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The Varchenko Determinant of a Coxeter Arrangement

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Abstract

The Varchenko determinant is the determinant of a matrix defined from an arrangement of hyperplanes. Varchenko proved that this determinant has a beautiful factorization. It is, however, not possible to use this factorization to compute a Varchenko determinant from a certain level of complexity. Precisely at this point, we provide an explicit formula of this determinant for the hyperplane arrangements associated to the finite Coxeter groups. The intersections of hyperplanes with the chambers of such arrangements have nice properties which play a central role for the calculation of their relating determinants.

Keywords: Varchenko Determinant, Coxeter Group, Hyperplane Arrangement

MSC Number: 05E15, 20C05

1 Introduction

Let $x = (x_1, \dots, x_n)$ be a variable of the Euclidean space $\mathbb{R}^n$, and $a_1, \dots, a_n, b$ real coefficients such that $(a_1, \dots, a_n) \neq (0, \dots, 0)$. A hyperplane H of $\mathbb{R}^n$ is a $(n-1)$–dimensional affine subspace $H := \{x \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = b\}$. An arrangement of hyperplanes in $\mathbb{R}^n$ is a finite set of hyperplanes. For example, the most known hyperplane arrangement is certainly $\mathcal{A}_{A_{n-1}} = \big\{\{x \in \mathbb{R}^n \mid x_i - x_j = 0\}\big\}_{1 \leq i < j \leq n}$ associated to the Coxeter group A_{n-1}.

A *chamber* of a hyperplane arrangement $\mathcal{A}$ is a connected component of the complement $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$. Denote the set of all chambers of $\mathcal{A}$ by $\mathfrak{C}(\mathcal{A})$.

Assign a variable a_H to each hyperplane H of an arrangement $\mathcal{A}$. Let $R_{\mathcal{A}} = \mathbb{Z}[a_H \mid H \in \mathcal{A}]$ be the ring of polynomials in variables a_H. The module of $R_{\mathcal{A}}$-linear combinations of chambers of the hyperplane arrangement $\mathcal{A}$ is

$$M_{\mathcal{A}} := \{ \sum_{C \in \mathfrak{C}(\mathcal{A})} x_C C \mid x_C \in R_{\mathcal{A}} \}.$$

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Let $\mathcal{H}(C, D)$ be the set of hyperplanes separating the chambers C and D in $\mathfrak{C}(\mathcal{A})$. The $R_{\mathcal{A}}$-bilinear symmetric form $\mathsf{B} : M_{\mathcal{A}} \times M_{\mathcal{A}} \to R_{\mathcal{A}}$ on the hyperplane arrangement $\mathcal{A}$ defined by Varchenko [11] is

$$\mathsf{B}(C, C) := 1, \text{ and } \mathsf{B}(C, D) := \prod_{H \in \mathcal{H}(C,D)} a_H \text{ if } C \neq D.$$

The Varchenko matrix of the hyperplane arrangement $\mathcal{A}$ is the matrix $\big(\mathsf{B}(C, D)\big)_{C,D \in \mathfrak{C}(\mathcal{A})}$ associated to the bilinear symmetric form B. In terms of Markov chains, it is the matrix of random walks on $\mathfrak{C}(\mathcal{A})$ whose walk probability from the chamber C to the chamber D is equal to $\mathsf{B}(C, D)$. The *Varchenko determinant* of the hyperplane arrangement $\mathcal{A}$ is the determinant

$$\det \mathcal{A} := \det \big(\mathsf{B}(C, D)\big)_{C,D \in \mathfrak{C}(\mathcal{A})}.$$

One of the first appearances of this bilinear form was in the work of Schechtman and Varchenko [10, 1. Quantum groups], in the implicit form of a symmetric bilinear form on a Verma module over a $\mathbb{C}$-algebra. It appeared more explicitly later, one year before its publication, however as a very special case, when Zagier studied a Hilbert space $\mathbb{H}$ together with a nonzero distinguished vector $|0\rangle$, and a collection of operators $a_k : \mathbb{H} \to \mathbb{H}$ satisfying the commutation relations $a(l)a^\dagger(k) - qa^\dagger(k)a(l) = \delta_{k,l}$, and the relation $a(k)|0\rangle = 0$. To demonstrate the realizability of its model, he defined an inner product space $\big(\mathbb{H}(q), \langle \cdot, \cdot \rangle\big)$ with basis B consisting of n-particle states $a^\dagger(k_1) \dots a^\dagger(k_n)|0\rangle$, and proved that [12, Theorem 1, 2]

$$\det \big(\langle u, v \rangle\big)_{u,v \in B} = \prod_{k=1}^{n-1} (1 - q^{k^2+k})^{\frac{n!(n-k)}{k^2+k}}.$$

It is the Varchenko determinant of $\mathcal{A}_{A_{n-1}}$ with all hyperplanes weighted by q. Using the diagonal solutions of the Yang-Baxter equation, Duchamp et al. computed [5, 6.4.2 A Decomposition of B_n]

$$\det \mathcal{A}_{A_{n-1}} = \prod_{\substack{I \in 2^{[n]} \\ |I| \geq 2}} \Big(1 - \prod_{\{i,j\} \in \binom{I}{2}} a_{H_{i,j}}^2\Big)^{(|I|-2)!\,(n-|I|+1)!}.$$

each hyperplane $\{x \in \mathbb{R}^n \mid x_i - x_j = 0\}\}$ having its own weight $a_{H_{i,j}}$ this time.
An *edge* of a hyperplane arrangement $\mathcal{A}$ is a nonempty intersection of some of its hyperplanes. Denote the set of all edges of $\mathcal{A}$ by $L(\mathcal{A})$. The *weight* $\mathsf{a}(E)$ of an edge E is

$$\mathsf{a}(E) := \prod_{\substack{H \in \mathcal{A} \\ E \subseteq H}} a_H.$$

The *multiplicity* $l(E)$ of an edge E is a positive integer computed as follows [4, 2. The Nullspace of the B Matrices]: First choose a hyperplane H of $\mathcal{A}$ containing E. Then $l(E)$ is half the number of chambers C of $\mathfrak{C}(\mathcal{A})$ which have the property that E is the minimal intersection containing $\overline{C} \cap H$.
The formula of the determinant of a hyperplane arrangement $\mathcal{A}$ [11, (1.1) Theorem] due to Varchenko is

$$\det \mathcal{A} = \prod_{E \in L(\mathcal{A})} \big(1 - \mathsf{a}(E)^2\big)^{l(E)}.$$

From this formula, we see that we can get a more explicit or computable value of the determinant of an arrangement if we have computable forms of the $\mathsf{a}(E)$'s and the $l(E)$'s. In this article, we prove that it is the case for the arrangements associated to finite Coxeter groups. Recall that a reflection in $\mathbb{R}^n$ is a linear map sending a nonzero vector α to its negative while fixing pointwise the hyperplane H orthogonal to α. The finite reflection groups are also called finite Coxeter groups, since they have been classified by Coxeter [3]. Coxeter groups find applications in pratically all mathematics areas. They are particularly studied in great depth in algebra [6], combinatorics [2], and geometry [1]. And they are the foundation ingredients of mathematical theories like the descent algebras, the Hecke algebras, or the Kazhdan–Lusztig polynomials. A finite Coxeter group W has the presentation

$$W := \langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle \quad \text{with} \quad m_{ii} = 1,\, m_{ij} \geq 2,\, m_{ij} = m_{ji}.$$

We begin with some definitions relating to the finite Coxeter group W.
The elements of the set $S := \{s_1, s_2, \ldots, s_n\}$ are called the simple reflections of W. The set of all reflections of W is denoted by $T := \{s_i^x \mid s_i \in S,\, x \in W\}$.
As usually, for a subset J of S, W_J is the parabolic subgroup $\langle J \rangle$ of W, T_J the set of reflections $T \cap W_J$ of W_J, X_J the set of coset representatives of minimal length of W_J, and $[J]$ its Coxeter class that is the set of W-conjugates of J which happen to be subsets of S.

One says that J is irreducible if the relating parabolic subgroup W_J is irreducible.
For a subset K of J, we write $N_{W_J}(W_K)$ for the normalizer of W_K in W_J, and $X(J,K)$ for the set of double coset representatives

$$X(J,K) := \{w \in W_J \cap X_K \cap X_K^{-1} \mid K^w = K\}.$$

The normalizers of the parabolic subgroups were determined by Howlett [7]. He also proved that $N_{W_J}(W_K) = W_J \cdot X(J,K)$ [7, Corollary 3].

Let $s_{i_1} \ldots s_{i_l}$ be a reduced expression of an element x of W. The support of x is

$$J(x) := \{s_{i_1}, \ldots, s_{i_l}\}$$

which is independent of the choice of the reduced expression [1, Proposition 2.16]. One says that x has full support if $J(x) = S$.
We finish with the set $\lfloor x \rceil := \{y \in W \mid J(y) = J(x) \text{ and } y \text{ is conjugate to } x\}$.

Now, we come to the hyperplane arrangement associated to a finite Coxeter group called Coxeter arrangement. Let H_t be the hyperplane $\ker(t-1)$ of $\mathbb{R}^n$ whose points are fixed by each element t of T. The hyperplane arrangement associated to the finite Coxeter group W is $\mathcal{A}_W := \{H_t\}_{t \in T}$. In this case, explicit formulas for $\mathsf{a}(E)$ and $l(E)$ can be given, and it is the aim of this article. For a subset U of T, we write E_U for the edge

$$E_U := \bigcap_{u \in U} H_u.$$

The aim of this article is to prove the following result.

Theorem 1.1. *Let E be an edge of $\mathcal{A}_W$. Then $l(E) \neq 0$ if and only if there exist an irreducible subset J of S and an element w of W such that $E = E_{T_J^w}$. In this case, we clearly have $\mathsf{a}(E_{T_J^w}) = \prod_{u \in T_J} a_{H_{u^w}}$. Moreover, let t_J be a reflection with support J, s_J a simple reflection and v an element of W such that a reduced expression of t_J is s_J^v. Then*

$$l(E_{T_J^w}) = |\lfloor t_J \rceil| \cdot |[J]| \cdot |X(S,J)| \cdot |X(J,\{s_J\})|.$$

Let Y_J be the set of the cosets of $N_W(W_J)$, and $\mathcal{I}(S)$ the set of the Coxeter classes $[J]$ of W such that J is irreducible. We deduce that the Varchenko determinant of the arrangement associated to a finite Coxeter group W is

$$\prod_{[J]\in\mathcal{I}(S)}\prod_{w\in Y_J}\Big(1-\prod_{u\in T_J} a^2_{H_{u^w}}\Big)^{|\lfloor t_J\rceil|\cdot|[J]|\cdot|X(S,J)|\cdot|X(J,\{s_J\})|}.$$

We compute the Varchenko determinants associated to the irreducible Coxeter groups in Section 6. We deduce the Varchenko determinant associated to a finite Coxeter group from the following remark: If $W = W_1W_2$, where W_1 and W_2 are two irreducible Coxeter groups, then

$$\det\mathcal{A}_W = (\det\mathcal{A}_{W_1})^{|W_2|}(\det\mathcal{A}_{W_2})^{|W_1|}.$$

Recall that there is a one-to-one correspondence between the elements of W and the chambers of $\mathcal{A}_W$ such that: if the chamber C corresponds to the neutral element e and C_x to another element x, then $C_x = Cx$ with $x = t_1 \dots t_r$, the H_{t_i}'s being the hyperplanes one goes through from C to C_x [1, Theorem 1.69].
Let $\langle\overline{C}_x \cap H_t\rangle$ be the subspace generated by the closed face $\overline{C}_x \cap H_t$ of the chamber C_x. Determining the multiplicity of the edge E contained in the hyperplane H_t consists of counting the half of the chambers C_x which have the property that E is the minimal edge containing $\langle\overline{C}_x \cap H_t\rangle$. For the proof of Theorem 1.1, we need to introduce the three propositions that we prove in the next three sections.

Proposition 1.2. *Let E be an edge of $\mathcal{A}$. Then $l(E) \neq 0$ if and only if there exist an irreducible subset J of S and an element w of W such that $E = E_{T_J^w}$.*

Let us introduce the set $L(E,t) := \big\{x \in W \mid \langle\overline{C}_x \cap H_t\rangle = E\big\}$.

Proposition 1.3. *Let J be an irreducible subset of S, t a reflection with support J, s a simple reflection, and v an element of W such that a reduced expression of t is s^v. For a conjugate K of J, let $c_{K,J}$ be an element of W such that $K^{c_{K,J}} = J$, and for a conjugate u of t with support J, let $c_{u,t}$ be an element of W such that $u^{c_{u,t}} = t$. Then,*

$$L(E_{T_J},t) = \bigsqcup_{K\in[J]} c_{K,J}X(S,J)\Big(\bigsqcup_{u\in\lfloor t\rceil} c_{u,t}N_{W_J}(W_{\{s\}})^v\Big).$$

Proposition 1.4. *Let $u, v \in T$ and let E be an edge contained in both H_u and H_v. Then*

$$|L(E,u)| = |L(E,v)|.$$

2 The Coxeter Complex

Not all edges are relevant, or in other words, there are some edges whose multiplicities are null. We develop the condition for an edge E to be relevant which means $l(E) \neq 0$.

Lemma 2.1. *A finite Coxeter group W is irreducible if and only if W has a reflection of full support.*

Proof. If W is irreducible, then W has a highest root [8, 2.10 Construction of root systems, 2.13 Groups of types H_3 and H_4], and the reflection corresponding to the highest root has full support.
Suppose that W is the product of nontrivial Coxeter groups W_1 and W_2, and let t be a reflection in W. Without loss of generality, we can suppose that t is a conjugate of a simple reflection s of W_1, hence t lies in W_1 and can not have full support. $\square$

We continue our investigation by using the Coxeter complex. Recall that the Coxeter complex $\mathcal{C}$ of W is the set $\{W_J x \mid J \subseteq S,\, x \in W\}$ of faces. The Coxeter complex is a combinatorial setup which permits to study the geometrical structure of $\mathcal{A}_W$. Indeed, $\mathbb{R}^n$ is partitioned by $\mathcal{C}$ [8, 1.15 The Coxeter complex].
The chamber C_x is identified with the singleton $\{x\}$. The coset $W_J w$ is a face of $\{x\}$ if and only if $W_J w = W_J x$. Then the closure of C_x is

$$\overline{C}_x := \{W_J x \mid J \subseteq S\}. \tag{1}$$

More generally, $W_K w$ is a face of $W_J x$ if and only if $W_J x \subseteq W_K w$ and $W_K w = W_K x$. Then the closure of a face $W_J x$ of C_x is

$$\overline{W_J x} := \{W_K x \mid J \subseteq K \subseteq S\}. \tag{2}$$

Hyperplanes, and more generally egdes, can be described as collections of the faces they consist of. We extend the definition of $J(x)$ to a subset X of W with the following way:

$$J(X) := \bigcup_{x \in X} J(x).$$

Lemma 2.2. *Let $t \in T$. Then $H_t = \{W_J w \mid J \subseteq S,\, w \in W,\, J(t^{w^{-1}}) \subseteq J\}$.*

Proof. As W acts by right multiplication, $W_J w \in H_t$ if and only if $W_J w t = W_J w$, i.e. $W_J^w t = W_J^w$, i.e. $t \in W_J^w$, i.e. $t^{w^{-1}} \in W_J$, i.e. $J(t^{w^{-1}}) \subseteq J$. $\square$

Lemma 2.3. *Let $x \in W$ and $t \in T$. Then, $\overline{C}_x \cap H_t = \overline{W_{J(t^{x^{-1}})} x}$.*

Proof. We have

$$\begin{aligned}
\overline{C}_x \cap H_t &= \{W_J x \mid J \subseteq S\} \cap \{W_J w \mid J \subseteq S,\, w \in W,\, J(t^{w^{-1}}) \subseteq J\} \\
&\quad \text{(Equation 1 and Lemma 2.2)} \\
&= \{W_J x \mid J(t^{x^{-1}}) \subseteq J\} \\
&= \overline{W_{J(t^{x^{-1}})} x} \quad \text{(Equation 2)}
\end{aligned}$$

$\square$

Lemma 2.4. *Let $W_J x$ be a face of the Coxeter complex. Then, the subspace $\langle W_J x\rangle$ generated by $W_J x$ is the edge $E_{T_J^x}$.*

Proof. It is clear that the subspace generated by $\overline{W_J x}$ is equal to the subspace generated by $W_J x$. From the proof of Lemma 2.2, we know that H_t is a hyperplane containing $W_J x$ is equivalent to $J(t^{x^{-1}}) \subseteq J$ which is equivalent to $t \in T_J^x$. $\square$

Let us take a reflection t of U. Recall that $l(E_U)$ is half the number of chambers C_x which have the property that E_U is the minimal intersection containing $\langle \overline{C}_x \cap H_t \rangle$. Minimality implies equality that is, for the chamber C_x to be counted, we must have $\langle \overline{C}_x \cap H_t \rangle = E_U$ i.e. $\langle \overline{W_{J(t^{x^{-1}})} x} \rangle = E_U$ (Lemma 2.3) i.e. $E_{T^x_{J(t^{x^{-1}})}} = E_U$ (Lemma 2.4). Moreover, since $t^{x^{-1}}$ is a full support reflection of the group $W_{J(t^{x^{-1}})}$, we know from Lemma 2.1 that $J(t^{x^{-1}})$ is an irreducible subset of S. So E_U is of the form E^x_J, where J is irreducible, otherwise $l(E_U) = 0$. That proves Proposition 1.2.

3 The Chambers to Consider

For a relevant edge E and a hyperplane H_t containing E, we determine the chambers C_x such that $\langle \overline{C}_x \cap H_t \rangle = E$.

Lemma 3.1. *Let J be an irreducible subset of S, t a reflection with support J, and E the relevant edge E_{T_J}. For each K in the Coxeter class of J, let $c_{K,J}$ be the coset of minimal length of $N_W(W_J)$ such that $K^{c_{K,J}} = J$. Then,*

$$L(E,t) = \bigsqcup_{K \in [J]} c_{K,J} \left\{ x \in N_W(W_J) \;\middle|\; J(t^{x^{-1}}) = J \right\}.$$

Proof. We have

$$\begin{aligned} L(E,t) &= \left\{ x \in W \mid \langle \overline{C}_x \cap H_t \rangle = E \right\} \\ &= \left\{ x \in W \mid \langle \overline{W_{J(t^{x^{-1}})} x} \rangle = E \right\} \quad \text{(Lemma 2.3)} \\ &= \left\{ x \in W \mid E_{T^x_{J(t^{x^{-1}})}} = E_{T_J} \right\} \quad \text{(Lemma 2.4)} \end{aligned}$$

Denoting $J(t^{x^{-1}})$ by K, the equality $E_{T^x_{J(t^{x^{-1}})}} = E_{T_J}$ means that

- $|J(t^{x^{-1}})| = |J|$,
- $K \in [J]$,
- and $x \in c_{K,J} N_W(W_J)$.

Hence

$$L(E,t) = \bigsqcup_{K \in [J]} \left\{ x \in c_{K,J} N_W(W_J) \;\middle|\; |J(t^{x^{-1}})| = |J| \right\}.$$

Let $x = c_{K,J} y$ with $y \in N_W(W_J)$. Since $|J(t^{x^{-1}})| = |J|$ if and only if $|J(t^{y^{-1}})| = |J|$ if and only if $J(t^{y^{-1}}) = J$, we obtain the result. □

Recall that $X(S,J) := \{x \in X_J \mid J^x = J\}$. We introduce the set

$$W_J(t) := \{x \in W_J \mid J(t^{x^{-1}}) = J\}.$$

Lemma 3.2. *Let J be a irreducible subset of S. Consider a reflection t of W_J with support J. Then,*

$$\left\{ x \in N_W(W_J) \;\middle|\; J(t^{x^{-1}}) = J \right\} \;=\; X(S,J) \cdot W_J(t).$$

Proof. We have $N_W(W_J) = W_J \cdot X(S,J)$ [7, Corollary 3]. So

$$\{x \in N_W(W_J) \mid J(t^{x^{-1}}) = J\} \;=\; \{yz \mid y \in W_J,\, z \in X(S,J),\, J(t^{z^{-1}y^{-1}}) = J\}.$$

Since we always have $t^{z^{-1}} = t$, the remaining condition is $J(t^{y^{-1}}) = J$. □

We write $C(x)$ for the centralizer of the element x of W.

Lemma 3.3. *Let J be a irreducible subset of S. Consider a reflection $t = s^v$ with support J where s is a simple reflection and v and element of W_J. For another reflection u of W_J with support J and conjugate to t, let $c_{u,t}$ be the coset of minimal length of $C(t)$ such that $u^{c_{u,t}} = t$. Then,*

$$W_J(t) = \bigsqcup_{u \in \lfloor t \rceil} c_{u,t} N_{W_J}(W_{\{s\}})^v.$$

Proof. The equality $J(t^{x^{-1}}) = J$ means that $x^{-1} \in C(t)c_{t,u}$ or $x \in c_{u,t}C(t)$, where u a conjugate of t with support J. Then,

$$W_J(t) = \bigsqcup_{u \in \lfloor t \rceil} c_{u,t} C(t).$$

Since $C(t) = C(s)^v$ and $C(s) = N_{W_J}(W_{\{s\}})$, we get the result. □

Proposition 1.3 is a combination of Lemma 3.1, Lemma 3.2, and Lemma 3.3.

4 Invariance of the Multiplicity

For the calculation of the multiplicity of an edge E, one has to choose a hyperplane containing E. We prove that the result of the calculation is independent of the choice of the hyperplane containing E.

Lemma 4.1. *Consider two hyperplanes H_u and H_v of $\mathcal{A}_W$ associated with the orthogonal vectors α_u and α_v respectively, of equal length and from the positive root system . Let r be the rotation of $\mathbb{R}^n$, not necessarily in W, which transforms α_u to α_v. Then, $r(\mathcal{A}_W) = \mathcal{A}_W$.*

Proof. Let $E_{u,v}$ be the 2-dimensional subspace $\langle \alpha_u, \alpha_v \rangle$:

- on $E_{u,v}$, the map r is the rotation of angle $\theta = \arccos \frac{\alpha_u \cdot \alpha_v}{\|\alpha_u\| \, \|\alpha_v\|}$,
- on $E_{u,v}^{\perp}$, the map r is the identity map.

We have $\mathcal{A}_W = \mathcal{A}_1 \sqcup \mathcal{A}_2$ with

$$\mathcal{A}_1 := \{H \in \mathcal{A} \mid E_{u,v} \subset H\} \quad \text{and} \quad \mathcal{A}_2 := \{H \in \mathcal{A} \mid \dim E_{u,v} \cap H = 1\}.$$

- ∘ For all H in $\mathcal{A}_1$, we have $r(H) = H$.
- ∘ Let p be projection on the subspace $E_{u,v}$. The arrangement $p(\mathcal{A}_2)$ is the arrangement of a dihedral group whose angle between two certain hyperplanes is θ. Then, $r\big(p(\mathcal{A}_2)\big) = p(\mathcal{A}_2)$. Hence, for any H in $\mathcal{A}_2$, we have

  $$r(H) = r\big(p(H) \oplus E_{u,v}^{\perp}\big) = r\big(p(H)\big) \oplus E_{u,v}^{\perp}$$

  which still belongs to $\mathcal{A}_2$.

□

We prove Proposition 1.4 now. Consider two hyperplanes H_u and H_v containing the edge E. Let α_u and α_v be the unit vectors of the positive root system associated to W which are orthogonal to H_u and H_v respectively. We use the rotation r of Lemma 4.1 transforming α_u to α_v, and leaving $\mathcal{A}$ invariant. For a given w in $L(E,u)$, we have

$$\begin{aligned}\overline{C}_w \cap H_u &= \overline{C}_w \cap E\\ (\overline{C}_w \cap H_u)r &= (\overline{C}_w \cap E)r\\ (\overline{C}_w)r \cap (H_u)r &= (\overline{C}_w)r \cap (E)r\\ (\overline{C}_w)r \cap H_v &= (\overline{C}_w)r \cap E.\end{aligned}$$

Hence $L(E,u)r = L(E,v)$, and $|L(E,u)| = |L(E,v)|$ which is Proposition 1.4.

5 The Multiplicity of an Edge

We establish a formula for the multiplicity of a relevant edge of $\mathcal{A}_W$ in this section.

We begin with the proof of Theorem 1.1. From Proposition 1.2, we know that the relevant edges are the intersections of hyperplanes $E_{T_J^w}$ with the condition that J is irreducible. The weight of $E_{T_J^w}$ is obviously $\prod_{u\in T_J} q_{u^w}$ so that the real problem concerns $l(E_{T_J^w})$. Fixing a hyperplane H_t containing $E_{T_J^w}$, the set of chambers taken into account to determine $l(E_{T_J^w})$ is $L(E_{T_J^w},t)$. But Proposition 1.4 allows us to choose any hyperplane containing $E_{T_J^w}$. Hence the multiplicity of $E_{T_J^w}$ is

$$l(E_{T_J^w}) = \frac{1}{2}|L(E_{T_J^w},t)|.$$

Lemma 5.1. *Let J be an irreducible subset of S, and w an element of W. Then,*

$$L(E_{T_J^w},t^w) = L(E_{T_J},t)w.$$

Proof. Let $x \in L(E_{T_J},t)$. We have

$$\begin{aligned}\overline{C}_x \cap H_t &= \overline{C}_x \cap E_{T_J}\\ (\overline{C}_x \cap H_t)w &= (\overline{C}_x \cap E_{T_J})w\\ (\overline{C}_x)w \cap (H_t)w &= (\overline{C}_x)w \cap (E_{T_J})w\\ \overline{C}_{xw} \cap H_{t^w} &= \overline{C}_{xw} \cap E_{T_J^w}\end{aligned}$$

Then $L(E_{T_J},t)w \subseteq L(E_{T_J^w},t^w)$. With the same argument, we obtain also $L(E_{T_J^w},t^w)w^{-1} \subseteq L(E_{T_J},t)$. Hence $L(E_{T_J^w},t^w) = L(E_{T_J},t)w$. □

We see in Lemma 5.1 that we just need to investigate $L(E_{T_J},t)$ for each Coxeter class $[J]$. From Proposition 1.3, we get

$$\begin{aligned}l(E_{T_J}) &= \frac{1}{2}|L(E_{T_J},t)|\\ &= \frac{1}{2}\Big|\bigsqcup_{K\in[J]} c_{K,J}X(S,J)\Big(\bigsqcup_{u\in\lfloor t\rceil} c_{u,t}N_{W_J}(W_{\{s\}})^v\Big)\Big|\\ &= |\lfloor t\rceil|\cdot|[J]|\cdot|X(S,J)|\cdot|X(J,\{s\})| \quad \text{since} \quad \frac{1}{2}|N_{W_J}(W_{\{s\}})| = |X(J,\{s\})|,\end{aligned}$$

which finishes the proof of Theorem 1.1.

Coxeter Groups	Number of Reflections	Number of Conjugacy Classes	Number of Full Support Reflections
A_{n-1}	$\binom{n}{2}$	1	1
B_n	n^2	2	$1 \mid n-1$
D_n	$n(n-1)$	1	$n-2$
E_6	36	1	7
E_7	63	1	16
E_8	120	1	44
F_4	24	2	$5 \mid 5$
H_3	15	1	8
H_4	60	1	42
$I_2(m)$ odd/even	m / m	1 / 2	$m-2$ / $\frac{m-2}{2} \mid \frac{m-2}{2}$

Table 1: Number of Full Support Reflections of the Irreducible Finite Coxeter Groups.

6 Computing the Determinants of Finite Coxeter Groups

Before computing the Varchenko determinants of the irreducible finite Coxeter groups, we first have to determine their numbers of full support reflections.
Let $\Gamma = (S, M)$ be the connected Coxeter graph of an irreducible finite Coxeter group. The set of vertices S is composed by the simple reflections, and every edge $(s_i\ s_j)$ of M is labelled by the order m_{ij} of $s_i s_j$.

Lemma 6.1. *The reflections of an irreducible finite Coxeter group form a single conjugacy class if and only if the labels of the edges in M are all odd.*

Proof. The reflections of the Coxeter group $I_2(m)$ form a single conjugacy class when m is odd, but two conjugacy classes when m is even. Since the conjugacy is an equivalence relation, two simple reflections are then conjugate if and only if the labels of the edges separating them are all odd. □

Proposition 6.2. *The number of conjugacy classes formed by the reflections of an irreducible finite Coxeter group W is $1 + \big|\big\{(s_i\ s_j) \in M \mid m_{ij} \text{ is even}\big\}\big|$.*

Proof. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_r$ the connected components of the subgraph obtained by deleting the even labelled edges of Γ. Choosing any reflection s_i of Γ_i, we know from Lemma 6.1 that the reflections of W are of the form $s_i^{w_i}$ such that $s_{i_1}^{w_{i_1}}$ is conjugate to $s_{i_2}^{w_{i_2}}$ if and only if $i_1 = i_2$. And the number of connected components is $1 + \big|\big\{(s_i\ s_j) \in M \mid m_{ij} \text{ is even}\big\}\big|$. □

We can see the numbers of reflections of the irreducible finite Coxeter groups in the book of Björner and Brenti [2, Appendix A1] for example. Using the Principle of Inclusion and Exclusion, applied to the irreducible maximal parabolic subgroups, and the Pascal's triangle in the form $\binom{n}{2} - \binom{n-1}{2} = n-1$, we get the number of full support reflections of the irreducible finite Coxeter groups in Table 1.

We are now able to compute the Varchenko determinant of a finite Coxeter group by using Theorem 1.1. The necessary ingredients mentioned to calculate this determinant are exposed in Table 2 for all irreducible finite Coxeter groups. They are obtained with tools of Table 1,

those in [6, Proposition 2.3.8, 2.3.10, 2.3.13], and [6, Table A.1, A.2].
Let $[\pm n] := \{-n, \dots, -2, -1, 1, 2, \dots, n\}$. We write $\overline{2^{[\pm n]}}$ for the subset of $2^{[\pm n]}$ having the following properties:

- the elements of $\overline{2^{[\pm n]}}$ are the elements $\{i_1, \dots, i_t\}$ of $2^{[\pm n]}$ such that $|i_r| \neq |i_s|$ if $r \neq s$,
- and if $\{i_1, \dots, i_t\} \in \overline{2^{[\pm n]}}$, then $\{-i_1, \dots, -i_t\} \notin \overline{2^{[\pm n]}}$.

Using Theorem 1.1 and Table 2, we refind, for example, the determinants

$$\det \mathcal{A}_{B_n} = \prod_{\substack{J \in \overline{2^{[\pm n]}} \\ |J| \geq 2}} \Big(1 - \prod_{\{i,j\} \in \binom{J}{2}} a_{H_{i,j}}^2\Big)^{2^{n-|J|+1}\,(|J|-2)!\,(n-|J|+1)!}$$
$$\prod_{\substack{I \in 2^{[n]} \\ |I| \geq 1}} \Big(1 - \prod_{i \in I} a_{H_i}^2 \prod_{\{i,j\} \in \binom{I}{2}} a_{H_{i,j}}^2\, a_{H_{-i,j}}^2\Big)^{2^{n-1}\,(|I|-1)!\,(n-|I|)!},$$

computed by Randriamaro [9] with combinatorial methods.

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Coxeter Groups	J	$\lvert\lfloor t_J \rceil\rvert$	$\lvert[J]\rvert$	$\lvert X(S,J)\rvert$	$\lvert X(J,\{s_J\})\rvert$
A_{n-1}	A_i	1	$n-i$	$(n-i-1)!$	$(i-1)!$
B_n	A_i $(i \leq n-1)$	1	$n-i$	$2^{n-i}(n-i-1)!$	$(i-1)!$
	B_j	1 / $j-1$	1	$2^{n-1}(n-j)!$	$(j-1)!$ / $(j-2)!$
D_n	A_i $(i \leq n-1)$	1	$n-i+1$	$2^{n-i-1}(n-i-1)!$	$(i-1)!$
	D_j	$j-2$	1	$2^{n-j}(n-j)!$	$2^{j-2}(j-2)!$
E_6	A_1	1	6	720	1
	A_2	1	5	72	1
	A_3	1	5	8	2
	A_4	1	4	2	6
	D_4	2	1	6	8
	A_5	1	1	2	24
	D_5	3	2	1	48
	E_6	7	1	1	720
E_7	A_1	1	7	23040	1
	A_2	1	6	1440	1
	A_3	1	6	96	2
	A_4	1	5	12	6
	D_4	2	1	48	8
	A_5'	1	1	12	24
	A_5''	1	1	4	24
	D_5	3	2	4	48
	A_6	1	1	2	120
	D_6	4	1	2	384
	E_6	7	1	2	720
	E_7	16	1	1	23040
E_8	A_1	1	8	2903040	1
	A_2	1	7	103680	1
	A_3	1	7	3840	2
	A_4	1	6	240	6
	D_4	2	1	1154	8
	A_5	1	4	24	24
	D_5	3	2	48	48
	A_6	1	3	4	120
	D_6	4	1	8	384
	E_6	7	1	12	720
	A_7	1	1	2	720
	D_7	5	1	2	3840
	E_7	16	1	2	23040
	E_8	44	1	1	2903040

F_4	A_1'	1	2	48	1
	A_1''	1	2	48	1
	A_2'	1	1	12	1
	A_2''	1	1	12	1
	B_2	2	1	8	2
	B_3'	1 / 2	1	2	8 / 4
	B_3''	1 / 2	1	2	8 / 4
	F_4	10	1	1	48
H_3	A_1	1	3	4	1
	A_2	1	1	2	1
	$I_2(5)$	3	1	2	1
	H_3	8	1	1	4
H_4	A_1	1	4	120	1
	A_2	1	2	12	1
	$I_2(5)$	3	1	20	1
	A_3	1	1	2	2
	H_3	8	1	2	4
	H_4	42	1	1	120
$I_2(m)$	A_1 odd/even	1 / 2	2 / 1	1	1
	$I_2(m)$	$m-2$	1	1	1

Table 2: Multiplicities of the Coxeter Classes.

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