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SHIBANANDA BISWAS, GARGI GHOSH, GADADHAR MISRA AND SUBRATA SHYAM ROY

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel   +49 7834 979 50
Fax   +49 7834 979 55
Email   admin@mfo.de
URL   www.mfo.de

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REDUCING SUB-MODULES OF THE BERGMAN MODULE $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ UNDER THE ACTION OF THE SYMMETRIC GROUP

SHIBANANDA BISWAS, GARGI GHOSH, GADADHAR MISRA, AND SUBRATA SHYAM ROY

Abstract. The weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, $\lambda > 0$, defined on the polydisc $\mathbb{D}^n$, split into orthogonal direct sum of subspaces $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ indexed by the partitions $\boldsymbol{p}$ of n, which are in one to one correspondence with the equivalence classes of the irreducible representations of the symmetric group on n symbols. In this paper, we prove that each sub-module $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ is a *locally free Hilbert module* of *rank* equal to square of the dimension $\chi_{\boldsymbol{p}}(1)$ of the corresponding irreducible representation. Given two partitions $\boldsymbol{p}$ and $\boldsymbol{q}$, we show that if $\chi_{\boldsymbol{p}}(1) \neq \chi_{\boldsymbol{q}}(1)$, then the sub-modules $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{\boldsymbol{q}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are not equivalent. For the trivial and the sign representation corresponding to the partitions $\boldsymbol{p} = (n)$ and $\boldsymbol{p} = (1, \ldots, 1)$, respectively, we prove that the sub-modules $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are inequivalent. In particular, for $n = 3$, we show that all the sub-modules in this decomposition are inequivalent.

1. Introduction

In this paper, we study the weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, $\lambda > 1$, of square integrable holomorphic functions defined on the polydisc $\mathbb{D}^n$ with respect to the measure $\big(\prod_{i=1}^n (1-|z_i|^2)^{\lambda-2}\big)dV(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{D}^n$. (In the sequel, we also consider the case of $\lambda > 0$.) The bi-holomorphic automorphism group $\mathrm{Aut}(\mathbb{D}^n)$ is easily seen to be the semi-direct product $\mathrm{Aut}(\mathbb{D})^n \rtimes \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the permutation group on n symbols. For $\Phi \in \mathrm{Aut}(\mathbb{D}^n)$, define $U : \mathrm{Aut}(\mathbb{D}^n) \to \mathcal{L}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ by the formula:

$$U(\Phi^{-1})h = \big(\det(D\Phi)\big)^{\lambda/2} h \circ \Phi,\ h \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n).$$

Since $\big(\det(D\Phi)\big)^{\lambda/2}(\boldsymbol{z}) : \mathrm{Aut}(\mathbb{D}^n) \times \mathbb{D}^n \to \mathbb{C}$ is a (projective) cocycle, it follows that the map U defines a (projective) unitary representation. The Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is also a module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$, namely,

$$m_p(h) = p \cdot h,\ p \in \mathbb{C}[\boldsymbol{z}],\ h \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n),$$

where $p \cdot h$ is the point-wise multiplication. Setting $(\Phi \cdot f)(\boldsymbol{z}) = f\big(\Phi^{-1}(\boldsymbol{z})\big)$, we have the relationship $m_{\Phi\cdot p} = U(\Phi)^* m_p U(\Phi)$, $\Phi \in \mathrm{Aut}(\mathbb{D}^n), p \in \mathbb{C}[\boldsymbol{z}]$, which is analogous to the imprimitivity introduced by Mackey (cf. [27, Chapter 6]). The imprimitivities of Mackey have been studied extensively and are related to induced representations, representations of the semi-direct product and homogeneous vector bundles, see Theorems 6.12, 6.20 and 6.24 in [27], respectively. However, the situation we have described is different in that the module action is defined over the ring of analytic polynomials rather than the algebra of continuous functions. This, we believe, merits a detailed investigation and the outcome, see [18, 21], so far is very encouraging. Also, the restriction of the representation U to the subgroup $\triangle := \{(\varphi, \ldots, \varphi) : \varphi \in \mathrm{Aut}(\mathbb{D})\}$ of $\mathrm{Aut}(\mathbb{D}^n)$ has a decomposition into irreducible components known as the Clebsch-Gordan decomposition. On the other hand, the symmetric group acts on $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ via the unitary map $R_{\sigma^{-1}} : h \to h \circ \sigma$, $\sigma \in \mathfrak{S}_n$. The Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is also a module over the ring of the symmetric polynomials $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, where the module map is given by the

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formula: $\mathfrak{m}_p(h) = p \cdot h,\ p \in \mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. Here, we propose to study the imprimitivity $\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n), \mathfrak{m}_p, R_\sigma\big)$ and obtain a decomposition of the Hilbert module $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ into sub-modules like in the more familiar Clebsch-Gordan decomposition mentioned above.

Let $\widehat{\mathfrak{S}_n}$ denote the equivalence classes of all irreducible representations of $\mathfrak{S}_n$. It is well known that these are finite dimensional and they are in one to one correspondence with partitions $\boldsymbol{p}$ of n [15, Theorem 4.3]. Recall that a partition $\boldsymbol{p}$ of n is a decreasing finite sequence $\boldsymbol{p} = (p_1, \ldots, p_k)$ of non-negative integers such that $\sum_{i=1}^k p_i = n$. A partition $\boldsymbol{p}$ of n is denoted by $\boldsymbol{p} \vdash n$. Let $\boldsymbol{\pi_p}$ be a unitary representation of $\mathfrak{S}_n$ in the equivalence class of $\boldsymbol{p} \vdash n$, that is, $\boldsymbol{\pi_p}(\sigma) = \big(\big(\boldsymbol{\pi}_{\boldsymbol{p}}^{ij}(\sigma)\big)\big)_{i,j=1}^m \in \mathbb{C}^{m\times m}$, $\sigma \in \mathfrak{S}_n$, where $m = \chi_{\boldsymbol{p}}(1)$ and $\chi_{\boldsymbol{p}}(\sigma) = \operatorname{trace}\big(\boldsymbol{\pi_p}(\sigma)\big)$, $\sigma \in \mathfrak{S}_n$, is the character of the representation $\boldsymbol{\pi_p}$.

A decomposition of $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ under the natural action of the group $\mathfrak{S}_n$, which is the restriction of U to $\mathfrak{S}_n$, is given by the formula (cf. [6] and [22]):

$$\mathbb{A}^{(\lambda)}(\mathbb{D}^n) = \bigoplus_{\boldsymbol{p}\vdash n} \mathbb{P}_{\boldsymbol{p}}\left(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\right),\ \boldsymbol{p} \vdash n \text{ is a partition of } n \in \mathbb{N},$$

where $\mathbb{P}_{\boldsymbol{p}} f = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma\in\mathfrak{S}_n} \overline{\chi_{\boldsymbol{p}}(\sigma)}(f\circ\sigma^{-1}), \sigma \in \mathfrak{S}_n$. On the right hand side, the irreducible representation of the group $\mathfrak{S}_n$ corresponding to the partition $\boldsymbol{p}$ is not multiplicity free. Both sides of the equation (1.1) happen to be modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, what is more, the explicit projection formula has been used extensively in [6] to study various properties of the Hilbert module $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$.

For the sake of concreteness, we have picked the Hilbert module $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ over the ring $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, however, the questions we raise here can be made up in similar but much more general context.

Let K be a $\mathfrak{S}_n$-invariant positive definite kernel on $\mathbb{D}^n$ and $\mathcal{H}_K$ be the corresponding reproducing kernel Hilbert space. Let $\boldsymbol{\pi_p}$ be the matrix representation of the finite dimensional unitary representation of $\mathfrak{S}_n$ corresponding to the partition $\boldsymbol{p} \vdash n$. Define the operators $\mathbb{P}_{\boldsymbol{p}}^{ij} : \mathcal{H}_K \to \mathcal{H}_K$, $1 \le i, j \le \chi_{\boldsymbol{p}}(1)$, by the formula

$$\mathbb{P}_{\boldsymbol{p}}^{ij} f = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma\in\mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})(f\circ\sigma^{-1}).$$

Also, $\mathbb{P}_{\boldsymbol{p}} = \sum_{i=1}^{\chi_{\boldsymbol{p}}(1)} \mathbb{P}_{\boldsymbol{p}}^{ii}$. Specializing to our situation, that is, when $K(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^n (1 - z_i\bar{w}_i)^{-\lambda}$ and $\mathcal{H}_K = \mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, we ask

1. if the sub-modules $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{\boldsymbol{q}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are inequivalent for distinct partitions $\boldsymbol{p}$ and $\boldsymbol{q}$ of n;
2. if the reducing sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{\boldsymbol{q}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are inequivalent whenever $(\boldsymbol{p}, i) \ne (\boldsymbol{q}, j)$, where $\boldsymbol{p}$, $\boldsymbol{q}$ are partitions of n, $1 \le i \le \chi_{\boldsymbol{p}}(1)$ and $1 \le j \le \chi_{\boldsymbol{q}}(1)$,
3. if the reducing sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$, $\boldsymbol{p}$ partition of n and $1 \le i \le \chi_{\boldsymbol{p}}(1)$, are minimal?

For any partition $\boldsymbol{p}$ of n, we have shown, see Corollary 2.15, that the Hilbert modules $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are locally free of rank $\chi_{\boldsymbol{p}}(1)^2$ on an open subset of $\mathbb{G}_n$. Furthermore, using similar arguments, we show that the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$, $1 \le i \le \chi_{\boldsymbol{p}}(1)$, are locally free of rank $\chi_{\boldsymbol{p}}(1)$. Therefore, if $\chi_{\boldsymbol{p}}(1) \ne \chi_{\boldsymbol{q}}(1)$, then the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{\boldsymbol{q}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are not equivalent, see Theorem 4.1. Although, we haven't been able to resolve this issue when $\chi_{\boldsymbol{p}}(1) = \chi_{\boldsymbol{q}}(1)$, in general, we have obtained the answer in one important special case, namely, for all partition $\boldsymbol{p}$ of n such that $\chi_{\boldsymbol{p}}(1) = 1$. For $n \ge 2$, there are only two such partitions: $\boldsymbol{p} = (n)$ or $(1, \ldots, 1)$. We show that the two sub-modules $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are inequivalent (there is no intertwining module map between them that is unitary) over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, see Theorem 4.5. Also these summands are locally free of rank 1, therefore they are irreducible and hence minimal. For $n = 2$, in the decomposition $\mathbb{A}^{(\lambda)}(\mathbb{D}^2) = \mathbb{P}_{(2)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^2)\big) \oplus \mathbb{P}_{(1,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^2)\big)$, the two summands are minimal and inequivalent. Therefore, in this case, we have answered the questions (1) - (3). Furthermore, for $n = 3$, it follows that all the submodules in the decomposition $\oplus_{\boldsymbol{p}\vdash 3}\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big)$ are inequivalent, see Corollary 4.18. Along the way, we give an explicit formula, see Theorem 4.11, for the weighted Bergman kernel of the symmetrized polydisc

$\mathbb{G}_n$ in the co-ordinates of $\mathbb{G}_n$ rather than that of the polydisc $\mathbb{D}^n$. In an earlier paper [22], the case of $n = 2$ was worked out.

For any partition $\boldsymbol{p}$ of n, we recall from [6] that the commuting n-tuple of multiplications $M_{\boldsymbol{s}}^{(\boldsymbol{p})} = (M_{s_1}, \ldots, M_{s_n})$ by the elementary symmetric functions $\boldsymbol{s}$ defined on the Hilbert space $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$, $\lambda \geq 1$, are examples of Γ_n-contractions. Since $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ admits a further decomposition into a direct sum of the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$, it follows that the n-tuple $M_{\boldsymbol{s}}^{(\boldsymbol{p})}$ acting on these reducing subspaces is also a Γ_n-contraction, which is Theorem 3.11 of this paper. What is more, we have shown that the Taylor joint spectrum of each of these n-tuples is Γ_n and thus, in these examples, the spectrum is a spectral set.

Since the Hilbert module $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$, as well as the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$, are locally free on some open subset of $\mathbb{G}_n$, it follows that these are in one to one correspondence with holomorphic hermitian vector bundles defined on some open subset of $\mathbb{G}_n$. The rank of this vector bundle is an invariant, albeit a very weak one. However, it is the rank which is used to distinguish the sub-modules $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ in this paper. We conclude the paper with an explicit realization of a spanning holomorphic cross-section for the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. This provides an invariant that we believe will be useful in our future work.

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2. Locally free Hilbert modules

First, we recall several useful definitions following [13, 8] and [7].

Definition 2.1. *A Hilbert space $\mathcal{H}$ is said to be a Hilbert module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ in n variables if the map $(p, h) \to p \cdot h$, $p \in \mathbb{C}[\boldsymbol{z}], h \in \mathcal{H}$, defines a homomorphism $p \mapsto T_p$, where T_p is bounded operator defined by $T_p h = p \cdot h$.*

Two Hilbert modules $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are said to be (unitarily) equivalent if there exists a unitary module map $U : \mathcal{H} \to \tilde{\mathcal{H}}$, that is, $UT_p = \tilde{T}_p U$, $p \in \mathbb{C}[\boldsymbol{z}]$.

Let $\mathbb{C}_{\boldsymbol{w}}$ be the one dimensional module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ defined by the evaluation, that is, $(p, c) \to p(\boldsymbol{w})c,\ c \in \mathbb{C}, p \in \mathbb{C}[\boldsymbol{z}]$. Following [13], we define the module tensor product of two Hilbert modules $\mathcal{H}$ and $\mathbb{C}_{\boldsymbol{w}}$ over $\mathbb{C}[\boldsymbol{z}]$ to be the quotient of the space Hilbert space tensor product $\mathcal{H} \otimes \mathbb{C}_{\boldsymbol{w}}$ by the subspace

$$\begin{aligned} \mathcal{N} &:= \{p \cdot f \otimes 1_{\boldsymbol{w}} - f \otimes p(\boldsymbol{w}) : p \in \mathbb{C}[\boldsymbol{z}], f \in \mathcal{H}\} \\ &= \{(p - p(\boldsymbol{w}))f : p \in \mathbb{C}[\boldsymbol{z}], f \in \mathcal{H}\}. \end{aligned}$$

Thus

$$\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]} \mathbb{C}_{\boldsymbol{w}} := (\mathcal{H} \otimes \mathbb{C})/\mathcal{N},$$

where the module action is defined by the compression of the operator $T_p \otimes 1_{\boldsymbol{w}},\ p \in \mathbb{C}[\boldsymbol{z}]$, to the subspace $(\mathcal{H} \otimes \mathbb{C})/\mathcal{N}$. We recall the notion of local freeness of a Hilbert module in accordance with [7, Definition 1.4].

Definition 2.2 (Definition 1.4, [7])**.** *Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Let Ω be a bounded open connected subset of $\mathbb{C}^n$. We say $\mathcal{H}$ is locally free of rank k at $\boldsymbol{w}$ in $\Omega^* := \{\boldsymbol{z} \in \mathbb{C}^n : \bar{\boldsymbol{z}} \in \Omega\}$ if there exists a neighbourhood Ω_0^* of $\boldsymbol{w}$ and holomorphic functions $\gamma_1, \gamma_2, \ldots, \gamma_k : \Omega_0^* \to \mathcal{H}$ such that the linear span of the set of k vectors $\{\gamma_1(\boldsymbol{z}), \ldots, \gamma_k(\boldsymbol{z})\}$ is the module tensor product $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]} \mathbb{C}_{\boldsymbol{z}}$. Following the terminology of* [7]*, we say that a module $\mathcal{H}$ is locally free on Ω of rank k if it is locally free of rank k at every $\boldsymbol{w}$ in Ω^*.*

Let $\mathbb{D}^n = \{\boldsymbol{z} : |z_1|, \ldots, |z_n| < 1\}$ be the polydisc in $\mathbb{C}^n$. For $\lambda > 0$, it is well known that the function $K^{(\lambda)} : \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{C}$ given by the formula

$$K^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{j=1}^{n} (1 - z_j \bar{w}_j)^{-\lambda},\ \ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n,$$

is positive definite. The function $K^{(\lambda)}$ uniquely determines a Hilbert space, say $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, consisting of holomorphic functions defined on $\mathbb{D}^n$ with the reproducing property

$$\langle f(\cdot), K^{(\lambda)}(\cdot, \boldsymbol{w})\rangle = f(\boldsymbol{w}),\ \ f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n),\ \ \boldsymbol{w} \in \mathbb{D}^n.$$

For $\lambda > 1$, this coincides with the usual notion of the weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ defined as the Hilbert space of square integrable holomorphic functions on $\mathbb{D}^n$ with respect to the measure $dV^{(\lambda)} = \left(\frac{\lambda-1}{\pi}\right)^n \Big(\prod_{i=1}^n (1 - r_i^2)^{\lambda-2} r_i dr_i d\theta_i\Big)$. The limiting case of $\lambda = 1$ is the Hardy space $H^2(\mathbb{D}^n)$. Throughout the rest of this paper, we will assume that $\lambda > 0$.

The natural action of the permutation group $\mathfrak{S}_n$ on $\mathbb{C}^n$. is given by the formula:

$$(\sigma, \boldsymbol{z}) \mapsto \sigma \cdot \boldsymbol{z} := (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)}),\ (\sigma, \boldsymbol{z}) \in \mathfrak{S}_n \times \mathbb{C}^n.$$

The induced action on the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is $f \mapsto f \circ \sigma^{-1}$, $\sigma \in \mathfrak{S}_n$. Let $\boldsymbol{s} : \mathbb{C}^n \to \mathbb{C}^n$ be the symmetrization map $\boldsymbol{s} = (s_1, \ldots, s_n)$, where $s_k(\boldsymbol{z}) = \sum_{1 \le i_1, \ldots, i_k \le n} z_{i_1} \cdots z_{i_k}$, $1 \le k \le n$. Let $(M_1, \ldots, M_n)$ denote the n-tuple of multiplication by the coordinate functions z_i, $1 \le i \le k$ on $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. Clearly, $(M_{s_1}, \ldots, M_{s_n})$ defines a commuting tuple of bounded linear operators on $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. Define $\Delta(\boldsymbol{z}) = \prod_{i<j}(z_i - z_j)$, for $\boldsymbol{z} \in \mathbb{C}^n$. Let

$$\mathscr{Z} = \{\boldsymbol{z} \in \mathbb{D}^n \mid \Delta(\boldsymbol{z}) = 0\} = \{\boldsymbol{z} \in \mathbb{D}^n \mid z_i = z_j \text{ for some } i \neq j, 1 \le i, j \le n\}$$

and $\mathbb{G}_n = \boldsymbol{s}(\mathbb{D}^n)$. For every $\boldsymbol{u} \in \mathbb{G}_n \setminus \boldsymbol{s}(\mathscr{Z})$, we note that the set $\boldsymbol{s}^{-1}(\{\boldsymbol{u}\})$ has exactly $n!$ elements. If M_ϕ is a multiplication operator on $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ by a holomorphic function ϕ, then $M_\phi^* K_{\boldsymbol{w}}^{(\lambda)} = \overline{\phi(\boldsymbol{w})} K_{\boldsymbol{w}}^{(\lambda)}$ for $\boldsymbol{w} \in \mathbb{D}^n$. Therefore we have the following lemma.

Lemma 2.3. *For $\sigma \in \mathfrak{S}_n$, $i = 1, \ldots, n$, $M_i^* K_{\boldsymbol{w}_\sigma}^{(\lambda)} = \bar{w}_{\sigma^{-1}(i)} K_{\boldsymbol{w}_\sigma}^{(\lambda)}$ and $M_{s_i}^* K_{\boldsymbol{w}_\sigma}^{(\lambda)} = \overline{s_i(\boldsymbol{w})} K_{\boldsymbol{w}_\sigma}^{(\lambda)}$.*

Let $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ be the ring of invariants under the action of $\mathfrak{S}_n$ on $\mathbb{C}[\boldsymbol{z}]$, that is,

$$\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n} = \{f \in \mathbb{C}[\boldsymbol{z}] : f(\sigma \cdot \boldsymbol{z}) = f(\boldsymbol{z}), \sigma \in \mathfrak{S}_n\}.$$

Furthermore, $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n} = \mathbb{C}[s_1, \ldots, s_n]$, see [23, p. 39]. We now state the main Theorem of this Section.

Theorem 2.4. *The Hilbert module $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ is locally free of rank $n!$ on $\mathbb{G}_n \setminus \boldsymbol{s}(\mathscr{Z})$.*

The proof is facilitated by breaking it up into several pieces. Some of these pieces make essential use of the fact that $\mathbb{C}[\boldsymbol{z}]$ is a finitely generated free module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of rank $n!$ [25, Theorem 1]. The motivation for the following lemma and some of the subsequent comments come from [9].

Lemma 2.5. *For any basis $\{p_\sigma\}_{\sigma \in \mathfrak{S}_n}$ of $\mathbb{C}[\boldsymbol{z}]$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, we have*

$$\det \big(\!\!\big(p_\sigma(\boldsymbol{w}_\tau)\big)\!\!\big)_{\sigma, \tau \in \mathfrak{S}_n} \not\equiv 0.$$

Proof. Let $L = \mathbb{C}(\boldsymbol{z})$ denote the field of rational functions and $K = \mathbb{C}(\boldsymbol{z})^{\mathfrak{S}_n}$ be the field of symmetric rational function. From [23, Example 2.22], it is known that L over K is a finite Galois extension with Galois group $\mathrm{Gal}(L/K) = \mathfrak{S}_n$. Let $f \in L$, that is, $f = \frac{p}{q}$ for some polynomials p and q. Pick $\tilde{q} = \prod_{\sigma \in \mathfrak{S}_n} q(\boldsymbol{z}_\sigma)$ and $\tilde{p} = p \prod_{\sigma \in \mathfrak{S}_n, \sigma \neq 1} q(\boldsymbol{z}_\sigma)$. Now, $f = \frac{\tilde{p}}{\tilde{q}}$, where $\tilde{q}$ is symmetric. Again, since $\{p_\sigma\}_{\sigma \in \mathfrak{S}_n}$ is a basis for $\mathbb{C}[\boldsymbol{z}]$ over the ring $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, we have $p = \sum_{\sigma \in \mathfrak{S}_n} p_\sigma h_\sigma$ where h_σ's are symmetric polynomial which in turn shows that $f = \sum_{\sigma \in \mathfrak{S}_n} p_\sigma \frac{h_\sigma}{\tilde{q}}$. Thus $\{p_\sigma\}_{\sigma \in \mathfrak{S}_n}$ forms a basis of L over K. Now we make use of the following basic result from Galois theory [10, Lemma 3.4]:

If N/F is a finite Galois extension with $\mathrm{Gal}(N/F) = \{g_1, \ldots, g_m\}$ *and* $\{e_1, \ldots, e_m\}$ *is a F-basis of N, then* $\big(g_1(e_j), \ldots, g_m(e_j)\big)_{j=1}^m$ *forms a basis of* F^m/F.

Consequently, $\big((p_\sigma \circ \tau^{-1})_{\sigma\in\mathfrak{S}_n}\big)_{\tau\in\mathfrak{S}_n}$ is a basis of $L^{n!}/L$. Hence we have the desired result. ∎

Recall that the length of permutation $\sigma \in \mathfrak{S}_n$ is the number of inversions in σ [17, p. 4]. Here, by an inversion in σ, we mean a pair (i, j) with $1 \leq i < j \leq n$ such that $\sigma(i) > \sigma(j)$. This is the smallest number of transpositions of the form $(i, i+1)$ required to write σ as a product of these transpositions.

Lemma 2.6. *Pick a basis for $\mathbb{C}[\boldsymbol{z}]$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ consisting of homogeneous polynomials p_σ, $\sigma \in \mathfrak{S}_n$, $\deg p_\sigma = \ell(\sigma)$. Then*

(i) *the determinant* $\det \big(\!\!\big(p_\sigma(\boldsymbol{w}_\tau)\big)\!\!\big)_{\sigma,\tau\in\mathfrak{S}_n}$ *is a homogeneous polynomial of degree* $\frac{n!}{2}\binom{n}{2}$,

(ii) $\det \big(\!\!\big(p_\sigma(\boldsymbol{w}_\tau)\big)\!\!\big)_{\sigma,\tau\in\mathfrak{S}_n}$ *is a non-zero constant multiple of* $\Delta(\boldsymbol{w})^{\frac{n!}{2}}$.

Proof. Clearly,

$$\det \big(\!\!\big(p_\sigma(\boldsymbol{w}_\tau)\big)\!\!\big)_{\sigma,\tau\in\mathfrak{S}_n} = \sum_{\nu\in\mathfrak{S}_{n!}} \prod_{\sigma\in\mathfrak{S}_n} p_\sigma(\boldsymbol{w}_{\nu\sigma}).$$

We note that

$$\deg \prod_{\sigma\in\mathfrak{S}_n} p_\sigma(\boldsymbol{w}_{\nu\sigma}) = \sum_{\sigma\in\mathfrak{S}_n} \deg p_\sigma(\boldsymbol{w}) = \sum_{\sigma\in\mathfrak{S}_n} \deg p_\sigma = \sum_{\sigma\in\mathfrak{S}_n} \ell(\sigma).$$

Let $I_n(k)$ denote the number of k-inversions in $\mathfrak{S}_n$ [20, p. 1]. Alternatively, $I_n(k) = \mathrm{card}\{\sigma \in \mathfrak{S}_n \mid \ell(\sigma) = k\}$. Note that

$$\sum_{\sigma\in\mathfrak{S}_n} \ell(\sigma) = \sum_{k=1}^{\binom{n}{2}} \sum_{\ell(\sigma)=k} \ell(\sigma) = \sum_{k=1}^{\binom{n}{2}} k I_n(k).$$

The generating function formula for $I_n(k)$ is given by [20, Theorem 1]

$$\sum_{k=1}^{\binom{n}{2}} I_n(k) z^k = \prod_{i=1}^{n-1} \sum_{j=0}^{i} z^j.$$

Differentiating with respect to z, we obtain

$$\sum_{k=1}^{\binom{n}{2}} k I_n(k) z^{k-1} = \sum_{i=1}^{n-1} (1 + \ldots + i z^{i-1}) \prod_{j=1, j\neq i}^{r-1} (1 + \ldots + z^j).$$

Putting $z = 1$, we have

$$\sum_{k=1}^{\binom{n}{2}} k I_n(k) = \sum_{i=1}^{n-1} \frac{i(i+1)}{2} \prod_{j=1, j\neq i}^{n-1} (j+1) = \frac{n!}{2} \sum_{i=1}^{n-1} i = \frac{n!}{2}\binom{n}{2}.$$

This proves part (i). For part (ii), let us choose i, j with $1 \leq i < j \leq n$. Consider the automorphism of $\mathfrak{S}_n$ given by $\tau \mapsto \tau(i, j)$, where (i, j) is the transposition. This automorphism maps an even permutation to an odd permutation and vice versa. For any polynomial p, clearly, $p(\boldsymbol{z}_\tau) = \sum_{m,n} a_{mn}(\boldsymbol{z}') z_i^m z_j^n \in \mathbb{C}[\boldsymbol{z}]$, where each $a_{mn}(\boldsymbol{z}')$ is a polynomial in the variables $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$. Thus $p(\boldsymbol{w}_\tau) - p(\boldsymbol{w}_{\tau(i,j)}) = \sum_{m,n} a_{mn}(\boldsymbol{w}')(w_i^m w_j^n - w_j^m w_i^n)$ is divisible by $w_i - w_j$. Thus for each even permutation τ, if we subtract the $\tau(i, j)$-th column $\big(p_\sigma(\boldsymbol{w}_{\tau(i,j)})\big)_{\sigma\in\mathfrak{S}_n}$ from τ-th column $(p_\sigma(\boldsymbol{w}_\tau))_{\sigma\in\mathfrak{S}_n}$, the determinant does not change. Consequently, we see that $w_i - w_j$ is a factor of the determinant. Since we have exactly $\frac{n!}{2}$ even permutations in $\mathfrak{S}_n$, it follows that $(w_i - w_j)^{\frac{n!}{2}}$ must divide the determinant. This is true for every pair of $i < j$ and $\mathbb{C}[\boldsymbol{z}]$ is a unique factorization domain. Hence $\Delta(\boldsymbol{w})^{\frac{n!}{2}}$ divides the determinant. From part (i) and Lemma 2.5, we see that the degree of the polynomial $\Delta(\boldsymbol{w})^{\frac{n!}{2}}$ is equal to $\frac{n!}{2}\binom{n}{2}$ completing the proof of part (ii). ∎

Remark 2.7. *The degree of the polynomials in a basis consisting of the Descent polynomials* [3, p. 6] *or the Schubert polynomials* [17, Theorem 2.16]*, meet the hypothesis made in Lemma 2.6.*

Lemma 2.8. *Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ consisting of holomorphic functions defined on the polydisc $\mathbb{D}^n$ possessing a reproducing kernel, say K. Assume that $\mathbb{C}[\boldsymbol{z}]$ is dense in $\mathcal{H}$. If v is in $\cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^*, \boldsymbol{w} \in \mathbb{D}^n \setminus \mathcal{Z}$, then there exists unique tuple $(c_\sigma)_{\sigma\in\mathfrak{S}_n}$, such that $v = \sum c_\sigma K(\cdot, \boldsymbol{w}_\sigma)$.*

Proof. Clearly, $M_{s_i}^* K(\cdot, w_\sigma) = \overline{s_i(w_\sigma)} K(\cdot, w_\sigma) = \overline{s_i(w)} K(\cdot, w_\sigma)$. To complete the proof, given a joint eigenvector v, it is enough to ensure the existence of a unique tuple $(c_\sigma)_{\sigma\in\mathfrak{S}_n}$ of complex numbers such that

$$\langle v, p\rangle = \langle \sum_{\sigma\in\mathfrak{S}_n} c_\sigma K(\cdot, \boldsymbol{w}_\sigma), p\rangle = \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \overline{p(\boldsymbol{w}_\sigma)},$$

for all polynomials p since $\mathbb{C}[\boldsymbol{z}]$ is dense in the Hilbert module $\mathcal{H}$. In particular, if there exists a unique solution for some choice of a basis, say $\{p_\tau\}_{\tau\in\mathfrak{S}_n}$, of $\mathbb{C}[\boldsymbol{z}]$ over the ring $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, then for any $p = \sum_{\tau\in\mathfrak{S}_n} p_\tau h_\tau \in \mathbb{C}[\boldsymbol{z}]$, we have

$$\begin{aligned}
\langle v, p\rangle &= \langle v, \sum_{\tau\in\mathfrak{S}_n} p_\tau h_\tau\rangle = \sum_{\tau\in\mathfrak{S}_n} \langle M_{h_\tau}^* v, p_\tau\rangle = \sum_{\tau\in\mathfrak{S}_n} \overline{h_\tau(\boldsymbol{w})}\langle v, p_\tau\rangle \\
&= \sum_{\tau\in\mathfrak{S}_n} \overline{h_\tau(\boldsymbol{w})} \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \overline{p_\tau(\boldsymbol{w}_\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \sum_{\tau\in\mathfrak{S}_n} \overline{h_\tau(\boldsymbol{w}_\sigma) p_\tau(\boldsymbol{w}_\sigma)} \\
&= \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \overline{p(\boldsymbol{w}_\sigma)}.
\end{aligned}$$

Thus choosing $\{p_\tau\}_{\tau\in\mathfrak{S}_n}$ as in the hypothesis of Lemma 2.6 and using part (ii) of that Lemma, we have a unique solution $(c_\sigma)_{\sigma\in\mathfrak{S}_n}$ for the system of equations

$$\langle v, p_\tau\rangle = \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \overline{p_\tau(\boldsymbol{w}_\sigma)}$$

as long as $\boldsymbol{w}$ is from $\mathbb{D}^n \setminus \mathcal{Z}$. ∎

As a consequence of the Lemma we have just proved, we see that the set of vectors $\{K_{\boldsymbol{w}_\sigma} \mid \sigma \in \mathfrak{S}_n\}$ are both linearly independent and spanning for the joint kernel $\cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^*, \boldsymbol{w} \in \mathbb{D}^n \setminus \mathcal{Z}$. Therefore, we have the following Corollary.

Corollary 2.9. *Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ consisting of holomorphic functions defined on the polydisc $\mathbb{D}^n$ possessing a reproducing kernel, say K. Assume that $\mathbb{C}[\boldsymbol{z}]$ is dense in $\mathcal{H}$. Then $\dim \cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^* = n!$.*

To complete the proof of Theorem 2.4, we need to relate the joint kernel $\cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^*$ to the module tensor product $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} \mathbb{C}_{\boldsymbol{w}}$. The following Lemma gives an isomorphism between these two. A special case of [13, Lemma 5.11], included in the Lemma below, is used in proving a generalization of Theorem 2.4 to $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$.

Lemma 2.10. *If $\mathcal{H}$ is a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ consisting of holomorphic functions defined on some bounded domain $\Omega \subseteq \mathbb{C}^n$, then we have*

(1) $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]} \mathbb{C}_{\boldsymbol{w}} \cong \cap_{p\in\mathbb{C}[\boldsymbol{z}]} \ker M_{p-p(\boldsymbol{w})}^*$;
(2) $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} \mathbb{C}_{\boldsymbol{w}} \cong \cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^*$;
(3) $p_1 \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} 1_{\boldsymbol{w}}, \ldots, p_t \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} 1_{\boldsymbol{w}}$ *spans* $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} \mathbb{C}_{\boldsymbol{w}}$, *for any set of generators* $p_1, \ldots, p_t$ *for* $\mathcal{H}$ *over* $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$.

Proof. We have to show that $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]} \mathbb{C}_{\boldsymbol{w}} = \cap_{p\in\mathbb{C}[\boldsymbol{z}]} \ker M^*_{p-p(\boldsymbol{w})}$. Recall that $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]} \mathbb{C}_{\boldsymbol{w}}$ is the orthocomplement of the subspace $\mathcal{N} = \{(p - p(\boldsymbol{w}))f : p \in \mathbb{C}[\boldsymbol{z}], f \in \mathcal{H}\}$ in $\mathcal{H} \otimes \mathbb{C}$. Therefore, we have

$$g \in \mathcal{N}^\perp \iff \langle g, \big(p - p(\boldsymbol{w})\big) f\rangle = 0 \text{ for all } p \in \mathbb{C}[\boldsymbol{z}], f \in \mathcal{H} \iff M^*_{\big(p-p(\boldsymbol{w})\big)} g = 0,\ p \in \mathbb{C}[\boldsymbol{z}].$$

Similarly, $\cap_{p\in\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} \ker M^*_{p-p(\boldsymbol{w})} \subseteq \cap_{i=1}^n \ker \big(M_{s_i} - s_i(\boldsymbol{w})\big)^*$. Also, if $f \in \cap_{i=1}^n \ker \big(M_{s_i} - s_i(\boldsymbol{w})\big)^*$, then $M^*_{s_i} f = \overline{s_i(\boldsymbol{w})} f$, $1 \le i \le n$. Since $p - p(\boldsymbol{w})$ is a symmetric polynomial, the existence of a polynomial q such that $p - p(\boldsymbol{w}) = q \circ \boldsymbol{s}$ follows. Thus

$$M^*_{q\circ \boldsymbol{s}} f = q(M_{s_1}, \ldots, M_{s_n})^* f = \overline{q(\boldsymbol{s}(\boldsymbol{w}))} f = 0.$$

To prove the last statement, consider the map $Q : \mathcal{H} \to \mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} \mathbb{C}_{\boldsymbol{w}}$ defined by $Qf = f \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} 1_{\boldsymbol{w}}$. Note that Q is the composition of a unitary map from $\mathcal{H}$ to $\mathcal{H} \otimes \mathbb{C}$ followed by the quotient map, hence it is onto and $\|Q\| \le 1$. Since $p_1\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n} + \cdots + p_t\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ is dense in $\mathcal{H}$, it follows that $Q(p_1\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n} + \cdots + p_t\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n})$ is dense in $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} \mathbb{C}_{\boldsymbol{w}}$. Now for any $\sum_{i=1}^t p_i f_i \in \mathcal{H}$, where f_i's are in $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, we have

$$Q\Big(\sum_{i=1}^t p_i f_i\Big) = \Big(\sum_{i=1}^t p_i f_i\Big) \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} 1_{\boldsymbol{w}} = \sum_{i=1}^t p_i \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} f_i \cdot 1_{\boldsymbol{w}} = \sum_{i=1}^t f_i(\boldsymbol{w}) p_i \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} 1_{\boldsymbol{w}}.$$

Therefore, $Q(p_1\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n} + \cdots + p_t\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n})$ is finite dimensional and hence $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} \mathbb{C}_{\boldsymbol{w}}$ is finite dimensional and is spanned by $p_1 \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} 1_{\boldsymbol{w}}, \ldots, p_t \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} 1_{\boldsymbol{w}}$. ∎

Proof of Theorem 2.4. Using Corollary 2.9, we show that the map $\boldsymbol{t} : \boldsymbol{u} \mapsto \operatorname{span}\{K^{(\lambda)}_{\boldsymbol{w}} \mid \boldsymbol{w} \in \boldsymbol{s}^{-1}(\boldsymbol{u})\}$ taking values in the Grassmannian $\mathrm{Gr}\big(n!, \mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ of the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ of rank $n!$ is anti-holomorphic. Given any $\boldsymbol{u}_0$, fixed but arbitrary, in $\mathbb{G}_n \setminus \boldsymbol{s}(\mathscr{Z})$, there exists a neighborhood of $\boldsymbol{u}_0$, say U, on which $\boldsymbol{s}$ admits $n!$ local inverses. Enumerate them as $\varphi_1, \ldots, \varphi_{n!}$. Then the linearly independent set

$$\big\{\gamma_i : \gamma_i(\boldsymbol{u}) = K^{(\lambda)}\big(\cdot, \varphi_i(\boldsymbol{u})\big), u \in U\big\}_{i=1}^{n!}$$

of anti-holomorphic $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$-valued functions spans the joint kernel $\cap_{i=1}^n \ker \big(M_{s_i} - s_i(\boldsymbol{w})\big)^*$. ∎

Remark 2.11. *We give a realization of $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ as a space of holomorphic functions on an open subset of $\big(\mathbb{G}_n \setminus \boldsymbol{s}(\mathscr{Z})\big)^*$ possessing a sharp (reproducing) kernel* [2, Definition 2.1], *we mimic here the construction in* [11, Remark 2.6]. *Define $\Gamma : \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \to \mathcal{O}(U^*)$ by $\big(\Gamma(f)(\bar{\boldsymbol{v}})\big)_i = \langle f, \gamma_i(\bar{\boldsymbol{v}})\rangle$. Let $\mathcal{H} = \Gamma\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) \subset \mathcal{O}(U^*)$. Let $\langle \Gamma f, \Gamma g\rangle = \langle f, g\rangle$, $f, g \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. Equipped with this inner product, $\mathcal{H}$ is a Hilbert space. Now, by definition Γ is a unitary from $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ to $\mathcal{H}$. Define $K_\Gamma : U^* \times U^* \to M_{n!}(\mathbb{C})$ by $\langle K_\Gamma(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}})e_j, e_i\rangle = \langle \gamma_j(\bar{\boldsymbol{v}}), \gamma_i(\bar{\boldsymbol{u}})\rangle = \langle \Gamma\big(\gamma_j(\bar{\boldsymbol{v}})\big)(\bar{\boldsymbol{u}}), e_i\rangle$, that is, $K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j = \Gamma\big(\gamma_j(\bar{\boldsymbol{v}})\big)$. The string of equalities*

$$\langle \Gamma f, K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j\rangle = \langle \Gamma f, \Gamma\big(\gamma_j(\bar{\boldsymbol{v}})\big)\rangle = \langle f, \gamma_j(\bar{\boldsymbol{v}})\rangle = \langle \Gamma(f)(\bar{\boldsymbol{v}}), e_j\rangle$$

shows that $\mathcal{H}$ is a reproducing kernel Hilbert space with K_Γ as reproducing kernel. Note that

$$\langle M_i^* K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j, \Gamma f\rangle = \langle K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j, \boldsymbol{u}_i \Gamma f\rangle = \overline{\langle \bar{\boldsymbol{v}}_i \Gamma f(\boldsymbol{v}), e_j\rangle} = \langle \boldsymbol{v}_i K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j, \Gamma f\rangle,$$

that is, $M_i^* K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j = \boldsymbol{v}_i K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j$. Thus

$$\begin{aligned} \Gamma M^*_{s_i} K^{(\lambda)}\big(\cdot, \phi_j(\bar{\boldsymbol{v}})\big) &= \Gamma\{\overline{s_i\big(\phi_j(\bar{\boldsymbol{v}})\big)} K\big(\cdot, \phi_j(\bar{\boldsymbol{v}})\big)\} = \boldsymbol{v}_i \Gamma K\big(\cdot, \phi_j(\bar{\boldsymbol{v}})\big) = \boldsymbol{v}_i \Gamma\big(\gamma_j(\bar{\boldsymbol{v}})\big) \\ &= \boldsymbol{v}_i K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j = M_i^* K_\Gamma(\cdot, \bar{\boldsymbol{v}})e_j \\ &= M_i^* \Gamma K^{(\lambda)}\big(\cdot, \phi_j(\bar{\boldsymbol{v}})\big). \end{aligned}$$

Since the linear span $K(\cdot, \boldsymbol{w})$, $\boldsymbol{w} \in U$, where $U \subseteq \mathbb{D}^n$ is any small open set, is dense in $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, it follows that $\Gamma M^*_{s_i} = M_i^* \Gamma$. Consequently, Γ is a module isomorphism between $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ and $\mathcal{H}$. So Γ is a unitary map from $\cap_{i=1}^n \ker \big(M_{s_i} - s_i(\boldsymbol{w})\big)^*$ to $\cap_{i=1}^n \ker(M_i - \boldsymbol{u}_i)^*$, where $\boldsymbol{s}(\boldsymbol{w}) = \boldsymbol{u}$. This shows that $ranK_\Gamma(\cdot, \boldsymbol{u})e_j = \cap_{i=1}^n \ker(M_i - \boldsymbol{u}_i)^*$, that is, K_Γ is sharp.

We would now make use of the following well known result, which is analogous to the statement: The polynomial ring $\mathbb{C}[\boldsymbol{z}]$ is a finitely generated free module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of rank $n!$.

Theorem 2.12. *The module $\mathbb{P}_{\boldsymbol{p}}\mathbb{C}[\boldsymbol{z}]$ is a finitely generated free module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of rank $\chi_{\boldsymbol{p}}(1)^2$.*

We are unable to locate a proof of this Theorem and therefore indicate a proof using results from [24].

Proof. Set $\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}} := \mathbb{P}_{\boldsymbol{p}}\mathbb{C}[\boldsymbol{z}]$. There exists a set of homogeneous polynomials in $\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}$, whose images in the quotient module $\mathcal{S}_{\boldsymbol{p}} = \mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}/\{s_1\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}} + \cdots + s_n\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}\}$ forms a $\mathbb{C}$-basis for $\mathcal{S}_{\boldsymbol{p}}$, see [24, Theorem 1.3]). Also, from [24, Theorem 3.10], it follows that $p_1, \ldots, p_\mu$ is a free basis for $\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. Now to see that $\mu = \chi_{\boldsymbol{p}}(1)^2$, we make use of [24, Theorem 4.9] and its proof along with [24, Corollary 4.9]. It says that the action of $\mathfrak{S}_n$ on the quotient ring $\mathbb{C}[\boldsymbol{z}]/\{s_1\mathbb{C}[\boldsymbol{z}] + \cdots + s_n\mathbb{C}[\boldsymbol{z}]\} \cong \oplus_{\boldsymbol{p}\vdash n}\mathcal{S}_{\boldsymbol{p}}$ is isomorphic to the regular representation of $\mathfrak{S}_n$, where the action on $\mathcal{S}_{\boldsymbol{p}}$ is isomorphic to the representation $\pi_{\boldsymbol{p}}$ corresponding to $\boldsymbol{p} \vdash n$ with multiplicity $\chi_{\boldsymbol{p}}(1)$. ∎

Recall that the rank of a Hilbert module $\mathcal{H}$ over a ring $\mathcal{R}$ is $\inf |\mathcal{F}|$, where $\mathcal{F} \subseteq \mathcal{H}$ is any subset with the property $\{r_1 f_1 + \cdots + r_k f_k : f_1, \ldots, f_k \in \mathcal{H}; r_1, \ldots, r_k \in \mathcal{R}\}$ is dense in $\mathcal{H}$ and $|\mathcal{F}|$ denotes the cardinality of $\mathcal{F}$ (cf. [8, Section 2.3]). The proof of the following Corollary is immediate from Theorem 2.12 and Lemma 2.10.

Corollary 2.13. *The Hilbert module $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ is of rank at most $\chi_{\boldsymbol{p}}(1)^2$.*

Let $M_{s_i}^{(\boldsymbol{p})} = M_{s_i}|_{\mathbb{P}_{\boldsymbol{p}}\left(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\right)}$. From [6, Remark 3.9], recall that each $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ is a reducing subspace of M_{s_i} for each $i, 1 \leq i \leq n$. Therefore, $M_{s_i}^* = \oplus_{\boldsymbol{p}\vdash n}\big(M_{s_i}^{(p)}\big)^*$ and we have

$$\cap_{i=1}^n \ker M^*_{s_i - s_i(\boldsymbol{w})} = \oplus_{\boldsymbol{p}\vdash n} \cap_{i=1}^n \ker\big(M^{(\boldsymbol{p})}_{s_i - s_i(\boldsymbol{w})}\big)^*.$$

Now we have the following useful Proposition.

Proposition 2.14. $\dim \cap_{i=1}^n \ker\big(M^{(\boldsymbol{p})}_{s_i - s_i(\boldsymbol{w})}\big)^* = \chi_{\boldsymbol{p}}(1)^2$, $w \in \mathbb{D}^n \setminus \mathcal{Z}$.

Proof. From Corollary 2.13 and Lemma 2.10, it follows that $\dim \cap_{i=1}^n \ker\big(M^{(\boldsymbol{p})}_{s_i - s_i(\boldsymbol{w})}\big)^* \leq \chi_{\boldsymbol{p}}(1)^2$. However if it is strictly less for some $\boldsymbol{p} \vdash n$ we have the following contradiction:

$$n! = \dim \cap_{i=1}^n \ker M^*_{s_i - s_i(\boldsymbol{w})} = \sum_{\boldsymbol{p}\vdash n} \dim \cap_{i=1}^n \ker\big(M^{(\boldsymbol{p})}_{s_i - s_i(\boldsymbol{w})}\big)^* < \sum_{\boldsymbol{p}\vdash n} \chi_{\boldsymbol{p}}(1)^2 = n!.$$

For the last equality, see [19, Theorem 3.4]. ∎

From the Proposition given above and the proof of Theorem 2.4, the following generalization to $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is evident.

Corollary 2.15. *The Hilbert module $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ is locally free of rank $\chi_{\boldsymbol{p}}(1)^2$ on $\mathbb{G}_n \setminus \boldsymbol{s}(\mathcal{Z})$.*

Remark 2.16. *Since $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ is assumed to be locally free at $\boldsymbol{w} \in \mathbb{G}_n \setminus \boldsymbol{s}(\mathcal{Z})$, it follows that $E_{\boldsymbol{p}} = \{(\boldsymbol{u}, x) \in U \times \mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) \mid x \in \cap_{i=1}^n \ker\big(M_{s_i - u_i})^*\}$ and $\pi(\boldsymbol{u}, x) = \boldsymbol{u}$ defines a rank $\chi_{\boldsymbol{p}}(1)^2$ hermitian anti-holomorphic vector bundle on some open neighbourhood W of $\boldsymbol{w}$. The equivalence class of this vector bundle $E_{\boldsymbol{p}}$ determines the isomorphism class of the module $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and conversely. The vector bundle E corresponding to the module $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is therefore the direct sum $\oplus_{\boldsymbol{p}\vdash n} E_{\boldsymbol{p}}$.*

Remark 2.17. *An alternative proof of the Corollary 2.9 is possible using Lemma 2.10. For this proof, which is indicated below, it is essential to use a non-trivial result from [11] rather than the direct proof that we have presented earlier. From Lemma 2.10, it follows that $\dim \cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^* \leq n!$. To prove the reverse inequality, we show that for $\boldsymbol{w} \in \mathbb{D}^n \setminus \mathcal{Z}$, the set of vectors $\{K_{\boldsymbol{w}_\sigma} \mid \sigma \in \mathfrak{S}_n\}$*

are linearly independent. Since the polynomial ring is dense in $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, the reproducing kernel K is non-degenerate. From [11, Lemma 3.6], *it then follows that K is strictly positive, that is, for all $k \geq 1$ the $k \times k$-operator matrix $\big(\!\big(K(\boldsymbol{z}_i, \boldsymbol{z}_j)\big)\!\big)_{1 \leq i,j \leq k}$ is injective for every collection $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k\}$ of distinct points in $\mathbb{D}^n \setminus \mathcal{Z}$. Since the set $\{\boldsymbol{w}_\sigma \mid \sigma \in \mathfrak{S}_n\}$ contains exactly $n!$ distinct points for every $\boldsymbol{w} \in \mathbb{D}^n \setminus \mathcal{Z}$, the matrix $\big(\!\big(\langle K_{\boldsymbol{w}_\sigma}, K_{\boldsymbol{w}_\tau} \rangle\big)\!\big)_{\sigma, \tau \in \mathfrak{S}_n}$ is injective and hence the nonsingularity of the grammian of $\{K_{\boldsymbol{w}_\sigma} \mid \sigma \in \mathfrak{S}_n\}$ gives the linear independence.*

3. $\mathfrak{S}_n$-invariant kernel

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain invariant under the action of $\mathfrak{S}_n$. Let K be a $\mathfrak{S}_n$-invariant reproducing kernel on Ω, that is,

$$K(\sigma \cdot \boldsymbol{z}, \sigma \cdot \boldsymbol{w}) = K(\boldsymbol{z}, \boldsymbol{w}) \text{ for all } \sigma \in \mathfrak{S}_n.$$

Let $\mathcal{H}_K$ denote the Hilbert space with K as reproducing kernel. Let $U : \mathfrak{S}_n \to \mathcal{B}(\mathcal{H}_K)$ be a unitary representation. Consider a function $f : \mathfrak{S}_n \to \mathbb{C}$ satisfying $f(\sigma^{-1}) = \overline{f(\sigma)}$. Define an operator on $\mathcal{H}_K$ by

$$T^f = \sum_{\sigma \in \mathfrak{S}_n} \overline{f(\sigma)} U(\sigma).$$

Since $U(\sigma)^* = U(\sigma^{-1})$, it follows that

$$(T^f)^* = \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) U(\sigma)^* = \sum_{\sigma \in \mathfrak{S}_n} \overline{f(\sigma^{-1})} U(\sigma^{-1}) = \sum_{\tau \in \mathfrak{S}_n} \overline{f(\tau)} U(\tau) = T^f.$$

Thus we have proved:

Lemma 3.1. *T^f is self adjoint on $\mathcal{H}_K$.*

As before, let $\boldsymbol{\pi}_{\boldsymbol{p}}$ be a unitary representation of $\mathfrak{S}_n$ in the equivalence class of $\boldsymbol{p} \vdash n$, that is, $\boldsymbol{\pi}_{\boldsymbol{p}}(\sigma) = \big(\!\big(\boldsymbol{\pi}_{\boldsymbol{p}}^{ij}(\sigma)\big)\!\big)_{i,j=1}^m \in \mathbb{C}^{m \times m}$, $\sigma \in \mathfrak{S}_n$, where $m = \chi_{\boldsymbol{p}}(1)$ and $\chi_{\boldsymbol{p}}$ is the character of the representation $\boldsymbol{\pi}_{\boldsymbol{p}}$. The following orthogonality relations [19, Proposition 2.9] play a central role in this section:

$$\sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ij}(\sigma^{-1}) \boldsymbol{\pi}_{\boldsymbol{q}}^{lm}(\sigma) = \frac{n!}{\chi_{\boldsymbol{p}}(1)} \delta_{\boldsymbol{p}\boldsymbol{q}} \delta_{im} \delta_{jl}, \tag{3.1}$$

where δ is the Kronecker symbol. Define the operators $\mathbb{P}_{\boldsymbol{p}}^{ij}, \mathbb{P}_{\boldsymbol{p}} : \mathcal{H}_K \to \mathcal{H}_K$, $1 \leq i, j \leq \chi_{\boldsymbol{p}}(1)$, by

$$\mathbb{P}_{\boldsymbol{p}}^{ij} = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1}) U(\sigma)$$

and

$$\mathbb{P}_{\boldsymbol{p}} = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi_{\boldsymbol{p}}(\sigma)} U(\sigma).$$

Clearly,

$$\sum_{i=1}^{\chi_{\boldsymbol{p}}(1)} \mathbb{P}_{\boldsymbol{p}}^{ii} = \mathbb{P}_{\boldsymbol{p}}.$$

The following lemma and some of the subsequent discussions are adapted from the properties of projection operators given in [19, p. 162]. We include this for sake of completeness.

Proposition 3.2. *For $1 \leq i, j \leq \chi_{\boldsymbol{p}}(1)$ and $1 \leq l, m \leq \chi_{\boldsymbol{q}}(1)$, $\mathbb{P}_{\boldsymbol{p}}^{ij} \mathbb{P}_{\boldsymbol{q}}^{lm} = \delta_{\boldsymbol{p}\boldsymbol{q}} \delta_{jl} \mathbb{P}_{\boldsymbol{p}}^{im}$.*

Proof. Since $\mathbb{P}_{\boldsymbol{p}}^{ij} = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1}) U(\sigma)$, we have that

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{p}}^{ij} \mathbb{P}_{\boldsymbol{q}}^{lm} &= \frac{\chi_{\boldsymbol{q}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1}) \mathbb{P}_{\boldsymbol{p}}^{ij} U(\sigma) \\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2} \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1}) \{ \sum_{\tau \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\tau^{-1}) U(\tau) \} U(\sigma) \\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\tau \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1}) \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\tau^{-1}) U(\tau) U(\sigma).
\end{aligned}$$

Let $\eta = \tau\sigma$. Then $\tau^{-1} = \sigma\eta^{-1}$ and

$$\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma\eta^{-1}) = (\boldsymbol{\pi}_{\boldsymbol{p}}(\sigma\eta^{-1}))_{ji} = (\boldsymbol{\pi}_{\boldsymbol{p}}(\sigma)\boldsymbol{\pi}_{\boldsymbol{p}}(\eta^{-1}))_{ji} = \sum_{k=1}^{\chi_{\boldsymbol{p}}(1)} \boldsymbol{\pi}_{\boldsymbol{p}}^{jk}(\sigma) \boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1}).$$

Thus, we also have

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{p}}^{ij} \mathbb{P}_{\boldsymbol{q}}^{lm} &= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\eta \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1}) \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma\eta^{-1}) U(\eta) \\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\eta \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1}) \sum_{k=1}^{\chi_{\boldsymbol{p}}(1)} \boldsymbol{\pi}_{\boldsymbol{p}}^{jk}(\sigma) \boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1}) U(\eta) \\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2} \sum_{\eta \in \mathfrak{S}_n} \sum_{k=1}^{\chi_{\boldsymbol{p}}(1)} \{ \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1}) \boldsymbol{\pi}_{\boldsymbol{p}}^{jk}(\sigma) \} \boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1}) U(\eta) \\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2} \sum_{\eta \in \mathfrak{S}_n} \sum_{k=1}^{\chi_{\boldsymbol{p}}(1)} \{ \delta_{\boldsymbol{pq}} \delta_{lj} \delta_{mk} \frac{n!}{\chi_{\boldsymbol{q}}(1)} \} \boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1}) U(\eta), \text{ (from Equation (3.1))} \\
&= \delta_{\boldsymbol{pq}} \delta_{jl} \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\eta \in \mathfrak{S}_n} \sum_{k=1}^{\chi_{\boldsymbol{p}}(1)} \delta_{mk} \boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1}) U(\eta) \\
&= \delta_{\boldsymbol{pq}} \delta_{jl} \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\eta \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{mi}(\eta^{-1}) U(\eta) \\
&= \delta_{\boldsymbol{pq}} \delta_{jl} \mathbb{P}_{\boldsymbol{p}}^{im}.
\end{aligned}$$

∎

Corollary 3.3. *For each partition $\boldsymbol{p}$ of n and $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$, $\mathbb{P}_{\boldsymbol{p}}^{ii}$ is an orthogonal projection and* $\sum_{\boldsymbol{p} \vdash n} \sum_{i=1}^{\chi_{\boldsymbol{p}}(1)} \mathbb{P}_{\boldsymbol{p}}^{ii} = \mathrm{id}$.

Proof. Since $\boldsymbol{\pi}_{\boldsymbol{p}}$ is a unitary representation, it follows that $\boldsymbol{\pi}_{\boldsymbol{p}}^{ii}(\sigma^{-1}) = \overline{\boldsymbol{\pi}_{\boldsymbol{p}}^{ii}(\sigma)}$. Thus from Lemma 3.1, we find that $\mathbb{P}_{\boldsymbol{p}}^{ii}$ is self adjoint. From the Proposition 3.2, it follows that $(\mathbb{P}_{\boldsymbol{p}}^{ii})^2 = \mathbb{P}_{\boldsymbol{p}}^{ii}$. Then we see that

$$\sum_{\boldsymbol{p} \vdash n} \sum_{i=1}^{\chi_{\boldsymbol{p}}(1)} \mathbb{P}_{ii}^{\boldsymbol{p}} = \sum_{\boldsymbol{p} \vdash n} \mathbb{P}_{\boldsymbol{p}} = \sum_{\boldsymbol{p} \vdash n} \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi_{\boldsymbol{p}}(\sigma) U(\sigma) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \Big(\sum_{\boldsymbol{p} \vdash n} \chi_{\boldsymbol{p}}(1) \chi_{\boldsymbol{p}}(\sigma) \Big) U(\sigma) = \mathrm{id},$$

where the last equality follows from the orthogonality relations [19, Proposition 3.8]. This completes the proof. ∎

For any $\mathfrak{S}_n$-invariant kernel K, we claim that the function $f \circ \sigma^{-1}$, $\sigma \in \mathfrak{S}_n$, is in $\mathcal{H}_K$, f in $\mathcal{H}_K$. To see this, recall that f is in $\mathcal{H}_K$ if and only if there exists a positive real number c such that $K_f(z,w) := \big(c^2 K(\boldsymbol{z},\boldsymbol{w}) - f(\boldsymbol{z})\overline{f(\boldsymbol{w})}\big)$ is positive definite, see [4, p. 194]. Since

$$\begin{aligned} K_{f\circ\sigma^{-1}}(z,w) &= c^2K(\boldsymbol{z},\boldsymbol{w}) - f\circ\sigma^{-1}(\boldsymbol{z})\overline{f\circ\sigma^{-1}(\boldsymbol{w})} \\ &= c^2K(\sigma\cdot\boldsymbol{u},\sigma\cdot\boldsymbol{v}) - f(\boldsymbol{u})\overline{f(\boldsymbol{v})} \\ &= c^2K(\boldsymbol{u},\boldsymbol{v}) - f(\boldsymbol{u})\overline{f(\boldsymbol{v})} \\ &= K_f(\boldsymbol{u},\boldsymbol{v}), \end{aligned}$$

where $\sigma\cdot\boldsymbol{u} = \boldsymbol{z}$ and $\sigma\cdot\boldsymbol{v} = \boldsymbol{w}$, it follows that $K_{f\circ\sigma^{-1}}$ is positive definite. Thus the operator $R_\sigma : \mathcal{H}_K \to \mathcal{H}_K$, $R_\sigma(f) = f\circ\sigma^{-1}$, is well defined.

Lemma 3.4. *The map $R : \sigma \mapsto R_\sigma$ is a unitary representation of $\mathfrak{S}_n$ on $\mathcal{H}_K$.*

Proof. Note that $R_{\sigma\tau}f(\boldsymbol{z}) = f\circ(\sigma\tau)^{-1}(\boldsymbol{z}) = f(\tau^{-1}\sigma^{-1}\cdot\boldsymbol{z}) = (R_\tau f)(\sigma^{-1}\cdot\boldsymbol{z}) = R_\sigma(R_\tau f)(\boldsymbol{z})$. Thus $R_{\sigma\tau} = R_\sigma R_\tau$. Since the set $\{K_{\boldsymbol{w}} \mid \boldsymbol{w}\in\Omega\}$ is total in $\mathcal{H}_K$, it is enough to check R_σ is unitary on $\{K_{\boldsymbol{w}} \mid \boldsymbol{w}\in\Omega\}$. Also,

$$R_\sigma K_{\boldsymbol{w}}(\boldsymbol{z}) = K_{\boldsymbol{w}}(\sigma^{-1}\cdot\boldsymbol{z}) = K(\sigma^{-1}\cdot\boldsymbol{z},\boldsymbol{w}) = K(\boldsymbol{z},\sigma\cdot\boldsymbol{w}) = K_{\sigma\cdot\boldsymbol{w}}(\boldsymbol{z}),$$

that is, $R_\sigma K_{\boldsymbol{w}} = K_{\sigma\cdot\boldsymbol{w}}$. Thus

$$\langle R_\sigma K_{\boldsymbol{w}}, R_\sigma K_{\boldsymbol{w}'}\rangle = \langle K_{\sigma\cdot\boldsymbol{w}}, K_{\sigma\cdot\boldsymbol{w}'}\rangle = K(\sigma\cdot\boldsymbol{w}',\sigma\cdot\boldsymbol{w}) = K(\boldsymbol{w}',\boldsymbol{w}) = \langle K_{\boldsymbol{w}}, K_{\boldsymbol{w}'}\rangle.$$

This completes the proof. ∎

For $\lambda > 0$, recall that $K^{(\lambda)} : \mathbb{D}^n\times\mathbb{D}^n \to \mathbb{C}$ is the reproducing kernel of $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. In the remaining portion of this section, we will specialize to the representation R. Now, the formula for $\mathbb{P}_{\boldsymbol{p}}^{ij}$ and $\mathbb{P}_{\boldsymbol{p}}$ simplifies to

$$\mathbb{P}_{\boldsymbol{p}}^{ij} f(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})(R_\sigma f)(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1}) f(\sigma^{-1}\cdot\boldsymbol{z}) \tag{3.2}$$

and

$$\mathbb{P}_{\boldsymbol{p}} f(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n} \overline{\chi_{\boldsymbol{p}}(\sigma)} R_\sigma f(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n} \overline{\chi_{\boldsymbol{p}}(\sigma)} f(\sigma^{-1}\cdot\boldsymbol{z}). \tag{3.3}$$

This is the projection formula used extensively earlier in [6, Equation (3.2)]. From Corollary 3.3, it follows that the subspace $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ of $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is a reproducing kernel Hilbert space for each $\boldsymbol{p}\vdash n$ and $1\le i\le \chi_{\boldsymbol{p}}(1)$. From Equation (3.2) and Proposition 3.2, we have

$$\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) = \bigoplus_{i=1}^{\chi_{\boldsymbol{p}}(1)} \mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) \tag{3.4}$$

and consequently, using Corollary 3.3, we obtain a finer decomposition of $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$:

$$\mathbb{A}^{(\lambda)}(\mathbb{D}^n) = \bigoplus_{\boldsymbol{p}\vdash n}\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) = \bigoplus_{\boldsymbol{p}\vdash n}\bigoplus_{i=1}^{\chi_{\boldsymbol{p}}(1)} \mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big). \tag{3.5}$$

The first of the two equalities was obtained in [6, p. 6237 - 6238], see also [22, p. 2368]. From Lemma 2.13, it follows that the orthogonal projection $\mathbb{P}_{\boldsymbol{p}}$ is non-trivial. In fact, the projections $\mathbb{P}_{\boldsymbol{p}}^{ii}$ are non-trivial as well. We record this as a separate Lemma. The main ingredient of the proof is borrowed from [19, p. - 166].

Lemma 3.5. *For each $\boldsymbol{p}\vdash n$ and $1\le i\le\chi_{\boldsymbol{p}}(1)$, $\mathbb{P}_{\boldsymbol{p}}^{ii}\neq 0$.*

Proof. From Proposition 3.2, we have

$$\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj} = \mathbb{P}_{\boldsymbol{p}}^{ij} = \mathbb{P}_{\boldsymbol{p}}^{ii}\mathbb{P}_{\boldsymbol{p}}^{ij},$$

and it then follows that $\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) \subseteq \mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. Also for $f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n)$,

$$\mathbb{P}_{\boldsymbol{p}}^{ii}f = \mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{ji}f = \mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}\mathbb{P}_{\boldsymbol{p}}^{ji}f$$

and thus $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) \subseteq \mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. Consequently, $\mathbb{P}_{\boldsymbol{p}}^{ij}$ is a surjective map from $\mathbb{P}_{\boldsymbol{p}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ onto $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. Now $\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}f = 0$ implies that $\mathbb{P}_{\boldsymbol{p}}^{ji}\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}f = 0$ and hence $\mathbb{P}_{\boldsymbol{p}}^{jj}f = (\mathbb{P}_{\boldsymbol{p}}^{jj})^2 f = 0$. This shows that $\mathbb{P}_{\boldsymbol{p}}^{ij}$ is injective on $\mathbb{P}_{\boldsymbol{p}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. The operator $\mathbb{P}_{\boldsymbol{p}}^{ij}$, being a finite linear combination of unitaries, is bounded and hence an invertible map (by Open mapping theorem) from $\mathbb{P}_{\boldsymbol{p}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ onto $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. Since each $\mathbb{P}_{\boldsymbol{p}}$ is non-trivial, from Equation (3.4), it follows that each $\mathbb{P}_{\boldsymbol{p}}^{ii}$ is non-trivial. ∎

Proposition 3.6. *For each $\boldsymbol{p} \vdash n$, $1 \le i \le \chi_{\boldsymbol{p}}(1)$ and $k = 1, \dots, n$, $M_{s_k}\mathbb{P}_{\boldsymbol{p}}^{ij} = \mathbb{P}_{\boldsymbol{p}}^{ij}M_{s_k}$.*

Proof. For $f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, from the Equation (3.2) we have

$$\begin{aligned}
\big(M_{s_k}\mathbb{P}_{\boldsymbol{p}}^{ij}f\big)(\boldsymbol{z}) &= \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})M_{s_k}f(\sigma^{-1}\cdot\boldsymbol{z})\\
&= \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})s_k(\sigma^{-1}\cdot\boldsymbol{z})f(\sigma^{-1}\cdot\boldsymbol{z})\\
&= \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})(s_kf)(\sigma^{-1}\cdot\boldsymbol{z})\\
&= \big(\mathbb{P}_{\boldsymbol{p}}^{ij}M_{s_k}f\big)(\boldsymbol{z}).
\end{aligned}$$

This completes the proof. ∎

In particular for each $\boldsymbol{p} \vdash n$ and i, $1 \le i \le \chi_{\boldsymbol{p}}(1)$, the projections $\mathbb{P}_{\boldsymbol{p}}^{ii}$ commute with M_{s_k} for each $k, 1 \le k \le n$ and we have the following corollary.

Corollary 3.7. *$\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ is a joint reducing subspace for $M_{s_k}, k = 1, \dots, n$, for every partition $\boldsymbol{p}$ of n and for each i, $1 \le i \le \chi_{\boldsymbol{p}}(1)$.*

3.1. **Γ_n - Contractions.** Set $M_{s_k}^{(\boldsymbol{p},i)} := M_{s_k}|_{\mathbb{P}_{\boldsymbol{p}}^{ii}\left(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\right)}$, $1 \le k \le n$. To find the spectrum of the commuting n-tuple $(M_{s_1}^{(\boldsymbol{p},i)}, \dots, M_{s_n}^{(\boldsymbol{p},i)})$, we first prove, following [26, Lemma 1.2], a Proposition giving a spectral inclusion for the direct sum of two commuting n-tuples.

Proposition 3.8. *Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be two commuting n-tuples of bounded linear operators acting the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then the Taylor joint spectrum $\sigma(\boldsymbol{S}_1)$ and $\sigma(\boldsymbol{S}_2)$ are contained in the Taylor joint spectrum $\sigma(\boldsymbol{S}_1 \oplus \boldsymbol{S}_2)$.*

Proof. Let $\iota : \mathcal{H}_1 \oplus \{0\} \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be the inclusion map, $(f, 0) \mapsto (f, 0)$ and $P : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \{0\} \oplus \mathcal{H}_2$ be the projection, $(f, g) \mapsto (0, g)$. Apply Lemma 1.2 of [26] to the short exact sequence

$$0 \to \mathcal{H}_1 \oplus \{0\} \xrightarrow{\iota} \mathcal{H}_1 \oplus \mathcal{H}_2 \xrightarrow{P} \{0\} \oplus \mathcal{H}_2 \to 0$$

and the direct sum $\boldsymbol{S}_1 \oplus \boldsymbol{S}_2$ to complete the proof. ∎

Since $\mathbb{P}_{\boldsymbol{p}}^{ii}K^{(\lambda)}(\cdot, w)$ is the reproducing kernel for $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$, it can vanish only on a set $X \subseteq \mathbb{D}^n$ such that the real dimension of X is at most $2n - 2$. Also,

$$M_{s_i}^{(\boldsymbol{p},i)}\mathbb{P}_{\boldsymbol{p}}^{ii}K^{(\lambda)}(\cdot, w) = \overline{s_i(w)}\mathbb{P}_{\boldsymbol{p}}^{ii}K^{(\lambda)}(\cdot, w), \tag{3.6}$$

and therefore, $\boldsymbol{s}(\mathbb{D}^n \setminus X) \subseteq \sigma(M_{s_1}^{(\boldsymbol{p},i)}, \dots, M_{s_n}^{(\boldsymbol{p},i)})$. Following the usual convention, set $\Gamma_n = \mathrm{clos}(\mathbb{G}_n)$ and note that $\Gamma_n = \boldsymbol{s}\big(\mathrm{clos}(\mathbb{D}^n)\big)$.

Theorem 3.9. *The Taylor joint spectrum of the n-tuple $(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)})$ is Γ_n.*

Proof. From Proposition 3.8, it follows that $\sigma(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}) \subseteq \sigma(M_{s_1}, \ldots, M_{s_n})$. The Taylor functional calculus shows that $\sigma(M_{s_1}, \ldots, M_{s_n}) = \boldsymbol{s}(\sigma(M_1, \ldots, M_n) = \Gamma_n$. Thus we have

$$\mathbb{G}_n \setminus \boldsymbol{s}(X) \subseteq \boldsymbol{s}(\mathbb{D}^n \setminus X) \subseteq \sigma(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}) \subseteq \Gamma_n.$$

Since $\operatorname{clos}\big(\mathbb{G}_n \setminus \boldsymbol{s}(X)\big) = \Gamma_n$ and the spectrum is compact, the proof is complete. ∎

The computation of the Taylor joint spectrum has some immediate applications. Commuting n -tuples of joint weighted shifts are discussed in [16]. They have shown (see [16, Corollary 3]), among other things, that the spectrum of a joint weighted shift must be Reinhardt (invariant under the action of the torus group). It is easy to see that Γ_n is not Reinhardt. Indeed $(1, \frac{1}{2}, \ldots, 0)$ is in Γ_n while $(1, -\frac{1}{2}, 0, \ldots, 0)$ is not in Γ_n. This follows from the observation that $(\mu_1, \ldots, \mu_k, 0, \ldots, 0)$ is in Γ_n if and only if $(\mu_1, \ldots, \mu_k)$ is in Γ_k. therefore we have proved the following Corollary.

Corollary 3.10. *The n- tuple $\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big)$ is not unitarily equivalent to any joint weighted shift.*

Let $X \subseteq \mathbb{C}^n$ be a polynomially convex set. A commuting n - tuple $\boldsymbol{T}$ of operators is said to admit X as a spectral set if $\|p(\boldsymbol{T})\| \leq \|p\|_{\infty,X} := \sup\{|p(\boldsymbol{z})| : \boldsymbol{z} \in X\}$. In the particular case of $X = \Gamma_n$, such a commuting n-tuple $\boldsymbol{T}$ is said to be a Γ_n-contraction. Since the restriction of a Γ_n-contraction to a reducing subspace is again a Γ_n-contraction, the proof of the following theorem is evident from [6, Proposition 2.13 and Corollary 3.11].

Theorem 3.11. *The commuting n-tuple $(M_{s_1}, \ldots, M_{s_n})$ acting on the Hilbert space $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ is a Γ_n-contraction for every partition $\boldsymbol{p}$ of n, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$ and all $\lambda \geq 1$.*

Remark 3.12. *It is observed in* [1, p. 47] *that the Taylor joint spectrum of a Γ_2-contraction is a subset of Γ_2. This is easily seen to be true of a Γ_n-contraction using polynomial convexity of Γ_n. Hence the Taylor joint spectrum of the commuting n-tuple $\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big)$ is contained in Γ_n. Here we emphasize that the n-tuple $\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big)$ is not only a Γ_n-contraction but admits its spectrum Γ_n as a spectral set.*

4. INEQUIVALENCE

Having obtained the decomposition (3.5) and having shown that each $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ is a reducing sub-module (Corollary 3.7) over the ring of symmetric polynomials $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of the Hilbert module $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, it is natural to ask whether these sub-modules are inequivalent for distinct pairs $(\boldsymbol{p}, i)$ of a partition $\boldsymbol{p}$ of n and i, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$. The following theorem provides a partial answer.

Theorem 4.1. *If $\boldsymbol{p}$ and $\boldsymbol{q}$ are two partitions of n such that $\chi_{\boldsymbol{p}}(1) \neq \chi_{\boldsymbol{q}}(1)$, then*

(a) *the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{\boldsymbol{q}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are not equivalent for any i, j, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$ and $1 \leq j \leq \chi_{\boldsymbol{q}}(1)$.*

(b) *the sub-modules $\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{\boldsymbol{q}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are not equivalent.*

Proof. From Corollary 3.7, it follows that

$$\bigcap_{k=1}^{n} \ker\big(M_{s_k}^{(\boldsymbol{p})} - s_k(\boldsymbol{w})\big)^* = \bigoplus_{i=1}^{\chi_{\boldsymbol{p}}(1)} \bigcap_{k=1}^{n} \ker\big(M_{s_k}^{(\boldsymbol{p},i)} - s_k(\boldsymbol{w})\big)^*.$$

Arguments similar to the ones given in the proof of Lemma 3.5 applied to the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ shows that $\cap_{k=1}^n \ker\big(M_{s_k}^{(\boldsymbol{p},i)} - s_k(\boldsymbol{w})\big)^*$ are isomorphic for all i, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$. Thus $\dim \cap_{k=1}^n \ker\big(M_{s_k}^{(\boldsymbol{p},i)} - s_k(\boldsymbol{w})\big)^* = \chi_{\boldsymbol{p}}(1)$ for all i. From the proof of Theorem 2.4, it follows that each of the sub-modules

$\mathbb{P}^{ii}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ is locally free of rank $\chi_{\boldsymbol{p}}(1)$ on $\mathbb{G}_n \setminus \boldsymbol{s}(\mathscr{Z})$. The rank being an invariant for locally free Hilbert modules, the proof of (a) is complete. The proof of (b) follows from Corollary 2.15. ∎

The theorem leaves open the question of equivalence when $\chi_{\boldsymbol{p}}(1) = \chi_{\boldsymbol{q}}(1)$. While we are not able to settle this question in its entirety, we answer it in the important special case of $\chi_{\boldsymbol{p}}(1) = 1 = \chi_{\boldsymbol{q}}(1)$, or equivalently, $\boldsymbol{p} = (n)$ and $\boldsymbol{q} = (1, \ldots, 1)$ since one dimensional representations of $\mathfrak{S}_n$ are the trivial and the sign representation.

We begin by setting up some notation which will be useful in the discussion to follow. The length $\ell(\boldsymbol{p})$ of a partition $\boldsymbol{p}$ of n is the number of positive summands of $\boldsymbol{p}$. For a positive integer n, we define the following two subsets of $\mathbb{Z}^n_+ := \{(m_1, \ldots, m_n) \in \mathbb{Z}^n : m_1, \ldots, m_n \geq 0\}$:

$$[n] = \{\boldsymbol{m} \in \mathbb{Z}^n_+ : m_i \geq m_j \text{ for } i < j\} \text{ and } [\![n]\!] = \{\boldsymbol{m} \in \mathbb{Z}^n_+ : m_i > m_j \text{ for } i < j\}.$$

If $\boldsymbol{p} \in [\![n]\!]$, then we can write $\boldsymbol{p} = \boldsymbol{m} + \boldsymbol{\delta}$, where $\boldsymbol{m} \in [n]$ and $\boldsymbol{\delta} = (n-1, n-2, \ldots, 1, 0)$. So,

$$[\![n]\!] = \{\boldsymbol{m} + \boldsymbol{\delta} : \boldsymbol{m} \in [n]\}.$$

Recall from equation (3.3) that for a partition $\boldsymbol{p}$ of n, the linear map $\mathbb{P}_{\boldsymbol{p}} : \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \to \mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ by

$$\mathbb{P}_{\boldsymbol{p}} f = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi_{\boldsymbol{p}}(\sigma)} f \circ \sigma^{-1}, \tag{4.1}$$

where $\chi_{\boldsymbol{p}}$ is the character of the representation corresponding to the partition $\boldsymbol{p}$ of n. Choosing the partition $\boldsymbol{p}$ of n to be $(n) := (n, 0, \ldots, 0)$ in Equation (4.1), it is easy to see that

$$\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) = \{f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n) : f \circ \sigma^{-1} = f \text{ for } \sigma \in \mathfrak{S}_n\},$$

that is, $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ consists of symmetric functions in $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. Thus $\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) = \mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. In view of [6, Equation (3.1)], the following proposition is a particular case of [6, Proposition 3.6] for $\boldsymbol{p} = (n)$.

Proposition 4.2. *The reproducing kernel $K^{(\lambda)}_{\text{sym}}$ of $\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n)$ is given explicitly by the formula:*

$$K^{(\lambda)}_{\text{sym}}(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{n!}\text{per}\Big(\!\big(\!\big((1 - z_j \bar{w}_k)^{-\lambda}\big)\!\big)^n_{j,k=1}\Big), \ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n,$$

where $\text{per}\Big(\!\big(\!\big(a_{ij}\big)\!\big)^n_{i,j=1}\Big) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{k=1}^n a_{k\sigma(k)}$.

The Hilbert space $\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n)$ can be thought of as a space of functions defined on the symmetrized polydisc $\mathbb{G}_n$ as follows. Recall that $\boldsymbol{s}$ is the symmetrization map and note that

$$\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) = \{f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n) : f = g \circ \boldsymbol{s} \text{ for some } g : \mathbb{G}_n \longrightarrow \mathbb{C} \text{ holomorphic } \}.$$

Let

$$\mathcal{H}^{(\lambda)}(\mathbb{G}_n) := \{g : \mathbb{G}_n \longrightarrow \mathbb{C} \text{ is holomorphic } : g \circ \boldsymbol{s} \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n)\}.$$

The inner product on $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ is given by $\langle f_1, f_2 \rangle_{\mathcal{H}^{(\lambda)}(\mathbb{G}_n)} := \langle f_1 \circ s, f_2 \circ s \rangle_{\mathbb{A}^{(\lambda)}(\mathbb{D}^n)}$. Now, the following corollary is immediate from Proposition 4.2.

Corollary 4.3. *The reproducing kernel $K^{(\lambda)}_{\mathbb{G}_n}$ of $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ is given explicitly by the formula:*

$$K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big) = \frac{1}{n!}\text{per}\Big(\!\big(\!\big((1 - z_j \bar{w}_k)^{-\lambda}\big)\!\big)^n_{j,k=1}\Big), \ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

Choosing the partition $\boldsymbol{p}$ of n to be $(1^n) := (1, \ldots, 1) \in [n]$, we see that

$$\mathbb{P}_{(1^n)}\big(\mathbb{A}^{(\mu)}(\mathbb{D}^n)\big) = \{f \in \mathbb{A}^{(\mu)}(\mathbb{D}^n) : f \circ \sigma^{-1} = \text{sgn}(\sigma) f \text{ for } \sigma \in \mathfrak{S}_n\}.$$

Since $\mathbb{P}_{(1^n)}\big(\mathbb{A}^{(\mu)}(\mathbb{D}^n)\big)$ consists of anti-symmetric functions, therefore $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n) = \mathbb{P}_{(1^n)}\big(\mathbb{A}^{(\mu)}(\mathbb{D}^n)\big)$. Appealing to [6, Proposition 3.8] for $\boldsymbol{p} = (n)$ and $\boldsymbol{p} = (1^n)$, we have a particular case of [6, Proposition 3.8], which we record below for future reference.

Lemma 4.4. *The Hilbert spaces $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ and $\mathbb{A}^{(\mu)}_{\rm anti}(\mathbb{D}^n)$ are Hilbert modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, under its natural action for $\lambda, \mu > 0$ and $n \geq 2$.*

The theorem below provides an affirmative answer to the question we raised in the beginning of this section.

Theorem 4.5. *The Hilbert modules $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ and $\mathbb{A}^{(\lambda)}_{\rm anti}(\mathbb{D}^n)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ are not equivalent for any $\lambda > 0$ and $n \geq 2$.*

We recall that $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n} = \mathbb{C}[s_1, \ldots, s_n]$. In view of this fact $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ is a Hilbert module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, under the natural action of $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. Consider the map from $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ to $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ defined by $f \mapsto f \circ \boldsymbol{s}$ and note that it is a unitary map which intertwines the n-tuple $(M_{s_1}, M_{s_2}, \ldots, M_{s_n})$ of multiplication operators by the coordinate functions $s_1, \ldots, s_n$ and the tuple $(M_{s_1(\boldsymbol{z})}, M_{s_2(\boldsymbol{z})}, \ldots, M_{s_n(\boldsymbol{z})})$, where $s_i(\boldsymbol{z})$ is the i-th elementary symmetric function in $z_1, \ldots, z_n$ for $i = 1, \ldots, n$. Therefore, there is a unitary module map between the Hilbert modules $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. We record this observation in the form of a lemma.

Lemma 4.6. *For $\lambda > 0$, the Hilbert modules $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ are equivalent as modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$.*

Now we describe the weighted Bergman space on the symmetrized polydisc $\mathbb{G}_n$ as a module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. For $\mu > 1$, let $dV^{(\mu)}$ be the probability measure $\left(\frac{\mu-1}{\pi}\right)^n \left(\prod_{i=1}^n (1 - r_i^2)^{\mu-2} r_i dr_i d\theta_i\right)$ on the polydisc $\mathbb{D}^n$. Let $dV^{(\mu)}_{\boldsymbol{s}}$ be the measure on the symmetrized polydisc $\mathbb{G}_n$ obtained by the change of variables formula [5, p. 106]:

$$\int_{\mathbb{G}_n} f dV^{(\mu)}_{\boldsymbol{s}} = \frac{1}{n!} \int_{\mathbb{D}^n} (f \circ \boldsymbol{s}) |J_{\boldsymbol{s}}|^2 dV^{(\mu)}, \ \mu > 1, \tag{4.2}$$

where $J_{\boldsymbol{s}}(\boldsymbol{z}) = \Delta(\boldsymbol{z})$ is the complex jacobian of the symmetrization map $\boldsymbol{s}$. The weighted Bergman space $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$, $\mu > 1$, on the symmetrized polydisc $\mathbb{G}_n$ is the subspace of $L^2(\mathbb{G}_n, dV^{(\mu)}_{\boldsymbol{s}})$ consisting of holomorphic functions. For $\mu > 1$, consider the map $\Gamma : \mathbb{A}^{(\mu)}(\mathbb{G}_n) \to \mathbb{A}^{(\mu)}(\mathbb{D}^n)$ defined by

$$\Gamma f = \frac{1}{\sqrt{n!}} J_{\boldsymbol{s}}(f \circ \boldsymbol{s}), \ f \in \mathbb{A}^{(\mu)}(\mathbb{G}_n). \tag{4.3}$$

It follows from Equation (4.2) that Γ is an isometry onto $\mathbb{A}^{(\mu)}_{\rm anti}(\mathbb{D}^n)$ [22, p. 2363]. One can easily check that $\|\boldsymbol{z}^{\boldsymbol{m}}\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \left\|z_1^{m_1} \ldots z_n^{m_n}\right\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \frac{m_1! \ldots m_n!}{(\mu)_{m_1} \cdots (\mu)_{m_n}}$. For a partition $\boldsymbol{m} = (m_1, \ldots, m_n) \in [\![n]\!]$, put $a_{\boldsymbol{m}}(\boldsymbol{z}) = a_{\boldsymbol{p}+\boldsymbol{\delta}}(\boldsymbol{z}) = \det\left(\!\left(z_i^{m_j}\right)\!\right)_{i,j=1}^n$, where $\boldsymbol{p} \in [n]$ and $\boldsymbol{m} = \boldsymbol{p} + \boldsymbol{\delta}$. The norm of $a_{\boldsymbol{m}}$ in $\mathbb{A}^{(\mu)}(\mathbb{D}^n)$ is easily calculated using orthogonality of distinct monomials in $\mathbb{A}^{(\mu)}(\mathbb{D}^n)$:

$$\|a_{\boldsymbol{m}}\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \Big\| \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{k=1}^n z_k^{m_{\sigma(k)}} \Big\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \sum_{\sigma \in \mathfrak{S}_n} \Big\| \prod_{k=1}^n z_k^{m_{\sigma(k)}} \Big\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \frac{n! \boldsymbol{m}!}{(\mu)_{\boldsymbol{m}}},$$

where $\boldsymbol{m}! = \prod_{j=1}^n m_j!$ and $(\mu)_{\boldsymbol{m}} = \prod_{j=1}^n (\mu)_{m_j}$. Here $(\mu)_{m_j}$ is the Pochhammer symbol $(\mu)_{m_j} = \mu(\mu+1) \ldots (\mu + m_j - 1)$. Putting $c_{\boldsymbol{m}} = \sqrt{\frac{(\mu)_{\boldsymbol{m}}}{n! \boldsymbol{m}!}}$, it follows from [22, p. 2364] that

$$\{e_{\boldsymbol{m}} = c_{\boldsymbol{m}} a_{\boldsymbol{m}} : \boldsymbol{m} \in [\![n]\!]\}$$

is an orthonormal basis of $\mathbb{A}^{(\mu)}_{\rm anti}(\mathbb{D}^n)$.

The determinant function $a_{\boldsymbol{p}+\boldsymbol{\delta}}$ is a polynomial and is divisible by each of the differences $z_i - z_j, 1 \leq i < j \leq n$ and hence by the product

$$\prod_{1 \leq i < j \leq n}^{n} (z_i - z_j) = \det\left(\!\left(z_i^{n-j}\right)\!\right)_{i,j=1}^n = a_{\boldsymbol{\delta}}(\boldsymbol{z}) = \Delta(\boldsymbol{z}).$$

For $\boldsymbol{p} \in [n]$, the quotient $S_{\boldsymbol{p}} := \frac{a_{\boldsymbol{p}+\boldsymbol{\delta}}}{a_{\boldsymbol{\delta}}}$, is therefore well-defined and is called the Schur polynomial [15, p. 454]. For $\boldsymbol{p} \in [n]$ and $\boldsymbol{m} = \boldsymbol{p} + \boldsymbol{\delta}$, recall that $c_{\boldsymbol{m}} = c_{\boldsymbol{p}+\boldsymbol{\delta}} = \sqrt{\frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{n!(\boldsymbol{p}+\boldsymbol{\delta})!}}$, now it follows from Equation (4.3) that

$$\Gamma\left(\sqrt{\frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{(\boldsymbol{p}+\boldsymbol{\delta})!}} S_{\boldsymbol{p}}\right) = \Gamma\left(\sqrt{n!} c_{\boldsymbol{p}+\boldsymbol{\delta}} S_{\boldsymbol{p}}\right) = c_{\boldsymbol{p}+\boldsymbol{\delta}} a_{\boldsymbol{p}+\boldsymbol{\delta}} = c_{\boldsymbol{m}} a_{\boldsymbol{m}},\ \boldsymbol{m} \in [\![n]\!].$$

Since the map $\Gamma : \mathbb{A}^{(\mu)}(\mathbb{G}_n) \to \mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ defined by Equation (4.3) is a unitary [22, p. 2363], the set

$$\{\gamma_{\boldsymbol{p}} S_{\boldsymbol{p}} : \boldsymbol{p} \in [n]\},\ \text{where } \gamma_{\boldsymbol{p}} = \sqrt{\frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{(\boldsymbol{p}+\boldsymbol{\delta})!}},$$

is an orthonormal basis for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$. Hence we have the following proposition,

Proposition 4.7. *The reproducing kernel $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}$ for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ is given by*

$$\mathbf{B}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big) = \sum_{\boldsymbol{p}\in[n]} \gamma^2_{\boldsymbol{p}} S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n, \mu > 1. \tag{4.4}$$

From [22, p. 2363], it follows that $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ and the weighted Bergman module $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ are unitarily equivalent as modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ for $\mu > 1$. The limiting case $\mu = 1$, is discussed in [22, p. 2367]. It is not difficult to show that the function $\mathbf{B}^{(\mu)}_{\mathbb{G}_n} : \mathbb{G}_n \times \mathbb{G}_n \to \mathbb{C}$, defined by the Equation (4.4), is positive definite for $\mu > 0$. For $0 < \mu < 1$, let $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ be the Hilbert space of holomorphic functions having $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}$ as its reproducing kernel. If we assume that the set $\{S_{\boldsymbol{p}}\}_{\boldsymbol{p}\in[n]}$ is orthogonal in $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ and $\|S_{\boldsymbol{p}}\|^2 = \frac{(\boldsymbol{p}+\boldsymbol{\delta})!}{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}$, then it is easy to verify that the injective linear map $\Gamma : \mathbb{A}^{(\mu)}(\mathbb{G}_n) \to \mathbb{A}^{(\mu)}(\mathbb{D}^n)$ defined in Equation (4.3) is an isometry. By similar arguments as in the case $\mu > 1$, we reach the desired conclusion for $0 < \mu < 1$ as well. This observation is recorded in the following Lemma.

Lemma 4.8. *For $\mu > 0$, the Hilbert modules $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ are equivalent, as modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$.*

In view of Lemma 4.6 and Lemma 4.8, proving Theorem 4.5 boils down to proving the following theorem.

Theorem 4.9. *The Hilbert modules $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ and $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ are not equivalent for any $\lambda > 0$ and $n \geq 2$.*

To prove this theorem, we recall the notion of a normalized kernel from [11]. Let $\Omega \subseteq \mathbb{C}^n$ be domain. A kernel function $K : \Omega \times \Omega \to \mathbb{C}$ is said to be normalized at $w_0 \in \Omega$ if $K(z, w_0) = 1$ for $z \in \Omega_0$, where $\Omega_0 \subseteq \Omega$, is a neighborhood of w_0. We note that $S_{\boldsymbol{p}}$ is a homogeneous symmetric polynomial of degree $|\boldsymbol{p}| := \sum_{i=1}^n p_i$, so $S_{\boldsymbol{0}} \equiv 1$ and $S_{\boldsymbol{p}}(\boldsymbol{0}) = 0$ for $\boldsymbol{p} \neq \boldsymbol{0}$, where $\boldsymbol{0} \in [n]$ with all components equal to 0. From Equation (4.4) and the discussion following Proposition 4.7, we see that $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{0}\big) = \gamma^2_{\boldsymbol{0}} = \frac{(\mu)_{\boldsymbol{\delta}}}{\boldsymbol{\delta}!}$ for $\boldsymbol{z} \in \mathbb{D}^n$ and $\mu > 0$. We record the following obvious corollary of Proposition 4.7 for future reference.

Corollary 4.10. *The normalized reproducing kernel $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}$ for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ is given by*

$$\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big) = \frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}} \sum_{\boldsymbol{p}\in[n]} \gamma^2_{\boldsymbol{p}} S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n, \mu > 0. \tag{4.5}$$

It is of independent interest to express the reproducing kernel $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}$ in terms of coordinates of $\mathbb{G}_n$, that is, in terms of elementary symmetric polynomials. In order to do that, we need to introduce some terminologies. To a partition $\boldsymbol{p} = (p_1, \ldots, p_n) \in [n]$ is associated a *Young diagram* [15, Section 4.1] with p_i boxes in the i-th row, the rows of boxes lined up on the left. The *conjugate partition* $\boldsymbol{p}' = (p'_1, \ldots, p'_r)$

to the partition $\boldsymbol{p}$ is defined by interchanging rows and columns in the Young diagram, that is, reflecting the diagram in the 45° line. For example, the conjugate partition to the partition $(3,3,2,1,1)$ is $(5,3,2)$. For the conjugate partition $\boldsymbol{p}'=(p_1',\ldots,p_r')$ to $\boldsymbol{p}$, let us require that $p_r'>0$ and call r the length of $\boldsymbol{p}'$. Let us agree to call s_k the k-th elementary symmetric polynomial in n variables for $k=0,1,\ldots,$ with the convention that $s_k\equiv 0$ if $k>n$. We are now ready to state the second of *Giambelli's formulas* expressing the Schur polynomials as functions of elementary symmetric polynomials. Here is Giambelli's second formula [15, p. 455]:

$$S_{\boldsymbol{p}}=\det\Big(\big(\!\big(s_{p_i'+j-i}\big)\!\big)_{i,j=1}^{r}\Big),\ \boldsymbol{p}\in[n], \tag{4.6}$$

where $\boldsymbol{p}'=(p_1',\ldots,p_r')$ is the conjugate partition to $\boldsymbol{p}$.

Combining Corollary 4.10 with the Equation (4.6), we obtain the following theorem.

Theorem 4.11. *The normalized reproducing kernel $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}$ for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ is given by*

$$\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}(\boldsymbol{s},\boldsymbol{t})=\frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}}\sum_{\boldsymbol{p}\in[n]}\gamma_{\boldsymbol{p}}^2\det\Big(\big(\!\big(s_{p_i'+j-i}\big)\!\big)_{i,j=1}^{r}\Big)\overline{\det\Big(\big(\!\big(t_{p_i'+j-i}\big)\!\big)_{i,j=1}^{r}\Big)},$$

for $\boldsymbol{s}=(s_1,\ldots,s_n),\boldsymbol{t}=(t_1,\ldots,t_n)\in\mathbb{G}_n,\mu>0$ and $\boldsymbol{p}'=(p_1',\ldots,p_r')$ is the conjugate partition to $\boldsymbol{p}\in[n]$.

Lemma 4.12. *Let $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}$ be the normalized reproducing kernel for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$. Then*

(i) *the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ is $\frac{\mu+n-1}{n}$,*

(ii) *the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ is $\frac{(\mu+n-1)(\mu+n)}{n(n+1)}$.*

Proof. Since the Schur polynomial $S_{\boldsymbol{p}}$ is a homogeneous symmetric polynomial of degree $|\boldsymbol{p}|:=\sum_{i=1}^n p_i$, therefore, it is a polynomial in the elementary symmetric polynomials $s_i(\boldsymbol{z})$ for $i=1,\ldots,n$. For a fixed $k,q\in\mathbb{Z}_+$, the term $s_k(\boldsymbol{z})^q\overline{s_k(\boldsymbol{w})^q}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ comes only from the terms which involves $S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})}$ in the series for $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ in Equation (4.5), where $\boldsymbol{p}=(p_1,\ldots,p_n)\in[n]$ such that $\sum_{i=1}^n p_i=kq$.

To get the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$, take $\boldsymbol{p}=(1,0,...,0)$. From Equation (4.5), the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ is

$$\frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}}\gamma_{\boldsymbol{p}}^2=\frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}}\cdot\frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{(\boldsymbol{p}+\boldsymbol{\delta})!}=\frac{\mu+n-1}{n},$$

where $\boldsymbol{p}=(1,0,\ldots,0)$. This proves (i).

Similarly, to obtain the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbf{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$, we need to consider terms corresponding to $\boldsymbol{p}=(2,0,\ldots,0)$ and $\boldsymbol{p}=(1,1,0,\ldots,0)$. From the Giambelli's formula (4.6), we get $S_{(2,0,\ldots,0,0)}(\boldsymbol{z})=({s_1}^2-s_2)(\boldsymbol{z})$ and $S_{(1,1,\ldots,0,0)}(\boldsymbol{z})=s_2(\boldsymbol{z})$.

Since s_1^2 appears only in $S_{(2,0,\ldots,0)}$, from Equation (4.5), it follows that the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ is

$$\frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}}\gamma_{\boldsymbol{p}}^2=\frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}}\cdot\frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{(\boldsymbol{p}+\boldsymbol{\delta})!}=\frac{(\mu+n-1)(\mu+n)}{n(n+1)},$$

where $\boldsymbol{p}=(2,0,\ldots,0)$. This proves (ii). ∎

Consider the restriction of the action of $\mathfrak{S}_n$ to $\mathbb{Z}_+^n$. Let $\mathfrak{S}_n\boldsymbol{m}$ denote the orbit of $\boldsymbol{m}\in\mathbb{Z}_+^n$. If $\boldsymbol{m}\in[n]$ has $k(\leq n)$ distinct components, that is, there are k distinct non-negative integers $m_1>\ldots>m_k$ such that

$$\boldsymbol{m}=(m_1,\ldots,m_1,m_2,\ldots,m_2,\ldots,m_k,\ldots,m_k),$$

where each m_i is repeated α_i times, for $i = 1, \ldots, k$, then $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is said to be the *multiplicity* of $\boldsymbol{m} \in [n]$. For any $\boldsymbol{m} \in \mathbb{Z}_+^n$ the components of $\boldsymbol{m}$ can be arranged in the decreasing order to obtain, say, $\widetilde{\boldsymbol{m}} \in [n]$. We say that $\boldsymbol{m} \in \mathbb{Z}_+^n$ is of *multiplicity* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ if $\widetilde{\boldsymbol{m}}$ has multiplicity $\boldsymbol{\alpha}$. In particular, the elements of $[\![n]\!]$ are of multiplicity (1^n), that is, 1 occurs n-times.

We recall that the number of distinct n-letter words with k distinct letters is $\frac{n!}{\boldsymbol{\alpha}!} = \frac{n!}{\alpha_1! \ldots \alpha_k!}$, where the k distinct letters $a_1, \ldots, a_k$ are repeated $\alpha_1, \ldots, \alpha_k$ times, respectively $(\alpha_1 + \ldots + \alpha_k = n)$. In other words, for a fixed $\boldsymbol{m} \in \mathbb{Z}_+^n$, we have $|\mathfrak{S}_n \boldsymbol{m}| = \frac{n!}{\boldsymbol{\alpha}!}$, where $|X|$ denotes the cardinality of a set X. Let $\mathbb{Z}_+^n / \mathfrak{S}_n$ denote the set of all orbits of $\mathbb{Z}_+^n$ under the action of $\mathfrak{S}_n$. We record the following as a lemma for later use.

Lemma 4.13. *The set $\mathbb{Z}_+^n / \mathfrak{S}_n$ is in one-one correspondence with the set $[n]$.*

Proof. First, we prove that each $\mathfrak{S}_n$ orbit of $\mathbb{Z}_+^n$ has exactly one n-tuple in decreasing order. To see this, observe that each orbit contains an n-tuple in decreasing order, and hence enough to prove it is unique. Suppose there are two n-tuples in decreasing order, say $\boldsymbol{m}, \boldsymbol{m}'$, in the same orbit. Since a permutation only changes the position of a component, it follows that all n-tuples in an orbit have the same multiplicity. Therefore the multiplicity of $\boldsymbol{m}$ and $\boldsymbol{m}'$ is the same and hence $\boldsymbol{m} = \boldsymbol{m}'$. Note that each element in $[n]$ is in some orbit and hence the proof is complete. ∎

Consider the monomial symmetric polynomials [15, p. 454]

$$\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = \sum_{\boldsymbol{\beta}} \boldsymbol{z}^{\boldsymbol{\beta}},$$

where the sum is over all distinct permutations $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ of $\boldsymbol{m} \in [n]$ and $\boldsymbol{z}^{\boldsymbol{\beta}} = z_1^{\beta_1} z_2^{\beta_2} ... z_n^{\beta_n}$. This definition of $\mathbb{M}_{\boldsymbol{m}}$ makes sense for $\boldsymbol{m} \in \mathbb{Z}_+^n$ as well and we use it in the sequel. Observe that $\mathfrak{S}_n \boldsymbol{m}$ is the set of all distinct permutations of $\boldsymbol{m}$, so,

$$\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = \sum_{\boldsymbol{\beta} \in \mathfrak{S}_n \boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{\beta}} = \mathbb{M}_{\boldsymbol{m}'}(\boldsymbol{z}) \text{ for } \boldsymbol{m}, \boldsymbol{m}' \in \mathfrak{S}_n \boldsymbol{m}. \tag{4.7}$$

The following lemma that gives us an expression for the reproducing kernel $K_{\mathbb{G}_n}^{(\lambda)}$ for $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ will play a significant role in the sequel.

Lemma 4.14. *The reproducing kernel $K_{\mathbb{G}_n}^{(\lambda)}$ for $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ is given by the formula:*

$$K_{\mathbb{G}_n}^{(\lambda)}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big) = \frac{1}{n!} \sum_{\boldsymbol{m} \in [n]} \frac{\boldsymbol{\alpha}! (\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} \mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) \overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}, \ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n,$$

where $\boldsymbol{m}$ is of multiplicity $\boldsymbol{\alpha}$.

Proof. If $\boldsymbol{m} \in [n]$ is of multiplicity $\boldsymbol{\alpha}$, then $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})$ is the sum of $|\mathfrak{S}_n \boldsymbol{m}| = \frac{n!}{\boldsymbol{\alpha}!}$ distinct monomials. We then observe that

$$\operatorname{per}\Big((\!(z_i^{m_j})\!)_{i,j=1}^n \Big) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n z_i^{m_{\sigma(i)}} = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n z_i^{m_{\sigma^{-1}(i)}} = \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{z}^{\sigma \cdot \boldsymbol{m}}, \tag{4.8}$$

is the sum of $n!$ monomials, from which exactly $|\mathfrak{S}_n \boldsymbol{m}| = \frac{n!}{\boldsymbol{\alpha}!}$ are distinct (since there can be only $\frac{n!}{\boldsymbol{\alpha}!}$ distinct permutations of a $\boldsymbol{m} \in [n]$ with multiplicity $\boldsymbol{\alpha}$). So, each distinct term must be repeated $\boldsymbol{\alpha}!$ times. Thus, from equation (4.7) we conclude that

$$\operatorname{per}\Big((\!(z_i^{m_j})\!)_{i,j=1}^n \Big) = \boldsymbol{\alpha}! \mathbb{M}_{\boldsymbol{m}'}(\boldsymbol{z}), \quad \text{for any } \boldsymbol{m}' \in \mathfrak{S}_n \boldsymbol{m}. \tag{4.9}$$

Since $\mathbb{Z}_+^n$ is the disjoint union of its $\mathfrak{S}_n$-orbits, from Lemma 4.13 we have

$$\mathbb{Z}_+^n = \cup_{\boldsymbol{m} \in [n]} \mathfrak{S}_n \boldsymbol{m}. \tag{4.10}$$

Therefore, from Corollary 4.3, we have

$$
\begin{aligned}
K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big) &= \frac{1}{n!}\operatorname{per}\Big(\big(\!\big((1-z_j\bar{w}_k)^{-\lambda}\big)\!\big)_{j,k=1}^{n}\Big)\\
&= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n}(1-z_i\bar{w}_{\sigma(i)})^{-\lambda}\\
&= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\boldsymbol{m}\in\mathbb{Z}^n_+}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\prod_{i=1}^{n}z_i^{m_i}\prod_{i=1}^{n}\bar{w}_{\sigma(i)}^{m_i}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in\mathbb{Z}^n_+}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\prod_{i=1}^{n}z_i^{m_i}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n}\bar{w}_{\sigma(i)}^{m_i}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in\mathbb{Z}^n_+}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\prod_{i=1}^{n}z_i^{m_i}\operatorname{per}\Big(\big(\!\big(\bar{w}_i^{m_j}\big)\!\big)_{i,j=1}^{n}\Big)\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\sum_{\boldsymbol{m}'\in\mathfrak{S}_n\boldsymbol{m}}\frac{(\lambda)_{\boldsymbol{m}'}}{\boldsymbol{m}'!}\prod_{i=1}^{n}z_i^{m'_i}\operatorname{per}\Big(\big(\!\big(\bar{w}_i^{m'_j}\big)\!\big)_{i,j=1}^{n}\Big)\ \text{(using (4.10))}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\operatorname{per}\Big(\big(\!\big(\bar{w}_i^{m_j}\big)\!\big)_{i,j=1}^{n}\Big)\sum_{\boldsymbol{m}'\in\mathfrak{S}_n\boldsymbol{m}}\prod_{i=1}^{n}z_i^{m'_i}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\boldsymbol{\alpha}!\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}\sum_{\boldsymbol{m}'\in\mathfrak{S}_n\boldsymbol{m}}\boldsymbol{z}^{\boldsymbol{m}'}\ \text{(using (4.9))}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\frac{\boldsymbol{\alpha}!(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})},
\end{aligned}
$$

where the last equality follows from Equation (4.7). ∎

Remark 4.15. *One could also write the reproducing kernel in terms of permanent using the equations* (4.9) *and* (4.10) *and the equality* $|\mathfrak{S}_n\boldsymbol{m}|=\frac{n!}{\boldsymbol{\alpha}!}$, *as follows:*

$$
\begin{aligned}
K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big) &= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\sum_{\boldsymbol{m}'\in\mathfrak{S}_n\boldsymbol{m}}\Big(\frac{n!}{\boldsymbol{\alpha}!}\Big)^{-1}\frac{\boldsymbol{\alpha}!(\lambda)_{\boldsymbol{m}'}}{\boldsymbol{m}'!}\frac{1}{\boldsymbol{\alpha}!}\operatorname{per}\Big(\big(\!\big(z_i^{m'_j}\big)\!\big)_{i,j=1}^{n}\Big)\frac{1}{\boldsymbol{\alpha}!}\operatorname{per}\Big(\big(\!\big(\bar{w}_i^{m'_j}\big)\!\big)_{i,j=1}^{n}\Big)\\
&= \frac{1}{(n!)^2}\sum_{\boldsymbol{m}\in\mathbb{Z}^n_+}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\operatorname{per}\Big(\big(\!\big(z_i^{m_j}\big)\!\big)_{i,j=1}^{n}\Big)\operatorname{per}\Big(\big(\!\big(\bar{w}_i^{m_j}\big)\!\big)_{i,j=1}^{n}\Big),
\end{aligned}
$$

for $\boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n$.

We note that the kernels $B^{(\lambda)}_{\mathbb{G}_n}$ and $K^{(\lambda)}_{\mathbb{G}_n}$ are defined on all of $\mathbb{G}_n$. Hence the Hilbert modules $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are locally free on all of $\mathbb{G}_n$ strengthening our earlier assertion (Corollary 2.15) that they are locally free only on $\mathbb{G}_n\setminus\boldsymbol{s}(\mathcal{Z})$. Thus we have proved the following Corollary.

Corollary 4.16. *The Hilbert modules* $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ *and* $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ *are locally free of rank* 1 *on* $\mathbb{G}_n$.

Lemma 4.17. *Let* $K^{(\lambda)}_{\mathbb{G}_n}$ *be the normalized reproducing kernel for* $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$. *Then*

(i) *the coefficient of* $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ *in* $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ *is* $\frac{\lambda}{n}$,

(ii) *the coefficient of* $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ *in* $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ *is* $\frac{\lambda(\lambda+1)}{2n}$.

Proof. Since the monomial symmetric polynomial $\mathbb{M}_{\boldsymbol{m}}$ is a homogeneous symmetric polynomial of degree $|\boldsymbol{m}| := \sum_{i=1}^{n} m_i$, therefore, it is a polynomial in the elementary symmetric polynomials $s_i(\boldsymbol{z})$ for $i = 1, \ldots, n$. For a fixed $k, q \in \mathbb{Z}_+$, the term $s_k(\boldsymbol{z})^q \overline{s_k(\boldsymbol{w})^q}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ comes only from the terms involving $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}$ in the series for $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ in Lemma 4.14, where $\boldsymbol{m} = (m_1, \ldots, m_n) \in [n]$ such that $\sum_{i=1}^{n} m_i = kq$.

To obtain the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$, we only need to consider the term $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}$, for $\boldsymbol{m} = (1, 0, ..., 0)$. Note that $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = s_1(\boldsymbol{z})$. Since $\boldsymbol{m} = (1, 0, \ldots, 0)$ has multiplicity $\alpha = (1, (n-1))$, it follows that the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ is

$$\frac{1}{n!} \cdot \frac{\boldsymbol{\alpha}!(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} = \frac{(n-1)!1!(\lambda)_1}{1!n!} = \frac{\lambda}{n}.$$

This proves (i).

Analogously, to find the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$, we need to consider terms corresponding to $\boldsymbol{m} = (2, 0, ..., 0)$ and $\boldsymbol{m} = (1, 1, 0, ..., 0)$. Note that $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = s_2(\boldsymbol{z})$ for $\boldsymbol{m} = (1, 1, 0, \ldots, 0)$, so the coefficient of the term $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}$ for $\boldsymbol{m} = (1, 1, 0, \ldots, 0)$, will not contribute here. Now $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = s_1(\boldsymbol{z})^2 - 2s_2(\boldsymbol{z})$ for $\boldsymbol{m} = (2, 0, \ldots, 0)$. Since $\boldsymbol{m} = (2, 0, \ldots, 0)$ has multiplicity $\boldsymbol{\alpha} = (1, n-1)$, it follows that the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ is

$$\frac{1}{n!} \cdot \frac{\boldsymbol{\alpha}!(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} = \frac{(n-1)!(\lambda)_2}{2!n!} = \frac{\lambda(\lambda+1)}{2n}.$$

This proves (ii). ∎

Proof of Theorem 4.9. If possible, let these two modules be unitarily equivalent. Recall that the reproducing kernels $\widetilde{\mathbf{B}}^{(\lambda)}_{\mathbb{G}_n}$ and $K^{(\lambda)}_{\mathbb{G}_n}$ have the property that

$$\widetilde{\mathbf{B}}^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), 0\big) = K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), 0\big) = 1 \ \text{ for } \boldsymbol{s}(\boldsymbol{z}) \in \mathbb{G}_n,$$

that is, these are the normalized reproducing kernels at 0 of the respective Hilbert spaces. Since by construction, the polynomial ring $\mathbb{C}[s_1, \ldots, s_n] = \mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ in n variables is dense in both $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$, it follows (cf. [12, Remark, p. 285]) that the dimension of the joint kernel is 1 for all $\boldsymbol{w} \in \mathbb{G}_n$. Therefore, by [11, Lemma 4.8(c)], we infer that

$$\widetilde{\mathbf{B}}^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big) = K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big) \ \text{ for } \boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w}) \in \mathbb{G}_n.$$

Equating the coefficients of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{z})}$ from Lemma 4.12 we see that $\lambda = \lambda + n - 1$. Thus we must have $n = 1$ completing the proof of the Theorem. ∎

Corollary 4.18. *In the decomposition of the Hilbert module* $\mathbb{A}^{(\lambda)}(\mathbb{D}^3)$ *:*

$$\mathbb{A}^{(\lambda)}(\mathbb{D}^3) = \mathbb{P}_{(3)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big) \oplus \mathbb{P}_{(2,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big) \oplus \mathbb{P}_{(1,1,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big),$$

all the sub-modules on the right hand side of the equality are inequivalent.

Proof. We have just proved that $\mathbb{P}_{(3)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big)$ cannot be equivalent to $\mathbb{P}_{(1,1,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big)$, in general. Since the rank of the sub-module $\mathbb{P}_{(2,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big)$ is $\chi_{(2,1)}(1)^2 = 4$, [15, Example 2.6], it cannot be equivalent to either of these. ∎

Remark 4.19. *The proof of Theorem 4.5 shows that we have proved a little more than what is claimed in the Theorem, namely: The Hilbert modules* $\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n)$ *and* $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ *over* $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ *are not equivalent for any* $\lambda, \mu > 0$ *and* $n \geq 2$. *To prove this more general claim, we merely note, as before, that equating the coefficients of* $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{z})}$ *and* $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{z})^2}$ *from Lemma 4.12 and Lemma 4.17, we obtain*

$$\lambda = \mu + n - 1 \quad \text{and} \quad \frac{\lambda(\lambda+1)}{2n} = \frac{(\mu+n-1)(\mu+n)}{n(n+1)}.$$

Combining these equations, we have that $n = 1$, which proves our claim. Indeed, the two modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{\boldsymbol{q}}^{jj}\big(\mathbb{A}^{(\mu)}(\mathbb{D}^n)\big)$ are not equivalent either for any $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$ and $1 \leq j \leq \chi_{\boldsymbol{q}}(1)$ for which $\chi_{\boldsymbol{p}}(1) \neq \chi_{\boldsymbol{q}}(1)$.

In cases where $\chi_{\boldsymbol{p}}(1) > 1$, we believe, the work of [28] and [14], may be useful in answering the question of mutual equivalence of the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. We intend to explore this possibility in our future work.

Let $\mathcal{H}$ be a locally free Hilbert module over $\Omega \subseteq \mathbb{C}^n$. Following [28] and [14], we define a holomorphic section $\gamma : \Omega \to \mathcal{H}$ to be a *spanning holomorphic cross-section* for $\mathcal{H}$ if

$$\bigvee\{\gamma(z) : z \in \Omega\} = \mathcal{H}.$$

Building on the work in [28], the existence of a spanning holomorphic cross-section for a large class of Hilbert modules over an admissible set was proved in [14]. However, in the case of the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$, the existence of a spanning holomorphic cross-section is easily established by exhibiting such a section. Indeed, we give an explicit realization of the spanning holomorphic cross-section for these sub-modules.

Let U be an open neighbourhood of of $\boldsymbol{u}_0$ in $\big(\mathbb{G}_n \setminus \boldsymbol{s}(\mathcal{Z})\big) \cap \boldsymbol{s}\big(\mathbb{D}^n \setminus X\big)$. The function $\boldsymbol{s}$ admits $n!$ local inverses on the open set U. Fix one such, say ϕ. Define $\gamma(\boldsymbol{u}) = \mathbb{P}_{\boldsymbol{p}}^{ii} K^{(\lambda)}(\cdot, \phi(\bar{\boldsymbol{u}}))$, $\boldsymbol{u} \in U^*$. From Equation (3.6), it follows that γ is a spanning holomorphic cross-section for $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. Let $E_{\boldsymbol{p}}^{(i)} = \{(\boldsymbol{u}, x) \in U^* \times \mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) \mid x = c\gamma(\boldsymbol{u}) \text{ for some } c \in \mathbb{C}\}$ denote the corresponding holomorphic hermitian line bundle and

$$\mathscr{K}_{\boldsymbol{p}}^{(i)}(\boldsymbol{u}) = -\sum_{j,k=1}^{n} \partial_j \bar{\partial}_k \log \|\gamma(\boldsymbol{u})\|^2 du_j \wedge d\bar{u_k},$$

be the curvature of $E_{\boldsymbol{p}}^{(i)}$. Now, we restate Theorem 5.2 of [14] using the spanning cross-sections we have found here.

Theorem 4.20. *For any two partitions $\boldsymbol{p}$ and $\boldsymbol{q}$ of n and for i, j, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$, $1 \leq j \leq \chi_{\boldsymbol{q}}(1)$, the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{\boldsymbol{q}}^{jj}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are equivalent if and only if $\mathscr{K}_{\boldsymbol{p}}^{(i)} = \mathscr{K}_{\boldsymbol{q}}^{(j)}$.*

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(Biswas, Ghosh and Shyam Roy) Indian Institute of Science Education and Research Kolkata, Mohanpur 741246, Nadia, West Bengal, India

E-mail address, Biswas: shibananda@gmail.com

E-mail address, Ghosh: gg13ip034@iiserkol.ac.in

E-mail address, Shyam Roy: ssroy@iiserkol.ac.in

(Misra) Department of Mathematics, Indian Institute of Science, Banaglore 560012, India

E-mail address, Misra: gm@math.iisc.ernet.in