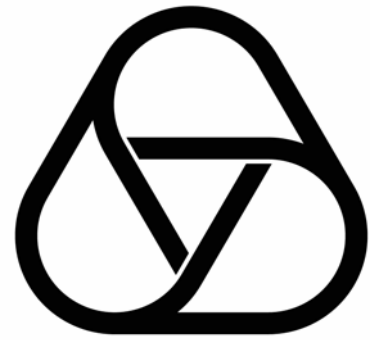


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ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

Monoid Valuations and Value Ordered
Supervaluations

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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MONOID VALUATIONS AND VALUE ORDERED SUPERVALUATIONS

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

ABSTRACT. We complement two papers on supertropical valuation theory ([IKR1], [IKR2]) by providing natural examples of m -valuations (= monoid valuations), after that of supervaluations and transmissions between them. The supervaluations discussed have values in **totally ordered supertropical semirings**, and the transmissions discussed respect the orderings. Basics of a theory of such semirings and transmissions are developed as far as needed.

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INTRODUCTION

The present paper is a complement to the papers [IKR1] and [IKR2] on supertropical valuation theory by the same authors. We deal with semirings which always are taken to be commutative. Generalizing Bourbaki's notion of a valuation on a commutative ring [B], we introduced in [IKR1] *m-valuations* (= monoid valuations) and then *supervaluations* on a (commutative) semiring R . These are certain maps from R to a “*bipotent semiring*” M and a “*supertropical semiring*” U , respectively.

Date: May 7, 2011.

2010 Mathematics Subject Classification. Primary: 13A18, 13F30, 16W60, 16Y60; Secondary: 03G10, 06B23, 12K10, 14T05.

Key words and phrases. Supertropical algebra, Ordered supertropical semirings, Bipotent semirings, Valuation theory, Monoid valuations, Supervaluations, Lattices, Transmissive and homomorphic equivalence relations.

The research of the first author has been supported by the Oberwolfach Leibniz Fellows Programme (OWLF), Mathematisches Forschungsinstitut Oberwolfach, Germany.

The research of the first and third authors have been supported by the Israel Science Foundation (grant No. 448/09).

The research of the second author was supported in part by the Gelbart Institute at Bar-Ilan University, the Minerva Foundation at Tel-Aviv University, the Department of Mathematics of Bar-Ilan University, and the Emmy Noether Institute at Bar-Ilan University.

To repeat, a semiring M is **bipotent** if M is a totally ordered monoid under multiplication with smallest element 0, and the addition is given by $x + y = \max(x, y)$. Then an **m-valuation** on R is a multiplicative map $v : R \rightarrow M$, which sends 0 to 0, 1 to 1, and obeys the rule $v(a + b) \leq v(a) + v(b)$. We call v a **valuation** if moreover the semiring M is cancellative. {In the classical case of a Krull valuation v , R is a field and $M = \mathcal{G} \cup \{0\}$, with \mathcal{G} the value group of v written in multiplicative notation.}

A **supertropical semiring** U is a semiring such that $e := 1 + 1$ is an idempotent of U and two more axioms hold ([IKR1, Definitions 3.5 and 3.9]), which imply in particular that the ideal $M := eU$ is a bipotent semiring. The elements of $M \setminus \{0\}$ are called **ghost** and those of $\mathcal{T}(U) := U \setminus M$ are called **tangible**. The zero element of U is regarded both as ghost and tangible. For $x \in U$ we call ex the **ghost companion** of x . For $x, y \in U$ we have the rule

$$x + y = \begin{cases} y & \text{if } ex < ey, \\ x & \text{if } ex > ey, \\ ex & \text{if } ex = ey. \end{cases}$$

Thus addition on U is uniquely defined by multiplication and the element e . We also mention that $ex = 0$ implies $x = 0$. We refer to [IKR1, §3] for all details.

Finally, a **supervaluation** on R is a multiplicative map $\varphi : R \rightarrow U$ to a supertropical semiring U sending 0 to 0 and 1 to 1, such that the map $e\varphi : R \rightarrow eU$, $a \mapsto e\varphi(a)$, is an m-valuation. We then say that φ **covers** the m-valuation $v := e\varphi$.

If $\varphi : R \rightarrow U$ is a supervaluation then $U' := \varphi(R) \cup e\varphi(R)$ is a sub-semiring of U and is again supertropical. In practice we nearly always may replace U by U' and then have a supervaluation at hand which we call **surjective**¹.

Given a surjective supervaluation $\varphi : R \rightarrow U$ and a map $\alpha : U \rightarrow V$ to a supertropical semiring V , the map $\alpha \circ \varphi$ is again a supervaluation iff α is multiplicative, sends 0 to 0, 1 to 1, e to e , and restricts to a semiring homomorphism from eU to eV . {We denote the elements $1 + 1$ in U and V both by “ e ”} We call such a map $\alpha : U \rightarrow V$ a **transmission**. Any semiring homomorphism from U to V is a transmission, but there exist others.

Transmissions are tied up with the relation of **dominance** defined in [IKR1, §5]. If $\varphi : R \rightarrow U$ and $\psi : R \rightarrow V$ are supervaluations and φ is surjective, then φ **dominates** ψ , which we denote by $\varphi \geq \psi$, iff there exists a transmission $\alpha : U \rightarrow V$ with $\psi = \alpha \circ \varphi$. If $\varphi \geq \psi$ we also say that ψ is a **coarsening** of the supervaluation φ .

A bipotent semiring M may be viewed as a supertropical semiring U with empty set $\mathcal{T}(U)$, i.e., $U = eU = M$. Then a transmission $\gamma : M \rightarrow N$ is just a semiring homomorphism. In other terms, γ is an order preserving monoid homomorphism with $\gamma(0) = 0$. If R is a field and $v : R \rightarrow M$, $w : R \rightarrow N$ are Krull valuations (in multiplicative notation), then the dominance relation $v \geq w$ means that w is a coarsening of v in the classical sense.

At crucial points in the paper [IKR1], [IKR2] we had to assume that the supervaluations in question cover a valuation $v : R \rightarrow M$ instead of just an m-valuation, i.e., M had to be assumed cancellative. On the other hand these papers contain few examples of true m-valuations. Thus a reader might suspect that it is better in supertropical valuation theory to focus from the beginning on valuations instead of m-valuations. The first goal of the present paper is to clarify this situation,

In §1 we study two very natural classes of m-valuations, the so-called **V -valuations** and **V^0 -valuations**. They have been introduced (on rings) by Harrison-Vitulli [HV1] and

¹Although this does not mean surjectivity in the usual sense, there is no danger of confusion since a supervaluation $\varphi : R \rightarrow U$ hardly ever can be surjective as a map except in the degenerate case $U = M$.

D. Zhang [Z], respectively. To our opinion these m-valuations, which often are not valuations, have not yet found the attention in the literature that they deserve.

In §1 it is proved that every nontrivial m-valuation dominates both a V -valuation and a V^0 -valuation (which may be different) in a canonical way. There are also given various instances of dominance $v \geq w$ with v a V^0 -valuation and w a V -valuation, or vice versa. Then in §2 we exhibit a canonical way to coarsen a given m-valuation to a valuation. If v is a V -valuation or a V^0 -valuation, almost always this coarsening is again a V -valuation or a V^0 -valuation. One gets the impression that a supertropical valuation theory excluding m-valuations would be very incomplete.

A second goal of the paper is to give natural explicit examples of supervaluations and dominance relations between them. For that reason we start in §3 a theory of **supertropical semirings** which are **totally ordered**. The total order on such a semiring U has to be compatible with addition and multiplication, and has to extend the order on $M = eU$ as a bipotent semiring.

A supervaluation with values in a totally ordered semiring U will be called a **value-ordered supervaluation** or **vo-supervaluation**, for short. Given two vo-valuations $\varphi : R \rightarrow U$ and $\psi : R \rightarrow U$ we establish in §5 a refined notion of dominance, called **total dominance** and written \geq_{tot} , which is sharper than the dominance relation $\varphi \geq \psi$ considered in [IKR1] and [IKR2]. If φ is surjective it means that the transmission $\alpha : U \rightarrow V$ with $\alpha \circ \varphi = \psi$ respects the orderings of U and V . We then say that the transmission α is **monotone**.

All examples of supervaluations in §3-§6 will be vo-supervaluations and all discussed transmissions between them will be monotone.

It seems desirable to have a theory of vo-supervaluations and monotone transmissions at hand which parallels the theory of supervaluations and transmissions in [IKR1] and [IKR2]. The present paper only takes first steps in such a theory, just enough to obtain a rich stock of examples of vo-supervaluations and transmissions. An advantage of the examples is that the total orderings ease the insight into the structure of such supervaluations and transmissions compared to cases where total orderings are not present or not respected.

An important point here is that every monotone transmission is a semiring homomorphism (cf. Theorem 5.3 below), while – as we know from [IKR1] and [IKR2] – there exist many transmissions which are not homomorphisms. Thus the examples do not reflect certain aspects of general supervaluation theory.

A full fledged theory of vo-supervaluations should embrace an analysis of the vo-supervaluations $\varphi : R \rightarrow U$ on a ring R equipped with a cone or prime cone T (cf. e.g. [BCR, Definitions 4.2.1, 4.3.1]) which are compatible with T and the total ordering of U in an appropriate sense. It should have relevance for real algebraic geometry. We have to leave these matters for future investigation.

Notations. Given sets X, Y we mean by $Y \subset X$ that Y is a subset of X , with $Y = X$ allowed. If E is an equivalence relation on X then X/E denotes the set of E -equivalence classes in X , and $\pi_E : X \rightarrow X/E$ is the map which sends an element x of X to its E -equivalence class, which we denote by $[x]_E$. If $Y \subset X$, we put $Y/E := \{[x]_E \mid x \in Y\}$.

$\mathcal{T}(U)$ and $\mathcal{G}(U)$ denote the sets of tangible and ghost elements of U , respectively, cf. [IKR1, Terminology 3.7].

If $v : R \rightarrow M$ is an m-valuation we call the ideal $v^{-1}(0)$ of R the **support** of v , and denote it by $\text{supp}(v)$.

1. V -VALUATIONS AND V^0 -VALUATIONS

Given any m -valuation $v : R \rightarrow M$ on a semiring R , we introduce the sets

$$A_v := \{x \in R \mid v(x) \leq 1\},$$

$$\mathfrak{p}_v := \{x \in R \mid v(x) < 1\}.$$

Clearly, A_v is a sub-semiring of R and \mathfrak{p}_v is a prime ideal of A_v . Moreover, the sets $R \setminus \mathfrak{p}_v$ and $R \setminus A_v$ are both closed under multiplication. The set $R \setminus A_v$ may be empty, but $R \setminus \mathfrak{p}_v$ is not, since $1 \notin \mathfrak{p}_v$.

Definition 1.1. We call A_v the **valuation semiring** of v , and \mathfrak{p}_v the **valuation ideal** of v .

If $R = F$ happens to be a semifield and $v : F \rightarrow M$ is a surjective m -valuation, hence a surjective valuation (cf. our terminology in [IKR1, §2]), then we meet a situation very similar to the classical case that F is a field. Now $A := A_v$ has the property that for any $x \in F^*$ either x or x^{-1} is an element of A , and for $x, y \in F^*$

$$v(x) \leq v(y) \Leftrightarrow \frac{x}{y} \in A^* \Leftrightarrow xA \subset yA.$$

Thus, the valuation v is determined up to equivalence by the sub-semiring $A_v = A$ of F . It is also uniquely determined by the set \mathfrak{p}_v since

$$A = \{z \in F \mid z\mathfrak{p}_v \subset \mathfrak{p}_v\}.$$

Notice that M is now a bipotent semifield, $M = \Gamma \dot{\cup} \{0\}$ with Γ an ordered abelian group. This group can be identified with the group F^*/A^* , since for $x, y \in F^*$ we have

$$v(x) = v(y) \Leftrightarrow xA^* = yA^*.$$

$\{A^*$ denotes the group of units of the semiring $A\}$ Thus, we may also write $M \cong F/A^*$, i.e., M is the quotient of the semifield F by the orbital equivalence relation on F given by A^* .

In the case that v is strong (which is automatic if F is a field), even the subgroup A^* of F determines v up to equivalence. Indeed, now

$$A = A^* \cup \{x \in R \mid 1 + x \in A^*\}.$$

In general, matters are much more complicated. In the present section our first goal is to coarsen a given surjective m -valuation $v : R \rightarrow N$ “slightly” in such a way that the m -valuation $w : R \rightarrow N$ has the same valuation ideal $\mathfrak{p}_w = \mathfrak{p}_v$ as v (as a subset of R closed under addition and multiplication), but w is determined by the set $\mathfrak{p} = \mathfrak{p}_w$ in a canonical way. $\{w$ is a so-called “ V^0 -valuation”, see below. $\}$ We then will pursue the same program based on the set A_v instead of \mathfrak{p}_v .

In the following R is always a (commutative) semiring.

Definition 1.2 (cf. [C, §1]). Let \mathfrak{p} be a subset of R with

$$0 \in \mathfrak{p}, \quad 1 \notin \mathfrak{p}, \quad \mathfrak{p} + \mathfrak{p} \subset \mathfrak{p},$$

and both \mathfrak{p} and $R \setminus \mathfrak{p}$ closed under multiplication. Then we call \mathfrak{p} a **prime** of R .

Example 1.3. If $v : R \rightarrow M$ is any m -valuation, then \mathfrak{p}_v is a prime of R .

Let \mathfrak{p} be a prime of R . For any $x \in R$ and subset L of R , we put

$$[L : x] = \{z \in R \mid zx \in L\}.$$

We define on R an equivalence relation \sim as follows:

$$x \sim y \iff [\mathfrak{p} : x] = [\mathfrak{p} : y].$$

We observe that this equivalence relation is multiplicative, i.e., $x \sim y$ implies $xz \sim yz$ for any $z \in R$. Indeed, suppose we have a triple x, y, z with $x \sim y$, but $xz \not\sim yz$, say, $[\mathfrak{p} : xz] \not\subset [\mathfrak{p} : yz]$. Then there exists some $u \in R$ with $uyz \in \mathfrak{p}$, but $uxz \notin \mathfrak{p}$. Thus $uz \in [\mathfrak{p} : y]$, but $uz \notin [\mathfrak{p} : x]$. This contradicts the equality $[\mathfrak{p} : x] = [\mathfrak{p} : y]$.

We introduce the monoid

$$M := M(R, \mathfrak{p}) := (R / \sim, \cdot)$$

with the multiplication

$$[x] \cdot [y] := [xy],$$

where $[x]$ denotes the equivalence class of x . Clearly, we have a partial ordering \leq on the set M given by

$$[y] \leq [x] \iff [\mathfrak{p} : x] \subset [\mathfrak{p} : y].$$

We start out to prove that this partial ordering is in fact total.

We will use the following lemma, now for the set $L = \mathfrak{p}$, but later also in other situations. Let L be a subset of R such that both L and $R \setminus L$ are closed under multiplication.

Lemma 1.4. *Let $x, y, s, t \in R$, and assume that $sx \in L$ and $ty \in L$. Then at least one of the elements sy, tx lies in L .*

Proof. Since L is closed under multiplication, we have

$$sy \cdot tx = sx \cdot ty \in L.$$

Since $R \setminus L$ is closed under multiplication, we conclude that $sy \in L$ or $tx \in L$. \square

Proposition 1.5. *Let $x, y \in R$ and $[\mathfrak{p} : x] \not\subset [\mathfrak{p} : y]$. Then $[\mathfrak{p} : y] \subset [\mathfrak{p} : x]$.*

Proof. We pick some $z \in R$ with $zx \in \mathfrak{p}$, but $zy \notin \mathfrak{p}$. Let $u \in [\mathfrak{p} : y]$ be given. We have $zx \in \mathfrak{p}$, $yu \in \mathfrak{p}$, but $zy \notin \mathfrak{p}$. We conclude by the lemma that $ux \in \mathfrak{p}$, i.e., $u \in [\mathfrak{p} : x]$. This proves the claim. \square

Thus, the ordering on M is total. The equivalence class $[0]$ is the smallest element of M , since

$$[\mathfrak{p} : 0] = R \supset [\mathfrak{p} : x]$$

for every $x \in R$. Observe also that our ordering is compatible with the multiplication on the monoid M . Indeed, $[\mathfrak{p} : x] \subset [\mathfrak{p} : y]$ implies $[\mathfrak{p} : xz] \subset [\mathfrak{p} : yz]$ for every $z \in R$, as is easily seen.

We regard M as a bipotent semiring, defining the addition on M in the usual way (cf. [IKR1, §1]):

$$\text{If } [x] \leq [y], \text{ then } [x] + [y] := [y].$$

We have $[0] = 0_M$, $[1] = 1_M$.

Theorem 1.6.

a) *The map*

$$v = v_{R, \mathfrak{p}} : R \rightarrow M, \quad v(x) := [x],$$

is an m -valuation on the semiring R .

b) *Its support is*

$$\mathfrak{q} := \{x \in R \mid Rx \subset \mathfrak{p}\},$$

and this is a prime ideal of R .

Proof. a): Clearly $v(0) = 0$, $v(1) = 1$, and $v(xy) = v(x)v(y)$ for any $x, y \in R$.

It remains to verify for any $x, y \in R$ with $v(x) \leq v(y)$ that $v(x+y) \leq v(y)$, i.e., $[\mathfrak{p} : y] \subset [\mathfrak{p} : x+y]$. Given $z \in [\mathfrak{p} : y]$, we have $zy \in \mathfrak{p}$. This implies $zx \in \mathfrak{p}$ and then $z(x+y) \in \mathfrak{p}$. {N.B. Here we use for the first time that $\mathfrak{p} + \mathfrak{p} \subset \mathfrak{p}$.} Thus $[\mathfrak{p} : y] \subset [\mathfrak{p} : x+y]$, i.e., $v(x+y) \leq v(y)$, as desired.

b): Given $x \in R$, we have $v(x) = 0$ iff $x \sim 0$, i.e., $[\mathfrak{p}; x] = [\mathfrak{p} : 0] = R$. Thus, v has the support

$$\mathfrak{q} := \{x \in R \mid Rx \subset \mathfrak{p}\}.$$

Now, if $x, y \in R \setminus \mathfrak{q}$, there exist elements $s, t \in R$ with $sx \notin \mathfrak{p}$, $ty \notin \mathfrak{p}$. It follows that $stxy \notin \mathfrak{p}$, and hence $xy \notin \mathfrak{q}$. Thus, \mathfrak{q} is a prime ideal of R . \square

Definition 1.7. *We say that*

$$v_{R,\mathfrak{p}} : R \rightarrow M(R, \mathfrak{p})$$

is the m -valuation associated to the prime \mathfrak{p} of R . We call any m -valuation equivalent to such a valuation $v_{R,\mathfrak{p}}$ a V^0 -valuation. Later (from §3 onward), we often write $v_{\mathfrak{p}}$ instead of $v_{R,\mathfrak{p}}$.

The construction of these m -valuations is in some sense dual to the construction of the “ V -valuations” in the paper [HV1] by Harrison and Vitulli; hence the label V^0 . We will discuss V -valuations below.

We compute the valuation semiring and valuation ideal of a V^0 -valuation $v_{R,\mathfrak{p}}$.

Proposition 1.8. *Let $v = v_{R,\mathfrak{p}}$ for a prime \mathfrak{p} of R . Then $A_v = \{x \in R \mid x\mathfrak{p} \subset \mathfrak{p}\}$ and $\mathfrak{p}_v = \mathfrak{p}$.*

Proof. Let $x \in R$.

a) $x \in A_v \Leftrightarrow v(x) \leq 1 \Leftrightarrow [\mathfrak{p} : x] \supset [\mathfrak{p} : 1] = \mathfrak{p} \Leftrightarrow x\mathfrak{p} \subset \mathfrak{p}$.

b) $x \in \mathfrak{p}_v \Leftrightarrow v(x) < 1 \Leftrightarrow [\mathfrak{p} : x] \not\supseteq [\mathfrak{p} : 1] = \mathfrak{p} \Leftrightarrow \mathfrak{p}x \subset \mathfrak{p}$, but there exists also some $s \in R \setminus \mathfrak{p}$ with $sx \in \mathfrak{p}$. Since both \mathfrak{p} and $R \setminus \mathfrak{p}$ are closed under multiplication, the last condition means that $x \in \mathfrak{p}$. We conclude that $\mathfrak{p}_v = \mathfrak{p}$. \square

Lemma 1.9. *Let $v : R \rightarrow M$ be a surjective V^0 -valuation. Then the bipotent semiring M has the following separation property:*

$$\begin{aligned} (\text{Sep}_V^0) : & \text{ If } \alpha, \beta \in M \text{ and } \alpha < \beta, \text{ there exists some } \gamma \in M \\ & \text{ with } \alpha\gamma < 1 \text{ and } \beta\gamma \geq 1. \end{aligned}$$

Proof. Choose $x, y \in R$ with $v(x) = \alpha$, $v(y) = \beta$. Then $[\mathfrak{p} : x] \not\supseteq [\mathfrak{p} : y]$. Thus, there exists some $z \in R$ with $zx \in \mathfrak{p}$ but $zy \notin \mathfrak{p}$. This means that $v(zx) < 1$ but $v(zy) \geq 1$. The element $\gamma := v(z)$ does the job. \square

Definition 1.10. *We call a bipotent semiring M having the separation property (Sep_V^0) a V^0 -semiring.*

We now can state a remarkable fact.

Theorem 1.11. *Assume that R is a semiring, M is a V^0 -semiring, and $v : R \rightarrow M$ is a surjective map with $v(0) = 0$, $v(1) = 1$, $v(xy) = v(x)v(y)$ for any $x, y \in R$ (i.e., v is a homomorphism from the monoid (R, \cdot) onto the monoid (M, \cdot)). Assume also that*

$$\forall x, y \in R: \quad v(x) < 1, v(y) < 1 \Rightarrow v(x + y) < 1. \quad (1.1)$$

Let $\mathfrak{p} := \{x \in R \mid v(x) < 1\}$. Then \mathfrak{p} is a prime of R , and v is a V^0 -valuation equivalent to $v_{R, \mathfrak{p}}$.

Proof. It is obvious that $0 \in \mathfrak{p}$, $1 \notin \mathfrak{p}$ and both \mathfrak{p} and $R \setminus \mathfrak{p}$ are closed under multiplication. The rule (1.1) tells us that $\mathfrak{p} + \mathfrak{p} \subset \mathfrak{p}$. Thus \mathfrak{p} is a prime of R . We will verify that

$$\forall x, y \in R: \quad v(x) \leq v(y) \Leftrightarrow [\mathfrak{p} : x] \supset [\mathfrak{p} : y]. \quad (1.2)$$

Then we will be done. Indeed, let $w := v_{R, \mathfrak{p}}$, $N := M(R, \mathfrak{p})$. We know by (1.2) that for any $x, y \in R$, $v(x) = v(y)$ iff $w(x) = w(y)$. Thus we have a well-defined bijection $\gamma : M \rightarrow N$ with $\gamma(v(x)) = w(x)$ for all $x \in R$. This map sends 0 to 0, 1 to 0, and is multiplicative. Further, (1.2) tells us that γ is order preserving. Thus γ is a semiring isomorphism and $\gamma \circ v = w$.

Instead of (1.2) we verify the equivalent property

$$\forall x, y \in R: \quad v(x) > v(y) \Leftrightarrow [\mathfrak{p} : x] \not\subseteq [\mathfrak{p} : y].$$

If $v(x) > v(y)$ then, of course, for every z with $v(xz) < 1$, i.e., $xz \in \mathfrak{p}$, we have $v(yz) < 1$ i.e., $yz \in \mathfrak{p}$. But since M is V^0 , there exists some $z' \in R$ with $v(xz') \geq 1$, $v(yz') < 1$, i.e., $xz' \notin \mathfrak{p}$, $yz' \in \mathfrak{p}$. Thus $[\mathfrak{p} : x] \not\subseteq [\mathfrak{p} : y]$. On the other hand, if $[\mathfrak{p} : x] \not\subseteq [\mathfrak{p} : y]$, we have some $z \in R$ with $v(xz) \geq 1$, $v(yz) < 1$, and hence $v(xz) > v(yz)$. Thus certainly $v(x) > v(y)$. \square

Theorem 1.12. *Assume that $v : R \rightarrow M$ is a surjective m -valuation. Let*

$$w := v_{R, \mathfrak{p}} : R \rightarrow N := M(R, \mathfrak{p})$$

denote the V^0 -valuation associated to the prime $\mathfrak{p} := \mathfrak{p}_v$.

Then v dominates w , i.e., there exists a (unique) semiring homomorphism $\gamma : M \rightarrow N$ such that $w = \gamma \circ v$. In other terms, w is a coarsening of v , cf. [IKR1, §2], [IKR2].

Proof. We only need to verify that, for any $x, y \in R$, $v(x) \leq v(y)$ implies $w(x) \leq w(y)$, cf. [IKR1, Definition 2.9]. But this is obvious. If $v(x) \leq v(y)$, then

$$[\mathfrak{p} : x] = \{z \in R \mid v(zx) \in \mathfrak{p}\} \supset \{z \in R \mid v(zy) \in \mathfrak{p}\} = [\mathfrak{p} : y].$$

\square

Definition 1.13. *We call w the V^0 -coarsening of v , and we write $w = v_{\downarrow}$.*

Example 1.14. *Let M be any bipotent semiring. The identity map $v = \text{id}_M$ may be viewed as a (strict) m -valuation on the semiring M . Thus it gives us a V^0 -valuation*

$$\gamma_V^0 := \gamma_{M, V}^0 := (\text{id}_M)_{\downarrow} : M \rightarrow M_{\downarrow}.$$

Since v is strict, its coarsening γ_V^0 is again strict, i.e., γ_V^0 is a semiring homomorphism. The associated homomorphic equivalence relation is given by

$$x \sim y \Leftrightarrow [\mathfrak{p}_M : x] = [\mathfrak{p}_M : y]$$

with

$$\mathfrak{p}_M := \{x \in M \mid x < 1\}.$$

The valuation semiring of γ_V^0 is

$$[\mathfrak{p}_M : \mathfrak{p}_M] := \{x \in M \mid x\mathfrak{p}_M \subset \mathfrak{p}_M\},$$

and the valuation ideal of γ_V^0 is \mathfrak{p}_M .

We retain the notations developed in this example. The map γ_V^0 allows us a fresh view of the V^0 -coarsening of any surjective m -valuation $v : R \rightarrow M$.

Proposition 1.15. *The V^0 -coarsening v_\downarrow of v is equivalent to the m -valuation*

$$\gamma_V^0 \circ v : R \rightarrow M_\downarrow.$$

Proof. Let $\mathfrak{p} := \mathfrak{p}_v$. We have $\gamma_V^0{}^{-1}(\mathfrak{p}_{M_\downarrow}) = \mathfrak{p}_M$ and $v^{-1}(\mathfrak{p}_M) = \mathfrak{p}$. Thus, $w := \gamma_V^0 \circ v : R \rightarrow M_\downarrow$ has the valuation ideal $\mathfrak{p}_w = \mathfrak{p}$. Moreover, M_\downarrow is a V^0 -semiring. Theorem 1.11 gives the claim. \square

We now turn to the construction of V -valuations, to be found in [HV1] (in the case that R is a ring). As before R may be any semiring.

Definition 1.16. [HV1]. *Let A be a subset of R with*

$$0 \in A, \quad 1 \in A, \quad A + A \subset A,$$

*and both A and $R \setminus A$ closed under multiplication. Then we call A a **CMC-subsemiring of R** .*

In other words, a set $A \subset R$ is a CMC-subsemiring of R iff A is a subsemiring of R and $R \setminus A$ is closed under multiplication. The label CMC (= complement multiplicatively closed) alludes to this latter property. Notice that, if R happens to be a ring, then the relation $(-1) \cdot (-1) = 1$ forces (-1) to be in A , hence A is a subring of R .

Let now A be a CMC-subsemiring of R , which is **proper**, i.e. $A \neq R$. Then, in complete analogy to the above, we define an equivalence relation \sim on R by

$$x \sim y \iff [A : x] = [A : y].$$

This equivalence relation is again multiplicative; hence we obtain a monoid

$$M := M(R, A) := (R / \sim, \cdot)$$

with the multiplication

$$[x] \cdot [y] := [xy],$$

and we can see as above (in particular use Lemma 1.4 for $L = A$) that this monoid M is totally ordered by the rule

$$[y] \leq [x] \iff [A : x] \subset [A : y].$$

One further verifies (cf. [HV1]) that the map

$$v := v_A := v_{R,A} : R \rightarrow M$$

is an m -valuation on the semiring A with support

$$\mathfrak{q} := \{x \in R \mid Rx \subset A\}$$

which again is a prime ideal of R .

Definition 1.17. [HV1]. *We call any m -valuation v on R which is equivalent to $v_{R,A}$ for some proper CMC-subsemiring A of R a **V-valuation** on R and we call $v_{R,A}$ the **V-valuation of R associated to A** .*

Historical comments: In [HV1] these valuations have been dubbed “finite V -valuations” by Harrison, to give credit to Marie A. Vitulli, who conveyed to him the idea of this construction and in addition a construction of “infinite V -valuations”, a type of absolute values having an archimedean flavour. We will not deal with infinite V -valuations here; hence we simply speak of finite V -valuations as “ V -valuations”.

Harrison and Vitulli report that already M. Griffin defined (finite) V -valuations in an unpublished paper [Gr], but then did not pursue this idea further [HV1, p. 269].

V^0 -valuations have been introduced - in the case of rings - by D. Zhang [Z]. \square

If $v = v_{R,A}$ for some proper CMC-subsemiring A of R , then it easily checked that v has the valuation semiring $A_v = A$. Further it turns out that the valuation ideal \mathfrak{p}_v of v is the set

$$P(A) := \{x \in R \mid \exists y \in R \setminus A : xy \in A\}. \quad (1.3)$$

In particular $P(A)$ is a prime of R , hence a prime ideal of A , a fact which for R a ring had been already been observed by P. Samuel in his seminal paper [S]. {Samuel’s direct proof also goes through verbatim for A a semiring.}

Definition 1.18. We call $P(A)$ the **central prime** (in R) of the proper CMC-subsemiring A .

Parallel to Lemma 1.9 we obtain the following:

Lemma 1.19. [HV1]. Let $v : R \rightarrow M$ be a surjective V -valuation. Then the bipotent semiring M has the following separation property:

$$(\text{Sep}_V) : \text{If } \alpha, \beta \in M \text{ and } \alpha < \beta, \text{ there exists some } \gamma \in M \\ \text{with } \alpha\gamma \leq 1 \text{ and } \beta\gamma > 1.$$

Definition 1.20. For any bipotent semiring M we introduce the CMC-subsemiring

$$A_M := \{x \in M \mid x \leq 1\},$$

and we call M a **proper bipotent semiring** if $A_M \neq M$. {N.B. If M is cancellative this means that M is “unbounded”, i.e., does not have a largest element.} We call the bipotent semiring M a **V-semiring** if M is proper and has the separation property (Sep_V) .

We now can deduce results parallel to Theorem 1.11 – Proposition 1.15 by arguing in exactly the same way as above. We first obtain

Theorem 1.21. Let R be a semiring, M a V -semiring, and $v : R \rightarrow M$ a surjective map with $v(0) = 0$, $v(1) = 1$, $v(xy) = v(x)v(y)$ for all $x, y \in R$ (i.e., v is a homomorphism from the monoid (R, \cdot) onto the monoid (M, \cdot)). Assume that

$$\forall x, y \in R : v(x) \leq 1, v(y) \leq 1 \Rightarrow v(x + y) \leq 1. \quad (1.4)$$

Let $A := \{x \in R \mid v(x) \leq 1\}$. Then A is a proper CMC-subsemiring of R , and v is a V -valuation equivalent to $v_{R,A}$.

Then, given any surjective m -valuation $v : R \rightarrow M$ with M proper, we obtain the **V-coarsening**

$$v^\uparrow : R \rightarrow M^\uparrow$$

of v , which is the finest coarsening of v to a V -valuation. This is associated to the CMC-subsemiring $A := A_v$ of R .

Starting with a proper bipotent semiring M we may apply this to $v = \text{id}_M$ and then get a surjective homomorphism

$$\gamma_V := \gamma_{M,V} := (\text{id}_M)^\uparrow : M \rightarrow M^\uparrow,$$

with M^\uparrow a V -semiring. We have

$$\gamma_V(x) \leq \gamma_V(y) \quad \text{iff} \quad [A_M : x] \supset [A_M : y]. \quad (1.5)$$

Finally we observe that the V -coarsening of any surjective m -valuation $v : R \rightarrow M$ with M proper is given by

$$v^\uparrow = \gamma_{M,V} \circ v. \quad (1.6)$$

2. TURNING m -VALUATIONS INTO VALUATIONS

In §1 we have seen classes of surjective m -valuations $v : R \rightarrow M$, where the bipotent semiring M in general has no reason to be cancellative, i.e., v is not a valuation. The problem arises how to handle such true m -valuations in a supertropical context.

Our way to do this is to find a coarsening $w : R \rightarrow N$ of v , as “slight” as possible, which is a valuation and has the same support as v , $w^{-1}(0) = v^{-1}(0)$. Then, as explained below in §3, we can interpret v as a tangible supervaluation covering w .

Given any bipotent semiring M , we look for a homomorphic equivalence relation C on M , as fine as possible, such that the semiring M/C is cancellative. If it happens that $\{0\}$ is a C -equivalence class in M , then for any surjective m -valuation $v : R \rightarrow M$ we will have the valuation

$$w := v/C := \pi_C \circ v : R \rightarrow M/C$$

at our disposal, and $w^{-1}(0) = v^{-1}(0)$, as desired.

We always assume that M is different from the zero ring $\{0\}$. Let \mathfrak{q} denote the nilradical of M , i.e.,

$$\mathfrak{q} := \text{Nil } M = \{x \in M \mid \exists n \in \mathbb{N} : x^n = 0\} = \sqrt{\{0\}}.$$

Lemma 2.1. \mathfrak{q} is a lower set and a prime ideal of M .

Proof. a) Assume that $x \leq y$ and $y^n = 0$. Clearly $y \leq 1$, hence $y^n \leq 1$ for all $n \in \mathbb{N}$. Thus

$$0 = y^n \geq y^{n-1}x \geq \dots \geq yx^{n-1} \geq x^n,$$

and we conclude that $x^n = 0$. Thus \mathfrak{q} is a lower set of M .

b) Clearly $y \cdot M \subset \mathfrak{q}$ for any $y \in \mathfrak{q}$. Also $1 \notin \mathfrak{q}$. Thus \mathfrak{q} is a proper ideal of M .

c) Let $x, y \in M$ be given with $xy \in \mathfrak{q}$, and assume that $x \leq y$. We have $x < 1$. Indeed, $x \geq 1$ would imply $xy \geq y \geq 1$, but the lower set \mathfrak{q} does not contain 1. It follows that $x^2 \leq xy$. Since \mathfrak{q} is a lower set, we infer that $x^2 \in \mathfrak{q}$, hence $x \in \mathfrak{q}$. Thus the ideal \mathfrak{q} is prime. \square

Notice that $\mathfrak{q} = \{0\}$ iff $M \setminus \{0\}$ is closed under multiplication, i.e., M is a semidomain.

Our ansatz for C is the following binary relation on the set M .

$$x \sim_C y \Leftrightarrow \begin{cases} \text{Either } x, y \in \mathfrak{q}, \\ \text{or there exists some } s \in M \setminus \mathfrak{q} \text{ with } sx = sy. \end{cases} \quad (2.1)$$

We verify that this is an equivalence relation on the set M . Only transitivity needs a proof. Let $x, y, z \in M$ be given with $x \sim_C y$ and $y \sim_C z$. If at least one of the elements x, y, z lies in \mathfrak{q} , then all are in \mathfrak{q} . Otherwise we have elements s, t in $M \setminus \mathfrak{q}$ with $sx = sy$ and $ty = tz$. This implies $stx = stz$ and $st \in M \setminus \mathfrak{q}$. Thus $x \sim_C z$ in both cases.

Theorem 2.2.

- a) C is a homomorphic equivalence relation on M . Thus we have a unique structure of a (bipotent) semiring on M/C such that the natural map

$$\pi_C : M \rightarrow M/C, \quad x \mapsto [x]_C$$

is a homomorphism.

- b) M/C is cancellative and $\pi_C^{-1}(0) = \mathfrak{q} := \text{Nil } M$.
 c) If $\gamma : M \rightarrow N$ is a homomorphism from M to a cancellative semiring N with $\gamma^{-1}(0) = \mathfrak{q}$, then γ factors through π_C in a unique way.

Proof. a): Let $x, y, z \in M$ be given with $x \sim_C y$. It is fairly obvious that $xz \sim_C yz$. We have to verify that also $x + z \sim_C y + z$.

Case 1: $x, y \in \mathfrak{q}$. If $z \in \mathfrak{q}$, then $x + z, y + z \in \mathfrak{q}$. If $z \notin \mathfrak{q}$, then $x < z$ and $y < z$, hence $x + z = z = y + z$.

Case 2: $x, y \notin \mathfrak{q}$. There exists some $s \in M \setminus \mathfrak{q}$ with $sx = sy$. It follows that $s(x + z) = s(y + z)$.

Thus $x + z \sim_C y + z$ in both cases. We have verified that the equivalence relation C is homomorphic.

b): By definition $x \sim_C 0$ iff $x \in \mathfrak{q}$. Thus the homomorphism $\pi_C : M \twoheadrightarrow M/C$ has the kernel $\pi_C^{-1}(0) = \mathfrak{q}$. We now verify that M/C is cancellative. Let $x, y, z \in M$ be given with $[z]_C \neq 0$ and $[x]_C \cdot [z]_C = [y]_C \cdot [z]_C$. In other words, $z \in M \setminus \mathfrak{q}$ and $xz \sim_C yz$. We want to prove that $[x]_C = [y]_C$, i.e., $x \sim_C y$.

If $x \in \mathfrak{q}$, we have $xz \in \mathfrak{q}$, hence $yz \in \mathfrak{q}$. Since $z \notin \mathfrak{q}$, this implies $y \in \mathfrak{q}$. Assume now that $x, y \notin \mathfrak{q}$. Then $xz, yz \notin \mathfrak{q}$. Since $xz \sim_C yz$ there exists some $s \in M \setminus \mathfrak{q}$ with $xzs = yzc$. We have $zs \in M \setminus \mathfrak{q}$, hence $x \sim_C y$ again.

c): Let $\gamma : M \rightarrow N$ be a homomorphism with N a cancellative semiring and $\gamma^{-1}(0) = \mathfrak{q}$. Given $x, y \in M$ with $x \sim_C y$ we want to verify that $\gamma(x) = \gamma(y)$.

Case 1: $x \in \mathfrak{q}$. Then $y \in \mathfrak{q}$. Since x, y are nilpotent and N is a semidomain, we conclude that $\gamma(x) = 0 = \gamma(y)$. {N.B. Here we did not yet need the hypothesis that $\gamma^{-1}(0) = \mathfrak{q}$.}

Case 2: $x \notin \mathfrak{q}$. Now $y \notin \mathfrak{q}$, and there exists some $s \in M \setminus \mathfrak{q}$ with $sx = sy$. It follows that $\gamma(s)\gamma(x) = \gamma(s)\gamma(y)$. Since $\gamma^{-1}(0) = \mathfrak{q}$, we have $\gamma(s) \neq 0$. Since N is cancellative, we conclude that $\gamma(x) = \gamma(y)$ again.

Thus γ induces a well-defined map $\bar{\gamma} : M/C \rightarrow N$, given by

$$\bar{\gamma}([x]_C) := \gamma(x)$$

for all $x \in M$. This is a homomorphism, and $\bar{\gamma} \circ \pi_C = \gamma$. Since π_C is surjective, we have no other choice for $\bar{\gamma}$. \square

Notations 2.3.

- (i) We call C the **minimal cancellative relation** on M . If necessary we more precisely write $C(M)$ instead of C .
 (ii) If $v : R \rightarrow M$ is an m -valuation on a semiring R , we denote the coarsening

$$\pi_C \circ v : R \rightarrow M \rightarrow M/C$$

by v/C . It is a valuation.

If v is a V^0 -valuation or a V -valuation, the question arises whether v/C is again V^0 or V . In order to attack this problem, it will be helpful to introduce two more classes of m -valuations.

Definition 2.4.

- (a) We say that an m -valuation $v : R \rightarrow M$ on a semiring R has **unit incapsulation** (abbreviated **UIC**), if for any $x, y \in R$ with $v(x) < v(y)$ there exists some $z \in R$ with

$$v(xz) \leq 1 \leq v(yz).$$

- (b) We say that v has **strict UIC**, if for any $x, y \in R$ with $v(x) < v(y)$ there exists some $z \in R$ with

$$v(xz) < 1 < v(yz).$$

- (c) We say that a bipotent semiring M has **UIC** (resp. **strict UIC**), if the m -valuation $\text{id}_M : M \rightarrow M$ has **UIC** (resp. **strict UIC**). In other words, if for any $\alpha < \beta$ in M there exists some $\gamma \in M$ with $\alpha\gamma \leq 1 \leq \beta\gamma$ (resp. $\alpha\gamma < 1 < \beta\gamma$).

We have the chart of implications

$$\begin{array}{ccccc} \text{strict UIC} & \implies & V \wedge V^0 & \implies & V^0 \\ & & \Downarrow & & \Downarrow \\ & & V & \implies & \text{UIC} . \end{array}$$

The class **UIC** is particularly useful for the following property not shared by the other classes.

Lemma 2.5. *If $v : R \rightarrow M$ has **UIC** and w is a coarsening of v , then also w has **UIC**.*

Proof. We may assume that v is surjective. Then $w = \gamma \circ v$ with $\gamma : M \rightarrow N$ a semiring homomorphism. If x, y are elements of R with $w(x) < w(y)$, then also $v(x) < v(y)$, hence there exists some $z \in R$ with $v(xz) \leq 1 \leq v(yz)$, and this implies $w(xz) \leq 1 \leq w(yz)$. \square

Lemma 2.6. *Assume that M is a bipotent semiring with **UIC**. Then there does not exist a saturated ideal of M different from M and $\{0\}$. In particular, $\text{Nil } M = \{0\}$, hence M is a semidomain.*

Proof. Suppose \mathfrak{q} is a saturated ideal of M different from $\{0\}$. We choose some $x \in \mathfrak{q}$ with $x \neq 0$. Applying **UIC** to the pair $0 < x$, we obtain some $z \in M$ with $1 \leq xz$. Since $xz \in \mathfrak{q}$ and \mathfrak{q} is saturated, the relation $1 + xz = xz$ implies that $1 \in \mathfrak{q}$, hence $\mathfrak{q} = M$. Applying this to $\text{Nil } M$, we see that $\text{Nil } M = \{0\}$. \square

It follows that, if the bipotent semiring M has **UIC**, we have the following simple description of the cancellation relation $C = C(M)$:

$$x \sim_C y \Leftrightarrow \exists z \in M \quad \text{with} \quad z \neq 0 \quad \text{and} \quad xz = yz. \quad (2.2)$$

Thus we can state

Proposition 2.7. *If $v : R \rightarrow M$ is an m -valuation with **UIC**, then $w := v/C$ can be characterized as follows: For any $x, y \in R$*

$$w(x) \leq w(y) \Leftrightarrow \exists z \in R \quad \text{with} \quad v(z) \neq 0 \quad \text{and} \quad v(xz) \leq v(yz).$$

Scholium 2.8. *Taking into account our explicit construction of V^0 - and V -valuations in §1, we see the following: Let R be any semiring.*

i) If v is the V^0 -valuation associated to a prime \mathfrak{p} of R , then for any $x, y \in R$

$$(v/C)(x) \leq (v/C)(y) \Leftrightarrow \exists z \in R \text{ with } Rz \not\subseteq \mathfrak{p} \text{ and } [\mathfrak{p} : xz] \supset [\mathfrak{p} : yz]. \quad (2.3)$$

ii) If v is the V -valuation associated to a proper CNC-subring A of R , then for any $x, y \in R$

$$(v/C)(x) \leq (v/C)(y) \Leftrightarrow \exists z \in R \text{ with } Rz \not\subseteq A \text{ and } [A : xz] \supset [A : yz]. \quad (2.4)$$

We want to exhibit good cases, in which v/C is a V^0 -valuation, or a V -valuation or even has strict UIC. For that reason we analyze under which additional assumption a cancellative bipotent semiring M has one of the properties V^0 , V , strict UIC. We will use a self-explanatory notation. For example, $M_{>1}$ denotes the set $\{x \in M \mid x > 1\}$.

Theorem 2.9. *Assume that M is a cancellative bipotent semiring with UIC and $M_{<1} \neq \{0\}$.*

- i) *If $M_{<1}$ has a biggest element, then M is a V^0 -semiring.*
- ii) *If $M_{>1}$ has a smallest element, then M is a V -semiring.*
- iii) *If $M_{>1} = \emptyset$, then trivially M is a V^0 -semiring.*
- iv) *Otherwise M has strict UIC.*

Proof. Given a pair $\alpha < \beta$ in M , there exists some $\gamma \in M$ with $\alpha\gamma \leq 1 \leq \beta\gamma$, since M has UIC. We need to modify γ in the various cases to get strict inequality at the appropriate places. Since M is cancellative, we know in advance that $\alpha\gamma < \beta\gamma$.

i): We want to find some $\delta \in M$ with $\alpha\delta < 1 \leq \beta\delta$. If $\alpha\gamma < 1$ we are done with $\delta = \gamma$. Assume now that $\alpha\gamma = 1$ and $M_{<1}$ has a biggest element $p_0 > 0$. Multiplying by p_0 we obtain

$$\alpha\gamma p_0 = p_0 < \beta\gamma p_0.$$

We conclude that $1 \leq \beta\gamma p_0$, hence

$$\alpha\gamma p_0 < 1 \leq \beta\gamma p_0.$$

ii): Now clear by the ‘‘dual’’ argument to the just given proof of i).

iii): obvious.

iv): Assume that $M_{<1} \neq \{0\}$, $M_{>1} \neq \emptyset$, and neither $M_{<1}$ has a biggest element nor $M_{>1}$ has a smallest element. We have $\alpha, \beta, \gamma \in M$ with $\alpha\gamma \leq 1 \leq \beta\gamma$ and $\alpha\gamma < \beta\gamma$. We want to find some $\delta \in M$ with $\alpha\delta < 1 < \beta\delta$.

Either $\alpha\gamma < 1$ or $\beta\gamma > 1$. By symmetry it suffices to study the case $\alpha\gamma < 1$. {The setting is not entirely symmetric, since α can be zero while β cannot be zero, but this does not matter.} Since $M_{<1}$ has no biggest element, we find $u, v \in M$ with $\alpha\gamma < u < v < 1$. Due to UIC we have some $\eta \in M$ with $u\eta \leq 1 \leq v\eta$. Again using that M is cancellative, we obtain

$$\alpha\gamma\eta < u\eta \leq 1 \leq v\eta < \eta \leq \beta\gamma\eta.$$

The element $\delta = \gamma\eta$ does the job. □

Scholium 2.10. *Let $v : R \rightarrow M$ be a surjective V^0 -valuation or V -valuation on a semiring R . Then v has UIC, hence also*

$$v/C : R \twoheadrightarrow \overline{M} := M/C$$

has UIC. Discarding the degenerate case that $\overline{M}_{<1} = \{0\}$, we read off from Theorem 2.9, that v/C is again a V^0 - or V -valuation. But in the cases that $\overline{M}_{<1}$ has a biggest element or $\overline{M}_{>1}$ has a smallest element, it may happen that v/C has the opposite type (V resp. V^0) to v .

3. TOTALLY ORDERED SUPERTROPICAL PREDOMAINS AND ULTRAMETRIC SUPERVALUATIONS

We will see that m -valuations, which are not necessarily valuations, can be interpreted as tangible supervvaluations with values in “totally ordered supertropical semirings”. By this we mean the following.

Definition 3.1. *Let U be a supertropical semiring. A **total ordering** of this semiring is a total ordering \leq of the set U which is compatible with addition and multiplication, i.e., for all $x, y \in U$*

$$x \leq y \Rightarrow x + z \leq y + z, \quad (3.1)$$

$$x \leq y \Rightarrow x \cdot z \leq y \cdot z, \quad (3.2)$$

and moreover satisfies

$$0 \leq 1 \quad (3.3)$$

(hence $0 \leq z$ for all $z \in U$).

{N.B. More generally, this definition can be formulated for any semiring R with $R \setminus \{0\}$ closed under addition.}

We know that every supertropical semiring U is *ub* (= upper bound), i.e., carries a partial ordering \leq , defined by

$$x \leq y \Leftrightarrow \exists z \in U : x + z = y,$$

[IKR1, Proposition 11.9]. It again obeys the rules (3.1)–(3.3). In the following, we call this partial ordering the **minimal ordering** of U . The reason for this terminology is that in this ordering any inequality $x \leq y$ is a formal consequence of the rules (3.1)–(3.3).

Any total ordering of U clearly refines the minimal ordering on U . It is further evident that the restriction of the minimal ordering of U to the subsemiring $M := eU$ is the minimal ordering of M , which is total. Since a total ordering cannot be further refined, it follows that any total ordering of U restricted to M gives the minimal ordering on the bipotent semiring M .

We now write down easy observations which tell us that the whole structure of a totally ordered supertropical semiring U can be understood in terms of the totally ordered monoid (U, \cdot) and the ghost map ν_U , regarded as a map from U to U (and denoted here p).

Scholium 3.2. *Assume that U is a totally ordered supertropical semiring.*

- a) (U, \cdot) is a totally ordered monoid with absorbing element 0.
- b) The map² $p : U \rightarrow U$, $x \mapsto ex$ has the following properties. {The label “Gh” alludes to “ghost map”.}

$$\text{(Gh 1) : } p \circ p = p.$$

$$\text{(Gh 2) : } p^{-1}(0) = \{0\}.$$

$$\text{(Gh 3) : } \forall x, y \in U : p(xy) = p(x)p(y).$$

$$\text{(Gh 4) : } \forall x, y \in U : x \leq y \Rightarrow p(x) \leq p(y).$$

$$\text{(Gh 5) : } \forall x \in U : x \leq p(x).$$

²The map p given here is analogous to the “ghost map” ν given for semirings with ghosts in [IR1].

- c) *The addition of the semiring U is determined by the ordering of U and the map p as follows: For all $x, y \in U$*

$$x + y = \begin{cases} y & \text{if } p(x) < p(y), \\ x & \text{if } p(y) < p(x), \\ p(x) & \text{if } p(x) = p(y). \end{cases}$$

- d) *$M := p(U)$ is an ideal of U . If $U \setminus M$ is closed under multiplication, then the totally ordered monoid $(M \setminus \{0\}, \cdot)$ is cancellative (cf. [IKR1, Proposition 3.13]).*

The observations just made lead us to a construction of all ordered supertropical predomains, i.e., order supertropical semirings U such that $U \setminus eU$ is not empty and closed under multiplication and eU is cancellative [IKR1, Definition 3.14]. (The case $U = eU$ could be included but lacks interest.)

Theorem 3.3. *Assume that (U, \cdot) is a totally ordered monoid with absorbing element 0 and $0 < 1$, and that $p : U \rightarrow U$ is a map obeying the rule (Gh1)–(Gh5) above in Scholium 3.2.b. Assume also that the submonoid $p(U) \setminus \{0\}$ of (U, \cdot) is cancellative. Define an addition $U \times U \xrightarrow{+} U$ by the rule given in Scholium 3.2.c. Then U , enriched by this addition, is a totally ordered supertropical semiring, and $p(x) = ex$ for all $x \in U$, with $e := e_U := 1_U + 1_U$. If $U \neq p(U)$ and $U \setminus p(U)$ is closed under multiplication, then U is a supertropical predomain.*

Proof. By [IKR2, Theorem 3.1] we have a unique structure of a supertropical semiring on the set U with multiplication as given, such that $p = \nu_U$, hence $p(U) = eU$, the addition being defined by the rule in Scholium 3.2.c. It is also clear from the assumptions that U is a supertropical predomain, if $U \neq p(U)$ and $U \setminus p(U)$ is closed under multiplication. Also the requirements (3.2) and (3.3) are covered by the assumptions. Notice that up to now (Gh4) and (Gh5) are not yet needed.

It remains to verify (3.1). Thus, given $x, y, z \in U$ with $x \leq y$, we have to prove that $x + z \leq y + z$. By (Gh4) we know that $p(x) \leq p(y)$. We distinguish the cases $p(x) < p(y)$ and $p(x) = p(y)$ and go through various subcases.

Case 1. $p(x) < p(y)$.

- a) If $p(z) < p(x)$, then $x + z = x$, $y + z = y$.
- b) If $p(z) = p(x)$, then $x + z = p(x)$, $y + z = y$. Suppose $p(x) > y$. Applying p , we obtain $p(x) \geq p(y)$, a contradiction. Thus $p(x) \leq y$. (In fact $p(x) < y$.)
- c) If $p(x) < p(z) < p(y)$, then $x + z = z$, $y + z = y$, and $z < y$, since $z \geq y$ would imply $p(z) \geq p(y)$.
- d) If $p(z) = p(y)$, then $x + z = z$, $y + z = p(z)$, and $z \leq p(z)$ by (Gh5).
- e) If $p(y) < p(z)$, then $x + z = z = y + z$.

Thus in all subcases $x + z \leq y + z$.

Case 2. $p(x) = p(y)$.

- a) If $p(z) < p(x)$, then $x + z = x$, $y + z = y$.
- b) If $p(z) = p(x)$, then $x + z = p(z) = y + z$.
- c) If $p(z) > p(x)$, then $x + z = z = y + z$.

Thus again in all subcases $x + z \leq y + z$. □

We will need two more observations about totally ordered supertropical semirings.

Lemma 3.4. *Let U be a totally ordered supertropical semiring.*

i) If $x \in \mathcal{T}(U)$, $y \in \mathcal{G}(U)$, then

$$x \leq y \Leftrightarrow ex \leq y.$$

ii) If $x \in \mathcal{G}(U)$, $y \in \mathcal{T}(U)$, then

$$x \leq y \Leftrightarrow x < ey.$$

Proof. i) (\Rightarrow): $x \leq y$ implies $ex \leq ey = y$.

(\Leftarrow): If $ex \leq y$ then $x \leq y$ (even $x < y$), because $x < ex$.

ii) (\Rightarrow): Clear, because $y < ey$.

(\Leftarrow): Let $x < ey$. Suppose $x > y$. Then $x = ex \geq ey$, a contradiction. Thus $x \leq y$ (even $x < y$). \square

We are ready to obtain an analogue of the construction of the supertropical semirings $\text{STR}(\mathcal{T}, \mathcal{G}, v)$ in [IKR1, Construction 3.16] for totally ordered supertropical semirings.

Assume that totally ordered monoids (\mathcal{T}, \cdot) , (\mathcal{G}, \cdot) and a monoid homomorphism $v : \mathcal{T} \rightarrow \mathcal{G}$ are given. Assume further that (\mathcal{G}, \cdot) is cancellative and v is order preserving, i.e.,

$$\forall x, y \in \mathcal{T} : x \leq_{\mathcal{T}} y \Rightarrow v(x) \leq_{\mathcal{G}} v(y).$$

(Here $\leq_{\mathcal{T}}$ denotes the ordering of \mathcal{T} and $\leq_{\mathcal{G}}$ denotes the ordering of \mathcal{G} .)

By [IKR1, Construction 3.16] we have the supertropical semiring

$$U := \text{STR}(\mathcal{T}, \mathcal{G}, v)$$

at our disposal. As a set, U is the disjoint union of \mathcal{T}, \mathcal{G} and $\{0\}$, and \mathcal{T} is the set of tangible elements of U , whereas \mathcal{G} is the set of ghost elements of U . The multiplication on U extends the multiplications on \mathcal{T} and on \mathcal{G} , has 0 as absorbing element, and obeys the rule $x \cdot y = v(x)y$ for $x \in \mathcal{T}$, $y \in \mathcal{G}$. Moreover, $e = v(1)$ and (hence) $ey = v(y)$ for all $y \in \mathcal{T}$.³

We want to define a total ordering on the set U which extends the given total orderings $\leq_{\mathcal{T}}$, $\leq_{\mathcal{G}}$ on \mathcal{T} and \mathcal{G} , and turns U into a totally ordered supertropical semiring. By Lemma 3.4 we are forced to define the relation \leq on U as follows. {The label ‘‘EO’’ alludes to ‘‘extended ordering’’ .}

$$\text{(EO 1)} : \quad \forall x \in U : \quad 0 \leq x.$$

$$\text{(EO 2)} : \quad \text{If } x, y \in \mathcal{T} : \quad x \leq y \Leftrightarrow x \leq_{\mathcal{T}} y.$$

$$\text{(EO 3)} : \quad \text{If } x, y \in \mathcal{G} : \quad x \leq y \Leftrightarrow x \leq_{\mathcal{G}} y.$$

$$\text{(EO 4)} : \quad \text{If } x \in \mathcal{T}, y \in \mathcal{G} : \quad x \leq y \Leftrightarrow v(x) \leq_{\mathcal{G}} y.$$

$$\text{(EO 5)} : \quad \text{If } x \in \mathcal{G}, y \in \mathcal{T} : \quad x \leq y \Leftrightarrow x <_{\mathcal{G}} v(y).$$

Theorem 3.5. *The relation \leq , defined by the rules (EO 1)–(EO 5) on $U := \text{STR}(\mathcal{T}, \mathcal{G}, v)$, is a total ordering of this supertropical semiring.*

Proof. We intend to apply Theorem 3.3. For that reason, we regard U as a multiplicative monoid, equipped with the map

$$p : U \rightarrow U, \quad x \mapsto ex.$$

Thus $p(x) = x$ if $x \in \mathcal{G} \cup \{0\}$, and $p(x) = v(x)$ if $x \in \mathcal{T}$. Clearly, p has the properties (Gh 1)–(Gh 3) listed in Scholium 3.2.b, and the addition of the semiring U is given by the rule listed in Scholium 3.2.c, cf. [IKR1, Construction 3.16].

³The construction of $\text{STR}(\mathcal{T}, \mathcal{G}, v)$ can be understood as a special case of [IKR2, Theorem 3.1].

Our main task now is to verify that the binary relation \leq on U defined by (EO 1)–(EO 5) is a total ordering of the set U and obeys the rules (Gh 4) and (Gh 5).

We read off from the list (EO 1)–(EO 5) that $x \leq x$ for every $x \in U$. Further

$$\begin{aligned} \forall x, y \in U : x \leq y, y \leq x &\Rightarrow x = y, \\ \forall x, y \in U : x \leq y \text{ or } y \leq x. \end{aligned}$$

Finally, we obtain (Gh 4) and (Gh 5), i.e.,

$$\begin{aligned} x \leq y &\Rightarrow p(x) \leq p(y), \\ x \leq p(x). \end{aligned}$$

If $a, b \in U$ then we mean by $a < b$ that $a \leq b$ and $a \neq b$. {N.B. It is not completely ridiculous to state this convention, since we do not yet know that \leq is an ordering.}

From (Gh 4) we conclude that

$$p(y) < p(x) \Rightarrow y < x. \quad (3.4)$$

Indeed, if $y < x$ does not hold then $x \leq y$, hence $p(x) \leq p(y)$ contradicting $p(y) < p(x)$.

We are now ready to prove the transitivity of our relation. Let $x, y, z \in U$ be given with $x \leq y$ and $y \leq z$. We have to verify that $x \leq z$. By (Gh 4) above we have $p(x) \leq p(y)$ and $p(y) \leq p(z)$, hence $p(x) \leq_{\mathcal{G}} p(y)$ and $p(y) \leq_{\mathcal{G}} p(z)$. This implies $p(x) \leq_{\mathcal{G}} p(z)$, hence $p(x) \leq p(z)$.

Case 1. $p(x) < p(z)$. We conclude by (3.4) that $x < z$.

Case 2. $p(x) = p(z)$. We conclude from $p(x) \leq_{\mathcal{G}} p(y) \leq_{\mathcal{G}} p(z)$, that also $p(x) = p(y)$. If $p(z) = 0$ then $x = 0 = z$. Henceforth we assume that $p(z) \neq 0$. Now $x, y, z \in \mathcal{G} \cup \mathcal{T}$. We proceed through several subcases.

- a) If $x, y, z \in \mathcal{T}$, then it is clear from the transitivity of the relation $\leq_{\mathcal{T}}$ that $x \leq z$.
- b) If $x \in \mathcal{G}$ then the relations $x \leq y$ and $p(x) = p(y)$ force $x = y$ by the rules (EO 5) and (EO 3). We conclude from $y \leq z$ that $x \leq z$.
- c) Similarly, if $y \in \mathcal{G}$ we obtain $y = z$ and then $x \leq z$.
- d) There remains the case that $x, y \in \mathcal{T}$ and $z \in \mathcal{G}$. Since $p(x) = p(z)$, we learn from (EO 4) that $x \leq z$ again.

Thus $x \leq z$ in all subcases.

We now know that our relation \leq is a total ordering on the set U obeying the rules (Gh 4) and (Gh 5). It extends the given orderings on \mathcal{G} and \mathcal{T} .

We check the compatibility of this ordering with multiplication. Let $x, y, z \in U$ be given with $x \leq y$. We have to verify that $x \cdot z \leq y \cdot z$.

If $x = 0$ or $z = 0$ this is obvious. Thus we may assume that all three elements x, y, z are in $\mathcal{T} \cup \mathcal{G}$. By (Gh 4) we have $p(x) \leq p(y)$. Since our ordering restricted to \mathcal{G} is known to be compatible with multiplication, we conclude, using (Gh 1), that

$$p(xz) = p(x)p(z) \leq p(y)p(z) = p(yz).$$

If $z \in \mathcal{G}$, or both x, y are in \mathcal{G} , then $xz, yz \in \mathcal{G}$ and thus

$$xz = p(xz) \leq p(yz) = yz.$$

If all three elements are in \mathcal{T} , then $xz \leq yz$, since the restriction of our ordering to \mathcal{T} is known to be compatible with multiplication.

We are left with the cases $x \in \mathcal{T}, y \in \mathcal{G}, z \in \mathcal{T}$ and $x \in \mathcal{G}, y \in \mathcal{T}, z \in \mathcal{T}$. In the first case, we have $xz \in \mathcal{T}, yz \in \mathcal{G}$, and we see by the rule (EO 4) that $xz \leq yz$. In the second

case, $xz \in \mathcal{G}$, $yz \in \mathcal{T}$, and we see by (EO5) that $xz \leq yz$, using in an essential way that the monoid \mathcal{G} is cancellative.

We now know that all assumptions made in Theorem 3.3 for the totally ordered monoid (U, \cdot) and the map $p : U \rightarrow U$ are valid in the present case. Thus by this theorem our ordered monoid U , together with the addition described in Scholium 3.2.c, is a totally ordered supertropical semiring. But this addition is the original one on the semiring $U = \text{STR}(\mathcal{T}, \mathcal{G}, v)$, cf. [IKR1, Constuction 3.16]. Our proof of Theorem 3.5 is complete. \square

Definition 3.6. *Given a triple $(\mathcal{T}, \mathcal{G}, v)$ consisting of totally ordered monoids \mathcal{T}, \mathcal{G} , with \mathcal{G} cancellative and an order preserving monoid homomorphism from \mathcal{T} to \mathcal{G} , we call the supertropical predomain $U = \text{STR}(\mathcal{T}, \mathcal{G}, v)$ together with the total ordering on U by the rules (EO1)–(EO5) the **ordered supertropical semiring associated to $(\mathcal{T}, \mathcal{G}, v)$** , and we denote this ordered supertropical predomain by $\text{OSTR}(\mathcal{T}, \mathcal{G}, v)$.*

We want to interpret m -valuations as a special kind of supervaluations with values in such semirings $\text{OSTR}(\mathcal{T}, \mathcal{G}, v)$. For that reason we need some more terminology.

Definition 3.7. *Let R be a semiring.*

- a) A **value-ordered supervaluation** on R , or **vo-supervaluation** for short, is a supervaluation $\varphi : R \rightarrow U$ with U a totally ordered supertropical semiring.
- b) We call a vo-supervaluation $\varphi : R \rightarrow U$ **ultrametric**, if

$$\forall a, b \in R : \varphi(a + b) \leq \max(\varphi(a), \varphi(b)).$$

- c) Let $\varphi : R \rightarrow U$ and $\psi : R \rightarrow V$ be vo-supervaluations, and let U', V' denote the subsemirings of U and V generated by $\varphi(R)$ and $\psi(R)$ respectively, i.e., $U' = \varphi(R) \cup e\varphi(R)$, $V' = \psi(R) \cup e\psi(R)$ (cf. [IKR1, Proposition 4.2]). We call φ and ψ **order-equivalent**, and write $\varphi \simeq_0 \psi$, if there exists an order-preserving isomorphism $\alpha : U' \xrightarrow{\sim} V'$ with $\psi(a) = \alpha(\varphi(a))$ for every $a \in R$.

Remarks 3.8.

- a) Given a vo-supervaluation $\varphi : R \rightarrow U$, let $v : R \rightarrow eU$ denote the m -valuation⁴ covered by the supervaluation φ . Then, for every $a \in R$,

$$\varphi(a) \leq e\varphi(a) = v(a).$$

The **support** of φ , defined as

$$\text{supp}(\varphi) := \{a \in R \mid \varphi(a) = 0\},$$

coincides with the support of v .

- b) If φ is ultrametric then

$$A_\varphi := \{a \in R \mid \varphi(a) \leq 1\}$$

is a CMC-subsemiring of R contained in A_v , and

$$\mathfrak{p}_\varphi := \{a \in R \mid \varphi(a) < 1\}$$

is a prime of R contained in \mathfrak{p}_v .

⁴In the examples below v will be a valuation.

Construction 3.9. Let $w : R \rightarrow N$ be an m -valuation and $\rho : N \rightarrow M$ a semiring homomorphism from N to a **cancellative** bipotent semiring M . Assume that $\rho^{-1}(0) = \{0\}$. Thus $v := \rho \circ w$ is a valuation coarsening w , and v, w have the same support $v^{-1}(0) = w^{-1}(0)$. Moreover, $M = \mathcal{G} \cup \{0\}$ and $N = \mathcal{T} \cup \{0\}$ with \mathcal{G} a totally ordered cancellative (multiplicative) monoid and also $\mathcal{T} := \rho^{-1}(\mathcal{G})$ a totally ordered monoid.

Abusing notation, we denote the monoid homomorphism $\mathcal{T} \rightarrow \mathcal{G}$ obtained from ρ by restriction again by ρ . It is order preserving. Then in the totally ordered supertropical predomain

$$U := \text{OSTR}(\mathcal{T}, \mathcal{G}, \rho)$$

we identify $\mathcal{T} = \mathcal{T}(U)$, $\mathcal{G} = \mathcal{G}(U)$ and then $N = \mathcal{T} \cup \{0\}$, $M = \mathcal{G} \cup \{0\}$ in the obvious way. Now $M = eU$ and $N = U \setminus \mathcal{G}$. The map

$$\varphi : R \rightarrow U, \quad a \mapsto w(a) \in N \subset U,$$

sends 0 to 0, 1 to 1, and is multiplicative. Further $e\varphi(a) = v(a)$ for all $a \in R$. Thus φ is a supervaluation covering the valuation v . It has values in $N = \mathcal{T}(U) \cup \{0\}$, and

$$\varphi(a + b) \leq \max(\varphi(a), \varphi(b))$$

for $a, b \in R$, since the ordering of U extends the ordering on N and w is an m -valuation. We conclude that $\varphi : R \rightarrow U$ is a tangible ultrametric supervaluation.

Conversely, given a tangible ultrametric supervaluation

$$\varphi : R \rightarrow \text{OSTR}(\mathcal{T}, \mathcal{G}, \rho) = U,$$

we may view φ as a map w from R to $N = \mathcal{T}(U) \cup \{0\}$, and this is an m -valuation. Moreover, $e\varphi = v$. Extending $\rho : \mathcal{T} \rightarrow \mathcal{G}$ to a semiring homomorphism $\rho : N \rightarrow M$ by $\rho(0) = 0$, we have $\rho \circ w = v$.

Now the following is fairly obvious.

Theorem 3.10. Given a valuation $v : R \rightarrow M$, the m -valuations $w : R \rightarrow N$ dominating v (cf. [IKR1, §2]) and having the same support as v correspond with the tangible ultrametric supervaluations $\varphi : R \rightarrow U$ covering v (i.e., $eU = M$, $e\varphi = v$) uniquely up to order equivalence in the way indicated by Construction 3.9.

Examples 3.11.

(i) Let R be a semiring and \mathfrak{p} a prime of R . In §1 we defined the associated V^0 -valuation

$$w := v_{\mathfrak{p}} : R \rightarrow N := M(R, \mathfrak{p}),$$

and in §2 we established the semiring homomorphism

$$\rho := \pi_C : N \rightarrow M := N/C.$$

We learned that $\rho^{-1}(0) = \{0\}$. Applying Construction 3.9 to these data, we obtain a tangible ultrametric supervaluation

$$\varphi := \varphi_{\mathfrak{p}} : R \rightarrow U(R, \mathfrak{p})$$

which is determined by the pair (R, \mathfrak{p}) above and covers the valuation

$$v := w/C : R \rightarrow M.$$

Here $U(R, \mathfrak{p})$ denotes the totally ordered supertropical semiring $\text{OSTR}(\mathcal{T}, \mathcal{G}, \rho)$ from Construction 3.9. We have

$$\mathfrak{p}_{\varphi} = \mathfrak{p}, \quad A_{\varphi} = [\mathfrak{p} : \mathfrak{p}].$$

{Recall the notations in Remark 3.8.}

- (ii) Similarly, given a proper CMC-subsemiring A of R , we obtain from the associated V -valuation

$$w := v_A : R \rightarrow N := M(R, A)$$

a tangible ultrametric supervaluation

$$\varphi := \varphi_A : R \rightarrow U(R, A)$$

covering w/C , with $A_\varphi = A$ and

$$\mathfrak{p}_\varphi = P(A) := \{x \in R \mid \exists y \in R \setminus A : xy \in A\}.$$

Remark 3.12. Assume that $\varphi : R \rightarrow U$ is a tangible ultrametric supervaluation covering a valuation $v : R \rightarrow M$. Now choose an MFCE-relation E on U which is also compatible with the ordering on U (cf. [IKR2, §4]). Then U/E is again a totally ordered supertropical semiring and

$$\varphi/E := \pi_E \circ \varphi : R \rightarrow U/E$$

is again an ultrametric supervaluation covering v (cf. §5 below for more details). But often φ/E will not be tangible. Then $\psi := \varphi/E$ cannot be interpreted as just an m -valuation covering v .

Of course, $\psi(R)$ is a multiplicative monoid with absorbing element 0 , and $\psi(R)$ is totally ordered by the ordering of U . Thus the map $\psi : R \rightarrow \psi(R)$ may be viewed as an m -valuation, but doing so we lose information about the supervaluation ψ .

We hasten to exhibit the “simplest” tangible ultrametric supervaluations.

Example 3.13. Let $v : R \rightarrow \mathcal{G} \cup \{0\} = M$ be a surjective valuation. Take $w = v$ in Construction 3.9. We obtain a tangible ultrametric supervaluation

$$\varphi : R \rightarrow U := \text{OSTR}(\mathcal{G}, \mathcal{G}, \text{id}_{\mathcal{G}}).$$

The supertropical domain

$$D(\mathcal{G}) := \text{STR}(\mathcal{G}, \mathcal{G}, \text{id}_{\mathcal{G}})$$

has been described in [IKR1, §3]. The minimal ordering of $D(\mathcal{G})$ is a total ordering. Thus we can identify $D(\mathcal{G}) = U$.

The vo -supervaluation φ coincides with the supervaluation $\check{v} : R \rightarrow D(\mathcal{G})$ in [IKR1, Example 9.16]. It is the minimal tangible supervaluation covering v .

More generally, it is fairly obvious that the minimal ordering of a supertropical pre-domain U is total iff every fiber of the ghost map ν_U contains at most one tangible element.

4. SUPERVALUATIONS FROM GENERALIZED CMC-SETS

In this section R is a semiring.

Definition 4.1.

- a) A **CMC-subset** of R is a set $A \subset R$ such that $0 \in A$, $1 \in A$, both A and $R \setminus A$ are closed under multiplication, and there exists a unit u of R with

$$u(A + A) \subset A.$$

- b) We call any such unit u an **exponent** of A .

- c) *The CMC-subsemirings of R are the CMC-subsets that have exponent 1. If A does not admit exponent 1, we call A a **true CMC-subset** of R . This means that A is not a subsemiring of R . In particular, then $A \neq R$.*

Essentially this is the terminology of Valente and Vitulli in their paper [VV], which in turn is rooted in the terminology of Harrison and Vitulli in [HV1], [HV2]. But we slightly deviate from [VV]. Valente and Vitulli call our CMC-subsets “weak CMC-subsets” and our exponents “weak exponents”. They define CMC-subsets (without “weak”) by including still one additional property of an archimedean flavor, following the route developed by Harrison and Vitulli in their quest for “infinite primes”, which are generalizations of the classical archimedean primes in number fields.

For our purposes here, to find interesting new examples of supervaluations, it will be amply clear that CMC-subsets as defined above should be the basic structure. {Consequently, in a planned extension of this paper we will call the CMC-subsets and exponents of [VV] “strong CMC-subsets” and “strong exponents”.}

Valente and Vitulli deal only with CMC-subsets in rings. They speak of “nonring CMC-subsets” instead of our “true CMC-subsets”. The analogous terms “nonsemiring CMC-subsets” would be simply too long.

In the papers [HV2], [VV], an exponent is most often denoted by the letter “e”. We have to deviate also from this habit due to our permanent use of “e” for the ghost unit element of a supertropical semiring.

Examples 4.2. *Let R be a totally ordered field.*

- i) *The closed unit interval*

$$[-1, 1]_R := \{x \in R \mid -1 \leq x \leq 1\}$$

is a true CMC-subset of R with exponent $\frac{1}{2}$.

- ii) *The closed unit disk*

$$\{x + iy \mid x^2 + y^2 \leq 1\}$$

of the field $R(i)$, $i := \sqrt{-1}$, is a true CMC-subset of $R(i)$, again with exponent $\frac{1}{2}$.

Example 4.3. *Let U be a totally ordered supertropical semiring which is not ghost, i.e., $U \neq eU$. Then*

$$A := A_U := \{x \in U \mid x \leq 1\}$$

is closed under multiplication and $0, 1 \in A$. Also $R \setminus A$ is closed under multiplication. Assume that U has a unit u (necessarily tangible) with $eu < 1$, a rather mild condition. Then $u(A + A) \subset A$. Indeed, for $x, y \in A$ we have $eux < 1$, $euy < 1$, hence

$$u(x + y) \leq eu(x + y) = \max(eux; euy) < 1.$$

But $1 + 1 \notin A$. Thus A is a true CMC-subset of U .

If such a unit u does not exist, then $A := A_U$ is not a CMC-subset of U . Indeed, for any unit u' of U with $u'(A + A) \subset A$, we infer that $eu' = u'(1 + 1) \in A$, i.e., $eu' \leq 1$, which forces $eu' < 1$.

In the following R is a semiring and A is a true CMC-subset of R . In §1 we constructed m -valuations

$$v_B : R \rightarrow M(R, B), \quad v_{\mathfrak{p}} : R \rightarrow M(R, \mathfrak{p})$$

for B a proper CMC-subsemiring of R and – with more detailed arguments – for \mathfrak{p} a prime of R .

We now will find, proceeding exactly in the same way, a map $v_A : R \rightarrow M(R, A)$ with $M(R, A)$ again a bipotent semidomain, but v_A will be a multiplicative map showing a behavior under addition somewhat weaker than m -valuations do.

Proposition 4.4. *If $x, y \in R$ and $[A : x] \not\subset [A : y]$, then $[A : y] \subset [A : x]$.*

Proof. Argue as in the proof of Proposition 1.5 by using Lemma 1.4 with $L = A$. \square

The proposition gives us an equivalence relation $\sim_{R,A}$ on the set R , defined by

$$x \sim_{R,A} y \Leftrightarrow [A : x] = [A : y],$$

and then a total ordering on the set

$$M(R, A) := R / \sim_{R,A}$$

of equivalence classes, given by

$$[x] \leq [y] \Leftrightarrow [A : x] \supset [A : y]$$

where we denote the equivalence class of an element $z \in R$ by $[z]$, or more precisely $[z]_{R,A}$ if necessary. The equivalence relation $\sim_{R,A}$ turns out to be multiplicative; hence we have a well-defined multiplication on $M(R, A)$ given by $[x] \cdot [y] := [xy]$. It has the unit element $[1]$, and the ordering is compatible with multiplication. Moreover, $[0]$ is the smallest element of $M(R, A)$. The set $M(R, A) \setminus \{[0]\}$ turns out to be closed under multiplication. Indeed, for $x \in R$ we have $[x] = [0]$ iff $[A : x] = [A : 0] = R$, i.e., $Rx \subset A$. Thus, if $[x] \neq [0]$, $[y] \neq [0]$, we have elements $s, t \in R$ with $sx \notin A$, $ty \notin A$, hence $(st)(xy) \notin A$, hence $[xy] \neq [0]$.

Thus we may regard $M(R, A)$ as a bipotent semidomain.

Theorem 4.5.

i) *The map*

$$v := v_A : R \rightarrow M(R, A), \quad x \mapsto [x],$$

is multiplicative, and $v(0) = 0$, $v(1) = 1$.

ii) $v^{-1}(0) = \{x \in R \mid Rx \subset A\}$.

iii) *If u is an exponent of A then, for all $x, y \in R$*

$$v(x + y) \leq v(u^{-1}) \cdot \max(v(x), v(y)). \quad (4.1)$$

iv) $\{x \in R \mid v(x) \leq 1\} = A$.

v) $\{x \in R \mid v(x) < 1\} = P(A)$ with

$$P(A) := \{x \in R \mid \exists y \in R \setminus A : xy \in A\}. \quad (4.2)$$

Proof. i) and ii) are obvious from the above.

iii): We may assume that $v(x) \leq v(y)$, i.e., $[A : x] \supset [A : y]$. Let $z \in [A : y]$, then also $z \in [A : x]$. Since both zx, zy are in A , we conclude that

$$uz(x + y) \in u(A + A) \subset A.$$

Thus $z \in [A : u(x + y)]$. This proves that $[A : y] \subset [A : u(x + y)]$, in other terms,

$$v(u(x + y)) \leq v(y).$$

Since v is multiplicative, we conclude that $v(x + y) \leq v(u^{-1})v(y)$.

iv): $v(x) \leq 1$ iff $[A : x] \supset [A : 1] = A$ iff $Ax \subset A$ iff $x \in A$.

v): $v(x) < 1$ iff $x \in A$ but $[A : x] \neq A$ iff there exists some $s \in R \setminus A$ with $sx \in A$. \square

In §1 we have seen that in much the same way primes and CMC-subsemirings of R give us m -valuations, namely V^0 - and V -valuations. In the present context we have a similar story dealing with CMC-subsets of R and “prime subsets” of R , to be defined now.

Definition 4.6. A **prime subset** of R is a set $\mathfrak{p} \subset R$ such that $0 \in \mathfrak{p}$, $1 \notin \mathfrak{p}$, both \mathfrak{p} and $R \setminus \mathfrak{p}$ are closed under multiplication, and there exists a unit u of R with

$$u(\mathfrak{p} + \mathfrak{p}) \subset \mathfrak{p}.$$

We call any such unit u an **exponent** of \mathfrak{p} .

The primes of R are the prime subsets of R which have exponent 1. The other prime subsets will be called the **true prime subsets** of R .

Example 4.7. Let A be a CMC-subset of R with exponent u . Then

$$P(A) := \{x \in R \mid \exists y \in R \setminus A : xy \in A\}$$

turns out to be a prime subset of R with exponent u . This follows easily from the description

$$P(A) = \{x \in R \mid v_{R,A}(x) < 1\}$$

in Theorem 4.5 and the properties of $v_{R,A}$ stated in parts i) and iii) of that theorem.

We call $P(A)$ the **central prime set** of the CMC-set A (in R).

Examples 4.8. Let R be a totally ordered field.

a) The open unit interval

$$]-1, +1[_R := \{x \in R \mid -1 < x < 1\}$$

is a true prime subset of R with exponent $\frac{1}{2}$. It is the central prime set of $[-1, 1]_R$.

b) The open unit disk

$$\{x + iy \mid x^2 + y^2 < 1\}$$

of the field $R(i)$, $i = \sqrt{-1}$, is a true prime subset of $R(i)$ with exponent $\frac{1}{2}$. It is the central prime set of the closed unit disk of $R(i)$.

Example 4.9. Let U be a totally ordered supertropical semiring. Assume that there exists a unit u of U with $eu < 1$. Then

$$\mathfrak{p}_U := \{x \in U \mid x < 1\}$$

is a prime subset of U with exponent u . It is the central prime subset of the CMC-subset A_U of U discussed in Examples 4.3.

Let R be a semiring, as before, and \mathfrak{p} a prime subset of R with exponent u . Then we see exactly as above that for any $x, y \in R$ either $[\mathfrak{p} : x] \subset [\mathfrak{p} : y]$ or $[\mathfrak{p} : y] \supset [\mathfrak{p} : x]$, and we obtain a map

$$v := v_{\mathfrak{p}} : R \rightarrow M(R, \mathfrak{p})$$

onto a bipotent semidomain $M(R, \mathfrak{p})$ such that

$$v(x) \leq v(y) \Leftrightarrow [\mathfrak{p} : x] \supset [\mathfrak{p} : y].$$

$M(R, \mathfrak{p})$ is obtained from R by dividing out the equivalence relation given by

$$x \sim y \Leftrightarrow [\mathfrak{p} : x] = [\mathfrak{p} : y],$$

and $v(x)$ is the equivalence class of x in this relation.

Parallel to Theorem 4.5, we have the following facts to be proved in an analogous way.

Theorem 4.10.

- i) The map $v := v_{\mathfrak{p}}$ is multiplicative and $v(0) = 0$, $v(1) = 1$.
- ii) $v^{-1}(0) = \{x \in R \mid Rx \subset \mathfrak{p}\}$.
- iii) If u is an exponent of \mathfrak{p} then, for all $x, y \in R$

$$v(x + y) \leq v(u^{-1}) \cdot \max(v(x), v(y)). \quad (4.3)$$

- iv) $\{x \in R \mid v(x) \leq 1\} = A(\mathfrak{p})$ with

$$A(\mathfrak{p}) := \{x \in R \mid yx \subset \mathfrak{p}\} = [\mathfrak{p} : \mathfrak{p}] \quad (4.4)$$

- v) $\mathfrak{p} = \{x \in R \mid v(x) < 1\} = \{x \in R \mid \exists y \in R \setminus \mathfrak{p} : xy \in \mathfrak{p}\}$.

Corollary 4.11. $A(\mathfrak{p})$ is a CMC-subset of R with exponent u .

Proof. This follows from points i), iii), iv) in the theorem. □

Definition 4.12. We call a map $v : R \rightarrow R'$ from R to a semiring R' **0-1-multiplicative** if v is multiplicative, i.e.,

$$\forall x, y \in R : \quad v(xy) = v(x) \cdot v(y),$$

and $v(0) = 0$, $v(1) = 1$.

In the following 0-1-multiplicative maps from R to bipotent semirings which are not m -valuations, will play a major role. We already met such maps in Theorem 4.5 and 4.10. The following remark is sometimes useful

Remark 4.13. Every surjective multiplicative map $v : R \rightarrow R'$ from R to a semiring R' is 0-1-multiplicative.

Proof. Let $z \in R'$ be given. We choose some $x \in R$ with $v(x) = z$. Then

$$\begin{aligned} v(1) \cdot z &= v(1) \cdot v(x) = v(1 \cdot x) = v(x) = z, \\ v(0) \cdot z &= v(0) \cdot v(x) = v(0 \cdot x) = v(0). \end{aligned}$$

From $v(1) \cdot z = z$ for every $z \in R'$ we conclude that that $v(1) = 1$. From $v(0) \cdot z = 0$ for every $z \in R'$ we conclude that $v(0) = 0$, since otherwise we would have $z \neq 0$ for every $z \in R'$, which is not true. □

We extend the notion of dominance for m -valuations on R [IKR1, §2] to 0-1-multiplicative maps from R to bipotent semirings.

Definition 4.14. If $v : R \rightarrow M$ and $w : R \rightarrow N$ are 0-1-multiplicative maps from R to bipotent semirings M, N , we say that v **dominates** w , or, that w is a **coarsening of** v , if

$$\forall a, b \in R : \quad v(a) \leq v(b) \Rightarrow w(a) \leq w(b).$$

We then write $v \geq w$.

The following proposition is now obvious.

Proposition 4.15. Assume that $v : R \rightarrow M$ and $w : R \rightarrow N$ are 0-1-multiplicative maps from R to bipotent semirings M, N . Assume further that v is surjective. Then $v \geq w$ iff there exists a (unique) semiring homomorphism $\gamma : M \rightarrow N$ with $\gamma \circ v = w$.

As in [IKR1, §2] we denote this homomorphism γ by $\gamma_{w,v}$ if necessary.

CMC-subsets and prime subsets of R will lead to many examples of dominance of 0-1-multiplicative maps to bipotent semirings. Since we will work with these subsets at an equal pace, we coin the following notion.

Definition 4.16. *A **generalized CMC-subset** L of R is either a CMC-subset A or a prime subset \mathfrak{p} of R . We call L a **true CMC-subset** of R if L is neither a subsemiring nor a prime of R .*

Given a true generalized CMC-subset L of R , we want to interpret the surjective multiplicative map v_L as a tangible value-ordered supervaluation in much the same way as we did this in §3 for L a CMC-subsemiring or a prime \mathfrak{p} of R .

Two strategies come to mind: Find either a proper CMC-subsemiring $B \supset L$ or a prime $Q \subset L$ of R such that v_L dominates the m -valuation $w := v_B$ or $w := v_Q$ respectively! Then v_L will also dominate the **valuation** w/C introduced in §2. If, in addition, $v^{-1}(0) = w^{-1}(0)$, we can obtain the desired vo-supervaluation by a straightforward generalization of Construction 3.9.

It will turn out that both strategies usually work well. Starting from our true generalized CMC-subset L of R with exponent u , we define the sets

$$B := B_u(L) := \bigcup_{n \in \mathbb{N}} u^{-n}L,$$

$$Q := Q_u(L) := \bigcap_{n \in \mathbb{N}} u^n L.$$

Theorem 4.17.

- i) B is a CMC-subsemiring of R containing L , and Q is a prime subset of R contained in L .
- ii) Q is a prime ideal of B .
- iii) The multiplicative map $v_L : R \rightarrow M(R, L)$ dominates the V^0 -valuation $v_Q : R \rightarrow M(R, Q)$, and $v_L^{-1}(0) = v_Q^{-1}(0)$.
- iv) If $B \neq R$, then v_L dominates the V -valuation $v_B : R \rightarrow M(R, B)$, and $v_L^{-1}(0) = v_B^{-1}(0)$.

Proof. a) It follows from $L \cdot L \subset L$ that $B \cdot B \subset B$. Given $n \in \mathbb{N}$ we have

$$u^{-n}L + u^{-n}L = u^{-n-1}u(L + L) \subset u^{-n-1}L.$$

Thus $B + B \subset B$. Further $1 = u^{-1} \cdot u \in u^{-1}L$, hence $1 \in B$. Of course, $L \subset B$ and in particular $0 \in B$.

Let $x, y \in R \setminus B$. For every $n \in \mathbb{N}$ we have $u^n x \notin L$, $u^n y \notin L$, hence $u^{2n}(xy) \notin L$, hence $xy \in R \setminus B$. Altogether this proves that B is a CMC-subsemiring of R containing L .

b) It follows from $L \cdot L \subset L$ that $L \cdot Q \subset Q$. Of course, $Q \subset L$. Thus certainly $Q \cdot Q \subset Q$. Let $x, y \in Q$ be given. For every $n \in \mathbb{N}$, we have $x \in u^{n+1}L$, $y \in u^{n+1}L$, hence

$$x + y \in u^{n+1}(L + L) \subset u^n L.$$

Thus $x + y \in Q$. Of course, $0 \in Q$. On the other hand, $1 \notin uL$, hence $1 \notin Q$.

Let $x, y \in R \setminus Q$. There exists some $n \in \mathbb{N}$ with $u^{-n}x \notin L$, $u^{-n}y \notin L$. Then $u^{-2n}(xy) \notin L$. Thus $xy \in R \setminus Q$. Altogether we have proved that Q is a prime of R , contained in the set L .

c) Let $x \in Q$ be given. For any $n \in \mathbb{N}$ we have $x \in u^{n+1}L$, hence $u^{-1}x \in u^nL$. Thus $u^{-1}x \in Q$. This proves that $u^{-1}Q \subset Q$. Since also $L \cdot Q \subset Q$, we see that $B \cdot Q \subset Q$. Because Q is a prime of R , it is now obvious that Q is a prime ideal of B .

d) For any $x \in R$ we have

$$[B : x] = \bigcup_n [u^{-n}L : x] = \bigcup_n u^{-n}[L : x].$$

Therefore

$$\forall x, y \in R : [L : x] \supset [L : y] \Rightarrow [B : x] \supset [B : y].$$

If $R \neq B$ this translates to

$$\forall x, y \in R : v_L(x) \leq v_L(y) \Rightarrow v_B(x) \leq v_B(y).$$

Thus v_L dominates v_B .

e) For any $x \in R$ we have

$$[Q : x] = \bigcap_n [u^nL : x] = \bigcap_n u^n[L : x].$$

It follows that

$$\forall x, y \in R : [L : x] \supset [L : y] \Rightarrow [Q : x] \supset [Q : y].$$

Thus v_L dominates v_Q .

f) As observed before (Theorems 4.5, 4.10),

$$v_L^{-1}(0) = \{x \in R \mid Rx \subset L\}.$$

Similarly $v_Q^{-1}(0) = \{x \in R \mid Rx \subset Q\}$ and if $B \neq R$, $v_B^{-1}(0) = \{x \in R \mid Rx \subset B\}$. Thus $v_Q^{-1}(0) \subset v_L^{-1}(0)$ and, if $B \neq R$, $v_L^{-1}(0) \subset v_B^{-1}(0)$.

We want to prove equality of these sets. Let $x \in R$ be given with $Rx \not\subset Q$. Choose some $z \in R$ with $zx \notin Q$. Then $u^{-n}zx \notin L$ for some n . Thus $Rx \not\subset L$. This proves that $v_Q^{-1}(0) = v_L^{-1}(0)$.

Now assume that $B \neq R$. Let $x \in R$ be given with $Rx \not\subset L$. We choose elements z, s of R with $xz \notin L$, $s \notin B$. Then $u^n s \notin L$ for all n , hence $u^n s z x \notin L$ for all n . It follows that $Rx \not\subset B$. This proves that $v_L^{-1}(0) = v_B^{-1}(0)$. \square

We describe all quantities occurring in Theorems 4.5, 4.10 and 4.17 in the perhaps simplest case of interest.

Examples 4.18. Let R be a real closed field. Associated to the ordering of R , we have the absolute value $|\cdot|_R$ on R with $|x|_R = x$ if $x \geq 0$ and $|x|_R = -x$ if $x \leq 0$. Also R contains (a unique copy of) the field \mathbb{R} of real numbers. We consider the CMC-subset

$$A := \{x \in R \mid |x|_R \leq 1\} = [-1, 1]$$

and the prime subset

$$\mathfrak{p} := \{x \in R \mid |x|_R < 1\}$$

(as in Examples 4.2.i) of R , both with exponent $u = \frac{1}{2}$. Then

$$B_u(A) = B_u(\mathfrak{p}) = \bigcup_n [-2^{-n}, 2^n] = \{x \in R \mid \exists n \in \mathbb{N} : |x| \leq n\}$$

and

$$Q_u(A) = Q_u(\mathfrak{p}) = \bigcap_n [-2^{-n}, 2^{-n}] = \left\{x \in R \mid \forall n \in \mathbb{N} : |x| \leq \frac{1}{n}\right\}.$$

Thus, $B := B_u(A)$ is a valuation domain with quotient field R , and is also the smallest convex subring of the ordered field R , and $Q := Q_u(A)$ is the maximal ideal of B . We have $B \neq R$ iff $\mathbb{R} \neq R$.

Observe that

$$[A : x] \supset [A : y] \text{ iff } |x|_R \leq |y|_R \text{ iff } [\mathfrak{p} : x] \supset [\mathfrak{p} : y].$$

Thus $v_A = v_{\mathfrak{p}}$, and we can identify this multiplicative map with the absolute value map

$$x \mapsto |x|_R, \quad R \rightarrow R_{\geq 0}.$$

Let w denote the canonical valuation associated to B . This is the natural map

$$w : R \rightarrow R/B^* = R^*/B^* \cup \{0\},$$

with R^*/B^* ordered by the rule

$$xB^* \leq yB^* \Leftrightarrow \frac{x}{y} \in B.$$

We have $B = \{x \in R \mid w(x) \leq 1\}$ and $Q = \{x \in R \mid w(x) < 1\}$, and obtain for $x, y \in R$ in the case $B \neq R$ that

$$[B : x] \supset [B : y] \text{ iff } w(x) \leq w(y) \text{ iff } [Q : x] \supset [Q : y].$$

Thus $v_B = v_Q \sim w$. If $B = R$ we may still identify v_Q with the now trivial valuation w .

It is easily checked that $A(\mathfrak{p}) = A$ and $P(A) = \mathfrak{p}$.

The coincidences $B_u(A) = B_u(\mathfrak{p})$, $Q_u(A) = Q_A(\mathfrak{p})$, $A(\mathfrak{p}) = A$, $P(A) = \mathfrak{p}$ in this example are typical for the case that the semiring R is a semifield, or at least has ‘‘many units’’ in an appropriate sense. We will pursue the case of semifields in [IKR3], while in the present paper we deal with the situation where such coincidences often fail.

Construction 4.19. Given a surjective multiplicative map $w : R \rightarrow N$ from R to a bipotent semiring N and a surjective valuation $v : R \rightarrow M$ such that w dominates v we have a semiring homomorphism $\rho : N \rightarrow M$ with $v = \rho \circ w$. If also $v^{-1}(0) = w^{-1}(0)$, we can repeat Construction 3.9 word by word. We obtain again an ordered supertropical predomain

$$U := \text{OSTR}(\mathcal{T}, \mathcal{G}, \rho)$$

with $\mathcal{T} = N \setminus \{0\}$, $\mathcal{G} = M \setminus \{0\}$, and $\rho : \mathcal{T} \rightarrow \mathcal{G}$ gained from $\rho : N \rightarrow M$ by restriction. As in §3 we identify $N = \mathcal{T} \cup \{0\} \subset U$ and $M = \mathcal{G} \cup \{0\} = eU$. The map

$$\varphi : R \rightarrow U, \quad a \mapsto w(a) \in N \subset U,$$

is a tangible supervaluation covering v , but now – in contrast to the situation in Construction 3.9 – v has no reason to be ultrametric. \square

Examples 4.20.

- a) As a consequence of Theorem 4.17 we can apply this construction for L a true generalized CMC-subset of R to $w := v_L : R \rightarrow M(R, L)$ and to

$$v := v_Q/C : R \rightarrow M(R, Q)/C.$$

In this case we denote the totally ordered supertropical semiring from above by $U(R, L, Q_u(L))$ and the arising tangible vo-supervaluation by

$$\varphi_{L,u} : R \rightarrow U(R, L, Q_u(L)).$$

b) Likewise, if $B_u(L) \neq R$, we can take $w := v_L$ and $v := v_B/C$ and obtain again a tangible vo-supervaluation, which we denote by

$$\psi_{L,u} : R \rightarrow U(R, L, B_u(L)).$$

If $\varphi : R \rightarrow U$ is any of these vo-supervaluations $\varphi_{L,u}, \psi_{L,u}$, then φ obeys a rule

$$\forall a, b \in R : \quad \varphi(a + b) \leq c \max(\varphi(a), \varphi(b)) \quad (4.5)$$

with c a unit of U , namely, $c = \varphi(u^{-1})$ for the chosen exponent u of L . This follows from Theorems 4.5.iii and 4.10.iii. {N.B. Any unit of U is a tangible element.}

Definition 4.21. *The rule (4.5) reminds us of the classical absolute values of Emil Artin (which he called “valuations”, cf. [A1, Chapter 1], [A2, Chapter 3]). Thus, we call a vo-valuation φ obeying (4.5) with a unit c of U an **artinian supervaluation (with constant c)**.*

The case $c = 1$ covers the ultrametric supervaluations (Definition 3.7), but for the supervaluations φ_L, ψ_L with L a true CMC-subset of R we have $c > 1$.

Artinian supervaluations abound among vo-valuations, and they are a source of prime subsets and CMC-subsets, due to the following facts.

Proposition 4.22. *If a totally ordered supertropical semiring U contains a unit $c > e$ then every vo-supervaluation $\varphi : R \rightarrow U$ is artinian with constant c . If in addition R contains a unit u with $\varphi(u) \leq c^{-1}$ then*

$$\mathfrak{p}_\varphi := \{a \in R \mid \varphi(a) < 1\}$$

is a prime subset of R with exponent u and

$$A_\varphi := \{a \in R \mid \varphi(a) \leq 1\}$$

is a CMC-subset of R with exponent u .

Proof. If $a, b \in R$ then

$$\varphi(a + b) \leq e\varphi(a + b) \leq \max(e\varphi(a), e\varphi(b)) \leq c \max(\varphi(a), \varphi(b)).$$

If in addition $u \in R$ and $\varphi(u) = c^{-1}$, then

$$\varphi(u(a + b)) \leq \max(\varphi(a), \varphi(b)).$$

This gives the claims about \mathfrak{p}_φ and A_φ . □

5. MONOTONE TRANSMISSIONS AND TOTAL DOMINANCE: SOME EXAMPLES

In the last section we introduced totally ordered supertropical semirings and then obtained a large stock of – as we believe – natural examples of supervaluations with values in such semirings, which we called *value ordered supervaluations*, or *vo-supervaluations* for short.

They call for a theory of dominance and transmissions adapted to this special class of supervaluations, which is parallel to our general theory in [IKR1] and [IKR2]. We give basic steps of such a theory. Here things seem to be easier than in the general theory, since our special transmissions, called “*monotone transmissions*”, turn out to be semiring homomorphisms, cf. Theorem 5.3.

Definition 5.1. *Let U_1, U_2 be totally ordered supertropical semirings. We call a transmission $\alpha : U_1 \rightarrow U_2$ (as defined in [IKR1, §5]) **monotone** if α is order preserving, i.e.,*

$$\forall x, y \in U_1 : \quad x \leq y \Rightarrow \alpha(x) \leq \alpha(y).$$

Example 5.2. Let $(\mathcal{T}_1, \mathcal{G}_1, v_1)$ and $(\mathcal{T}_2, \mathcal{G}_2, v_2)$ be triples consisting of totally ordered monoids \mathcal{T}_i , cancellative totally ordered monoids \mathcal{G}_i , and order preserving homomorphisms $v_i : \mathcal{T}_i \rightarrow \mathcal{G}_i$ ($i = 1, 2$). In §3 we associated to such triples ordered supertropical predomains

$$U_i := \text{OSTR}(\mathcal{T}_i, \mathcal{G}_i, v_i) = \mathcal{T}_i \dot{\cup} \mathcal{G}_i \dot{\cup} \{0\}.$$

Now assume that also order preserving monoid homomorphisms $\beta : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ and $\gamma : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ are given with $\gamma v_1 = v_2 \beta$. Assume in addition that γ is injective. Then the map $\alpha : U_1 \rightarrow U_2$ with $\alpha(0) = 0$, $\alpha(x) = \beta(x)$ for $x \in \mathcal{T}_1$, $\alpha(y) = \gamma(y)$ for $y \in \mathcal{G}_1$ is a monotone transmission.

Indeed, α clearly obeys the rules TM1-TM5 from [IKR1, §5], hence is a transmission, and looking at the rules (EO 1)–(EO 5) in §3, which describe the ordering of the U_i , one checks that α is also order preserving. Notice that compatibility with the rule (EO 5) demands that γ is injective.

Example 5.2 gives us, up to isomorphism, all monotone transmissions between totally ordered supertropical predomains which map tangible elements to tangible elements.

We state a fundamental fact about monotone transmissions in general.

Theorem 5.3. Assume that U and V are totally ordered supertropical semirings and that $\alpha : U \rightarrow V$ is an order preserving map, which is multiplicative, i.e., $\alpha(xy) = \alpha(x)\alpha(y)$ for any $x, y \in U$. The following are equivalent.

- (1) $\alpha(0) = 0$, $\alpha(1) = 1$, $\alpha(e) = e$.
- (2) α is a semiring homomorphism.
- (3) α is a (monotone) transmission.

Proof. The implications (2) \Rightarrow (3) and (3) \Rightarrow (1) are trivial.

(1) \Rightarrow (2): Given $x, y \in U$, we have to verify that $\alpha(x + y) = \alpha(x) + \alpha(y)$. We may assume that $ex \leq ey$.

Case 1: $e\alpha(x) < e\alpha(y)$, hence $ex < ey$.

Now $x + y = y$, $\alpha(x) + \alpha(y) = \alpha(y)$.

Case 2: $ex = ey$, hence $e\alpha(x) = e\alpha(y)$.

Now $x + y = ex$, $\alpha(x) + \alpha(y) = e\alpha(y) = \alpha(ey)$.

Case 3: $ex < ey$, but $e\alpha(x) = e\alpha(y)$.

Now $ex < y < ey$, hence $e\alpha(y) = e\alpha(x) \leq \alpha(y) \leq \alpha(ey) = e\alpha(y)$, hence $\alpha(y) = e\alpha(y)$. We have $x + y = y$, $\alpha(x) + \alpha(y) = e\alpha(y) = \alpha(y)$.

Thus $\alpha(x) + \alpha(y) = \alpha(x + y)$ in all cases. □

We coin a notion of “total dominance” for vo-valuations refining the definition of dominance for arbitrary supervaluations in [IKR1, §5], and relate this to monotone transmissions.

First recall from [IKR1, §5] that we defined for supervaluations $\varphi : R \rightarrow U$, $\psi : R \rightarrow V$, that φ dominates ψ , and wrote $\varphi \geq \psi$, if for all $a, b \in R$,

$$D1 : \quad \varphi(a) = \varphi(b) \quad \Rightarrow \quad \psi(a) = \psi(b),$$

$$D2 : \quad e\varphi(a) \leq e\varphi(b) \Rightarrow e\psi(a) \leq e\psi(b),$$

$$D3 : \quad \varphi(a) \in eU \quad \Rightarrow \quad \psi(a) = eV.$$

Definition 5.4. Assume that $\varphi : R \rightarrow U$, $\psi : R \rightarrow V$ are vo-valuations. Then we say that φ **dominates** ψ **totally**, and write $\varphi \geq_{\text{tot}} \psi$, if φ and ψ obey axiom D3 but instead of D1,

D2 obey the modified axioms

$$D1' : \quad \varphi(a) \leq \varphi(b) \quad \Rightarrow \quad \psi(a) \leq \psi(b),$$

$$D2' : \quad e\varphi(a) = e\varphi(b) \Rightarrow e\psi(a) = e\psi(b).$$

Here axiom D1' is stronger than D1 but D2' is weaker than D2. But notice that D1' and D2' together imply D2. Indeed, if $e\varphi(a) < e\varphi(b)$, then $\varphi(a) < \varphi(b)$, since $\varphi(a) \geq \varphi(b)$ would imply $e\varphi(a) \geq e\varphi(b)$. From this we obtain by D1' that $\psi(a) \leq \psi(b)$ and then $e\psi(a) \leq e\psi(b)$. On the other hand, if $e\varphi(a) = e\varphi(b)$, then we have $e\psi(a) = e\psi(b)$ by D2', hence again $e\psi(a) \leq e\psi(b)$. We conclude

Remark 5.5. *If $\varphi \leq_{\text{tot}} \psi$, then $\varphi \leq \psi$.*

We look for examples of total dominance between the artinian supervvaluations constructed at the end of §4 (Examples 4.20). First an obvious remark.

Remark 5.6. *Assume that $\varphi : R \rightarrow U$ is an artinian supervvaluation with constant c . If φ totally dominates $\psi : R \rightarrow V$, then ψ is again artinian with constant $\alpha_{\psi, \varphi}(c)$. In particular, if φ is ultrametric then ψ is ultrametric.*

Examples 5.7. *Let R be a semiring and L a true generalized CMC-subset of L . Then the artinian supervvaluation*

$$\varphi_{L,u} : R \rightarrow U(R, L, Q_u(L))$$

(cf. Example 4.20.a) totally dominates the ultrametric supervvaluation $\varphi_{Q_u(L)}$. Likewise, if $B_n(L) \neq R$, the artinian supervvaluation

$$\psi_{L,u} : R \rightarrow U(R, L, B_u(L))$$

(cf. Example 4.20.b) totally dominates the ultrametric supervvaluation $\varphi_{B_u(L)}$.

In these examples the associated transmissions restrict to the identity on the ghost ideals.

Examples 5.8. *Assume again that L is a true generalized CMC-subset of a semiring R and that u is an exponent of L . We choose some $g \in R^* \cap L$. Then also $f := ug$ is an exponent of L . Let $B := B_u(L)$, $B' := B_f(L)$, $Q := Q_u(L)$, and $Q' := Q_f(L)$. We have $u^{-1} = f^{-1}g \in B'$, $g^{-1} = f^{-1}u \in B'$, and we conclude easily that $B' \supset B$ and then $B' = \bigcup_n g^n B$. We have*

$$Q' = \bigcap_n g^n u^n L \subset \bigcap_n u^n g^m L = g^m Q$$

for any fixed $m \in \mathbb{N}$, hence $Q' \subset \bigcap_n g^n Q$. On the other hand,

$$Q' = \bigcap_n g^n u^n B' \supset \bigcap_n g^n u^n B = \bigcap_n g^n B \supset \bigcap_n g^n Q,$$

and we conclude that $Q' = \bigcap_n g^n Q$. For $x \in R$ we have

$$[B' : x] = \bigcup_n [g^{-n} B : x] = \bigcup_n g^{-n} [B : x].$$

$$[Q' : x] = \bigcap_n [g^n Q : x] = \bigcap_n g^n [Q : x].$$

From these formulas it is evident that for any $x, y \in R$

$$[B : x] \supset [B : y] \quad \Rightarrow \quad [B' : x] \supset [B' : y]$$

and

$$[Q : x] \supset [Q : y] \quad \Rightarrow \quad [Q' : x] \supset [Q' : y].$$

The second implication tells us that the m -valuation v_Q dominates $v_{Q'}$. It is now essentially trivial to verify for $\varphi_{L,u}$ and $\varphi_{L,f}$ the axioms D1', D2', D3. Thus the artinian supervaluation $\varphi_{L,u}$ dominates $\varphi_{L,f}$ totally.

Likewise, if $B' = R$, the artinian supervaluation $\psi_{L,u}$ dominates $\psi_{L,f}$ totally.

Notice that in these examples it would not be possible to resort to the construction in Example 5.2 for gaining directly the appropriate monotone transmissions, since the transmissions from v_B/C to $v_{B'}/C$ and from v_Q/C to $v_{Q'}/C$ usually are not injective.

We add two examples of total dominance between ultrametric supervaluations.

Example 5.9. Let \mathfrak{p} be a prime of a semiring R , but not a prime ideal of R . Then $A := A(\mathfrak{p}) := [\mathfrak{p} : \mathfrak{p}]$ is a proper CMC-subsemiring of R . For any $x \in R$ we have

$$[A : x] = [[\mathfrak{p} : \mathfrak{p}] : x] = [[\mathfrak{p} : x] : \mathfrak{p}].$$

Thus for $x, y \in R$ with $[\mathfrak{p} : x] \supset [\mathfrak{p} : y]$ we have $[A : x] \supset [A : y]$. This tells us that the V^0 -valuation $v_{\mathfrak{p}} : R \rightarrow M(R, \mathfrak{p})$ introduced in §1 dominates the V -valuation $v_A : R \rightarrow M(R, A)$ also introduced there. It is easily verified that the associated ultrametric supervaluation

$$\varphi_{\mathfrak{p}} : R \rightarrow U(R, \mathfrak{p})$$

(cf. Example 3.11.i) totally dominates the ultrametric supervaluation

$$\varphi_A : R \rightarrow U(R, A)$$

(cf. Example 3.11.ii) by checking the axioms D1', D2', D3. Alternatively we can construct explicitly a monotone transmission $\alpha : U(R, \mathfrak{p}) \rightarrow U(R, A)$ with $\varphi_A = \alpha \circ \varphi_{\mathfrak{p}}$, by resorting to a natural commuting square of order preserving semiring homomorphisms

$$\begin{array}{ccc} M(R, \mathfrak{p}) & \xrightarrow{\gamma} & M(R, A) \\ \downarrow & & \downarrow \\ M(R, \mathfrak{p})/C & \xrightarrow{\gamma/C} & M(R, A)/C \end{array}$$

with γ the transmission from $v_{\mathfrak{p}}$ to v_A , γ/C the transmission from $v_{\mathfrak{p}}/C$ to v_A/C , and canonical vertical arrows.

Example 5.10. Let A be a proper CMC-subsemiring of R and

$$\mathfrak{p} := P(A) := \{x \in R \mid \exists y \in R \setminus A : xy \in A\},$$

which is a prime of R . We study again the associated m -valuations v_A and $v_{\mathfrak{p}}$. We have

$$\mathfrak{p} = \{x \in R \mid v_A(x) < 1\}.$$

Consequently, given $x \in R$, an element z of R lies in $[\mathfrak{p} : x]$ iff $v_A(x) + v_A(z) < 1$. Thus for elements x, y of R with $v_A(x) < v_A(y)$ we have $[\mathfrak{p} : x] \supset [\mathfrak{p} : y]$, hence $v_{\mathfrak{p}}(x) \leq v_{\mathfrak{p}}(y)$. This shows that v_A dominates $v_{\mathfrak{p}}$. One now can verify in the same way as in the preceding example that the ultrametric supervaluation φ_A dominates $\varphi_{\mathfrak{p}}$ totally.

6. ORDER COMPATIBLE TE-RELATIONS

We now look at monotone transmissions via equivalence relations.

Definition 6.1. *Let U be a totally ordered supertropical semiring. We call an equivalence relation E on the set U an **order compatible TE-relation**, or **OCTE-relation** for short, if the following holds:*

- (1) E is multiplicative, i.e., $\forall x, y, z \in E: x \sim_E y \Rightarrow xz \sim_E yz$.
- (2) E is order compatible, i.e., obeys the axiom (OC) form [IKR2, §4]; equivalently, if all E -equivalence classes are convex subsets of the totally ordered set U .

Comment. This terminology is related to the definition of “TE-Relations” (= transmissive equivalence relation) in [IKR2, §4]. To repeat, an equivalence relation E on a supertropical semiring U with ghost ideal $M := eU$ is called a **TE-relation**, if E is multiplicative (Axiom TE1 in [IKR2, §4]), the restriction $E|M$ is order compatible (Axiom TE2), and $x \in U$ with $ex \sim_E 0$ is itself E -equivalent to 0 (Axiom TE3).

If U is a totally ordered and E is an OCTE-relation on U , then clearly TE1 and TE2 are valid. But also TE3 holds: If $x \in U$ then $0 \leq x \leq ex$, and thus $ex \sim_E 0$ implies $x \sim_E 0$ since E is order compatible. Thus an OCTE-relation on U is certainly a TE-relation. \square

If E is an OCTE-relation, then it is obvious that the set U/E of E -equivalence classes has a (unique) well defined structure of a totally ordered monoid, such that the map

$$\pi_E : U \rightarrow U/E, \quad x \mapsto [x]_E,$$

is multiplicative and order preserving (cf. the arguments in the beginning of [IKR2, §4]). This structure is given by the rules ($x, y \in U$)

$$[x]_E \cdot [y]_E = [xy]_E, \tag{6.1}$$

$$[x]_E \leq [y]_E \iff x \leq y. \tag{6.2}$$

The monoid U/E has the unit element $[1_U]_E$ and the absorbing element $[0_U]_E$. Further it is easily checked that the map

$$p : U/E \rightarrow U/E, \quad [x]_E \mapsto [ex]_E,$$

obeys the rules (Gh1)–(Gh5) from §3. Thus we have on U/E the structure of a totally ordered supertropical semiring, as indicated in Theorem 3.3.

The ghost ideal of U/E is

$$p(eU) = \{[x]_E \mid x \in M\}.$$

Its unit element $e_{U/E}$ is the class $[e]_E$. We see that the map $\pi_E : U \rightarrow U/E$ fulfills condition (1) in Theorem 5.3, and we conclude by that theorem that π_E is a monotone transmission. Thus for $x, y \in U$ we have the rule

$$[x]_E + [y]_E = [x + y]_E. \tag{6.3}$$

We have arrived at the following theorem.

Theorem 6.2. *If E is an OCTE-relation on a totally ordered supertropical semiring U , then the set U/E carries a (unique) structure of a totally ordered supertropical semiring such that the map $\pi_E : U \rightarrow U/E$ is a monotone transmission, and hence a semiring homomorphism. The structure of the ordered supertropical semiring U/E is given by the rules (6.1)–(6.3).*

Remark 6.3. *Conversely, if $\alpha : U \rightarrow V$ is a surjective monotone transmission, then the equivalence relation $E := E(\alpha)$ on U (i.e., $x \sim_E y$ iff $\alpha(x) = \alpha(y)$) is clearly an OCTE-relation, and $\alpha = \rho \circ \pi_E$ with ρ an isomorphism from U/E to the semiring V . Thus knowing the OCTE-relation on U gives us a hold on all surjective monotone transmissions starting from U . \square*

In [IKR2, §4] we pursued the question, when a given TE-relation E on a supertropical semiring U is **transmissive**, i.e., the monoid U/E admits the structure of a supertropical semiring such that π_E is transmission. The main result there [IKR2, Theorem 4.7] stated that this happens if the monoid $(M/E) \setminus \{0\}$ is cancellative ($M := eU$). The present Theorem 6.2 exhibits another class of TE-relations, which are transmissive. Here no cancellation hypothesis is needed.

In [IKR2] a transmissive equivalence relation E on supertropical semiring is called **homomorphic**, if the transmission $\pi_E : U \rightarrow U/E$ is a semiring homomorphism. §5 and §6 of [IKR2] contain various explicit examples of homomorphic equivalence relations.

Assume now that U is totally ordered. As we know, all OCTE-relations on U are homomorphic. We search for cases where the homomorphic equivalence relations described in [IKR2, §5 and §6] are OCTE-relations.

Example 6.4. *If \mathfrak{a} is any ideal of U then the homomorphic equivalence relation $E(\mathfrak{a})$ on U (cf. [IKR2, §5]) is order compatible, hence is an OCTE-relation.*

Proof. We prove that the $E(\mathfrak{a})$ equivalence classes are convex in U and then will be done. Let $x, y, z \in U$ be given with $x \leq z \leq y$ and $x \sim_{\mathfrak{a}} y$. We verify that $y \sim_{\mathfrak{a}} z$ by using the description of $E(\mathfrak{a})$ in [IKR2, Theorem 5.4]. We have $ex \leq ez \leq ey$.

Case 1: $ex \leq ea$ for some $a \in \mathfrak{a}$. Then $x \sim_{\mathfrak{a}} 0$, hence $y \sim_{\mathfrak{a}} 0$. Thus $ey \leq eb$ for some $b \in \mathfrak{a}$, and from $ez \leq ey$ we infer that $z \sim_{\mathfrak{a}} 0$.

Case 2: $ex > ea$ for every $a \in \mathfrak{a}$. Now $x = y$, hence also $x = z$.

Thus $z \sim_{\mathfrak{a}} x$ in both cases. \square

Example 6.5. *Let Φ be a homomorphic equivalence relation on $M := eU$, and let \mathfrak{A} be an ideal of U containing M . We study the equivalence relations $E := E(U, \mathfrak{A}, \Phi)$ defined in [IKR2, §6] (cf. Definition 6.11). Thus for $x, y \in \mathfrak{A}$,*

$$x \sim_E y \iff \begin{cases} \text{either} & x = y, \\ \text{or} & x, y \in \mathfrak{A} \text{ and } ex \sim_{\Phi} ey. \end{cases}$$

We know from [IKR2, Theorem 4.13.i] that E is multiplicative. It is easily verified that every equivalence class of E is convex in U if the following two conditions hold:

- (1) $\nu_U^{-1}(x) \cap \mathfrak{A}$ for every $x \in M$ such that $x \sim_{\Phi} y$ for some $y \in M$ with $y < x$.
- (2) $\nu_U^{-1}(x) \cap \mathfrak{A}$ is convex in $\nu_U^{-1}(x)$ for the other $x \in M$ ⁵.

Looking at [IKR2, Theorem 6.14] we see that the conditions (1), (2) imply that E is homomorphic. Thus E is an OCTE-relation precisely if these conditions (1), (2) are valid.

The case that $\Phi = \text{diag}(M)$, i.e., $x \sim_{\Phi} y$ iff $x = y$, gives us

Subexample 6.6. *Let \mathfrak{A} be an ideal of U containing M . The MFCE-relation $E(U, \mathfrak{A})$ (cf. [IKR2, Definition 6.15]) is order compatible (hence an OCTE-relation) iff for every $x \in M$ the set $\mathfrak{A} \cap \nu_U^{-1}(x)$ is convex in $\nu_U^{-1}(x)$.*

⁵This means that $\nu_u^{-1}(x) \cap \mathfrak{A}$ is an upper set in the totally ordered set $\nu_u^{-1}(x)$.

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