varphi_{\gamma v}/(1 + \mathfrak{m}_{\gamma v}),$$

and $\mathfrak{m}_{\gamma v}$ is definitely smaller than $\mathfrak{m}_v$. Thus $\overline{\varphi}_{\gamma v} \gneqq \gamma_*(\overline{\varphi}_v)$. Question 2.15.(1) has a negative answer.

Returning to the general situation, but still with $v : R \to M$ strong, we should expect that $\overline{\varphi}_{\gamma v} \gneqq \gamma_*(\overline{\varphi}_v)$ except in rather pathological cases. Indeed, it seems often possible to pass from $v : R \to M$ to a strong valuation $\tilde{v} : \widetilde{R} \to \widetilde{M}$, with $\widetilde{R}$ a semifield by a localization process (which we did not discuss), and to argue in $\mathrm{Cov}_{\mathrm{t}}(\tilde{v})$.

Concerning applications, the strong supervaluations seem to be more important than the others. But the fact that $\gamma_*(\overline{\varphi}_v)$ differs from $\overline{\varphi}_{\gamma v}$, while $\gamma_*(\varphi_v) = \varphi_{\gamma v}$, indicates that it would not be advisable in supervaluation theory to restrict the study to strong supervaluations from the start, as said already in the Introduction.

3. Supertropical predomains with prescribed ghost map

For later use we give a generalization of Construction 3.16 in [IKR1] of supertropical predomains. It merits independent interest.

Theorem 3.1. *Assume that M is a cancellative bipotent semidomain. Assume further that $U = (U, \cdot\ )$ is an abelian monoid, and $(M, \cdot\ )$ is a monoid ideal of U (i.e., M is a subsemigroup of U and $UM \subset M$). Assume finally that a monoid homomorphism $p : U \to M$ is given (i.e., p is multiplicative and $p(1_U) = 1_M$) with $p(x) = x$ for every $x \in M$ and $p^{-1}(0) = \{0\}$. Then the following hold:*

i) $0 \cdot x = 0$ *for every $x \in U$, and $U \backslash \{0\}$ is closed under multiplication.*

ii) *On U there exists a unique addition $(+)$ extending the addition on M such that $(U, +, \cdot\ )$ is a supertropical semiring with M the ghost ideal and p the ghost map of $U = (U, +, \cdot\ )$.*

iii) *$U = (U, +, \cdot\ )$ is a supertropical predomain, and for $x_1, x_2 \in U$ we have the rule*[6]

$$x_1 + x_2 = \begin{cases} x_1 & \text{if} \;\; p(x_1) > p(x_2), \\ x_2 & \text{if} \;\; p(x_1) < p(x_2), \\ p(x_1) & \text{if} \;\; p(x_1) = p(x_2). \end{cases} \tag{3.1}$$

Proof. We proceed in several steps.

(a) If $x \in U$, then $p(x \cdot 0) = p(x)p(0) = p(x) \cdot 0 = 0$. Thus $x \cdot 0 = 0$.

(b) If $x, y \in U \backslash \{0\}$, then $p(x) \neq 0$, $p(y) \neq 0$; hence $p(xy) = p(x)p(y) \neq 0$, and $xy \neq 0$. Thus, $U \backslash \{0\}$ is closed under multiplication.

[6] Recall that every bipotent semiring has a natural total ordering [IKR1, §2].

(c) We are forced to *define* addition on U by the rule (3.1) above (cf. [IKR1, Theorem 3.11]). Clearly this extends the given addition on M. We have $1_U + 1_U = p(1_U) = 1_M$.

(d) Write $1_M = e$, $1_U = 1$. For $x \in U$ we have
$$e \cdot x = p(e \cdot x) = p(e) \cdot p(x) = e \cdot p(x) = p(x).$$
Thus, $p(x) = e \cdot x$ for every $x \in U$.

(e) We start out to verify that U is a semiring. Obviously, the addition on U is commutative, and it is easily checked that the addition is also associative. For $x \in U$ we have $x + 0 = x$ if $p(x) > 0$, and $x + 0 = 0$ if $p(x) = 0$ iff $x = 0$. Thus, $0 = 0_M$ is the neutral element of the addition on U.

(f) It remains to verify distributivity. Let $x_1, x_2, z \in U$ be given. If $x_1 = 0$ then $x_1 z = 0$, $x_1 + x_2 = x_2$; hence
$$x_1 z + x_2 z = 0 + x_2 z = x_2 z,$$
and thus
$$x_1 z + x_2 z = (x_1 + x_2) z.$$
The same holds if $x_2 = 0$, and clearly also if $z = 0$.

Assume now that $x_1, x_2, z \in G := M \backslash \{0\}$. If $p(x_1) < p(x_2)$ then $p(x_1 z) < p(x_2 z)$ since $p(x_i z) = p(x_i) p(z)$ and the monoid G is cancellative. Thus, $x_1 + x_2 = x_2$, $x_1 z + x_2 z = x_2 z$, and we see again that
$$(x_1 + x_2) z = x_1 z + x_2 z.$$
By symmetry this also holds if $p(x_1) > p(x_2)$. In the case $p(x_1) = p(x_2)$, we have $p(x_1 z) = p(x_2 z)$, $x_1 + x_2 = p(x_1)$, and
$$x_1 z + x_2 z = p(x_1 z) = e x_1 z = p(x_1) z = (x_1 + x_2) z.$$
Now distributivity is proved in all cases.

(g) We have proved that U is a semiring with $x + x = ex = p(x)$ for every $x \in U$, and thus $M = p(U) = eU$. The axioms (3.3′), (3.3″), (3.4) from [IKR1, §3] are now evident. Thus, U is supertropical and $\nu_U = p$. The semiring U is a supertropical predomain.

□

Theorem 3.1 supersedes Construction 3.16 in [IKR1] since here we do not need to assume that $U \backslash M$ is closed under multiplication. Every supertropical semiring U with eU a cancellative bipotent semidomain arises in the way indicated in the theorem.

Example 3.2. *We discuss again the construction of the supertropical semiring $U = U(v)$ for a valuation $v : R \to M$, given in* [IKR1, Example 4.5]*. Let $\mathfrak{q} := v^{-1}(0)$ the support of v, and let U denote the disjoint union of the sets $R \backslash \mathfrak{q}$ and M. We introduce on U a multiplication $\odot$ as follows: For $x, y \in R \backslash \mathfrak{q}$ and $z, w \in M$, put*
$$x \odot y = xy, \quad x \odot z = z \odot x = v(x) z, \quad z \odot w = zw.$$
It is immediate that in this way U becomes an abelian monoid with $U \odot M \subset M$. The map $p : U \to M$ given by $p(x) = v(x)$ for $x \in R \backslash \mathfrak{q}$, $p(z) = z$ for $z \in M$ is a monoid

homomorphism and $p^{-1}(0) = \{0\}$. *Theorem 3.1 tells us that with the addition*

$$x \oplus y := \begin{cases} x & \text{if} \quad p(x) > p(y) \\ y & \text{if} \quad p(x) < p(y) \\ p(x) & \text{if} \quad p(x) = p(y) \end{cases}$$

the monoid U *becomes a supertropical semiring. The map* $\varphi : R \to U$ *with* $\varphi(a) = a$ *for* $a \in R \backslash \mathfrak{q}$, $\varphi(a) = 0$ *for* $a \in \mathfrak{q}$ *turns out to be a supervaluation covering* v.

4. Transmissive equivalence relations

If a surjective transmission $\alpha : U \to V$ is given, V can be identified with the set $U/E(\alpha)$ of equivalence classes of the equivalence relation $E(\alpha)$ [7] in such a way that $\alpha = \pi_{E(\alpha)}$. We now pose the following problem: For which equivalence relations E on a supertropical semiring U can the set U/E be equipped with the structure of a (supertropical) semiring in such a way that $\pi_E : U \to U/E$ is a transmission?

We first study the case $U = eU$.

U is a bipotent semiring, in other words, U is a totally ordered monoid with absorbing smallest element 0, cf. [IKR1, §1].

Assume more generally that M is a totally ordered set and E is an equivalence relation on M. We want to install a total ordering on the set M/E in such a way that the map

$$\pi_E : M \to M/E, \qquad x \mapsto [x]_E,$$

is order preserving (in the weak sense; $x \leqslant y \Rightarrow \pi_E(x) \leqslant \pi_E(y)$). Thus we want that, if $\xi_1, \xi_2 \in M/E$ and $x_1 \in \xi_1$, $x_2 \in \xi_2$, then

$$x_1 \leqslant x_2 \;\Rightarrow\; \xi_1 \leqslant \xi_2,$$

or, equivalently,

$$\xi_1 > \xi_2 \;\Rightarrow\; x_1 > x_2.$$

It is clear that such a total ordering on M/E exists iff the following holds. Given $\xi_1, \xi_2 \in M/E$, either $x_1 < x_2$ for all $x_1 \in \xi_1$, $x_2 \in \xi_2$, or $x_1 > x_2$ for all $x_1 \in \xi_1$, $x_2 \in \xi_2$, or $\xi_1 = \xi_2$. More succinctly, this condition can be written as follows:

(OC) : If $x_1, x_2, x_3, x_4 \in M$, and $x_1 \leqslant x_2$, $x_3 \leqslant x_4$, $x_1 \sim_E x_4$, $x_2 \sim_E x_3$, then $x_1 \sim x_2$.

(Hence all x_i are E-equivalent.)

If an equivalence relation E on the totally ordered set M obeys the rule (OC), we call E **order compatible**.

It is sometimes useful to view order compatibility as a convexity property. A subset Y of M is called **convex** (in M), if for any $y_1, y_2 \in Y$ and $x \in M$ with $y_1 < x < y_2$, also $x \in M$.

Remark 4.1. *An equivalence relation* E *on the totally ordered set* M *is order compatible iff every equivalence class of* E *is convex in* M.

[7] Recall that $E(\alpha)$ is defined by $x \sim_{E(\alpha)} y$ iff $\alpha(x) = \alpha(y)$.

Proof. a) If $y_1 < x < y_2$ and $y_1 \sim_E y_2$, then (OC) implies $y_1 \sim_E x$. (Take there $x_2 = x_3$.)

b) Assume that the equivalence classes of E are convex. We verify (OC). Let $x_1, x_2, x_3, x_4 \in M$ be given with $x_1 \leqslant x_2$, $x_3 \leqslant x_4$, and $x_1 \sim_E x_4$, $x_2 \sim_E x_3$.

Case 1. $x_2 \leqslant x_4$. Now $x_1 \leqslant x_2 \leqslant x_4$, and hence $x_1 \sim_E x_2$.

Case 2. $x_2 > x_4$. Now $x_3 \leqslant x_4 \leqslant x_2$, and hence $x_4 \sim_E x_2$, and thus again $x_1 \sim_E x_2$. □

We present a proposition which is quite obvious from the initial considerations on order compatibility given above.

Proposition 4.2. *Let M be a bipotent semiring and E an equivalence relation on the set M. There exists a (unique) structure of a (bipotent) semiring on the set* [8] *M/E such that the natural map $\pi_E : M \to M/E$, $x \mapsto [x]_E$ is a semiring homomorphism iff E is multiplicative and order compatible. In this case the multiplication on M/E is given by the rule $(x, y \in M)$*

$$[x]_E \cdot [y]_E = [x \cdot y]_E,$$

and the ordering by the rule $(\xi, \eta \in M/E)$

$$\xi \leqslant \eta \quad \Leftrightarrow \quad \exists x \in \xi,\ y \in \eta \quad \text{with} \quad x \leqslant y.$$

Proof. Just notice that a map between bipotent semirings is a semiring homomorphism iff it is multiplicative, sends 0 to 0, 1 to 1, and is compatible with the orderings (cf. [IKR1, §1]). □

We turn to equivalence relations on supertropical semirings instead of just bipotent semirings.

Definition 4.3. *Let U be a supertropical semiring. We call an equivalence relation E on U* ***transmissive*** *if on the set U/E there exists a semiring structure such that U/E is supertropical and the map $\pi_E : U \to U/E$, $x \mapsto [x]_E$ is transmissive.*

We point out that, if E is transmissive, the semiring structure on U/E is uniquely determined by the semiring structure of U and the relation E. This is clear from the following reasoning.

Assume a surjective transmission $\alpha : U \to V$ is given. Let $E := E(\alpha)$. Since the map α is multiplicative, the equivalence relation E has to be multiplicative, and the multiplication on V is determined by U and α, since $\alpha(x) \cdot \alpha(y) = \alpha(xy)$. We have $\alpha(e_U) = e_V$, and α restricts to a surjective homomorphism $eU \to eV$ of bipotent semirings. Thus, the restricted equivalence relation $E|eU$ is order compatible, and the ordering on eV is determined by the ordering of eU and the map α.

It follows that the addition on V is also determined by U and α, since it can be expressed in terms of the multiplication on V, the element $e_V = \alpha(e_U)$ and the ordering of eV (cf. [IKR1, Theorem 3.11]).

Notice also that, if $x \in U$ and $ex \sim_E 0$, then $x \sim_E 0$, since $e\alpha(x) = \alpha(ex) = 0$ implies $\alpha(x) = 0$.

We summarize these considerations as follows:

[8] Recall that, for any set $Y \subset U$ we write $Y/E := \{[y]_E \mid y \in Y\}$.

Proposition 4.4. *Let U be a supertropical semiring, $M := eU$, and assume that E is a transmissive equivalence relation on U. Then the following is true:*

$$TE1: \ E \quad \text{is multiplicative.}$$
$$TE2: \ \text{The equivalence relation } E|M \text{ is order compatible.}$$
$$TE3: \ \text{If} \quad x \in U \quad \text{and} \quad ex \sim_E 0, \quad \text{then} \quad x \sim_E 0.$$

The structure of the supertropical semiring U/E is uniquely determined by the following data.

a) *If $x, y \in U$, then $[x]_E \cdot [y]_E = [xy]_E$.*
b) *The ghost ideal of U/E is*

$$M/E := \{[x]_E \mid x \in U\}.$$

c) *If $x, y \in M$, then*

$$x \leqslant y \quad \Rightarrow \quad [x]_E \leqslant [y]_E.$$

Definition 4.5. *We call an equivalence relation on U which has the properties TE1-TE3 a **TE-relation**.*

Not every TE-relation is transmissive as will be clear from [IKR3]. Something "non-universal" has to be added to guarantee that a given TE-relation is transmissive. We now show one such condition.

Definition 4.6. *We call a multiplicative equivalence relation E on U **ghost-cancellative** if the following holds.*

$$\forall x, y, z \in eU: \ \text{If } xz \sim_E yz, \quad \text{and} \quad z \nsim_E 0, \quad \text{then} \quad x \sim_E y. \qquad \text{(Canc)}$$

This means that the monoid $(M/E)\setminus\{0\}$ is cancellative. {If $U = M$, we usually say "cancellative" for "ghost-cancellative".}

We arrive at the main result of this section.

Theorem 4.7. *Let U be a supertropical semiring and $M := eU$ its ghost ideal. Assume that E is a TE-relation on U. Assume also that E is ghost-cancellative. Then E is transmissive.*

Proof. Let $\overline{U}$ denote the set U/E, and, for any $x \in U$, let $\bar{x} = [x]_E$. Proposition 4.2 tells us that, due to $TE1$ and $TE2$, we have the structure of a bipotent semiring on the set

$$\overline{M} := M/E := \{\bar{x} \mid x \in M\},$$

such that the map $M \to \overline{M}$, $x \mapsto \bar{x}$, is a semiring homomorphism. It has the unit element $\bar{e}$ ($e := e_U$) and the zero element $\bar{0}$. The assumption (Canc) means that $\overline{M}$ is cancellative. We have $\overline{U} \cdot \overline{M} \subset \overline{M}$. The map $p : \overline{U} \to \overline{M}$, $p(\bar{x}) := \bar{e}\bar{x} = e\bar{x}$ is a monoid homomorphism with $p(\bar{x}) = \bar{x}$ for $x \in M$. The assumption $TE3$ means that $p^{-1}(\bar{0}) = \{\bar{0}\}$. Thus, Theorem 3.1 applies and gives us the structure of a supertropical semidomain on the set $\overline{U}$ with ghost map $\nu_{\overline{U}} = p$ and ghost ideal $\overline{M}$.

It remains to prove that the map $\pi_E : U \to \overline{U}$, $x \mapsto \bar{x}$, is a transmission. We have to check the axioms $TM1$-$TM5$ in [IKR1, §5]. The first four axioms $TM1$-$TM4$ are evident. $TM5$ holds, since indeed the map $M \to \overline{M}$, $x \mapsto \bar{x}$, is a semiring homomorphism. □

This theorem allows a second approach to the key result of §1, Theorems 1.11 and 1.13, which seems to be faster than the route taken in §1 (but perhaps gives less insight).

Example 4.8. *We return to the assumptions of Theorems 1.11 and 1.13: U is a supertropical semiring, and γ is a surjective homomorphism from $M := eU$ to a cancellative bipotent semidomain M'. We **define** a binary relation $F := F(U,\gamma)$ on U, decreeing*

$$x_1 \sim_F x_2 \quad \Leftrightarrow \quad \text{either} \quad x_1 = x_2, \quad \text{or} \quad \gamma(ex_1) = \gamma(ex_2),\ x_1 = ex_1,\ x_2 = ex_2,$$
$$\text{or} \quad \gamma(ex_1) = \gamma(ex_2) = 0.$$

One verifies directly in an easy way that F is an equivalence relation. Clearly F is multiplicative. The restriction

$$F|M := F \cap (M \times M)$$

is order compatible, since γ preserves the ordering (in the weak sense). For $x \in U$ we have $x \sim_F 0$ iff $\gamma(ex) = 0$ iff $ex \sim_E 0$. Thus axioms $TE1$–$TE3$ are valid. The semiring M/F is isomorphic to M' via γ, and hence is a cancellative semidomain. Now Theorem 4.7 tells us that the map π_F is transmissive.

Then the proof of Theorem 1.13 gives us that π_F is a pushout transmission. {One does not need to know for this that π_F is initial.} Alternatively, one may use a more general result on pushout transmissions given below (Theorem 4.13). In particular, in Notation 1.7,

$$F(U,\gamma) = E(U,\gamma).$$

In [IKR1, §8] we introduced *orbital* equivalence relations. Typically a relation $F(U,\gamma)$, as just considered, is almost never orbital. We now ask for those orbital equivalence relations which are transmissive.

Lemma 4.9. *Let M be a totally ordered set and H an (abelian) semigroup*[9] *which operates on M in an order preserving way. {If $x, y \in M$, $h \in H$, and $x \leqslant y$, then $hx \leqslant hy$.} We introduce on M an equivalence relation $E := E(H)$ as follows:*

$$x \sim_E y \quad \Leftrightarrow \quad \exists g, h \in H : gx = hy.$$

Assume that for every $x \in M$ the orbit Hx is convex in M. Then E is order compatible.

Proof. We verify that every equivalence class of E is convex, and then will be done (cf. Remark 4.1). Let $x_1, x_2, y \in M$ be given with $x_1 < y < x_2$, and $x_1 \sim_E x_2$. There exist elements h_1, h_2 in H with $h_1x_1 = h_2x_2$. This implies

$$h_2x_1 \leqslant h_2y \leqslant h_2x_2 = h_1x_1.$$

Since Hx_1 is convex, there exists some $h_3 \in H$ with $h_2y = h_3x_1$; hence $y \sim_E x_1$. □

If G is a (totally) ordered (abelian) cancellative semigroup, we denote the group envelope of G (given in the well-known way by fractions $\frac{g_1}{g_2}$ with $g_1, g_2 \in G$) by $\langle G\rangle$. We equip $\langle G\rangle$ with the unique ordering which extends the given ordering of G and is compatible with multiplication.

[9] All semigroups occurring in this paper are assumed to be abelian.

Theorem 4.10. *Let U be a supertropical semiring with ghost ideal $M := eU$, and let H be a submonoid of*

$$S(U) := \{x \in U \mid x\mathcal{T}(U) \subset \mathcal{T}(U)\}.$$

Finally, let

$$\mathfrak{q} := \{x \in M \mid \exists h \in H : hx = 0\} = \{x \in M \mid x \sim_H 0\},$$

which is an ideal of M. Assume that M is a semidomain.

a) *The semigroup H operates on M, and hence on $M\backslash\mathfrak{q}$, by multiplication in an order preserving way. Either $\mathfrak{q}$ is a lower set and a prime ideal of M, or $\mathfrak{q} = M$.*
b) *If $\mathfrak{q} \neq M$, and the monoid $M\backslash\mathfrak{q}$ is cancellative, and the submonoid $\nu_U(H) = He$ of $M\backslash\mathfrak{q}$ is convex in the ordered abelian group $\langle M\backslash\mathfrak{q}\rangle$, then $E(H)$ is transmissive.*
c) *If $\mathfrak{q} = M$, then U/E is the null ring, and hence $E(H)$ is again transmissive.*

Proof. a) If $x_1, x_2 \in M$, $h \in H$, and $x_1 \leqslant x_2$, then $x_1 + x_2 = x_2$; hence $hx_1 + hx_2 = hx_2$, and thus $hx_1 \leqslant hx_2$. If $x \in \mathfrak{q}$, $y \in M$ and $y \leqslant x$, there exists some $h \in H$ with $hx = 0$. We have $hy \leqslant hx$; hence $hy = 0$, and thus $y \in \mathfrak{q}$. Thus $\mathfrak{q}$ is a lower set of M. Clearly, $h(M\backslash\mathfrak{q}) \subset M\backslash\mathfrak{q}$ for every $h \in H$.

If $x, y \in M$ are given with $xy \in \mathfrak{q}$, then there exists some $h \in H$ with $hxy = 0$. Since M is a semidomain, it follows that $hx = 0$ or $y = 0$, and hence $x \in \mathfrak{q}$ or $y \in \mathfrak{q}$. This proves that the ideal $\mathfrak{q}$ of M is prime.

b) We will use Theorem 4.7. The equivalence relation $E(H)$ is multiplicative. For any $x \in U$ with $ex \sim_H 0$, there exists some $h \in H$ with $e(hx) = h(ex) = 0$. This implies $hx = 0$, and hence $x \sim_H 0$. Thus $E(H)$ obeys $TE1$ and $TE3$.

We verify $TE2$ by proving that every equivalence class of $E(H)|M$ is convex. Let $x_1, x_2, x_3 \in M$ be given with $x_1 < x_2 < x_3$ and $x_1 \sim_H x_3$. We need to be convinced that $x_1 \sim_H x_2$.

Case 1. $x_1 \in \mathfrak{q}$, i.e., $x_1 \sim_H 0$. Then $x_3 \sim_H 0$. Since $\mathfrak{q}$ is a lower set, we conclude that $x_2 \sim_H 0$, and hence $x_1 \sim_H x_2$.

Case 2. $x_1 \notin \mathfrak{q}$. Now all x_i lie in $M\backslash\mathfrak{q}$, since $M\backslash\mathfrak{q}$ is an upper set. We verify that for every $x \in M\backslash\mathfrak{q}$ the orbit Hx is convex in $M\backslash\mathfrak{q}$. Then Lemma 4.9 will tell us that the restriction of $E(H)$ to $M\backslash\mathfrak{q}$ is order compatible. This will imply that $x_1 \sim_H x_2$, as desired.

Let $x, y \in M\backslash\mathfrak{q}$ and $h_1, h_2 \in H$ be given with $h_1x \leqslant y \leqslant h_2x$. In the ordered abelian group $\langle M\backslash\mathfrak{q}\rangle$, we have $h_1 \leqslant yx^{-1} \leqslant h_2$. By our convexity hypothesis, this implies $yx^{-1} = h_3 \in H$. Thus $y = h_3x \in Hx$, as desired. $TE2$ is verified.

It remains to check that $E(H)$ is ghost-cancellative. Let $x, y, z \in M$ be given with $xz \sim_H yz$, $z \not\sim_H 0$. Thus $z \notin \mathfrak{q}$. We have elements h_1, h_2 in H with $h_1xz = h_2yz$.

If $x \in \mathfrak{q}$, then $h_2yz \in \mathfrak{q}$, and hence $y \in \mathfrak{q}$, since $\mathfrak{q}$ is prime. Thus $x \sim_H y$ in this case. The same holds if $y \in \mathfrak{q}$. Assume finally that $x, y \in M\backslash\mathfrak{q}$. The assumption that the monoid $M\backslash\mathfrak{q}$ is cancellative implies that $h_1x = h_2y$; hence, $x \sim_H y$ again.

Now Theorem 4.7 tells us that indeed $E(H)$ is transmissive.

c) If $\mathfrak{q} = M$ then $ex \sim_H 0$ for every $x \in U$, and hence $x \sim_H 0$ by an argument from (b) above. Thus $U/E(H) = \{0\}$. □

Example 4.11. *In the case that U is a supertropical semifield, $M = \Gamma \cup \{0\}$ with Γ an ordered abelian group, the situation addressed in Theorem 4.10 reads as follows:*

Let H be a subgroup of $\mathcal{T}(U)$ whose image $\Delta := He$ in Γ is convex in Γ. Then $U/E(H)$ is just the orbit space U/H (in the traditional sense), and $\mathfrak{q} = \{0\}$. We have

$$\mathcal{T}(U/H) = \mathcal{T}(U)/H, \quad \mathcal{G}(U/H) = \Gamma/\Delta, \quad e_{U/H} = He.$$

The map π_H from U to $U/E(H)$ sends an element x of U to Hx. It is a transmission. It covers the semiring homomorphism

$$\gamma_\Delta : \Gamma \cup \{0\} \to \Gamma/\Delta \cup \{0\},$$

which sends an element g of Γ to $g\Delta$ and 0 to 0.

If $\Delta \neq \{e\}$, then π_H is not a semiring homomorphism. Indeed, we can choose elements $x, y \in \mathcal{T}(U)$ with $Hx = Hy$, but $ex < ey$. Then $x + y = y$; hence $\pi_H(x + y) = Hy$, while $\pi_H(x) + \pi_H(y) = eHy = \Delta(ey)$. Notice also that the transmission π_H is not initial, since $E(H)$ is different from the relation $E(U, \gamma_H)$ described in Example 4.8.

We return to transmissive equivalence relations in general.

Definition 4.12. *We call a transmissive equivalence relation E on a supertropical semiring U **initial** (resp. **pushout**) if the transmission $\pi_E : U \to U/E$ is initial (resp. pushout) (cf. Definitions 1.2 and 1.3).*

We now bring a condition which guarantees that a given transmissive equivalence relation E is pushout. The proof will follow essentially the same arguments as used in Theorem 1.13 in the case considered there and reconsidered in Example 4.8.

Theorem 4.13. *Assume that E is a transmissive equivalence relation on a supertropical semiring U with the following additional property:*

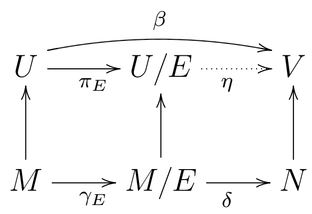
If $x \in \mathcal{T}(U)$, $y \in U$, and $x \sim_E y$, then either $x = y$ or $x \sim_E 0$ (and hence $y \sim_E 0$).

Then E is pushout.

Proof. Let $M := eU$, and let $\gamma_E : M \to M/E$ denote the ghost component of the transmission $\pi_E : U \to U/E$.

In order to verify the pushout property of π_E, assume that $\delta : M/E \to N$ is a homomorphism from M/E to a bipotent semiring N and $\beta : U \to V$ is a transmission covering $\delta \circ \gamma_E$. {In particular, $eV = N$}.

We look for a transmission $\eta : U/E \to V$ covering δ with $\eta \circ \pi_E = \beta$.



We are forced to define the map η by the formula

$$\eta([x]_E) = \beta(x) \qquad (x \in U).$$

In order to prove that η is a well-defined map, we have to verify for $x, y \in U$ with $x \sim_E y$ that $\beta(x) = \beta(y)$.

Case 1. $x \in M$, $y \in M$. Now

$$\beta(x) = \delta\gamma_E(x) = \delta([x]_E)$$

and $\beta(y) = \delta([y]_E)$. Since $x \sim_E y$, we conclude that $\beta(x) = \beta(y)$.

Case 2. $x \in \mathcal{T}(U)$. If $x = y$, then, of course, $\beta(x) = \beta(y)$. Otherwise $x \sim_E 0$, $y \sim_E 0$ by the hypothesis of the theorem; hence $ex \sim_E 0$, $ey \sim_E 0$. By the settled first case, we conclude that $e\beta(x) = \beta(ex) = 0$, which implies $\beta(x) = 0$. In the same way, $\beta(y) = 0$. Thus $\beta(x) = \beta(y)$ again.

The case that $y \in \mathcal{T}(U)$ is now settled, too. Thus, η is indeed a well-defined map. We have $\eta\pi_E = \beta$.

Since both β and π_E are transmissions, and π_E is surjective, we know by [IKR1, Proposition 6.1.ii] that η is a transmission. By assumption $\beta(x) = \delta([x]_E)$ for every $x \in M$. But also $\beta(x) = \eta([x]_E)$. Thus η covers δ. The pushout property of π_E is verified. □

5. The equivalence relations $E(\mathfrak{a})$

We study a class of transmissive equivalence relations which turns out to be particularly well accessible.

If R is a ring and $\mathfrak{a}$ is an ideal of R we have the well-known equivalence relation "mod $\mathfrak{a}$" at our disposal. We write down the obvious analogue of this relation for semirings.

Definition 5.1. *Let R be a semiring and $\mathfrak{a}$ an ideal of R. We define an equivalence relation $E(\mathfrak{a})$ on R as follows, writing $\sim_{\mathfrak{a}}$ instead of $\sim_{E(\mathfrak{a})}$.*

$$x \sim_{\mathfrak{a}} y \quad \Leftrightarrow \quad \exists a, b \in \mathfrak{a} : x + a = y + b.$$

For $x \in R$ we denote the equivalence class $[x]_{E(\mathfrak{a})}$ more briefly by $[x]_{\mathfrak{a}}$, and denote the map $x \mapsto [x]_{\mathfrak{a}}$ from R to the set $R/E(\mathfrak{a})$ usually by $\pi_{\mathfrak{a}}$ instead of $\pi_{E(\mathfrak{a})}$.

If $x, y, z \in R$ and $x \sim_{\mathfrak{a}} y$, then clearly $x + z \sim_{\mathfrak{a}} y + z$ and $xz \sim_{\mathfrak{a}} yz$. Thus, we have a well-defined addition and multiplication on the set $R/E(\mathfrak{a})$, given by the rules $(x, y \in R)$

$$[x]_{\mathfrak{a}} + [y]_{\mathfrak{a}} := [x + y]_{\mathfrak{a}},$$
$$[x]_{\mathfrak{a}} \cdot [y]_{\mathfrak{a}} := [xy]_{\mathfrak{a}}.$$

With these compositions $R/E(\mathfrak{a})$ is a semiring and $\pi_{\mathfrak{a}}$ is a homomorphism from R onto $R/E(\mathfrak{a})$, cf. [RS].

Theorem 5.2. *If R is supertropical, then for any ideal $\mathfrak{a}$ of R the relation $E(\mathfrak{a})$ is transmissive.*

Proof. Any homomorphism between supertropical semirings clearly obeys the axioms TM1–TM5 from [IKR1, §5], hence is a transmissive map. Thus our task is only to prove that the semiring $U/E(\mathfrak{a})$ is supertropical.

We verify directly the axioms (3.3′), (3.3"), (3.3) from [IKR1, §3] for the semiring $U/E(\mathfrak{a})$, i.e.,

$$(3.3')_{\mathfrak{a}}: \quad 1 + 1 + 1 + 1 \sim_{\mathfrak{a}} 1 + 1,$$
$$(3.3'')_{\mathfrak{a}}: \quad x + x \sim_{\mathfrak{a}} y + y \Rightarrow x + x \sim_{\mathfrak{a}} x + y,$$
$$(3.3)_{\mathfrak{a}}: \quad \pi_{\mathfrak{a}}(x) \neq \pi_{\mathfrak{a}}(y) \Rightarrow \pi_{\mathfrak{a}}(x) + \pi_{\mathfrak{a}}(y) \in \{\pi_{\mathfrak{a}}(x), \pi_{\mathfrak{a}}(y)\}.$$

Clearly $(3.3')_{\mathfrak{a}}$ holds since (3.3′) of [IKR1] holds for R, and $(3.3)_{\mathfrak{a}}$ holds since (3.3) of [IKR1] holds for R and $\pi_{\mathfrak{a}}(x) + \pi_{\mathfrak{a}}(y) = \pi_{\mathfrak{a}}(x + y)$.

We turn to $(3.3)_{\mathfrak{a}}$. We are given $a, b \in \mathfrak{a}$ with $x+x+a = y+y+b$. We add $c := e(a+b)$ to both sides and obtain $x + x + c = y + y + c$. Since $c + c = c$ it follows that

$$(x + c) + (x + c) = (y + c) + (y + c).$$

Now $(3.3'')$ for R gives us

$$(x + c) + (x + c) = (x + c) + (y + c).$$

Thus $x + x \sim_{\mathfrak{a}} x + y$, as desired. □

Let again R be any semiring. In contrast to the case of rings, different ideals $\mathfrak{a}, \mathfrak{b}$ of R may give the same relation $E(\mathfrak{a}) = E(\mathfrak{b})$, but this ambiguity can be tamed.

Clearly $\mathfrak{a}_1 := [0]_{\mathfrak{a}}$ is again an ideal of the semiring R. It consists of all $x \in R$ with $x + a \in \mathfrak{a}$ for some $a \in \mathfrak{a}$. We call $\mathfrak{a}_1$ the **saturum** of $\mathfrak{a}$, and we write $\mathfrak{a}_1 = \operatorname{sat} \mathfrak{a}$. We call $\mathfrak{a}$ **saturated** (in U), if $\mathfrak{a} = \mathfrak{a}_1$.

Proposition 5.3. *Let R be any semiring and $\mathfrak{a}, \mathfrak{b}$ ideals of R.*

i) $E(\mathfrak{a}) = E(\operatorname{sat} \mathfrak{a})$*;*
ii) $E(\mathfrak{a}) \subset E(\mathfrak{b})$ *iff* $\operatorname{sat} \mathfrak{a} \subset \operatorname{sat} \mathfrak{b}$;
iii) $\operatorname{sat} \mathfrak{a}$ *is the unique biggest ideal $\mathfrak{a}'$ of R with $E(\mathfrak{a}') = E(\mathfrak{a})$.*

Proof. a) If $\mathfrak{a} \subset \mathfrak{b}$ then $E(\mathfrak{a}) \subset E(\mathfrak{b})$. Conversely, if $E(\mathfrak{a}) \subset E(\mathfrak{b})$, then $[0]_{\mathfrak{a}} \subset [0]_{\mathfrak{b}}$, i.e., $\operatorname{sat} \mathfrak{a} \subset \operatorname{sat} \mathfrak{b}$.

b) Let $\mathfrak{a}_1 := \operatorname{sat} \mathfrak{a}$. If $x \sim_{\mathfrak{a}_1} y$, then there exist $z, w \in \mathfrak{a}_1$ with $x + z = y + w$, and there exist $a, b \in \mathfrak{a}$ with $z + a \in \mathfrak{a}$, $w + b \in \mathfrak{a}$. It follows that

$$x + (z + a) + b = y + (w + b) + a,$$

which tells us that $x \sim_{\mathfrak{a}} y$. Thus $E(\mathfrak{a}_1) = E(\mathfrak{a})$.

c) If $\operatorname{sat} \mathfrak{a} \subset \operatorname{sat} \mathfrak{b}$, then

$$E(\mathfrak{a}) = E(\operatorname{sat} \mathfrak{a}) \subset E(\operatorname{sat} \mathfrak{b}) = E(\mathfrak{b}).$$

Now the claims i) and ii) are evident.

d) If $E(\mathfrak{a}') = E(\mathfrak{a})$, then it follows from ii) that $\operatorname{sat} \mathfrak{a}' = \operatorname{sat} \mathfrak{a}$, and hence $\mathfrak{a}' \subset \operatorname{sat} \mathfrak{a}$. □

Assume now that U is a supertropical semiring with ghost ideal $M := eU$. Then we can give a very precise description of the relation $E(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of U.

Theorem 5.4. *Let $\mathfrak{a}$ be an ideal of U. The equivalence classes of the relation $E(\mathfrak{a})$ are the set $[0]_{\mathfrak{a}} = \operatorname{sat} \mathfrak{a}$ and the one-point sets $\{x\}$ with $x \in U \setminus \operatorname{sat} \mathfrak{a}$. More precisely the following holds:*

i) *If $ex > e\mathfrak{a}$ (i.e., $ex > ea$ $\forall a \in \mathfrak{a}$), then $[x]_{\mathfrak{a}} = \{x\}$.*
ii) *If $ex \leqslant ea$ for some $a \in \mathfrak{a}$, then $x \sim_{\mathfrak{a}} 0$.*

Proof. i) Assume that $ex > e\mathfrak{a}$ and $x \sim_{\mathfrak{a}} y$. There exist elements a, b in $\mathfrak{a}$ with $x+a = y+b$. Now $ex > ea$, and hence $x + a = x$. From $ex = ey + eb$ we conclude that $ex = \max(ey, eb)$. But $ex > eb$. Thus $ex = ey$, and $y + b = y$. We have $x = y$.

ii) If $ex < ea$ for some $a \in \mathfrak{a}$, then $x + a = a$, and hence $x \sim_{\mathfrak{a}} 0$. If $ex = ea$ for some $a \in \mathfrak{a}$, then $x + a = ea$, and hence again $x \sim_{\mathfrak{a}} 0$. □

The set $e\mathfrak{a}$ is an ideal of both U and M; hence, it gives us a relation $E_U(e\mathfrak{a})$ on U and a relation $E_M(e\mathfrak{a})$ on M. It further gives us ideals $\mathrm{sat}_U(e\mathfrak{a})$ and $\mathrm{sat}_M(e\mathfrak{a})$ of U and M, respectively.

Corollary 5.5. *Let $\mathfrak{a}$ be an ideal of U.*

i) $\mathrm{sat}_U\,\mathfrak{a}$ *is the set of all $x \in U$ with $ex \leqslant c$ for some $c \in e\mathfrak{a}$.*
ii) $\mathfrak{a}$ *is saturated in U iff $e\mathfrak{a}$ is a lower set of M and every $x \in U$ with $ex \in \mathfrak{a}$ is itself an element of $\mathfrak{a}$.*
iii) $\mathrm{sat}_U(e\mathfrak{a}) = \mathrm{sat}_U(\mathfrak{a})$.
iv) $\mathrm{sat}_M(e\mathfrak{a}) = \mathrm{sat}_U(\mathfrak{a}) \cap M = e\,\mathrm{sat}_U(\mathfrak{a})$.
v) $E_U(\mathfrak{a}) = E_U(e\mathfrak{a})$.
vi) *The restriction $E_U(\mathfrak{a})|M = E_U(\mathfrak{a}) \cap (M \times M)$ of the relation $E_U(\mathfrak{a})$ to M coincides with $E_M(e\mathfrak{a})$.*

Proof. (i) is evident from Theorem 5.4, since $\mathrm{sat}_U(\mathfrak{a}) = [0]_{\mathfrak{a}}$, and (ii), (iii) are evident from (i). We then obtain (iv) by applying (i) to both U and M. {More generally, $e\mathfrak{b} = \mathfrak{b} \cap M$ for any ideal $\mathfrak{b}$ of U.} Claim (v) is clear, because the description of $E_U(\mathfrak{a})$ does not change if we replace $\mathfrak{a}$ by $e\mathfrak{a}$. Finally, we read off (vi) by applying Theorem 5.4 to both U and M. □

Corollary 5.6. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of U, then $E_U(\mathfrak{a}) \subset E_U(\mathfrak{b})$ or $E_U(\mathfrak{b}) \subset E_U(\mathfrak{a})$.*

Proof. We may assume from the start that $\mathfrak{a}$ and $\mathfrak{b}$ are saturated. Now $e\mathfrak{a}$ and $e\mathfrak{b}$ are lower sets of M. Thus, $e\mathfrak{a} \subset e\mathfrak{b}$ or $e\mathfrak{b} \subset e\mathfrak{a}$. This implies that $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$ (cf. Corollary 5.5.i), hence $E(\mathfrak{a}) \subset E(\mathfrak{b})$ or $E(\mathfrak{b}) \subset E(\mathfrak{a})$. □

Example 5.7. *The unique maximal saturated proper ideal of U is*

$$\mathfrak{a} := \{x \in U \mid ex < e\}.$$

It is easily seen to be a prime ideal (provided U is not the null ring), but perhaps $\mathfrak{a}$ is not a maximal ideal of U. Take for example $U = M = \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the ordered monoid $\{1, 2, 3, \dots\}$ with standard multiplication and standard ordering. Now $\mathfrak{a} = \{0\}$, but $M \backslash \{1\}$ is the only maximal ideal of M.

From Corollary 5.5 we can read off further facts about saturated ideals, which will be needed later on.

Scholium 5.8. *As before, U is a supertropical semiring, and $M := eU$.*

a) *An ideal of U is saturated, iff $e\mathfrak{a}(= \mathfrak{a} \cap M)$ is saturated in M, and moreover every $x \in U$ with $ex \in \mathfrak{a}$ is an element of $\mathfrak{a}$.*
b) *If $\mathfrak{c}$ is a saturated ideal of M, then $\mathfrak{a} := \{x \in U \mid ex \in \mathfrak{c}\}$ is a saturated ideal of U and $e\mathfrak{a} = \mathfrak{c}$.*
c) *The saturated ideals $\mathfrak{a}$ of U correspond uniquely with the ideals $\mathfrak{c}$ of M which are lower sets via*

$$\mathfrak{c} = e\mathfrak{a}\ (= \mathfrak{a} \cap M), \qquad \mathfrak{a} = \{x \in U \mid ex \in \mathfrak{c}\}.$$

Proof. a) Clear from Corollary 5.5.ii,iv.

b) We have $\mathfrak{c} \subset \mathfrak{a}$, and hence $\mathfrak{c} = e\mathfrak{a}$. Now use a).

c) Now evident, taking into account Corollary 5.5.ii. □

The saturated ideals of U form a chain (Corollary 5.6). We ask: which of these ideals are prime ideals? In particular, given a saturated ideal $\mathfrak{a} \neq U$, does there exist a saturated prime ideal $\mathfrak{p} \supset \mathfrak{a}$? If "Yes", which is the smallest one?

These questions can be pushed to the ghost level by the following simple observation.

Lemma 5.9. *Assume that $\mathfrak{a}$ is an ideal of U with $e \notin \mathfrak{a}$. Then $\mathfrak{a}$ is a prime ideal of U, iff $e\mathfrak{a}(= \mathfrak{a} \cap M)$ is a prime ideal of M and every $x \in U$ with $ex \in e\mathfrak{a}$ is an element of $\mathfrak{a}$.*

Proof. a) If $\mathfrak{a}$ is prime in U, then $e\mathfrak{a} = \mathfrak{a} \cap M$ is prime in M. Moreover, if $x \in U$ and $ex \in e\mathfrak{a}$, then $ex \in \mathfrak{a}$. Since $e \notin \mathfrak{a}$, it follows that $x \in \mathfrak{a}$.

b) Assume that $e\mathfrak{a}$ is prime in M and $x \in \mathfrak{a}$ for every $x \in U$ with $ex \in e\mathfrak{a}$. Let $y, z \in U$ be given with $yz \in \mathfrak{a}$. Then $(ey)(ez) \in e\mathfrak{a}$; hence, $ey \in \mathfrak{a}$ or $ez \in \mathfrak{a}$, implying $y \in \mathfrak{a}$ or $z \in \mathfrak{a}$. Thus $\mathfrak{a}$ is prime. □

N.B. The condition $e \notin \mathfrak{a}$ is important here. For example, if $\mathcal{T}(U)$ is closed under multiplication, then $\mathfrak{a} := eU$ is prime in U, but $\mathfrak{a} \cap M = M$ is not prime in M.

Proposition 5.10.

i) *The prime ideals $\mathfrak{a}$ of U with $e \notin \mathfrak{a}$ correspond uniquely with the prime ideals $\mathfrak{c}$ of M via $\mathfrak{c} = e\mathfrak{a}(= \mathfrak{a} \cap M)$ and*

$$\mathfrak{a} = \{x \in U \mid ex \in \mathfrak{c}\}.$$

ii) *$\mathfrak{a}$ is a saturated prime ideal of U iff $e\mathfrak{a}$ is a saturated prime ideal of M.*

Proof. i) is clear by Lemma 5.9. Now ii) follows by Scholium 5.8.a. (Notice that if $\mathfrak{a}$ is a saturated ideal of U and $\mathfrak{a} \neq U$, then $e \notin \mathfrak{a}$, since $1 + e = e$.) □

Theorem 5.11. *Let $\mathfrak{a}$ be a saturated ideal of U and $\mathfrak{a} \neq U$. Then*

$$\mathfrak{b} := \{x \in U \mid \exists n \in \mathbb{N} : ex^n \in \mathfrak{a}\}$$

is a prime ideal of U. It is the smallest prime ideal containing $\mathfrak{a}$, and it coincides with the radical $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$, defined by

$$\sqrt{\mathfrak{a}} := \{x \in U \mid \exists n \in \mathbb{N} : x^n \in \mathfrak{a}\}.$$

Proof. a) If $\mathfrak{c}$ is an ideal of M, let

$$\sqrt{\mathfrak{c}} := \{x \in M \mid \exists n \in \mathbb{N} : x^n \in \mathfrak{c}\}.$$

In this notation

$$\mathfrak{b} = \{x \in U \mid ex \in \sqrt{e\mathfrak{a}}\}.$$

By Proposition 5.10 it is clear that it suffices to prove that $\sqrt{e\mathfrak{a}}$ is the smallest saturated prime ideal of M containing $e\mathfrak{a}$. We have $e \notin \mathfrak{a}$, since otherwise the relation $1 + e = e$ would imply that $1 \in \operatorname{sat} \mathfrak{a} = \mathfrak{a}$. Thus $e \notin \mathfrak{c}$, hence $e \notin \sqrt{\mathfrak{c}}$.

b) Let $\mathfrak{c} := e\mathfrak{a}$. This is a saturated ideal of M, i.e., an ideal of M which is a lower set of M (cf. Scholium 5.8). Clearly $M \cdot \sqrt{\mathfrak{c}} \subset \sqrt{\mathfrak{c}}$, and hence $\sqrt{\mathfrak{c}}$ is an ideal of M. Let $x \in \sqrt{\mathfrak{c}}$, $y \in M$ and $y < x$. Choosing some $n \in \mathbb{N}$ with $x^n \in \mathfrak{c}$, we have $y^n \leqslant x^n$; hence, $y^n \in \mathfrak{c}$, and $y \in \sqrt{\mathfrak{c}}$. Thus $\sqrt{\mathfrak{c}}$ is a lower set of M. The ideal $\sqrt{\mathfrak{c}}$ is saturated in M.

c) Let $x, y \in M$ be given with $xy \in \sqrt{\mathfrak{c}}$. Assume that $y \leqslant x$. We have $y^2 \leqslant xy$, and hence $y^2 \in \sqrt{\mathfrak{c}}$, implying $y \in \sqrt{\mathfrak{c}}$. This proves that $\sqrt{\mathfrak{c}}$ is prime in M.

d) Let $\mathfrak{p}$ be a prime ideal of M containing $\mathfrak{c}$. If $x \in \sqrt{\mathfrak{c}}$ then $x^n \in \mathfrak{c} \subset \mathfrak{p}$ for some $n \in \mathbb{N}$, and hence $x \in \mathfrak{p}$.

e) If $x \in U$ and $ex^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$, then $x^n \in \mathfrak{a}$ since $x^n + ex^n = ex^n$ and $\mathfrak{a}$ is saturated. Thus $\mathfrak{b} = \sqrt{\mathfrak{a}}$. □

Our proof that $E_U(\mathfrak{a})$ is transmissive (Proposition 4.4) does not rely on the criterion Theorem 4.7 (nor on any other theory). In particular, it is not necessary to assume that $E_U(\mathfrak{a})$ is ghost-cancellative (i.e., the ghost ideal $M/E_U(\mathfrak{a})$ of $U/E_U(\mathfrak{a})$ is cancellative, cf. §2). In fact, the following theorem tells us that this often does not hold.

Theorem 5.12. *Assume that $M = eU$ is a cancellative semidomain. Let $\mathfrak{a}$ be a saturated ideal of U with $\mathfrak{a} \neq U$. The following are equivalent:*

(1) *The ghost ideal $M/E_U(\mathfrak{a})$ of $U/E_U(\mathfrak{a})$ is a cancellative semidomain.*
(2) *$e\mathfrak{a}$ is a prime ideal of M.*
(3) *$\mathfrak{a}$ is a prime ideal of U.*

Proof. a) We first study the case that U is ghost, i.e., $U = M$. Condition (1) means the following.

$$\forall x, y, x \in M: \quad xz \sim_{\mathfrak{a}} yz,\ z \notin \mathfrak{a} \ \Rightarrow\ x \sim_{\mathfrak{a}} y.$$

If this holds, then taking $y = 0$ we see that $\mathfrak{a}$ is a prime ideal. This proves (1) ⇒ (2).

Assume now that $\mathfrak{a}$ is prime. Let $x, y, z \in M$ be given with $xz \sim_{\mathfrak{a}} yz$ and $z \notin \mathfrak{a}$.

Case 1. $x \in \mathfrak{a}$. Then $yz \in \mathfrak{a}$. Since $\mathfrak{a}$ is prime, we conclude that $y \in \mathfrak{a}$. Thus, $x \sim_{\mathfrak{a}} 0 \sim_{\mathfrak{a}} y$.

Case 2. $x \notin \mathfrak{a}$. Now $xz \notin \mathfrak{a}$. Taking into account Theorem 5.4 we obtain $xz = yz$. Since M is cancellative, this implies $x = y$. Thus $x \sim_{\mathfrak{a}} y$ in both cases. This proves (2) ⇒ (1).

b) Let now U be any supertropical semiring. The ideal $e\mathfrak{a}$ is saturated in M (cf. Scholium 5.8), and $M/E_U(\mathfrak{a}) = M/E_M(e\mathfrak{a})$ (cf. Corollary 5.5.vi).

Applying what has been proved to M and $e\mathfrak{a}$, we see that $M/E_U(\mathfrak{a})$ is cancellative iff $e\mathfrak{a}$ is prime in M. By Proposition 5.10.ii this is equivalent to $\mathfrak{a}$ being prime in U. □

Example 5.13. *Let $M := [0, 1]$ be the closed unit interval of $\mathbb{R}$ with the usual multiplication and the addition $x + y := \max(x, y)$. M is a cancellative bipotent semidomain. We choose some $\theta \in]0, 1[$. Then $\mathfrak{a} := [0, \theta]$ is an ideal and a lower set of M, and hence is a saturated ideal of M. But $\mathfrak{a}$ is not prime, since the half open interval $]\theta, 1]$ is not closed under multiplication. In fact, the only saturated prime ideals of M are $\{0\}$ and $[0, 1[$.*

The bipotent semiring $M/E(\mathfrak{a})$ can be identified with the subset $\{0\} \cup]\theta, 1]$ of $[0, 1]$ equipped with the new multiplication

$$x \odot y = \begin{cases} xy & \text{if } \ xy > \theta \\ 0 & \text{if } \ xy \leqslant \theta \end{cases}$$

and the addition

$$x \oplus y = \max(x, y).$$

Theorem 5.14. *If $\mathfrak{a}$ is any ideal of a supertropical semiring U, then the transmissive equivalence relation $E(\mathfrak{a})$ is pushout (i.e., the transmission $\pi_{\mathfrak{a}}$ is pushout, cf. Definition 4.12).*

Proof. We may assume that $\mathfrak{a}$ is saturated. Looking at the description of $E(\mathfrak{a})$ in Theorem 5.4, we realize that the hypothesis in Theorem 4.13 holds for $E = E(\mathfrak{a})$. Thus, $E(\mathfrak{a})$ is pushout. □

It follows that, in the terminology of Notation 1.7,

$$E(\mathfrak{a}) = E(U, \gamma_{\mathfrak{a}})$$

with $\gamma_{\mathfrak{a}}$ the map $x \mapsto [x]_{\mathfrak{a}}$ from M to $M/E(\mathfrak{a})$ covered by $\pi_{\mathfrak{a}}$. Notice that Theorem 5.14 is not covered by the central result Theorem 1.13 in §1, since we do not assume cancellation for $M/E(\mathfrak{a})$.

We draw a connection from the relations $E(\mathfrak{a})$ to other equivalence relations.

Theorem 5.15. *Let E be a TE-relation (e.g., E is a transmissive equivalence relation). The set $\mathfrak{q} := [0]_E$ is a saturated ideal of U with $E(\mathfrak{q}) \subset E$. Moreover, $\mathfrak{q}$ is the biggest ideal $\mathfrak{a}$ of U with $E(\mathfrak{a}) \subset E$.*

Proof. a) If $x \sim_E 0$, then $zx \sim_E 0$ for any $z \in U$. Thus $U \cdot \mathfrak{q} \subset \mathfrak{q}$.

b) From $e\mathfrak{q} \subset \mathfrak{q}$ we conclude that $e\mathfrak{q} = \mathfrak{q} \cap M = [0]_E \cap M$. This is convex in M and contains 0, hence is a lower set of M.

c) By axiom $TE3$ every $x \in U$ with $ex \in \mathfrak{q}$ is an element of $\mathfrak{q}$.

d) Let $x, y \in \mathfrak{q}$ be given, and assume without loss of generality that $ex \leqslant ey$. Then $e(x + y) = ey \in \mathfrak{q}$, and hence $x + y \in \mathfrak{q}$. This completes the proof that $\mathfrak{q}$ is an ideal of U. We conclude from c) and Scholium 5.8.a that this ideal is saturated.

e) The equivalence classes of $E(\mathfrak{q})$ are $\mathfrak{q} = [0]_E$ and one-point sets (Theorem 5.4). Thus, certainly $E(\mathfrak{q}) \subset E$. If $E(\mathfrak{a}) \subset E$ then

$$\mathfrak{a} \subset \operatorname{sat} \mathfrak{a} = [0]_{\mathfrak{a}} \subset [0]_E = \mathfrak{q}.$$

□

Notation 5.16. *If E, F are equivalence relations on a set X with $E \subset F$, we denote by F/E the equivalence relation induced by F on the set X/E. Thus, for $x, y \in X$,*

$$[x]_E \sim_{F/E} [y]_E \quad \Leftrightarrow \quad x \sim_F y.$$

Proposition 5.17. *Let E be a transmissive equivalence relation on U and $\mathfrak{q} := [0]_E$. We know by Theorem 5.15 that $\mathfrak{q}$ is an ideal of U and $E(\mathfrak{q}) \subset E$. Let*

$$\overline{E} := E/E(\mathfrak{q}).$$

i) *$\overline{E}$ is transmissive.*
ii) *$\overline{E}$ is pushout iff E is pushout.*

Proof. i) We have the factorization $\pi_E = \pi_{\overline{E}} \circ \pi_{\mathfrak{q}}$. Since π_E and $\pi_{\mathfrak{q}}$ are transmissive and $\pi_{\mathfrak{q}}$ is surjective, we conclude that $\pi_{\overline{E}}$ is transmissive (cf. [IKR1, Proposition 6.1.ii or Corollary 6.2]).

ii) We have a natural commuting diagram of transmissions

$$\begin{array}{ccccc} U & \xrightarrow[\pi_{\mathfrak{q}}]{} & U/E(\mathfrak{q}) & \xrightarrow[\pi_{\overline{E}}]{} & U/E \\ \uparrow & & \uparrow & & \uparrow \\ M & \xrightarrow[\gamma_{\mathfrak{q}}]{} & M/E(\mathfrak{q}) & \xrightarrow[\gamma_{\overline{E}}]{} & M/E \end{array}$$

with $\gamma_{\mathfrak{q}}$ and $\gamma_{\overline{E}}$ the ghost components of $\pi_{\mathfrak{q}}$ and $\gamma_{\overline{E}}$, respectively. Theorem 4.13 tells us that the left square is pushout in the category STROP. Since $\gamma_{\mathfrak{q}}$ is surjective, it follows that the outer rectangle is pushout iff the right square is pushout (e.g. [ML, p. 72, Exercise 8]). This gives the second claim. □

6. Homomorphic equivalence relations

Let R be a semiring.

Definition 6.1. *We call an equivalence relation E on R* ***additive****, if*

$$\forall x, y, z \in R : x \sim_E y \Rightarrow x + z \sim_E y + z,$$

and ***multiplicative****, if*

$$\forall x, y, z \in R : x \sim_E y \Rightarrow xz \sim_E yz.$$

We call E ***homomorphic****, if E is both additive and multiplicative.*

If E is homomorphic, we have a well-defined addition and multiplication on the set R/E, given by the rules $(x, y \in R)$:

$$[x]_E + [y]_E = [x + y]_E, \quad [x]_E \cdot [y]_E = [xy]_E,$$

and these make R/E a semiring. Moreover, we can say that an equivalence relation E on R is homomorphic, iff there exists a (unique) semiring structure on the set R/E, such that $\pi_E : R \to R/E$, $x \mapsto [x]_E$, is a homomorphism.

In the following, U is always a supertropical semiring and $M := eU$ is its ghost ideal.

Examples 6.2. *We have already seen two instances of homomorphic equivalence relations on U, namely, the MFCE-relations and the relations $E(\mathfrak{a})$ with $\mathfrak{a}$ an ideal of U.*

On the other hand, if $\gamma : M \to M'$ is a homomorphism from M to a cancellative bipotent semiring M', the transmissive equivalence relation $E := E(U, \gamma)$ (cf. Theorem 1.11) will usually not be additive, hence not homomorphic. Indeed, if $x_1, x_2 \in M$, $z \in \mathcal{T}(U)$ and $ex_1 < ez < ex_2$, $x_1 \sim_E x_2$, i.e., $\gamma(x_1) = \gamma(x_2)$, but $\gamma(x_1) \neq 0$, then $x_1 + z = z \in \mathcal{T}(U)$ and $x_2 + z = x_2 \in M$; hence, $x_1 + z \nsim_E x_2 + z$.

We have the following remarkable fact, a special case of which occurred already in Theorem 5.2.

Theorem 6.3. *Every homomorphic equivalence relation on U is transmissive. {In other terms, every homomorphic image of a supertropical semiring is again supertropical.}*

Proof. As in the proof of Theorem 5.2, we see that the only problem is to prove that the semiring U/E is supertropical. For this only the axiom (3.3″) from [IKR1, §3] needs serious consideration.

Given $x, y \in U$ with $ex \sim_E ey$, we have to verify that $ex \sim_E x + y$. We may assume that $ex \leqslant ey$. Now, if $ex = ey$, then $ex = x + y$. If $ex < ey$, then $x + y = y$ and $ex + y = y$, hence

$$x + y = ex + y \sim_E ey + y = ey \sim_E ex.$$

Thus, indeed $ex \sim_E x + y$ in both cases. □

We seek a more detailed understanding of the homomorphic equivalence relations on a supertropical semiring U. As an intermediate step we analyze the additive equivalence relations on U.

Proposition 6.4. *Let E be an equivalence relation on U. The following are equivalent.*

(1) *E is additive.*
(2) *E obeys the following rules.*

$$\begin{aligned} &\text{AE1}: && x \sim_E y \Rightarrow ex \sim_E ey. \\ &\text{AE2}: && E|M \text{ is order compatible.} \\ &\text{AE3}: && \text{If } ex < ey \text{ and } ex \sim_E ey, \text{ then } ex \sim_E y. \end{aligned}$$

Proof. We write $\sim$ for $\sim_E$. (1) $\Rightarrow$ (2):

a) If $x \sim y$, then

$$ex = x + x \sim y + x \sim y + y = ey.$$

b) We verify that every equivalence class of $E|M$ is convex, which will prove order compatibility of $E|M$ (cf. Remark 4.1). Let $x_1, x_2, y \in M$ and assume that $x_1 \sim x_2$ and $x_1 < y < x_2$. Then

$$y = x_1 + y \sim x_2 + y = x_2;$$

hence also $y \sim x_1$.

c) Assume that $ex < ey$ and $ex \sim_E ey$. Then

$$y = ex + y \sim ey + y = ey \sim ex.$$

(2) $\Rightarrow$ (1) : Given $x_1, x_2, z \in U$ with $x_1 \sim x_2$, we have to verify that $x_1 + z \sim x_2 + z$. We may assume that $ex_1 \leqslant ex_2$.

We distinguish six cases.

1) If $ez < ex_1$, we have $z + x_i = x_i$ $(i = 1, 2)$.
2) If $ez > ex_2$, we have $z + x_i = z$ $(i = 1, 2)$.
3) If $ex_1 = ez < ex_2$, then $z + x_1 = ex_1$, $z + x_2 = x_2$. By AE3, we have $ex_1 \sim x_2$.
4) If $ex_1 < ez < ex_2$, then $z + x_1 = z$, $z + x_2 = x_2$. By AE3, we have $ex_1 \sim z$, $ex_1 \sim x_2$.
5) If $ex_1 < ez = ex_2$, then $z + x_1 = z$, $z + x_2 = ex_2$. By AE3, $ex_1 \sim z$. By AE1, $ex_2 \sim ex_1$.
6) If $ex_1 = ez = ex_2$, then $z + x_1 = ez$ and $z + x_2 = ez$.

We see that in all six cases indeed $z + x_1 \sim z + x_2$. □

Example 6.5. *Assume that E is fiber conserving, i.e., $x \sim_E y$ implies $ex = ey$ (*[IKR1, Definition 6.3]*. Then E is additive. Indeed, the conditions* AE1–AE3 *hold trivially,* AE3 *being empty.*

Theorem 6.6. *Every additive equivalence relation E on U arises in the following way. Choose a partition $(M_i \mid i \in I)$ into non-empty convex subsets of M. Let J denote the set of all indices $i \in I$ such that M_i has a smallest element a_i and $a_i \neq 0$. Choose for every $i \in J$ an equivalence relation E_i on the fiber $\{x \in U \mid ex = a_i\}$. If x, y are elements of U with $ex \leqslant ey$, define*

$$\begin{aligned} x \sim_E y :\Leftrightarrow\ & \text{There exists some } i \in I \text{ with } ex, ey \in M_i; \\ & \text{and in case } i \in J, \text{ either } ex > a_i \\ & \text{or } ex = a_i \text{ and } x \sim_{E_i} ex, \\ & \text{or } ex = ey = a_i \text{ and } x \sim_{E_i} y. \end{aligned}$$

If $x, y \in U$ and $ex > ey$, define, of course, $x \sim_E y :\Leftrightarrow y \sim_E x$.

Proof. Given an additive equivalence relation E on U, this description of E holds with $(M_i \mid i \in I)$ the set of equivalence classes of $E \mid M$, indexed in some way, and $E_i := E \mid M_i$ for $i \in J$, due to the properties AE1–AE3 stated in the Proposition 6.4. Conversely, if data $(M_i \mid i \in I)$ and $(E_i \mid j \in J)$ are given, as indicated in the theorem, it is fairly obvious that the binary relation defined there is an equivalence relation obeying AE1–AE3. {Notice that the fiber U_0 over $0 \in M$ is the one-point set $\{0\}$. Thus, we may omit the index i with $0 \in M_i$ in the set J.} Proposition 6.4 tells us that E is additive. ☐

When dealing with additive equivalence relations, we now strive for a more intrinsic notation than the one used in Theorem 6.6.

As noticed above (Remark 4.1), an additive (= order compatible) equivalence relation Φ on M is the same thing as a partition of M into convex subsets, namely, the partition of M into the equivalence classes of Φ,

$$\Phi \mathrel{\hat{=}} (\xi \mid \xi \in M/\Phi).$$

Notation 6.7.

a) *Given an additive equivalence relation Φ on M, define*

$$L(\Phi) := \{x \in M \mid x \neq 0 \text{ and } x \leqslant y \text{ for every } y \in M \text{ with } x \sim_\Phi y\}.$$

Thus, $L(\Phi)$ consists of those $x \in L$ which are the smallest element of $[x]_\Phi$.

b) *If E is an additive equivalence relation on U, define*

$$L(E) := L(E|M).$$

Of course, $L(\Phi)$ and $L(E)$ may be empty. Clearly, $[0]_\Phi \cap L(\Phi) = \varnothing$ and $[0]_E \cap L(E) = \varnothing$.

We can rewrite Theorem 6.6 as follows:

Theorem 6.6′. *Given an additive equivalence relation Φ on M and for every $a \in L := L(\Phi)$ an equivalence relation E_a on the set*

$$U_a := \{x \in U \mid ex = a\},$$

there exists a unique additive equivalence relation E on U with $E|M = \Phi$ and $E|U_a = E_a$ for every $a \in L$. It can be described as follows:

Let $x, y \in U$ and $ex \leqslant ey$.

1) *If $ex \notin L$, then*

$$x \sim_E y \iff ex \sim_\Phi ey.$$

2) *If $ex = a \in L$, but $ey > a$, then*

$$x \sim_E y \iff ex \sim_\Phi ey \quad \text{and} \quad x \sim_{E_a} ex.$$

3) *If $ex = ey = a \in L$, then*

$$x \sim_E y \iff x \sim_{E_a} y.$$

We want to analyze under which conditions on the data Φ and $(E_a \mid a \in L(\Phi))$ the additive relation E will also be multiplicative, hence homomorphic. For this we need still another preparation, namely, a study of the set

$$A(E) := \{x \in U \mid x \sim_E ex\}.$$

It turns out that it is appropriate to start with an even weaker property of E than additivity.

Definition 6.8. *We call an equivalence relation E on the supertropical semiring U* ***ghost compatible****, if the condition* AE1 *from above holds, i.e.,*

$$\forall x, y \in U : x \sim_E y \Rightarrow ex \sim_E ey.$$

Clearly, every multiplicative and every additive equivalence relation is ghost compatible.

Lemma 6.9.

a) *If E is any equivalence relation on U, then $M \subset A(E)$ and $A(E) + A(E) \subset A(E)$.*

b) *If E is ghost compatible, then*

$$A(E) = \{x \in U \mid \exists z \in M : x \sim_E z\}.$$

c) *If E is multiplicative, then $A(E)$ is an ideal of U.*

Proof. a): It is trivial that $M \subset A(E)$. Let $x, y \in A(E)$ be given with $ex \leqslant ey$ (without loss of generality). If $ex < ey$, then $x+y = y \in A(E)$. If $ex = ey$, then $x+y = ey \in M \subset A(E)$.

b): Assume that $x \in U$, $z \in M$, and $x \sim_E z$. Then $ex \sim_E ez = z$, since E is ghost compatible. It follows that $x \sim_E ex$.

c): If $x \sim_E ex$, then $zx \sim_E ezx$ for every $z \in U$, since E is multiplicative. Thus $U \cdot A(E) \subset A(E)$. It follows by a) that $A(E)$ is an ideal of U. □

Remark 6.10. *If E is additive, then, using the data from Theorem 6.6′, we have*

$$A(E) = \{x \in U \mid ex \notin L\} \cup \bigcup_{a \in L} \{x \in U_a \mid x \sim_{E_a} a\}.$$

Theorem 6.11. *Assume that E is an additive equivalence relation on U with the data*

$$\Phi := E|M, \quad \mathfrak{A} := A(E), \quad L := L(\Phi), \quad E_a := E|U_a$$

for $a \in L$. The following are equivalent:

a) *E is multiplicative (hence homomorphic).*

b) *Φ is multiplicative. $\mathfrak{A}$ is an ideal of U. For any $a \in L$, $x, y \in U \setminus \mathfrak{A}$ with $ex = ey = a$, and $z \in U$ with $za \in L$:*

$$x \sim_{E_a} y \ \Rightarrow\ zx \sim_{E_a} zy.$$

Proof. a) ⇒ b): evident.

b) ⇒ a): Let $x, y, z \in U$ be given with $x \sim_E y$. We have to verify that $xz \sim_E yz$. Since E is ghost compatible and Φ is multiplicative, $exz \sim_E eyz$.

Case 1. $x \in \mathfrak{A}$ or $y \in \mathfrak{A}$. Due to Lemma 6.9.b, the set $\mathfrak{A} = A(E)$ is a union of equivalence classes of E. Thus both x and y are in $\mathfrak{A}$. Since $\mathfrak{A}$ is assumed to be an ideal, zx and zy are in $\mathfrak{A}$, and then

$$zx \sim_E ezx \sim_E ezy \sim_E zy.$$

Case 2. $x \notin \mathfrak{A}$ and $y \notin \mathfrak{A}$. Now $ex \in L$, $ey \in L$. Since $ex \sim_\Phi ey$, it follows that $ex = ey =: a \in L$. Thus $x \sim_{E_a} y$. If $za \notin L$, then zx and zy are in $\mathfrak{A}$, and we conclude as above that $zx \sim_E zy$. If $za \in L$, we conclude from $x \sim_{E_a} y$ by assumption b) that $zx \sim_{E_{za}} zy$.

Thus, $zx \sim_E zy$ in all cases. □

We introduce a special class of ghost-compatible equivalence relations, and then will identify the homomorphic relations among these.

Definition 6.12. *Let Φ be an equivalence relation on the set M, and let $\mathfrak{A}$ be a subset of U containing M. We define an equivalence relation $E := E(U, \mathfrak{A}, \Phi)$ on U as follows:*

$$x_1 \sim_E x_2 \quad \Leftrightarrow \quad \begin{array}{l} \textit{Either } x_1 = x_2, \\ \textit{or } x_1 \in \mathfrak{A},\ x_2 \in \mathfrak{A},\ \textit{and } ex_1 \sim_\Phi ex_2. \end{array}$$

□

The equivalence classes of $E = E(U, \mathfrak{A}, \Phi)$ are the sets $\{x \in \mathfrak{A} \mid ex \in \xi\}$, with ξ running through M/Φ, and the one point sets $\{x\}$ with $x \in U \backslash \mathfrak{A}$. Clearly E is ghost compatible and $E|M = \Phi$.

There is a structural characterization of $E(U, \mathfrak{A}, \Phi)$.

Proposition 6.13. *Let $E := E(U, \mathfrak{A}, \Phi)$ with Φ an equivalence relation on M and $\mathfrak{A}$ a subset of U containing M.*

i) *$A(E) = \mathfrak{A}$.*
ii) *E is the finest ghost compatible equivalence relation on U with $E|M \supset \Phi$ and $A(E) \supset \mathfrak{A}$.*

Proof. i): If $x \in \mathfrak{A}$, then clearly $x \sim_E ex$. But, if $x \notin \mathfrak{A}$, then $x \neq ex$, and hence $x \not\sim_E ex$.

ii): Let F be a ghost compatible equivalence relation on U with $F|M \supset \Phi$ and $A(F) \supset \mathfrak{A}$. Let $x \in U$ be given. We verify that $[x]_E \subset [x]_F$.

Case 1. $x \notin \mathfrak{A}$. Now $[x]_E = \{x\} \subset [x]_F$.

Case 2. $x \in \mathfrak{A}$. Let $y \sim_E x$. Then $y \in \mathfrak{A}$ and $ex \sim_\Phi ex$. Thus, $x \sim_F ex$, $y \sim_F ey$, $ex \sim_F ey$. We conclude that $y \sim_F x$. Thus again $[x]_E \subset [x]_F$. □

Theorem 6.14. *Let again $E := E(U, \mathfrak{A}, \Phi)$ with Φ an equivalence relation on U and $\mathfrak{A}$ a subset of U containing M.*

i) *E is multiplicative iff Φ is multiplicative and $\mathfrak{A}$ is an ideal of U.*
ii) *E is additive, iff Φ is order compatible and $\mathfrak{A}$ contains every $x \in U$ with $ex \notin L(\Phi)$.*
iii) *Thus, E is homomorphic, iff Φ is homomorphic and $\mathfrak{A}$ is an ideal containing $\nu_U^{-1}(M \backslash L(\Phi))$.*

Proof. a) We know that

$$\mathfrak{A} = A(E) = \{x \in U \mid \exists z \in M : x \sim_E z\},$$

and that $\mathfrak{A} + \mathfrak{A} \subset \mathfrak{A}$.

b) If E is multiplicative, then, of course, Φ is multiplicative, and $\mathfrak{A}$ is an ideal by Lemma 6.9.c. If E is additive, then Φ is additive, which means that Φ is order compatible. Also then $\mathfrak{A}$ contains every $x \in U$ with $ex \notin L(\Phi)$ by Property AE3 in Proposition 6.4. If E is homomorphic, then all these properties hold.

c) Assume now that Φ is multiplicative, and $\mathfrak{A}$ is an ideal of U. We want to prove that E is multiplicative. Let $x, y, z \in U$ be given with $x \sim_E y$. We want to verify that $xz \sim_E yz$. If $x \in \mathfrak{A}$, then $y \in \mathfrak{A}$ and $ex \sim_\Phi ey$; hence, $exz \sim_\Phi eyz$. Since $xz, yz \in \mathfrak{A}$, we conclude that $xz \sim_E yz$. If $x \notin \mathfrak{A}$, then $x = y$, and hence $xz = yz$.

d) Assume that Φ is order compatible and $x \in \mathfrak{A}$ for every $x \in U$ with $ex \notin L(\Phi)$. We want to prove that E is additive, and we use the criterion of Proposition 6.4 for this.

Clearly, E obeys the axioms AE1 and AE2 there. It remains to check AE3. Let $x, y \in U$ be given with $ex < ey$ and $ex \sim_E ey$, i.e., $ex \sim_\Phi ey$. Then $ey \notin L(\Phi)$. By our assumption on $\mathfrak{A} = A(E)$ it follows that $y \in \mathfrak{A}$, i.e., $y \sim_E ey$. We conclude that $ex \sim_E y$, as desired. Thus E is indeed additive.

e) We have proved claims i) and ii) of the theorem. They implies iii). □

We discuss the special case that Φ is the diagonal of M, $\Phi = \operatorname{diag} M$. In other words, $x \sim_\Phi y$ iff $x = y$. We write more briefly $E(U, \mathfrak{A})$ for $E(U, \mathfrak{A}, \operatorname{diag} M)$. Repeating Definition 6.12 in this case we have

Definition 6.15. *Let $\mathfrak{A}$ be any ideal of the supertropical semiring U containing the ghost ideal M of U. The equivalence relation $E := E(U, \mathfrak{A})$ on U is defined as follows: Let $x, y \in U$.*

If $x \notin \mathfrak{A}$: $x \sim_E y \Leftrightarrow x = y$.
If $x \in \mathfrak{A}$: $x \sim_E y \Leftrightarrow y \in \mathfrak{A},\ ex = ey$. □

Clearly $L(\operatorname{diag} M) = M \setminus \{0\}$. Thus, Theorem 6.14 tells us that the equivalence relation $E(U, \mathfrak{A})$ is homomorphic. This also follows from [IKR1, §6], since $E(U, \mathfrak{A})$ is obviously an MFCE-relation.

Thus, the set U/E with $E := E(U, \mathfrak{A})$ is a supertropical semiring, the addition and multiplication being given by $(x, y \in U)$:

$$[x]_E + [y]_E := [x + y]_E, \quad [x]_E \cdot [y]_E := [xy]_E.$$

Every equivalence class $[x]_E$ of E contains a unique element of the set

$$V := (U \setminus \mathfrak{A}) \cup M,$$

namely, the element x, for $x \notin \mathfrak{A}$, and the element ex, for $x \in \mathfrak{A}$. Notice that V is closed under addition (Remark 6.10.b).

Identifying the set U/E of equivalence classes of E with the set of representatives V, we arrive at the following theorem.

Theorem 6.16. *Let $\mathfrak{A}$ be an ideal of U containing M and $V := (U \setminus \mathfrak{A}) \cup M$. On V we define an addition $+$ and multiplication $\odot$ as follows:*

$$x + y \text{ is the sum of } x \text{ and } y \text{ in } U.$$

$$x \odot y := \begin{cases} xy & \text{if } xy \notin \mathfrak{A}, \\ exy & \text{if } xy \in \mathfrak{A}. \end{cases}$$

Then $V = (V, +, \odot)$ is a supertropical semiring, and the map $\alpha : U \to V$ with $\alpha(x) = x$ for $x \in U \setminus \mathfrak{A}$, $\alpha(x) = ex$ for $x \in \mathfrak{A}$ is a surjective semiring homomorphism. It gives the equivalence relation $E(\alpha) = E(U, \mathfrak{A})$.

Of course, this can also be verified in a direct straightforward way.

Remarks 6.17.

(i) *The sub-semiring M of U is also a sub-semiring of V (in its given semiring structure). In particular, $e_U = e_V$.*
(ii) *M is also the ghost ideal of V, and the ghost map ν_V is the restriction of ν_U to V.*
(iii) *We have $1_U = 1_V$ if $1_U \notin \mathfrak{A}$, and $1_M = 1_V$ if $1_U \in \mathfrak{A}$. In the latter case $V = M$.*

Example 6.18. *Let L be a subset of M with $M \backslash L$ an ideal of M. Define*

$$\mathfrak{A} = \mathfrak{A}_L := \{x \in U \mid ex \in M\backslash L\} \cup M = \nu_U^{-1}(M\backslash L) \cup L.$$

Then $\mathfrak{A}$ is an ideal of U containing M. It is easily checked that $E(U, \mathfrak{A})$ is the equivalence relation on U which we considered in [IKR1, Example 6.13]. *We have*

$$V = (M\backslash L) \cup \nu_U^{-1}(L).$$

If $L \cdot L \subset L$, then $V \cdot V \subset V$; hence, the supertropical semiring V is a sub-semiring of U,. This is the case considered in [IKR1, Example 6.12].

Definition 6.19. *We call an equivalence relation E on U* ***strictly ghost separating*** *if no $x \in \mathcal{T}(U)$ is E-equivalent to an element y of M. Under the very mild assumption that E is ghost compatible, this means that $A(E) = M$ (cf. Lemma 6.9.b).*[10]

The restriction of $E(U, \mathfrak{A})$ to the supertropical semiring $V = (U\backslash\mathfrak{A}) \cup M$ from above is always ghost separating. Moreover, we have the following facts.

Proposition 6.20. *Assume that F is a multiplicative equivalence relation (and hence $A(F)$ is an ideal of U), and $\mathfrak{A}$ is an ideal of U with $M \subset \mathfrak{A} \subset A(F)$.*

i) *$E(U, \mathfrak{A}) \subset F$.*
ii) *The equivalence relation $\overline{F} := F/E(U, \mathfrak{A})$ on $\overline{U} := U/E(U, \mathfrak{A})$ is again multiplicative, and $A(\overline{F})$ is the image of $A(F)$ in $\overline{U}$, i.e., $A(\overline{F}) = A(F)/E(U, \mathfrak{A})$.*
iii) *$\overline{F}$ is strictly ghost separating iff $\mathfrak{A} = A(F)$.*
iv) *If we identify $\overline{U}$ with the semiring $V := (U\backslash\mathfrak{A}) \cup M$, as explicated above, then $\overline{F} = F|V$.*
v) *$\overline{F}$ is transmissive iff F is transmissive.*
vi) *$\overline{F}$ is homomorphic iff F is homomorphic.*

Proof. Let $E := E(U, \mathfrak{A})$.

a) We claim that for any $x, y \in U$ with $x \sim_E y$ also $x \sim_F y$. Now, if $x \notin \mathfrak{A}$, then $x = y$. If $x \in \mathfrak{A}$, then $y \in \mathfrak{A}$ and $ex = ey$. Since $\mathfrak{A} \subset A(F)$, it follows that $x \sim_F ex$, $y \sim_F ey$, and then that $x \sim_F y$. Thus $x \sim_F y$ in both cases. This proves $E \subset F$.

b) Claims ii) – iv) of the proposition are fairly obvious. v) follows from [IKR1, Corollary 6.2] since $\pi_F = \pi_{\overline{F}} \circ \pi_E$, and π_E is a surjective homomorphism. vi) is again obvious. □

We now exhibit a case where we have met the equivalence relation $E(U, \mathfrak{A}, \Phi)$ before. First a very general observation.

Remark 6.21. *Every $\mathfrak{a}$ of U with $e \cdot \mathfrak{a} \subset \mathfrak{a}$ is closed under addition. The reason is, that for any $x, y \in U$ the sum $x + y$ is either x or y or ex. Thus every subset $\mathfrak{a}$ of U with $U \cdot \mathfrak{a} \subset \mathfrak{a}$ (i.e., $\mathfrak{a}$ a monoid ideal of U) is an ideal of U. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of U then $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{a} + \mathfrak{b}$.*

Assume that Φ is a homomorphic equivalence relation on M. It gives us the homomorphism π_Φ from M to the bipotent semiring M/Φ. We define

$$\mathfrak{a}_\Phi := \{x \in U \mid ex \sim_\Phi 0\}$$

[10] We reserve the label "ghost separating" for a slightly broader class of equivalence relations to be introduced in [IKR3].

which is an ideal on U, and define

$$\mathfrak{A} := M \cup \mathfrak{a}_\Phi = M + \mathfrak{a}_\Phi,$$

which is an ideal of U containing M. It is the set of all $x \in U$ with $x = ex$ or $ex \sim_\Phi 0$. If necessary we more precisely write $\mathfrak{a}_{U,\Phi}$, $\mathfrak{A}_{U,\Phi}$ instead of $\mathfrak{a}_\Phi$, $\mathfrak{A}_\Phi$. Starting from Definition 6.12 it can be checked in a straightforward way that the multiplicative equivalence relation

$$E := E(U, \mathfrak{A}_\Phi, \Phi)$$

has the following description ($x, y \in U$):

$$x \sim_E y \quad \Leftrightarrow \quad \begin{array}{l} \text{either } x = y \\ \text{or } x = ex,\ y = ey,\ ex \sim_\Phi ey \\ \text{or } ex \sim_\Phi ey \sim_\Phi 0. \end{array}$$

Thus E is the equivalence relation $F(U, \gamma)$ defined in Example 4.8 with $\gamma := \pi_\Phi$, If M/Φ is cancellative the we know from Theorem 1.11 and Example 4.8 that $E(U, \mathfrak{A}_\Phi, \Phi)$ is transmissive. There are other cases where this also holds, cf. Remark 6.23 below.

We now apply Proposition 6.20 to the relation

$$F := E(U, \mathfrak{A} \cup \mathfrak{a}_\Phi, \Phi)$$

for $\mathfrak{A}$ any ideal of U containing M. Let $\overline{U}$ denote the supertropical semiring $U/E(U, \mathfrak{A})$, whose ghost ideal has been identified above with $M = eU$. It again can be checked in a straightforward way that the equivalence relation $F/E(U, \mathfrak{A})$ on $\overline{U}$ is just the relation

$$E(\overline{U}, \mathfrak{A}_{\overline{U},\Phi}, \Phi) = F(\overline{U}, \pi_\Phi),$$

in the notation of Example 4.8. Thus we arrive at the following result.

Theorem 6.22. *Let Φ be a homomorphic equivalence relation on $M := eU$ and $\mathfrak{A}$ an ideal of U which contains M. Let $\mathfrak{D} := \mathfrak{A} \cup \mathfrak{a}_\Phi = \mathfrak{A} + \mathfrak{a}_\Phi$ with $\mathfrak{a}_\Phi := \{x \in U | ex \sim_\Phi 0\}$.*

(a) *Then $\overline{U} := U/E(U, \mathfrak{A})$ is a supertropical semiring (as we know for long) and $E(U, \mathfrak{D}, \Phi)/E(U, \mathfrak{A})$ is the multiplicative equivalence relation $F(\overline{U}, \pi_\Phi)$.*
(b) *$E(U, \mathfrak{D}, \Phi)$ is transmissive iff $F(\overline{U}, \pi_\Phi)$ is transmissive.*
(c) *Tn particular $E(U, \mathfrak{D}, \Phi)$ is transmissive if M/Φ is cancellative.*

Remark 6.23. *Looking at Theorem 6.14 and Proposition 6.20.vi we can also state the following: $F(U, \mathfrak{D}, \Phi)$ is homomorphic iff $F(\overline{U}, \pi_\Phi)$ is homomorphic iff $\nu_U^{-1}(M\ L(\Phi)) \subset \mathfrak{D}$.*

Remark 6.24. *The question might arise whether the $E(U, \mathfrak{A}, \Phi)$ is transmissive for* ***any*** *ideal $\mathfrak{A} \supset M$ of U if, say, M/Φ is cancellative. The answer in general is "No": If $E(U, \mathfrak{A}, \Phi)$ is transmissive then $\mathfrak{A}$ must contain the ideal $\mathfrak{a}_\Phi$. The reason is that for any transmission $\alpha : U \to V$ and $x \in U$ with $\alpha(ex) = 0$ we have $\alpha(x) = 0$ since $\alpha(ex) = e\alpha(x)$.*

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