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Duško Jojić, Ilya Nekrasov, Gaiane Panina and Rade Živaljević

Alexander r-Tuples and Bier Complexes

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Imprint:

ALEXANDER r-TUPLES AND BIER COMPLEXES

DUŠKO JOJIĆ, ILYA NEKRASOV, GAIANE PANINA, AND RADE ŽIVALJEVIĆ

ABSTRACT. We introduce and study *Alexander r-tuples* $\mathcal{K} = \langle K_i \rangle_{i=1}^r$ of simplicial complexes, as a common generalization of pairs of Alexander dual complexes (Alexander 2-tuples) and r-unavoidable complexes of [BFZ-1]. In the same vein, the *Bier complexes*, defined as the deleted joins $\mathcal{K}_\Delta^*$ of Alexander r-tuples, include both standard *Bier spheres* and *optimal multiple chessboard complexes* (Section 2.2) as interesting, special cases.

Our main results are Theorem 4.3 saying that (1) the r-fold deleted join of Alexander r-tuple is a pure complex homotopy equivalent to a wedge of spheres, and (2) the r-fold deleted join of a collective unavoidable r-tuple is $(n - r - 1)$-connected, and a classification theorem (Theorem 5.1 and Corollary 5.2) for Alexander r-tuples and Bier complexes.

1. INTRODUCTION

Topological combinatorics utilizes methods from algebraic (combinatorial) topology to solve problems in combinatorics and discrete geometry. Among the highlights (that strongly influenced the subsequent developments), and early achievements of topological combinatorics are the solution of Kneser conjecture (L. Lovász, 1978), topological Tverberg theorem (I. Bárány, S.B. Shlosman, A. Szűcs, 1981), N. Alon's 'Splitting necklace theorem' (1987), and many others, see [Bjö95, Mat, Ž04] for an overview and introduction.

Simplicial complexes are among the central objects of study in topological combinatorics. Their role in this subject can be compared to the role of manifolds in differential geometry and topology, for illustration R. Forman's 'Discrete Morse theory' (Section 2.4) exemplifies a fruitful interplay of ideas and techniques from these areas.

In this paper we introduce "Alexander r-tuples of simplicial complexes" and closely related "collective r-unavoidable complexes" (Section 3), as unifying concepts that bring together *Alexander pairs* of mutually dual complexes, and *r-unavoidable complexes* of Blagojević, Frick, and Ziegler [BFZ-1, Definition 4.1].

The deleted join operation, applied to an Alexander pair (K, K°), yields a combinatorial sphere $Bier(K) = K *_\Delta K^\circ$, known as the Bier sphere associated to K, see [Mat, Section 5.6]. The special case of a self-dual complex $K = K^\circ \subset 2^{[n]}$ is of particular importance. In this case the Bier

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sphere $Bier(K) = K *_\Delta K$ is a $\mathbb{Z}_2$-complex and its equivariant $\mathbb{Z}_2$-index is $\mathrm{Ind}_{\mathbb{Z}_2}(Bier(K)) = \mathrm{Ind}_{\mathbb{Z}_2}(S^{n-2}) = n - 2$. This fact alone has many interesting consequences, including the Van Kampen-Flores theorem [Mat, Theorem 5.1.1] which says that the d-skeleton $(\sigma^{2d+2})^{\leq d}$ of a $(2d + 2)$-dimensional simplex is non-embeddable in $\mathbb{R}^{2d}$.

The *r-unavoidable complexes* [BFZ-1] play the central role in applications of the 'constraint method' of Blagojević, Frick, and Ziegler. This method, also known under the name 'Gromov-Blagojević-Frick-Ziegler reduction', has found numerous applications to theorems of Tverberg-Van Kampen-Flores type. We refer the reader to [Gr10, Section 2.9(c)] and [BFZ-1] for the original exposition of this beautiful technique (see also our Section 2.5 for a brief overview).

The 2-unavoidable complexes are easily identified as superdual complexes $K \supseteq K^\circ$. From here it easily follows that self-dual complexes are precisely (inclusion) minimal 2-unavoidable complexes.

Moreover, it was shown in [JVZ-3] (Theorem 3.6) that if K is an r-unavoidable complex, then the associated r-fold deleted join $K_\Delta^{*r} = K *_\Delta \cdots *_\Delta K$ is a S_r-complex such that the equivariant G-index $\mathrm{Ind}_G(K_\Delta^*) \geq n - r$ (where $r = p^k$ is a prime power and $G = (\mathbb{Z}_r)^k \subset S_r$ is an elementary abelian group).

The outline above leads to the conclusion that r-unavoidable complexes can be interpreted as r-fold analogues (relatives) of Alexander self-dual complexes, with many nice properties preserved. It may be tempting to extend this analogy further, to include r-fold generalization of (not necessarily symmetric) Alexander dual pairs. The following research problem summarizes the desirable properties of such an extension.

Problem 1.1. Describe a property $\mathcal{P}_r$ of collections $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle$ of simplicial complexes on the same vertex set, $K_i \subset 2^{[n]}$, such that:

(1) If $r = 2$ then a pair of complexes $\langle K_1, K_2 \rangle$ satisfies $\mathcal{P}_2$ if and only if $\langle K_1, K_2 \rangle$ is an Alexander superdual pair in the sense that $K_1 \supseteq K_2^\circ$ (equivalently $K_2 \supseteq K_1^\circ$);

(2) If $K_1 = \cdots = K_r = K$ then $\mathcal{K}$ satisfies $\mathcal{P}_r$ if and only if K is an r-unavoidable complex;

(3) If $\mathcal{K} \in \mathcal{P}_r$ then the deleted join $\mathcal{K}_\Delta^* = K_1 *_\Delta \cdots *_\Delta K_r$ is an $(n - r - 1)$-connected complex.

Moreover, it is desirable to describe a stronger property $\mathcal{P}_r^\sharp \subset \mathcal{P}_r$ such that:

$(1^\sharp)$ $\langle K_1, K_2 \rangle \in \mathcal{P}_2^\sharp$ if and only if $K_1 = K_2^\circ$;

$(2^\sharp)$ If $K_1 = \cdots = K_r = K$ and $\mathcal{K} \in \mathcal{P}_r^\sharp$, then K is an (inclusion) minimal r-unavoidable complex;

$(3^\sharp)$ If $\mathcal{K} \in \mathcal{P}_r^\sharp$ then the deleted join $\mathcal{K}_\Delta^* = K_1 *_\Delta \cdots *_\Delta K_r$ has the homotopy type of a wedge of $(n - r)$-dimensional spheres.

Motivated by Problem 1.1, we describe (Definition 3.1) the class CU_r of "collective r-unavoidable complexes", as our primary candidate for the class

$\mathcal{P}_r$. Individual r-unavoidable complexes often arise from the 'pigeonhole principle' (see [BFZ-1, Lemma 4.2]). For this reason we may occasionally say that an ordered collection $\mathcal{K} = \langle K_i \rangle_{i=1}^{r} \in CU_r$ has the *pigeonhole property,* or that $\mathcal{K}$ itself is a *pigeonhole r-tuple.*

We introduce the class $\mathcal{A}_r$ of "Alexander r-tuples of simplicial complexes" (Definition 3.4), as the most regular class of "collective r-unavoidable complexes", and as our primary candidate for the class $\mathcal{P}_r^{\sharp}$.

Finally, *Bier complexes* (Section 4) arise as the deleted joins of Alexander r-tuples, in perfect analogy with the case of standard Bier spheres, which arise as deleted joins of Alexander pairs of complexes.

1.1. **Summary of the main results.** The core of the paper are the results showing that the collective r-unavoidable complexes (and their deleted joins) as well as the Alexander r-tuples (and the associated Bier complexes) indeed satisfy the properties listed in Problem 1.1. Perhaps the most interesting among them are the following (see Sections 3 and 4).

If $\mathcal{K} = \langle K_i \rangle_{i=1}^{r} = \langle K_1, \ldots, K_r \rangle$ is a collective r-unavoidable collection of subcomplexes of $2^{[n]}$ (Definition 3.1), then (by Problem 1.1 (3)) the associated deleted join $\mathcal{K}_{\Delta}^{*} = K_1 *_{\Delta} \cdots *_{\Delta} K_r$ is expected to be $(n-r-1)$-connected. This is indeed the case, as shown in the first part of Theorem 4.3. In particular we recover the result that $K *_{\Delta} K^{\circ}$ is an $(n-2)$-dimensional homotopy sphere, whenever $K \neq 2^{[n]}$ is *superdual* in the sense that $K \supseteq K^{\circ}$.

In the special case when $\mathcal{K} = \langle K_i \rangle_{i=1}^{r}$ is an Alexander r-tuple (Definition 3.4), we have a stronger result (see the second half of Theorem 4.3), that the associated Bier complex $\mathcal{K}_{\Delta}^{*}$ is a wedge of spheres of the same dimension $n - r$ (Property ($3^{\sharp}$) in Problem 1.1). We describe an algorithm how the number of these spheres can be explicitly calculated (Corollary 4.4) and illustrate the calculation in the case of 'optimal chessboard complexes' (Section 8).

A classification theorem for Alexander r-tuples (Theorem 5.1) is proved in Section 5. It turns out, somewhat unexpectedly and as a pleasant surprise, that the 'optimal chessboard complexes' (introduced in Section 2.2) are the central examples of Bier complexes (Section 4) for $r \geq 3$.

Among the corollaries of our results are exact connectivity bounds for some classes of generalized chessboard complexes (including the main case of Theorem 3.2 from [JVZ-1]). These results are highly relevant for applications to the results of Tverberg-Van Kampen-Flores type. As illustrated by the results in Section 3, our alternative methods provide some new insight complementing both the 'constraint method' of [BFZ-1] and the 'index theory' approach [Mat, JVZ-3].

The rest of the paper is organized as follows. Section 2 is an overview of basic notions and facts, including a brief exposition of the discrete Morse theory [Fo02] (which is our central tool in this paper). We develop a version of this

method which appears to be particularly well adapted for the analysis of Bier spheres (Section 6). We show in Section 7 how the method can be extended to the case of deleted joins of collective r-unavoidable complexes and general Bier complexes (introduced in Section 4). The highlights include the construction of a perfect discrete Morse function in the case of 'optimal chessboard complexes' (Section 8) and their relatives 'long chessboard complexes' (Section 9).

2. An overview of basic definitions and facts

In this section we collect some standard definitions and facts, as a reminder for the reader. This is also an opportunity to introduce some less standard notation and concepts, used in the rest of the paper. For other standard facts and definitions the reader is referred to [Mat].

2.1. **Simplicial complexes.** A simplicial complex on a set V of vertices is a subset $K \subset 2^V$ such that (1) $\emptyset \in K$ and (2) if $A \subset B \in K$ then $A \in K$. By definition it is possible that $\{v\} \notin K$ for some $v \in V$, however $K \neq \emptyset$ (since $\emptyset \in K$ by property (1)).

The complex 2^V is often referred to as the simplex spanned by V, and denoted by $\Delta(V)$. We use, side by side, topological and combinatorial language (and notation). For example,

$$\binom{[n]}{\leqslant k},$$

is the $(k-1)$-skeleton of the $(n-1)$-dimensional simplex $\Delta([n])$.

The *deleted join* [Mat, Section 6] of a family $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle$ of subcomplexes of $2^{[n]}$ is the complex $\mathcal{K}_\Delta^* = K_1 *_\Delta \cdots *_\Delta K_r \subset (2^{[n]})^{*r}$ where $A = A_1 \uplus \cdots \uplus A_r \in \mathcal{K}_\Delta^*$ if and only if A_j are pairwise disjoint and $A_i \in K_i$ for each $i = 1, \ldots, r$.

2.2. **Multiple chessboard complexes.** A 'chessboard complex', in a very broad sense, is any subcomplex $K \subset 2^{([n] \times [r])}$ of the simplex $\Delta([n] \times [r])$ spanned by elementary squares of an $(n \times r)$-chessboard. Following [JVZ-1, Section 2.1], the multiple chessboard complex

$$\Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}} = \Delta_{n,r}^{m_1,\ldots,m_r;1,\ldots,1}$$

is described by the condition that $S \in \Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$ if and only if the cardinality of the set $S \cap ([n] \times \{i\})$ is at most m_i for each $i = 1, \ldots, r$, and the cardinality of the set $S \cap (\{j\} \times [r])$ is at most 1 for each $j = 1, \ldots, n$.

A moment's reflections reveals that there is a relation,

$$\Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}} \cong \begin{pmatrix} [n] \\ \leqslant m_1 \end{pmatrix} *_\Delta \cdots *_\Delta \begin{pmatrix} [n] \\ \leqslant m_r \end{pmatrix},$$

which says that the multiple chessboard complex can be always expressed as the deleted join of skeletons of the simplex $\Delta([n]) \cong \Delta^{n-1}$.

One of the central results of [JVZ-1, Theorem 3.2] says that $\Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$ is $(\nu-2)$-connected where $\nu = m_1+\cdots+m_r$, provided $n \geq m_1+\cdots+m_r+r-1$. For this reason the chessboard complex $\Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$ is often called *optimal*, if $n = m_1+\cdots+m_r+r-1$. Similarly we say that a multiple chessboard complex is *long* if $n > m_1 + \cdots + m_r + r - 1$.

2.3. **Alexander duality and Bier spheres.**

The Alexander dual [Mat, Section 5.6] of $K \subset 2^V$ is the set K° of all complements of non-simplices in K,

$$K^\circ = \{F \subset V \mid V \setminus F \notin K\}.$$

In order to rule out the possibility $K^\circ = \emptyset$, we tacitly assume throughout the paper that $K \neq 2^V$, whenever we are dealing with Alexander pairs (K, K°) of complexes.

For a given simplicial complex $K \subset 2^{[n]}$, the associated Bier sphere $Bier(K) = K *_\Delta K^\circ$ is described as the deleted join of K with its Alexander dual K°. The simplices of the deleted join $K *_\Delta K^\circ$ are by definition disjoint unions $A_1 \uplus A_2 \subset [n] \uplus [n] \cong [n] \times [2]$, where $A_1 \in K, A_2 \in K^\circ$ and $A_1 \cup A_2 \neq \emptyset$. They can be also described as ordered partitions of the set $[n]$ into three parts $(A_1, A_2; B)$ (where $B := [n] \setminus (A_1 \cup A_2)$).

Note that a partition $(A_1, A_2; B)$ corresponds to a simplex in the deleted join $K *_\Delta K^\circ$ if and only if:

  (1) $A_1 \in K$,
  (2) $A_2 \in K^\circ$ (or equivalently $[n] \setminus A_2 \notin K$);
  (3) $\emptyset \neq B \neq [n]$ (equivalently $\emptyset \neq A_1 \cup A_2 \neq [n]$).

  The incidence relation of the simplices is described by the rule:
  $(A_1, A_2; B) \subseteq (A_1', A_2'; B')$ iff $A_1 \subseteq A_1'$, and $A_2 \subseteq A_2'$.

2.4. **Discrete Morse theory.**

Robin Forman's discrete Morse theory [Fo02] is, as a tool, as powerful as the smooth Morse theory. It has been used in computations of the homology, the cup-product, Novikov homology, and other topological and combinatorial computations and applications. Major advantage of discrete Morse theory (compared to smooth Morse theory) is its applicability to a considerably larger class of objects which include simplicial and cellular complexes (and not only smooth manifolds).

In our paper we make use of a relatively small and quite reduced piece of the general theory. For our purposes it suffices to think of a 'Morse function' as a special kind of matching on the set of simplices. Here is a brief overview of some of the central definitions and results of discrete Morse theory.

Let K be a simplicial complex. Its p-dimensional simplices (p-simplices for short) are denoted by α^p, β^p, σ^p, etc. A *discrete vector field* D is a set of pairs (α^p, β^{p+1}) (called a matching) such that:

(1) each simplex of the complex participates in at most one pair, and
(2) in each pair, the simplex α^p is a facet of β^{p+1}.

The pair (α^p, β^{p+1}) can be informally thought of as a vector in the vector field D. For this reason it is occasionally denoted by $\alpha^p \to \beta^{p+1}$ (and in this case β^{p+1} is referred to as *the end* of the arrow $\alpha^p \to \beta^{p+1}$).

Given a discrete vector field D, a *gradient path* in D is a sequence of simplices

$$\alpha_0^p, \ \beta_0^{p+1}, \ \alpha_1^p, \ \beta_1^{p+1}, \ \alpha_2^p, \ \beta_2^{p+1}, ..., \alpha_m^p, \ \beta_m^{p+1}, \ \alpha_{m+1}^p,$$

which satisfies the following conditions:

(1) $p \geq 0$, that is, the empty set $\emptyset \in K$ is never matched,
(2) $(\alpha_i^p, \ \beta_i^{p+1})$ is a pair in D for each i,
(3) for each $i = 0, \ldots, m$ the simplex α_{i+1}^p is a facet of β_i^{p+1}.
(4) $\alpha_i \neq \alpha_{i+1}$.

A path is *closed* if $\alpha_{m+1}^p = \alpha_0^p$. A *discrete Morse function* (DMF for short) is a discrete vector field without closed paths.

Assuming that a discrete Morse function is fixed, the *critical simplices* are those simplices of the complex that are not matched. The Morse inequality [Fo02] states that critical simplices cannot be completely avoided.

A discrete Morse function is a *perfect Morse function* whenever the number of critical k-simplices equals the k-th Betty number of the complex. It is equivalent to the condition that the number of all critical simplices equals the sum of Betty numbers.

Perhaps the main idea of discrete Morse theory, as summarized in the following theorem of R. Forman, is to contract all matched pairs of simplices and to reduce the simplicial complex K to a cell complex (where critical simplices correspond to the cells).

Theorem 2.1. [Fo02] *Assume that a discrete Morse function on a simplicial complex K has a single zero-dimensional critical simplex σ^0 and that all other critical simplices have the same dimension $N > 1$. Then K is homotopy equivalent to a wedge of N-dimensional spheres.*

More generally, if all critical simplices, aside from σ^0, have dimension $\geq N$, then the complex K is $(N-1)$-connected. □

2.5. **The 'constraint method' and 'unavoidable complexes'.** The Gromov-Blagojević-Frick-Ziegler reduction, or the *constraint method*, is an elegant and powerful method for proving results of Tverberg-Van Kampen-Flores type. It relies on the concept of 'unavoidable' or more precisely r-unavoidable complex, where r is a positive integer. The property of being 'unavoidable' is one of the central themes of our paper. For this reason we briefly review the 'constraint method' where this concept originally appeared.

$$(1) \qquad \begin{array}{ccc} K & \xrightarrow{\;f\;} & \mathbb{R}^d \\ e\big\downarrow & & i\big\downarrow \\ \Delta^N & \xrightarrow{\;F\;} & \mathbb{R}^{d+1} \end{array}$$

Suppose that the continuous Tverberg theorem holds for the triple $(\Delta^N, r, \mathbb{R}^{d+1})$ in the sense that for each continuous map $F : \Delta^N \to \mathbb{R}^{d+1}$ there exists a collection of r vertex disjoint simplices $\Delta_1, \ldots, \Delta_r$ of Δ^N such that $F(\Delta_1) \cap \ldots \cap F(\Delta_r) \neq \emptyset$. For example the *Topological Tverberg theorem* [Mat, Section 6] (proved by Bárány, Shlosman, and Szüs for primes, and Özaydin for prime powers) says that this is the case if $r = p^k$ is a prime power and $N = (r-1)(d+2)$. Suppose that $K \subset \Delta^N$ is a simplicial complex which is r-*unavoidable* in the sense that if $A_1 \uplus \ldots \uplus A_r = [N+1]$ is a partition of the set $[N+1]$ (of vertices of Δ), then at least one of the simplices A_i of Δ^N is in K. Then for each continuous map $f : K \to \mathbb{R}^d$ there exists vertex disjoint simplices $\sigma_1, \ldots, \sigma_r \in K$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

Indeed, let F' be an extension ($F' \circ e = f$) of the map f to Δ^N. Suppose that $\rho : \Delta^N \to \mathbb{R}$ is the function $\rho(x) := \mathrm{dist}(x, K)$, measuring the distance of the point $x \in \Delta^N$ from K. Define $F = (F', \rho) : \Delta^N \to \mathbb{R}^{d+1}$ and assume that $\Delta_1, \ldots, \Delta_r$ is the associated family of vertex disjoint simplices of Δ^N, such that $F(\Delta_1) \cap \ldots \cap F(\Delta_r) \neq \emptyset$. More explicitly suppose that $x_i \in \Delta_i$ such that $F(x_i) = F(x_j)$ for each $i, j = 1, \ldots, r$. Since K is r-unavoidable, $\Delta_i \in K$ for some i. As a consequence $\rho(x_i) = 0$, and in turn $\rho(x_j) = 0$ for each $j = 1, \ldots, r$. If Δ_i' is the minimal simplex of Δ^N containing x_i then $\Delta_i' \in K$ for each $i = 1, \ldots, r$ and $f(\Delta_1') \cap \ldots \cap f(\Delta_r') \neq \emptyset$.

The reader is referred to [BFZ-1] for a more complete exposition and numerous examples of applications of the 'constraint method', see also [Gr10, Section 2.9(c)] for the historically first appearance of the idea.

3. Collectively unavoidable r-tuples and Alexander r-tuples of complexes

In this section we introduce the central objects of our paper. Our tacit assumption is that all complexes K are *proper subcomplexes* of $2^{[n]}$ in the sense that $K \subsetneq 2^{[n]}$.

Definition 3.1. An ordered r-tuple $\mathcal{K} = \langle K_1, ..., K_r \rangle$ of subcomplexes of $2^{[n]}$ is *collective r-unavoidable* (we also say that $\mathcal{K}$ is a *pigeonhole r-tuple* on $[n]$), if for each ordered collection $(A_1, ..., A_r)$ of disjoint sets in $[n]$ there exists i such that $A_i \in K_i$. The class of collective r-unavoidable complexes is denoted by $CU_{r,n}$, or by CU_r if n is fixed or clear from the context.

On closer inspection, the definition can be usefully rephrased as follows. For the ordered r-tuple $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$ and for an ordered disjoint collection $(A_1, \ldots, A_r)$ of subsets of $[n]$, we construct a bipartite graph $\Gamma \subset K_{r,r}$, where

by definition there is an edge $(i, j) \in \Gamma$ if and only if $A_i \notin K_j$. Then the collective r-unavoidability of $\mathcal{K}$ is equivalent to the condition that the graph Γ does not contain a complete matching (does not satisfy the *marriage condition* of the classical Hall's theorem). We therefore conclude that the pigeonhole property does not depend on the ordering of simplicial complexes.

Remark 3.2. The bipartite graph $\Gamma = \{(i, j) \in [r]^2 \mid A_i \notin K_j\}$ interpretation naturally leads to an extension of Definition 3.1 to the case of collections $\mathcal{K} = \langle K_1, ..., K_s \rangle$ where s is not necessarily equal to r. Note however that the symmetric case $s = r$ is somewhat exceptional. For example the classical 'Hilfssatz' of Frobenius [Sch] implies that $\mathcal{K} = \langle K_1, ..., K_r \rangle$ is collective r-unavoidable if and only if for each ordered collection $(A_1, ..., A_r)$ of disjoint sets in $[n]$ there exists a pair (S, T) of subsets of $[r]$, such that $|S| + |T| = r + 1$, and $A_i \in K_j$ for each $i \in S$ and $j \in T$.

It is easy to characterize all pigeonhole 2-tuples: (K_1, K_2) is collective unavoidable if and only if $K_1^\circ \subset K_2$ (or equivalently $K_2^\circ \subset K_1$).

For an r-tuple of complexes $\langle K_1, ..., K_r \rangle$ we shall use a natural partial ordering on the set of all set of pairwise disjoint r-tuples $(A_1, ..., A_r)$ with $A_i \in K_i$: say that $(A_1, ..., A_r) \leqslant (A_1', ..., A_r')$ whenever $\forall i : A_i \subseteq A_i'$.

We also put a partial ordering on the set of all r-tuples of complexes by the same rule. So we automatically have the notion of *minimal unavoidable r-tuple of complexes* $\langle K_1, ..., K_r \rangle$.

Lemma 3.3. *Suppose that the r-tuple $\mathcal{K} = \langle K_1, ..., K_r \rangle$ is collective r-unavoidable. Then for each maximal disjoint collection $(A_1, ..., A_r)$ with $A_i \in K_i$, the set $[n] \setminus \bigcup_{i=1}^{r} A_i$ contains at most $r - 1$ elements.*

Proof. Suppose that $\langle K_1, ..., K_r \rangle$ is collective r-unavoidable. Let $(A_1, ..., A_r)$ be a maximal disjoint collection satisfying the condition $A_i \in K_i$ for each $i = 1, ..., r$. Suppose (for contradiction) that $\{a_1, ..., a_r\} \subset [n] \setminus \bigcup_{i=1}^{r} A_i$. Then $A_i' = A_i \cup \{a_i\} \notin K_i$ (by the maximality of the collection $(A_1, ..., A_r)$) and the collection $(A_1', ..., A_r')$ clearly violates the collective r-unavoidability condition for $\langle K_1, ..., K_r \rangle$. $\square$

Definition 3.4. An r-tuple of complexes $\mathcal{K} = \langle K_1, ..., K_r \rangle$ on one and the same set of vertices $[n]$ is an *Alexander r-tuple* if,

    (1) it is collective r-unavoidable, and
    (2) for each r-tuple of sets $A_1, ..., A_r$ with $A_i \in K_i$ the set $[n] \setminus \bigcup_{i=1}^{r} A_i$ has at least $r - 1$ elements.

The class of Alexander r-tuples of subcomplexes of $2^{[n]}$ is denoted by $\mathcal{A}_r$ (or by $\mathcal{A}_{r,n}$ if the set $[n]$ of vertices should be emphasized).

Proposition 3.5. *Given an Alexander r-tuple on $[n]$, for each maximal r-tuple of disjoint sets $(A_1, ..., A_r)$ with $A_i \in K_i$ the set $[n] \setminus \bigcup_{i=1}^{r} A_i$ has exactly $r - 1$ elements.*

Proof. This follows from Lemma 3.3 and the property (2) from the definition of the Alexander r-tuple (Definition 3.4). $\square$

Proposition 3.6. *An Alexander r-tuple of complexes is always a minimal pigeonhole r-tuple of complexes.*

Proof. Assume $\langle K_1, \ldots, K_r \rangle$ is an Alexander r-tuple which is not a minimal collective r-unavoidable collection of complexes. This means that (possibly after a re-enumeration) the collection $\langle K_1 \setminus \{A_1\}, K_2, \ldots, K_r \rangle$ is also collective r-unavoidable for some maximal simplex $A_1 \in K_1$. As a consequence the restrictions $\langle K_2|_{[n]\setminus A_1}, \ldots, K_r|_{[n]\setminus A_1} \rangle$ form a collective $(r-1)$-unavoidable family of complexes. Lemma 3.3 implies that for any maximal disjoint collection $(A_2, \ldots, A_r)$ such that $A_j \in K_j|_{[n]\setminus A}$ for each $j = 2, \ldots, r$, the set $[n] \setminus \bigcup_{i=1}^{r} A_i$ contains strictly less than $r - 1$ elements. Then $(A_1, \ldots, A_r)$ is a maximal family satisfying $A_i \in K_i$ for each $i = 1, \ldots, r$, which is in contradiction with the condition (2) from Definition 3.4. $\qquad\square$

The converse of Proposition 3.6 is in general not true.

Example 3.7.

$$K_1 = K_2 = K_3 = \binom{[10]}{\leqslant 2} \sqcup \binom{[9]}{\leqslant 3}$$

is a minimal collective unavoidable 3-tuple which is not an Alexander 3-tuple.

Example 3.8. *A 2-tuple of complexes is an Alexander 2-tuple iff it is a pair of mutually dual complexes (K, K°).*

The following examples describes the Alexander r-tuples $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$ where each of the complexes K_i is a skeleton of the simplex $2^{[n]}$.

Example 3.9. *The collection of subcomplexes of $2^{[n]}$,*

$$\left(\binom{[n]}{\leqslant m_1}, \ldots, \binom{[n]}{\leqslant m_r} \right)$$

is always an Alexander r-tuple, provided $n = \sum_{i=1}^{r} m_i + r - 1$.

Example 3.10. *Define a simplicial complex $K \subset 2^{[6]}$ as the cone with apex 1 over the five-element set $\{2, 3, 4, 5, 6\}$. The complex K is essentially a graph with five edges $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}$. It is not difficult to see that $\langle K, K, K \rangle$ is indeed an Alexander 3-tuple.*

This example is the simplest case of a more general construction. For given integers $m_1, m_2, \ldots, m_r$, let $n = m_1 + m_2 + \cdots + m_r + r - 1$. Choose a simplex $\Delta(C)$ spanned by $C \neq \emptyset$ (where $[n] \cap C = \emptyset$) and define the complexes,

$$K_i = \binom{[n]}{\leqslant m_i} * \Delta(C) \text{ , for all } i = 1, 2, \ldots, r.$$

It can be easily seen that $\langle K_1, K_2, \ldots, K_r \rangle$ is an Alexander r-tuple.

3.1. **Operations generating collective r-unavoidable complexes.** As demonstrated by the classification theorem (Theorem 5.1), Alexander r-tuples are scarce, and a very special class of simplicial complexes. The situation with the collective r-unavoidable complexes is quite the opposite, as illustrated by the following construction.

Let $r \geq 2$ and let $K_i \subset 2^{[n]}$ be a collection of not necessarily distinct simplicial complexes. Assume that the $(r-1)$-tuple $\mathcal{K} = \langle K_1, K_2, \ldots, K_{r-1} \rangle$ is NOT collective $(r-1)$-unavoidable on $[n]$.

Define $R(\mathcal{K}) = R_r(\mathcal{K}) = R_r(K_1, K_2, \ldots, K_{r-1})$ as the subcomplex of $2^{[n]}$ where $F \in R(\mathcal{K})$ if and only if there exists an ordered partition $F_1 \uplus \cdots \uplus F_{r-1} = F^c$ of the complement of F such that $F_i \notin K_i$ for each $i = 1, \ldots, r-1$.

Observe that $R(\mathcal{K})$ is generated by the sets $(F_1 \cup \cdots \cup F_{r-1})^c$ where F_j are pairwise disjoint and F_j is a minimal non-face in K_j for each $j = 1, \ldots, r-1$.

Note that $R_r(\mathcal{K})$ can be described as the unique minimal simplicial complex Z such that $\langle K_1, \ldots, K_{r-1}, Z \rangle$ is a collective unavoidable r-tuple on $[n]$. Observe that $\emptyset \in R_r(\mathcal{K})$ follows from the assumption that $\mathcal{K}$ is not $(r-1)$-unavoidable.

Definition 3.11. The complex $R_r(\mathcal{K})$ is referred to as the *residual complex* of the $(r-1)$-tuple $\mathcal{K} = \langle K_1, K_2, \ldots, K_{r-1} \rangle$.

Observe that in the case $r = 2$ the residual complex of $K \subset 2^{[n]}$ is precisely the Alexander dual, $R(K) = K^\circ$. More generally, for a complex $K \subseteq 2^{[n]}$ we define the associated r^{th} residual complex $R_r(K) = R(K_1, \ldots, K_{r-1})$ where $K_1 = \cdots = K_{r-1} = K$. Note that K is a minimal r-unavoidable complex if and only if $R_r(K) = K$.

Problem 3.12. Find interesting examples of ordered collections of complexes $\mathcal{K} = \langle K_1, K_2, \ldots, K_{r-1} \rangle$ such that $\langle K_1, K_2, \ldots, K_{r-1}, R(\mathcal{K}) \rangle$ satisfies the condition $(3^\sharp)$ (in Problem 1.1).

4. Bier complexes

For each Alexander 2-tuple $\langle K_1, K_2 \rangle = \langle K_1, K_1^\circ \rangle = \langle K_2^\circ, K_2 \rangle$, the associated deleted join $K_1 *_\Delta K_2$ is the standard Bier sphere $Bier(K_1) \cong Bier(K_2)$ (Example 3.8). This observation motivates the following definition.

Definition 4.1. Suppose that $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$ is an Alexander r-tuple of complexes $K_i \subset 2^{[n]}$. Then the associated *Bier complex* is defined as the deleted join,

$$Bier(\mathcal{K}) := \mathcal{K}_\Delta^{*r} = K_1 *_\Delta K_2 *_\Delta \ldots *_\Delta K_r.$$

It is well known that the 'join' and the 'deleted join' operations commute (see for example Lemma 6.4.3. in [Mat]). The following lemma is a natural generalization.

Lemma 4.2. *Let $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$ and $\mathcal{L} = \langle L_1, \ldots, L_r \rangle$ be two collections of simplicial complexes where $K_i \subset 2^{[m]}$ and $L_i \subset 2^{[n]}$ for each $i = 1, \ldots, r$. Then,*

$$(2) \qquad\qquad (\mathcal{K} * \mathcal{L})^{*r}_\Delta \cong \mathcal{K}^{*r}_\Delta * \mathcal{L}^{*r}_\Delta$$

where by definition $\mathcal{K} * \mathcal{L} := \langle K_1 * L_1, \ldots, K_r * L_r \rangle$.

Proof. If $A = A_1 \uplus \cdots \uplus A_r \in \mathcal{K}^{*r}_\Delta$ and $B = B_1 \uplus \cdots \uplus B_r \in \mathcal{L}^{*r}_\Delta$ then $A * B \in \mathcal{K}^{*r}_\Delta * \mathcal{L}^{*r}_\Delta$ corresponds to the simplex $C_1 \uplus \cdots \uplus C_r \in (\mathcal{K} * \mathcal{L})^{*r}_\Delta$ where $C_i := A_i \uplus B_i$ for each $i = 1, \ldots, r$. $\qquad\square$

The following theorem is one of the main results of our paper. It says that the classes CU_r and $\mathcal{A}_r$ respectively, satisfy the central properties (3) and $(3^\sharp)$, listed in Problem 1.1.

Theorem 4.3. *Let $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$ be a collection of subcomplexes of $2^{[n]}$.*

(1) *The deleted join $\mathcal{K}^{*r}_\Delta = K_1 *_\Delta K_2 *_\Delta \ldots *_\Delta K_r$ of a collective r-unavoidable collection $\mathcal{K}$ of complexes is always $(n - r - 1)$-connected.*

(2) *The Bier complex $Bier(\mathcal{K}) = K_1 *_\Delta K_2 *_\Delta \ldots *_\Delta K_r$, associated to an Alexander r-tuple $\mathcal{K}$, is a pure complex of dimension $n - r$, homotopy equivalent to a wedge of $(n - r)$-dimensional spheres.*

The following corollary of the proof of Theorem 4.3 emphasizes the computational efficiency of the approach based on the discrete Morse function described in Section 7.

Corollary 4.4. *For an Alexander r-tuple $\mathcal{K}$ the number of spheres in the wedge $Bier(\mathcal{K})$ can be efficiently calculated as the number of critical simplices of the discrete Morse function D constructed in Section 7.*

The efficiency of the method is illustrated in Section 8 by the calculation of the number of spheres in the important particular case of the optimal multiple chessboard complex (Section 2.2).

Recall that the number of spheres in a wedge decomposition can be in principle calculated as the reduced Euler characteristic of the complex. This calculation is typically very slow and inefficient, as it is based on an 'inclusion-exclusion' type formula which involves enumeration of all simplices in $Bier(\mathcal{K})$.

One of important motivations for introducing (collective) r-unavoidable complexes are applications to problems of Tverberg-Van Kampen-Flores type. By emphasizing the role of Theorem 4.3, the following corollaries provide some initial evidence illustrating this interesting and important connection.

Corollary 4.5. *([Mat, Theorem 5.5.5], [JVZ-3, Theorem 3.6]) Suppose that K is an r-unavoidable complex with vertices in $[n]$. Suppose that $r = p^k$ is a prime power and let $G = (\mathbb{Z}_p)^k$ be an elementary abelian p-group acting freely on the set $[r]$. Let K^{*r}_Δ be the r-fold deleted join of K. Then,*

$$(3) \qquad\qquad \mathrm{Ind}^\sharp_G(K^{*r}_\Delta) \geq n - r,$$

where $\mathrm{Ind}^\sharp_G$ is the equivariant index function described in [JVZ-3, Section 3].

Proof. If $K \subset 2^{[n]}$ is r-unavoidable then the collection $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$, where $K_1 = \cdots = K_r = K$, is a collective r-unavoidable collection of complexes. By Theorem 4.3 the deleted join K_Δ^{*r} is $(n-r-1)$-connected. The inequality (3) follows from this observation and the basic properties of the index function $\mathrm{Ind}_G^\sharp$, see for example Proposition 3.3 (inequality (5)) in [JVZ-3]. $\square$

The following result is a simplest example illustrating the role of r-unavoidable complexes in Tverberg type problems. For a more general theorem of this type the reader is referred to [BFZ-1, Theorem 4.4], see also [JVZ-3, Theorem 4.6] for a related result.

Corollary 4.6. ([BFZ-1]) *Suppose that $K \subset 2^{[n]}$ is an r-unavoidable complex. Assume that $r = p^k$ is a prime power and let d be the integer satisfying the inequality $(r-1)(d+2)+1 \leq n$. Then K is* globally r-non-embeddable in $\mathbb{R}^d$ *in the sense that for each continuous map $f : K \to \mathbb{R}^d$ there exist r vertex-disjoint simplices $\Delta_1, \ldots, \Delta_r$ of K such that,*

$$f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset.$$

Proof. The most elegant proof of this result is by the 'constraint method' [BFZ-1] (see Section 2.5 for an outline). The 'index theory proof', in the spirit of [Mat, Section 6] and [JVZ-3], is based on Corollary 4.5. $\square$

Remark 4.7. Let us observe that the 'Gromov-Blagojević-Frick-Ziegler reduction' (the 'constrain method') reduces a Van Kampen-Flores (or Tverberg) type question, to another result of that type. More explicitly (and more generally) the method says that the question if there exists a map $f : K \to \mathbb{R}^d$ without (global) r-fold points (Tverberg r-tuples) can be reduced to a similar problem for an appropriate map $F : \Sigma \to \mathbb{R}^D$. Here $K \subset \Sigma$ is a complex which is *relatively r-unavoidable subcomplex of Σ* in the sense of [JVZ-3, Definition 2.5].

This reasoning illustrates why the 'index theory methods' (which rely on results of Dold and Volovikov, see [Mat, Section 6.2.6]) retain their importance. This also explains why the results like Theorem 4.3 may be interesting since both the Dold's and the Volovikov's theorem are based on the homotopical (respectively homological) connectivity of the associated configuration space (deleted join).

For illustration, Theorem 2.1 from [JVZ-2], that needs such a connectivity result for its proof, is possibly a good candidate for a Tverberg-Van Kampen-Flores type result that cannot be obtained directly by the 'constraint method'.

4.1. **Bier complexes and discrete Morse theory.** The proof of Theorem 4.3 (Section 7) and the proofs of other connectivity results in this paper rely on Discrete Morse theory (Theorem 2.1). All our discrete Morse functions (DMF) are defined on deleted joins $\mathcal{K}_\Delta^{*r} = K_1 *_\Delta \cdots *_\Delta K_r$ of complexes $K_i \subset 2^{[n]}$ and they all have some common features.

A simplex $\beta \in \mathcal{K}_\Delta^{*r}$ is usually recorded as a disjoint sum $\beta = A_1 \uplus \cdots \uplus A_r$, see [Mat, Sections 5 and 6]. We find it convenient (for bookkeeping purposes)

to use an alternative 'partition notation' $\beta = (A_1, \ldots, A_r; B)$ where $B = [n] \setminus \cup_{i=1}^{r} A_i$. To match a p-simplex $\alpha^p = (A_1', \ldots, A_r'; B')$ with a $(p+1)$-simplex $\beta^{p+1} = (A_1, \ldots, A_r; B)$ is the same as to 'migrate' an element $i \in B'$ to one of the sets A_j'. This is possible if α^p is a facet of β^{p+1} i.e. if $B' = B \uplus \{i\}$ for some $i \in B'$.

Caveat: In the paper, we simplify the notation by omitting the braces and by writing simply $B \cup i$ instead of $B \cup \{i\}$ (with the tacit assumption that $i \notin B$). We also write $j < B$ ($j > B$) if $j < i$ for each $i \in B$ (respectively if $j > i$ for each $i \in B$).

5. CLASSIFICATION THEOREM FOR ALEXANDER r-TUPLES

Theorem 5.1. *If $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$ is an Alexander r-tuple then,*

(1) *$r = 2$ and $(K_1, K_2) = (K, K^\circ)$ is an Alexander pair of dual complexes (Example 3.8), or*

(2) *$r \geq 3$ and $K_i = \binom{[n]}{\leq m_i}$ (Example 3.9) where $n = m_1 + \cdots + m_r + r - 1$, or*

(3) *$r \geq 3$ and $K_i = \binom{[n]}{\leq m_i} * \Delta(C)$ (Example 3.10) where $n = m_1 + \cdots + m_r + r - 1$ and $\Delta(C) = 2^C$ is the simplex spanned by a non-empty set C such that $C \cap [n] = \emptyset$.*

Proof. Suppose that $r \geq 3$. A minimal non-simplex of a simplicial complex $K \subset 2^{[n]}$ is called a K-blocker. Equivalently, $A \subset [n]$ is a K-blocker if $A \notin K$ and $\partial(A) \subset K$.

Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a maximal disjoint r-tuple of sets in $[n]$ such that $A_i \in K_i$ for each $i = 1, \ldots, r$. Moreover, we assume that A_r has the maximal size possible in all such r-tuples.

Since $\mathcal{K}$ is an Alexander r-tuple the set $[n] \setminus \bigcup_{i=1}^{r} A_i = \{t_1, \ldots, t_{r-1}\}$ has exactly $(r-1)$ elements (Proposition 3.5).

Let $X_{\mathcal{A}} = X_1 \uplus \cdots \uplus X_{r-1} \uplus X_r$ be the associated 'blocker partition' where $X_i := A_i \cup \{t_i\}$ for each $i = 1, \ldots, r-1$ and $X_r := A_r = [n] \setminus \bigcup_{i=1}^{r-1} X_i$. The name is justified by the fact that X_i is a K_i-blocker for each $i = 1, \ldots, r-1$.

Indeed, suppose that X_ν is not a K_ν-blocker for some $\nu = 1, \ldots, r-1$, which means that there exists $x \in A_\nu$ such that $X_\nu \setminus \{x\} \notin K_\nu$. The maximality of A_r implies that $A_r \cup \{x\} \notin K_r$. This is a contradiction since the partition $\mathcal{Z} = \langle Z_1, \ldots, Z_r \rangle$ where $Z_\nu := X_\nu \setminus \{x\}, Z_r := A_r \cup \{x\}$ and $Z_j = X_j$ for $j \notin \{\nu, r\}$ clearly violates the condition that $\mathcal{K}$ is collective r-unavoidable.

Let $V \subset [n]$. We say that a simplicial complex $K \subset 2^{[n]}$ is V-homogeneous if $S \in K \Leftrightarrow \phi(S) \in K$ for each permutation $\phi : V \to V$ and each $S \subset V$.

Claim 1. Each of the complexes $\{K_j\}_{j=1}^{r-1}$ is X-homogeneous where $X = \bigcup_{j=1}^{r-1} X_j = [n] \setminus A_r$.

Proof of the Claim 1: Let us show for illustration that K_1 is X-homogeneous. This is deduced from the observation that for each bijection $\phi : X \to X$,

(4)   (a) $\phi(X_1) \notin K_1$ and   (b) $\phi(\partial(X_1)) \subset K_1$.

This is obvious if $\phi(X_1) = X_1$. Moreover, it is sufficient to establish (4) in the case when ϕ is a transposition, say $\phi(x_1) = x_2$ where $x_1 \in X_1$ and $x_2 \in X_2$ (the case $x_2 \in X_j$ for $j > 2$ is treated similarly).

(a) is equivalent to $(X_1 \setminus \{x_1\}) \cup \{x_2\} \notin K_1$. This is true since otherwise,

$$(X_1 \setminus \{x_1\}) \cup \{x_2\} \in K_1, \ X_2 \setminus \{x_2\} \in K_2, \ A_3 \in K_3, \ldots, A_{r-1} \in K_{r-1}, \ A_r \in K_r$$

would be a disjoint family of sets covering all but $(r-2)$ elements of $[n]$ (contradicting (2) in Definition 3.4).

In order to prove (b) let $X_1 \setminus \{y\}$ be a facet of $\partial(X_1)$ (the interesting case is $y \neq x_1$). Then, $X_1' := \phi(X_1 \setminus \{y\}) = (X_1 \setminus \{x_1, y\}) \cup \{x_2\} \in K_1$. Otherwise the disjoint collection,

$$X_1' \notin K_1, \ (X_2 \setminus \{x_2\}) \cup \{x_1\} \notin K_2, \ X_3 \notin K_3, \ldots, X_{r-1} \notin K_{r-1}, \ X_r \cup \{y\} \notin K_r$$

would violate the collective r-unavoidability of $\mathcal{K}$. (Note that $(X_2 \setminus \{x_2\}) \cup \{x_1\} = \phi(X_2) \notin K_2$ follows from (a).)   □

Summarizing, we have so far established that for each $j = 1, \ldots, r-1$ the restriction of K_j on X is the complex $\binom{X}{\leq m_j}$ where m_i is the cardinality of the set A_i. In particular the sets $A_1, A_2, \ldots, A_{r-1}$ can be replaced by any disjoint family $A_1', A_2', \ldots, A_{r-1}'$ of subsets of X such that $|A_i'| = m_i$ for each i.

Claim 2. If $x \in X$ then $X_r \cup \{x\} = A_r \cup \{x\} \notin K_r$.

Proof of the Claim 2: By Claim 1 we can assume that $x \in X \setminus \bigcup_{i=1}^{r-1} A_i$. Then the assumption $A_r' := A_r \cup \{x\} \in K_r$ would contradicts the fact that $\mathcal{A} = (A_1, \ldots, A_r)$ is a maximal disjoint r-tuple of sets in $[n]$ such that $A_i \in K_i$ for each $i = 1, \ldots, r$.   □

It follows from Claim 2 that either $X_r \cup \{x\} = A_r \cup \{x\}$ is a K_r-blocker (this corresponds to the case (2) of the theorem) or there exists a proper subset $S \subset X_r$ such that $S \cup \{x\}$ is a K_r-blocker. The following claim makes this observation more precise by showing (eventually) that $S \subset X_r$ is unique with this property (and in particular independent of x).

Claim 3. Choose $x \in X$. Let $X_r = S \uplus C$ be a partition of X_r such that $\{x\} \cup S$ is a K_r-blocker, i.e. such that $\{x\} \cup S \notin K_r$ and $\partial(\{x\} \cup S) \subset K_r$. Then $T \cup C \in K_i$ for each $T \subset X \setminus \{x\}$ of cardinality m_i where $i = 1, \ldots, r-1$. Moreover, $T \cup C$ is a facet (maximal simplex) of K_i.

Remark. The case $C = \emptyset$ is NOT ruled out. As it will turn out from the proof if $X_r = S' \uplus C'$ is another decomposition such that $\{x\} \cup S'$ is a K_r-blocker then $S' = S$ and $C' = C$.

Proof of the Claim 3: Assume that $i = 1$ (the proof in other cases is analogous). Since the sets $A_1, \ldots, A_{r-1}$ can be replaced by any disjoint family $A_1', \ldots, A_{r-1}'$

of subsets of X such that $|A_i'| = m_i$ for each i, we assume that $T = A_1$. For a similar reason we can assume that $x \notin \bigcup_{i=1}^{r-1} A_i$.

Then $A_1 \cup C \in K_1$ since otherwise,

$$A_1 \cup C \notin K_1, \ X_2 \notin K_2, \ldots, X_{r-1} \notin K_{r-1}, \ \{x\} \cup S \notin K_r$$

would violate the collective r-unavoidability of $\mathcal{K}$.

Suppose that $A_1 \cup C$ is not a facet of K_1. It follows that $A_1 \cup C \cup \{z\} \in K_1$ for some $z \in S$, hence $(S \cup \{x\}) \setminus \{z\} \in K_r$. This is a contradiction since the family,

$$A_1 \cup C \cup \{z\} \in K_1, \ A_2 \in K_2, \ A_3 \in K_3, \ldots, A_{r-1} \in K_{r-1}, \ (S \cup \{x\}) \setminus \{z\} \in K_r$$

is a disjoint family of sets covering all but $(r-2)$ elements of $[n]$ (contradicting (2) in Definition 3.4). $\qquad\square$

It follows from Claim 3 that K_1 and K_r can interchange roles. More explicitly $B_r := S$ can be included in a disjoint family $\{B_j\}_{j=1}^r$ (replacing the family $\{A_i\}_{i=1}^r$) where $B_j = A_j$ for each $j = 2, \ldots, r-1$ and $B_1 := A_1 \cup C$.

In light of Claim 1 each of the complexes $\{K_j\}_{j=2}^r$ is Y-homogeneous where $Y = [n] \setminus (\{x\} \cup B_1)$ and x is an arbitrary element in $[n] \setminus B_1$. Moreover the decomposition $B_1 = A_1 \uplus C$ corresponds to the decomposition $X_r = A_r = S \uplus C$ in Claim 3.

From here it is not difficult to conclude that for each $i = 1, \ldots, r$ there is a decomposition $K_i \cong W_i * F$ where $W_i \cong \binom{[n]}{\leqslant m_i}$ and F is either empty or $F = \Delta(C)$ is the simplex spanned by a finite, non-empty set C. $\qquad\square$

Corollary 5.2. *If $K = K_1 *_\Delta \cdots *_\Delta K_r$ is a Bier complex (Definition 4.1) then either,*

(1) *$r = 2$ and $K = K_1 *_\Delta K_2 = K *_\Delta K^\circ$ is a Bier sphere, or*
(2) *$r \geq 3$ and $K = \Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$ is an optimal chessboard complex where $n = m_1 + \cdots + m_r + r - 1$ (Section 2.2), or*
(3) *$r \geq 3$ and $K = \Delta * [r]^{*k}$ where $\Delta = \Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$ is an optimal chessboard complex and $[r]^{*k} = [r] * \cdots * [r]$ is the join of $k \geq 1$ copies of the 0-dimensional complex $[r]$.*

(Note that (2) is a formal 'consequence' of (3) if we allow $k = 0$.)

Proof. Assume $r \geq 3$. It follows from Theorem 5.1 and Lemma 4.2 that $K \cong E * F$ where (Section 2.2) $E = \Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$ is an optimal chessboard complex and F is either empty (the case (2)) or $F = (\Delta(C))_\Delta^{*r}$ for a non-empty set C of cardinality $|C| = k$. The proof is completed by the observation that,

$$(\Delta(C))_\Delta^{*r} \cong (\{p\}^{*k})_\Delta^{*r} \cong (\{p\}_\Delta^{*r})^{*k} \cong [r]^{*k}.$$

6. Two perfect discrete Morse functions on the Bier sphere

We illustrate the method of constructing DMF on deleted joins (by the method of 'migrating elements') first in the case of classical Bier spheres.

It is known that Bier spheres are always shellable, see [BPSZ]. A method of Chari [Cha] can be used to turn this shelling into a perfect DMF on a Bier sphere. The construction of the 'first perfect DMF' on a Bier sphere (Section 6.1) essentially follows this path.

The 'second perfect DMF' (Section 6.2) differs from the first DMF, although the 'migration rules' look very similar. The advantage of the second DMF is that it can be generalized to Alexander r-tuples and the associated Bier complexes.

6.1. **First perfect DMF.** We construct a discrete vector field D_1 on the Bier sphere $Bier(K)$ in two steps:

(1) We match the simplices

$$\alpha = (A_1, A_2; B \cup i) \text{ and } \beta = (A_1, A_2 \cup i; B)$$

iff the following holds:
   (a) $i < B$, $i < A_2$
       (that is, i is smaller than all the entries of B and A_2).
   (b) $A_2 \cup i \in K^\circ$.

Before we pass to step 2, let us observe that the non-matched simplices are labelled by $(A_1, A_2; B \cup i)$ such that $A_2 \in K^\circ$, but $A_2 \cup i \notin K^\circ$. As a consequence, for non-matched simplices $A_1 \cup B \in K$.

(2) In the second step we match together the simplices

$$\alpha = (A_1, A_2; B \cup j) \text{ and } \beta = (A_1 \cup j, A_2; B)$$

iff the following holds:
   (a) None of the simplices α and β is matched in the first step.
   (b) $j > B$, $j > A_1$.
   (c) $A_1 \cup j \in K$.

Observe that the condition (c) always holds (provided that the condition (a) is satisfied), except for the case $B = \emptyset$.

Lemma 6.1. *The discrete vector field D_1 is a discrete Morse function on the Bier sphere $Bier(K)$.*

Proof. Since D_1 is (by construction) a discrete vector field, it remains to check that there are no closed gradient paths. Observe that in each pair of simplices in the discrete vector field D_1 there is exactly one *migrating element*. More precisely, in the case (1) the element i migrates to A_2, and in the case (2) the element j migrates to A_1.

The lemma follows from the observation that (along a gradient path) the values of the migrating element that move to A_2 strictly decreases. Similarly, the values of migrating elements that move to A_1 can only increase.

Let us illustrate this observation by an example. Assume we have a fragment of a gradient path that contains two matchings of type 1. We have:

$$(A_1 \cup k, A_2; B \cup i) \to (A_1 \cup k, A_2 \cup i; B) \to$$
$$(A_1, A_2 \cup i; B \cup k) \to (A_1, A_2 \cup k \cup i; B)$$

The migrating elements here are i and k. The definition of the matching D_1 implies $k < i$. Otherwise $(A_1, A_2 \cup i; B \cup k)$ is matched with $(A_1, A_2; B \cup k \cup i)$, and the path would terminate after its second term. $\square$

It is not difficult to see that there are precisely two critical simplices in D_1:

(1) An $(n-2)$-dimensional simplex,

$$(A_1, A_2; i)$$

where $A_1 < i < A_2$, (this condition describes this simplex uniquely, in light of the fact that $A_1 \in K$ and $A_2 \in K^\circ$),

(2) and the 0-dimensional simplex,

$$(\emptyset, \{1\}; \{2, 3, 4, ..., n\}).$$

(Here we make a simplifying assumption that $\{1\} \in K^\circ$, which can be always achieved by a re-enumeration, except in the trivial case $K^\circ = \{\emptyset\}$.)

6.2. **Second perfect DMF.** The construction of the second discrete vector field D_2 is also in two steps:

The first step remains the same:

(1) We match the simplices

$$\alpha = (A_1, A_2; B \cup i) \text{ and } \beta = (A_1, A_2 \cup i; B)$$

iff the following holds:
  (a) $i < B$, $i < A_2$
      (that is, i is smaller than all elements in B and A_2).
  (b) $A_2 \cup i \in K^\circ$.

  Before we pass to the second step, let us remind ourselves that the non-matched simplices are labelled by $(A_1, A_2; B \cup i)$ such that $A_2 \in K^\circ$, but $A_2 \cup i \notin K^\circ$. As a consequence, for non-matched simplices $A_1 \cup B \in K$.

(2) We match together the simplices

$$\alpha = (A_1, A_2, B \cup i \cup j) \text{ and } \beta = (A_1 \cup j, A_2, B \cup i)$$

iff the following holds:
  (a) None of the simplices α and β was matched in the first step, i.e.
      $i < j$, $i < B$, $i < A_2$, and $i \cup A_2 \notin K^\circ$.

    (b) $j < B$, $j < A_1 \setminus [1, i]$.
    (c) $A_1 \cup j \in K^\circ$.

Note that the condition (c) is always satisfied (provided that the condition (a) above holds).

We omit the proof that D_2 is indeed a discrete Morse function since a more general fact will be established in the proof of Theorem 4.3 (Section 7).

Finally we observe that, with the same simplifying assumption $\{1\} \in K^\circ$, the discrete vector fields D_2 and D_1 have the same critical simplices:

    (1) (A_1, A_2, i) such that $A_1 < i < A_2$
    (2) and $(\emptyset, \{1\}; \{2, 3, 4, ..., n\})$.

7. Proof of Theorem 4.3

The proof of Theorem 4.3 is based on the construction of a discrete Morse function D on the deleted join $\mathcal{K}_\Delta^{*r}$, where $\mathcal{K} = \langle K_1, ..., K_r \rangle$ is a collective r-unavoidable collection of complexes.

We will demonstrate that:

- If the r-tuple $\mathcal{K}$ is collective r-unavoidable, then the critical simplices of the discrete Morse field D may appear only starting with dimension $n - r$ (except for the unique 0-dimensional simplex). This observation immediately implies the connectivity bound in Theorem 4.3, part (1).
- Under the stronger hypothesis that $\mathcal{K}$ is an Alexander r-tuple, the discrete vector field D has a single 0-dimensional critical simplex, while all other critical simplices have one and the same dimension $n - r$. Theorem 4.3 (part (2)) is an immediate consequence. Moreover, a direct dimension count will establish the purity of the complex $\mathcal{K}_\Delta^{*r}$.

As in Section 3, a simplex $\beta = A_1 \uplus \cdots \uplus A_r \in \mathcal{K}_\Delta^{*r}$ is in the 'partition notation' recorded as $\beta = (A_1, \ldots, A_r; B)$ where $B = [n] \setminus \cup_{i=1}^r A_i$. More explicitly, an ordered partition $(A_1, A_2, ..., A_r; B)$ of $[n]$ into $r + 1$ parts, corresponds to a simplex in $\mathcal{K}_\Delta^{*r}$ if and only if,

    (1) $A_i \in K_i$ for each $i = 1, \ldots, r$;
    (2) $\cup \{A_i\}_{i=1}^r \neq \emptyset$, meaning that the partition $(\emptyset, \ldots, \emptyset, [n])$ is excluded.

Observe that the dimension of a simplex $\beta = (A_1, \ldots, A_r; B)$ is determined by the cardinality of B, indeed $\dim(\beta) = n - |B| - 1$.

Moreover, a facet of a simplex $\beta = (A_1, A_2, ..., A_r; B)$ is obtained by moving (we also say 'migrating') an element from one of the sets A_i to B. For example, $(\{1, 2\}, \{6\}, \{5\}; \{3, 4, 7\})$ is a facet of $(\{1, 2\}, \{6, 7\}, \{5\}; \{3, 4\})$ obtained by the migration of the element $7 \in A_2$.

Construction of the discrete Morse function D. The discrete vector field D is described by a step-by-step construction, generalizing the construction of the discrete vector field D_2 from Section 6.2.

In the first step we match the simplices,

$$\alpha = (A_1, A_2, ..., A_r; B \cup i_1) \text{ and } \beta = (A_1 \cup i_1, A_2, ..., A_r; B)$$

iff the following holds:

(1) $i_1 < B$, $i_1 < A_1$;
(2) $A_1 \cup i_1 \in K_1$.

In other words a simplex $\alpha = (A_1, A_2, ..., A_r; B')$ is matched (if possible) with the simplex $\beta = (A'_1, A_2, ..., A_r; B)$ obtained from α by migrating the minimum i_1 of the set $B' = B \cup i$ into $A'_1 = A_1 \cup i$ (provided $i_1 < A_1$ and $A'_1 \in K_1$).

Observe that many simplices are matched already in this step. Indeed, for $\alpha = (A'_1, A_2, ..., A_r; B')$ let $i_1 = \min(A'_1 \cup B')$. If $i_1 \in B'$ then α is matched with the simplex obtained by migrating i_1 from B' to A'_1. If $i_1 \in A'_1$, α is obtained by the migration of i_1 from its facet $\gamma = (A_1, A_2, ..., A_r; B)$, where $A_1 = A'_1 \setminus i_1$ and $B = B' \cup i_1$.

The remaining non-matched simplices $(A_1, A_2, ... A_r; B \cup i_1)$ fall into two types:

(1) The first type:
$$i_1 < B, i_1 < A_1 \text{ and } A_1 \cup i_1 \notin K_1.$$
(2) The second type:
$$B = \emptyset \text{ and } A_1 = \emptyset.$$

Here we declare that the non-matched simplices **of the second type** *will not participate* in matching in later steps of the construction, i.e. they will contribute to the critical simplices of D.

There is a single 0-dimensional non-matched simplex, $(\{1\}, \emptyset, ..., \emptyset; \{2, ..., n\})$. Here (as in Section 6) we make a simplifying (non-essential) assumption that $\{1\} \in K_1$. (This condition can be easily satisfied by choosing a different linear order on $\mathcal{K}$ and $[n]$, if necessary.)

We continue the construction by trying to migrate elements from B into A_2 (in the second step), into A_3 (in the third step), etc. Assume, inductively, that the first $(k-1)$-steps of the construction are completed.

In the k-th step of the construction we match the simplices,

$$\alpha = (..., A_k, ..., A_r; B \cup i_1 ... \cup i_{k-1} \cup i_k) \text{ and } \beta = (..., A_k \cup i_k, ..., A_r; B \cup i_1 ... \cup i_{k-1})$$

iff the following holds:

(1) α and β are non-matched simplices of **the first type** in all preceding steps,
(2) $i_k < B$, $i_k < A_k \setminus [1, i_{k-1}]$,
(3) $A_k \cup i_k \in K_k$.

The remaining non-matched simplices $(A_1, A_2, ..., A_k, ...A_r; B \cup i_1...\cup i_{k-1} \cup i_k)$ again fall into two types:

(1) The first type:
    (a) the simplex is a first type non-matched simplex on steps $1, ..., k-1$,
    (b) $i_k < B$, $i_k < A_k \setminus [1, i_{k-1}]$,
    (c) $A_k \cup i_k \notin K_k$.
(2) The second type:
    (a) the simplex is a first type non-matched simplex on steps $1, ..., k-1$,
    (b) $B = \emptyset$, $A_k \subset [1, i_{k-1}]$.

(As before we declare that the non-matched simplices of the second type never participate in subsequent matchings.)

From the assumption that $\mathcal{K}$ is a collective r-unavoidable collection of complexes we conclude that on the r-th step there are no non-matched simplices of the first type. From here we deduce that the cardinality of B for critical simplices can vary from 0 to $r - 1$, and in particular the dimension of any critical simplex is greater or equal than $n - r$. (The only exception being of course the 0-dimensional critical simplex $(\{1\}, \emptyset, ..., \emptyset; \{2, ..., n\})$.)

An alternative description of the DMF. It may be instructive to summarize the construction of the discrete Morse function D in the form of an 'algorithm' which describes the matching and lists the critical simplices.

For this purpose we introduce an operator **a** which takes simplices

$$(A_1, \ldots, A_r; B) \in K_1 *_\Delta \cdots *_\Delta K_r$$

and maps them to strictly increasing r-tuples,

$$\mathbf{a} = (a_1 < a_2 < \cdots < a_r) \in (\mathbb{N} \cup \{\infty\})^r,$$

by the following rule:

1: $a_1 := \min(B \cup A_1)$; if $B \cup A_1 = \emptyset$ then $a_1 = ... = a_r := \infty$.
2: $a_2 := \min((B \cup A_2)\setminus[1, a_1])$;
           if $(B \cup A_2)\setminus[1, a_1] = \emptyset$ then $a_k := \infty$ for all $k \geq 2$.

                       . . .

i: $a_i := \min((B \cup A_i)\setminus[1, a_{i-1}])$;
           if $((B \cup A_i)\setminus[1, a_{i-1}]) = \emptyset$ then $a_k := \infty$ for all $k \geq i$.

                       . . .

r: $a_r := \min\{(B \cup A_r)\setminus[1, a_{r-1}]\}$;
           if $((B \cup A_r)\setminus[1, a_{r-1}]) = \emptyset$ then $a_r := \infty$.

An element $a_j \neq \infty$ of the r-tuple $\mathbf{a}(A_1, \ldots, A_r; B)$ is **movable** if:
(1) either $a_j \in B$ and $A_j \cup a_j \in K_j$,
(2) or $a_j \in A_j$.
The **standard move** of a movable element a_j is the matching of:

(1) either $(A_1, \ldots, A_j, \ldots, A_r; B) \to (A_1, \ldots, A_j \cup a_j, \ldots, A_r; B \backslash a_j)$,
(2) or $(A_1, \ldots, A_j \setminus a_j, \ldots, A_r; B \cup a_j) \to (A_1, \ldots, A_j, \ldots, A_r; B)$.

The following procedure finds the corresponding pair (if any) for each simplex. If the simplex is not matched, the algorithm reports that it is critical.

Matching Algorithm. A simplex $(A_1, \ldots, A_r; B) \in K_1 *_\Delta \cdots *_\Delta K_r$ is matched with the simplex obtained by the standard move of the *minimal movable element* in $\mathbf{a}(A_1, \ldots, A_r; B)$. If there are no movable elements, the simplex is critical.

Proposition 7.1. *The "Matching Algorithm" describes a discrete Morse function D.*

It is clear that D is a discrete vector field. The proof of the acyclicity follows from the following lemmas.

Lemma 7.2. *Assume the lexicographic order on the set $(\mathbb{N} \cup \{\infty\})^r$. Then the function $\mathbf{a}$ decreases (non-strictly) along any gradient path of the discrete vector field D described by the "Matching Algorithm".*

Proof. For any gradient path,

$$\alpha_0^p \to \beta_0^{p+1} \to \alpha_1^p \to \beta_1^{p+1} \to \ldots,$$

we observe that $\mathbf{a}(\alpha_i^p) = \mathbf{a}(\beta_i^{p+1})$ and $\mathbf{a}(\beta_i^{p+1}) \leq \mathbf{a}(\alpha_{i+1}^p)$. $\qquad\square$

Lemma 7.3. *If the function $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ is constant along a gradient path, then the path has at most 2 elements.*

Proof. Let us inspect a typical fragment of the gradient path,

$$(\ldots, A_r; B \cup i) \to (\ldots, A_k \cup i, \ldots, A_r; B) \to (\ldots, A_k \cup i, \ldots, A_m \setminus j, \ldots, A_r; B \cup j).$$

Observe that if $j > i$, the simplex $(A_1 \ldots, A_k \cup i, \ldots, A_m \setminus j, \ldots, A_r; B \cup j)$ is matched with $(A_1 \ldots, A_k, \ldots, A_m \setminus j, \ldots, A_r; B \cup j \cup i)$. Therefore it turns out to be "the end of an arrow", and the path terminates here. If $j < i$, the value of a_k must change. $\qquad\square$

To establish the second statement in Theorem 4.3, we need the second half of Theorem 2.1. By Proposition 3.5 if $\mathcal{K}$ is an Alexander r-tuple, then the complex $\mathcal{K}_\Delta^{*r}$ is pure $(n - r)$-dimensional.

We end the proof with an efficient, combinatorial description of critical cells of the discrete Morse function D. The Corollary 4.4 is a consequence of the well known fact that the spheres in the wedge decomposition of $Bier(\mathcal{K})$ are in one-to-ne correspondence with the critical cells of D.

    1. An $(n - r)$-dimensional simplex

       $(A_1, A_2, \ldots, A_r; i_1 \cup i_2 \cup \cdots \cup i_{r-1})$ with $i_1 < i_2 < \ldots < i_{r-1}$

       is a critical simplex of the discrete Morse function D if and only if:
       (a) A_1 avoids the segment $[1, i_1]$,

(b) A_k avoids the segment $[i_{k-1}, i_k]$ for $k \in [2, r-1]$,
(c) A_{r-1} avoids the segment $[i_{r-2}, i_{r-1}]$,
(d) $A_k \cup i_k \notin K_k$ for $k \in [1, r-1]$,
(e) A_r avoid the segment $[i_{r-1}, n]$.

2. There is a single 0-dimensional simplex:

$$(\{1\}, \emptyset, \dots, \emptyset; \{2, 3, 4, \dots, n\}).$$

With this observation we complete the proof of Theorem 4.3 (Corollary 4.4).

Example 7.4. The chessboard complex $\Delta_{5,3}^{1,1,1;1}$ is the Bier complex associated to the Alexander 3-tuple $\mathcal{K} = \langle K_1, K_2, K_3 \rangle$, where $K_1 = K_2 = K_3$ is the 0-dimensional skeleton of the 4-dimensional simplex $\Delta([5])$. Then the critical simplices of the discrete Morse function D constructed in the proof of Theorem 4.3 are the following:

$(A_1, A_2, A_3; B) = (4, 5, 2; \{1, 3\}), (5, 4, 2; \{1, 3\}), (2, 5, 3; \{1, 4\}), (3, 5, 3; \{1, 4\}),$
$(4, 5, 1; \{2, 3\}),\ (5, 4, 1; \{2, 3\}),\ (3, 5, 1; \{2, 4\}),\ (5, 1, 3; \{2, 4\}),\ (3, 1, 4; \{2, 5\}),$
$(4, 1, 3; \{2, 5\}), (5, 2, 1; \{3, 4\}), (5, 1, 2; \{3, 4\}), (4, 2, 1; \{3, 5\})\ (4, 1, 2; \{3, 5\}),$ and
$(1, \emptyset, \emptyset; \{2, 3, 4, 5\}).$

8. ENUMERATION OF CRITICAL SIMPLICES
FOR OPTIMAL CHESSBOARD COMPLEXES

Optimal chessboard complexes (see Section 2.2 and Example 3.9 in Section 3) are our key examples of Alexander r-tuples for $r \geq 3$. In this section we enumerate critical simplices of $\Delta_{n,r}^{m_1,\dots,m_r;1}$ for $n = m_1 + \dots + m_r + r - 1$.

For a given simplex $\beta \in \Delta_{n,r}^{m_1,\dots,m_r;1}$, let us encode the set of free columns $1 \leqslant x_1 < \dots < x_{r-1} \leqslant n$ as an $(r-1)$-tuple $\mathbf{x} = (x_1, x_2, \dots, x_{r-1})$. Observe that if (in the notation of Section 7) $\beta = (A_1, \dots, A_n; B)$ then $B = \{x_i\}_{i=1}^{r-1}$.

Let $\mathbf{b} = (b_1, b_2, \dots, b_r)$ denote the sequence that counts the number of rooks (in all rows) between consecutive free columns, i.e.

$$b_1 = x_1 - 1, b_2 = x_2 - x_1 - 1, \dots, b_r = r - x_r.$$

Let $b_{i,j}$ denote the number of rooks in the jth column between columns x_{i-1} and x_i (with the obvious interpretation of numbers b_{i1} and b_{ir}). We know from the "critical simplices criterion" (found at the end of the proof of Theorem 4.3 in Section 7), that $b_{ii} = 0$ for all i. For a given $\mathbf{x}$, all possible numbers of rooks between free columns in critical simplices (we ignore for a moment the order or rooks), corresponds to all non-negative $r \times r$ matrices,

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ b_{r1} & b_{r2} & \cdots & b_{rr} \end{pmatrix}$$

with non-negative integers such that:

$$b_{11} = \cdots = b_{rr} = 0, \ B \cdot \mathbf{1} = (m_1, \ldots, m_r), \ \mathbf{1}^t \cdot B = (b_1, b_2, \ldots, b_r).$$

Denote the number of such matrices by $R_{\mathbf{x}}$. If the number of rooks between two consecutive free columns for each row is fixed, the number of all configurations is

$$\binom{b_1}{b_{11}, b_{21}, \ldots, b_{r1}} \binom{b_2}{b_{12}, b_{22}, \ldots, b_{r2}} \cdots \binom{b_r}{b_{1r}, b_{2r}, \ldots, b_{rr}}.$$

Therefore, the number of all critical simplices of $\Delta_{n,r}^{m_1,\ldots,m_r;1}$ is

$$\sum_{1 \leqslant x_1 < \ldots < x_{r-1} \leqslant r} R_{\mathbf{x}} \binom{b_1}{b_{11}, b_{21}, \ldots, b_{r1}} \binom{b_2}{b_{12}, b_{22}, \ldots, b_{r2}} \cdots \binom{b_r}{b_{1r}, b_{2r}, \ldots, b_{rr}}$$

9. Discrete Morse function for a long chessboard complex

The 'long' chessboard complexes (described in Section 2.2) are not collective r-unavoidable complexes, let alone Alexander r-tuples. However, the construction of the discrete Morse function, described in Section 7, is sufficiently general and versatile to be applied in this case as well. This is very interesting since the existence of a perfect Morse function on this complex provides an alternative proof of the (critical case) of Theorem 3.2 from [JVZ-1]. Recall that this result paved the way for some new Tverberg-Van Kampen-Flores type results, including the Theorem 1.2 from [JVZ-2].

Recall that a multiple chessboard complex $K = \Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$ is 'long' if $n > m_1 + \cdots + m_r + r - 1$. Following essentially the construction of the matching described in Section 7, one obtains the discrete Morse function D on K which has the following critical simplices.

With the exception of the unique 0-dimensional critical simplex, all other critical simplices are described as the configurations $(A_1, A_2, \ldots, A_r; B)$ for which there exist elements $i_1 < i_2 < \cdots < i_r$ in B such that the following conditions are satisfied:

(1) A_1 avoids the segment $[1, i_1]$,
(2) A_k avoids the segment $[i_{k-1}, i_k]$ for $k \in [2, r]$,
(3) $A_k \cup i_k \notin K_k$ for $k \in [1, r-1]$,
(4) $B \backslash \{i_1, \ldots, i_r\} > i_r$.

The condition (3) implies that all the critical simplices have one and the same dimension $(m_1 + \cdots + m_r - 1)$.

Example 9.1. The complex $\Delta_{4,2}^{1,1;\mathbf{1}}$ has 5 critical simplices of dimension 1:
  $(4, 3, \{1, 2\}) \ (3, 4, \{1, 2\}) \ (2, 4, \{1, 3\}) \ (4, 1, \{3, 2\}) \ (3, 1, \{4, 2\}).$

The existence of a perfect discrete Morse function on the long, multiple chessboard complex $\Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$ provides an alternative proof of the following theorem from [JVZ-1]. (Two other proofs, both of them comparatively complex and non-trivial, relied respectively on a shelling construction, and the Nerve Lemma.)

Theorem 9.2. ([JVZ-1, Theorem 3.2]) *The long chessboard complex is homotopy equivalent to a wedge of $(m_1 + \cdots + m_r - 1)$-dimensional spheres.*

9.1. Enumeration of critical simplices for a long chessboard complex.
In this section we enumerate the critical simplices in the long chessboard complex $\Delta_{n,r}^{m_1,\ldots,m_r;1}$.

We use the notation as in Section 8. Recall that (for all $i = 1, 2, \ldots, r$) the integer b_{ij} evaluates the number of rooks in the ith row between $(j-1)^{th}$ and j^{th} free column (for $j = 1, 2, \ldots, r$). The distributions of rooks between columns (we again ignore for a moment the exact positions of the rooks) is encoded by the matrix $B \in Mat_{r,r+1}(\mathbb{N}_0)$ where

$$b_{11} = \cdots = b_{rr} = 0, \ B \cdot \mathbf{1} = (m_1, \ldots, m_r).$$

Also, in this case we have $n - r - m_1 - m_2 - \cdots - m_r$ free columns, and all of them are positioned behind the rth free column.

Simply by counting all partitions of the corresponding multisets, we obtain the following formula for the number of all critical simplices in $\Delta_{n,r}^{m_1,\ldots,m_r;1}$,

$$\sum_{\substack{B \in \mathbb{M}_{r,r+1}(\mathbb{N}_0), \\ B \cdot \mathbf{1} = \mathbf{m}, b_{11} = \cdots = b_{rr} = 0}} \binom{n - r - \sum_i m_i + \sum_j b_{jr}}{\sum_j b_{jr}} \prod_{i=1}^{r} \binom{b_{11} + b_{21} + \cdots + b_{r1}}{b_{11}, b_{21}, \ldots, b_{r1}}$$

For example, if $r = 2$ we have that the number of critical simplices is

$$\sum_{b_{31}=1}^{m_1} \sum_{b_{32}=1}^{m_2} \binom{b_{13} + b_{23}}{b_{13}} \binom{n - 2 - m_1 - m_2 + b_{13} + b_{23}}{b_{13} + b_{23}}.$$

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(D. Jojić) FACULTY OF SCIENCE, UNIVERSITY OF BANJA LUKA

(I. Nekrasov) CHEBYSHEV LABORATORY, ST. PETERSBURG STATE UNIVERSITY

(G. Panina) MATHEMATICS & MECHANICS DEPARTMENT, ST. PETERSBURG STATE UNIVERSITY

(R. Živaljević) MATHEMATICAL INSTITUTE SASA, BELGRADE