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KARSTEN BOHLEN AND ELMAR SCHROHE

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
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Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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GETZLER RESCALING VIA ADIABATIC DEFORMATION AND A RENORMALIZED LOCAL INDEX FORMULA

KARSTEN BOHLEN, ELMAR SCHROHE

Abstract. We prove a local index theorem of Atiyah-Singer type for Dirac operators on manifolds with a Lie structure at infinity (*Lie manifolds* for short). After introducing a renormalized supertrace on Lie manifolds with spin structure, defined on a suitable class of rapidly decaying functions, the proof of the index theorem relies on a rescaling technique similar in spirit to Getzler's rescaling. With a given Lie manifold we associate an appropriate integrating Lie groupoid. We then describe the heat kernel of a geometric Dirac operator via a functional calculus with values in the convolution algebra of sections of the rescaled bundle over the adiabatic groupoid and introduce a rescaling of the heat kernel encoded in a vector bundle over the adiabatic groupoid. Finally, we calculate the right coefficient in the heat kernel expansion using the Lichnerowicz theorem on the fibers of the groupoid and the Lie manifold.

1. Introduction

There are various routes to the Atiyah-Singer index theorem (cf. [4], [5], [6]) for the Fredholm index of elliptic operators on a closed manifold. Different proofs in turn have often given rise to profound generalizations, in particular to the index theory of elliptic operators on non-compact manifolds modeled on manifolds with singularities, manifolds with boundary or manifolds with corners. A particularly fruitful approach is based on the deformation groupoid (the *tangent groupoid*) introduced by A. Connes, [10]. It has given rise to a number of extensions and generalizations, see e.g. [8], [14], [32]. In the analysis of non-compact manifolds modeling different types of singular manifolds, Lie groupoids enter naturally as models for singular spaces, an observation first made by A. Connes. The problem then is to find ellipticity conditions implying the Fredholm property of a suitable class of differential operators acting between appropriate Sobolev spaces as the most natural condition, namely the pointwise invertibility of the invariantly defined principal symbol, is no longer sufficient. If the noncompact manifold is the interior of a compact manifold with corners and the boundary strata are embedded submanifolds of the same dimension, the index theory of foliations initiated by A. Connes and G. Skandalis [12] provides a basis for the formulation of an index problem. In general, however, the dimension of the strata will vary. Moreover, Connes realized that the natural receptacle for the foliation index is the K-theory of the C^*-algebra of the holonomy groupoid of the foliation. Similarly, for a manifold with corners, the corresponding generalized analytic index maps into the K-theory of the C^*-algebra of the Lie groupoid which desingularizes the manifold. The task therefore is to find a purely topological interpretation of the generalized analytic index. This has been achieved for several cases of singular manifolds, see e.g. [32] for Lie manifolds. A significant drawback of this approach is that the generalized analytic index almost never equals the Fredholm index. In fact, both agree for closed manifolds, since in this

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case the groupoid under consideration is the pair groupoid whose C^*-algebra is the algebra of the compact operators. In other interesting cases the Fredholm index does not equal the generalized analytic index. The more difficult problem is therefore to calculate the *Fredholm index* in topological terms, thus generalizing the Atiyah-Singer index theorem to a large class of non-compact manifolds.

In this article we focus on such a large class, namely, the *manifolds with a Lie structure at infinity* or *Lie manifolds* for short. While many special instances of these manifolds have been considered in the literature and index theorems of the above type have been proven by different techniques, a general Fredholm index theorem, valid for any Lie manifold, has not yet been obtained. We refer to the excellent survey [36] for more information. The problem lies in the more complicated Fredholm conditions on non-compact manifolds and the fact that the boundary strata give rise to non-local invariants in the resulting index theorem. Only in the simplest case of asymptotically flat Lie structures is a direct analogue of Atiyah-Singer possible, cf. [11].

In this article we will follow the strategy to first establish a *local index theorem* via the heat kernel and then use this local index theorem to prove the Atiyah-Singer index formula. On the other hand we adhere to the program, started by A. Connes and continued by other authors, of using deformation groupoids in order to extract the Fredholm index and to express it in topological terms. The particular technique, however, is different from the tangent groupoid proof in [10], because this proof would a priori only calculate the generalized analytic index. (Note, however, that at least for manifolds with boundary, the authors in [8] have obtained a groupoid version of the Atiyah-Patodi-Singer index theorem by modifying Connes' technique.) We describe instead a proof which combines the rescaling technique of Getzler with the adiabatic groupoid.

Early proofs of the local index theorem are due to Atiyah-Bott-Patodi [3], Gilkey [17] and Patodi [39]. Our proof is based on Getzler's rescaling proof, see [16] and also [7] for a very good exposition. We think that it is possible to use the idea of Getzler of replacing the heat kernel k by a *rescaled* heat kernel $k(u,t,x) = u^{\frac{n}{2}} k(ut, u^{\frac{1}{2}} x)$, $0 < u \leq 1$, subsequent calculation of the asymptotic expansion of the rescaled kernel and application of the Lichnerowicz theorem in the limit $u \to 0^+$, and adapt it to our more general case. Nevertheless, we have chosen to apply a deformation groupoid argument. The idea for such an argument in the standard case, using the tangent groupoid, can be found already in Quillen's notebooks, [40]. We partly rely on unpublished notes by P. Siegel [43] and the expository account of Getzler's argument by J. Roe [41]. Siegel gives an account of a rescaling technique using the tangent groupoid, deriving the local index formula for a smooth closed manifold. In our more general case one has to confront a number of difficulties which we will explain in the sequel.

1.1. **Overview.** Manifolds with a Lie structure at infinity have been introduced by Ammann, Lauter and Nistor, [1]. They can be used to model many types of singular manifolds. A Lie manifold is a tuple $(M, \mathcal{A}, \mathcal{V})$ where M is a compact manifold with corners and $\mathcal{V} \subset \Gamma(TM)$ is a Lie algebra of smooth vector fields. Moreover, $\mathcal{V}$ is assumed to be a subset of the Lie algebra $\mathcal{V}_b$ of all vector fields tangent to the boundary strata and to be a finitely generated projective $C^{\infty}(M)$-module. Also the compact manifold with corners M is thought of as a compactification of a non-compact manifold with a degenerate, singular metric which is of product type at infinity. We denote by ∂M the (stratified) boundary of M and $M_0 = M \setminus \partial M$ the interior. By the Serre-Swan theorem there exists a vector bundle $\mathcal{A} \to M$ such that $\Gamma(\mathcal{A}) \cong \mathcal{V}$. The bundle $\mathcal{A}$

has the structure of a Lie algebroid. A further piece of information we need is a Lie groupoid $\mathcal{G} \rightrightarrows M$. It is known that for any Lie structure there is an s-connected Lie groupoid $\mathcal{G}$ such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$. For general Lie algebroids, Crainic and Fernandes [13] obtained computable obstructions for the integrability. In our case we have to construct a holonomy groupoid $\mathcal{G}$ with good topological properties (e.g. amenable and Hausdorff) for the given Lie structure, see e.g. [33]. We make the common assumption that $\mathcal{A}_{|M_0} \cong TM_0$ and $\mathcal{G}_{|M_0} \cong M_0 \times M_0$ are the tangent bundle and pair groupoid on the interior, respectively. Additionally, we say that the Lie manifold is *non-degenerate*, if we can find an integrating Lie groupoid which is Hausdorff. We recall particular examples of such groupoids in the main body of the paper.

We also assume that the Lie manifold $(M, \mathcal{A}, \mathcal{V})$ is *spin*, i.e. there is a spin structure $S \to M$, cf. [2]. We let W be a $\mathrm{Cl}(\mathcal{A})$-module, where $\mathrm{Cl}(\mathcal{A}) \to M$ denotes the bundle of Clifford algebras on the fibers of $\mathcal{A}$. By D we denote a geometric Dirac operator obtained from an *admissible* connection ∇^W, cf. [26]. We will call a bilinear form $g = g_{\mathcal{A}}$ defined on $\mathcal{A}$, which yields a Riemannian metric when restricted to M, a *compatible metric*, cf. [2]. The heat kernel κ_t of e^{-tD^2} will not be of trace class in general, because the trace does a priori only exist on the interior of the manifold with corners M. We therefore introduce the *renormalized super-trace* ${}^{\mathcal{V}}\mathrm{Tr}_{\mathrm{s}}$ which relies on a regularization at infinity. In addition, we introduce a suitable class $\mathbf{S}(\mathcal{G})$ of rapidly decaying functions or distributions over the integrating groupoid and a corresponding class ${}^{\mathcal{V}}\mathrm{S}(M)$ over the Lie manifold. If the Lie structure is non-degenerate, we can assume $\mathcal{G}$ to be Hausdorff. In this case we prove that ${}^{\mathcal{V}}\mathrm{S}(M)$ can be identified with $\mathbf{S}(\mathcal{G})$ via the *vector representation* $\varrho\colon \mathrm{End}(C^\infty(\mathcal{G})) \to \mathrm{End}(C^\infty(M))$. The vector representation is characterized by the equality: $\varrho(P)(f \circ r) = (Pf) \circ r$, where r is the range map of the groupoid (a surjective submersion), $P \in \mathrm{End}(C^\infty(\mathcal{G}))$ and $f \in C^\infty(M)$, see also [1], [38].

In the classical setting, the tangent groupoid deforms the pair groupoid over the manifold M into its tangent bundle TM. In our case we deform the integrating groupoid $\mathcal{G}$, rather than just the pair groupoid, and consider the *adiabatic groupoid* $\mathcal{G}^{ad} = \mathcal{G} \times (0,1] \cup \mathcal{A}(\mathcal{G}) \times \{0\}$ which deforms $\mathcal{G}$ into the Lie algebroid $\mathcal{A}(\mathcal{G})$. Then we perform the rescaling over the adiabatic groupoid adapted to a formal Ansatz for the asymptotic expansion of the heat kernel. The geometric admissible Dirac operator D on a Lie manifold is realized as the vector representation of a corresponding geometric admissible Dirac operator $\not{D}$ on the Lie groupoid, see [26]. Hence given the heat kernel k_t on the Lie groupoid whose vector representation is the heat kernel κ_t on the Lie manifold, we consider the Ansatz for the asymptotic expansion of k_t. We also gather from [45] and [46] the proof of the approximation of the heat kernel on Lie groupoids as described for complete Riemannian manifolds in [7]. We use this approximation and the estimates from [45] to show that the heat kernel is contained in the Schwartz class $\mathbf{S}(\mathcal{G})$. The asymptotic expansion Ansatz is $e^{-t\not{D}^2} \sim (4\pi)^{-\frac{n}{2}} t^{-\frac{n}{2}} \sum_{i=0}^{\infty} a_i t^i$. We next describe a way to extract the coefficient $a_{n/2}$ in the asymptotic expansion of the heat kernel. For this we deform $\not{D}$ into a smooth equivariant family of operators on the Lie algebroid $\mathcal{A}(\mathcal{G})$ associated to M. The rescaling deforms $\not{D}$ in such a way that the Clifford multiplication is taken into account and at the same time the right coefficient in the Ansatz is extracted. This is done by a rescaling of the Clifford algebra. The result is that $\not{D}$ is deformed into a polynomial coefficient operator whose supertrace has the right asymptotics. We then study the groupoid convolution algebra $C_c^\infty(\mathcal{G}^{ad}, \mathrm{Hom}(S))$ where $\mathrm{hom}(S) \to M$ given by $\mathrm{hom}(S)_x \cong \mathrm{hom}(S_x, S_x) \cong \mathrm{Cl}(\mathcal{A}_x \otimes \mathbb{C})$ is lifted to an equivariant bundle $\mathrm{Hom}(S) \to \mathcal{G}^{ad}$. Given a Clifford filtration by degree

$\mathrm{Cl}_0 \subseteq \mathrm{Cl}_1 \subseteq \cdots \subseteq \mathrm{Cl}(\mathcal{A} \otimes \mathbb{C})$, we can extend this filtration to a neighborhood of $\mathcal{A}$ within the adiabatic groupoid $\mathcal{G}^{ad}$. Here we view $\mathcal{A}$ as an embedded boundary stratum of the manifold $\mathcal{G}^{ad}$ and use the accompanying tubular neighborhood to extend the filtration. Subsequent to this we introduce an equivariant *rescaling bundle* $\tilde{S} \to \mathcal{G}^{ad}$ extending $\mathrm{Hom}(S)$ such that the sections of this bundle have a polynomial coefficient expansion, where the coefficients are contained in the sections of the extended Clifford filtration. We define a functional calculus which realizes the groupoid heat kernel as an element of the convolution algebra of smooth sections of the rescaled bundle $C_c^{\infty}(\mathcal{G}^{ad}, \tilde{S})$. The final calculation of the coefficient in the asymptotic expansion is then performed using the Lichnerowicz theorem applied to the fibers of the Lie groupoid and, by $\mathcal{G}$-invariance, to the Lie manifold.

1.2. **The main theorem.** We will prove the following result:

Theorem 1.1. *Let $(M, \mathcal{A}, \mathcal{V})$ be an n-dimensional non-degenerate Lie manifold, $S \to M$ a spin structure, $\mathrm{Cl}(\mathcal{A}) \to M$ the bundle of Clifford algebras and $W \in \mathrm{Cl}(\mathcal{A}) - mod$ a Clifford module. Given a compatible Riemannian metric $g = g_{\mathcal{A}}$ fix an admissible connection ∇^W and the corresponding Dirac operator $D = D^W \in \mathrm{Diff}_{\mathcal{V}}^1(M; W)$. Then we have the formula for the renormalized super trace*

$$\lim_{t \to \infty} {}^{\mathcal{V}}\mathrm{Tr}_s(e^{-tD^2}) = {}^{\mathcal{V}}\!\!\fint {}^{\mathcal{V}}\mathbb{A} \wedge \exp F^{W/S} \, d\mu + {}^{\mathcal{V}}\eta(D) \tag{1}$$

where $F^{W/S}$ is the twisting curvature and ${}^{\mathcal{V}}\mathbb{A} = h(R)$, for the curvature tensor R obtained from the compatible metric, denotes the n-form given by the formal power series

$$h(R) = \left(-\frac{i}{2\pi}\right)^{\frac{n}{2}} \det\left(\frac{\frac{1}{2}R}{\sinh(\frac{1}{2}R)}\right)^{\frac{1}{2}}.$$

The function ${}^{\mathcal{V}}\eta$ is the renormalized η-invariant which is given by the integrated trace defect

$${}^{\mathcal{V}}\eta(D) := \frac{1}{2}\int_0^{\infty} {}^{\mathcal{V}}\mathrm{Tr}_s([D, De^{-tD^2}])\, dt.$$

The left hand side of (1) has been shown to converge to the Fredholm index in special cases (cf. [30], Section 7.8) and the *trace defect* ${}^{\mathcal{V}}\eta$ on the right hand side can be calculated in terms of restrictions to the boundary strata (cf. [30], Section 5.5), though this calculation is rather complicated in general. We refer to [20], [21] and [30] for the discussion in the case of b-manifolds. A local index formula in the special case of cusp vector fields has been obtained in [24] and for the case of a fibered cusp Lie structure in [25] as well as in [22] by using the method of deformation of the metrics of b-, cusp and fibered cusp type. We also refer to [31] for a K-theoretic index theorem on manifolds with fibered cusp structure.
The paper is organized as follows. In the second section we give the definition of the geometric Dirac operators for Lie groupoids and Lie manifolds. We also prove the Lichnerowicz theorem for the generalized Laplacian on a Lie manifold defined with respect to an admissible connection. In the third section we study the groupoid heat kernel and its approximation. We introduce a class of rapidly decaying functions on a Lie groupoid and show that under suitable conditions the heat kernel is contained in this class. In Section four we define a functional calculus for the convolution algebra over the adiabatic groupoid. The fifth section contains the definition of the renormalized super trace on Lie manifolds as well as the class of rapidly decaying functions on Lie

manifolds. Finally, in section six we introduce the rescaling and give the proof of the main theorem.

2. Dirac operators on Lie manifolds

Geometric Dirac operators on Lie manifolds are given as vector representations of operators on Lie groupoids integrating the Lie structure. In this section we will outline some details of their construction, following [1], and state the corresponding Lichnerowicz theorem. To this end we will assume that the given Lie manifold carries a spin structure, which we will keep fixed, and an *admissible* connection.

Let $(M, \mathcal{A}, \mathcal{V})$ be an n-dimensional Lie manifold. Denote by $P_{\mathrm{SO}}(\mathcal{A}) \to M$ the bundle of oriented orthonormal frames. This is a principal $\mathrm{SO}(n)$-bundle. According to [2], a *spin structure* over M is a tuple $(P_{\mathrm{Spin}}(\mathcal{A}), \alpha)$, where $P_{\mathrm{Spin}}(\mathcal{A})$ is a principal $\mathrm{Spin}(n)$-bundle and $\alpha \colon P_{\mathrm{Spin}}(\mathcal{A}) \to P_{\mathrm{SO}}(\mathcal{A})$ is a fiber map over the identity of M, compatible with the double covering $\theta \colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ and the corresponding group actions, i.e., the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{Spin}(n) \times P_{\mathrm{Spin}}(\mathcal{A}) & \longrightarrow & P_{\mathrm{Spin}}(\mathcal{A}) & \searrow & \\
{\scriptstyle \theta\times\alpha}\big\downarrow & & {\scriptstyle \alpha}\big\downarrow & & M, \\
\mathrm{SO}(n) \times P_{\mathrm{SO}}(\mathcal{A}) & \longrightarrow & P_{\mathrm{SO}}(\mathcal{A}) & \nearrow &
\end{array}$$

where the horizontal arrows are induced by the group actions.

The spinor bundle is defined as $S := P_{\mathrm{Spin}}(\mathcal{A}) \times_{\sigma_n} \Sigma_n$, where $\sigma_n \colon \mathrm{Spin}(n) \to \mathrm{SU}(\Sigma_n)$ is the complex spinor representation (e.g. the restriction of an odd complex irreducible representations of the Clifford algebra on n-dimensional space). Here Σ_n denotes an irreducible spin-representation of $\mathrm{Cl}_n(\mathcal{A}) \otimes \mathbb{C}$. If n is odd there are two distinct irreducible representations. For n even, there is one irreducible representation which splits into two non equivalent sub-representations. See also Lawson and Michelsohn[27, Section II.3].

A Clifford module $W \to M$ is a complex vector bundle together with a positive definite inner product $\langle \cdot, \cdot \rangle$, anti-linear in the second component, an $\mathcal{A}^*$-valued connection $\nabla^W \in \mathrm{Diff}_{\mathcal{V}}(M, W, W \otimes \mathcal{A}^*)$ and a linear bundle map $\mathcal{A} \otimes W \to W,\ c \colon X \otimes \varphi \mapsto X \cdot \varphi$ called Clifford multiplication, such that the following holds.

(1) $(X \cdot Y + Y \cdot X + 2g(X, Y)) \cdot \varphi = 0$ for each $X, Y \in \Gamma(\mathcal{A}),\ \varphi \in \Gamma(W)$.

(2) ∇^W is *metric*
$$\partial_X \langle \psi, \varphi \rangle = \langle \nabla_X^W \psi, \varphi \rangle + \langle \psi, \nabla_X^W \varphi \rangle,\ X \in \Gamma(\mathcal{A}),\ \varphi, \psi \in \Gamma(W).$$

(3) Clifford multiplication with vectors is skew-symmetric, i.e.,
$$\langle X \cdot \psi, \varphi \rangle = \langle \psi, X\varphi \rangle,\ \varphi, \psi \in \Gamma(W),\ X \in \Gamma(\mathcal{A}).$$

(4) The connection is *admissible*, i.e.
$$\nabla_X^W (Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + Y(\nabla_X^W \varphi),\ X, Y \in \Gamma(\mathcal{A}),\ \varphi \in \Gamma(W).$$

Here ∇ is the Levi-Civita connection with respect to the compatible metric.

Definition 2.1. Let $W \to M$ be a Clifford bundle and g a compatible metric. Then the geometric Dirac operator D^W is defined by the composition $D^W = c \circ (\mathrm{id} \otimes \sharp) \circ \nabla^W$, acting on $\Gamma(W)$,

$$\Gamma(W) \xrightarrow{\nabla^W} \Gamma(W \otimes \mathcal{A}^*) \xrightarrow{\mathrm{id}\otimes\sharp} \Gamma(W \otimes \mathcal{A}) \xrightarrow{c} \Gamma(W),$$

where c denotes Clifford multiplication and $\sharp$ is the conjugate-linear isomorphism $\mathcal{A} \cong \mathcal{A}^*$ induced by the metric g.

Following [26], we next outline the construction of a geometric Dirac operator $\not{D}$ on $\mathcal{G}$ as a $\mathcal{G}$-invariant family of operators on the s-fibers $(\mathcal{G}_x)_{x \in M}$ of a given Lie groupoid $\mathcal{G} \rightrightarrows M$ integrating the Lie structure, i.e. $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}_{\mathcal{V}}$. Fix the spinor bundle $S \to M$ as above, the bundle $\mathrm{Cl}(\mathcal{A}) \to M$ of Clifford algebras and a Clifford module $W \to M$. The Clifford multiplication defines a map $c \colon \mathrm{Cl}(\mathcal{A}) \to \mathrm{End}(W)$.
The Levi-Civita connection on $\mathcal{G}$ is obtained as follows. Let $X \in \Gamma(\mathcal{A})$ and let $\tilde{X}$ denote a lift to a $\mathcal{G}$-invariant s-vertical vector field (i.e. a smooth section of $T_s\mathcal{G} := \ker ds$). For $x \in M$ let g_x be the metric on $\mathcal{G}_x$ induced by the fixed compatible Riemannian metric g. Denote by $\nabla^x \colon \Gamma(T_s\mathcal{G}_x) \to \Gamma(T_x\mathcal{G}_x \otimes T_s^*\mathcal{G}_x)$ the Levi-Civita connection associated to g_x. We obtain a smooth and $\mathcal{G}$-invariant family of differential operators $\nabla_{\tilde{X}}^x \colon \Gamma(T_s\mathcal{G}_x) \to \Gamma(T_s\mathcal{G}_x)$ that descends to $\nabla_X \in \mathrm{Diff}(\mathcal{G}, \mathcal{A})$.
According to [26, Proposition 6.1], we find a $\mathcal{G}$-invariant connection $\not{\nabla}^W$ on $\mathcal{G}$ satisfying the following condition of *admissibility*,

$$\not{\nabla}_X^W(c(Y)\xi) = c(\nabla_X Y)\xi + c(Y)\not{\nabla}_X^W(\xi), \ \xi \in \Gamma(r^*W), \ X, Y \in \Gamma(r^*\mathcal{A}). \tag{2}$$

Definition 2.2. Let $W \to M$ be a Clifford module and $\mathcal{G} \rightrightarrows M$ a Lie groupoid. The geometric Dirac operator $\not{D}^W$ is defined by $\not{D}^W := c \circ (\mathrm{id} \otimes \sharp) \circ \nabla^W$ where $\sharp$ denotes the conjugation isomorphism induced by the fixed compatible metric g, $c \in \mathrm{Hom}(W \otimes \mathcal{A}^*)$ Clifford multiplication and $\nabla^W \in \mathrm{Diff}(\mathcal{G}; r^*W, r^*W \otimes \mathcal{A}^*)$ an admissible connection.

It is shown in [26] that, with the above definition, the Dirac operator on the Lie manifold M is the vector respresentation of the Dirac operator on the Lie groupoids. We now state the Lichnerowicz theorem for the generalized Laplacian D^2 of a Dirac operator D on a Lie manifold. The proof is similar as in [7], but we provide the details to make the paper more self-contained.

Theorem 2.3 (Lichnerowicz formula)**.** *Let $(M, \mathcal{V}, \mathcal{A})$ be a Lie manifold, $S \to M$ a spin structure and $g = g_{\mathcal{A}}$ a compatible Riemannian metric. Denote by $\mathrm{Cl}(\mathcal{A}) \to M$ the Clifford bundle and let $W \in \mathrm{Cl}(\mathcal{A}) - mod$ be a Clifford module. Let ∇^W be an admissible connection and D the corresponding Dirac operator. Then we have the formula*

$$D^2 = \Delta^W + c(F^{W/S}) + \frac{\kappa}{4}$$

where κ is scalar curvature, $F^{W/S} \in \Lambda^2(\mathrm{End}_{\mathrm{Cl}(\mathcal{A})} W)$ is the twisting curvature.

Proof. Let R be the Riemannian curvature tensor induced by the fixed compatible metric g. We give the construction of the twisting curvature in the proof of the following assertion.
Claim: The curvature $R^{\nabla^W} \in \Lambda^2(\mathrm{End}(W))$ decomposes under the isomorphism

$$\mathrm{End}(W) \cong \mathrm{Cl}(\mathcal{A}^*)) \otimes \mathrm{End}_{\mathrm{Cl}(\mathcal{A}^*)}(W)$$

as $R^W + F^{W/S}$, where R^W is the action of the Riemannian curvature R on W given by

$$R^W(e_i,e_j) = \frac{1}{4}\sum_{k,l=1}^{n} g(R(e_i,e_j)e_k,e_l)c(e^k)c(e^l), \tag{3}$$

for an arbitrary orthonormal frame $\{e_1,\cdots,e_n\}$ of $\mathcal{A}$ and the dual frame $\{e^1,\cdots,e^n\}$.
Proof of the claim: Note that $R^W \in \Lambda^2(\mathrm{Cl}(\mathcal{A}))$. We set $F^{W/S} := R^{\nabla^W} - R^W$ and show that $F^{W/S} \in \Lambda^2(\mathrm{End}_{\mathrm{Cl}(\mathcal{A}^*)}W)$. To this end we will prove that the exterior multiplication $\epsilon(F^{W/S})$ acting on $\Gamma(W)$ commutes with Clifford multiplication $c(a)$ by an element $a \in \mathcal{A}^*$.
Since ∇^W is admissible we have $[\nabla^W, c(a)] = c(\nabla a)$ where ∇ is the connection obtained from the fixed compatible metric g. Hence we get

$$[R^{\nabla^W}, c(a)] = [\nabla^W,[\nabla^W,c(a)]] = [\nabla^W, c(\nabla a)] = c(\nabla^2 a) = c(Ra).$$

We need to show that R^W also satisfies the commutator property $[R^W, c(a)] = c(Ra)$. For then

$$[F^{W/S}, c(a)] = [R^{\nabla^W}, c(a)] - [R^W, c(a)] = 0$$

and hence $F^{W/S}$ is an element of $\Lambda^2(\mathrm{End}_{\mathrm{Cl}(\mathcal{A}^*)}W)$.
We identify $\mathcal{A} \cong \mathcal{A}^*$ via g and write $a = \sum_{l=1}^n e^k(a)e_k$. Then

$$R(e_i,e_j)a = \sum_{l=1}^{n} g(R(e_i,e_j)a,e_l)e^l = \sum_{k,l=1}^{n} g(R(e_i,e_j)e_k,e_l)e^k(a)e^l. \tag{4}$$

It is sufficient to check the commutator property for $a = e^s$, $s = 1,\ldots,n$. We first recall the identity

$$\begin{aligned} &c(e^i)c(e^j)c(e^k) \\ &= \frac{1}{3!}\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)c(e^{\sigma(i)})c(e^{\sigma(j)})c(e^{\sigma(k)}) - \delta^{ij}c(e^k) - \delta^{jk}c(e^i) + \delta^{ki}c(e^j). \end{aligned} \tag{5}$$

According to (3) we then obtain

$$\begin{aligned} R^W(e_i,e_j)c(e^s) - c(e^s)R^W(e_i,e_j) &= \frac{1}{4}\sum_{k=1}^{n} g(R(e_i,e_j)e_k,e_s)c(e^k)c(e^s)c(e^s) \\ - c(e^s)\frac{1}{4}\sum_{k=1}^{n} g(R(e_i,e_j)e_k,e_s)c(e^k)c(e^s) &+ \frac{1}{4}\sum_{l=1}^{n} g(R(e_i,e_j)e_s,e_l)c(e^s)c(e^l)c(e^s) \\ + c(e^s)\frac{1}{4}\sum_{l=1}^{n} g(R(e_i,e_j)e_s,e_l)c(e^s)c(e^l). \end{aligned}$$

By Clifford multiplication all four terms take the form $\frac{1}{4}\sum_{l=1}^n g(R(e_i,e_j)e_s,e_l)c(e^l)$. Together with (4) we find that $[R^W(e_i,e_j), c(e^k)] = c(Re_k)$, and this proves the claim.
If c denotes the quantization map $\Lambda \to \mathrm{Cl}$, then $F^{W/S} \in \Lambda^2(\mathrm{End}_{\mathrm{Cl}(\mathcal{A}^*)}(W))$ has the image under c

$$c(F^{W/S}) = \sum_{i<j} F^{W/S}(e_i,e_j)c(e^i)c(e^j).$$

The scalar curvature κ is given by

$$\kappa = \sum_{ik} R_{ikik},\ \ R_{ijkl} := g(R(e_i,e_j)e_k,e_j).$$

Write $D = \sum_i c(e^i)\nabla_i^W$ for ∇_i^W the covariant derivative in direction e_i. This gives

$$D^2 = \frac{1}{2}\sum_{i,j}[c(e^i), c(e^j)]\nabla_i^W \nabla_j^W + \sum_{i,j} c(e^i)[\nabla_i^W, c(e^j)] + \frac{1}{2}\sum_{i,j} c(e^i)c(e^j)[\nabla_i^W, \nabla_j^W].$$

By Clifford multiplication we have $[c(e^i), c(e^j)] = -2g^{ij}$, hence the first formula becomes $-\sum_{i,j} g^{ij}\nabla_i^W \nabla_j^W$. Secondly, by admissibility of ∇^W it follows $[\nabla_i^W, c(e^j)] = c(\nabla_i e^j)$. Write $\nabla_i e^j = -\sum_k \Gamma_{ik}^j e^k$ in terms of Christoffel symbols. Then $[\nabla_i^W, c(e^j)] = -\sum_k \Gamma_{ik}^j c(e^k)$. Using the symmetry of Γ_{ik}^j in i and k rewrite the second term

$$\sum_{ij} c(e^i)[\nabla_i^W, c(e^j)]\nabla_j^W = \frac{1}{2}\sum_{i,k}[c(e^i), c(e^k)]\sum_k \Gamma_{ik}^j \nabla_j^W = -\sum_{i,k} g^{ik}\sum_k \Gamma_{ik}^j \nabla_j^W.$$

For the third term consider the curvature tensor R^{∇^W} and use $[e_i, e_j] = 0, \ i \neq j$ to obtain

$$[\nabla_i^W, \nabla_j^W] = R^{\nabla^W}(e_i, e_j).$$

Putting everything together D^2 is rewritten as

$$D^2 = -\sum_{i,j} g^{ij}(\nabla_i^W \nabla_j^W - \sum_k \Gamma_{ij}^k \nabla_k^W) + \frac{1}{2}\sum_{i,j} c(e^i)c(e^j)R^{\nabla^W}(e_i, e_j).$$

Notice that the first term on the right is Δ^W. By the claim, the second term is rewritten

$$\frac{1}{2}\sum_{i,j} c(e^i)c(e^j)R^{\nabla^W}(e_i, e_j) = -\frac{1}{8}\sum_{ijkl} R_{ijkl}c(e^i)c(e^j)c(e^k)c(e^l) + c(F^{W/S}).$$

Rewrite $c(e^i)c(e^j)c(e^k)$ as in (5), recall the Bianchi identity $R_{ijkl} + R_{kijl} + R_{jkil} = 0$ and apply it together with (5) to obtain

$$\sum_{ijkl} R_{ijkl}c(e^i)c(e^j)c(e^k)c(e^l) = -\sum_{ijkl} R_{ijkl}(-\delta^{ij}c(e^k) - \delta^{jk}c(e^i) + \delta^{ki}c(e^j))c(e^l) = -\sum_{ilk} R_{iikl}c(e^k)c(e^l) - \sum_{ikl} R_{ikkl}c(e^i)c(e^l) + \sum_{jkl} R_{kjkl}c(e^j)c(e^l).$$

Since R is antisymmetric in the first two entries, the first term on the right hand side vanishes. Renaming indices we obtain

$$\sum_{ijkl} R_{ijkl}c(e^i)c(e^j)c(e^k)c(e^l) = 2\sum_{ijk} R_{jkik}c(e^j)c(e^i).$$

Since $\sum_{ij} R_{jkik}c(e^j)c(e^i) = -\sum_i R_{ikik}$ we obtain the result. □

3. Heat kernel approximation for Lie groupoids

The heat kernel of a groupoid Laplacian is a convolution kernel which has the properties expected of the heat kernel. We recall the approximation of the heat kernel on Riemannian manifolds from Berline, Getzler and Vergne, [7] and the corresponding approximation on Lie groupoids. Note that if $\mathcal{G} \rightrightarrows M$ is a Lie groupoid over a Lie manifold $(M, \mathcal{A}, \mathcal{V})$ such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$, then an admissible metric g yields a C^∞-family of

Riemannian metrics $(g_x)_{x\in M}$ on the s-fibers $(\mathcal{G}_x)_{x\in M}$. We fix such a g and note that, by the definition of submersions on manifolds with corners, the s-fibers are smooth manifolds without corners, cf. [26]. Additionally, $(\mathcal{G}_x, g_x)$ is a Riemannian manifold with uniformly bounded geometry and we refer to [46] for a proof of this.
The class of Lie manifolds $(M, \mathcal{A}, \mathcal{V})$ we consider in this section will be non-degenerate Lie structures whose integrating groupoid is Hausdorff.
We will give examples of such Lie structures below.
Let us fix for the moment a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a Haar system $\{\mu_x\}_{x\in M}$ on $\mathcal{G}$ such that there is a *length function*, i.e. a function $\varphi\colon \mathcal{G} \to \overline{\mathbb{R}}_+$ which has the following properties:
i) $\varphi(\gamma_1\gamma_2) \leq \varphi(\gamma_1) + \varphi(\gamma_2)$ for $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$.
ii) $\varphi(\gamma^{-1}) = \varphi(\gamma)^{-1},\ \gamma \in \mathcal{G}$.
iii) φ is proper.
In [23] the authors require in addition:
iv) φ is of polynomial growth, i.e. there is a $C > 0$ and $N \in \mathbb{N}$ such that for each $r \in \mathbb{R}_+$ we have $\mu_x(\varphi^{-1}([0,r])) \leq C(r^N + 1)$.
Condition *iv)* guarantees that for k sufficiently large the integral $\int_{\mathcal{G}_x} \frac{1}{(1+\varphi(\gamma))^k}\, d\mu_x(\gamma)$ remains uniformly bounded. We will recall below some of the consequences of this additional property. Though we remark at the outset that the Schwartz spaces we will consider do not need this assumption.
A vector field v in $\Gamma(\mathcal{A}(\mathcal{G})) = \mathcal{V}$ can be regarded as a $\mathcal{G}$-invariant first order differential operator on $\mathcal{G}$ (by lifting v to the s-vertical tangent bundle of the groupoid). We denote by $(v_1, \cdots, v_l) \mapsto \omega_{\overline{v},i}$ the distributional action $\omega_{\overline{v},i}(f) = v_1 \cdots v_i f v_{i+1} \cdots v_l$ for $f \in C_0(\mathcal{G})$. Define

$$S_\varphi^{k,0}(\mathcal{G}) := \{f \in C_0(\mathcal{G}) : \sup_{\gamma\in\mathcal{G}} |\mathbb{P}(\varphi(\gamma))f(\gamma)| < \infty,\ \mathbb{P} \in \mathbb{R}[X],\ \deg\mathbb{P} = k\}.$$

Also define the spaces

$$S_\varphi^{k,l}(\mathcal{G}) := \{f \in C^0(\mathcal{G}) : \|f\|_{\mathbb{P},l} < \infty,\ \mathbb{P} \in \mathbb{R}[X],\ \deg(\mathbb{P}) = k\}.$$

Here we denote by $\|\cdot\|_{\mathbb{P},l}$ for $l \in \mathbb{N}$ and a given polynomial $\mathbb{P} \in \mathbb{R}[X]$ of degree k, the seminorms

$$\|f\|_{\mathbb{P},l} := \sup_{1\leq i\leq l} \sup_{\|v_j\|\leq 1,\ \overline{v}=(v_1,\cdots,v_l)\in\mathcal{V}} \sup_{\gamma\in\mathcal{G}} |\mathbb{P}(\varphi(\gamma))\omega_{\overline{v},i}(f)|.$$

Proposition 3.1. *The spaces $\{S_\varphi^{k,l}(\mathcal{G})\}_{k,l\in\mathbb{N}}$ form a dense projective system of Banach spaces.*

Proof. We have the semi-norm system $\|\cdot\|_{\mathbb{P},l}$ given by

$$f \mapsto \sup_{1\leq i\leq l} \sup_{\|v_j\|\leq 1,\ \overline{v}=(v_1,\cdots,v_l)\in\mathcal{V}} \sup_{\gamma\in\mathcal{G}} |\mathbb{P}(\varphi(\gamma))\omega_{\overline{v},i}(f)|$$

and parametrized by $\mathbb{P} \in \mathbb{R}[X]$ and $l \in \mathbb{N}$. We also have the system $\{\|\cdot\|_{k,l}\}_{k,l\in\mathbb{N}}$ where

$$f \mapsto \|f\|_{k,l} := \sup_{1\leq i\leq l} \sup_{\|v_j\|\leq 1,\ \overline{v}=(v_1,\cdots,v_l)} \sup_{\gamma\in\mathcal{G}} (1+\varphi(\gamma))^k |\omega_{\overline{v},i}(f)|.$$

It is easy to check that the two systems are equivalent.
For the projectivity we observe that if l is fixed and $k_1 \geq k_2$ then $\|\cdot\|_{k_1,l} \leq \|\cdot\|_{k_2,l}$. Secondly, if k is fixed and $l_1 \geq l_2$ then $\|\cdot\|_{k,l_1} \leq \|\cdot\|_{k,l_2}$. The density of the inclusions is immediate. We obtain that $\{S_\varphi^{l,k}(\mathcal{G})\}_{(l,k)\in\mathbb{N}^2}$ forms a dense projective system of Banach spaces. □

Definition 3.2. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with length function $\varphi\colon \mathcal{G} \to \mathbb{R}_+$. Then define the space of rapidly decaying distributions as the dense projective limit

$$\mathbf{S}_\varphi(\mathcal{G}) := \varprojlim_{k,l\in\mathbb{N}} S_\varphi^{k,l}(\mathcal{G}).$$

If the length function is of polynomial growth the class is closed under holomorphic functional calculus, see [23, Theorem 7.5].

Proposition 3.3. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with polynomial length function φ. Then $\mathbf{S}_\varphi(\mathcal{G})$ is a $*$-subalgebra of $C_r^*(\mathcal{G})$, stable with regard to holomorphic functional calculus.*

In fact, it is shown in [23, Lemma 7.8] that $S_\varphi^{k,l}(\mathcal{G})$ is closed under holomorphic functional calculus in $C_r^*(\mathcal{G})$ for large k, hence so is $\mathbf{S}_\varphi(\mathcal{G})$.

Example 3.4. *i)* Let M be a compact manifold with embedded corners and $\{\rho_i\}_{i=1}^N$ a set of boundary defining functions. The boundary of M is stratified by the closed, codimension one hyperfaces $F_i = \{\rho_i = 0\}$:

$$\partial M = \bigcup_{1\le i\le N} F_i.$$

We consider the Lie structure $\mathcal{V}_b := \{V \in \Gamma^\infty(TM) : V \text{ tangent to } F_i,\ 1 \le i \le N\}$. The Lie algebroid $\mathcal{A} \to M$ is the b-tangent bundle such that $\Gamma(\mathcal{A}) \cong \mathcal{V}_b$. Following Monthubert [34], we find a Lie groupoid $\mathcal{G}_b(M)$ integrating $\mathcal{A}$ which is s-connected, Hausdorff and amenable: We start with the set

$$\Gamma_b(M) = \{(x, y, \lambda) \in M \times M \times (\mathbb{R}_+)^N : \rho_i(x) = \lambda_i\rho_i(y),\ 1 \le i \le N\}$$

endowed with the structure $(x, y, \lambda)\circ(y, z, \mu) = (x, z, \lambda\cdot\mu)$, $(x, y, \lambda)^{-1} = (y, x, \lambda^{-1})$ and $r(x, y, \lambda) = x$, $s(x, y, \lambda) = y$, $u(x) = (x, x, 1)$. Here multiplication $\lambda \cdot \mu$ and inversion λ^{-1} are componentwise.
We then define the b-groupoid $\mathcal{G}_b(M)$ as the s-connected component (the union of the connected components of the s-fibers of $\Gamma_b(M)$), i.e. $\mathcal{G}_b(M) := \mathcal{C}_s\Gamma_b(M)$. The b-groupoid has the polynomial length function $\varphi(x, y, \lambda) = |\ln(\lambda)|$, cf. [23].
ii) Let M be a compact manifold with corners as in the previous example. Fix the Lie structure $\mathcal{V}_{c_n}$ of generalized cusp vector fields for $n \ge 2$ given by the local generators in a tubular neighborhood of a boundary hyperface: $\{x_1^n\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}\}$. Let us recall the construction of the associated Lie groupoid $\mathcal{G}_n(M)$, the so-called cusp groupoid, given in [23] for the benefit of the reader. We set

$$\Gamma_n(M) := \{(x, y, \mu) \in M \times M \times (\mathbb{R}_+)^N : \mu\rho_i(x)^n\rho_i(y)^n = \rho_i(x)^n - \rho_i(y)^n\}$$

with structure $r(x, y, \lambda) = x$, $s(x, y, \lambda) = y$, $u(x) = (x, x, 0)$ and $(x, y, \lambda)(y, z, \mu) = (x, z, \lambda + \mu)$. We then define $\mathcal{G}_n(M)$ as the s-connected component of $\Gamma_n(M)$. There exists a homeomorphism $\Theta_n\colon \mathcal{G}_b(M) \to \mathcal{G}_n(M)$ given by $(x, y, \lambda) \mapsto (u, v, \mu)$ as follows. Assume first that M has only one boundary hyperface, i.e. M is a manifold with boundary. The generalization to arbitrarily many hyperfaces is easy. We then partition M into $M = \mathcal{U} \cup (M \setminus \mathcal{U})$ where $\mathcal{U}$ is a tubular neighborhood of the boundary. Then

$$u = \begin{cases} x,\ x \in M \setminus \mathcal{U} \\ \pi^{-1} \circ \tau_n \circ \pi(x),\ x \in \mathcal{U} \end{cases}$$

and

$$v = \begin{cases} y, \ y \in M \setminus \mathcal{U} \\ \pi^{-1} \circ \tau_n \circ \pi(y), \ y \in \mathcal{U} \end{cases}$$

Here $\tau_n \colon \mathbb{R}_+ \to \mathbb{R}_+$ is the continuous and strictly increasing function given by

$$t \mapsto \begin{cases} \frac{1}{e}(-\ln(t))^{-\frac{1}{n}}, \ t \in \left(0, \frac{1}{e}\right), \\ 0, \ t = 0, \\ t, \ t \geq \frac{1}{e} \end{cases} .$$

Set $\mu = \log(\lambda)$ and check with the above that $\mu\rho(u)^n\rho(v)^n = \rho(u)^n - \rho(v)^n$. This transformation in a tubular neighborhood of the boundary motivates the definition of the cusp groupoid. The polynomial length function on the cusp groupoid is then obtained by $\varphi_n := \varphi \circ \Theta_n^{-1}$ where φ denotes the polynomial length function of the b-groupoid and Θ_n is the homeomorphism constructed above. We obtain that $\varphi(x, y, \mu) = |\mu|$.
iii) The following example of the fibered cusp calculus is from Mazzeo and Melrose [28] and we use the formulation and notation for manifolds with fibered corners as given in [15]. We briefly recall the definition of the associated groupoid and refer to loc. cit. for the details. See also [18] for the precise geometric construction of the fibered cusp groupoid and the associated polynomial length function in this case. Let M be a manifold with embedded and fibered corners. We denote by $\{F_i\}_{i=1}^N$ the boundary hyperfaces of M with boundary defining functions ρ_i and write $\pi = (\pi_1, \cdots, \pi_N)$, where $\pi_i \colon F_i \to B_i$ are fibrations; B_i is the base, which is a compact manifold with corners. Define the Lie structure

$$\mathcal{V}_\pi := \{V \in \mathcal{V}_b : V_{|F_i} \text{ tangent to the fibers } \pi_i \colon F_i \to B_i, \ V\rho_i \in \rho_i^2 C^\infty(M)\}.$$

Then $\mathcal{V}_\pi$ is a finitely generated $C^\infty(M)$-module and a Lie sub-algebra of $\Gamma^\infty(TM)$. The corresponding groupoid is amenable [15, Lemma 4.6]; as a set it is defined as

$$\mathcal{G}_\pi(M) := (M_0 \times M_0) \cup \left(\bigcup_{i=1}^N (F_i \times_{\pi_i} T^\pi B_i \times_{\pi_i} F_i) \times \mathbb{R}\right),$$

where $T^\pi B_i$ denotes the algebroid of B_i.

Reduced metric distance. In view of the right invariance of the action of $\mathcal{G}$ on itself we consider the family of metrics $(g_x)_{x \in M}$ on the s-fibers of the groupoid. We denote the family of induced metric distances by $(d_x)_{x \in M}$ and note that this is a $\mathcal{G}$-invariant family as well, i.e.,

$$d_{s(\gamma)}(\gamma_1\gamma, \gamma_2\gamma) = d_{r(\gamma)}(\gamma_1, \gamma_2).$$

Given $\gamma, \eta \in \mathcal{G}_{s(\gamma)}$ we see from this that $d_{s(\gamma)}(\gamma, \eta) = d_{r(\gamma)}(\mathrm{id}_{r(\gamma)}, \eta\gamma^{-1})$. Hence we can define a *reduced metric distance* by $\psi(\gamma) := d_{s(\gamma)}(\mathrm{id}_{s(\gamma)}, \gamma)$.

Lemma 3.5. *The reduced metric distance $\psi(\gamma) = d_{s(\gamma)}(\mathrm{id}_{s(\gamma)}, \gamma)$ is a length function, i.e. if $(\gamma, \eta) \in \mathcal{G}^{(2)}$ then $\psi(\gamma\eta) \leq \psi(\gamma) + \psi(\eta)$ and for each $\gamma \in \mathcal{G}$ we have $\psi(\gamma^{-1}) = \psi(\gamma)$.*

Proof. First apply the triangle inequality, the $\mathcal{G}$-invariance and the fact that $r(\eta) = s(\gamma)$ by composability to obtain

$$\begin{aligned}\psi(\gamma\eta) &= d_{s(\gamma\eta)}(\mathrm{id}_{s(\gamma\eta)}, \gamma\eta) \\ &\leq d_{s(\gamma\eta)}(\mathrm{id}_{s(\gamma\eta)}, \eta) + d_{s(\gamma\eta)}(\eta, \gamma\eta) \\ &= d_{s(\eta)}(\mathrm{id}_{s(\eta)}, \eta) + d_{s(\gamma)}(\mathrm{id}_{s(\gamma)}, \gamma).\end{aligned}$$

Secondly, by right invariance

$$\begin{aligned}\psi(\gamma^{-1}) &= d_{s(\gamma^{-1})}(\mathrm{id}_{s(\gamma^{-1})}, \gamma^{-1}) = d_{r(\gamma^{-1})}(\gamma, \mathrm{id}_{r(\gamma^{-1})}) \\ &= d_{s(\gamma)}(\gamma, \mathrm{id}_{s(\gamma)}) = \psi(\gamma).\end{aligned}$$

□

Definition 3.6. Given a Hausdorff Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$, we write $\mathbf{S}(\mathcal{G}) := \mathbf{S}_\psi(\mathcal{G})$, where ψ is the reduced metric distance of $\mathcal{G}$.

On the Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)} = M$ we introduce the heat kernel for the generalized (twisted) Laplacian $\not{D}^2$ depending on the admissible connection $\not{\nabla}^W$ (see Section 2 for the definitions and notation). If $g = g_{\mathcal{A}}$ is a compatible metric induced on $(M, \mathcal{A}, \mathcal{V})$, then (M, g) is also a manifold with bounded geometry, see [2]. Let us fix an invariant connection ∇ on $\mathcal{G}$ which is obtained from the $\mathcal{G}$-invariant family of connections $(\nabla_x)_{x\in\mathcal{G}^{(0)}}$ associated to the $\mathcal{G}$-invariant family of metric $(g_x)_{x\in\mathcal{G}^{(0)}}$. Hence by varying $x \in \mathcal{G}^{(0)}$ we obtain from the family of exponential mappings $\exp_x \colon T\mathcal{G}_x \to \mathcal{G}_x$ an exponential mapping $\exp^\nabla \colon \mathcal{A} \to \mathcal{G}$, see [38, p. 128f]. Let $r_0 > 0$ be the bounded injectivity radius. Then the induced exponential mapping $\exp^\nabla$ maps $(\mathcal{A})_{r_0} := \{v \in \mathcal{A} : \|v\|_g < r_0\}$ diffeomorphically onto its image $\mathcal{B}_{r_0} := \{\gamma \in \mathcal{G} : d_{s(\gamma)}(\gamma, s(\gamma)) < r_0\}$. We fix polar coordinates (p, θ) on $\mathcal{A}_x$ such that $d(\exp^\nabla(p, \theta), x) = p$. Define the radial vector field $\partial_{\mathcal{R}} := d(\gamma, s(\gamma)) d\exp^\nabla(\partial_p)$, $s(\gamma) = x$, and set $J := \det(d\exp^\nabla) \circ \exp^\nabla)^{-1}$.
Consider the pullbacks $r^*W \to \mathcal{G}$ and $s^*W \to \mathcal{G}$ of the Clifford module $W \to M$. The parallel transport for $\gamma = \exp^\nabla v \in \mathcal{B}_{r_0}$, $w \in W_{s(\gamma)}$, $v \in \mathcal{A}_{s(\gamma)}$ is given by $\tau(\gamma)(w) \in r^*W_\gamma$ of w to γ along $\exp^\nabla(tv)$, $t \in [0, 1]$. Hence we have defined a map

$$\tau \colon \{(\gamma, w) : \gamma \in \mathcal{B}_{r_0},\ w \in W_{s(\gamma)}\} \to r^*W_{|\mathcal{B}_{r_0}}$$

The inverse is given by $\tau^{-1} \colon r^*W_{|\mathcal{G}_x \cap \mathcal{B}_r} \to W_x$.
Denote by $r^*W \otimes s^*W^* \times (0, \infty)$ the pullback of the vector bundle $r^*W \otimes s^*W^* \to \mathcal{G}$ along the projection $\mathcal{G} \times (0, \infty) \to \mathcal{G}$.
The groupoid heat kernel is a C^0-section $Q \in \Gamma^0(r^*W \otimes s^*W^* \times (0, \infty))$ such that for $Q_t = Q(t, \cdot)$
i) the heat equation $(\partial_t + \not{D}^2)Q_t(\gamma) = 0$ and
ii) the initial condition $\lim_{t\to 0} Q_t * u = u$ hold for each $u \in \Gamma_c^\infty(r^*W \otimes s^*W^*)$.
Since the generalized Laplacian $\not{D}^2$ on $\mathcal{G}$ comes from an equivariant family (compare the remarks in Section 2) the map $\mathcal{G}_x \times \mathcal{G}_x \ni (\gamma, \eta) \mapsto Q_t(\gamma\eta^{-1})$ defines a heat kernel for $\not{D}_x^2$ on $\mathcal{G}_x$ for each $x \in M$. Since $\mathcal{G}_x$ has bounded geometry the heat kernel of $\not{D}_x^2$ is unique (cf. [7]). Hence, by $\mathcal{G}$-invariance, Q must be unique as well.
We repeat the formal heat kernel approximation from [7, Section 2.5] and more specifically from [45].
Let $q \colon \mathcal{B}_{r_0} \times (0, \infty) \to \mathbb{R}$ denote the Gaussian

$$q(\gamma, t) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(\gamma, s(\gamma))^2}{4t}}.$$

According to [7] and [46] there is a formal power series

$$\Phi(\gamma, t) = \sum_{i=0}^{\infty} t^i \Phi_i(\gamma), \ \Phi_i \in \Gamma^\infty(r^* W \otimes s^* W^*) \tag{6}$$

such that $q(\gamma, t)\Phi(\gamma, t) \in \Gamma^\infty(r^* W \otimes s^* W^* \times (0, \infty))$ and

$$(\partial_t + \not{D}^2) q(\gamma, t)\Phi(\gamma, t) = q(\gamma, t) \left(\partial_t + \not{D}^2 + r^* \nabla^E_{\partial_{\mathcal{R}}} + \frac{\mathcal{L}_{\partial_{\mathcal{R}}} J}{2tJ} \right) \Phi(\gamma, t). \tag{7}$$

Moreover, we have the recursive relations

$$\Phi_0(\exp V) = J^{-\frac{1}{2}} \tau(\exp V),$$
$$\Phi_i(\exp V) = -J^{-\frac{1}{2}} \tau \int_0^1 J^{\frac{1}{2}} \tau^{-1}((\partial_t + \not{D}^2)\Phi_{i-1}(\exp(tV)) t^{i-1} \, dt.$$

Theorem 3.7. *Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be a Hausdorff Lie groupoid and let $\not{D}$ be a geometric Dirac operator adapted to an admissible connection. Then for $t > 0$ we have $e^{-t\not{D}^2} \in \mathbf{S}(\mathcal{G})$.*

Proof. We use the heat kernel approximation and the bounded geometry of the groupoid fibers. First note that the groupoid heat kernel is smooth by the results of [45].
Secondly, we recall from [46, Proposition 4.12] the following off-diagnoal estimate of the heat kernel.
Claim: Let $\epsilon > 0$ be given. If $t > 0$ is fixed we have the estimate:

$$\forall_{\lambda > 0} \ \exists_{C > 0} \ |e^{-t\not{D}^2}(\gamma)| \leq C e^{-\lambda \psi(\gamma)}, \ \gamma \in \mathcal{G}, \ \psi(\gamma) > 2\epsilon. \tag{8}$$

We make use of this fact to estimate the heat groupoid kernel Q_t in the semi-norms of $\mathbf{S}(\mathcal{G})$. By the formal solution, the heat kernel Q_t takes the form $Q_t(\gamma) = q(\gamma, t)\Phi(\gamma, t)$. Then we obtain that

$$\|Q_t\|_{k,l} = \sup_{1 \leq i \leq l} \sup_{\overline{v} = (v_1, \cdots, v_l) \in \mathcal{V}^l, \|v_j\| \leq 1} \sup_{\gamma \in \mathcal{G}} (1 + \psi(\gamma))^k |\omega_{\overline{v}, i}(Q_t)|$$
$$= (4\pi t)^{-\frac{n}{2}} \sup_{1 \leq i \leq l} \sup_{\|v_j\| \leq 1} \sup_{\gamma \in \mathcal{G}} (1 + d_{s(\gamma)}(s(\gamma), \gamma))^k |v_1 \cdots v_i e^{-\frac{d(s(\gamma), \gamma)^2}{4t}} \Phi(\gamma, t) v_{i+1} \cdots v_l|.$$

Using the recursive definition of Φ_i and the bounded geometry we obtain that the uniform norm over $\Phi(\gamma, t)$ is bounded by (8). See also [46], Prop. 4.12. The boundedness of $e^{-\frac{d(\gamma, s(\gamma))^2}{4t}}$ and all its derivatives follows also by the uniformly bounded geometry of the s-fibers of the groupoid. Secondly, we can estimate the exponential by noting that it decays faster than any polynomial in the length function $\psi(\gamma)$. The vector fields v_j are $\mathcal{G}$-invariant differential operators of first order and the action of these vector fields leaves $Q_t(\gamma)$ bounded by the previous remarks. It is an elementary computation to check that after application of the action of the vector fields we obtain a term of the form polynomial in $\psi(\gamma)$ and $e^{-\psi(\gamma)^2}$. Hence by (8) the heat kernel is bounded in the semi-norms defining the Schwartz class and the assertion follows. ☐

4. Adiabatic deformation and functional calculus

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ over a smooth manifold M with corners we define the *adiabatic groupoid* $\mathcal{G}^{ad} \rightrightarrows M \times I$ where I is either the closed interval $[0, 1]$ or the real numbers $\mathbb{R}$. We also write $I^* := I \setminus \{0\}$. Formally, the groupoid $\mathcal{G}^{ad}$ is defined as $\mathcal{G}^{ad} = \mathcal{G} \times I^* \cup \mathcal{A}(\mathcal{G}) \cup \{0\}$. The groupoid structure is defined over $t \neq 0$ to be the structure of $\mathcal{G}$ and I^*, where the latter is simply viewed as the trivial set groupoid

with units I^*. Over $t = 0$ the structure is given by that of $\mathcal{A}(\mathcal{G})$, where we view $\mathcal{A}(\mathcal{G}) = \bigcup_{x \in M} T_{u(x)}\mathcal{G}_x$ as a bundle with fiberwise defined Lie group structure.
Note right away the most important special case of this definition: If M is a smooth manifold without boundary or corners, and $M \times M \rightrightarrows M$ is the pair groupoid, we recover with $(M \times M)^{ad}$ the *tangent groupoid* due to A. Connes, [10]. In the more general situation we are in, where M may have a boundary or corners, we need an integrating groupoid $\mathcal{G}$ which is more general, in particular longitudinally smooth.
Most importantly for us is the smooth structure defined on $\mathcal{G}^{ad}$, which we will need later. In the special case of the tangent groupoid we fix a Riemannian metric on M with Levi-Civita connection and define the topology of $(M \times M)^{ad}$ via a glueing using the exponential mapping $\exp\colon TM \to M$, cf. [10] for the tangent groupoid. In the more general case of the adiabatic groupoid at hand we need the so-called *generalized exponential* $\mathrm{Exp}\colon \mathcal{A}(\mathcal{G}) \to \mathcal{G}$ from [19]. This definition together with the *groupoid parametrization* defined in the same paper will be most convenient for the calculation of the Lichnerowicz formula on the fibers of the adiabatic groupoid in the final section of this article.

Definition 4.1. Write $T^s\mathcal{G} := \ker ds \subseteq T\mathcal{G}$ and let $\pi : T\mathcal{G} \to \mathcal{G}$ be the canonical projection. The right *generalized exponential map* $\mathrm{Exp}^R\colon \mathcal{A}(\mathcal{G}) \to \mathcal{G}$ is defined by

$$\mathrm{Exp}^R(V) := \pi_{\ker ds \to \mathcal{G}}(V(1))$$

where $V(t)$ is the solution to the flow equation $V'(t) = l_{V(t)}V(t),\ V(0) = V \in \mathcal{A}(\mathcal{G})$ (provided it is defined in $t = 1$) and $l_W : T_{\pi W}\mathcal{G} \to T_W T^s\mathcal{G}$ is the so-called adapted horizontal lift, which will be defined below.

Note that a left exponential map can be defined analogously with $\ker dr$ in place of $\ker ds$ in the above statement.
We start with the following observation:

Lemma 4.2. *We have an isomorphism* $\ker ds \cong r^*\mathcal{A}(\mathcal{G})$ *implemented by the right multiplication* dR_γ *applied fiberwise.*

Proof. By definition $(r^*\mathcal{A}(\mathcal{G}))_\gamma = \left(\bigcup_{x \in \mathcal{G}^{(0)}} T_{u(x)}\mathcal{G}_x\right)_\gamma = T_{u(r(\gamma))}\mathcal{G}_{r(\gamma)}$. The application of the right multiplication $R_\gamma\colon \mathcal{G}_{r(\gamma)} \to \mathcal{G}_{s(\gamma)}$ yields

$$\begin{array}{ccc} T_{u(r(\gamma))}\mathcal{G}_{r(\gamma)} & --- & \mathcal{A}(\mathcal{G})_{r(\gamma)} \\ {\scriptstyle dR_\gamma}\downarrow & & \downarrow \\ T_{u(s(\gamma))}\mathcal{G}_{s(\gamma)} & --- & \mathcal{A}(\mathcal{G})_{s(\gamma)} \end{array}$$

This yields a well-defined map $dR_\gamma\colon (r^*\mathcal{A}(\mathcal{G}))_\gamma \to (\ker ds)_\gamma$ for each $\gamma \in \mathcal{G}$ since $ds \circ dR_\gamma = d(s \circ R_\gamma) = d(s(\gamma)) = 0$. The inverse is given by $(dR_\gamma)^{-1} = dR_\gamma^{-1} = dR_{\gamma^{-1}}$. ☐

Given a smooth manifold B and a vector bundle $\pi\colon E \to B$ with connection ∇^E, we obtain a splitting $TE = T^{vert}E \oplus T^{hor}E$ of TE with $T^{vert}E = \ker d\pi$. Associated with the decomposition we have a lift of vectors: for $b \in B$ and $e \in E_b$ we have a lift

$$l_e^E : T_bB \to T_e^{hor}E$$

via parallel transport. We can also lift a curve $\gamma\colon [0,1] \to B$. Let $\gamma(0) = b_0$ and $v_0 \in E_{b_0}$. Then we obtain the lift $\Gamma = l_{v_0}^E(\gamma)$ of γ by solving the initial value problem

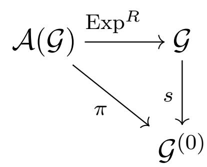
$$\dot{\Gamma}(t) = l_{\Gamma(t)}^E(\dot{\gamma}(t)), \quad \Gamma(0) = v_0.$$

In order to define the adapted horizontal lift, we recall that $\mathcal{A}(\mathcal{G})$ is the restriction of $\ker ds$ to the units or the image under the pullback of the unit map, i.e. $\mathcal{A}(\mathcal{G}) = u^* T^s\mathcal{G}$. Hence the connection ∇ on $\mathcal{A}(\mathcal{G})$ can be lifted to a connection $\tilde{\nabla}$ on $r^*\mathcal{A}(\mathcal{G}) \cong T^s\mathcal{G}$. Applied to our setting, we denote by $l_{\bullet}^{\mathcal{A}(\mathcal{G})}$ the horizontal lift to the bundle $\mathcal{A}(\mathcal{G}) \to \mathcal{G}^{(0)}$. The *adapted horizontal lift* l of a tangent vector $V = \frac{d\gamma(t)}{dt}_{|t=0}$ in $T_\gamma\mathcal{G}$ to $W \in T_\gamma^s\mathcal{G}$ is defined by

$$l_W(V) = \frac{d}{dt}\left[dR_{\gamma(t)^{-1}} l_{dR_\gamma(W)}^{\mathcal{A}(\mathcal{G})}(s(\gamma(t)))\right]_{t=0}.$$

Here, $l^{\mathcal{A}(\mathcal{G})}$ lifts the curve $t \mapsto s(\gamma(t))$. This defines a map $l_W \colon T_\gamma\mathcal{G} \to T_W(\ker ds)$, where $\gamma(t)$ is a geodesic in $\mathcal{G}$ with $V = \dot{\gamma}(0)$ and $\gamma(0) = \gamma$ defined via parallel transport $V \mapsto V(t)$ using the connection $\tilde{\nabla}$ such that $\dot{V}(t) = l_{V(t)}V(t)$ holds.

Lemma 4.3. *The following diagram commutes*



i.e. $s(\mathrm{Exp}^R(V)) = \pi(V)$ *holds.*

Proof. Fix a geodesic $\gamma_V(t)$ in $\mathcal{G}$. By definition we have $s(\gamma_V(0)) = \pi(V)$ and $\frac{ds\gamma_V(t)}{dt} = ds(V(t)) = 0$. ☐

Finally, we recall the *tubular neighborhood theorem* of the generalized exponential from [19]. We refer to loc. cit. for the proof. There is an open neighborhood $\mathcal{G}^{(0)} \subset V \subset \mathcal{A}(\mathcal{G})$ of the zero section in $\mathcal{A}(\mathcal{G})$ and an associated open neighborhood $\mathcal{G}^{(0)} \subset W \subset \mathcal{G}$ of the unit space in $\mathcal{G}$ such that $\mathrm{Exp}^R(x) = x$ for each $x \in \mathcal{G}^{(0)}$ and Exp^R induces a diffeomorphism of V and W.
¿From now on we simply write $\mathrm{Exp} := \mathrm{Exp}^R$, where it is understood that all our constructions are right invariant. We are in a position to define the smooth structure of the adiabatic groupoid $\mathcal{G}^{ad} \rightrightarrows \mathcal{G}^{(0)} \times I$.
It is defined by the glueing

$$\mathcal{A}(\mathcal{G}) \times \{0\} \supset \mathcal{O} \ni (x, v, t) \mapsto \begin{cases} (x, v), \ t = 0 \\ (\mathrm{Exp}(-tv), t), \ t > 0 \end{cases}$$

where $\mathcal{O}$ denotes an open neighborhood of $\mathcal{A}(\mathcal{G}) \times \{0\}$ in $\mathcal{G}^{ad}$.

Let $(\mathcal{M}, g)$ denote a complete Riemannian manifold and $S \to \mathcal{M}$ the bundle of spinors. The next goal is to define the functional calculus from a suitable class of rapidly decaying functions into the groupoid convolution algebra. Note that for complete Riemannian manifolds there is a spectral theorem for such operators, see also [9], [35].
Given a Dirac operator D acting on smooth sections of S, we define for a given $f \in \mathbf{S}(\mathbb{R})$ the operator $f(D) = \frac{1}{2\pi}\int \hat{f}(t)e^{itD}\,dt$ in the weak sense, i.e. there is an $f(D)$ acting on $L^2(S)$ such that for each $s, \tilde{s} \in L^2$ we have

$$\langle f(D)s, \tilde{s}\rangle = \frac{1}{2\pi}\int \hat{f}(t)\langle e^{itD}s, \tilde{s}\rangle\,dt, \ \ s, \tilde{s} \in L^2(S).$$

Another notion we need to recall here is that of *finite propagation speed.*

If D is an operator of first order on $\mathcal{M}$ and σ_1 its principal symbol, we denote by

$$\mathbf{c}(x) := \sup\{\|\sigma_1(x,\xi)\| : \|\xi\| = 1\}$$

the *propagation speed* of D.

Definition 4.4. A first order differential operator D has *finite propagation speed* if there is a constant $C > 0$ such that we have the uniform bound $\mathbf{c}(x) \leq C$.

We recall the following theorem due to Chernoff from [9].

Theorem 4.5 (P. R. Chernoff, 1973)**.** *Let $D \colon \Gamma(E) \to \Gamma(E)$ be a first order differential operator over a non-compact complete manifold and assume that D is formally self-adjoint and has finite propagation speed. Then D^k is essentially self-adjoint for $k \in \mathbb{N}_0$.*

Proposition 4.6. *Let D be a Dirac operator acting on L^2-sections of the spinor bundle $S \to \mathcal{M}$.*
i) *The wave equation $\partial_t s = iDs$ with initial data $s_0 \in \Gamma_c^\infty(S)$ has a unique solution which preserves the L^2-norm.*
ii) *The operator $f(D)$ is well-defined and bounded on $L^2(S)$.*
iii) *The assignment $\mathbf{S}(\mathbb{R}) \to \mathcal{L}(L^2(S))$ is a ring-homomorphism such that $\|f(D)\| \leq \|f\|_\infty$.*
iv) *If $\hat{f}$ has compact support, then $f(D)$ is a smoothing operator with finite propagation speed, and $f(D)$ is essentially self-adjoint.*

Proof. i): See Proposition 7.4 of [41].
ii)-iii): Use the fact that the Fourier transform maps isomorphically $\mathbf{S}(\mathbb{R})$ into itself and the Cauchy-Schwarz inequality. The homomorphism property follows from the linearity of the Fourier transform, that pointwise multiplication is converted into convolution as well as the identity $e^{itD} = e^{isD}e^{i(t-s)D}$ which follows by uniqueness of solutions of the wave equation. The inequality follows by a reduction to the case of compact manifolds. We refer to the proof of Proposition 9.20 in [41].
iv): For the finite propagation speed property we refer to [9] as well as [41], p. 104 for the proof. See also [42]. The essential self-adjointness follows from the quoted theorem of Chernoff, Theorem 4.5. □

The following theorem is the generalization of the theorem for the tangent groupoid given by Siegel [43], Corollary 2 and Roe [41], Proposition 5.30, 5.31.
Denote by $\mathcal{P}$ the set of functions in the Schwartz class $\mathbf{S}(\mathbb{R})$ which have compactly supported Fourier transform.

Theorem 4.7. *Let $(M, \mathcal{A}, \mathcal{V})$ be a Lie manifold with spin structure $S \to M$ and $\mathcal{G} \rightrightarrows M$ a Lie groupoid such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$. Denote by $\mathbb{D} := (\not{D}_{x,t})_{(x,t) \in M \times I}$ an equivariant family of Dirac operators on $\mathcal{G}^{ad}$. Then there exists a ring homomorphism $\Psi_{\mathbb{D}} \colon \mathcal{P} \to C_c^\infty(\mathcal{G}^{ad}, \mathrm{Hom}(S))$ such that the regular representation $\pi_{x,t} \colon C_c^\infty(\mathcal{G}^{ad}, \mathrm{Hom}(S)) \to \mathcal{L}(L^2(\mathcal{G}_{(x,t)}^{ad}))$ fulfills the identity*

$$\pi_{x,t}(\Psi_{\mathbb{D}}(f)) = f(\not{D}_{x,t}), \quad f \in \mathcal{P}.$$

Proof. Applying Proposition 4.6(i) we fix the solution operator $e^{i\tau \not{D}_{x,t}}$ to the wave equation for $\not{D}_{x,t}$. For given $f \in \mathcal{P}$ we use the functional calculus to define $f(\not{D}_{x,t}) = \frac{1}{2\pi} \int \hat{f}(\tau) e^{i\tau \not{D}_{x,t}} \, d\tau$. By the part *iv)* of Proposition 4.6, $f(\not{D}_{x,t})$ is a smoothing operator with finite propagation speed. The equivariant family $(f(\not{D}_{x,t}))$ has a reduced kernel

which we denote by k^f, obtained from the equivariant family of Schwarz kernels $k^f_{x,t}$ defined on the fibers and smooth with regard to (x,t). By the finite propagation speed property k^f is a compactly supported distribution. We therefore define $\Psi_{\mathbb{D}}(f)$ via the assignment $\gamma \mapsto k^f_{s(\gamma)}$ in $C_c^\infty(\mathcal{G}^{ad})$. This is a continuous ring homomorphism by part Proposition 4.6(iii). The equation for the regular representation follows because k^f is the reduced kernel of the operator $f(\mathbb{D})$ on $\mathcal{G}^{ad}$.
Recall the definition of the representation $\pi_{x,t} \colon C_c^\infty(\mathcal{G}^{ad}, \mathrm{Hom}(S)) \to \mathcal{L}(L^2(\mathcal{G}^{ad}_{x,t}))$ given by

$$\pi_{x,t}(f)(\xi)(\gamma,t) = \int_{\mathcal{G}^{ad}_{x,t}} f(\eta,t)\xi(\gamma\eta^{-1},t)\, d\mu_{(x,t)}(\eta),\ \xi \in L^2(\mathcal{G}^{ad}).$$

This yields by definition of $\Psi_{\mathbb{D}}$ the L^2-action

$$f(\not{D}_{x,t})g(\gamma) = \pi_{x,t}(\Psi_{\mathbb{D}}(f))g(\gamma) = (\Psi_{\mathbb{D}}(f) * g)(\gamma)$$

and hence the last identity is proven. □

5. Renormalized Super Trace

We fix in the following a Lie manifold $(M, \mathcal{A}, \mathcal{V})$.
First we define a class of rapidly decaying distributions on the Lie manifold. This will be the class which contains the heat kernel and on which we define the renormalized super trace.
Let g be any compatible metric on M, i.e. a bilinear form on $\mathcal{A}$. Denote by the $d = d_g$ the metric distance induced by g. Note that the interior M_0 of M with the metric g restricted to it is a complete Riemannian manifold, cf. [2]. Fix an arbitrary point $\wr \in M_0$ and set $p(x) := d(\wr, x)$. We define the spaces ${}^{\mathcal{V}}S^{k,l}(M) := \{f \in C^\infty(M) : \|f\|_{k,l} < \infty\}$, where the semi-norm system $\|\cdot\|_{k,l}$ is defined for $k, l \in \mathbb{N}$ by

$$\|f\|_{k,l} := \sup_{1 \le i \le l} \sup_{\overline{v}=(v_1,\cdots,v_l)\in\mathcal{V}^l,\ \|v_i\|\le 1} \sup_{x\in M} |(1+p(x))^k \omega_{\overline{v},i}(f)(x)|.$$

We note that if $\overline{v} := (v_1, \cdots, v_l) \in \mathcal{V}^l$ then each v_i can be regarded as a differential operator of first order in $\mathrm{Diff}^1_{\mathcal{V}}(M)$. Secondly, we define $\omega_{\overline{v},i}(f) := v_1 \cdots v_i f v_{i+1} \cdots v_l$. In the same way as the proof of Proposition 3.1 we show that if $k_1 \ge k_2$, l is fixed we have $\|f\|_{k_1,l} \le \|f\|_{k_2,l}$ and if $l_1 \ge l_2$ with k fixed we have $\|f\|_{k,l_1} \le \|f\|_{k,l_2}$. Hence the spaces $\left({}^{\mathcal{V}}S^{k,l}(M)\right)_{(k,l)\in\mathbb{N}^2}$ form a dense projective system of Banach spaces.

Definition 5.1. The Schwartz space of rapidly decaying functions on the Lie manifold $(M, \mathcal{A}, \mathcal{V})$ is defined as the space ${}^{\mathcal{V}}\mathrm{S}(M)$ given by the projective limit

$${}^{\mathcal{V}}\mathrm{S}(M) := \varprojlim_{k,l\in\mathbb{N}} {}^{\mathcal{V}}S^{k,l}(M).$$

In the definition of a generalized trace class for the given Lie structure we face the problem that the density from the Lie structure is not integrable as we approach the boundary. See e.g. [20] for the b-case and the example below. The remedy is a regularization procedure. Similarly, one could define the canonical (KV) trace, the Wodzicki residue trace and the $\mathcal{V}$-determinant, but this is outside the scope of the present work.
Introduce the following notation: $\dot{C}^\infty(M, {}^{\mathcal{V}}\Omega^1)$ is the space of smooth functions vanishing to all orders at the boundary. For $F \in \mathcal{F}_1(M)$ let ρ_F be a boundary defining

function. Given a weight system $w : \mathcal{F}_1(M) \to \mathbb{N}_0$ we write $\rho^w = \prod_{F \in \mathcal{F}_1(M)} \rho_F^{w(F)}$ and let

$$\Psi_{\mathcal{V}}^{m,-w}(M) := \rho^{-w} \Psi_{\mathcal{V}}^m(M)$$

for the weighted pseudodifferential calculus, see [1].

In order to regularize the density defined on the Lie algebroid $\mathcal{A}$ via g we note that by definition $\mathcal{V} \subset \mathcal{V}_b$. On an arbitrary boundary face $F \in \mathcal{F}_1(M)$ fix the local coordinates $\{x_1, \cdots, x_n\}$ in a small tubular neighborhood $[0, \epsilon) \times F$. Then $\{x_1 \partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}\}$ is a local basis of $\mathcal{V}_b$. Consider the boundary defining function $\rho_F \colon M \to \overline{\mathbb{R}}_+$ and let $\nu \colon (-\epsilon, \epsilon) \times F \xrightarrow{\sim} \mathcal{U} \subset M$ be the isomorphism from the tubular neighborhood theorem such that

$$(\rho_F \circ \nu)(x_1, x') = x_1, \ (x_1, x') \in [0, \epsilon) \times F.$$

Consider the local generators of vector fields $\{V_1, \cdots, V_n\}$ such that

$$\mathcal{V}_{|\mathcal{U}} = \operatorname{span}_{C^\infty(\mathcal{U})}\{V_1, \cdots, V_n\}.$$

Also assume that V_1 is chosen so that $V_1 = x_1^k \partial_{x_1}$ and such k is the highest power of degeneracy among the V_i. Note that the *degeneracy index* $k = k_F$ is in fact invariantly defined.

We intend to extend the integral

$$\int_M : \dot{C}^\infty(M, {}^{\mathcal{V}}\Omega^1) \to \mathbb{C}$$

to the space $\rho^w C^\infty(M, {}^{\mathcal{V}}\Omega^1)$. Let $f \in \rho^w C^\infty(M, {}^{\mathcal{V}}\Omega^1)$ and set $G(f) \colon z \mapsto \int_M \rho^z f$ which yields a function holomorphic in

$$\{z \in \mathbb{C}^{\mathcal{F}_1(M)} : \Re z_F + w(F) > k_F - 1, \ F \in \mathcal{F}_1(M)\}.$$

Following Lauter and Moroianu [24, Section 4] ,[25] we obtain a meromorphic extension $G(f) \colon \mathbb{C}^{\mathcal{F}_1(M)} \to \mathbb{C}$ with at most simple poles in $z_F = k - 1 - w(F) - j$ for $j \in \mathbb{N}_0$. Hence we can define

$$^{\mathcal{V}}\!\!\int f : \text{ regularized value at } z_F = 0 \text{ of } G(f).$$

This yields the desired extension

$$^{\mathcal{V}}\!\!\int : \rho^w C^\infty(M, {}^{\mathcal{V}}\Omega^1) \to \mathbb{C}$$

which does, however, depend on the choice of boundary defining function for $w(F) \leq 1$. Let $\omega_{\mathcal{V}}$ be a fixed degenerate symplectic form on $\mathcal{A}^*$.

Definition 5.2. Define the $\mathcal{V}$-trace for $m < -n$ as the functional

$$^{\mathcal{V}}\mathrm{Tr} \colon \Psi_{\mathcal{V}}^{m,-w}(M) \to \mathbb{C}, \ A \mapsto {}^{\mathcal{V}}\!\!⨍ \kappa_{A|\Delta_{\mathcal{V}}}$$

where κ_A denotes the Schwartz kernel of A which restricts to an element of $\rho^w C^\infty(M, {}^{\mathcal{V}}\Omega^1)$.

We can alternatively write

$$^{\mathcal{V}}\mathrm{Tr}(A) = {}^{\mathcal{V}}\!\!⨍_{\mathcal{A}^*} a \omega_{\mathcal{V}}^n.$$

Here $a \in S_{cl}^m(\mathcal{A}^*)$ such that $\mathcal{F}_{\mathrm{f}}^{-1} a = \kappa_A$ near $\Delta_{\mathcal{V}}$ with the fiberwise Fourier transform $\mathcal{F}_{\mathrm{f}}$ defined in [44, Chapter 1.5]. The correspondence

$$⨍_M \leftrightarrow ⨍_{\mathcal{A}^*}$$

is obtained via the Fourier transform identity

$$f(0) = \int_{\mathbb{R}^n} \mathcal{F}(f)(\zeta)\, d\zeta$$

which is being applied fiberwise.
The finite part integral $⨍_{\mathcal{A}^*}$ is defined as in [24], i.e. we have two interpretations

- Using growth conditions at infinity.
- Via radial compactification of $\mathcal{A}^*$ (i.e. compactify $\mathcal{A}$ to a manifold with corners $\widehat{\mathcal{A}} \to M$ which is fibered over M such that $\widehat{\mathcal{A}}_x$ is a closed ball of dimension n).

Example 5.3. Consider the case of the b vector fields $\mathcal{V} = \mathcal{V}_b$ and a manifold M_0 with cylindrical end $(-\infty, 0]_s \times Y$, see also [20] for further details on this special case. We set $\widehat{M_0} = M$ for the compactification which is a manifold with boundary. The correspondence is obtained via the Kondratiev transform $x = e^s$ such that for $s \to -\infty$ we obtain $x \to 0$. This yields close to the boundary the density $ds = \frac{dx}{x}$ with $\partial_s = x\partial_x$. The singular structure is encoded in a Riemannian metric g (a compatible metric on the b-tangent bundle $\mathcal{A}^b \to M$) which is product type close to the boundary

$$g = ds^2 + h = \left(\frac{dx}{x}\right)^2 + h.$$

Notice that $\frac{dx}{x}$ is not integrable over $[0,1]_x$ and therefore the heat kernel $e^{-t\Delta_g}$ is not of trace class. We use therefore the regularization by observing that for $\Re z > 0$ the function x^z is integrable with regard to $\frac{dx}{x}$ over $[0,1]_x$. Hence $x^z e^{-t\Delta_g}$ is trace class and by setting

$$G(f)(z) = \int_M x^z f\, dg, \ f \in C^\infty(M), \ \Re(z) > 0$$

we define the b-trace as the regularized value of $G(f)(z)$ in $z = 0$.

We will show later that the heat kernel of a generalized Laplacian on a Lie manifold is actually contained in the class ${}^{\mathcal{V}}\mathrm{S}(M)$. By the above example this does however not imply that the heat kernel is of trace class. Nevertheless, we readily see that ${}^{\mathcal{V}}\mathrm{S}(M) \subset {}^{\mathcal{V}}\mathcal{L}^1(M)$ for ${}^{\mathcal{V}}\mathcal{L}^1(M)$ denoting the class of operators with bounded renormalized trace, i.e. the renormalized trace class. Hence the renormalized trace extends to a well-defined function on the ${}^{\mathcal{V}}\mathrm{Tr}\colon \Psi_{\mathcal{V}}^{-w,m}(M) + {}^{\mathcal{V}}\mathrm{S}(M) \to \mathbb{C}$.
We recall next the definition of the supertrace functional, which in our case acts on the homomorphism bundle $\hom(S) \to M$. Hence assume that the Lie manifold $(M, \mathcal{V}, \mathcal{A})$ is spin with given spin structure $S \to M$. Decompose the spinor bundle $S = S^+ \oplus S^-$ into elements of even and odd degree. Assume the Dirac operator is odd graded with regard to this decomposition. According to [7] 1.3., the bundle is realized as a super-bundle (a bundle consisting of super spaces, i.e. $\mathbb{Z}_2$ graded spaces). The super bundle $\hom(S)$ decomposes as

$$\hom(S) = \begin{pmatrix} \hom(S^+, S^+) & \hom(S^+, S^-) \\ \hom(S^-, S^+) & \hom(S^-, S^-) \end{pmatrix}.$$

Likewise, each element $T \in \hom(S)$ decomposes

$$T = \begin{pmatrix} T^{++} & T^{+-} \\ T^{-+} & T^{--} \end{pmatrix}.$$

We note that $\hom(S)_x = \hom(S_x, S_x) \cong \mathrm{Cl}(\mathcal{A}_x \otimes \mathbb{C})$, hence $\hom(S)$ can be viewed as a bundle of superalgebras. Given a super algebra A denote by $[\cdot,\cdot]_s\colon A \times A \to A$ the

supercommutator given by $[a,b]_s := ab - (-1)^{|a||b|}ba$. Here by $|a|$ we denote the parity, i.e. 0 for even degree and 1 for odd degree. A supertrace by definition is a linear form $\mathrm{tr}_s \colon A \to \mathbb{R}$ such that $\mathrm{tr}_s[a,b]_s = 0$. In our context we define $\mathrm{tr}_s \colon \hom(S) \to \mathbb{R}$ by $\mathrm{tr}_s(T) := \mathrm{tr}(T^{++}) - \mathrm{tr}(T^{--})$. This yields a supertrace by Proposition 1.31 of [7].
Denote by $\mathrm{Cl}_0 \subseteq \mathrm{Cl}_1 \subseteq \cdots \subseteq \mathrm{Cl}(\mathcal{A} \otimes \mathbb{C})$ the Clifford filtration by degree. The we have the following Lemma from Roe [41], Prop. 11.4 which we need later for the construction of a suitable rescaling.

Lemma 5.4. *We have* $\mathrm{tr}_{s|\mathrm{Cl}_{n-1}} = 0$ *and* $\mathrm{tr}_s(e_1, \cdots, e_n) = 2^{\frac{n}{2}}(-1)^{\frac{n}{2}}$ *for any oriented orthonormal frame* $\{e_i\}_{i=1}^n$ *of* $\mathcal{A}$.

Proof. Any odd element T^{odd} has the form $\begin{pmatrix} 0 & T^{+-} \\ T^{-+} & 0 \end{pmatrix}$ and hence $\mathrm{tr}_s(T^{odd}) = 0$. The even elements of $\hom(S)$ are generated by $T^{ev} = e_{i_1} \wedge \cdots \wedge e_{i_{2k}}$ with $i_1 < i_2 < \cdots < i_{2k}$, $2k < n$. For $m \in \{1, \cdots, n\} \setminus \{i_1, \cdots, i_{2k}\}$ we notice that e_m is an involution, i.e. $c(e_m)^2 = 1$. If $c(e_m)$ denotes the endomorphism by Clifford multiplication then $T^{ev} = \begin{pmatrix} T^{++} & 0 \\ 0 & c(e_m)T^{++}c(e_m) \end{pmatrix}$. Hence $\mathrm{tr}_s(T^{ev}) = 0$ follows by conjugation invariance. Finally we check the value of tr_s on the volume element $\omega = e_1 \wedge \cdots \wedge e_n$. ¿From $\mathrm{tr}_s(c) = \mathrm{tr}_V(i^{\frac{n}{2}}\omega c)$ where V is the spin representation and with $\omega^2 = (-1)^{\frac{n}{2}}$ we obtain $\mathrm{tr}_s(\omega) = \mathrm{tr}_V((-i)^{\frac{n}{2}}) = (-i)^{\frac{n}{2}} \dim(V) = (-2i)^{\frac{n}{2}}$. □

As a final remark we give the definition of the renormalized super trace of an operator defined on a spin Lie manifold acting on a Clifford module. Let $(M, \mathcal{V}, \mathcal{A})$ be a spin Lie manifold with spin structure $S \to M$ and $W \to M$ be a $\mathrm{Cl}(\mathcal{A})$-module. By the previous discussion we define the *renormalized super-trace* to be the functional ${}^{\mathcal{V}}\mathrm{Tr_s} \colon {}^{\mathcal{V}}\mathrm{S}(M) \otimes \mathrm{Hom}(W, W \otimes \mathcal{A}^*) \to \mathbb{C}$ given by ${}^{\mathcal{V}}\mathrm{Tr_s}(T) = {}^{\mathcal{V}}\!\!\fint \mathrm{tr}_s(k_T)\,d\mu$ for $T \in {}^{\mathcal{V}}\mathrm{S}(M) \otimes \mathrm{Hom}(W, W \otimes \mathcal{A}^*)$.

6. Rescaling

In this section we finish the proof of the index formula given in Theorem 1.1. We write the deformed Dirac operator on the adiabatic groupoid using the parametrization of Lie groupoids as defined in [19]. The Lichnerowicz theorem yields an expression for $\not{D}_{x,t}^2$ in normal coordinates. Then we calculate the renormalized super-trace and extract the right coefficient using the rescaling as defined previously.
We first establish the following representation theorem:

Theorem 6.1. *Let* $(M, \mathcal{A}, \mathcal{V})$ *be a non-degenerate Lie manifold. If* $\mathcal{G} \rightrightarrows M$ *is an integrating Hausdorff Lie groupoid, then there is a canonical isomorphism* ${}^{\mathcal{V}}\mathrm{S}(M) \cong \mathbf{S}(\mathcal{G})$ *implemented by the vector representation* ϱ.

Proof. The surjectivity of ϱ follows by an argument completely analogous to the proof of Theorem 3.2. in [1] (considering the action of the isotropy group $\Gamma := \mathcal{G}_x^x$ on the family of spaces $(\mathbf{S}(G_x))_{x \in M}$ and a diagram chase).
We prove the injectivity using the Hausdorff condition on the groupoid, see also [37]. Let $z_0 \in M_0$ be fixed and denote by $e_z \colon \mathbf{S}(\mathcal{G}) \to \mathbf{S}(\mathcal{G}_z)$ the evaluation $T = (T_x)_{x \in M} \mapsto T_z$. To see the injectivity of e_z let $T_z = 0$. We need to prove that $T_w = 0$ for each $w \in M$, i.e. $T = 0$. Since $\mathcal{G}_{|M_0} \cong M_0 \times M_0$ and by $\mathcal{G}$-invariance of the family T it follows that $T_w = 0$ for each $w \in M_0$. Let $w \in M$ be arbitrary, then $\langle T, \psi \rangle = 0$ for each $\psi \in C^\infty(\mathcal{G}_w)$. In order to see this let $\varphi \in C^\infty(\mathcal{G})$ be such that $\varphi_w = \psi$, which is possible since $\mathcal{G}_w \subset \mathcal{G}$ is closed in the locally compact Hausdorff space $\mathcal{G}$. We choose a

Haar system, then by the smoothness of the Haar system and the Hausdorff property of $\mathcal{G}$, the function $w \mapsto \|\langle T_w, \varphi_w\rangle\|$ is continuous and on $w \in M_0$ the function vanishes. By density of M_0 in M it follows $T_w\varphi_w = 0$ for each $w \in M$. Hence e_z is injective. The bijection $j\colon \mathbf{S}(\mathcal{G}_z) \to {}^{\mathcal{V}}\mathrm{S}(M)$ is obtained using the canonical diffeomorphism $\mathcal{G}_z \cong M_0$. Since ϱ equals by definition $j \circ e_z$, it is injective. □

From Theorem 3.7 we then obtain:

Corollary 6.2. *Let $(M, \mathcal{A}, \mathcal{V})$ be a non-degenerate Lie manifold with spin structure $S \to M$ and Clifford module W over $\mathrm{Cl}(\mathcal{A})$ such that D denotes the Dirac operator induced by an admissible connection ∇^W. Then the heat kernel of the generalized Laplacian e^{-tD^2} is for $t > 0$ contained in ${}^{\mathcal{V}}\mathrm{S}(M) \otimes \mathrm{Hom}(W, W \otimes \mathcal{A}^*)$.*

Remark 6.3. Given a Lie manifold $(M, \mathcal{A}, \mathcal{V})$ assume that the Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$ is Hausdorff and in addition has a length function of polynomial growth. Then by Theorem 6.1 together with Proposition 3.3 we obtain that the Lie calculus $\Psi_{\mathcal{V}}^m(M) + {}^{\mathcal{V}}\mathrm{S}(M)$, with the smoothing ideal given by the Schwartz class ${}^{\mathcal{V}}\mathrm{S}(M)$, is closed under holomorphic functional calculus. This is therefore in particular true for the examples given in Example 3.4.

We introduce the rescaling bundle and the method of extracting the right coefficient in the asymptotic expansion Ansatz for the heat kernel. As usual $\not{D}$ denotes the Dirac operator on the groupoid $\mathcal{G}$ and D its vector representation, the Dirac operator on the Lie manifold $(M, \mathcal{A}, \mathcal{V})$. Recall first the following notions. An *equivariant bundle* $\mathbb{E} \to \mathcal{G}$ over a Lie groupoid $\mathcal{G}$ is a vector bundle such that $R_\gamma\colon \mathcal{G}_{r(\gamma)} \to \mathcal{G}_{s(\gamma)}$ induces a vector bundle isomorphism $R_\gamma^*\colon \mathbb{E}_{\mathcal{G}_{r(\gamma)}} \to \mathbb{E}_{\mathcal{G}_{s(\gamma)}}$. Given a vector bundle $E \to M$ we define $\mathrm{Hom}(E)$ to be the pullback bundle $s^*(E^*) \otimes r^*(E^*)$ obtaining a bundle $\mathrm{Hom}(E) \to \mathcal{G}$ over $\mathcal{G}$.
In the following we describe the structure of the rescaling approach to the local index theorem as given by Siegel in [43]. Assume we are given a non-degenerate Lie manifold with spin structure $S \to M$ and let $\mathcal{G} \rightrightarrows M$ be an integrating Lie groupoid which is Hausdorff. We obtain from the above a bundle $\mathrm{Hom}(S) \to \mathcal{G}^{ad}$ as a lifting. Let $j\colon \mathcal{A}(\mathcal{G}) \hookrightarrow \mathcal{G}^{ad}$ be the natural embedding as a submanifold. Denote by $\mathrm{hom}(S) \to M$ the bundle with fibers $\mathrm{hom}(S)_x = \mathrm{hom}(S_x, S_x) \cong \mathrm{Cl}(\mathcal{A}_x \otimes \mathbb{C})$, $x \in M$. Since on $\mathcal{A}(\mathcal{G})$ source equals range we have

$$\mathrm{Hom}(S)_{|\mathcal{A}} \cong j^* \mathrm{hom}(S) \cong \mathrm{Cl}(\mathcal{A} \otimes \mathbb{C}).$$

The basic idea for the definition of the rescaling bundle $\tilde{S} \to \mathcal{G}^{ad}$ is to extend a Clifford filtration by degree to a neighbhorhood of $\mathcal{A}$ inside the adiabatic groupoid. More precisely, note the following.

- The *rescaling* will be adapted to the Clifford filtration $\mathrm{Cl}_0 \subseteq \mathrm{Cl}_1 \subseteq \cdots \subseteq \mathrm{Cl}(\mathcal{A} \otimes \mathbb{C}) = \mathrm{Hom}(S)_{|\mathcal{A}}$ filtered by degree.
- The bundle $\mathrm{Hom}(S)$ is endowed with a connection via pullback:

$$\mathcal{G}^{ad} \xrightarrow{\ s\ } M \times I \xrightarrow{\mathrm{pr}_1} M.$$

  So that $(\mathrm{pr}_1 \circ s)^*\nabla$ is a connection on $\mathrm{Hom}(S)$, where ∇ is the Levi-Civita connection on M from a fixed compatible metric on M.
- Extend the filtration $\{\mathrm{Cl}_k\}$ to a filtration $\{\tilde{\mathrm{Cl}}_k\}$ on a neighborhood of $\mathcal{A}$.

We define the *rescaled* sections

$$^{\mathcal{V}}\mathcal{D} := \left\{ u \in C_c^\infty(\mathcal{G}^{ad}, \mathrm{Hom}(S)) : u = \sum_{j=0}^{n} t^{n-j} u^j + t^{n+1} u' \text{ near } \mathcal{A} \right\}$$

with $u^j \in C_c^\infty(\mathcal{G}^{ad}, \tilde{\mathrm{Cl}}_j)$ and $u' \in C_c^\infty(\mathcal{G}^{ad}, \mathrm{Hom}(S))$.
By the Serre-Swan theorem there is a bundle $\tilde{S} \to \mathcal{G}^{ad}$ such that $C_c^\infty(\mathcal{G}^{ad}, \tilde{S}) = i_{\mathrm{Cl}}^* \mathcal{D}$, where $i_{\mathrm{Cl}} \colon \tilde{S} \to \mathrm{Hom}(S)$ is a bundle map and an isomorphism over $\mathcal{G}^{ad}_{|(0,1]}$, see also Prop. 8.4 in [30]. We refer to $\tilde{S}$ as the *rescaled bundle*.
An alternative description of $^{\mathcal{V}}\mathcal{D}$ can be obtained by Taylor expansion and is given by

$$^{\mathcal{V}}\mathcal{D} = \{ u \in C_c^\infty(\mathcal{G}^{ad}, \mathrm{Hom}(S)) : \nabla_{ad}^p u_{|\mathcal{A}} \in C_c^\infty(\mathcal{A}, \mathrm{Cl}_{k-p}),\ 0 \leq p \leq n \}.$$

Here ∇_{ad} denotes the connection obtained from parallel transport along the vector fields ∂_t.
Finally, we want to extend the supertrace functional to the rescaled bundle $\tilde{S}$. Note first the following Lemma.

Lemma 6.4. *We have the canonical isomorphism of Clifford algebra* $\tilde{S}_{|\mathcal{A}} \cong \Lambda \mathcal{A}^*$.

Proof. Note first that the filtration of the Clifford algebra $\mathrm{Cl}(\mathcal{A} \otimes \mathbb{C})$ by degree has associated to it a graded algebra which identifies with the exterior algebra $\Lambda \mathcal{A}^*$, cf. [7]. The rescaled bundle $\tilde{S}$ associated to the filtration $\{\mathrm{Cl}_k\}$ by Clifford degree restricts to the graded bundle associated to $\{\mathrm{Cl}_k\}$. By combining these two facts the assertion follows. □

The more direct proof would rely on the intuition that $\tilde{S}$ is just the bundle obtained from $\mathrm{Hom}(S)$ by replacing over $t \neq 0$ the spinor bundle S (lifted to $\mathcal{G}$) with the spinor bundle S_t which is a $\mathcal{G}$-invariant bundle such that over each fiber $\mathcal{G}_x$ it is the spinor bundle S_x^t constructed from the Riemannian metric $t g_x(\cdot)$.
By an application of Lemma 5.4 and the definition of the rescaling we obtain the following Lemma.

Lemma 6.5. *Let* $\mathcal{G}_\Delta := \{\gamma \in \mathcal{G}^{ad} : s(\gamma) = r(\gamma)\} \subset \mathcal{G}^{ad}$, *then for* $t \neq 0$ *the supertrace functional maps* $\mathrm{tr}_s \colon C_c^\infty(\mathcal{G}_\Delta, \tilde{S}_{|\mathcal{G}_\Delta}) \to t^n C_c^\infty(\mathcal{G}, \mathrm{hom}(S))$.

Proof. By the Lemma 5.4 the supertrace functional maps Cl_0 to Cl_n and by our rescaling this yields the assertion. □

The previous lemma ensures that the right coefficient is extracted when we apply the supertrace functional to the vector representation of the groupoid heat kernel.
Consider the Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$. We fix $x_0 \in \mathcal{G}^{(0)}$. Then a *parametrization* of $\mathcal{G}$ at x_0 is given by a tuple (φ, ψ) where $\varphi \colon U \to \mathcal{G}^{(0)}$ and $\psi \colon U \times V \to \mathcal{G}$ are homeomorphisms where U is a 0-neighborhood in $\mathbb{R}^n$ and V is a 0-neighborhood in $\mathbb{R}^m$. The following conditions should hold:
i) $\psi(0,0) = x_0$,
ii) $r(\psi(u,v)) = \varphi(u)$,
iii) $\psi(U \times \{0\}) = \psi(U \times V) \cap \mathcal{G}^{(0)}$.
Note that r is a submersion at x_0. Condition *i)* and *ii)* imply $\varphi(u) = \psi(u,0)$.
This induces a parametrization of $\mathcal{A}(\mathcal{G})$, more precisely of the neighborhood $\mathcal{A}(\mathcal{G})_{\varphi(U)}$ of the fiber $\mathcal{A}_{x_0}(\mathcal{G})$, which is given by $\theta \colon U \times \mathbb{R}^m \to \mathcal{A}(\mathcal{G})$, $\theta(u,v) = \left(\varphi(u), \frac{\partial \psi}{\partial v}(u,0) v\right)$.

Using these parametrizations we can formulate a local diffeomorphism theorem. Then using the generalized exponential functions we can describe a geodesic coordinate system. This will be needed in the proof of the main theorem.
For each $x_0 \in \mathcal{G}^{(0)}$ there is a neighborhood $\psi(U \times V)$ of x_0 in $\mathcal{G}$ such that $\alpha = \psi \circ \theta^{-1}$ implements a diffeomorphism of the neighborhood of $(x_0, 0)$ given by $\theta(U \times V)$ with $\psi(U \times V)$. Additionally, $\alpha(\mathcal{A}_x(\mathcal{G})) \subset \mathcal{G}_x$ holds for each $x \in \varphi(U)$.
The induced parametrization and the local diffeomorphism property can be improved if we make use of the generalized exponential map $\mathrm{Exp}\colon \mathcal{A}(\mathcal{G}) \to \mathcal{G}$, cf. Section 4. Let $\alpha = \mathrm{Exp}_{|V}$, then $\alpha(\mathcal{A}_x(\mathcal{G}) \cap V) = \mathcal{G}_x \cap W$, $\alpha_x'(0) = \mathrm{id}_{\mathcal{A}_x(\mathcal{G})}$ where $\alpha_x = \alpha_{|\alpha_x(\mathcal{G}) \cap W}$.
We are now in a position to give a proof of the main theorem.

Proof of Theorem 1.1. For the proof we apply the previously constructed functional calculus $\Psi_{\mathbb{D}}$ adapted to a $\mathcal{G}$-invariant family of Dirac operators $(\not{D}_{x,t})_{(x,t) \in M \times I}$. Here we consider the family given by $\not{D}_{x,t} = t\not{D}_x$ and set $\mathbb{D} := (t\not{D}_x)_{(x,t) \in M \times I}$. For $f \in \mathcal{P}$, $\Psi_{\mathbb{D}}(f) \in C_c^\infty(\mathcal{G}^{ad}, \tilde{S})$ by the construction of the functional calculus. Here $\tilde{S} \to \mathcal{G}^{ad}$ is the rescaled bundle introduced above. Recall the action of the functional calculus

$$f(\not{D}_{x,t})g(\gamma) = \pi_{x,t}(\Psi_{\mathbb{D}}(f))g(\gamma) = (\Psi_{\mathbb{D}}(f) * g)(\gamma), \quad g \in L^2(\mathcal{G}_{x,t}^{ad}).$$

In our case this yields

$$f(t\not{D})g(\gamma) = \int_{\mathcal{G}_{s(\gamma)}} \Psi_{\mathbb{D}}(f)(\gamma\eta^{-1})g(\eta)t^{-n}\, d\mu_{s(\gamma)}(\eta).$$

Hence roughly speaking the function $f(x) = e^{-x^2}$ yields $\Psi_{\mathbb{D}}(f) = t^n k_{t^2}$. Technically, we have to first convolve f with a rapidly decaying function whose Fourier transform has large compact support, since the Fourier transform of f does not have compact support. Nevertheless we loosely identify $\Psi_{\mathbb{D}}(f)$ with $t^n k_{t^2}$.
Let $\mathcal{G}_\Delta = \{\gamma \in \mathcal{G}^{ad} : s(\gamma) = r(\gamma)\}$ denote the diagonal in $\mathcal{G}^{ad}$. Define $l_t := \Psi_{\mathbb{D}}(e^{-x^2})_{|\mathcal{G}_\Delta}$, then $l_t(\gamma) = t^n k_{t^2}(\gamma)$ for $t \neq 0$ and $\gamma \in \mathcal{G}_{\Delta_t}$. We have now on the Lie manifold for $D = \varrho(\not{D})$ that

$$^{\mathcal{V}}\mathrm{Tr_s}(e^{-tD^2}) = {}^{\mathcal{V}}\!\!⨍\, \mathrm{tr}_s(\kappa_t(x,x))\, d\mu(x)$$

where $\mu = \mu_g$ is the density defined by the fixed compatible metric g and κ_t denotes the heat kernel of e^{-tD^2}. By the representation Theorem 6.1 and Theorem 3.7 we obtain that $\kappa_t \in {}^{\mathcal{V}}\mathrm{S}(M)$ and the definition is therefore $\mathcal{G}$-invariant. Denote by $\tilde{l}_t$ the vector representation of l_t. The equation above makes sense for $t \neq 0$. We have that $\kappa_{t^2|\Delta}$ identifies with $t^{-n}\mathrm{tr}_s(\tilde{l}_t)$. Since $t^{-n}\mathrm{tr}_s(\tilde{l}_t)$ extends smoothly to $t = 0$, we have $t^{-n}\mathrm{tr}_s(\tilde{l}_t) = \mathrm{tr}_s(\tilde{l}_0) + o(t)$. From $\Psi_{\mathbb{D}}(e^{-x^2})_{|t \neq 0} = t^n k_{t^2}$ and $\Psi_{\mathbb{D}}(e^{-x^2})_{|t=0} = k_{u|u=1}$ we obtain

$$\begin{aligned} {}^{\mathcal{V}}\mathrm{Tr_s}(e^{-tD^2}) &= {}^{\mathcal{V}}\!\!⨍\, t^{-\frac{n}{2}}\mathrm{tr}_s(\tilde{l}_{t^{\frac{1}{2}}})\, d\mu \\ &= {}^{\mathcal{V}}\!\!⨍\, \mathrm{tr}_s(\tilde{l}_0)\, d\mu + o(t^{\frac{1}{2}}). \end{aligned}$$

Hence we have reduced the task to calculating $\mathrm{tr}_s(\tilde{l}_0)$. We calculate the kernel l_0 on the groupoid using the Lichnerowicz theorem applied to the fibers of the integrating groupoid.
Denote by $\varphi\colon U \to \mathcal{G}^{(0)}$, $\psi\colon U \times V \to \mathcal{G}$ a parametrization of $\mathcal{G}$ around a fixed $x_0 \in M$ such that for $\alpha = \mathrm{Exp}_{|V}$ we have $\alpha(\mathcal{A}_x(\mathcal{G}) \cap V) = \mathcal{G}_x \cap \tilde{V}$ for some open subset $\tilde{V}$ and $x \in \varphi(U)$.
Let $\alpha_x = \alpha_{|\mathcal{A}_x(\mathcal{G}) \cap V}$ which is by definition induced by the exponential map $\exp_x$ on the fiber $\mathcal{G}_x$. Let $\alpha_x(\gamma) = (a_1, \cdots, a_m) =: a$ be the corresponding geodesic coordinates.

Consider the induced parametrization of $\mathcal{G}^{ad}$ given by $\Phi\colon U^{ad} \times V \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$, where $U^{ad} = U \times \mathbb{R}$. Restrict this map to the chart $V \times \{x\} \times \{t\}$ and call the restriction $\Phi_{x,t}$. An elementary calculation yields $\Phi_{x,t}(\eta) = \frac{1}{t}(\alpha_x(\eta) - a)$. Then the Lichnerowicz theorem on the complete manifold $(\mathcal{G}_x, g_x)$ yields for $b = \Phi_{x,t}(\eta)$

$$\begin{aligned}
&\not{D}_{x,t}^2 f(\eta) = \not{D}_{x,t}^2 f(\Phi_{x,t}^{-1}(b_1, \cdots, b_m)) \\
&= \ t^2 \not{D}_x^2 f(\alpha_x^{-1}((tb_1, \cdots, tb_m) + a)) \\
&= \ -t^2 \sum_i \left(\frac{1}{t}\partial_i^x + \frac{1}{4}\sum_j \frac{1}{t}\left(R_{ij}^x(a_j + tb_j)\right)\right)^2 f(\eta) \\
&\quad + \left(\sum_{i<j} F^{W_x/S}(e_i, e_j)(a_j + tb_j)(a_j + tb_j) + \frac{t^2}{4}\kappa\right) f(\eta) \\
&= \ -\sum_i \left(\partial_i^x + \frac{1}{4}\sum_j R_{ij}^x(a_j + tb_j)\right)^2 f(\eta) \\
&\quad + \left(\sum_{i<j} F^{W_x/S}(e_i, e_j)(a_j + tb_j)(a_j + tb_j) + \frac{t^2}{4}\kappa\right) f(\eta).
\end{aligned}$$

The right hand side depends smoothly on t up to and including $t = 0$. In the limit as $t \to 0$ we obtain

$$\not{D}_{x,0}^2 = -\sum_i \left(\partial_i^x + \frac{1}{4}\sum_j R_{ij}^x a_j\right)^2 + \sum_{i<j} F^{W_x/S}(e_i, e_j)(a_j)(a_j).$$

The remainder of the argument consists in the solution of the differential equation of the heat kernel of $\not{D}_{x,0}^2$, which one recognizes as the equation for the harmonic oscillator with twisting. We can therefore use the analysis in [7] to obtain the solution in terms of a formal power series in the scalar curvature R_{ij}^x and the exponential of the twisting bundle $\exp F^{W_x/S}$. By the $\mathcal{G}$-invariance of the curvature tensor as well as the twisting curvature and the Lichnerowicz theorem for Lie manifolds given in Theorem 2.3, it follows from [7], p. 164 and [41], Proposition 12.25, 12.26 that we obtain the integrand ${}^{\mathcal{V}}\mathbb{A} \wedge \exp F^{S/W}$ in the trace formula.
Thus we have shown that

$$\lim_{t \to 0^+} {}^{\mathcal{V}}\mathrm{Tr_s}(e^{-tD^2}) = {}^{\mathcal{V}}\!\!\fint {}^{\mathcal{V}}\mathbb{A} \wedge \exp F^{W/S}\, d\mu.$$

To obtain the limit $t \to \infty$ consider

$$\lim_{t \to \infty} {}^{\mathcal{V}}\mathrm{Tr_s}(e^{-tD^2}) - \lim_{t \to 0^+} {}^{\mathcal{V}}\mathrm{Tr_s}(e^{-tD^2}) = \int_0^\infty \partial_t {}^{\mathcal{V}}\mathrm{Tr_s}(e^{-tD^2})\, dt.$$

We have $\partial_t {}^{\mathcal{V}}\mathrm{Tr_s}(e^{-tD^2}) = {}^{\mathcal{V}}\mathrm{Tr_s}(\partial_t e^{-tD^2})$. The latter equals $-\frac{1}{2}{}^{\mathcal{V}}\mathrm{Tr_s}([D, De^{-tD^2}]_s)$ since by the odd grading of D we have $D^2 e^{-tD^2} = \frac{1}{2}[D, De^{-tD^2}]_s$, where $[\cdot,\cdot]_s$ denotes the super commutator. Setting ${}^{\mathcal{V}}\eta(D) := \frac{1}{2}\int_0^\infty {}^{\mathcal{V}}\mathrm{Tr_s}([D, De^{-tD^2}]_s)\, dt$ this finishes the proof of the index theorem. □

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[1] Bergische Universität Wuppertal, Germany
E-mail address: bohlen@uni-wuppertal.de

[2] Leibniz Universität Hannover, Germany
E-mail address: schrohe@math.uni-hannover.de