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OVERLAP SYNCHRONISATION IN MULTIPARTITE RANDOM ENERGY MODELS

GIUSEPPE GENOVESE AND DANIELE TANTARI

Abstract. In a multipartite random energy model, made of coupled GREMs, we determine the joint law of the overlaps in terms of the ones of the single GREMs. This provides the simplest example of the so-called *synchronisation* of the overlaps.

1. Introduction

In this note we concern on a multipartite random energy model, originally studied in the bipartite case in [1], obtained coupling each level of M distinct generalised random energy models (GREMs). We show the joint law of the overlaps to have a simple expression in terms of the ones of the single GREMs. This provides a plain example of the so-called *synchronisation* of the overlaps, recently introduced by Panchenko as a relevant property of multipartite systems [2].

The model is defined as follows. Let $N, M \in \mathbb{N}$, $\kappa \in \{1, \dots, M\}$ and $N_\kappa \in \mathbb{N}$ with $\sum_\kappa N_\kappa = N$, $\alpha^{(\kappa)} := N_\kappa/N$, $n_0 = 0$, $n_\kappa - n_{\kappa-1} = 2^{N_\kappa}$. For each configuration $\sigma \in \Sigma_N := \{1, \dots, 2^N\}$ we can write $\sigma = (\mu_{(1)}, \dots, \mu_{(M)})$, $\mu_{(\kappa)} \in \{n_{\kappa-1}+1, \dots, n_\kappa\}$. We divide each part respectively into $K_1, \dots K_M$ hierarchical levels. For each level j of the hierarchy, each group of configurations is divided in $2^{N_{\kappa,j}}$ further subgroups indexed by $\mu_{(\kappa,j)}$, with of course $\sum_j N_{\kappa,j} = N_\kappa$ and $\varsigma_{\kappa,j} := N_{\kappa,j}/N_\kappa$, $j \in \{1, \dots, K_\kappa\}$. Each configuration can be thought of as a M-ple $\sigma = (\mu_{(1)}, \dots, \mu_{(M)})$ or as a $\prod_\kappa K_\kappa$-ple $\sigma = (\mu_{(1,1)} \dots \mu_{(1,K_1)}, \dots, \mu_{(M,1)}, \dots \mu_{(M,K_M)})$. This multipartite setting brings a somewhat heavy notation. To lighten it a little we let

$$\ell_{\kappa,j} := \mu_{(\kappa,1)} \dots \mu_{(\kappa,j)}$$

label the configurations in the j-th level of the κ-th tree. With a slight abuse of notation we will denote with the same symbol also the set of such configurations. However the correct meaning will be always clear from the context.

We attach to each couple of levels Gaussian centred r.vs $J^{(\kappa,j)}_{\ell_{\kappa,j}}$, and $J^{(\kappa_1,j_1)(\kappa_2,j_2)}_{\ell_{\kappa_1,j_1}\ell_{\kappa_2,j_2}}$ with

$$\begin{aligned} E\left[J^{(\kappa,j)}_{\ell_{\kappa,j}} J^{(\kappa,j)}_{\ell'_{\kappa,j}}\right] &= \delta_{\ell_{\kappa,j},\ell'_{\kappa,j}}\,, \\ E\left[J^{(\kappa_1,j_1)(\kappa_2,j_2)}_{\ell_{\kappa_1,j_1}\ell_{\kappa_2,j_2}} J^{(\kappa_1,j_1)(\kappa_2,j_2)}_{\ell'_{\kappa_1,j_1}\ell'_{\kappa_2,j_2}}\right] &= \delta_{\ell_{\kappa_1,j_1},\ell'_{\kappa_1,j_1}} \delta_{\ell_{\kappa_2,j_2},\ell'_{\kappa_2,j_2}}\,. \end{aligned}$$

The levels interact via the following Hamiltonian

$$H_N(\sigma) := -\sqrt{\frac{N}{2}}\left[\sum_{\kappa=1}^M \alpha_\kappa \sum_{j=1}^{K_\kappa} a^{(\kappa)}_j J^{(\kappa,j)}_{\ell_{\kappa,j}} + \sqrt{2\alpha_1 \dots \alpha_M} \sum_{(\kappa_1,\kappa_2)} \sum_{j_1=1}^{K_{\kappa_1}} \sum_{j_2=1}^{K_{\kappa_2}} c^{(\kappa_1,\kappa_2)}_{j_1,j_2} J^{(\kappa_1,j_1)(\kappa_2,j_2)}_{\ell_{\kappa_1,j_1}\ell_{\kappa_2,j_2}}\right] \tag{1.1}$$

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with

$$\sum_{j}^{K_\kappa} a_j^{(\kappa)} = \sum_{j_1=1}^{K_{\kappa_1}} \sum_{j_2=1}^{K_{\kappa_2}} c_{j_1,j_2}^{(\kappa_1,\kappa_2)} = 1\,, \quad \forall\, \kappa, \kappa_1, \kappa_2 \in \{1,\dots,M\}\,.$$

We define as customary for $\beta > 0$ $(-\frac{1}{\beta})$ the free energy to be

$$A_N(\beta) := \frac{1}{N} \log \sum_{\sigma} e^{-\beta H_N(\sigma)}\,, \qquad A(\beta) := \lim_N A_N(\beta)\,. \tag{1.2}$$

Of course as a consequence of Talagrand inequality $A_N(\beta)$ is self-averaging as $N \to \infty$, so we can always take the expectation w.r.t. the disorder, when needed.

Given two configurations σ, σ' we can define the partial overlaps. We introduce M sequences of numbers in $[0,1]$

$$0 = q_0^{(\kappa)} < q_1^{(\kappa)} < \cdots < q_{K_\kappa}^{(\kappa)} < q_{K_\kappa+1}^{(\kappa)} = 1\,, \qquad \kappa \in \{1,\dots,M\}$$

and

$$\tau_{\mu_{(\kappa)}\mu'_{(\kappa)}} := \inf \left\{ j \,:\, \mu_{(\kappa,j+1)} \neq \mu'_{(\kappa,j+1)} \right\}\,. \tag{1.3}$$

So we define the overlaps through

$$q_{\mu_{(\kappa)}\mu'_{(\kappa)}} := q^{(\kappa)}_{\tau_{\mu_{(\kappa)}\mu'_{(\kappa)}}}\,. \tag{1.4}$$

Here and further we denote by $P_{N,\beta}$ the Gibbs distribution associated to the model and by $\langle \cdot \rangle_{N,\beta}$ the quenched average of observables (we drop the subscript N in the thermodynamic limit).

We let $x_\kappa(q) := P_\beta \left(q_{\mu_{(\kappa)}\mu'_{(\kappa)}} \leq q \right)$. The main result of this note is

Theorem. *Let υ be a random variable uniformly distributed in $[0,1]$. Then*

$$\left(q^{(1)}_{\mu_{(1)}\mu'_{(1)}}, \dots, q^{(M)}_{\mu_{(M)}\mu'_{(M)}} \right) \overset{d}{=} (x_1^{-1}(\upsilon), \dots, x_M^{-1}(\upsilon))\,. \tag{1.5}$$

A larger class of non-hierarchical random energy models which includes the one under consideration was studied by Bolthausen and Kistler in [3, 4]. We shall make use of some crucial ideas from those two papers, in which the so-called *Parisi picture* (which could be also fairly named the Derrida-Ruelle picture) *i.e.* variational principle for the free energy and the ultrametricity of the overlap is proved. A precise form of their statement will be given below.

2. More on the Model

Prior to embark the proof of the Theorem, it is convenient to discuss a little more the model. What follows is in a good part heuristics and rigorous proofs can be found in [3, 4].

First consider for simplicity the bipartite model with $K_1 = K_2 = 1$, defined by the Hamiltonian (we set $a^{(1)} = a$, $a^{(2)} = b$ and $\alpha^{(1)} = \alpha$)

$$H_N(\sigma) := -\sqrt{\frac{N}{2}} \left[\alpha a J^{(1)}_{\mu_1} + (1-\alpha) b J^{(2)}_{\mu_2} + \sqrt{2\alpha(1-\alpha)}\, c J^{(1,2)}_{\mu_1\mu_2} \right]\,. \tag{2.1}$$

If we assume for definiteness $\alpha a^2 > (1-\alpha) b^2$, there are two possibilities: either $\alpha a^2 \leq (1-\alpha)b^2 + 2\alpha c^2$ or $\alpha a^2 > (1-\alpha) b^2 + 2\alpha c^2$. As the first case is less rich, we focus on the second one. At very high temperature everything is ergodic and the free energy coincides with the annealed one. As $\beta > \beta_1 := 2\sqrt{\log 2}/a\sqrt{\alpha}$, the μ_1-subset *freezes*, *i.e.* its relative entropy goes to zero analogously as in the first transition in a GREM and one can show that

$$P^{(\mu_1)}_{N,\beta}(\mu;\beta) := Z^{-1} \sum_{\nu} e^{-\beta H_1(\sigma)} \xrightarrow{w} PD(0, x_1)\,,$$

where Z is a normalisation factor, $x_1 := \beta_1/\beta$, $PD(0,x)$ denotes the normalised Poisson point process with intensity $\rho(t) = xt^{-x-1}$ or Poisson-Dirichlet distribution. In this regime $P_\beta(q_{\mu\mu'} \geq q) = 1 - x_1$, while for any $q, p > 0$ $P_\beta(p_{\mu\mu'} \geq p) = P_\beta(q_{\mu\mu'} \geq q, p_{\nu\nu'} \geq p) = 0$. The free energy is a convex combination (with α) of two REMs, one on the μ subset at low temperature and the other on the rest of the system at high temperature. As β increases further, the total entropy vanishes for $\beta > \beta_2 := 2\sqrt{\log 2}/\sqrt{(1-\alpha)b^2 + 2\alpha c^2}$ and the whole Gibbs measure converges toward a Poisson-Dirichlet process

$$P_{N,\beta}(\sigma;\beta) \xrightarrow{w} PD(0,x_2)\,,$$

with $x_2 := \beta_2/\beta$. The free energy is the convex combination of two REMs at low temperature. $P_\beta(q_{\mu\mu'} \geq q)$ is unchanged, but $P_\beta(p_{\mu\mu'} \geq p) = P_\beta(q_{\mu\mu'} \geq q, p_{\nu\nu'} \geq p) = 1 - x_2$. Note

$$P_\beta(q_{\mu\mu'} \geq q, p_{\nu\nu'} \geq p) = \min\left(P_\beta(q_{\mu\mu'} \geq q)\,,\ P_\beta(p_{\nu\nu'} \geq p)\right)$$

for any β. We remark that, since $q, p \in \{0,1\}$ in this simple case, $P_\beta(q_{\mu\mu'} \geq q) = P_\beta(q_{\mu\mu'} = 1)$ and $P_\beta(p_{\mu\mu'} \geq p) = P_\beta(p_{\nu\nu'} = 1)$ (as $q, p > 0$, otherwise they are trivially one). Therefore the first system starts freezing at higher temperature and so if the second systems is frozen then also the first one is so (as in a two-level GREM). The whole picture is summarised as follows

$$\begin{cases} \beta < \beta_1 & A^{1,1}(\beta) = \log 2\left(1 + \alpha\frac{\beta^2}{\beta_1^2} + (1-\alpha)\frac{\beta^2}{\beta_2^2}\right)\,, & \langle q\rangle_\beta = \langle p\rangle_\beta = \langle Q\rangle_\beta = 0\,; \\ \beta_1 < \beta < \beta_2 & A^{1,1}(\beta) = \log 2\left(2\alpha\frac{\beta}{\beta_1} + (1-\alpha)\left(1 + \frac{\beta^2}{\beta_2^2}\right)\right)\,, & \langle q\rangle_\beta = 1 - x_1\,,\ \langle p\rangle_\beta = \langle Q\rangle_\beta = 0\,; \\ \beta > \beta_2 & A^{1,1}(\beta) = 2\beta\log 2\left(\alpha\frac{\beta}{\beta_1} + (1-\alpha)\frac{\beta}{\beta_2}\right)\,, & \langle q\rangle_\beta = 1 - x_1\,,\ \langle p\rangle_\beta = \langle Q\rangle_\beta = 1 - x_2\,. \end{cases}$$

Therefore, albeit not inbuilt in the model, a GREM-like hierarchical structure naturally emerges. A way to visualise that in the general model defined by the Hamiltonian (1.1) is as follows. Recall that $\ell_{\kappa,j}$, $\kappa \in \{1,\dots,M\}$, $j \in \{0,\dots,K_\kappa\}$ denote the configurations up to the j-th level of the κ-th GREM. Then the phase space is naturally coarse-grained by the class of sets $\{\ell_{\kappa_1,j_1}, \ell_{\kappa_2,j_2}\}_{j_2=1,\dots,K_{\kappa_2}}^{j_1=1,\dots,K_{\kappa_1}}$. We think of each level now as an atom and we can consider the power set

$$\wp := \wp\{\ell_{1,1},\dots,\ell_{1,K_1},\dots,\ell_{M,1},\dots,\ell_{M,\kappa_M}\}\,.$$

According to [3, 4] a *chain* Γ is defined to be an increasing (finite) sequence of sets in $\wp$: $\Gamma = \{\Gamma_n\}_{n=0,\dots,K}$, for a given $K \leq \sum_\kappa K_\kappa$, with $\Gamma_n \in \wp$, $\Gamma_n \subset \Gamma_{n+1}$ and $\Gamma_0 = \emptyset$, $\Gamma_K = \{\ell_{1,0},\dots,\ell_{1,K_1},\dots,\ell_{M,1},\dots,\ell_{M,\kappa_M}\}$. To each Γ we associate two sequences $\{\alpha_n\}_{n=1,\dots,K}$ and $\gamma := \{\gamma_n\}_{n=1,\dots,K}$. The α_n represent the relative sizes of the Γ_n

$$\alpha_n := \frac{\log_2\left|\bigcup_{i,j:\ell_j^i\in\Gamma_n}\ell_j^i\right|}{N}\,,$$

easily computed from the numbers $\alpha^{(\kappa)}$ and $\varsigma_{\kappa,j}$; the γ_n are variances defined by

$$\gamma_n^2 := \sum_{\kappa=1}^M {\alpha^{(\kappa)}}^2 \sum_{j\,:\,\ell_{\kappa,j}\in\Gamma_n/\Gamma_{n-1}} (a_j^{(\kappa)})^2 + \sum_{\substack{(\kappa_1,\kappa_2)\,,\,(j_1,j_2)\,:\\ :\,\{\ell_{\kappa_1,j_1},\ell_{\kappa_2,j_2}\}\in\Gamma_n/\Gamma_{n-1}}} 2\alpha_{\kappa_1}\alpha_{\kappa_2}(c_{j_1,j_2}^{(\kappa_1,\kappa_2)})^2\,.$$

From α_n and γ_n we can define another sequence of critical inverse temperatures $\{\beta_n\}_{n=1,\dots,K}$, $\beta_n := \sqrt{\alpha_n\log 2}\,\gamma_n^{-1}$. Of course for a generic chain $\{\beta_n\}_{n=1,\dots,K}$ is not monotone, but we can conveniently confine our attention to those chains for which $\beta_1 \leq \beta_2 \leq \dots \leq \beta_K$. We denote by $\mathcal{T}$ the set of such chains.

To fix the ideas, let us consider again a bipartite REM with K_1, K_2 levels. The Hamiltonian reads (recall $\alpha^{(1)} = \alpha$, $\alpha^{(2)} = 1-\alpha$)

$$H_N(\sigma) == -\sqrt{\frac{N}{2}}\left[\alpha\sum_{j=1}^{K_1} a_j^{(1)} J^{(1,j)}_{\ell_{1,j}} + (1-\alpha)\sum_{j=1}^{K_2} a_j^{(2)} J^{(2,j)}_{\ell_{2,j}} + \sqrt{2\alpha(1-\alpha)}\sum_{j_1=1}^{K_1}\sum_{j_2=1}^{K_2} c_{j_1,j_2} J^{(1,j_1)(2,j_2)}_{\ell_{1,j_1}\ell_{2,j_2}}\right] \tag{2.2}$$

For a given $\Gamma \in \mathcal{T}$ of length K, we set for $n = 1,\dots,K$

$$\begin{aligned} H_n \quad &:= \quad -\sqrt{\frac{N}{2}}\left[\alpha \sum_{j\,:\,\ell_{1,j}\in\Gamma_n/\Gamma_{n-1}} a_j^{(1)} J^{(1,j)}_{\ell_{1,j}} + (1-\alpha)\sum_{j\,:\,\ell_{2,j}\in\Gamma_n/\Gamma_{n-1}} a_j^{(2)} J^{(2,j)}_{\ell_{2,j}}\right. \\ &+ \quad \left.\sqrt{2\alpha(1-\alpha)} \sum_{(j_1,j_2)\,:\,\ell_{1,j_1},\ell_{2,j_2}\in\Gamma_n/\Gamma_{n-1}} c_{j_1,j_2} J^{(1,j_1)(2,j_2)}_{\ell_{1,j_1}\ell_{2,j_2}}\right], \end{aligned}$$

so that we can decompose the Hamiltonian (2.2) according to

$$H_N(\sigma) = \sum_{n=1}^{K} H_n\,, \tag{2.3}$$

and the partition function can be written as

$$Z_N(\beta) = \sum_{\{\Gamma_1\}} e^{-\beta H_1} \sum_{\{\Gamma_2/\Gamma_1\}} e^{-\beta H_2} \cdots \sum_{\{\Gamma_n/\Gamma_{n-1}\}} e^{-\beta H_n}\,.$$

Now we see the following scenario. At β small enough the annealed approximation holds and the overlaps are set to zero. Then β increases, $\beta > \beta_1$, and the configurations in Γ_1 *freeze*. Then H_2 depends in fact on configurations in Γ_2/Γ_1, *i.e.* H_1 and all the other addenda in the r.h.s. of (2.3) become independent as $N \to \infty$. Thus the partition function asymptotically factorises

$$Z_N(\beta) \simeq \sum_{\{\Gamma_1\}} e^{-\beta H_1} \sum_{\{\Sigma/\Gamma_1\}} e^{-\beta(H-H_1)}$$

as two independent REMs: the first one on the space of configurations Γ_1 is at low temperature, the second one on the remaining configuration space is at high temperature (with the right variance $\sqrt{\sum_{n\geq 2}\gamma_n^2}$). The free energy is a convex combination w.r.t. α_1 (*i.e.* the relative size of Γ_1) of these two REMs. As in the previous example, we have convergence of the marginalised Gibbs measure to a Poisson-Dirichlet distribution

$$P^{(1)}_{N,\beta}(\Gamma_1;\beta) := Z_1^{-1}\sum_{\Sigma/\Gamma_1} e^{-\beta H(\sigma)} \xrightarrow{w} PD(0,x_1)\,,$$

with $x_1 := 1-\beta_1/\beta$ and Z_1 an opportune normalisation. Since H_1 and H_2 remain independent for all $\beta > \beta_1$ we can iterate this procedure: for instance as $\beta > \beta_2$ also Γ_2 *freezes* and H_2 becomes asymptotically independent on H_3; thus the partition function is factorised as

$$Z_N(\beta) \simeq \sum_{\{\Gamma_1\}} e^{-\beta H_1} \sum_{\{\Gamma_2/\Gamma_1\}} e^{-\beta H_2} \sum_{\{\Sigma/\Gamma_2\}} e^{-\beta(H-H_1-H_2)}\,.$$

These are three independent REMs on configurations Γ_1, Γ_2/Γ_1 and Σ/Γ_2, the associated free energy is given by a convex combination of the low temperature free energy of the first two REMs and the high temperature free energy of the third one and

$$P^{(2)}_{N,\beta}(\Gamma_2;\beta) := Z_2^{-1}\sum_{\Sigma/\Gamma_2} e^{-\beta H(\sigma)} \xrightarrow{w} PD(0,x_2)\,,$$

with $x_2 := 1 - \beta_2/\beta$ and again Z_2 a normalisation. Going on this way we recover the free energy and the Gibbs measure as a GREM-like structure along the chain. At zero temperature the free energy of the model is just the convex combination of those of REMs at low temperature, each defined on an element of the chain. This construction can be made for every chain in $\mathcal{T}$. Of course for fixed β, the more REMs are at low temperature, the higher is the free energy. According to this criterion one can select the chain along which the free energy is maximal. By the above construction it should be clear that such a chain, here denoted by Γ^*, is unique.

Therefore the results of [3, 4] (for the case of our interest) can be precisely formulated as follows. Let $\gamma \in \mathcal{T}$ and $A_{GREM}(\gamma;\beta)$ denote the GREM pressure computed on the hierarchical structure γ. We have

Theorem (Bolthausen and Kistler)**.** *It holds*

$$\lim_N A_N(\beta) = \lim_N \mathbb{E}[A_N(\beta)] = A(\beta) = \min_{\gamma\in\mathcal{T}} \left(A_{GREM}(\gamma;\beta)\right) ,$$

Moreover there is a β^* such that for each triad of configurations $(\sigma,\sigma',\sigma'')\in\Sigma^3$

$$\lim_N P_{N,\beta}\left(d(\sigma,\sigma') \le \max\{d(\sigma,\sigma''), d(\sigma',\sigma'')\}\right) = 1 \tag{2.4}$$

holds for $\beta > \beta^*$.

3. Proof

Now we are ready to give the proof of our statement. For simplicity we keep working mostly in the bipartite case. We convey to fix the optimal chain Γ^* once for all. The sequences $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{\beta_n\}$ will be always referred to Γ^*.

A direct computation from (1.1) and (1.2) yields

$$\begin{aligned}
P_{N,\beta}(q_{\mu_{(1)}\mu'_{(1)}} \ge q^{(1)}_j) &= \left\langle 1_{\{\ell_{1,j}=\ell_{1,j}{}'\}}\right\rangle_{N,\beta} = 1 - \frac{2}{a^{(1)}_j\beta^2\alpha^2}\partial_{a^{(1)}_j}A_N \,,\\
P_{N,\beta}(q_{\mu_{(2)}\mu'_{(2)}} \ge q^{(2)}_j) &= \left\langle 1_{\{\ell_{2,j}=\ell_{2,j}{}'\}}\right\rangle_{N,\beta} = 1 - \frac{2}{a^{(2)}_j\beta^2(1-\alpha)^2}\partial_{a^{(2)}_j}A_N \,,\\
P_{N,\beta}(q_{\mu_{(1)}\mu'_{(1)}} \ge q^{(1)}_{j_1},\, q_{\mu_{(2)}\mu'_{(2)}} \ge q^{(2)}_{j_2}) &= \left\langle 1_{\{(\ell_{1,j_1},\ell_{2,j_2})=(\ell'_{1,j_1},\ell'_{2,j_2})\}}\right\rangle_{N,\beta} = 1 - \frac{1}{c_{jk}\beta^2\alpha(1-\alpha)}\partial_{c_{jk}}A_N \,.
\end{aligned}$$

On the other hand we know that the free energy is a convex combination of REM ones along Γ^*. Therefore its derivatives can be explicitly computed. We set

$$n_1(j) := \min\{n \,:\, \ell_{1,j}\in\Gamma^*_n\}\,, \quad n_2(j) := \min\{n \,:\, \ell_{2,j}\in\Gamma^*_n\}\,, \quad n(j_1,j_2) := \max(n_1(j), n_2(k))\,.$$

Then

$$\partial_{a^{(1)}_j}A = \begin{cases} \beta^2\frac{\alpha^2 a^{(1)}_j}{2} & \beta < \beta_{n_1(j)} \\ \beta\sqrt{\alpha_{n_1(j)}\log 2}\,\alpha^2\frac{a^{(1)}_j}{\gamma_{n_1(j)}} & \beta \ge \beta_{n_1(j)}\,, \end{cases} \tag{3.1}$$

$$\partial_{a^{(2)}_j}A = \begin{cases} \beta^2\frac{(1-\alpha)^2 a^{(2)}_j}{2} & \beta < \beta_{n_2(j)} \\ \beta\sqrt{\alpha_{n_2(j)}\log 2}\,(1-\alpha)^2\frac{a^{(2)}_j}{\gamma_{n_2(j)}} & \beta \ge \beta_{n_2(j)}\,, \end{cases} \tag{3.2}$$

$$\partial_{c_{jk}}A = \begin{cases} \beta^2\alpha(1-\alpha)c_{jk} & \beta < \beta_{n(j_1,j_2)} \\ \beta\sqrt{\alpha_{n(j_1,j_2)}\log 2}2\alpha(1-\alpha)\frac{c_{jk}}{\gamma_{n(j_1,j_2)}} & \beta \ge \beta_{n(j_1,j_2)}\,. \end{cases} \tag{3.3}$$

As $N \to \infty$ the two expressions for the derivatives have to be equal. Therefore we see at once that if $n_1(j_1) = n_2(j_2) = \bar{n}$, then

$$
\begin{aligned}
& P_\beta(q_{\mu_{(1)}\mu_{(1)}'} \geq q_j^{(1)}) = P_\beta(q_{\mu_{(2)}\mu_{(2)}'} \geq q_j^{(2)}) \\
& = P_\beta(q_{\mu_{(1)}\mu_{(1)}'} \geq q_{j_1}^{(1)},\, q_{\mu_{(2)}\mu_{(2)}'} \geq q_{j_2}^{(2)}) = \begin{cases} 0 & \beta < \beta_{\bar{n}} \\ 1 - \frac{\beta_{\bar{n}}}{\beta} & \beta \geq \beta_{\bar{n}}\,. \end{cases}
\end{aligned} \tag{3.4}
$$

Otherwise we have

$$
P_\beta(q_{\mu_{(1)}\mu_{(1)}'} \geq q_j^{(1)}) = \begin{cases} 0 & \beta < \beta_{n_1(j)} \\ 1 - \frac{\beta_{n_1(j)}}{\beta} & \beta \geq \beta_{n_1(j)}\,, \end{cases} \tag{3.5}
$$

$$
P_\beta(q_{\mu_{(2)}\mu_{(2)}'} \geq q_j^{(2)}) = \begin{cases} 0 & \beta < \beta_{n_2(j)} \\ 1 - \frac{\beta_{n_2(j)}}{\beta} & \beta \geq \beta_{n_2(j)}\,, \end{cases} \tag{3.6}
$$

and

$$
P_\beta(q_{\mu_{(1)}\mu_{(1)}'} \geq q_{j_1}^{(1)},\, q_{\mu_{(2)}\mu_{(2)}'} \geq q_{j_2}^{(2)}) = \begin{cases} 0 & \beta < \beta_{n(j_1,j_2)} \\ 1 - \frac{\beta_{n(j_1,j_2)}}{\beta} & \beta \geq \beta_{n(j_1,j_2)}\,. \end{cases} \tag{3.7}
$$

As $1 - \beta_n/\beta$ is decreasing in n, formulas (3.4) and (3.5), (3.6), (3.7) establish directly

$$
P_\beta(q_{\mu_{(1)}\mu_{(1)}'} \geq q^{(1)},\, q_{\mu_{(2)}\mu_{(2)}'} \geq q^{(2)}) = \min\left(P_\beta(q_{\mu_{(1)}\mu_{(1)}'} \geq q^{(1)})\,,\ P_\beta(q_{\mu_{(2)}\mu_{(2)}'} \geq q^{(2)})\right)\,. \tag{3.8}
$$

In the multipartite case the above formula immediately generalises as

$$
P_\beta\left(q^{(1)}_{\mu_{(1)}\mu'_{(1)}} \geq q^{(1)}, \dots, q^{(M)}_{\mu_{(M)}\mu'_{(M)}} \geq q^{(M)}\right) = \min\left[\left\{P_\beta\left(q^{(\kappa)}_{\mu_{(\kappa)}\mu'_{(\kappa)}} \geq q^{(\kappa)}\right)\right\}_{\kappa=1,\dots,M}\right] \tag{3.9}
$$

This is the synchronisation property from which we are going to readily deduce (1.5). Let $\upsilon \sim U(0,1)$. We have

$$
\begin{aligned}
P_\beta\left(q_{\mu_{(1)}\mu'_{(1)}} \geq q^{(1)}, \dots, q_{\mu_{(M)}\mu'_{(M)}} \geq q^{(M)}\right) &= \min\left[\left\{P_\beta(q_{\mu(\kappa)\mu'(\kappa)} \geq q^{(\kappa)})\right\}_{\kappa\in\{1,\dots,M\}}\right] \\
&= P_\beta\left(\upsilon \leq \min_{\kappa\in\{1,\dots,M\}} P_\beta(q_{\mu_{(\kappa)}\mu'_{(\kappa)}} \geq q^{(\kappa)})\right) \\
&= P_\beta(x_1^{-1}(1-\upsilon) \geq q^{(1)} \dots x_M^{-1}(1-\upsilon) \geq q^{(M)})\,,
\end{aligned}
$$

which concludes the proof.

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