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Prony's method: an old trick for new problems

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In 1795, French mathematician Gaspard de Prony invented an ingenious trick to solve a recovery problem, aiming at reconstructing functions from their values at given points, which arose from a specific application in physical chemistry. His technique became later useful in many different areas, such as signal processing, and it relates to the concept of sparsity that gained a lot of well-deserved attention recently. Prony's contribution, therefore, has developed into a very modern mathematical concept.

1 Recovery

1.1 Recovery problems

The type of problem that we will consider here is nowadays called a *recovery problem*. The task is to reconstruct a function f from measurements or samples $f_n := f(x_n)$, where $x_n \in X$ and X is a *finite* set of points where the value of the function is known. In this overly general form, the problem has potentially infinitely many solutions. For example, given two points x_1 and x_2 in $\mathbb{R}^2$, we can fix values $f(x_1)$ and $f(x_2)$ and try to draw all possible curves with those values at x_1 and x_2 – it may take a while.[1]

Therefore, in order to get a unique, or at least "well-defined" (reasonable), solution, one must make some assumptions on the function f.[2] The simplest assumption one can make is that f can be written as a *linear combination*[3] of several known *basis functions* f_j, which reads

$$f = \sum_{j=1}^{n} a_j \, f_j, \qquad a_j \in \mathbb{R}, \tag{1}$$

and then one only has to compute the *coefficients* a_j. The basis functions f_j are known functions that generally have certain properties. For example, any real polynomial $f(x)$ can be decomposed as $f(x) = \sum_n a_n \, x^n$ for appropriate coefficients a_n, where $n \geq 0$. Here, the polynomials x^n play the role of the basis functions f_n.

Recovery on linear spaces has a long history and already Newton (1642–1726/27) gave recipes for how to compute such a representation (1) for polynomials from values at given points [5, Liber III, Lemma V].[4] is turned into a system of linear equations

$$\begin{array}{ccccc} f(x_1) & = & a_1 f_1(x_1) & + \cdots + & a_n f_n(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_m) & = & a_1 f_1(x_m) & + \cdots + & a_n f_n(x_m), \end{array} \tag{2}$$

whose solvability depends on properties of the following *matrix*

$$\boldsymbol{M} = \begin{pmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_m) & \cdots & f_n(x_m) \end{pmatrix}, \tag{3}$$

also defined in a compact form by $M_{jk} := f_j(x_k)$. Given the existence of a natural matrix associated to the linear recovery problem, it is natural to relay the problem to linear algebra. Of course, all this is well-known and is standard material in any course of numerical analysis, see [10].

[1] Footnote added In fact, there are infinitely many curves that pass through two points in a plane.

[2] By well-defined solution we mean a solution that can be computed easily from the given set of data.

[3] Footnote added This snapshot relies on linear algebra in the more detailed explanations. If you have not studied much linear algebra yet, you may find it easier to skip the details.

[4] "Linear" because we recover a *linear combination* of the basis functions $f_1, \ldots, f_n$.

1.2 Recovery problems: change of approach

Now, we change the perspective and consider a different class of functions, namely, in the notation of Prony, functions of the form

$$z = f(x) = \sum_{j=1}^{n} \mu_j \, \rho_j^x, \tag{4}$$

depending on the coefficients $\mu_1, \dots, \mu_n \in \mathbb{R}$ *and* the positive real numbers[5] $\rho_1, \dots, \rho_n \in \mathbb{R}_+$. The aim is to recover the values of these parameters from sample values $z_k := f(x_k)$ of f. This problem was originally considered 1795 by Gaspard Clair François Marie Riche de Prony, see [6]. Compared to linear recovery this different approach has the new quality that it also requires to find those ρ_j, or equivalently the correct basis functions $f_j(x) = \rho_j^x$ that *explain* the measurements $z_k = f(x_k)$.

Let us consider an example. A *musical tone*, which could be a combination of oscillations of a violin string, is usually written as a combination of the periodic functions sine or cosine as a function of the time t, of a basic frequency ω and its multiples. Formally, one has

$$f(t) = \sum_{j=1}^{n} a_j \cos(j\omega\, t), \qquad t \in \mathbb{R}, \tag{5}$$

see [4]. The *amplitude parameters* a_j determine the sound of the instrument and allow us to distinguish between, say, a violin and a clarinet. Now assume that two instruments play two different tones at the same time, but we neither know the pitches nor the instruments. Can we do the same as the human ear and recognize the instruments and their pitches from a digital record, say, a `wav` file? The answer is "yes" as soon as we know how many instruments are playing and how many partial tones each of them has.[6] By using the complex representation $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ of the cosine function, our musical problem can indeed be cast in the form (4), where we have $\rho_j = e^{\pm i\omega}$. Prony himself was interested in a problem of vaporization of liquids, see [6]. In his model μ_j was the amount of the ingredient, ρ_j a material constant and x the temperature.

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[5] Negative numbers are generally unpleasant, but can be handled easily in the complex version of the formalism. The case $\rho_j = 0$ would give trivial contributions and is therefore irrelevant.

[6] A partial tone is one of the summands in (5).

In the end, it does not matter to the mathematician which particular physical problem is really modelled by employing (4). The more interesting questions are

i) Under which assumptions can one recover the μ_j and the ρ_j from finitely many measurements?
ii) What is the minimum number of samples z_k that we need to take?
iii) Should the samples have any particular "structure"?

1.3 Recovery problems: approaches to solutions

As a matter of fact, we take the simplest possible sampling method of f here, namely *equidistant sampling*[7] with distance $h > 0$:

$$z_k = f(hk) = \sum_{j=1}^{n} \mu_j \, \rho_j^{hk} = \sum_{j=1}^{n} \mu_j \, \hat{\rho}_j^{k}, \qquad \hat{\rho}_j := \rho_j^{h}. \tag{6}$$

We note that the sampling distance h does not affect the "structure" of the problem, but just the effective values ρ_j that these parameters take (that is, if we choose a different h, we will have a different amount of ρ_j constants with in general different values. This change is "quantitative" – the amount of terms – but not "qualitative" – the sampling technique itself). However, once h is known, this is a simple and well-defined 1-to-1 relationship and we can easily compute one from the other.

Our ability to recover f from the measurements, on the other hand, will rely on the fact that the problem (4) is *sparse*. This means that the number n of terms in f is relatively small compared to the number of measurements z_k. To avoid counting unnecessary terms, we will always assume that $\mu_j \neq 0$ in the future. We therefore say that the function f has a "simple explanation" with only a few parameters.

Sparsity has become an important concept in applied mathematics in recent years. To some extent, it seems to be arguably the only way to deal with the continuously growing amount of data that we collect about all phenomena around us when we try to give simple explanations to each phenomenon, given an overwhelming amount of observations available. The new discipline called *compressive sensing* was born to try to provide concepts for, and tools to deal with, the mathematical aspects of such questions. For an introduction to the topic see [3].

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[7] Accidentally, the core mechanisms behind this method are the same as those that occur in sampling audio signals as well. For example, a `wav` file just records amplitude values with a certain rate, for example 44000 times per second for CD-quality resolution.

2 An old trick

We now move on to describe a different way to tackle the problem at hand.

Given an undetermined number of measurements or sampling points $z_0, z_1, \ldots$, Prony first computed numbers $p_0, \ldots, p_n$ by requesting that[8]

$$\begin{aligned} 0 &= z_0\, p_0 + \cdots + z_n\, p_n \\ 0 &= z_1\, p_0 + \cdots + z_{n+1}\, p_n \\ &\vdots \qquad\qquad \vdots \\ 0 &= z_k\, p_0 + \cdots + z_{k+n}\, p_n, \qquad k = 0, 1, 2, \ldots \end{aligned} \tag{7}$$

This is a condition on the *correlation* between the sequence of the measurements and the sequence p. It is a correlation because there are measurements z_k that appear in more than one line. We indicate the correlation between the elements of the system by $z \star p$, which can be formally expressed through its elements $(z \star p)_k$. These are

$$(z \star p)_k := \sum_{j=0}^{n} z_{k+j}\, p_j, \qquad k = 0, 1, 2, \ldots. \tag{8}$$

Therefore, the shorthand for Prony's requirement (7) is $z \star p = 0$ and our aim is to solve this equation for the p_js. The beauty of this approach is that the p_js indeed encode the numbers $\hat{\rho}_j$. To see this, we substitute the definition (4) of the function into (8) and get

$$0 = \sum_{j=0}^{n} z_{k+j}\, p_j = \sum_{j=0}^{n} p_j \sum_{\ell=0}^{n} \mu_\ell \hat{\rho}_\ell^{k+j} = \sum_{\ell=0}^{n} \mu_\ell \hat{\rho}_\ell^{k} \sum_{j=0}^{n} p_j\, \hat{\rho}_\ell^{j} = \sum_{\ell=0}^{n} \mu_\ell \hat{\rho}_\ell^{k}\, p(\hat{\rho}_\ell), \tag{9}$$

with the *polynomial* $p(x)$ defined by

$$p(x) := \sum_{j=0}^{n} p_j\, x^j. \tag{10}$$

In the language of signal processing, our condition $z \star p = 0$ corresponds to finding a *filter* p with n *taps* that *annihilates* the measurement z.[9] Also in this language, the polynomial $p(x)$ would be called the *generating function*, or *symbol*, or z*-transform* of the *magic filter* p.

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[8] This tableau can be found with slightly different notation in the original [6, p. 30], and Prony gives some explicit way to find the numbers $p_0, \ldots, p_n$.

[9] We note that, strictly speaking, a filter is usually defined as a *convolution* which would correspond to reversing the order of p, but note that we are glossing over some details here.

Equation (9) is also a one-line proof of the following fundamental fact: the coefficients of the *polynomial*

$$p(x) = (x - \hat{\rho}_1) \cdots (x - \hat{\rho}_n) \tag{11}$$

are a solution of (8), and any annihilating filter, that is, any solution of (8) is the coefficient vector of a multiple of the polynomial p from (11).[10] Hence, if we find the *shortest* (with the smallest possible n) filter p such that $z \star p = 0$, then the zeros of the associated polynomial are exactly the values $\hat{\rho}_j$.

In an algorithmic sense, what we have to do is to solve the linear system (8) for p and then find the zeros[11] of the polynomial $p(x)$ to obtain the numbers $\hat{\rho}_j$. There are, fortunately, various methods to find the zeros of a polynomial although this is in general not a well-conditioned problem. Furthermore, this is one of the "bad" kind of problems that triggered a rigorous mathematical analysis of the effects of *roundoff errors*, see the highly recommendable paper [13]. Once the zeros are known, we are left with a linear *interpolation problem* which is yet another system of linear equations.

The interpolation problem is the linear system obtained to determine the factors μ_j. We can just write (6) as a linear system where k denotes the index of the respective equation. The system is uniquely solvable since the $\hat{\rho}_j^k$ give rise to a uniquely solvable polynomial interpolation problem (back to Newton's ideas). The matrix of such a linear system is then called a "Vandermonde matrix".

A careful inspection of the linear system (8) shows that only the equations with indices $k = 0, \ldots, n$ have to be considered, which means that only the $2n+1$ measurements $z_0, \ldots, z_{2n}$ are really needed.[12] This appears quite efficient as we need only one measurement more than the number of parameters that need to be fixed. It turns out, however, that the computations become much more reliable and numerically stable if the data is *oversampled*, that is, if we can use more than $2n+1$ samples. In other words, the "sparser" the problem is relative to the number of samples, the better the numerical behavior becomes.

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[10] Can you see why this is the case?

[11] Prony only says "solve $p(x) = 0$". The zeros of a polynomial are those numbers x_j such that $p(x_j) = 0$. Can you see why (11) says that the zeroes of $p(x)$ are given by the numbers $\hat{\rho}_j$?

[12] Can you see this? Equation (9) shows that $\sum_{\ell=0}^{n} \mu_\ell \hat{\rho}_\ell^k \, p(\hat{\rho}_\ell) = 0$ for all $k \geq 0$. This can be viewed as a system of linear equations for $\mu_\ell \hat{\rho}_\ell^k$ with coefficients $p(\hat{\rho}_l)$. Since $\mu_\ell \neq 0$, it follows that $\mu_\ell \hat{\rho}_\ell^k$ are linearly independent and we have that $p(\hat{\rho}_l) = 0$.

3 More sparsity

So far, we have not really made use of sparsity, and neither did Prony. But, in those times, the application he was interested in automatically restricted him to $n \leq 3$. By using one of the most important techniques in mathematics, namely "adding zero" (which does not change the quantity it is added to), we can trivially rewrite (4) for any $N > n$ as

$$f(x) = \sum_{j=1}^{N} \mu_j \, \rho_j^x, \qquad \mu_{n+1} = \cdots = \mu_N = 0, \tag{12}$$

where $\rho_{n+1}, \ldots, \rho_N$ can be just *any* positive numbers different from the original $\rho_1, \ldots, \rho_n$. Therefore, the "good" representation in (4) is the one where all terms are really necessary, which means that $\mu_j \neq 0$, for all $j = 1, \ldots, n$. On the other hand, if $\mu_\ell = 0$ for some ℓ then (9) shows that the zero of p at the respective ρ_ℓ would simply be irrelevant, so a shorter (smaller n) filter p would do as well. Of course, if all $\mu_j \neq 0$, then there is no shorter filter p and no "simpler" explanation of the measurements (that is, *all* of the numbers $\mu_j \neq 0$ will constitute the filter).

This observation on sparsity can be used to formulate the recovery problem as an *optimization problem*, see for example [2]. To that end we consider, for some number N, *all* possible explanations (μ, ρ), where $\mu = (\mu_1, \ldots, \mu_N)$, $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_j > 0$, and *grade* them with respect to their sparsity by counting the number of nonzero coefficients μ_j:

$$\|\mu\|_0 := \#\{j : \mu_j \neq 0\}. \tag{13}$$

We say that equation (13) is a measure of the sparsity of the explanation (μ, ρ).

Following the motto "sparser is better", we then can formulate the following optimization problem with the measurements $z_0, \ldots, z_M$, $M > 0$, as side conditions:

$$\min \|\mu\|_0 \qquad \text{subject to } z_k = f(x_k) = \sum_{j=1}^{N} \mu_j \, \rho_j^{x_k}, \quad k = 0, \ldots, M. \tag{14}$$

In plain words this means "find the simplest explanation of the form (4) for the measurements" (that is, smallest $\|\mu\|_0$) and the unique solution of this problem is exactly the recovery with nonzero μ_j as long as N is chosen sufficiently large, $N \geq n$.

Unfortunately, there is just a little difficulty: the problem (14) is an *NP hard* combinatorial optimization problem, which means that for nontrivial values of N the time a computer algorithm needs to find a solution is simply unacceptable (scales exponentially with the number N, see [12]).

This is a typical dilemma in compressive sensing: the sparsest solution, or the simplest explanation, is a really great thing, but impossible to compute practically. Due to that, the problem (14) is *relaxed* by choosing a different, simpler "norm" or measure, such as

$$\|\mu\|_1 = \sum_{j=1}^{N} |\mu_j|. \tag{15}$$

Surprisingly, it can be shown, and this is the essence of compressive sensing, that under certain circumstances the solution of

$$\min \|\mu\|_1 \qquad \text{subject to } z_k = f(x_k) = \sum_{j=1}^{N} \mu_j \, \rho_j^{x_k}, \quad k = 0, \ldots, M, \tag{16}$$

is indeed the simplest explanation in the sense that it is also a solution of the minimization problem (14), with the main difference that the solution of the relaxed problem can be *computed* much more efficiently (or faster, see [4])!

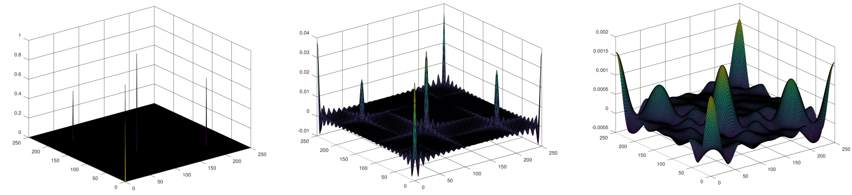
4 Applications, old and new

Prony developed the method we have described not for its own sake, but for a concrete application in physical chemistry.[13] Later, Prony's method was used in "multisource radar signal processing", leading to algorithmic realizations with nice names like MUSIC [9] and ESPRIT [7] which have been developed further ever since with a focus on numerical stability and efficiency.

More recently Prony's problem, extended to functions of two variables x and y, became popular in the context of *superresolution* [1]. For example, in microscopy, an image often consists of few localized spots whose brightness values can be visualized as spikes over the plane of the image, as shown in the left image in Figure 1. The optical system (which is typically the system of lenses and mirrors in a telescope or microscope, so really the hardware used for taking the picture), on the other hand, usually acts as a *low pass filter*[14] and turns these spikes or points on the image plane into so-called *point spread functions*. A still well-localized example can be seen in the middle image of Figure 1, but more often the system is really retaining only quite low frequencies, resulting in a recorded image like the one on the right hand side of Figure 1. We would appreciate is you could please provide larger version of the figures with better resolution.

[13] Therefore resulting in the only math paper I am aware of with alcohol in it.

[14] Briefly, a low-pass filter is a filter that passes signals with a frequency lower than a certain cutoff frequency and attenuates signals with frequencies higher than the cutoff frequency.



(a) (b) (c)

Figure 1: Spikes (a) and point spread functions with low pass (b) and very low pass (c) filtering.

The task is simple: reconstruct the spikes in Figure 1(a) from the blurred low pass filtered Figure 1(c).

We now discuss how this can be done. The plane of the image is part of $\mathbb{R}^2$. Suppose that the spikes are at positions $X \subset \mathbb{R}^2$, with amplitude a_x, where again X is a *sparse* set of locations with small size. Then, the *Fourier transform*[15] of the spikes gives a doubly-labeled sequence

$$z_\alpha := z_{\alpha_1,\alpha_2} = \sum_{x \in X} a_x e^{\alpha_1 x_1 + \alpha_2 x_2}, \qquad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2, \tag{17}$$

where the variables α_1, α_2 take values on all of $\mathbb{Z}^2$.[16]

The low pass filter cuts the amount of involved pairs α_1, α_2 down to the finitely many measurements

$$z_\alpha = \sum_{x \in X} a_x \rho_x^\alpha, \qquad |\alpha_1|, |\alpha_2| \le n, \tag{18}$$

where $\rho_x^\alpha = \rho_{x,1}^{\alpha_1} \rho_{x,2}^{\alpha_2}$ and $\rho_{x,j} = e^{i x_j}$, $j = 1, 2$, so that (18) is exactly the two-dimensional version of Prony's problem (4). The special flavour in the superresolution approach is that the low pass structure of the optical system determines the number n of measurements.

If the points in X are well separated, then a relaxed optimization approach gives a sparsest solution in the sense of (14) and therefore recovers the spikes and their intensity, but nevertheless the case with two variables discussed here is significantly more complex than the one with one variable discussed in Section 2.

[15] See Snapshot 3/2014 *The ternary Goldbach problem* by Harald Helfgott

[16] That is, a set with point labeled by two integers.

On the one hand, as shown in[17] [8], Prony's method can be extended to several variables. However, the idea needs to be generalized: in these cases one does not need one filter and thus one polynomial, but instead one requires several filters p, and the *ideal* generated by the associated polynomials[18]. Also the sampling set is not completely arbitrary and a matter of counting evaluations, it must be related to the geometry of the ρ_x. Indeed: despite the fact that we have $4n^2$ measurements in (18), the number of spikes that can be reconstructed ranges between n when all the spikes lie on a straight line to n^2 when the spikes are on a rectangular grid, while the separation distances are actually quite comparable. In other words: the sparsity factor now also depends on the (unknown) geometry of the location of the spikes. On the other hand, once the underlying concepts are understood in this more abstract sense, any number of variables is possible and Prony's method works surprisingly well, especially in high dimensions, see [8].

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