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Second Main Theorems and Unicity of Meromorphic Mappings with Moving Hypersurfaces

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Second main theorems and unicity of meromorphic mappings with moving hypersurfaces

Si Duc Quang

Abstract. In this article, we establish some new second main theorems for meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and moving hypersurfaces with truncated counting functions. As an application, we prove a uniqueness theorem for these mappings sharing few moving hypersurfaces without counting multiplicity. This result is an improvement of the results of Dulock - Min Ru [2] and Dethloff - Tan [4]. Moreover the meromorphic mappings maybe algebraically degenerate.

1 Introduction

In 2004, Min Ru [8] showed a second main theorem for algebraically nondegenerate meromorphic mappings and a family of hypersurfaces in weakly general position. After that, with the same assumptions, T. T. H. An and H. T. Phuong [1] improved the result of Min Ru by giving an explicit truncation level for counting functions. Applying the second main theorem of An - Phuong, Dulock and Min Ru [2] proved a uniqueness theorem for meromorphic mappings sharing a family of hypersurfaces in weakly general position.

Recently, in [3] Dethloff and Tan generalized and improved the second main theorems of Min Ru and An - Phuong to the case of moving hypersurfaces. They proved that

Theorem A (Dethloff - Tan [3]) *Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let $\{Q_i\}_{i=1}^q$ be a set of slow (with respect to f) moving hypersurfaces in weakly general position with* $\deg Q_j = d_j$ $(1 \le i \le q)$. *Assume that f is algebraically nondegenerate over $\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}$. Then for any $\epsilon > 0$ there exist positive integers L_j $(j = 1, ..., q)$, depending only on n, ϵ and d_j $(j = 1, ..., q)$ in an explicit way such that*

$$|| \ (q-n-1-\epsilon)T_f(r) \le \sum_{i=1}^{q} \frac{1}{d_i} N^{[L_j]}_{Q_i(f)}(r) + o(T_f(r)).$$

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Here, the truncation level L_j is estimated by

$$L_j \leq \frac{d_j \cdot \binom{n+M}{n} t_{p_0+1} - d_j}{d} + 1,$$

where d is the least common multiple of the d_j's, $d = lcm(d_1, ..., d_q)$, and

$$\begin{aligned} M =& d \cdot [2(n+1)(2^n - 1)(nd+1)\epsilon^{-1} + n + 1], \\ p_0 =& [\frac{(\binom{n+M}{n}^2 \cdot \binom{q}{n} - 1) \cdot \log(\binom{n+M}{n}^2 \cdot \binom{q}{n})}{\log(1 + \dfrac{\epsilon}{2\binom{n+M}{n} N})} + 1]^2, \\ \textit{and } t_{p_0+1} <& \left(\binom{n+M}{n}^2 \cdot \binom{q}{n} + p_0 \right)^{\left(\binom{n+M}{n}^2 \cdot \binom{q}{n} - 1\right)}, \end{aligned}$$

where $[x] = \max\{k \in \mathbf{Z} \ ; \ k \leq x\}$ for a real number x.

By using this second main theorem, Dethloff and Tan proved a uniqueness theorem for meromorphic mappings sharing slow moving hypersurfaces (see Theorem 3.1 [4]). However in their result, the number of moving hypersurfaces is too big. An essential reason comes from the fact that the truncation levels given in Theorem B actually are very weak. Morever their proof of the uniqueness theorem is too complicate.

We also would like to note that, in all mentioned results on second main theorem of Min Ru, An - Phuong and Dethloff - Tan the algebraically nondegeneracy condition of the meromorphic mappings can not be removed.

Our aim in the present paper is to show some new second main theorems for meromorphic mappings and slow moving hypersurfaces with better truncation levels for counting functions. Moreover the mappings maybe algebraically degenerate. Namely, we prove the following theorems.

Theorem 1.1. *Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let Q_i $(i = 1, ..., q)$ be slow (with respect to f) moving hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ in weakly general position with $\deg Q_i = d_i$, $q \geq nN + n + 1$, where $N = \binom{n+d}{n} - 1$ and $d = lcm(d_1, ..., d_q)$. Assume that $Q_i(f) \not\equiv 0$ $(1 \leq i \leq q)$. Then we have*

$$|| \ \frac{q}{nN+n+1} T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[N]}_{Q_i(f)}(r) + o(T_f(r)).$$

Theorem 1.2. *Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let Q_i $(i = 1, ..., q)$ be slow (with respect to f) moving hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ in weakly general position with $\deg Q_i = d_i$, $q \geq N + 2$, where $N = \binom{n+d}{n} - 1$ and $d = lcm(d_1, ..., d_q)$. Assume that f is algebraically nondegenerate over $\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}$. Then we have*

$$|| \ \frac{q}{N+2} T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[N]}_{Q_i(f)}(r) + o(T_f(r)).$$

As an application of these second main theorems, we prove the following uniqueness theorem for meromorphic mappings sharing slow moving hypersurfaces without counting multiplicity.

Theorem 1.3. *Let f and g be nonconstant meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let Q_i $(i = 1, ..., q)$ be a set of slow (with respect to f and g) moving hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ in weakly general position with $\deg Q_i = d_i$. Put $d = lcm(d_1, ..., d_q)$ and $N = \binom{n+d}{n} - 1$. Let k $(1 \le k \le n)$ be an integer. Assume that*

(i) $\dim\big(\bigcap_{j=0}^{k} \mathrm{Zero}Q_{i_j}(f)\big) \le m - 2$ *for every* $1 \le i_0 < \cdots < i_k \le q$,

(ii) $f = g$ *on* $\bigcup_{i=1}^{q}\big(\mathrm{Zero}Q_i(f) \cup \mathrm{Zero}Q_{i_j}(g)\big)$.

Then the following assertions hold:

a) If $q > \dfrac{2kN(nN + n + 1)}{d}$ *then* $f = g$.

b) In addition to the assumptions (i)-(ii), we assume that both f and g are algebraically nondegenerate over $\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}$. *If* $q > \dfrac{2kN(N + 2)}{d}$ *then* $f = g$.

We note that the numbers of hypersurfaces in our results are smaller than that in the results of Dulock - Min Ru [2] and Dethloff - Tan [4]. Also by introducing some new techniques, we simplify their proofs.

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2 Basic notions and auxiliary results from Nevanlinna theory

2.1. We set $||z|| = \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$ and define

$$B(r) := \{z \in \mathbf{C}^m : ||z|| < r\}, \quad S(r) := \{z \in \mathbf{C}^m : ||z|| = r\} \ (0 < r < \infty).$$

Define

$$v_{m-1}(z) := \big(dd^c||z||^2\big)^{m-1} \qquad \text{and}$$

$$\sigma_m(z) := d^c\log||z||^2 \wedge \big(dd^c\log||z||^2\big)^{m-1} \text{on} \quad \mathbf{C}^m \setminus \{0\}.$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbf{C}^m$. For a set $\alpha = (\alpha_1, ..., \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + ... + \alpha_m$ and $\mathcal{D}^\alpha F = \dfrac{\partial^{|\alpha|}F}{\partial^{\alpha_1}z_1...\partial^{\alpha_m}z_m}$. We define the map $\nu_F : \Omega \to \mathbf{Z}$ by

$$\nu_F(z) := \max\ \{k : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < k\} \ (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbf{C}^m$ a map $\nu : \Omega \to \mathbf{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\le m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\le m-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \ne 0\}}$, which is either a purely $(m-1)$-dimensional analytic subset of Ω or an empty set.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbf{C}^m$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \le m-2$, and we define the divisors ν_φ^0, ν_φ^∞ by $\nu_\varphi^0 := \nu_F$, $\nu_\varphi^\infty := \nu_G$, which are independent of choices of F and G and so globally well-defined on Ω.

2.3. For a divisor ν on $\mathbf{C}^m$ and for a positive integer M or $M = \infty$, we define the counting function of ν by

$$\nu^{[M]}(z) = \min\,\{M, \nu(z)\},$$

$$n(t) = \begin{cases} \int\limits_{|\nu|\,\cap B(t)} \nu(z) v_{m-1} & \text{if } m \ge 2, \\ \sum\limits_{|z| \le t} \nu(z) & \text{if } m = 1. \end{cases}$$

Similarly, we define $n^{[M]}(t)$.

Define

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}} dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, \nu^{[M]})$ and denote it by $N^{[M]}(r, \nu)$.

Let $\varphi : \mathbf{C}^m \longrightarrow \mathbf{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r, \nu_\varphi^0), \ N_\varphi^{[M]}(r) = N^{[M]}(r, \nu_\varphi^0).$$

For brevity we will omit the character $^{[M]}$ if $M = \infty$.

2.4. Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbf{P}^n(\mathbf{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbf{C}^m$ and $f(z) = \big(f_0(z) : \cdots : f_n(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_n|^2\big)^{1/2}$.

The characteristic function of f is defined by

$$T_f(r) = \int\limits_{S(r)} \log \|f\| \sigma_m - \int\limits_{S(1)} \log \|f\| \sigma_m.$$

2.5. Let φ be a nonzero meromorphic function on $\mathbf{C}^m$, which are occasionally regarded as a meromorphic map into $\mathbf{P}^1(\mathbf{C})$. The proximity function of φ is defined by

$$m(r,\varphi) := \int_{S(r)} \log\max\ (|\varphi|, 1)\sigma_m.$$

The Nevanlinna's characteristic function of φ is defined as follows

$$T(r,\varphi) := N_{\frac{1}{\varphi}}(r) + m(r,\varphi).$$

Then

$$T_\varphi(r) = T(r,\varphi) + O(1).$$

The function φ is said to be small (with respect to f) if $||\ T_\varphi(r) = o(T_f(r))$. Here, by the notation $''||\ P''$ we mean the assertion P holds for all $r \in [0,\infty)$ excluding a Borel subset E of the interval $[0,\infty)$ with $\int_E dr < \infty$.

We denote by $\mathcal{M}$ (resp. $\mathcal{K}_f$) the field of all meromorphic functions (resp. small meromorphic functions) on $\mathbf{C}^m$.

2.6. Denote by $\mathcal{H}_{\mathbf{C}^m}$ the ring of all holomorphic functions on $\mathbf{C}^m$. Let Q be a homogeneous polynomial in $\mathcal{H}_{\mathbf{C}^m}[x_0,\dots,x_n]$ of degree $d \geq 1$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbf{C}$ obtained by substituting a specific point $z \in \mathbf{C}^m$ into the coefficients of Q. We also call a moving hypersurface in $\mathbf{P}^n(\mathbf{C})$ each homogeneous polynomial $Q \in \mathcal{H}_{\mathbf{C}^m}[x_0,\dots,x_n]$ such that the common zero set of all coefficients of Q has codimension at least two.

Let Q be a moving hypersurface in $\mathbf{P}^n(\mathbf{C})$ of degree $d \geq 1$ given by

$$Q(z) = \sum_{I\in\mathcal{I}_d} a_I\omega^I,$$

where $\mathcal{I}_d = \{(i_0,...,i_n) \in \mathbf{N}_0^{n+1}\ ;\ i_0+\cdots+i_n = d\}$, $a_I \in \mathcal{H}_{\mathbf{C}^m}$ and $\omega^I = \omega_0^{i_0}\cdots\omega_n^{i_n}$. We consider the meromorphic mapping $Q' : \mathbf{C}^m \to \mathbf{P}^N(\mathbf{C})$, where $N = \binom{n+d}{n}$, given by

$$Q'(z) = (a_{I_0}(z) : \cdots : a_{I_N}(z))\ (\mathcal{I}_d = \{I_0,...,I_N\}).$$

The moving hypersurfaces Q is said to be "slow" (with respect to f) if $||\ T_{Q'}(r) = o(T_f(r))$. This is equivalent to $||T_{\frac{a_{I_i}}{a_{I_j}}}(r) = o(T_f(r))$ for every $a_{I_j} \not\equiv 0$.

Let $\{Q_i\}_{i=1}^q$ be a family of moving hypersurfaces in $\mathbf{P}^n(\mathbf{C})$, $\deg Q_i = d_i$. Assume that

$$Q_i = \sum_{I\in\mathcal{I}_{d_i}} a_{iI}\omega^I.$$

We denote by $\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}$ the smallest subfield of $\mathcal{M}$ which contains $\mathbf{C}$ and all $\frac{a_iI}{a_{iJ}}$ with $a_{iJ} \not\equiv 0$. We say that $\{Q_i\}_{i=1}^q$ are in weakly general position if there exists $z \in \mathbf{C}^m$ such that all a_{iI} $(1 \le i \le q,\ I \in \mathcal{I})$ are holomorphic at z and for any $1 \le i_0 < \cdots < i_n \le q$ the system of equations

$$\begin{cases} Q_{i_j}(z)(w_0, \dots, w_n) = 0 \\ 0 \le j \le n \end{cases}$$

has only the trivial solution $w = (0, \dots, 0)$ in $\mathbf{C}^{n+1}$.

2.7. Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Denote by $\mathcal{C}_f$ the set of all non-negative functions $h : \mathbf{C}^m \setminus A \longrightarrow [0, +\infty] \subset \overline{\mathbf{R}}$, which are of the form

$$h = \frac{|g_1| + \cdots + |g_l|}{|g_{l+1}| + \cdots + |g_{l+k}|},$$

where $k, l \in \mathbf{N},\ g_1, ..., g_{l+k} \in \mathcal{K}_f \setminus \{0\}$ and $A \subset \mathbf{C}^m$, which may depend on $g_1, ..., g_{l+k}$, is an analytic subset of codimension at least two. Then, for $h \in \mathcal{C}_f$ we have

$$\int\limits_{S(r)} \log h \sigma_m = o(T_f(r)).$$

Lemma 2.8 (Lemma 2 [3]). *Let $\{Q_i\}_{i=0}^n$ be a set of homogeneous polynomials of degree d in $\mathcal{K}_f[x_0, ..., x_n]$. Then there exists a function $h_1 \in \mathcal{C}_f$ such that, outside an analytic set of $\mathbf{C}^m$ of codimension at least two,*

$$\max_{i \in \{0,\dots,n\}} |Q_i(f_0, ..., f_n)| \le h_1 ||f||^d.$$

If, moreover, this set of homogeneous polynomials is in weakly general position, then there exists a nonzero function $h_2 \in \mathcal{C}_f$ such that, outside an analytic set of $\mathbf{C}^m$ of codimension at least two,

$$h_2 ||f||^d \le \max_{i \in \{0,\dots,n\}} |Q_i(f_0, ..., f_n)|.$$

2.9. Lemma on logarithmic derivative (Lemma 3.11 [9]) . *Let f be a nonzero meromorphic function on $\mathbf{C}^m$. Then*

$$\Big|\Big| \quad m\left(r, \frac{\mathcal{D}^\alpha(f)}{f}\right) = O(\log^+ T(r, f)) \ (\alpha \in \mathbf{Z}_+^m).$$

2.10. Assume that $\mathcal{L}$ is a subset of a vector space V over a field $\mathcal{R}$. We say that the set $\mathcal{L}$ is *minimal* over $\mathcal{R}$ if it is linearly dependent over $\mathcal{R}$ and each proper subset of $\mathcal{L}$ is linearly independent over $\mathcal{R}$.

Repeating the argument in (Prop. 4.5 [5]), we have the following:

Proposition 2.11. *Let $\Phi_0, ..., \Phi_k$ be meromorphic functions on $\mathbf{C}^m$ such that $\{\Phi_0, ..., \Phi_k\}$ are linearly independent over $\mathbf{C}$. Then there exists an admissible set*

$$\{\alpha_i = (\alpha_{i1}, ..., \alpha_{im})\}_{i=0}^k \subset \mathbf{Z}_+^m$$

with $|\alpha_i| = \sum_{j=1}^m |\alpha_{ij}| \le k$ $(0 \le i \le k)$ *such that the following are satisfied:*

(i) $\{\mathcal{D}^{\alpha_i}\Phi_0, ..., \mathcal{D}^{\alpha_i}\Phi_k\}_{i=0}^k$ *is linearly independent over* $\mathcal{M}$, *i.e.,* $\det(\mathcal{D}^{\alpha_i}\Phi_j) \not\equiv 0$.

(ii) $\det\big(\mathcal{D}^{\alpha_i}(h\Phi_j)\big) = h^{k+1} \cdot \det\big(\mathcal{D}^{\alpha_i}\Phi_j\big)$ *for any nonzero meromorphic function* h *on* $\mathbf{C}^m$.

3 Second main theorems for moving hypersurfaces

In order to prove Theorem 1.1 we need the following.

Lemma 3.1. *Let* f *be as in Theorem 1.1. Let* $\{Q_i\}_{i=0}^{n(N+1)}$ *be a set of homogeneous polynomials in* $\mathcal{K}_f[x_0, ..., x_n]$ *of common degree* d *in weakly general position, where* $N = \binom{n+d}{n} - 1$. *Assume that* $Q_i(f) \not\equiv 0$ $(0 \le i \le n(N+1))$. *Then there exist a subset* B *of* $\{Q_i(f)\ ;\ 0 \le i \le n(N+1)\}$ *and subsets* $I_1, ..., I_k$ *of* B *such that the following are satisfied:*

(i) I_1 *is minimal,* I_i *is independent over* $\mathcal{K}_f$ $(2 \le i \le k)$.

(ii) $B = \bigcup_{i=1}^k I_i$, $I_i \cap I_j = \emptyset$ $(i \ne j)$ *and* $\sharp B \ge n+1$.

(iii) For each $1 \le i \le k$, *there exist meromorphic functions* $c_\alpha \in \mathcal{K}_f \setminus \{0\}$ *such that*

$$\sum_{Q_\alpha(f) \in I_i} c_\alpha Q_\alpha(f) \in \left(\bigcup_{j=1}^{i-1} I_j\right)_{\mathcal{K}_f}.$$

Proof. Denote by V_f^d the vector space of all homogeneous polynomials of degree d in $\mathcal{K}_f[x_0, ..., x_n]$. It is seen that $\dim V_f^d = \binom{n+d}{n} = N+1$.

• We set $A_0 = \{Q_i(f)\ ;\ 0 \le i \le n(N+1)\}$. We are going to construct the subset B_0 of A_0 as follows:

Since $\sharp A_0 > N+1 = \dim V_f^d$, the set A_0 is linearly independent over $\mathcal{K}_f$. Therefore, there exists a minimal subset I_1^0 over $\mathcal{K}_f$ of A_0. If $\sharp I_1^0 \ge n+1$ or $(I_1^0)_{\mathcal{K}_f} \cap (A_0 \setminus I_1^0)_{\mathcal{K}_f} = \{0\}$ then we stop the process and set $B_0 = I_1^0, A_1 = A_0 \setminus B_0$.

Otherwise, since $(I_1^0)_{\mathcal{K}_f} \cap (A_0 \setminus I_1^0)_{\mathcal{K}_f} \ne \{0\}$, we now choose a subset I_2^0 of $A_0 \setminus I_1^0$ such that I_2^0 is the minimal subset of $A_0 \setminus I_1^0$ satisfying $(I_1^0)_{\mathcal{K}_f} \cap (I_2^0)_{\mathcal{K}_f} \ne \{0\}$. By the minimality, the subset I_2^0 is linearly independent over $\mathcal{K}_f$. If $\sharp(I_1^0 \cup I_2^0) \ge n+1$ or $(I_1^0 \cup I_2^0)_{\mathcal{K}_f} \cap (A_0 \setminus (I_1^0 \cup I_2^0))_{\mathcal{K}_f} = \{0\}$ then we stop the process and set $B_0 = I_1^0 \cup I_2^0, A_1 = A_0 \setminus B_0$.

Otherwise, by repeating the above argument, we have a subset I_3^0 of $A_0 \setminus (I_1^0 \cup I_2^0)$.

Continuiting this process, there exist subsets $I_1^0, ..., I_k^0$ such that: I_i^0 is a subset of $A_0 \setminus \bigcup_{j=1}^{i-1} I_j^0$, I_j^0 is linearly independent over $\mathcal{K}_f$ $(2 \le j \le k)$, $(I_i^0)_{\mathcal{K}_f} \cap \left(\bigcup_{j=1}^{i-1} I_j^0\right)_{\mathcal{K}_f} \ne \{0\}$, $\sharp B_0 \ge n+1$ or $(B_0)_{\mathcal{K}_f} \cap (A_0 \setminus B_0)_{\mathcal{K}_f} = \{0\}$. Also, by the minimality of each subset I_i^0 $(2 \le i \le k)$, there exist nonzero meromorphic functions $c_\alpha^0 \in \mathcal{K}_f$ such that

$$\sum_{Q_\alpha(f) \in I_i^0} c_\alpha^0 Q_\alpha(f) \in \left(\bigcup_{j=1}^{i-1} I_j^0\right)_{\mathcal{K}_f}.$$

• If $\sharp B_0 \geq n+1$, by setting $B = B_0, I_i = I_i^0$ then the proof is finished.

Otherwise, we have $(B_0)_{\mathcal{K}_f} \cap (A_0 \setminus B_0)_{\mathcal{K}_f} = \{0\}$. We set $A_1 = A_0 \setminus B_0$. Then $\dim(A_1)_{\mathcal{K}_f} \leq N+1-\dim(B_0)_{\mathcal{K}_f} \leq N$ and $\sharp A_1 \geq nN+1 > N \geq \dim(A_1)_{\mathcal{K}_f}$. Similarly, we construct the subset B_1 of A_1 with the same properties as B_0.

• If $\sharp B_1 \geq n+1$ then the proof is finished. Otherwise, by repeating the same argument we have subsets A_3, B_3 and I_i^3.

Continuiting this process, we have the following two cases:

Case 1. By this way, we may construct subsets $B_1, ..., B_N$ with $\sharp B_i \leq n$ $(1 \leq i \leq N)$. We set $B_{N+1} = A_0 \setminus \bigcup_{i=0}^N B_i$. Then $\sharp B_{N+1} \geq n(N+1)+1-n(N+1) = 1$. Then $\dim(B_{N+1})_{\mathcal{K}_f} \geq 1$. On the other hand, it is easy to see that

$$\dim(B_{N+1})_{\mathcal{K}_f} = \dim(A_0)_{\mathcal{K}_f} - \sum_{i=0}^{N} \dim(B_i)_{\mathcal{K}_f} \leq N+1-(N+1) = 0.$$

This is a contradiction. Hence this case is impossible.

Case 2. At the step $k-th$ $(k \leq N)$, we get $\sharp B_k \geq n+1$. Then similarly as above, the proof is finished. □

Lemma 3.2. *Let f be as in Theorem 1.1. Let $\{Q_i\}_{i=0}^{n(N+1)}$ be a set of homogeneous polynomials in $\mathcal{K}_f[x_0, ..., x_n]$ of common degree d in weakly general position, where $N = \binom{n+d}{n} - 1$. Assume that $Q_i(f) \not\equiv 0$ $(0 \leq i \leq n(N+1))$. Then we have*

$$|| \ T_f(r) \leq \sum_{i=0}^{n(N+1)} \frac{1}{d} N_{Q_i(f)}^{[N]}(r) + o(T_f(r)).$$

Proof. By Lemma 3.1, we may assume that there exist the subsets

$$I_i = \{Q_{t_i+1}(f), ..., Q_{t_{i+1}}(f)\} \ \ (1 \leq i \leq k)$$

and functions $c_i \in \mathcal{K}_f \setminus \{0\}$ $(t_2+1 \leq i \leq t_{k+1})$, where $t_1 = -1$, which satisfy the assertions of Lemma 3.1.

Since I_1 is minimal over $\mathcal{K}_f$, there exist $c_{1j} \in \mathcal{R} \setminus \{0\}$ such that

$$\sum_{j=0}^{t_2} c_{1j} Q_j(f) = 0.$$

Define $c_{1j} = 0$ for all $j > t_1$. Then $\sum_{j=0}^{t_{k+1}} c_{1j} Q_j(f) = 0$.

Since $\{c_{1j} Q_j(f)\}_{j=1}^{t_2}$ is linearly independent over $\mathcal{K}_f$, there exists an admissible set

$\{\alpha_{11}, ..., \alpha_{1t_2}\} \subset \mathbf{Z}_+^m$ $(|\alpha_{1j}| \le t_2 - 1 \le N)$ such that

$$A_1 \equiv \begin{vmatrix} \mathcal{D}^{\alpha_{11}}(c_{11}Q_1(f)) & \cdots & \mathcal{D}^{\alpha_{11}}(c_{1t_2}Q_{t_2}(f)) \\ \mathcal{D}^{\alpha_{12}}(c_{11}Q_1(f)) & \cdots & \mathcal{D}^{\alpha_{12}}(c_{1t_2}Q_{t_2}(f)) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_2}}(c_{11}Q_1(f)) & \cdots & \mathcal{D}^{\alpha_{1t_2}}(c_{1t_2}Q_{t_2}(f)) \end{vmatrix}$$

$$\equiv f_0^{t_1} \cdot \begin{vmatrix} \mathcal{D}^{\alpha_{11}}\left(\dfrac{c_{11}Q_1(f)}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{11}}\left(\dfrac{c_{1t_2}Q_{t_2}(f)}{Q_0(f)}\right) \\ \mathcal{D}^{\alpha_{12}}\left(\dfrac{c_{11}Q_1(f))}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{12}}\left(\dfrac{c_{1t_2}Q_{t_2}(f)}{Q_0(f)}\right) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_2}}\left(\dfrac{c_{11}Q_1(f)}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{1t_1}}\left(\dfrac{c_{1t_2}Q_{t_2}(f)}{Q_0(f)}\right) \end{vmatrix} \equiv (Q_0(f))^{t_2} \cdot \tilde{A}_1 \not\equiv 0.$$

Now consider $i \ge 2$. We set $c_{ij} = c_j \not\equiv 0$ $(t_i + 1 \le j \le t_{i+1})$, then $\sum_{j=t_i+1}^{t_{i+1}} c_{ij}Q_j(f) \in \left(\bigcup_{j=1}^{i-1} I_j\right)_{\mathcal{K}_f}$. Therefore, there exist meromorphic functions $c_{ij} \in \mathcal{K}_f$ $(0 \le j \le t_i)$ such that $\sum_{j=0}^{t_{i+1}} c_{ij}Q_j(f) = 0$.

Define $c_{ij} = 0$ for all $j > t_{i+1}$. Then $\sum_{j=0}^{t_{k+1}} c_{ij}Q_j(f) = 0$.

Since $\{c_{ij}Q_j(f)\}_{j=t_i+1}^{t_{i+1}}$ is linearly independent over $\mathcal{K}_f$, there exists $\{\alpha_{ij}\}_{j=t_i+1}^{t_{i+1}} \subset \mathbf{Z}_+^m$ $(|\alpha_{ij}| \le t_{i+1} - t_i - 1 \le N)$ such that

$$\begin{aligned} A_i =& \det\left(\mathcal{D}^{\alpha_{ij}}\left(c_{is}Q_s(f)\right)\right)_{j,s=t_i+1}^{t_{i+1}} = (Q_0(f))^{t_{i+1}-t_i} \cdot \det\left(\mathcal{D}^{\alpha_{ij}}\left(\frac{c_{is}Q_s(f)}{Q_0(f)}\right)\right)_{j,s=t_i+1}^{t_{i+1}} \\ =& Q_0(f)^{t_{i+1}-t_i} \cdot \tilde{A}_i \not\equiv 0. \end{aligned}$$

Consider an $t_{k+1} \times (t_{k+1}+1)$ minor matrixes $\mathcal{T}$ and $\tilde{\mathcal{T}}$ given by

$$
\mathcal{T} = \begin{bmatrix}
\mathcal{D}^{\alpha_{11}}(c_{10}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{11}}(c_{1t_{k+1}}Q_{t_{k+1}}(f)) \\
\mathcal{D}^{\alpha_{12}}(c_{10}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{12}}(c_{1t_{k+1}}Q_{t_{k+1}}(f)) \\
\vdots & \vdots & \vdots \\
\mathcal{D}^{\alpha_{1t_2}}(c_{10}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{1t_2}}(c_{1t_{k+1}}Q_{t_{k+1}}(f)) \\
\mathcal{D}^{\alpha_{2t_2+1}}(c_{20}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{2t_2+1}}(c_{2t_{k+1}}Q_{t_{k+1}}(f)) \\
\mathcal{D}^{\alpha_{2t_2+2}}(c_{20}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{2t_2+2}}(c_{2t_{k+1}}Q_{t_{k+1}}(f)) \\
\vdots & \vdots & \vdots \\
\mathcal{D}^{\alpha_{2t_3}}(c_{20}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{2t_3}}(c_{2t_{k+1}}Q_{t_{k+1}}(f)) \\
\vdots & \vdots & \vdots \\
\mathcal{D}^{\alpha_{kt_k+1}}(c_{k0}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{kt_k+1}}(c_{kt_{k+1}}Q_{t_{k+1}}(f)) \\
\mathcal{D}^{\alpha_{kt_k+2}}(c_{k0}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{kt_k+2}}(c_{kt_{k+1}}Q_{t_{k+1}}(f)) \\
\vdots & \vdots & \vdots \\
\mathcal{D}^{\alpha_{kt_{k+1}}}(c_{k0}Q_0(f)) & \cdots & \mathcal{D}^{\alpha_{kt_{k+1}}}(c_{kt_{k+1}}Q_{t_{k+1}}(f))
\end{bmatrix}
$$

$$
\tilde{\mathcal{T}} = \begin{bmatrix}
\mathcal{D}^{\alpha_{11}}\left(\frac{c_{10}Q_0(f)}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{11}}\left(\frac{c_{1t_{k+1}}Q_{t_{k+1}}(f)}{Q_0(f)}\right) \\
\vdots & \vdots & \vdots \\
\mathcal{D}^{\alpha_{1t_2}}\left(\frac{c_{10}Q_0(f)}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{1t_2}}\left(\frac{c_{1t_{k+1}}Q_{t_{k+1}}(f)}{Q_0(f)}\right) \\
\mathcal{D}^{\alpha_{2t_2+1}}\left(\frac{c_{20}Q_0(f)}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{2t_2+1}}\left(\frac{c_{1t_{k+1}}Q_{t_{k+1}}(f)}{Q_0(f)}\right) \\
\vdots & \vdots & \vdots \\
\mathcal{D}^{\alpha_{2t_3}}\left(\frac{c_{20}Q_0(f)}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{2t_3}}\left(\frac{c_{2t_{k+1}}Q_{t_{k+1}}(f)}{Q_0(f)}\right) \\
\vdots & \vdots & \vdots \\
\mathcal{D}^{\alpha_{kt_k+1}}\left(\frac{c_{k0}Q_0(f)}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{kt_k+1}}\left(\frac{c_{kt_{k+1}}Q_{t_{k+1}}(f)}{Q_0(f)}\right) \\
\vdots & \vdots & \vdots \\
\mathcal{D}^{\alpha_{kt_{k+1}}}\left(\frac{c_{k0}Q_0(f)}{Q_0(f)}\right) & \cdots & \mathcal{D}^{\alpha_{kt_{k+1}}}\left(\frac{c_{kt_{k+1}}Q_{t_{k+1}}(f)}{Q_0(f)}\right)
\end{bmatrix}.
$$

Denote by D_i (resp. $\tilde{D}_i$) the determinant of the matrix obtained by deleting the $(i+1)$-th column of the minor matrix $\mathcal{T}$ (resp. $\tilde{\mathcal{T}}$). It is clear that the sum of each row of $\mathcal{T}$ (resp.$\tilde{\mathcal{T}}$) is zero, then we have

$$
\begin{aligned}
D_i = (-1)^i D_0 = (-1)^i \prod_{i=1}^{k} A_i &= (-1)^i (Q_0(f))^{t_{k+1}} \prod_{i=1}^{k} \tilde{A}_i \\
&= (-1)^i (Q_0(f))^{t_{k+1}} \tilde{D}_0 = (Q_0(f))^{t_{k+1}} \tilde{D}_i.
\end{aligned}
$$

Since $\sharp(\bigcup_{i=1}^k I_i) \geq n+1$ and $Q_0, ..., Q_{t_{k+1}}$ are in weakly general position, by Lemma 2.8 there exists a function $\Psi \in \mathcal{C}_f$ such that

$$||f(z)||^d \leq \Psi(z) \cdot \max_{0 \leq i \leq t_{k+1}} \big(|Q_i(f)(z)|\big) \ (z \in \mathbf{C}^m).$$

Fix $z_0 \in \mathbf{C}^m$. Take i $(0 \leq i \leq t_k)$ such that $|Q_i(f)(z_0)| = \max_{0 \leq j \leq t_k} |Q_j(f)(z_0)|$. Then

$$\frac{|D_0(z_0)| \cdot ||f(z_0)||^d}{\prod_{j=0}^{t_{k+1}} |Q_j(f)(z_0)|} = \frac{|D_i(z_0)|}{\prod_{\substack{j=0 \\ j \neq i}}^{t_{k+1}} |Q_j(f)(z_0)|} \cdot \left(\frac{||f(z_0)||^d}{|Q_i(f)(z_0)|}\right) \leq \Psi(z_0) \cdot \frac{|D_i(z_0)|}{\prod_{\substack{j=0 \\ j \neq i}}^{t_{k+1}} |Q_j(f)(z_0)|}.$$

This implies that

$$\begin{aligned}\log \frac{|D_0(z_0)|.||f(z_0)||^d}{\prod_{j=0}^{t_{k+1}} |Q_j(f)(z_0)|} &\leq \log^+\left(\Psi(z_0) \cdot \left(\frac{|D_i(z_0)|}{\prod_{j=0, j\neq i}^{t_{k+1}} |Q_j(f)(z_0)|}\right)\right) \\ &\leq \log^+\left(\frac{|D_i(z_0)|}{\prod_{j=0, j\neq i}^{t_k} |Q_j(f)(z_0)|}\right) + \log^+ \Psi(z_0).\end{aligned}$$

Thus, for each $z \in \mathbf{C}^m$, we have

$$\begin{aligned}\log \frac{|D_0(z)|.||f(z)||^d}{\prod_{i=0}^{t_{k+1}} |Q_i(f)(z)|} &\leq \sum_{i=0}^{t_{k+1}} \log^+\left(\frac{|D_i(z)|}{\prod_{j=0, j\neq i}^{t_k} |Q_j(f)(z)|}\right) + \log^+ \Psi(z) \\ &= \sum_{i=0}^{t_{k+1}} \log^+\left(\frac{|\tilde{D}_i(z)|}{\prod_{j=0, j\neq i}^{t_k} \left|\frac{Q_j(f)(z)}{Q_0(f)(z)}\right|}\right) + \log^+ \Psi(z).\end{aligned} \tag{3.3}$$

Note that $\dfrac{\tilde{D}_i}{\prod_{j=0, j\neq i}^{t_{k+1}} \dfrac{Q_j(f)}{Q_0(f)}} = \det \begin{bmatrix} \dfrac{\mathcal{D}^{\alpha_{11}}\left(\dfrac{c_{10}Q_0(f)}{Q_0(f)}\right)}{\dfrac{Q_0(f)}{Q_0(f)}} & \cdots & \dfrac{\mathcal{D}^{\alpha_{11}}\left(\dfrac{c_{1t_{k+1}}Q_{t_{k+1}}(f)}{Q_0(f)}\right)}{\dfrac{Q_{t_{k+1}}(f)}{Q_0(f)}} \\ \vdots & \vdots & \vdots \\ \dfrac{\mathcal{D}^{\alpha_{kt_{k+1}}}\left(\dfrac{c_{k0}Q_0(f)}{Q_0(f)}\right)}{\dfrac{Q_0(f)}{Q_0(f)}} & \cdots & \dfrac{\mathcal{D}^{\alpha_{kt_{k+1}}}\left(\dfrac{c_{kt_{k+1}}Q_{t_{k+1}}(f)}{Q_0(f)}\right)}{\dfrac{Q_{t_{k+1}}(f)}{Q_0(f)}} \end{bmatrix}$

(The determinant is counted after deleting the i-th column in the above matrix)

By the lemma on logarithmic derivative, for each i and $c \in \mathcal{K}_f$ we have

$$\begin{aligned}\Big\| \quad m\left(r, \frac{\mathcal{D}^{\alpha}\left(\dfrac{cQ_j(f)}{Q_0(f)}\right)}{\dfrac{Q_j(f)}{Q_0(f)}}\right) &\leq m\left(r, \frac{\mathcal{D}^{\alpha}\left(\dfrac{cQ_j(f)}{Q_0(f)}\right)}{\dfrac{cQ_j(f)}{Q_0(f)}}\right) + m(r,c) \\ &\leq O\left(\log^+ T_{\frac{cQ_j(f)}{Q_0(f)}}(r)\right) + T_c(r) = o(T_f(r))\end{aligned}$$

Therefore, we have

$$\Big|\Big| \quad m\left(r, \frac{\tilde{D}_i}{\prod_{j=0,j\neq i}^{t_{k+1}} \dfrac{Q_j(f)}{Q_0(f)}}\right) = o(T_f(r)) \ (0 \le i \le t_k).$$

Integrating both sides of the inequality (3.3), we get

$$\begin{aligned}\Big|\Big| \int_{S(r)} \log ||f||^d \sigma_m + \int_{S(r)} \log\left(\frac{|D_0|}{\prod_{i=0}^{t_{k+1}} |Q_i(f)|}\right)\sigma_m \\ \le \sum_{i=0}^{t_{k+1}} \int_{S(r)} \log^+\left(\frac{|\tilde{D}_i|}{\prod_{j=0,j\neq i}^{t_{k+1}} |\dfrac{Q_j(f)}{Q_0(f)}|}\right)\sigma_m + \int_{S(r)} \log^+ \Psi(z)\sigma_m \\ \le \sum_{i=0}^{t_{k+1}} m\left(r, \frac{\tilde{D}_i}{\prod_{j=0,j\neq i}^{t_{k+1}} \dfrac{Q_j(f)}{Q_0(f)}}\right) + o(T_f(r)) = o(T_f(r)).\end{aligned}$$

By Jensen formula, the above inequality implies that

$$(3.4) \qquad || \ dT_f(r) + N_{D_0}(r) - N_{\frac{1}{D_0}}(r) - \sum_{i=0}^{t_{k+1}} N_{Q_i(f)}(r) \le o(T_f(r)).$$

We see that a pole of D_0 must be pole of some c_{is} or pole of some nonzero coefficients a_{iI} of Q_i and

$$N_{\frac{1}{D_0}}(r) \le O(\sum_{i,s} N_{\frac{1}{c_{is}}}(r) + \sum_{a_{iI}\not\equiv 0} N_{\frac{1}{a_{iI}}}(r)) = o(T_f(r)).$$

Therefore, the inequality (3.4) implies that

$$(3.5) \qquad || \ dT_f(r) \le \sum_{i=0}^{t_{k+1}} N_{Q_i(f)}(r) - N_{D_0}(r) + o(T_f(r)).$$

Here we note that $D_i = (-1)^i D_0$, then $\nu^0_{D_i} = \nu^0_{D_0}$.

We now assume that z is a zero of some functions $Q_i(f)$. Since $t_{k+1} + 1 \ge n + 1$ and z can not be zero of more than n functions $Q_i(f)$, without loss of generality we may assume that z is not zero of $Q_0(f)$. Then

$$\begin{aligned}&\nu^0_{\mathcal{D}^{\alpha_{st_{s-1}+j}}(c_{si}Q_i(f))}(z) \ge \min_{\beta\in\mathbf{Z}^m_+ \text{ with } \alpha_{st_{s-1}+j}-\beta\in\mathbf{Z}^m_+} \{\nu^0_{\mathcal{D}^\beta c_{si}\mathcal{D}^{\alpha_{st_{s-1}+j}-\beta}Q_i(f)}(z)\}\\ &\ge \min_{\beta\in\mathbf{Z}^m_+ \text{ with } \alpha_{st_{s-1}+j}-\beta\in\mathbf{Z}^m_+} \{\max\{0, \nu^0_{Q_i(f)}(z) - |\alpha_{st_{s-1}+j} - \beta|\} - (\beta+1)\nu^\infty_{c_{si}}(z)\}\\ &\ge \max\{0, \nu^0_{Q_i(f)}(z) - N\} - (N+1)\nu^\infty_{c_{si}}(z)\end{aligned}$$

for each $1 \le i \le t_{k+1}, 1 \le j \le t_s - t_{s-1}, 1 \le s \le k+1$, where $t_0 = 0$..

Put $I(z) = (N+1)\sum_{s=1}^{k}\sum_{i=0}^{t_k}(t_s - t_{s-1})\nu^{\infty}_{c_{si}}(z)$. Then

$$\nu_{D_0}(z) \geq \sum_{i=0}^{t_{k+1}} \max\{0, \nu^0_{Q_i(f)}(z) - N\} - I(z). \tag{3.6}$$

We note that if z is not zero of a function $Q_i(f)$ with $i \neq 0$, replacing D_0 by D_i and repeating the same above argument we again get the inequality (3.6). Hence (3.6) holds for all $z \in \mathbf{C}^m$. It follows that

$$\sum_{i=0}^{t_{k+1}} \nu^0_{Q_i(f)}(z) - \nu_{D_0}(z) \leq \sum_{i=0}^{t_k - 1} \min\{N, \nu^0_{Q_i(f)}(z)\} + I(z).$$

Integrating both sides of the above inequality, we get

$$\sum_{i=0}^{t_{k+1}} N_{Q_i(f)}(r) - N_{D_0}(r) \leq \sum_{i=0}^{t_{k+1}} N^{[N]}_{Q_i(f)}(r) + o(T_f(r)).$$

Combining this and (3.5), we get

$$|| \ T_f(r) \leq \sum_{i=0}^{n(N+1)} \frac{1}{d} N^{[N]}_{Q_i(f)}(r) + o(T_f(r)).$$

The lemma is proved. $\square$

Proof of Theorem 1.1.

We first prove the theorem for the case where all Q_i $(i = 1, ..., q)$ have the same degree d. By changing the homogeneous coordinates of $\mathbf{P}^n(\mathbf{C})$ if necessary, we may assume that $a_{iI_1} \not\equiv 0$ for every $i = 1, ..., q$. We set $\tilde{Q}_i = \dfrac{1}{a_{iI_1}} Q_i$. Then $\{\tilde{Q}_i\}_{i=1}^q$ is a set of homogeneous polynomials in $\mathcal{K}_f[x_0, ..., x_n]$ in weakly general position.

Consider $(nN+n+1)$ polynomials $\tilde{Q}_{i_1}, ..., \tilde{Q}_{i_{nN+n+1}}$ $(1 \leq i_j \leq q)$. Applying Lemma 3.2, we have

$$|| \quad T_f(r) \leq \sum_{j=1}^{nN+n+1} \frac{1}{d} N^{[N]}_{\tilde{Q}_i(f)}(r) + o(T_f(r)) \ \leq \sum_{j=1}^{nN+n+1} \frac{1}{d} N^{[N]}_{Q_i(f)}(r) + o(T_f(r)).$$

Taking summing-up of both sides of this inequality over all combinations $\{i_1, ..., i_{nN+n+1}\}$ with $1 \leq i_1 < ... < i_{nN+n+1} \leq q$, we have

$$\Big|\Big| \quad \frac{q}{nN+n+1} T_f(r) \leq \sum_{j=1}^{nN+n+1} \frac{1}{d} N^{[N]}_{Q_i(f)}(r) + o(T_f(r)).$$

The theorem is proved in this case.

We now prove the theorem for the general case where $\deg Q_i = d_i$. Then, applying the above case for f and the moving hypersurfaces $Q_i^{\frac{d}{d_i}}$ $(i = 1, ..., q)$ of common degree d, we have

$$\Big|\Big| \quad \frac{q}{nN+n+1} T_f(r) \le \sum_{j=1}^{q} \frac{1}{d} N^{[N]}_{Q_i^{d/d_i}(f)}(r) + o(T_f(r))$$
$$\le \sum_{j=1}^{q} \frac{1}{d} \frac{d}{d_i} N^{[N]}_{Q_i(f)}(r) + o(T_f(r))$$
$$= \sum_{j=1}^{q} \frac{1}{d_i} N^{[N]}_{Q_i(f)}(r) + o(T_f(r)).$$

The theorem is proved. □

Proof of Theorem 1.2.

By repeating the argument as in the proof of Theorem 1.1, it suffices to prove the theorem for the case where all Q_i have the same degree.

By changing the homogeneous coordinates of $\mathbf{P}^n(\mathbf{C})$ if necessary, we may assume that $a_{iI_1} \not\equiv 0$ for every $i = 1, ..., q$. We set $\tilde{Q}_i = \dfrac{1}{a_{iI_1}} Q_i$. Then $\{\tilde{Q}_i\}_{i=1}^q$ is a set of homogeneous polynomials in $\mathcal{K}_f[x_0, ..., x_n]$ in weakly general position.

Consider $(N+2)$ polynomials $\tilde{Q}_{i_1}, ..., \tilde{Q}_{i_{N+2}}$ $(1 \le i_j \le q)$. We see that $\dim(\tilde{Q}_{i_j}\ ;\ 1 \le j \le N+2)_{\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}} \le N+1 < N+2$. Then the set $\{Q_{i_1}, ..., Q_{i_{N+2}}\}$ is linearly independent over $\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}$. Hence, there exists a minimal subset over $\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}$, for instance that is $\{\tilde{Q}_{i_1}, ..., \tilde{Q}_{i_t}\}$, of $\{\tilde{Q}_{i_1}, ..., \tilde{Q}_{i_{N+2}}\}$. Then, there exist nonzero functions c_j $(1 \le j \le t)$ in $\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}$ such that

$$c_1\tilde{Q}_{i_1} + \cdots + c_t\tilde{Q}_{i_t} = 0.$$

Since $Q_{i_1}, ..., Q_{i_{N+2}}$ are in weakly general position, $t \ge n+2$. Setting $F_j = c_jQ_j(f)$, we have

$$F_1 + \cdots F_{t-1} = -F_t.$$

Choose a meromorphic functions h so that $F = (hF_1 : \cdots : hF_{t-1})$ is a reduced representation of a meromorphic mapping F from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. It is seen that

$$N_h(r) \le \sum_{j=1}^{t-1} (N_{\frac{1}{c_j}}(r) + N_{a_{i_j I_1}}(r)) = o(T_f(r)).$$

On the other hand, by the minimality of the set $\{\tilde{Q}_{i_1}, ..., \tilde{Q}_{i_t}\}$, then F is linearly nondegenerate over $\mathbf{C}$. Applying the second main theorem for fixed hyperplanes, we get

$$\begin{aligned}
\| \ T_F(r) &\le \sum_{j=1}^{t} N^{[t-2]}_{hF_j}(r) + o(T_F(r)) \\
&\le \sum_{j=1}^{t} (N^{[t-2]}_{\tilde{Q}_{i_j}(f)}(r) + N^{[t-2]}_{c_j}(r)) + tN^{[t-2]}_{h}(r) + o(T_F(r)) \\
&= \sum_{j=1}^{t} N^{[t-2]}_{Q_{i_j}(f)}(r) + o(T_f(r)) \le \sum_{j=1}^{N+2} N^{[N]}_{Q_{i_j}(f)}(r) + o(T_f(r)).
\end{aligned}$$

It follows that

$$\| \ T_f(r) = \frac{1}{d} T_F(r) + o(T_f(r)) \le \sum_{j=1}^{N+2} \frac{1}{d} N^{[N]}_{Q_{i_j}(f)}(r) + o(T_f(r)).$$

Taking summing-up of both sides of this inequality over all combinations $\{i_1, ..., i_{N+2}\}$ with $1 \le i_1 < ... < i_{N+2} \le q$, we have

$$\Big\| \quad \frac{q}{N+2} T_f(r) \le \sum_{j=1}^{q} \frac{1}{d} N^{[N]}_{Q_i(f)}(r) + o(T_f(r)).$$

The theorem is proved. $\square$

4 Uniqueness theorem for meromorphic mappings sharing moving hypersurfaces

In order to prove Theorem 1.3 we need the following.

Lemma 4.1. *Let f and g be nonconstant meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let Q_i $(i = 1, ..., q)$ be slow (with respect to f and g) moving hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ in weakly general position with $\deg Q_i = d_i$. Put $d = lcm(d_1, ..., d_q)$ and $N = \binom{n+d}{n} - 1$. Then the following assertions hold:*

(i) If $q > \frac{2N(nN+n+1)}{d}$ then $\| \ T_f(r) = O(T_g(r))$ and $\| \ T_g(r) = O(T_f(r))$.

(ii) If both f and g are algebraically nondegenerate over $\tilde{\mathcal{K}}_{\{Q_i\}_{i=1}^q}$ and $q > \frac{2N(N+2)}{d}$ then $\| \ T_f(r) = O(T_g(r))$ and $\| \ T_g(r) = O(T_f(r))$.

Proof. (i) It is clear that $q > nN + n + 1$. Then applying Theorem 1.1 for f, we have

$$
\begin{aligned}
\| \ \frac{q}{nN+n+1} T_g(r) &\le \sum_{i=1}^{q} \frac{1}{d_i} N^{[N]}_{Q_i(g)}(r) + o(T_g(r)) \\
&\le \sum_{i=1}^{q} \frac{N}{d_i} \ N^{(1)}_{Q_i(g)}(r) + o(T_g(r)) \\
&\le \sum_{i=1}^{q} \frac{N}{d_i} \ N^{(1)}_{Q_i(f)}(r) + o(T_g(r)) \\
&\le qN \ T_f(r) + o(T_g(r)).
\end{aligned}
$$

Hence $\| \ \ T_g(r) = O(T_f(r))$. Similarly, we get $\| \ \ T_f(r) = O(T_g(r))$.

(ii) By applying Theorem 1.2 instead of Theorem 1.1 in the proof of the first assertion, we will get the proof of the second one. □

Proof of Theorem 1.3. We assume that f and g have reduced representations $f = (f_0 : \cdots : f_n)$ and $g = (g_0 : \cdots : g_n)$ respectively.

a) By Lemma 4.1 (i) , we have $\| \ T_f(r) = O(T_g(r))$ and $\| \ T_g(r) = O(T_f(r))$. Suppose that f and g are two distinct maps. Then there exist two index s, t $(0 \le s < t \le n)$ satisfying

$$H := f_s g_t - f_t g_s \not\equiv 0.$$

Set $S = \bigcup \{ \bigcap_{j=0}^{k} \mathrm{Zero} Q_{i_j}(f) \ ; \ 1 \le i_0 < \cdots < i_k \le q \}$. Then S is either an analytic subset of codimension at least two or an empty set.

Assume that z is a zero of some $Q_i(f)$ $(1 \le i \le q)$ and $z \notin S$. Then the condition (iii) yields that z is a zero of the function H. Also, since $z \notin S$, z can not be zero of more than k functions $Q_i(f)$. Therefore, we have

$$\nu^0_H(z) = 1 \ge \frac{1}{k} \sum_{i=1}^{q} \min\{1, \nu^0_{Q_i(f)}(z)\}.$$

This inequality holds for every z outside the analytic subset S of codimension at least two. Then, it follows that

$$N_H(r) \ge \frac{1}{k} \sum_{i=1}^{q} N^{[1]}_{Q_i(f)}(r). \tag{4.2}$$

On the other hand, by the definition of the characteristic function and Jensen formula, we have

$$
\begin{aligned}
N_H(r) &= \int_{S(r)} \log |f_s g_t - f_t g_s| \sigma_m \\
&\le \int_{S(r)} \log ||f|| \sigma_m + \int_{S(r)} \log ||f|| \sigma_m \\
&= T_f(r) + T_g(r).
\end{aligned}
$$

Combining this and (4.2), we obtain

$$T_f(r) + T_g(r) \geq \frac{1}{k}\sum_{i=1}^{q} N^{[1]}_{Q_i(f)}(r).$$

Similarly, we have

$$T_f(r) + T_g(r) \geq \frac{1}{k}\sum_{i=1}^{q} N^{[1]}_{Q_i(g)}(r).$$

Summing-up both sides of the above two inequalities, we have

$$\begin{aligned}
2(T_f(r) + T_g(r)) &\geq \frac{1}{k}\sum_{i=1}^{q} N^{[1]}_{Q_i(f)}(r) + \frac{1}{k}\sum_{i=1}^{q} N^{[1]}_{Q_i(g)}(r)\\
&= \frac{1}{k}\sum_{i=1}^{q} N^{[1]}_{Q_i^{d/d_i}(f)}(r) + \frac{1}{k}\sum_{i=1}^{q} N^{[1]}_{Q_i^{d/d_i}(g)}(r)\\
&\geq \sum_{i=1}^{q} \frac{1}{kN} N^{[N]}_{Q_i^{d/d_i}(f)}(r) + \sum_{i=1}^{q} \frac{1}{kN} N^{[N]}_{Q_i^{d/d_i}(g)}(r). \qquad (4.3)
\end{aligned}$$

From (4.3) and applying Theorem 1.1 for f and g, we have

$$\begin{aligned}
2(T_f(r) + T_g(r)) &\geq \sum_{i=1}^{q} \frac{1}{kN} N^{[N]}_{Q_i^{d/d_i}(f)}(r) + \sum_{i=1}^{q} \frac{1}{kN} N^{[N]}_{Q_i^{d/d_i}(g)}(r)\\
&\geq \frac{d}{kN}\frac{q}{nN+n+1}(T_f(r) + T_g(r)) + o(T_f(r) + T_g(r)).
\end{aligned}$$

Letting $r \longrightarrow +\infty$, we get $2 \geq \frac{d}{kN}\frac{q}{nN+n+1} \Leftrightarrow q \leq \frac{2kN(nN+n+1)}{d}$. This is a contradiction.

Hence $f = g$. The assertion a) is proved.

b) By Lemma 4.1 (ii) , we have $\|\ T_f(r) = O(T_g(r))$ and $\|\ T_g(r) = O(T_f(r))$. Suppose that f and g are two distinct maps. Repeating the same argument as in a), we get the following inequality, which is similar to (4.3),

$$2(T_f(r) + T_g(r)) \geq \sum_{i=1}^{q} \frac{1}{kN} N^{[N]}_{Q_i^{d/d_i}(f)}(r) + \sum_{i=1}^{q} \frac{1}{kN} N^{[N]}_{Q_i^{d/d_i}(g)}(r). \qquad (4.4)$$

From (4.4) and applying Theorem 1.2 for f and g, we have

$$\begin{aligned}
2(T_f(r) + T_g(r)) &\geq \sum_{i=1}^{q} \frac{1}{kN} N^{[N]}_{Q_i^{d/d_i}(f)}(r) + \sum_{i=1}^{q} \frac{1}{kN} N^{[N]}_{Q_i^{d/d_i}(g)}(r)\\
&\geq \frac{d}{kN}\frac{q}{N+2}(T_f(r) + T_g(r)) + o(T_f(r) + T_g(r)).
\end{aligned}$$

Letting $r \longrightarrow +\infty$, we get $2 \geq \frac{d}{kN}\frac{q}{N+2} \Leftrightarrow q \leq \frac{2kN(N+2)}{d}$. This is a contradiction.

Hence $f = g$. The assertion b) is proved. □

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