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News on quadratic polynomials

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Many problems in mathematics have remained unsolved because of missing links between mathematical disciplines, such as algebra, geometry, analysis, or number theory. Here we introduce a recently discovered result concerning quadratic polynomials, which uses a bridge between algebra and analysis. We study the iterations of quadratic polynomials, obtained by computing the value of a polynomial for a given number and feeding the outcome into the exact same polynomial again. These iterations of polynomials have interesting applications, such as in fractal theory.

1 Introduction

Around the year 825, the Persian mathematician Muhammad al-Khwarizmi wrote his book *Al-Kitāb al-muhtaṣar fī ḥisāb al-ğabr wa-'l-muqābala*[1] (The Compendious Book on Calculation by Completion and Balancing). In his work he explains in detail how to solve equations of the form

$$3\,x^2 + 5\,x = 1. \tag{1}$$

The ability to solve such equations was important already at that time. For example, these equations arose in disputes of inheritance and legacies. Nowadays,

[1] You may be familiar with some of these words: The word *al-ğabr* is the origin of the word *algebra* and the name *al-Khwarizmi* gave rise to the word *algorithm*.

every high school student is familiar with the general formula for the solution of (1). The two solutions of (1) are $x_{\pm} = -5/6 \pm \sqrt{(5/6)^2 + 1/3}$.

Given that we have been able to solve these equations without much difficulties for almost 1200 years, it may be surprising that we want to study something (seemingly) simple as quadratic equations in a report on modern mathematics. However, we note that also prime numbers have been studied for at least 2300 years and they have not, by far, revealed all of their secrets. It should be clear that, regardless of how simple or well studied a problem is, it can always hide extremely important secrets.

In the following we will define more rigorously the main objects of this snapshot.

1.1 Quadratic polynomials

A *quadratic polynomial* is a polynomial $f(z)$ of the generic form

$$f(z) = a\,z^2 + b\,z + c, \tag{2}$$

but in this snapshot we only consider polynomials $f_c(z)$ of the form $f_c(z) = z^2 + c$ (for instance $z^2 + 2$, $z^2 - 1$, $z^2 + \sqrt{3}$, $z^2 + 7\,i, \dots$).[2] The parameter c is a complex number.[3] We visualize $\mathbb{C}$ as a plane, where the element $a + b\,i$ has the coordinates a and b, see Figure 1.

Every polynomial of the form $f_c(z) = z^2 + c$, where c is a complex number, describes a map f_c from $\mathbb{C}$ to $\mathbb{C}$. This means that, if we substitute any complex number in place of the variable z, we obtain another complex number as the outcome. In order to illustrate the fact that we regard f_c as a map $f_c : \mathbb{C} \longrightarrow \mathbb{C}$ we formally write

$$f_c : z \longmapsto z^2 + c.$$

Of course, it is easy to calculate $f_c(z)$ for given complex numbers c and z.

As a concrete example we consider $c = -{}^{29}\!/_{16}$ and we have $f_{-29/16}(z) = z^2 - {}^{29}\!/_{16}$. Therefore, starting with $z = {}^{3}\!/_{4}$ one finds

$$f_{-29/16}({}^{3}\!/_{4}) = ({}^{3}\!/_{4})^2 - {}^{29}\!/_{16} = -{}^{20}\!/_{16} = -{}^{5}\!/_{4}.$$

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[2] For an introduction to polynomials see Snapshot 3/2016 *On the containment problem* by Tomasz Szemberg and Justyna Szpond.

[3] We will assume that the reader is familiar with the set $\mathbb{R}$ of real numbers. You can see $\mathbb{R}$ as a set of all decimal numbers like 2 or $\frac{1}{3}$ or $\pi = 3.1415926\dots$. For a gentle introduction to the complex numbers, see Snapshot 4/2014 *What does ">" really mean?* by Bruce Reznick. A quick reminder: Take an *imaginary* element i with $i^2 = -1$. Then, the *complex numbers* $\mathbb{C}$ are numbers of the form $a + b\,i$, where a and b are real numbers. In $\mathbb{C}$, we can calculate as we please all four operations $+$, $-$, $\times$, and $\div$ (the only restriction is that we cannot divide by zero).

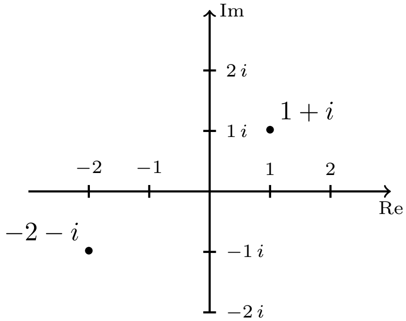


Figure 1: The complex numbers $-2-i$ and $1+i$ visualized in the complex plane. Re stands for the *real part* of the complex number and Im for its *imaginary part*.

But what happens if we apply the map f_c to $f_c(z)$, and then to $f_c(f_c(z))$, and so on?[4] To ease notation we will call the n-th iteration $f_c^n(z)$, which reads

$$f_c^n(z) = \underbrace{f_c(f_c(\cdots f_c}_{n\text{-times}}(z)\cdots)).$$

The sequence of complex numbers $z, f_c(z), f_c^2(z), f_c^3(z), \ldots$ is called the f_c*-orbit* of z. A natural question to ask at this point is, what happens with the f_c-orbit of z for given numbers c and z? Without much thinking, we can anticipate two different scenarios to occur and we distinguish them as follows. Either the f_c-orbit of z consists of infinitely many different complex numbers, or the f_c-orbit of z only contains finitely many different complex numbers. In the latter case z is called a *preperiodic point* of f_c. We will explain this naming in an example.

We work again with the map $f_{-29/16}$.[5] In the following, we will use $\mapsto$ to denote the application of the map $f_{-29/16}$. Starting with $z = 3/4$ we get:

$$\frac{3}{4} \mapsto -\frac{5}{4} \mapsto -\frac{1}{4} \mapsto -\frac{7}{4} \mapsto \frac{5}{4} \mapsto -\frac{1}{4} \mapsto -\frac{7}{4} \mapsto \cdots$$

[4] These continued iterations of a given function are nothing unusual. For example, they appear in the mathematical description of chaos, as used by meteorologists to forecast the weather. In this snapshot, however, we will enjoy the amenity of pure mathematics, which allows us to study mathematical problems without any real life application in mind.

[5] This map is not chosen arbitrarily. It has exactly 8 different preperiodic points in the rational numbers. It is still an open problem to find a rational number c such that f_c has more than eight preperiodic points!

Now we are in a loop, and the element $-\frac{1}{4}$ appears in the above f-orbit *periodically*. Since the element $\frac{3}{4}$ is in the f-orbit *prior* to the first number that initiates a periodical (or repetitive) behaviour, it is called *preperiodic*.

This example helps us to introduce the following mathematical statement, which we will formulate as a mathematical "helping theorem", also called *Lemma*.

Lemma 1. *Given any complex numbers c and z. Then z is a preperiodic point of f_c if and only if there are different integers n and m such that $f_c^n(z) = f_c^m(z)$.*

In the example considered above, we have $f^2_{-29/16}(3/4) = f^5_{-29/16}(3/4)$.

Among all possible preperiodic points that can exist for a specific quadratic polynomial $f_c(z)$ for a complex number c, we are particularly interested in studying polynomials $f_c(z)$ of which 0 or 1 are preperiodic points. Given that the choice could have fallen on any number to start with, we note that the numbers 0 and 1 are not arbitrary numbers. They are special numbers in the field of both real and complex numbers. This fact stems from the properties they enjoy when an arbitrary number is added to 0, or multiplied by either of them. We note that we can find an arbitrary number of complex c's for which 0 is a preperiodic point of f_c. Similarly, we note that we can find an arbitrary number of b's for which 1 is a preperiodic point of f_b. The fact that (infinitely) many complex numbers c and b that have these properties exist is not very surprising per se. What is more challenging, and by far less obvious, is to be able to answer the following question, which is the core of this snapshot:

Question 1. *For which complex numbers c are 0 and 1 preperiodic points of $f_c(z) = z^2 + c$?*

Surprisingly, this question is still open! However, we will discuss the major result due to Matthew Baker and Laura DeMarco [1], which states that there are only finitely many of such complex numbers c.

2 The case of integer parameters

One of the main difficulties in answering Question 1 is that the set of complex numbers is huge. The Question becomes much simpler to answer if we just ask for *integers* c such that 0 and 1 are both preperiodic for $f_c(z) = z^2 + c$. It is true that there are infinitely many integers in the set of integers $\mathbb{Z}$, but – in contrast to the complex numbers – the set $\mathbb{Z}$ is *discrete*. This just means that the integers are a set of points where each point stays away from the others: the distance between any two integers is always greater than or equal to 1.

Using this discreteness, we can prove that any integer c such that 0 is a preperiodic point of f_c must be one of the integers $-2, -1, 0, 1, 2$. The argument is as follows:

Given any integer c with $|c| \geq 3$, we show that the sequence $f_c(0)$, $f_c^2(0)$, $f_c^3(0), \ldots$ is strictly growing, which in turn implies that no value can be attained twice. To see this, note that $|c| \geq 3$ implies $c^2 + c > |c|$.[6] We have $f_c(0) = c$ and $f_c^2(0) = f_c(c) = c^2 + c$, hence $f_c^2(0) > f_c(0)$. It remains to show $(f_c^n(0))^2 + c > f_c^n(0)$ for $n > 2$. We inductively assume $f_c^n(0) > |c|$. If $c > 0$, $(f_c^n(0))^2 + c > f_c^n(0)$ follows. If instead $c < 0$, we have $f_c^n(0) > |c| = -c \Rightarrow c > -f_c^n(0)$. We conclude the argument by calculating $f_c^n(0) > 2 \Rightarrow (f_c^n(0))^2 + c > 2f_c^n(0) + c \Rightarrow (f_c^n(0))^2 + c > f_c^n(0)$. With this argument we have transformed the problem to a finite computation. This means, one has only to check the elements -2, -1, 0, 1, and 2, which we have done in Table 1.

map	orbit of 0 / orbit of 1	orbit finite?
$f_{-2}(z) = z^2 - 2$	$0 \mapsto -2 \mapsto 2 \mapsto 2 \mapsto \cdots$	Yes!
	$1 \mapsto -1 \mapsto -1 \mapsto \cdots$	Yes!
$f_{-1}(z) = z^2 - 1$	$0 \mapsto -1 \mapsto 0 \mapsto -1 \mapsto \cdots$	Yes!
	$1 \mapsto 0 \mapsto -1 \mapsto 0 \mapsto \cdots$	Yes!
$f_0(z) = z^2$	$0 \mapsto 0 \mapsto \cdots$	Yes!
	$1 \mapsto 1 \mapsto \cdots$	Yes!
$f_1(z) = z^2 + 1$	$0 \mapsto 1 \mapsto 2 \mapsto 5 \mapsto 26 \mapsto \cdots$	No!
	$1 \mapsto 2 \mapsto 5 \mapsto 26 \mapsto \cdots$	No!
$f_2(z) = z^2 + 2$	$0 \mapsto 2 \mapsto 6 \mapsto 38 \mapsto 1446 \mapsto \cdots$	No!
	$1 \mapsto 3 \mapsto 11 \mapsto 123 \mapsto \cdots$	No!

Table 1: The orbits of 0 and 1 for some quadratic polynomials.

We have given some initial considerations and examples above, regarding our main Question 1. These considerations provide us with a strong partial result regarding integers, which we now state as a Theorem.

Theorem 1. *The only integers c for which* 0 *and* 1 *are preperiodic points of $f_c(z) = z^2 + c$ are -2, -1, and* 0.

In the following, we proceed to extend this analysis to cases where the numbers c are not integers. We anticipate that this will provide a nice characterisation of some important mathematical structures, known as fractals.

[6] The calculations are, $c > 2 \Rightarrow c^2 + c > 4 + c > c = |c|$, and $c < -2 \Rightarrow c(c+2) > 0 \Rightarrow c^2 + 2c > 0 \Rightarrow c^2 + c > -c = |c|$.

3 Non-integer parameters: algebra and analysis

In the abstract we anticipated that we would use a bridge between analysis and algebra. In this section we will describe briefly the algebraic and the analytic side of Question 1.

Roughly speaking, *algebra* is the theory of solving polynomial equations. By a fundamental theorem[7], for every polynomial $f(z) = z^d + a_{d-1}\, z^{d-1} + a_{d-2}\, z^{d-2} + \ldots + a_1\, z + a_0$, where $a_0, \ldots, a_d$ are complex numbers, there exist complex numbers $\alpha_1, \ldots, \alpha_d$ such that

$$f(z) = (z - \alpha_1)\, \cdots\, (z - \alpha_d). \tag{3}$$

The numbers $\alpha_1, \ldots, \alpha_d$ are known as *the zeroes* or *the roots* of the polynomial $f(z)$. The theorem guarantees that the roots $\alpha_1, \ldots, \alpha_d$ are unique and there can be no others. For instance, the polynomial $z^2 + 2$ is zero for $z = \sqrt{2}\, i$ and $z = -\sqrt{2}\, i$. It follows, $z^2 + 2 = (z - \sqrt{2}i)\,(z + \sqrt{2}i)$.

We have noticed in Lemma 1 that 0 is a preperiodic point of $f_c(z) = z^2 + c$, if for some integers n and m we have $f_c^n(0) = f_c^m(0)$. It is not obvious at first, but if we regard c for the moment as a variable, then this equation is a polynomial equation! For example, for $n = 2$ and $m = 4$, we have

$$f_c^2(0) = c^2 + c = c^8 + 4c^7 + 6c^6 + 6c^5 + 5c^4 + 2c^3 + c^2 + c = f_c^4(0).$$

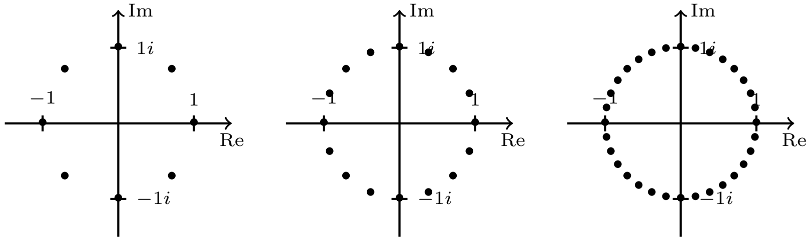
Therefore, 0 is a preperiodic point for f_c, whenever c satisfies the equation $c^8 + 4c^7 + 6c^6 + 6c^5 + 5c^4 + 2c^3 = 0$, or in other words, whenever c is a zero of the polynomial $p(z)$, that is, $p(c) = 0$, where

$$p(z) = z^8 + 4z^7 + 6z^6 + 6z^5 + 5z^4 + 2z^3.$$

This shows, that we can find (all) complex numbers c for which 0 is a preperiodic point by algebraic methods; namely, by finding the zeroes of the polynomials(!) $f_z^n(0) - f_z^m(0)$. Therefore, we could answer Question 1 by solving infinitely many polynomial equations. However, unfortunately we do not have infinite time to do mathematics. Therefore, we need another strategy. More precisely, we need some theory which deals with numbers that can be arbitrarily close to zero. The name of this theory is *analysis*. The idea is to study the behaviour of the zeroes of $f_z^n(0) - f_z^m(0)$ for growing n (or growing m) using tools from analysis.

We will illustrate this idea by a simple example. When we draw the roots of the polynomials $z^2 - 1$, $z^3 - 1$, $z^4 - 1$ ... into the complex plane, the picture looks more and more like a circle of radius 1 around zero, see Figure 2.

[7] This theorem is called the *fundamental theorem of algebra*. For more information, see, for example, Wikipedia: `https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra`.



(a) roots of $z^8 - 1$. (b) roots of $z^{16} - 1$. (c) roots of $z^{32} - 1$.

Figure 2: The distribution of complex roots of polynomials of the form $z^n - 1$. Note the increase in "density" of the roots distributed along the circle of unit radius.

So, with increasing n, we can regard the set of roots of $z^n - 1$ as a circle. This means, we can apply analytic methods (like "taking derivatives" or "integrate") on the point set of these polynomials. The important thing is that this will never work for any fixed number n. Even if n is a number greater than anything one can possibly count (like neutrons in the observable universe), the set of roots of $z^n - 1$ is discrete, and therefore not in the least like a smooth circle. But if we have a sequence $n_1, n_2, n_3, \ldots$ of growing integers, then in the *limit* of infinite points, the roots of the polynomials $z^{n_1} - 1, z^{n_2} - 1, \ldots$ will be a circle. We say that the sequence of roots of these polynomials is *equidistributed around the circle of radius* 1 *around* 0.

The distribution of the roots of the polynomials $f_z^n(0) - f_z^m(0)$ and $f_z^n(1) - f_z^m(1)$[8] when n and/or m grow is of course much more complicated. But the fantastic – and by far not obvious – result is that such a distribution exists! This was proven by Baker and DeMarco using a deep result on equidistribution that was independently discovered by several groups of mathematicians in [2], [3], [4].

4 Mandelbrot sets

We have seen that, for any integer c, the f_c-orbit of 0 is either finite (for c equals -2, -1, or 0) or grows to infinity (for all other integers c). If we allow arbitrary complex numbers as values of c, there is a third thing that can happen (can

[8] Recall that these are exactly the complex numbers we aimed at classifying in Question 1.

you guess what?). As an example let $c = -1/2$. Then the f_c-orbit of 0 is

$$0 \mapsto -\frac{1}{2} \mapsto \left(-\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4} \mapsto -\frac{7}{16} \mapsto -\frac{79}{256} \mapsto \cdots$$

It turns out that this orbit consists of infinitely many different numbers, but the modulus of every element in this orbit is at most $1/2$. We say that the $f_{-1/2}$-orbit of 0 is bounded.[9]

These considerations allow us to introduce the *classical Mandelbrot set* M_0. This important mathematical set is defined as the set of all those complex numbers c for which the f_c-orbit of 0 is bounded. The Mandelbrot set was defined by Benoît Mandelbrot in the 1970s. It is a beautifully shaped set of complex numbers, and early graphical computer programs generating this set became quite popular among the general audience.

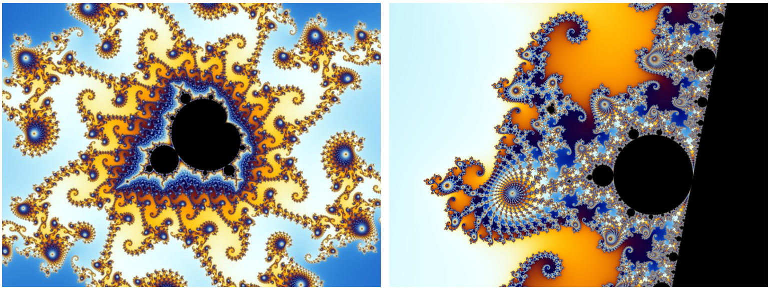


Figure 3: Left: The black area is a sketch of the classical Mandelbrot set M_0. Right: This is a zoom-in of the area around an edge of M_0.

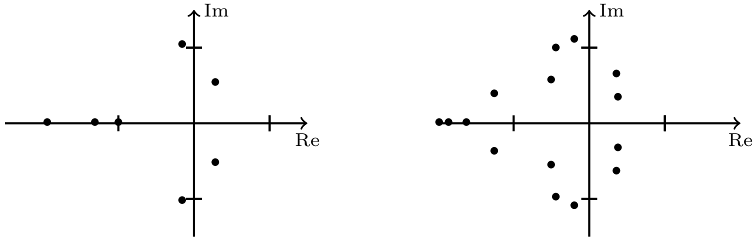
The image of the Mandelbrot set M_0 in Figure 3 is only a very rough sketch of the true shape. In fact M_0 is fractal-like, which means that you can find infinitely many copies of the shape of M_0 if you zoom closer to the border. Moreover, by zooming and analysing this set, you can find shapes of rabbits, airplanes, seahorses, ...[10]

What does all of this have to do with Question 1? By Lemma 1, all complex numbers c for which 0 is a preperiodic point of f_c satisfy the equation $f_c^n(0) = f_c^m(0)$ for some n and m. But if we draw all roots of such equations where n and/or m tend to infinity, or become larger and larger, the picture

[9] More precisely we could say that this orbit is bounded by $1/2$.

[10] These are indeed the common names for some of the shades you can find in the Mandelbrot set! There is a nice tool by Prof. Dr. Edmund Weitz which allows you to zoom through the Mandelbrot set as you please. This tool can be downloaded from `http://weitz.de/mandelbrot/`. For a nice introduction to fractals we refer to the book [5].

will look almost as the boundary[11] of the Mandelbrot set M_0. We say that these sets of points are equidistributed around the boundary of M_0. This would be extremely hard to guess by actually drawing examples as in Figure 4. In particular, since the boundary of M_0 is an object that is very difficult to describe and represent graphically.



(a) All complex numbers c for which $f_c^5(0) = f_c(0)$.

(b) All complex numbers c for which $f_c^6(0) = f_c(0)$.

Figure 4: Some sets of roots of polynomials of the form $f_z^n(0) - f_z(0)$.

Recall that we are also looking for complex numbers c for which 1 is a preperiodic point of f_c. In this case we can repeat everything above with 0 replaced by 1. The *generalized Mandelbrot set* M_1 is defined as the set of complex numbers c for which the f_c-orbit of 1 is bounded. Again the sets of roots of $f_z^n(1) - f_z^m(1)$ are equidistributed around the boundary of M_1, as n and/or m tend to infinity.

The set M_1 is similar in shape to M_0, but they are not equal! Recall that the imaginary element i satisfies $i^2 = -1$. So the f_i-orbits of 0 and 1 are

$$0 \mapsto i \mapsto -1 + i \mapsto -i \mapsto -1 + i \mapsto \cdots$$
$$1 \mapsto 1 + i \mapsto 3i \mapsto -9 + i \mapsto 80 - 17i \mapsto \cdots$$

Therefore, i is in M_0 but not in M_1. Applying analytic tools one can prove the following result:

Lemma 2. *The boundary of the Mandelbrot set M_0 is not the same as the boundary of the generalized Mandelbrot set M_1.*

By classical algebraical methods we also get:

[11] There is, of course, a mathematical definition for *boundary*. But here we intuitively understand what is meant by the word *boundary*!

Lemma 3. *If some $c \neq 0$ satisfies $f_c^n(0) - f_c^m(0) = 0$ and $f_c^k(1) - f_c^l(1) = 0$ for integers m, n, k, l, then most roots of $f_z^n(0) - f_z^m(0)$ are also roots of $f_z^k(1) - f_z^l(1)$.*[12]

These two lemmas appear to have nothing in common. But the link between them is the equidistribution of the roots of the polynomials in Lemma 3. This equidistribution is the promised bridge between algebra and analysis. In the next section we will walk over this bridge step by step.

5 Summary and conclusions

We promised in the title of this snapshot to present some new results on quadratic polynomials. In particular, we wanted to give some partial answers to the following question:

Question. *For which complex numbers c are 0 and 1 preperiodic points of $f_c(z) = z^2 + c$?*

We summarize the results that we have presented above:

- The numbers -2, -1 and 0 are the only *integers* satisfying the requirement of the above question (see Theorem 1).
- Any c satisfying this requirement is a root of a polynomial of the form $f_z^n(0) - f_z^m(0)$ and a root of a polynomial of the form $f_z^k(1) - f_z^l(1)$ for some integers n, m, k, l (see Lemma 1).
- Then, it follows that $f_z^n(0) - f_z^m(0)$ and $f_z^k(1) - f_z^l(1)$ have most roots in common (see Lemma 3).
- If there were infinitely many complex numbers c satisfying the requirement above, then there would be a sequence of polynomials $f_z^{n_1}(0) - f_z^{m_1}(0)$, $f_z^{n_2}(0) - f_z^{m_2}(0)$, ... with m_1, m_2, ... and/or n_1, n_2, ... growing, and a sequence of polynomials $f_z^{k_1}(1) - f_z^{l_1}(1)$, $f_z^{k_2}(1) - f_z^{l_2}(1)$, ... with k_1, k_2, ... and/or l_1, l_2, ... growing, such that for any index i the polynomials $f_z^{n_i}(0) - f_z^{m_i}(0)$ and $f_z^{k_i}(1) - f_z^{l_i}(1)$ have most roots in common.
- These common roots would be equidistributed around the boundary of M_0 and around the boundary of M_1. This means, that if we would visualize these roots as a subset of $\mathbb{C}$, they would form the shape of the boundaries of M_0 and M_1 at the same time. This implies that the boundaries of M_0 and M_1 look exactly the same. But these boundaries are different (see Lemma 2), and hence they have a different shape. This is a contradiction!
- Therefore, there cannot be infinitely many complex numbers c such that 0 and 1 are preperiodic points of f_c.

[12] This statement is not very precise, but it should satisfy for the purpose of this snapshot.

- It remains open, to find all these complex numbers. Conjecturally, -2, -1, and 0 are the only complex numbers satisfying the assumption.

We conclude by noting that, although the numbers 0 and 1 enjoy a special"status" among real numbers, there is no special reason in starting from 0 and 1. In fact, Baker and DeMarco proved that there are only finitely many complex numbers c such that z_1 and z_2 are preperiodic points of f_c, for any complex numbers z_1 and z_2 with $z_1 \neq \pm z_2$. However, the proof becomes much harder if z_1 and z_2 are so called "transcendental" numbers like π.

Image credits

Figure 3 "Mandelbrot set pictures" were created by Wolfgang Beyer.

`http://www.misterx.ca/Mandelbrot_Set/M_Set-IMAGES_&_WALLPAPER.html`, visited on July 5, 2017.

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