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GENO P. NIKOLOV AND ALEXEI SHADRIN

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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On the Markov inequality in the L_2-norm with the Gegenbauer weight

G. Nikolov, A. Shadrin

Abstract

Let $w_\lambda(t) := (1-t^2)^{\lambda-1/2}$, where $\lambda > -\frac{1}{2}$, be the Gegenbauer weight function, let $\|\cdot\|_{w_\lambda}$ be the associated L_2-norm,

$$\|f\|_{w_\lambda} = \left\{\int_{-1}^{1} |f(x)|^2 w_\lambda(x)\,dx\right\}^{1/2},$$

and denote by $\mathcal{P}_n$ the space of algebraic polynomials of degree $\le n$. We study the best constant $c_n(\lambda)$ in the Markov inequality in this norm

$$\|p_n'\|_{w_\lambda} \le c_n(\lambda)\|p_n\|_{w_\lambda}\,, \qquad p_n \in \mathcal{P}_n\,,$$

namely the constant

$$c_n(\lambda) := \sup_{p_n\in\mathcal{P}_n} \frac{\|p_n'\|_{w_\lambda}}{\|p_n\|_{w_\lambda}}\,.$$

We derive explicit lower and upper bounds for the Markov constant $c_n(\lambda)$, which are valid for all n and λ.

MSC 2010: 41A17

Key words and phrases: Markov type inequalities, Gegenbauer polynomials, matrix norms

1 Introduction

Let $w_\lambda(t) := (1-t^2)^{\lambda-1/2}$, where $\lambda > -\frac{1}{2}$, be the Gegenbauer weight function, let $\|\cdot\|_{w_\lambda}$ be the associated L_2-norm,

$$\|f\|_{w_\lambda} = \left\{\int_{-1}^{1} |f(x)|^2 w_\lambda(x)\,dx\right\}^{1/2},$$

and denote by $\mathcal{P}_n$ the space of algebraic polynomials of degree $\le n$. In this paper, we study the best constant $c_n(\lambda)$ in the Markov inequality in this norm

$$\|p_n'\|_{w_\lambda} \le c_n(\lambda)\|p_n\|_{w_\lambda}\,, \qquad p_n \in \mathcal{P}_n\,, \tag{1.1}$$

namely the constant

$$c_n(\lambda) := \sup_{p_n\in\mathcal{P}_n} \frac{\|p_n'\|_{w_\lambda}}{\|p_n\|_{w_\lambda}}\,.$$

Our goal is to derive *good* and *explicit* lower and upper bounds for the Markov constant $c_n(\lambda)$ which are valid for *all* n and λ, i.e., to find constants $\underline{c}(n,\lambda)$ and $\overline{c}(n,\lambda)$ such that

$$\underline{c}(n,\lambda) \le c_n(\lambda) \le \overline{c}(n,\lambda)\,,$$

with a small ratio $\frac{\overline{c}(n,\lambda)}{\underline{c}(n,\lambda)}$.

It is known that, for a fixed λ, $c_n(\lambda)$ grows like $\mathcal{O}(n^2)$, and that the asymptotic value

$$c_*(\lambda) := \lim_{n\to\infty} \frac{c_n(\lambda)}{n^2}$$

is equal to $1/(2\, j_{\frac{2\lambda-3}{4}})$, with j_ν being the first positive zero of the Bessel function J_ν, see [2, Thms. 1.1–1.3], whereby it can be shown that $c_*(\lambda)$ behaves like $\mathcal{O}(\lambda^{-1})$. There is also a number of more precise results.

For $\lambda = \frac{1}{2}$ (the constant weight $w_{\frac{1}{2}} \equiv 1$), it follows from the Schmidt result [4] that

$$\frac{1}{\pi}(n+\tfrac{3}{2})^2 \le c_n(\tfrac{1}{2}) \le \frac{1}{\pi}(n+2)^2 \,.$$

For $\lambda = 0, 1$ (the Chebyshev weights $w_0(x) = \frac{1}{\sqrt{1-x^2}}$ and $w_1(x) = \sqrt{1-x^2}$, respectively), Nikolov [3] proved that

$$\begin{aligned} 0.472135 n^2 &\le c_n(0) \le 0.478849(n+2)^2\,, \\ 0.248549 n^2 &\le c_n(1) \le 0.256861(n+\tfrac{5}{2})^2\,. \end{aligned} \tag{1.2}$$

In [1], we obtained an upper bound valid for all n and λ,

$$c_n(\lambda) \le \frac{(n+1)(n+2\lambda+1)}{2\sqrt{2\lambda+1}}\,, \tag{1.3}$$

however, the already mentioned asymptotics $c_*(\lambda) = \mathcal{O}(\lambda^{-1})$ shows that this result is not optimal.

The main result of this paper is lower and upper bounds for $c_n(\lambda)$ which are uniform with respect to n and λ. They show, in particular, that

$$[c_n(\lambda)]^2 \asymp \frac{1}{\lambda^2} n(n+2\lambda)^3\,.$$

For $n = 1, 2$ the exact values of the Markov constant are easily computable:

$$[c_1(\lambda)]^2 = 2(1+\lambda), \qquad [c_2(\lambda)]^2 = \frac{4(2+\lambda)(2+2\lambda)}{2\lambda+1}\,. \tag{1.4}$$

Therefore, we consider below the case $n \ge 3$. Our main result is

Theorem 1.1 *For all $\lambda > -\frac{1}{2}$ and $n \ge 3$, the best constant $c_n(\lambda)$ in the Markov inequality*

$$\|p_n'\|_{w_\lambda} \le c_n(\lambda) \|p_n\|_{w_\lambda}\,, \qquad p_n \in \mathcal{P}_n\,,$$

admits the estimates

$$\frac{1}{4}\frac{n^2(n+\lambda)^2}{(\lambda+1)(\lambda+2)} \;<\; [c_n(\lambda)]^2 \;<\; \frac{n(n+2\lambda+2)^3}{(\lambda+2)(\lambda+3)}, \qquad \lambda \ge 2\,; \tag{1.5}$$

$$\frac{(n+\lambda)^2(n+2\lambda')^2}{(2\lambda+1)(2\lambda+5)} \;<\; [c_n(\lambda)]^2 \;<\; \frac{(n+\lambda+\lambda''+2)^4}{2(2\lambda+1)\sqrt{2\lambda+5}}, \qquad \lambda > -\tfrac{1}{2}\,, \tag{1.6}$$

where $\lambda' = \min\{0, \lambda\}$, $\lambda'' = \max\{0, \lambda\}$.

As a consequence, we can specify the following bounds for the asymptotic value $c_*(\lambda)$:

Corollary 1.2 *For any $\lambda > -\frac{1}{2}$, the asymptotic Markov constant $c_*(\lambda) = \lim\limits_{n\to\infty} n^{-2} c_n(\lambda)$ satisfies the inequalities*

$$\frac{1}{(2\lambda+1)(2\lambda+5)} < [c_*(\lambda)]^2 < \begin{cases} \dfrac{1}{2(2\lambda+1)\sqrt{2\lambda+5}}\,, & -\frac{1}{2} < \lambda \le \lambda^*\,, \\[2ex] \dfrac{1}{(\lambda+2)(\lambda+3)}\,, & \lambda > \lambda^*\,, \end{cases}$$

where $\lambda^* \approx 25$.

The lower bound in (1.5) follows from that in (1.6) and is less accurate, we put it in this form to make the comparison between the two bounds in (1.5) more obvious.

The upper bound in (1.6) does not have the right order $\mathcal{O}(n^4/\lambda^2)$ in λ (for λ fixed), however this bound serves not only for the case $-\frac{1}{2} < \lambda < 2$, but for a fixed $\lambda \in [2, \lambda^*]$ and $n \ge n_0(\lambda)$ it is also better than the one in (1.5).

In the next corollary, we set $\lambda = 0, 1$ in the upper estimate (1.6), and that improves the upper estimates in (1.2) for the Chebyshev weights. When coupled with the lower estimate from (1.2), this gives rather tight bounds.

Corollary 1.3 *For the Chebyshev weights $w_0(x) = \frac{1}{\sqrt{1-x^2}}$ and $w_1(x) = \sqrt{1-x^2}$, we have*

$$0.472135\, n^2 \le c_n(0) \le 0.472871\, (n+2)^2\,,$$
$$0.248549\, n^2 \le c_n(1) \le 0.250987\, (n+4)^2\,.$$

The lower and upper estimates in (1.5) have different orders with respect to λ. However we can get a perfect match with slightly less accurate constants.

Theorem 1.4 *For all $\lambda \ge 7$ and $n \ge 3$, the best constant $c_n(\lambda)$ in the Markov inequality satisfies*

$$\frac{1}{16}\frac{n(n+2\lambda)^3}{\lambda^2} \le [c_n(\lambda)]^2 \le \frac{n(n+2\lambda)^3}{\lambda^2}\,. \tag{1.7}$$

Corollary 1.5 *For the Markov constant $c_n(\lambda)$ we have the following asymptotic estimates:*

i) $\sqrt{n} \le \lim\limits_{\lambda\to\infty} \dfrac{c_n(\lambda)}{\sqrt{2\lambda}} \le \sqrt{3n}\,;$

ii) $(n-\frac{1}{2})(n-1) \le \lim\limits_{\lambda\to -\frac{1}{2}} c_n(\lambda)\cdot 2\sqrt{2\lambda+1} \le (n+\frac{3}{2})^2\,.$

Part ii) follows from (1.6). Though part i) does not formally follow from Theorem 1.4, it follows from a part of its proof.

Let us describe briefly how these results are obtained.

It is well-known that the squared best constant in the Markov inequality in the L_2-norm with arbitrary (and possibly different) weights for p and p' is equal to the largest eigenvalue of a certain positive definite matrix, in our case we have

$$[c_n(\lambda)]^2 = \mu_{\max}(\mathbf{B}_n)\,, \tag{1.8}$$

where the matrix $\mathbf{B}_n$ is specified in Sect. 2. We obtain then lower and upper bounds for $\mu_{\max}(\mathbf{B}_n)$ using three values associated with the matrix $\mathbf{B}_n$ and its eigenvalues (μ_i) (note that $\mu_i > 0$):

a) the trace

$$\mathrm{tr}\,(\mathbf{B}_n) := \sum b_{ii} = \sum \mu_i\,;$$

b) the max-norm

$$\|\mathbf{B}_n\|_\infty = \max_i \sum_j |b_{ij}|\,;$$

c) the Frobenius norm

$$\|\mathbf{B}_n\|_F^2 := \sum_{i,j} |b_{ij}|^2 = \mathrm{tr}\,(\mathbf{B}_n\mathbf{B}_n^T) = \sum \mu_i^2\,.$$

Clearly, we have

$$\text{i)}\quad \mu_{\max} \le \mathrm{tr}\,(\mathbf{B}_n)\,, \qquad \text{ii)}\quad \mu_{\max} \le \|\mathbf{B}_n\|_\infty\,, \qquad \text{iii)}\quad \mu_{\max} \le \|\mathbf{B}_n\|_F\,, \tag{1.9}$$

and generally $\mu_{\max} \le \|\mathbf{B}_n\|_*$, where $\|\cdot\|_*$ is any matrix norm. The upper estimate (1.3) cited from [1] is exactly the first inequality $\mu_{\max} \le \mathrm{tr}\,(\mathbf{B}_n)$, and as we noted, this estimate is not optimal. The better upper bounds (1.5)-(1.6) in Theorem 1.1 are obtained from (1.9.ii) and (1.9.iii), respectively.

For the lower bounds we use the inequalities

$$\text{i}')\quad \mu_{\max} \ge \frac{\sum \mu_i^2}{\sum \mu_i} = \frac{\|\mathbf{B}_n\|_F^2}{\mathrm{tr}\,(\mathbf{B}_n)}\,, \qquad \text{ii}')\quad \mu_{\max}(\mathbf{B}_n) \ge \max_i b_{ii}\,. \tag{1.10}$$

Inequality (i′) gives the lower estimates in (1.5)-(1.6), and combination of (i′) and (ii′) yields the lower bound in (1.7).

The paper is organised as follows. In Sect. 2, following our previous studies [1], we give an explicit form of the matrix $\mathbf{B}_n$ appearing in (1.8). Sects. 2-4 contain some auxiliary inequalities. In Sect. 5, we find an upper bound for the max-norm $\|\mathbf{B}_n\|_\infty$, and in Sect .6 we give both lower and upper estimates for the Frobinuis norm $\|\mathbf{B}_n\|_F$. Finally, in Sect. 7 we prove the upper and the lower estimates in Theorems 1.1-1.4 using inequalities (1.9)-(1.10) and relation (1.8). Here we have used the expression for $\mathrm{tr}\,(\mathbf{B}_n)$ and for diagonal elements b_{ii} found in [1].

The formulas for the trace, the max-norm and the Frobenius norm of a matrix are straightforward once the matrix elements are known, so the main technical issues are, firstly, in finding reasonable upper and lower bounds for the entries of the matrix $\mathbf{B}_n = (b_{ij})$ which are expressed initially in terms of the Gamma function Γ, and, secondly, in finding reasonable estimates for their sums. The first issue is dealt with in Sect. 3, where we show that

$$b_{jk} \asymp \frac{f_\sigma(j)}{f_\tau(k)}\,, \qquad f_\alpha(x) = x^{\alpha_1}(x+\tfrac{\lambda}{2})^{\alpha_2}(x+\lambda)^{\alpha_3}\,,$$

and the second one in Sect. 4 , where we give elementary but effective upper and lower bounds for the integrals of the type

$$\int_{x_0}^{x} f(t)\,dt, \qquad f(x) = (x+\gamma_1)^{\alpha_1}(x+\gamma_2)^{\alpha_2}\cdots(x+\gamma_r)^{\alpha_r}.$$

2 Preliminaries

In this section, we quote a result obtained earlier in [1], which equate the Markov constant $c_n(\lambda)$ with the largest eigenvalue of a specific matrix $\mathbf{B}_n$.

Definition 2.1 For $n \in \mathbb{N}$, set $m := \lfloor \frac{n+1}{2} \rfloor$ and define symmetric positive definite matrices $\mathbf{A}_m, \widetilde{\mathbf{A}}_m \in \mathbb{R}^{m\times m}$ with entries a_{kj} and $\widetilde{a}_{kj}$ given by

$$a_{kj} := \Big(\sum_{i=1}^{\min(k,j)} \alpha_i^2\Big)\beta_k\beta_j\,, \qquad \widetilde{a}_{kj} := \Big(\sum_{i=1}^{\min(k,j)} \widetilde{\alpha}_i^2\Big)\widetilde{\beta}_k\widetilde{\beta}_j\,, \tag{2.1}$$

so that

$$\mathbf{A}_m := \begin{pmatrix} \alpha_1^2\beta_1^2 & \alpha_1^2\beta_1\beta_2 & \alpha_1^2\beta_1\beta_3 & \cdots & \alpha_1^2\beta_1\beta_m \\ \alpha_1^2\beta_1\beta_2 & \Big(\sum_{i=1}^2 \alpha_i^2\Big)\beta_2^2 & \Big(\sum_{i=1}^2 \alpha_i^2\Big)\beta_2\beta_3 & \cdots & \Big(\sum_{i=1}^2 \alpha_i^2\Big)\beta_2\beta_m \\ \alpha_1^2\beta_1\beta_3 & \Big(\sum_{i=1}^2 \alpha_i^2\Big)\beta_2\beta_3 & \Big(\sum_{i=1}^3 \alpha_i^2\Big)\beta_3^2 & \cdots & \Big(\sum_{i=1}^3 \alpha_i^2\Big)\beta_3\beta_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^2\beta_1\beta_m & \Big(\sum_{i=1}^2 \alpha_i^2\Big)\beta_2\beta_m & \Big(\sum_{i=1}^3 \alpha_i^3\Big)\beta_3\beta_m & \cdots & \Big(\sum_{i=1}^m \alpha_i^2\Big)\beta_m^2 \end{pmatrix}, \tag{2.2}$$

with the same outlook for $\widetilde{\mathbf{A}}_m$. The numbers α_k, β_k and $\widetilde{\alpha}_k, \widetilde{\beta}_k$ are given by

$$\alpha_k := (2k-1+\lambda)h_{2k-1}, \quad \beta_k := \frac{1}{h_{2k}}\,; \tag{2.3}$$

$$\widetilde{\alpha}_k := (2k-2+\lambda)h_{2k-2}, \quad \widetilde{\beta}_k := \frac{1}{h_{2k-1}}\,, \tag{2.4}$$

where

$$h_i^2 := h_{i,\lambda}^2 := \frac{\Gamma(i+2\lambda)}{(i+\lambda)\Gamma(i+1)}\,. \tag{2.5}$$

Note that

$$\widetilde{\alpha}_k = \alpha_{k-\frac{1}{2}}\,, \qquad \widetilde{\beta}_k = \beta_{k-\frac{1}{2}}\,. \tag{2.6}$$

Definition 2.2 For $n \in \mathbb{N}$, set

$$\mathbf{B}_n := \begin{cases} 4\mathbf{A}_m, & n = 2m; \\ 4\widetilde{\mathbf{A}}_m, & n = 2m-1. \end{cases} \tag{2.7}$$

Theorem 2.3 ([1], Theorem 3.2) *Let $c_n(\lambda)$ be the best constant in the Markov inequality (1.1). Then*

$$[c_n(\lambda)]^2 = \mu_{\max}(\mathbf{B}_n)\,,$$

where $\mu_{\max}(\mathbf{B}_n)$ is the largest eigenvalue of the matrix $\mathbf{B}_n$.

Remark 2.4 Appearance of two matrices $\mathbf{A}_m$ and $\widetilde{\mathbf{A}}_m$ reflects the fact that the extreme polynomial $\widehat{p}_n$ for the Markov inequality with an even weight function $w(x) = w(-x)$ is either odd or even. The latter is a relatively simple conclusion, what is not obvious though is whether $\widehat{p}_n$ is of degree exactly n and not $n-1$. In [1], we proved that for the Gegenbauer weights w_λ,

$$\mu_{\max}(\widetilde{\mathbf{A}}_m) < \mu_{\max}(\mathbf{A}_m) < \mu_{\max}(\widetilde{\mathbf{A}}_{m+1})$$

and this implies that $\deg \widehat{p}_n = n$, hence $[c_n(\lambda)]^2$ is the largest eigenvalue of $\mathbf{A}_m$ or $\widetilde{\mathbf{A}}_m$ for $n = 2m$ or $n = 2m-1$, respectively.

We finish this section by simplifying the expressions for a_{kj} and thus for the matrix $\mathbf{A}_m$ as follows. From (2.1), we derive

$$a_{kj} := \Big(\sum_{i=1}^{\min(k,j)} \alpha_i^2\Big)\beta_k\beta_j = \begin{cases} \frac{\beta_k}{\beta_j}\left(\beta_j^2 \sum_{i=1}^{j} \alpha_i^2\right), & j < k\,, \\ \frac{\beta_j}{\beta_k}\left(\beta_k^2 \sum_{i=1}^{k} \alpha_i^2\right), & j > k\,, \end{cases}$$

so that

$$a_{jj} = \beta_j^2 \sum_{i=1}^{j} \alpha_i^2\,, \qquad a_{kj} = \begin{cases} \frac{\beta_k}{\beta_j}\, a_{jj}, & j < k\,, \\ \frac{\beta_j}{\beta_k}\, a_{kk}, & j > k\,. \end{cases} \tag{2.8}$$

Respectively,

$$\mathbf{A}_m = \begin{pmatrix} a_{11} & \frac{\beta_2}{\beta_1}a_{11} & \frac{\beta_3}{\beta_1}a_{11} & \cdots & \frac{\beta_m}{\beta_1}a_{11} \\ \frac{\beta_2}{\beta_1}a_{11} & a_{22} & \frac{\beta_3}{\beta_2}a_{22} & \cdots & \frac{\beta_m}{\beta_2}a_{22} \\ \frac{\beta_3}{\beta_1}a_{11} & \frac{\beta_3}{\beta_2}a_{22} & a_{33} & \cdots & \frac{\beta_m}{\beta_3}a_{33} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_m}{\beta_1}a_{11} & \frac{\beta_m}{\beta_2}a_{22} & \frac{\beta_m}{\beta_3}a_{33} & \cdots & a_{mm} \end{pmatrix}.$$

Note that $\mathbf{A}_m$ and $\mathbf{A}_{m+1}$ are embedded. An analogous representation and embedding hold for $\widetilde{\mathbf{A}}_m$.

3 Estimates for a_{kk} and $\frac{\beta_k}{\beta_j}$

We will need upper and lower estimates for the elements of matrices $\mathbf{A}_m$ and $\widetilde{\mathbf{A}}_m$, namely

$$a_{kk} = \beta_k^2 \sum_{i=1}^{k} \alpha_i^2 \,, \qquad a_{kj} = \begin{cases} \frac{\beta_k}{\beta_j}\, a_{jj}, & j < k\,, \\ \frac{\beta_j}{\beta_k}\, a_{kk}, & j > k\,. \end{cases}$$

We found expression for a_{kk} and $\widetilde{a}_{kk}$ in [1, Lemmas 2.1(ii) and 2.2(ii)], those are quoted in Proposition 3.1, and in this section we obtain inequalities for the ratios $\frac{\beta_k}{\beta_j}$.

Proposition 3.1 ([1]) *The following identities hold:*

$$(i) \quad a_{kk} := \beta_k^2 \sum_{i=1}^{k} \alpha_i^2 \;\; = \;\; c_0 f_0(k), \tag{3.1}$$

$$(ii) \quad \widetilde{a}_{kk} := \widetilde{\beta}_k^2 \sum_{i=1}^{k} \widetilde{\alpha}_i^2 \;\; = \;\; c_0 f_0(k - \tfrac{1}{2}), \tag{3.2}$$

where

$$c_0 := \frac{4}{2\lambda + 1}, \qquad f_0(x) := x(x + \tfrac{\lambda}{2})(x + \lambda)\,.$$

Proposition 3.2 *Let $j, k \in \mathbb{N}$, $j < k$. Then the coefficients β_k in (2.3) satisfy the following relations:*

(i) If $-\frac{1}{2} < \lambda \le 0$ or $\lambda \ge 1$, then

$$\Big(\frac{j}{k}\Big)^{2\lambda-2} \le \frac{\beta_k^2}{\beta_j^2} \le \Big(\frac{j+\lambda}{k+\lambda}\Big)^{2\lambda-2}. \tag{3.3}$$

(ii) If $0 < \lambda \le 1$, then

$$\Big(\frac{j}{k}\Big)^{2\lambda-2} \ge \frac{\beta_k^2}{\beta_j^2} \ge \Big(\frac{j+\lambda}{k+\lambda}\Big)^{2\lambda-2}. \tag{3.4}$$

Proof. Denote the left-hand, the middle and the right-hand side terms in (3.3)-(3.4) by $\ell(\lambda)$, $m(\lambda)$ and $r(\lambda)$, respectively. From definitions (2.3) and (2.5) we have

$$m(\lambda) := \frac{\beta_k^2}{\beta_j^2} = \frac{\Gamma(2j+2\lambda)}{(2j+\lambda)\Gamma(2j+1)} \Big(\frac{\Gamma(2k+2\lambda)}{(2k+\lambda)\Gamma(2k+1)}\Big)^{-1}, \tag{3.5}$$

and using the functional equation $\Gamma(t+1) = t\,\Gamma(t)$ we see that

$$m(\lambda) = \begin{cases} \big(\frac{k}{j}\big)^2, & \lambda = 0, \\ 1, & \lambda = 1, \end{cases} \quad \Rightarrow \quad \ell(\lambda) = m(\lambda) = r(\lambda), \quad \lambda = 0, 1\,. \tag{3.6}$$

We shall prove inequalities (3.3)-(3.4) for the logarithms of the values involved.

1) Let us start with the proof of the left-hand side inequalities in (3.3)-(3.4). Consider the difference of the logarithms of the middle and the left-hand side terms,

$$g(\lambda) := \log m(\lambda) - \log \ell(\lambda) = \log m(\lambda) - (2\lambda - 2)\log\frac{j}{k}$$

We need to prove that $g(\lambda) \le 0$ for $\lambda \in [0, 1]$ and that $g(\lambda) > 0$ otherwise. Since $g(0) = g(1) = 0$ by (3.6), it suffices to show that $g''(\lambda) > 0$ for all $\lambda > -\frac{1}{2}$, i.e., that $[\log m(\lambda)]'' > 0$.

From (3.5), we have

$$\log m(\lambda) = \log\Gamma(2j+2\lambda) - \log\Gamma(2k+2\lambda) - \log\frac{2j+\lambda}{2k+\lambda} - \log\frac{\Gamma(2j+1)}{\Gamma(2k+1)}\,,$$

therefore, using the digamma function $\psi(t) := \Gamma'(t)/\Gamma(t)$, we obtain

$$[\log m(\lambda)]' = 2\left[\psi(2j+2\lambda) - \psi(2k+2\lambda)\right] - \Big[\frac{1}{2j+\lambda} - \frac{1}{2k+\lambda}\Big].$$

From the equation $\Gamma(t+1) = t\,\Gamma(t)$ it follows that $\psi(t+1) = \psi(t) + 1/t$, and the latter implies

$$[\log m(\lambda)]' = -2\sum_{i=2j}^{2k-1}\frac{1}{i+2\lambda} - \Big[\frac{1}{2j+\lambda} - \frac{1}{2k+\lambda}\Big], \tag{3.7}$$

whence

$$[\log m(\lambda)]'' = 4\sum_{i=2j}^{2k-1}\frac{1}{(i+2\lambda)^2} + \Big[\frac{1}{(2j+\lambda)^2} - \frac{1}{(2k+\lambda)^2}\Big] > 0\,,$$

and that proves the left-hand inequalities in (3.3)-(3.4).

2) We approach in the same way to the proof of the right-hand inequalities in (3.3) and (3.4), by taking the difference of the logarithms of the middle and the right-hand terms,

$$h(\lambda) := \log m(\lambda) - \log r(\lambda) = \log m(\lambda) - (2\lambda-2)\log\frac{j+\lambda}{k+\lambda}\,. \tag{3.8}$$

We need to show that $h(\lambda) \ge 0$ for $\lambda \in [0,1]$ and that $h(\lambda) < 0$ otherwise. Since $h(0) = h(1) = 0$ by (3.6), it suffices to show that $h'(\lambda) < 0$ for $\lambda > 1$ and that $h''(\lambda) < 0$ for $\lambda \in (-\frac{1}{2}, 1]$.

2a) Let us show that $h'(\lambda) \le 0$ for $\lambda \ge 1$. From (3.8) using (3.7), we obtain

$$h'(\lambda) = -2\sum_{i=2j}^{2k-1}\frac{1}{i+2\lambda} - \Big[\frac{1}{2j+\lambda} - \frac{1}{2k+\lambda}\Big] - 2\log\frac{j+\lambda}{k+\lambda} - (2\lambda-2)\Big[\frac{1}{j+\lambda} - \frac{1}{k+\lambda}\Big]. \tag{3.9}$$

For the sum, since the function $f(x) = (x+2\lambda)^{-1}$ is decreasing, we have

$$-2\sum_{i=2j}^{2k-1}\frac{1}{i+2\lambda} < -2\int_{2j}^{2k}\frac{1}{x+2\lambda}\,dx = 2\,\log\frac{j+\lambda}{k+\lambda}\,,$$

hence

$$h'(\lambda) < -\Big[\frac{1}{2j+\lambda} - \frac{1}{2k+\lambda}\Big] - (2\lambda-2)\Big[\frac{1}{j+\lambda} - \frac{1}{k+\lambda}\Big], \tag{3.10}$$

and for $\lambda > 1$ and $j < k$, the right-hand side is negative. Thus, $h'(\lambda) < 0$ for $\lambda > 1$.

2b) Next, we prove that if $\lambda \in (-\frac{1}{2}, 1]$, then $h''(\lambda) < 0$. From (3.9), we derive

$$h''(\lambda) = 4\sum_{i=2j}^{2k-1}\frac{1}{(i+2\lambda)^2} + \Big[\frac{1}{(2j+\lambda)^2} - \frac{1}{(2k+\lambda)^2}\Big] \tag{3.11}$$

$$-4\Big[\frac{1}{j+\lambda} - \frac{1}{k+\lambda}\Big] + (2\lambda-2)\Big[\frac{1}{(j+\lambda)^2} - \frac{1}{(k+\lambda)^2}\Big]\,. \tag{3.12}$$

The first term in the right-hand side is estimated as follows

$$\begin{aligned} 4\sum_{i=2j}^{2k-1}\frac{1}{(i+2\lambda)^2} &= 4\sum_{i=2j+1}^{2k}\frac{1}{(i+2\lambda)^2} + \Big[\frac{1}{(j+\lambda)^2} - \frac{1}{(k+\lambda)^2}\Big] \\ &< 2\Big[\frac{1}{j+\lambda} - \frac{1}{k+\lambda}\Big] + \Big[\frac{1}{(j+\lambda)^2} - \frac{1}{(k+\lambda)^2}\Big], \end{aligned}$$

where for the sum we have used the inequality $\sum_{i=2j+1}^{2k}(i+2\lambda)^{-2} < \int_{2j}^{2k}(x+2\lambda)^{-2}\,dx$.

Next, for $\lambda \in (-\frac{1}{2}, 1]$ and $x \geq \frac{1}{2}$ the function $f(x) = (2x+\lambda)^{-2} - (x+\lambda)^{-2}$ is increasing, hence for the second term in (3.11) we have

$$\Big[\frac{1}{(2j+\lambda)^2} - \frac{1}{(2k+\lambda)^2}\Big] < \Big[\frac{1}{(j+\lambda)^2} - \frac{1}{(k+\lambda)^2}\Big].$$

Substituting the above upper bounds in the expression (3.11)-(3.12) for $h''(\lambda)$, we obtain

$$h''(\lambda) < -2\Big[\frac{1}{j+\lambda} - \frac{1}{k+\lambda}\Big] + 2\lambda\Big[\frac{1}{(j+\lambda)^2} - \frac{1}{(k+\lambda)^2}\Big] = -\frac{2(k-j)(kj-\lambda^2)}{(j+\lambda)^2(k+\lambda)^2} < 0\,, \qquad (3.13)$$

since $1 \leq j < k$ and $\lambda \in (-\frac{1}{2}, 1]$. □

Proposition 3.3 *Let $j, k \in \mathbb{N}$, $j < k$. Then the coefficients $\widetilde{\beta}_k$ in (2.4) satisfy the following relations.*

(i) If $-\frac{1}{2} < \lambda \leq 0$ or $\lambda \geq 1$, then

$$\Big(\frac{j-\frac{1}{2}}{k-\frac{1}{2}}\Big)^{2\lambda-2} \leq \frac{\widetilde{\beta}_k^2}{\widetilde{\beta}_j^2} \leq \Big(\frac{j-\frac{1}{2}+\lambda}{k-\frac{1}{2}+\lambda}\Big)^{2\lambda-2}. \qquad (3.14)$$

(ii) If $0 < \lambda \leq 1$, then

$$\Big(\frac{j-\frac{1}{2}}{k-\frac{1}{2}}\Big)^{2\lambda-2} \geq \frac{\widetilde{\beta}_k^2}{\widetilde{\beta}_j^2} \geq \Big(\frac{j-\frac{1}{2}+\lambda}{k-\frac{1}{2}+\lambda}\Big)^{2\lambda-2}. \qquad (3.15)$$

Proof. By equality (2.6), we have

$$\widetilde{\beta}_j = \beta_{j-\frac{1}{2}}\,, \qquad \widetilde{\beta}_k = \beta_{k-\frac{1}{2}}\,.$$

Then all the relations throughout (3.5)-(3.13) remain valid with the substitution

$$j \to j - \tfrac{1}{2}, \qquad k \to k - \tfrac{1}{2}\,.$$

The only exception is inequality (3.13) which fails for $j = 1$, $k = 2$, and $\lambda \in [\frac{\sqrt{3}}{2}, 1]$, since the factor $\big[(k-\frac{1}{2})(j-\frac{1}{2}) - \lambda^2\big]$ is not positive then.

Let us prove that $\widetilde{h}(\lambda) \geq 0$ in this case as well. Since $\widetilde{h}(1) = 0$, it is sufficient to prove that $\widetilde{h}'(\lambda) < 0$ for $\lambda \in [\frac{\sqrt{3}}{2}, 1]$ and $j = 1$, $k = 2$. We have

$$\widetilde{h}'(\lambda)\Big|_{j,k} = h'(\lambda)\Big|_{j-\frac{1}{2},k-\frac{1}{2}}$$

so substituting $j = \frac{1}{2}$, $k = \frac{3}{2}$ into (3.10), we find that for $\lambda \in [\frac{3}{4}, 1] \supset [\frac{\sqrt{3}}{2}, 1]$

$$\begin{aligned} \widetilde{h}'(\lambda)\Big|_{1,2} = h'(\lambda)\Big|_{\frac{1}{2},\frac{3}{2}} &< -\Big[\frac{1}{1+\lambda} - \frac{1}{3+\lambda}\Big] - (2\lambda-2)\Big[\frac{1}{\frac{1}{2}+\lambda} - \frac{1}{\frac{3}{2}+\lambda}\Big] \\ &\leq -\Big[\frac{1}{1+\lambda} - \frac{1}{3+\lambda}\Big] + \frac{1}{2}\Big[\frac{1}{\frac{1}{2}+\lambda} - \frac{1}{\frac{3}{2}+\lambda}\Big] \\ &= -\frac{2}{(1+\lambda)(3+\lambda)} + \frac{2}{(1+2\lambda)(3+2\lambda)} < 0. \end{aligned}$$

4 Three lemmas

In the next two sections, we deal with lower and upper estimates for the sums $\sum_{j=1}^{\ell} f(j)$, in particular for $f = F_\nu$, where F_1, F_2 are given in (4.1) below. For that purpose, we need the following three lemmas.

We use the following notation:

$$\sum_{i=1}^{\ell}{}'' f(i) = \tfrac{1}{2} f(1) + \sum_{i=2}^{\ell-1} f(i) + \tfrac{1}{2} f(\ell)\,.$$

Lemma 4.1 *For a convex integrand f, we have*

$$\sum_{i=1}^{\ell} f(i) \le \int_{\frac{1}{2}}^{\ell+\frac{1}{2}} f(x)\,dx, \qquad \sum_{i=1}^{\ell}{}'' f(i) \ge \int_{1}^{\ell} f(x)\,dx\,.$$

Proof. The inequalities reveal well-known properties of the midpoint and the trapezoidal quadrature formulas relative to the corresponding integrals. □

Lemma 4.2 *For $\lambda > -\frac{1}{2}$, the functions*

$$F_1(x) = x^{2\lambda}(x + \tfrac{\lambda}{2})^2(x+\lambda)^2, \qquad F_2(x) = x^2(x+\tfrac{\lambda}{2})^2(x+\lambda)^{2\lambda} \tag{4.1}$$

are convex on $[\frac{1}{2}, \infty)$ and increasing on $[1, \infty)$.

Proof. 1) For $\lambda \ge 1$, all the factors of F_1, F_2 in (4.1) are convex, positive and increasing on $[0, \infty)$, hence the statement.

2) For $\lambda \in [0, 1]$ the functions

$$u_1(x) := x^\lambda(x+\lambda), \qquad u_2(x) := x(x+\lambda)^\lambda$$

are non-negative and increasing on $[0, \infty)$. Further, u_2 is convex on $[0, \infty)$, because it can be written in the form

$$u_2(x) = (x+\lambda)^{\lambda+1} - \lambda(x+\lambda)^\lambda\,,$$

where both terms are convex for $\lambda \in [0, 1]$, whereas u_1 is convex on $[\frac{1}{2}, \infty]$ because

$$u_1''(x) = [x^{\lambda+1} + \lambda x^\lambda]'' = \lambda x^{\lambda-2}\Big[(\lambda+1)x + \lambda(\lambda-1)\Big] > \lambda x^{\lambda-2}\Big[x - \tfrac{1}{4}\Big] \ge 0, \quad x \ge \tfrac{1}{2}\,.$$

Therefore, both $F_1(x) = [u_1(x)]^2(x+\frac{\lambda}{2})^2$ and $F_2(x) = [u_2(x)]^2(x+\frac{\lambda}{2})^2$ are convex on $[\frac{1}{2}, \infty)$ and increasing on $(0, \infty)$.

3) Let $\lambda \in (-\frac{1}{2}, 0]$. Then

$$u_1'(x) = x^{\lambda-1}\Big[(\lambda+1)x + \lambda^2\Big] > 0, \quad x > 0,$$

and

$$u_2'(x) = (x+\lambda)^{\lambda-1}\Big[(\lambda+1)(x+\lambda) - \lambda^2\Big] \ge (x+\lambda)^{\lambda-1}\Big[(\lambda+1)^2 - \lambda^2\Big] > 0, \quad x \ge 1,$$

hence F_1 and F_2 are increasing on $[1, \infty)$. Further, the function

$$v_1(x) := x^\lambda(x+\tfrac{\lambda}{2})(x+\lambda) = x^{\lambda+2} + \tfrac{3\lambda}{2}\,x^{\lambda-1} + \tfrac{\lambda^2}{2}x^\lambda$$

is convex for $x > 0$ because all the terms are convex for $\lambda \in (-\frac{1}{2}, 0]$, hence $F_1(x) = [v_1(x)]^2$ is convex whenever v_1 is nonnegative, i.e., for $x > -\lambda$, thus for $x \geq \frac{1}{2}$. Finally, for

$$v_2(x) := x(x + \tfrac{\lambda}{2})(x + \lambda)^\lambda = y^{\lambda+2} - \tfrac{3\lambda}{2}\, y^{\lambda+1} + \tfrac{\lambda^2}{2} y^\lambda\,, \quad y = x + \lambda,$$

we obtain

$$v_2''(x) = y^{\lambda-2}\Big[(\lambda+2)(\lambda+1)y^2 - \tfrac{3}{2}\lambda^2(\lambda+1)y + \tfrac{1}{2}\lambda^3(\lambda-1)\Big] =: y^{\lambda-2}p_2(y)\,,$$

and it is easy to check that, for $\lambda \in (-\frac{1}{2}, 0]$, the quadratic polynomial p_2 has no real zeros. Hence, v_2 is convex and so is $F_2(x) = [v_2(x)]^2$ for $x \geq \frac{1}{2}$. □

Lemma 4.3 *Let* $\alpha_i > 0$, $\gamma_{\min} \leq \gamma_i \leq \gamma_{\max}$, $1 \leq i \leq r$, *and let*

$$f(x) := (x+\gamma_1)^{\alpha_1}(x+\gamma_2)^{\alpha_2}\cdots(x+\gamma_r)^{\alpha_r}, \qquad s := \sum_{i=1}^r \alpha_i\,.$$

Then, for any $x > x_0$, *where* $x_0 + \gamma_{\min} \geq 0$, *we have*

$$\frac{1}{s+1}\Big[(t+\gamma_{\min})f(t)\Big]_{x_0}^x < \int_{x_0}^x f(t)\,dt < \frac{1}{s+1}\,(x+\gamma_{\max})f(x)\,. \tag{4.2}$$

Proof. Set

$$G(x) := \frac{1}{s+1}\,(x+\gamma_{\min})f(x), \qquad F(x) := \frac{1}{s+1}\,(x+\gamma_{\max})f(x)\,.$$

It suffices to show that $G'(t) < f(t) < F'(t)$ for $x_0 \leq t \leq x$. We have

$$G'(t) = \frac{1}{s+1}\Big[1 + \sum_{i=1}^r \alpha_i \frac{t+\gamma_{\min}}{t+\gamma_i}\Big]f(t) \leq \frac{1}{s+1}\Big[1+\sum_{i=1}^r \alpha_i\Big]f(t) = f(t)\,,$$

and similarly

$$F'(t) = \frac{1}{s+1}f(t)\Big[1 + \sum_{i=1}^s \alpha_i \frac{t+\gamma_{\max}}{t+\gamma_i}\Big] \geq \frac{1}{s+1}\Big[1+\sum_{i=1}^s \alpha_i\Big]f(t) = f(t)\,. \qquad \square$$

Remark 4.4 We can refine the upper estimate as follows:

$$\int_{x_0}^x f(t)\,dt < \frac{1}{s+1}[f(x)]^{\frac{s+1}{s}}\,.$$

Indeed, with $F(x) := \frac{1}{s+1}[f(x)]^{\frac{s+1}{s}}$, it suffices to show that $F'(t) \geq f(t)$ for every $t > x_0$. We have the equivalent relations

$$F'(t) = \frac{1}{s}\,\big[f(t)\big]^{\frac{1}{s}}\, f'(t) \,\geq f(t) \quad\Leftrightarrow\quad \big[f(t)\big]^{\frac{1}{s}} \geq \frac{s}{\frac{f'(t)}{f(t)}}\,,$$

and the latter is simply the inequality between the geometric and harmonic means

$$\Big(\prod (x+\gamma_i)^{\alpha_i}\Big)^{\frac{1}{\sum \alpha_i}} \geq \frac{\sum \alpha_i}{\sum \frac{\alpha_i}{x+\gamma_i}}\,.$$

5 An upper bound for $\|\mathbf{A}_m\|_\infty$ for $\lambda > 2$

Proposition 5.1 *For $\lambda > 2$, we have*

$$\|\mathbf{A}_m\|_\infty \le \frac{4}{(\lambda+2)(\lambda+3)}\, m(m+\tfrac{\lambda}{2})(m+\lambda)(m+\tfrac{3\lambda}{2}+3)\,. \tag{5.1}$$

Proof. Let us recall that

$$\|\mathbf{A}_m\|_\infty = \max_k \sum_j |a_{kj}|\,,$$

and, as is seen from (2.2), $a_{k,j} > 0$.

For a fixed k, $1 \le k \le m$, we consider the sum of the elements in the k-th row of $\mathbf{A}_m$,

$$\sum_{j=1}^m a_{kj} = \sum_{j=1}^{k-1} \frac{\beta_k}{\beta_j}\, a_{jj} + a_{kk} + \sum_{j=k+1}^m \frac{\beta_j}{\beta_k}\, a_{kk}.$$

By (3.1) and by (3.3),

$$a_{jj} = c_0 f_0(j), \qquad \frac{\beta_k}{\beta_j} \le \Big(\frac{j+\lambda}{k+\lambda}\Big)^{\lambda-1}, \quad j<k, \quad \lambda > 0\,, \tag{5.2}$$

where

$$c_0 := \frac{4}{2\lambda+1}\,, \qquad f_0(x) := x(x+\tfrac{\lambda}{2})(x+\lambda)\,,$$

hence

$$\sum_{j=1}^m a_{kj} \le c_0 \Big[\sum_{j=1}^{k-1} f_0(j)\Big(\frac{j+\lambda}{k+\lambda}\Big)^{\lambda-1} + f_0(k) + f_0(k)\sum_{j=k+1}^m \Big(\frac{k+\lambda}{j+\lambda}\Big)^{\lambda-1}\Big]\,. \tag{5.3}$$

For the first sum, since $f(x) = f_0(x)(x+\lambda)^{\lambda-1}$ is increasing, we apply an integral estimate and then Lemma 4.3 to obtain

$$\sum_{j=1}^{k-1} f_0(j)\Big(\frac{j+\lambda}{k+\lambda}\Big)^{\lambda-1} \le \int_1^k f_0(x)\Big(\frac{x+\lambda}{k+\lambda}\Big)^{\lambda-1} dx \le \frac{1}{\lambda+3}\,(k+\lambda)f_0(k)\,.$$

For the second sum, since $g(x) = 1/(x+\lambda)^{\lambda-1}$ is decreasing (and $\lambda > 2$), an integral estimate gives

$$f_0(k)\sum_{j=k+1}^m \Big(\frac{k+\lambda}{j+\lambda}\Big)^{\lambda-1} \le f_0(k)\int_k^m \Big(\frac{k+\lambda}{x+\lambda}\Big)^{\lambda-1} dx = \frac{1}{\lambda-2}(k+\lambda)f_0(k)\Big[1-\Big(\frac{k+\lambda}{m+\lambda}\Big)^{\lambda-2}\Big]\,.$$

Replacement in the right-hand of (5.3) yields

$$\sum_{j=1}^m a_{kj} < c_0\,(k+\lambda)f_0(k)\left[\frac{1}{\lambda+3} + \frac{1}{\lambda-2} - \frac{1}{\lambda-2}\Big(\frac{k+\lambda}{m+\lambda}\Big)^{\lambda-2}\right] + c_0 f_0(k) =: A + B\,. \tag{5.4}$$

1) We estimate A as follows.

$$\begin{aligned}
A &= \frac{4}{2\lambda+1}\,(k+\lambda)f_0(k)\left[\frac{1}{\lambda+3} + \frac{1}{\lambda-2} - \frac{1}{\lambda-2}\Big(\frac{k+\lambda}{m+\lambda}\Big)^{\lambda-2}\right] \\
&= \frac{4}{2\lambda+1}\,(k+\lambda)f_0(k)\left[\frac{2\lambda+1}{(\lambda+3)(\lambda-2)} - \frac{1}{\lambda-2}\Big(\frac{k+\lambda}{m+\lambda}\Big)^{\lambda-2}\right] \\
&= \frac{4(m+\lambda)^4}{(\lambda+3)(\lambda-2)}\,\frac{f_0(k)}{(k+\lambda)^3}\,\Big(\frac{k+\lambda}{m+\lambda}\Big)^4\left[1 - \frac{\lambda+3}{2\lambda+1}\Big(\frac{k+\lambda}{m+\lambda}\Big)^{\lambda-2}\right] \\
&= \frac{4(m+\lambda)^4}{(\lambda+3)(\lambda-2)}\,\psi_\lambda(k)\,\phi_\lambda(y)\,,
\end{aligned} \tag{5.5}$$

where in the last line we set

$$\phi_\lambda(y) := y^4 - \frac{\lambda+3}{2\lambda+1}\, y^{\lambda+2}\,, \qquad y = \frac{k+\lambda}{m+\lambda} \in [0,1]\,, \qquad \psi_\lambda(k) := \frac{f_0(k)}{(k+\lambda)^3}\,.$$

Let us evaluate $\phi_\lambda(y)$ and $\psi_\lambda(k)$. On $[0,1]$, for a fixed $\lambda > 2$, the function ϕ_λ has a unique local extremum, a maximum, which is attained at

$$y_\lambda = \Big(\frac{4(2\lambda+1)}{(\lambda+2)(\lambda+3)}\Big)^{\frac{1}{\lambda-2}} = \Big(1 - \frac{(\lambda-1)(\lambda-2)}{(\lambda+2)(\lambda+3)}\Big)^{\frac{1}{\lambda-2}} \in (0,1)\,.$$

Then

$$\phi_\lambda(y) \le \phi_\lambda(y_\lambda) = \frac{\lambda-2}{\lambda+2}\, y_\lambda^4 < \frac{\lambda-2}{\lambda+2}\,. \tag{5.6}$$

The function $\psi_\lambda(x) = \frac{x(x+\frac{\lambda}{2})(x+\lambda)}{(x+\lambda)^3}$ is increasing (since $h(x) = \frac{x+a}{x+b}$ is increasing for $a < b$), thus

$$\psi_\lambda(k) = \frac{f_0(k)}{(k+\lambda)^3} \le \frac{f_0(m)}{(m+\lambda)^3}\,. \tag{5.7}$$

Consequently, putting the estimates (5.6)-(5.7) into (5.5), we obtain

$$A \le \frac{4}{(\lambda+2)(\lambda+3)}\,(m+\lambda) f_0(m)\,.$$

2) For B in (5.4) we use the trivial upper estimate

$$B = c_0 f_0(k) \le c_0 f_0(m) = \frac{4}{2\lambda+1}\, f_0(m)\,.$$

3) Thus, from (5.4), we derive

$$\begin{aligned}\sum_{k=j}^{m} a_{kj} \le A + B \quad &\le \quad \frac{4}{(\lambda+2)(\lambda+3)}\Big(m + \lambda + \frac{(\lambda+2)(\lambda+3)}{2\lambda+1}\Big) f_0(m)\\ &\le \quad \frac{4}{(\lambda+2)(\lambda+3)}\,(m + \tfrac{3\lambda}{2} + 3) f_0(m)\,,\end{aligned}$$

where we have used that $\frac{(\lambda+2)(\lambda+3)}{2\lambda+1} < \frac{\lambda}{2} + 3$ for $\lambda > 2$. Hence,

$$\|\mathbf{A}_m\|_\infty \le \frac{4}{(\lambda+2)(\lambda+3)}\, m(m+\tfrac{\lambda}{2})(m+\lambda)(m+\tfrac{3\lambda}{2}+3)\,, \tag{5.8}$$

and (5.1) is proved. □

Proposition 5.2 *For $\lambda > 2$, and $n \in \mathbb{N}$, we have*

$$\begin{aligned}\|\mathbf{B}_n\|_\infty \quad &\le \quad \frac{1}{(\lambda+2)(\lambda+3)}\, n(n+\lambda)(n+2\lambda)(n+3\lambda+6) &(5.9)\\ &\le \quad \frac{1}{(\lambda+2)(\lambda+3)}\, n(n+2\lambda+2)^3\,. &(5.10)\end{aligned}$$

Proof. Recall that

$$\mathbf{B}_n = \begin{cases} 4\mathbf{A}_m, & n = 2m\\ 4\widetilde{\mathbf{A}}_m, & n = 2m-1\,. \end{cases} \tag{5.11}$$

Let us rewrite (5.8) as

$$\|\mathbf{A}_m\|_\infty \le 4c_1 K_1(m) = 4c_1 K_1(\tfrac{n}{2}), \qquad n = 2m\,, \tag{5.12}$$

where

$$c_1 := \frac{1}{(\lambda+2)(\lambda+3)}, \qquad K_1(m) := m(m+\tfrac{\lambda}{2})(m+\lambda)(m+\tfrac{3\lambda}{2}+3).$$

We derived this upper bound from two estimates in (5.2), namely

$$a_{jj} = c_0\, f_0(k), \qquad \frac{\beta_k}{\beta_j} \le \Big(\frac{j+\lambda}{k+\lambda}\Big)^{\lambda-1}, \quad j<k.$$

Now, we note that, by (3.14) and (3.2), we have similar estimates

$$\widetilde{a}_{jj} = c_0\, f_0(k-\tfrac12), \qquad \frac{\widetilde{\beta}_k}{\widetilde{\beta}_j} \le \Big(\frac{j-\frac12+\lambda}{k-\frac12+\lambda}\Big)^{\lambda-1}, \quad j<k,$$

and it is easy to see that all the inequalities for the sum $\sum_j \widetilde{a}_{kj}$ throughout (5.3)-(5.8) remain valid with the substitution $(j,k)\to(j-\frac12, k-\frac12)$, hence

$$\|\widetilde{\mathbf{A}}_m\|_\infty \le 4c_1K_1(m-\tfrac12) = 4c_1K_1(\tfrac{n}{2}), \qquad n=2m-1. \tag{5.13}$$

Now, from (5.11), (5.12) and (5.13), we obtain that for any $n\in\mathbb{N}$

$$\begin{aligned}\|\mathbf{B}_n\|_\infty \le 16c_1K_1\big(\tfrac{n}{2}\big) &= c_1n(n+\lambda)(n+2\lambda)(n+3\lambda+6)\\ &\le c_1n(n+2\lambda+2)^3,\end{aligned}$$

where the last inequality follows by relation between geometric and arithmetic means, namely $abc \le (\frac{a+b+c}{3})^3$, with $(a,b,c)=(n+\lambda, n+2\lambda, n+3\lambda+6)$. This proves (5.9)-(5.10). □

6 Lower and upper estimates for $\|\mathbf{A}_m\|_F$ for $\lambda > -\frac12$

Proposition 6.1 *For $\lambda > -\frac12$, we have*

$$c_4\,(m+\lambda')(m+\lambda'+4)F_0(m) \;\le\; \|\mathbf{A}_m\|_F^2 \;\le\; c_4\,(m+\lambda+\lambda''+\tfrac52)^2F_0(m+\tfrac12), \tag{6.1}$$

where $\lambda' := \min\{0,\lambda\}$, $\lambda'' := \max\{0,\lambda\}$, and

$$c_4 := \frac{4}{(2\lambda+1)^2(2\lambda+5)}, \qquad F_0(x) := [f_0(x)]^2 = x^2(x+\tfrac{\lambda}{2})^2(x+\lambda)^2.$$

Proof. By the definition of the Frobenius norm,

$$\|\mathbf{A}_m\|_F^2 := \sum_{j,k=1}^m a_{kj}^2.$$

Since matrices $\{\mathbf{A}_m\}$ are symmetric and embedded, we have

$$\|\mathbf{A}_k\|_F^2 - \|\mathbf{A}_{k-1}\|_F^2 \;=\; 2{\sum_{j=1}^k}' a_{kj}^2 \;=\; 2{\sum_{j=1}^k}' \frac{\beta_k^2}{\beta_j^2}\, a_{jj}^2. \tag{6.2}$$

where $\sum'$ means that the last summand is halved. Recall that by (3.1)

$$a_{jj}^2 = c_0^2\,[f_0(j)]^2 =: c_2F_0(j), \qquad c_2 := \frac{16}{(2\lambda+1)^2}.$$

1) *The case* $\lambda \in (-\frac{1}{2}, 0] \cup [1, \infty)$. In that case, by (3.3),

$$\Big(\frac{j}{k}\Big)^{2\lambda-2} \le \frac{\beta_k^2}{\beta_j^2} \le \Big(\frac{j+\lambda}{k+\lambda}\Big)^{2\lambda-2},$$

so we obtain from (6.2)

$$2\,c_2 \sum_{j=1}^{k}{}' f_1(j) \le \|\mathbf{A}_k\|_F^2 - \|\mathbf{A}_{k-1}\|_F^2 \le 2\,c_2 \sum_{j=1}^{k}{}' f_2(j)\,, \tag{6.3}$$

where

$$f_1(x) \;:=\; F_0(x)\Big(\frac{x}{k}\Big)^{2\lambda-2} = \frac{1}{k^{2\lambda-2}}\, x^{2\lambda}(x+\tfrac{\lambda}{2})^2(x+\lambda)^2, \tag{6.4}$$

$$f_2(x) \;:=\; F_0(x)\Big(\frac{x+\lambda}{k+\lambda}\Big)^{2\lambda-2} = \frac{1}{(k+\lambda)^{2\lambda-2}}\, x^2(x+\tfrac{\lambda}{2})^2(x+\lambda)^{2\lambda}\,. \tag{6.5}$$

Note that, by Lemma 4.2, both functions are convex on $[\frac{1}{2}, \infty)$ and monotonely increasing on $[1, \infty)$, and that

$$f_1(k) = f_2(k) = F_0(k)\,.$$

Set

$$\lambda' := \min\{0, \tfrac{\lambda}{2}, \lambda\} = \min\{0, \lambda\}\,, \qquad \lambda'' := \max\{0, \tfrac{\lambda}{2}, \lambda\} = \max\{0, \lambda\}\,.$$

Those will play the roles of $\gamma_{\max}$ and $\gamma_{\min}$ when we apply Lemma 4.3.

1a) For the upper estimate, since f_2 is convex and increasing, we have by Lemmas 4.1 and 4.3 for $k \ge 2$,

$$\sum_{j=1}^{k-1} f_2(j) \le \int_{\frac{1}{2}}^{k-\frac{1}{2}} f_2(x)dx \le \frac{k-\frac{1}{2}+\lambda''}{2\lambda+5} f_2(k-\tfrac{1}{2}) \le \frac{k-\frac{1}{2}+\lambda''}{2\lambda+5} f_2(k)\,,$$

so that, for $k \ge 1$,

$$\sum_{j=1}^{k}{}' f_2(j) \le \frac{k-\frac{1}{2}+\lambda''}{2\lambda+5} f_2(k) + \frac{1}{2} f_2(k) = c_3\,(k+\lambda''+\lambda+2) f_2(k), \qquad c_3 := \frac{1}{2\lambda+5}\,,$$

hence

$$\|\mathbf{A}_k\|_F^2 - \|\mathbf{A}_{k-1}\|_F^2 \le 2\,c_2 c_3\,(k+\lambda''+\lambda+2) f_2(k) = 2\,c_2 c_3\,(k+\lambda''+\lambda+2) F_0(k)\,. \tag{6.6}$$

Then,

$$\|\mathbf{A}_m\|_F^2 = \sum_{k=1}^{m} \Big(\|\mathbf{A}_k\|_F^2 - \|\mathbf{A}_{k-1}\|_F^2\Big) \le 2\,c_2 c_3 \sum_{k=1}^{m} g_2(k)\,,$$

where $g_2(x) = (x+\lambda''+\lambda+2)F_0(x)$ is convex, and by Lemmas 4.1 and 4.3 we obtain

$$\sum_{k=1}^{m} g_2(k) \le \int_{\frac{1}{2}}^{m+\frac{1}{2}} g_2(x)\,dx \le \frac{1}{8}\,(m+\lambda+\lambda''+\tfrac{5}{2})\,g_2(m+\tfrac{1}{2}) = \frac{1}{8}(m+\lambda+\lambda''+\tfrac{5}{2})^2 F_0(m+\tfrac{1}{2})\,, \tag{6.7}$$

and this proves the upper estimates in (6.1) for $\lambda \in (-\frac{1}{2}, 0] \cup [1, \infty)$, with the constant $c_4 = \frac{1}{4} c_2 c_3$.

1b) For the lower estimate, we get by Lemmas 4.1 and 4.3,

$$\begin{aligned}\sum_{j=1}^{k}{}' f_1(j) \;&=\; \frac{1}{2} f_1(1) + \sum_{j=1}^{k}{}'' f_1(j) \;\ge\; \frac{1}{2} f_1(1) + \int_1^k f_1(x)\,dx \\ &\ge\; \frac{1}{2} f_1(1) + \frac{k+\lambda'}{2\lambda+5} f_1(k) - \frac{1+\lambda'}{2\lambda+5} f_1(1) \;\ge\; \frac{k+\lambda'}{2\lambda+5} f_1(k)\,,\end{aligned}$$

hence

$$\|\mathbf{A}_k\|_F^2 - \|\mathbf{A}_{k-1}\|_F^2 \geq 2c_2c_3(k+\lambda')f_1(k) = 2c_2c_3(k+\lambda')F_0(k)\,. \tag{6.8}$$

Then,

$$\|\mathbf{A}_m\|_F^2 \geq 2c_2c_3 \sum_{k=1}^{m} g_1(k)\,,$$

where $g_1(x) = (x+\lambda')F_0(x)$ is convex, therefore, by Lemmas 4.1 and 4.3,

$$\begin{aligned}
\sum_{k=1}^{m} g_1(k) &= \frac{1}{2}g_1(1) + {\sum_{j=1}^{k}}'' g_1(j) + \frac{1}{2}g_1(m) \geq \frac{1}{2}g_1(1) + \int_1^m g_1(x)\,dx + \frac{1}{2}g_1(m) \\
&\geq \frac{1}{2}g_1(1) + \frac{m+\lambda'}{8}g_1(m) - \frac{1+\lambda'}{8}g_1(1) + \frac{1}{2}g_1(m) \geq \frac{m+\lambda'}{8}g_1(m) + \frac{1}{2}g_1(m) \\
&\geq \frac{1}{8}(m+\lambda'+4)\,g_1(m) = \frac{1}{8}(m+\lambda')(m+\lambda'+4)\,F_0(m)\,,
\end{aligned} \tag{6.9}$$

and the lower estimate in (6.1) follows, with $c_4 = \frac{1}{4}c_2c_3$.

2) *The case* $\lambda \in [0,1]$. In that case, by (3.14), we have

$$\Big(\frac{j+\lambda}{k+\lambda}\Big)^{2\lambda-2} \leq \frac{\beta_k^2}{\beta_j^2} \leq \Big(\frac{j}{k}\Big)^{2\lambda-2},$$

so we obtain

$$2c_2 {\sum_{j=1}^{k}}' f_2(j) \leq \|\mathbf{A}_k\|_F^2 - \|\mathbf{A}_{k-1}\|_F^2 \leq 2c_2 {\sum_{j=1}^{k}}' f_1(j)\,,$$

i.e., the same inequality as in (6.3), but with f_1 and f_2 interchanged.

2a) Then the upper estimates will run in the same way only with f_1 instead of f_2, and because

$$f_1(k) = f_2(k) = F_0(k) \tag{6.10}$$

we arrive at the same inequality (6.6), so that the final upper estimate for $\|\mathbf{A}_m\|_F^2$ for $\lambda \in [0,1]$ is the same as (6.7).

2b) Similarly, the lower estimates for $\lambda \in [0,1]$ will run in the same way only with f_2 instead of f_1, and because of (6.10) we arrive at the same inequality (6.8), so that the final lower estimate for $\|\mathbf{A}_m\|_F^2$ for $\lambda \in [0,1]$ is also the same as (6.9). □

Proposition 6.2 *For $n \in \mathbb{N}$ and $\lambda > -\frac{1}{2}$, we have*

$$c_5^2\,(n+8)n^3(n+\lambda)^2(n+2\lambda)^2 \;\leq\; \|\mathbf{B}_n\|_F^2 \;\leq\; c_5^2\,(n+2\lambda+2)^8, \quad \lambda \geq 0; \tag{6.11}$$

$$c_5^2\,(n+2\lambda+8)n^2(n+\lambda)^2(n+2\lambda)^3 \;\leq\; \|\mathbf{B}_n\|_F^2 \;\leq\; c_5^2\,(n+\lambda+2)^8, \quad \lambda \in (-\tfrac{1}{2}, 0] \tag{6.12}$$

where

$$c_5^2 := \frac{1}{16}\,c_4 = \frac{1}{4(2\lambda+1)^2(2\lambda+5)}\,.$$

Proof. Recall again that

$$\mathbf{B}_n = \begin{cases} 4\mathbf{A}_m, & n = 2m; \\ 4\widetilde{\mathbf{A}}_m, & n = 2m-1, \end{cases} \tag{6.13}$$

and rewrite (6.1) as

$$c_4 K_{2,\lambda}(m) \leq \|\mathbf{A}_m\|_F^2 \leq c_4 K_{3,\lambda}(m), \qquad n = 2m\,. \tag{6.14}$$

Then, for odd $n = 2m-1$, by the same arguments as in the proof of Proposition 5.2, we obtain

$$c_4 K_{2,\lambda}(m-\tfrac{1}{2}) \le \|\widetilde{\mathbf{A}}_m\|_F^2 \le c_4 K_{3,\lambda}(m-\tfrac{1}{2}), \qquad n = 2m-1\,, \tag{6.15}$$

so that, for all $n \in \mathbb{N}$,

$$16c_4 K_{2,\lambda}\Big(\frac{n}{2}\Big) \le \|\mathbf{B}_n\|_F^2 \le 16 c_4 K_{3,\lambda}\Big(\frac{n}{2}\Big)\,. \tag{6.16}$$

Simplifying $K_{2,\lambda}(\frac{n}{2})$ we obtain

$$K_{2,\lambda}\Big(\frac{n}{2}\Big) \;=\; \frac{1}{2^8} n^2(n+\lambda)^2(n+2\lambda)^2(n+2\lambda')(n+2\lambda'+8)\,,$$

and this gives the lower bounds in (6.11)-(6.12) with the constant

$$c_5^2 = \frac{16}{2^8}\, c_4 = \frac{1}{16}\, c_4\,.$$

For the upper bounds we get

$$\begin{aligned} K_{3,\lambda}\Big(\frac{n}{2}\Big) &= \frac{1}{2^8}(n+1)^2(n+\lambda+1)^2(n+2\lambda+1)^2(n+2\lambda+2\lambda''+5)^2 \\ &\le \frac{1}{2^8}(n+\tfrac{5}{4}\lambda+\tfrac{1}{2}\lambda''+2)^8\,, \end{aligned}$$

where we used the inequality $abcd \le (\frac{a+b+c+d}{4})^4$. The last term does not exceed $2^{-8}(n+2\lambda+2)^8$, if $\lambda \ge 0$, and $2^{-8}(n+\lambda+2)^8$, if $\lambda \in (-\frac{1}{2}, 0)$.

That proves the upper bounds in (6.11)-(6.12). □

7 Proof of the main results

Firstly, we will prove Theorem 1.1 by establishing separately the lower and the upper bounds therein.

Theorem 7.1 *For the upper bounds, we have*

$$[c_n(\lambda)]^2 \le \begin{cases} \frac{1}{(\lambda+2)(\lambda+3)}\, n(n+2\lambda+2)^3, & \lambda > 2; \\ \frac{1}{2(2\lambda+1)\sqrt{2\lambda+5}}\, (n+\lambda+\lambda''+2)^4, & \lambda > -\frac{1}{2}. \end{cases} \tag{7.1}$$

where $\lambda'' = \max\{0, \lambda\}$.

Proof. We proved in Propositions 5.2 and 6.2 that

$$\|\mathbf{B}_n\|_\infty \le L_1(n,\lambda), \quad \lambda > 2, \qquad \|\mathbf{B}_n\|_F \le L_2(n,\lambda), \quad \lambda > -\frac{1}{2},$$

where L_ν is the ν-th line in (7.1), and since $[c_n(\lambda)]^2 = \mu_{\max}(\mathbf{B}_n)$, and the largest eigenvalue $\mu_{\max}(\mathbf{B}_n)$ is smaller than any matrix norm, the upper bounds (7.1) follow. □

Theorem 7.2 *For the lower bounds, we have*

$$[c_n(\lambda)]^2 \ge \begin{cases} \frac{1}{4(\lambda+1)(\lambda+2)}\, n^2(n+\lambda)^2, & \lambda > 2; \\ \frac{1}{(2\lambda+1)(2\lambda+5)}\, (n+\lambda)^2(n+2\lambda')^2, & \lambda > -\frac{1}{2}\,, \end{cases} \tag{7.2}$$

where $\lambda' = \min\{0, \lambda\}$.

Proof. 1) The first inequality in (7.2) follows from second, since

$$\frac{1}{4(\lambda+1)(\lambda+2)} < \frac{1}{(2\lambda+1)(2\lambda+5)}, \qquad n+2\lambda' = n \quad (\lambda > 0)\,.$$

2) Let us prove the second inequality in (7.2) splitting the cases $\lambda > 0$ and $-\frac{1}{2} < \lambda \le 0$. We proved in Proposition 6.2 that

$$\|\mathbf{B}_n\|_F^2 \ge \begin{cases} c_5^2 n^3(n+8)(n+\lambda)^2(n+2\lambda)^2, & \lambda \ge 0; \\ c_5^2 n^2(n+\lambda)^2(n+2\lambda)^3(n+2\lambda+8), & \lambda \in (-\frac{1}{2}, 0]\,, \end{cases} \tag{7.3}$$

where

$$c_5^2 = \frac{1}{4(2\lambda+1)^2(2\lambda+5)}\,.$$

Next, we will need an expression for the trace of $\mathbf{B}_n$, which we obtained in [1, p. 17],

$$\mathrm{tr}\,(\mathbf{B}_n) = \begin{cases} c_6\, n(n+2)(n+2\lambda)(n+2\lambda+2)\,, & n = 2m; \\ c_6 \Big[[(n+1)(n+2\lambda+1)]^2 - 2[(n+1)(n+2\lambda+1)]\Big]\,, & n = 2m-1\,, \end{cases} \tag{7.4}$$

where

$$c_6 = \frac{1}{4(2\lambda+1)}\,.$$

From (7.4) we can get a common upper bound for both odd and even n as follows. For odd n, we obtain from (7.4)

$$\begin{aligned} \mathrm{tr}\,(\mathbf{B}_n) &< c_6\big[(n+1)^2(n+2\lambda+1)^2 - (n+1)^2\big] \\ &= c_6(n+1)^2(n+2\lambda)(n+2\lambda+2), \qquad \lambda \ge 0, \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} \mathrm{tr}\,(\mathbf{B}_n) &\le c_6\,[(n+1)^2(n+2\lambda+1)^2] - (n+2\lambda+1)^2 \\ &= c_6\,(n+2\lambda+1)^2 n(n+2), \qquad \lambda \in (-\tfrac{1}{2}, 0]\,, \end{aligned} \tag{7.6}$$

and it is clear the both estimates (7.5)-(7.6) give upper bounds for $\mathrm{tr}\,(\mathbf{B}_n)$ for even $n = 2m$ in (7.4) as well.

Set

$$c_7 = \frac{c_5^2}{c_6} = \frac{1}{(2\lambda+1)(2\lambda+5)}\,.$$

2a) Then, for $\lambda \ge 0$, from (1.10), (7.3) and (7.5) we have

$$\begin{aligned} \mu_{\max}(\mathbf{B}_n) \ge \frac{\|\mathbf{B}_n\|_F^2}{\mathrm{tr}\,(\mathbf{B}_n)} &\ge c_7 \frac{n^3(n+8)(n+\lambda)^2(n+2\lambda)^2}{(n+1)^2(n+2\lambda)(n+2\lambda+2)} \\ &=: c_7 n^2(n+\lambda)^2 \phi_\lambda(n) \\ &> c_7 n^2(n+\lambda)^2\,, \end{aligned}$$

since for $\lambda \ge 0$ and $n \ge 3$

$$\phi_\lambda(n) := \frac{n(n+8)}{(n+1)^2}\frac{n+2\lambda}{n+2\lambda+2} \ge \frac{n(n+8)}{(n+1)^2}\frac{n}{n+2} \ge 1\,.$$

2b) Similarly, for $\lambda \in (-\frac{1}{2}, 0]$, from (1.10), (7.3) and (7.6), we have

$$\begin{aligned} \mu_{\max}(\mathbf{B}_n) \ge \frac{\|\mathbf{B}_n\|_F^2}{\mathrm{tr}\,(\mathbf{B}_n)} &\ge c_7 \frac{n^2(n+\lambda)^2(n+2\lambda)^3(n+2\lambda+8)}{n(n+2)(n+2\lambda+1)^2} \\ &= c_7(n+\lambda)^2(n+2\lambda)^2\psi_\lambda(n) \\ &> c_7(n+\lambda)^2(n+2\lambda)^2\,, \end{aligned}$$

since for $\lambda \in (-\frac{1}{2}, 0]$ and $n \geq 3$

$$\psi_\lambda(n) := \frac{n}{n+2}\frac{(n+2\lambda)(n+2\lambda+8)}{(n+2\lambda+1)^2} \geq \frac{n}{n+2}\frac{n(n+8)}{(n+1)^2} \geq 1\,.$$

This proves the lower estimates (7.3). □

For the proof of Theorem 1.4, we need yet one more lower bound.

Lemma 7.3 *For all $n \in \mathbb{N}$ and $\lambda > -\frac{1}{2}$, we have*

$$[c_n(\lambda)]^2 \geq \frac{2}{2\lambda+1} n(n+\lambda)(n+2\lambda)\,. \tag{7.7}$$

Proof. For any symmetric matrix $\mathbf{C} \in \mathbb{R}^{m\times m}$, its largest eigenvalue $\mu_{\max}(\mathbf{C})$ satisfies the inequality $\mu_{\max}(\mathbf{C}) = \sup_{\|\mathbf{x}\|=1}(\mathbf{C}\mathbf{x}, \mathbf{x}) \geq (\mathbf{C}\mathbf{e}_i, \mathbf{e}_i) = c_{ii}$, $1 \leq i \leq m$. Therefore,

$$[c_n(\lambda)]^2 = \mu_{\max}(\mathbf{B}_n) \geq b_{mm} = 4a_{mm}$$

and by (3.1)-(3.2), with $f_0(x) = x(x+\frac{\lambda}{2})(x+\lambda)$, we have

$$4a_{mm} = \frac{16}{2\lambda+1} f_0\Big(\frac{n}{2}\Big) = \frac{2}{2\lambda+1} n(n+\lambda)(n+2\lambda)\,.$$

□

We will prove Theorem 1.4 by establishing a slightly stronger statement.

Theorem 7.4 *For $n \geq 3$ and $\lambda > 2$, we have*

$$\frac{1}{8}F(n,\lambda) \leq [c_n(\lambda)]^2 \leq F(n,\lambda) \tag{7.8}$$

where

$$F(n,\lambda) = \frac{n(n+\lambda)(n+2\lambda)(n+3\lambda)}{(\lambda+1)(\lambda+2)} \tag{7.9}$$

Proof. 1) For the upper bound, using the upper bound in (5.9), we have

$$[c_n(\lambda)]^2 \quad \leq \quad \frac{n(n+\lambda)(n+2\lambda)(n+3\lambda+6)}{(\lambda+2)(\lambda+3)} =: F(n,\lambda)\phi(n,\lambda)$$

where

$$\phi(n,\lambda) := \frac{\lambda+1}{\lambda+3}\cdot\frac{n+3\lambda+6}{n+3\lambda} \leq \frac{\lambda+1}{\lambda+3}\cdot\frac{3+3\lambda+6}{3+3\lambda} = 1\,, \qquad n \geq 3.$$

2) For the lower bound, we consider two cases.

2a) If $n > 5\lambda$, we use the lower estimate (1.5)

$$[c_n(\lambda)]^2 \geq \frac{1}{4}\frac{n^2(n+\lambda)^2}{(\lambda+1)(\lambda+2)} =: \frac{1}{4}F(n,\lambda)\psi_1(n,\lambda)\,,$$

where

$$\psi_1(n,\lambda) := \frac{n(n+\lambda)}{(n+2\lambda)(n+3\lambda)} = \frac{1}{(1+\frac{2\lambda}{n})(1+\frac{2\lambda}{n+\lambda})} > \frac{1}{(1+\frac{2}{5})(1+\frac{2}{6})} = \frac{5}{7}\cdot\frac{6}{8} > \frac{1}{2}$$

2b) For $n \leq 5\lambda$, we use the estimate (7.7),

$$[c_n(\lambda)]^2 \geq \frac{2}{2\lambda+1} n(n+\lambda)(n+2\lambda) \geq \frac{1}{\lambda+1} n(n+\lambda)(n+2\lambda) = F(n,\lambda)\psi_2(n,\lambda)\,.$$

where

$$\psi_2(n,\lambda) := \frac{\lambda+2}{n+3\lambda} > \frac{\lambda}{n+3\lambda} \geq \frac{1}{5+3} = \frac{1}{8}\,.$$

□

Proof of Theorem 1.4. Since

$$\frac{3}{4}(n+2\lambda)^2 < (n+\lambda)(n+3\lambda) < (n+2\lambda)^2$$

and

$$\frac{1}{\frac{3}{2}\lambda^2} < \frac{1}{(\lambda+1)(\lambda+2)} < \frac{1}{\lambda^2}, \qquad \lambda \geq 7,$$

we derive from (7.8) that

$$\frac{1}{16}\,\frac{n(n+2\lambda)^3}{\lambda^2} < [c_n(\lambda)]^2 < \frac{n(n+2\lambda)^3}{\lambda^2}\,, \quad \lambda \geq 7\,,$$

and that proves (1.7). □

Proof of Corollary 1.5. Claim i) is equivalent to

$$n \leq \lim_{\lambda\to\infty} \frac{c_n(\lambda)^2}{2\lambda} \leq 3n\,.$$

The upper estimate follows from (7.8), while the lower estimate follows from (7.7). Claim ii) follows from estimates (1.6). □

Remark 7.5 The approach proposed here is applicable for derivation of tight two sided estimates for the best constant in the Markov L_2 inequality with the Laguerre weight $w_\alpha(x) = x^\alpha e^{-x}$. The results will appear in a forthcoming paper.

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Geno Nikolov

Department of Mathematics and Informatics
Universlty of Sofia
5 James Bourchier Blvd.
1164 Sofia
BULGARIA
E-mail: geno@fmi.uni-sofia.bg

Alexei Shadrin

Department of Applied Mathematics and Theoretical Physics (DAMTP)
Cambridge University
Wilberforce Road
Cambridge CB3 0WA
UNITED KINGDOM
E-mail: a.shadrin@damtp.cam.ac.uk