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Homogenization of a Nonlinear Monotone Problem with Nonlinear Signorini Boundary Conditions in a Domain with Highly Rough Boundary

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Homogenization of a nonlinear monotone problem with nonlinear Signorini boundary conditions in a domain with highly rough boundary

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Abstract

We consider a domain $\Omega_\varepsilon \subset \mathbb{R}^N$, $N \geq 2$, with a very rough boundary depending on ε. For instance, if $N = 3$ the domain Ω_ε has the form of a brush with an ε-periodic distribution of thin cylinders with fixed height and a small diameter of order ε. In Ω_ε a nonlinear monotone problem with nonlinear Signorini boundary conditions, depending on ε, on the lateral boundary of the cylinders is considered. We study the asymptotic behavior of this problem, as ε vanishes, i.e. when the number of thin attached cylinders increases unboundedly, while their cross sections tend to zero. We identify the limit problem which is a nonstandard homogenized problem. Namely, in the region filled up by the thin cylinders the limit problem is given by a variational inequality coupled to an algebraic system.

Keywords: Homogenization of rough boundaries, nonlinear monotone problems, nonlinear Signorini boundary conditions.

2010 AMS subject classifications: 35B27, 35J60, 35R35.

Contents

1 Introduction

This paper is devoted to studying the asymptotic behavior, as ε vanishes, of a nonlinear monotone problem with nonlinear Signorini boundary conditions, depending on ε, in a domain $\Omega_\varepsilon \subset \mathbb{R}^N$, $N \geq 2$, whose boundary contains a very rough part depending on ε. The geometry of Ω_ε is rigorously introduced in Section 2. Roughly speaking, Ω_ε has the form of a brush in $3D$ (see Figure 1) or the form of a comb in $2D$. It is composed of two parts: a fixed box Ω^b and a "forest" Ω^a_ε of cylinders with fixed height and small cross section of diameter of order ε, ε-periodically distributed in the first $N-1$ directions on the upper basis of Ω^b.

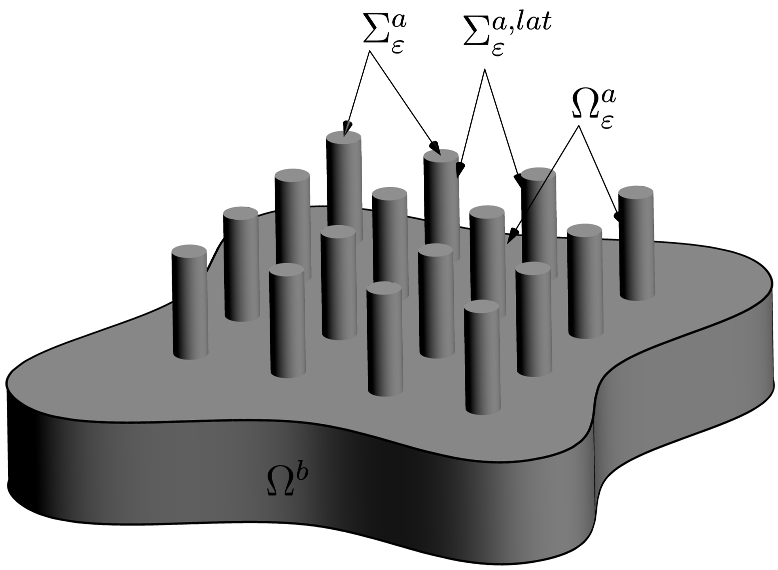

Figure 1: Ω_ε

The upper boundary and the lateral boundary of these cylinders are denoted by Σ^a_ε and $\Sigma^{a,lat}_\varepsilon$, respectively. Here as well as in the whole of the present paper, the superscripts a, b, and lat refer to "above", "below", and "lateral", respectively. Moreover, Ω^a denotes the "smallest" box containing Ω^a_ε for every ε, Σ^a and Σ^0 its upper basis and its lower basis, respectively, and $\Omega = \Omega^a \cup \Sigma^0 \cup \Omega^b$ (see Figure 2).

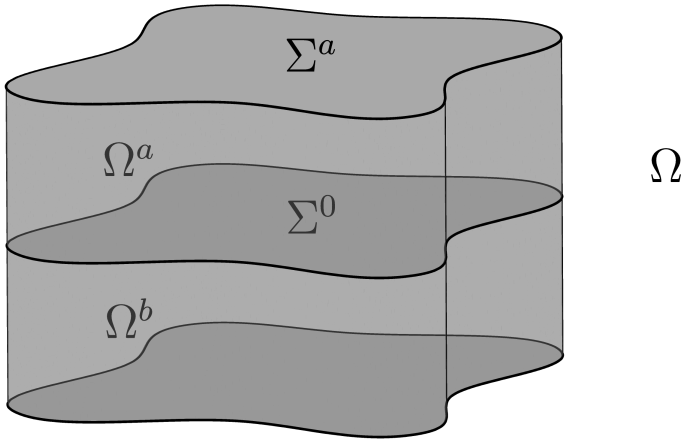

Figure 2: Ω

Boundary-value problems in a domain with rough boundary arise in many fields of biology, physics and engineering sciences. For instance, for understanding the motion of ciliated microorganisms, the flow in a channel with rugose boundary, heat transmission through winglets, propagation of electromagnetic waves in regions with rough boundaries, air flow through compression system in turbo machine such as a jet engine, the vibrations of foundations of buildings, etc. (for instance, see [6], [14], [18], [33], and [41]). It is often impossible to approach these problems directly with numerical methods, because the rough boundary requires a large number of mesh points in its neighborhood. Thus, the computational cost associated with such a problem grows rapidly when ε gets smaller. Moreover, it can occur that the required discretization step becomes too small for the machine precision. Then, the goal is to approach the problem on Ω_ε, when the periodicity ε gets smaller, with a fictious problem on Ω which can be numerically solved.

In this paper, the following Signorini free boundary value problem is considered

$$\begin{cases} -\mathrm{div}(a(x, Du_\varepsilon(x))) + a_0(x, u_\varepsilon(x)) = f(x), \text{ in } \Omega_\varepsilon, \\ \\ u_\varepsilon = 0, \text{ on } \Sigma^a_\varepsilon, \\ \\ \begin{cases} u_\varepsilon(x) \le g(x), \quad a(x, Du_\varepsilon(x))\nu_\varepsilon(x) + \varepsilon^\lambda h(x, u_\varepsilon(x)) \le 0, \\ \\ (u_\varepsilon(x) - g(x))(a(x, Du_\varepsilon(x))\nu_\varepsilon(x) + \varepsilon^\lambda h(x, u_\varepsilon)) = 0, \end{cases} \quad \text{on } \Sigma^{a,lat}_\varepsilon, \\ \\ a(x, Du_\varepsilon(x))\nu_\varepsilon(x) = 0, \text{ on } \partial\Omega_\varepsilon \setminus \left(\Sigma^a_\varepsilon \cap \Sigma^{a,lat}_\varepsilon\right), \end{cases} \tag{1.1}$$

where a, a_0, and h are Carathéodory functions, satisfying monotone and usual p-growth conditions in the second variable (see assumptions (3.1)-(3.12)), $p \in [2,+\infty[$, $f \in L^{\frac{p}{p-1}}(\Omega)$, $g \in W^{1,p}(\Omega^a)$ is a non negative function with $g_{|_{\Sigma^a \cup \Sigma^0}} = 0$, $\lambda \in [1,+\infty[$, and ν_ε denotes the unit outer normal on $\partial\Omega_\varepsilon$. The Signorini boundary conditions in the third and fourth lines in (1.1) mean that on the lateral boundary of the cylinders $\Sigma_\varepsilon^{a,lat}$ one can distinguish two *a priori* unknown subsets where u_ε satisfies the complementary boundary conditions:

$$u_\varepsilon(x) = g(x), \qquad \text{or} \qquad a(x, Du_\varepsilon(x))\nu_\varepsilon(x) = -\varepsilon^\lambda h(x, u_\varepsilon).$$

Problem (1.1) can modelize chemical activity in a multi-structure with thick absorbers (for instance, the adsorption of nutrients on the tissues of the stomach wall and intestine lining). The impossibility to control physical processes on the lateral boundary of the teeth suggests to use Signorini boundary conditions which seem more realistic for describing real phenomena. Some experiments in thick absorbers are described in [32]. See also [44] about nonlinear boundary conditions in chemical engineering.

The weak formulation of problem (1.1) is given by the following variational inequality (for instance, see [37] or our Appendix)

$$\begin{cases} u_\varepsilon \in \mathcal{K}_\varepsilon = \left\{ v \in W^{1,p}(\Omega_\varepsilon) : v \le g \text{ on } \Sigma_\varepsilon^{a,lat}, \quad v_{|_{\Sigma_\varepsilon^a}} = 0 \right\}, \\[2ex] \displaystyle\int_{\Omega_\varepsilon} a(x, Du_\varepsilon) D(v - u_\varepsilon)dx + \int_{\Omega_\varepsilon} a_0(x, u_\varepsilon)(v - u_\varepsilon)dx \\[2ex] \displaystyle +\varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} h(x, u_\varepsilon)(v - u_\varepsilon)d\sigma \ge \int_{\Omega_\varepsilon} f(v - u_\varepsilon)dx, \quad \forall v \in \mathcal{K}_\varepsilon. \end{cases} \tag{1.2}$$

The existence, the uniqueness, and *a priori* $H^1(\Omega_\varepsilon)$-estimates (independent of ε) of the solution u_ε to problem (1.2) will be proved in Section 5. The goal of this paper is to study the asymptotic behaviour of u_ε as ε vanishes, i.e. when the number of thin attached cylinders increases unboundedly, while their thickness tends to zero. In addition, the passage to the limit is accompanied by the perturbed coefficient ε^λ in the nonlinear Signorini conditions. The influence of this perturbation on the asymptotic behavior of the solution will be also studied.

As said above, to studying the asymptotic behavior of problem (1.2), as ε vanishes, one works in the fixed domain Ω. To this aim, one introduces the zero extension operator to Ω^a, i.e. $\widetilde{v}$ denotes the zero extension to Ω^a of any function v defined in Ω_ε^a (see definition (2.3)). The main result of this paper is stated in the following theorem.

Theorem 1.1. *Let Ω^a, Ω^b, Σ^0, Ω_ε, Σ_ε^a, and $\Sigma_\varepsilon^{a,lat}$ be defined in Section 2. Assume* (3.1)-(3.15). *For every ε, let u_ε be the unique solution to problem* (1.2) *and set $u_\varepsilon^a = u_{\varepsilon|_{\Omega_\varepsilon^a}}$, $u_\varepsilon^b = u_{\varepsilon|_{\Omega^b}}$. Let $\mathcal{K}$ be defined by*

$$\mathcal{K} = \left\{ \begin{array}{l} v = (v^a, v^b) \in L^p(\Omega^a) \times W^{1,p}(\Omega^b) \; : \; D_{x_N} v^a \in L^p(\Omega^a), \\[1ex] v^a \le g \textit{ a.e. in } \Omega^a, \quad v^a_{|_{\Sigma^a}} = 0, \quad v^a_{|_{\Sigma^0}} = v^b_{|_{\Sigma^0}} \end{array} \right\}. \tag{1.3}$$

Then,

$$\widetilde{u_\varepsilon^a} \rightharpoonup |\omega'| u^a \textit{ weakly in } L^p(\Omega^a), \tag{1.4}$$

$$\widetilde{D_{x_N} u^a_\varepsilon} = D_{x_N} \widetilde{u^a_\varepsilon} \rightharpoonup |\omega'| D_{x_N} u^a \text{ weakly in } L^p(\Omega^a), \tag{1.5}$$

$$\widetilde{D_{x'} u^a_\varepsilon} \rightharpoonup |\omega'| d' \text{ weakly in } (L^p(\Omega^a))^{N-1}, \tag{1.6}$$

$$u^b_\varepsilon \rightharpoonup u^b \text{ weakly in } W^{1,p}(\Omega^b), \tag{1.7}$$

$$\widetilde{a(x, Du^a_\varepsilon)} \rightharpoonup |\omega'| \left(0, \cdots, 0, a_N\left(x, (d', D_{x_N} u^a)\right)\right) \text{ weakly in } (L^{\frac{p}{p-1}}(\Omega^a))^N, \tag{1.8}$$

$$a(x, Du^b_\varepsilon) \rightharpoonup a(x, Du^b) \text{ weakly in } (L^{\frac{p}{p-1}}(\Omega^b))^N, \tag{1.9}$$

$$\widetilde{a_0(x, u^a_\varepsilon)} \rightharpoonup |\omega'| a_0(x, u^a) \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^a), \tag{1.10}$$

$$a_0(x, u^b_\varepsilon) \rightharpoonup a_0(x, u^b) \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^b), \tag{1.11}$$

$$\widetilde{h(x, u^a_\varepsilon)} \rightharpoonup |\omega'| h(x, u^a) \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^a), \tag{1.12}$$

as ε tends to zero, and (u^a, u^b) and d' is the unique solution to the following system

$$\begin{cases} (u^a, u^b) \in \mathcal{K}, \quad d' \in (L^p(\Omega^a))^{N-1}, \\ a'\left(x, (d', D_{x_N} u^a)\right) = 0, \text{ a.e. in } \Omega^a, \\ |\omega'| \displaystyle\int_{\Omega^a} \left(a_N\left(x, (d', D_{x_N} u^a)\right)(D_{x_N} v - D_{x_N} u^a) + a_0(x, u^a)(v - u^a)\right) dx \\ +\delta_{\lambda,1} |\partial\omega'| \displaystyle\int_{\Omega^a} h(x, u^a)(v - u^a) dx \\ + \displaystyle\int_{\Omega^b} \big(a(x, Du^b)(Dv - Du^b) + a_0(x, u^b)(v - u^b)\big)\, dx \\ \geq |\omega'| \displaystyle\int_{\Omega^a} f(v - u^a) dx + \int_{\Omega^b} f(v - u^b) dx, \quad \forall v \in \mathcal{K}, \end{cases} \tag{1.13}$$

where $a' = (a_1, \cdots, a_{N-1})$, $a = (a', a_N)$, $|\omega'|$ and $|\partial\omega'|$ denote the $(N-1)$-Lebesgue measure and the $(N-2)$-Hausdorff measure of the cross-section ω' of the reference cylinder (see Section 2) and of its boundary $\partial\omega'$, respectively, and $\delta_{\lambda,1}$ is the Kronecker delta.

Problem (1.13) admits a unique solution $(u^a, u^b) \in \mathcal{K}$, $d' \in (L^p(\Omega^a))^{N-1}$. This problem is composed by the algebraic system

$$a'\left(x, (d'(x), D_{x_N} u^a(x))\right) = 0, \text{ a.e. in } \Omega^a, \tag{1.14}$$

with $N-1$ equations and N unknowns (d', u^a), coupled to a variational inequality involving (d', u^a, u^b), with u^a and u^b satisfying a transmission condition on Σ^0. In the case of the p-laplacian, i.e. $a(x, \xi) = |\xi|^{p-2}\xi$, the algebraic system (1.14) implies $d' = 0$ and consequently in problem (1.13) $a_N\left(x, (d', D_{x_N} u^a)\right) = |D_{x_N} u^a|^{p-2} D_{x_N} u^a$. In general, d' is not the zero valued function, also in the linear case $a(x, \xi) = A(x)\xi$ (see Reamark 7.2).

If $\lambda > 1$, the Signorini boundary condition does not give any contribution to the limit problem; while if $\lambda = 1$, the Signorini boundary condition becomes the volume integral

$$|\partial\omega'| \int_{\Omega^a} h(x, u^a)(v - u^a) dx,$$

in the limit problem. If $\lambda < 1$, the existence and the uniqueness of the solution to problem (1.2) holds agains true, but one can not expect to obtain *a priori* estimates independent of ε, without additional assumptions on h (see Remark 5.4).

If in problem (1.2) one adds the term

$$\varepsilon^\beta \int_{\Sigma_\varepsilon^{a,lat}} w(v - u_\varepsilon^a) d\sigma, \quad \forall v \in \mathcal{K}_\varepsilon,$$

with $\beta \in [1, +\infty[$ and $w \in W^{1,\frac{p}{p-1}}(\Omega_\varepsilon^a)$, it is easy to prove that the following term

$$\delta_{\beta,1} |\partial\omega'| \int_{\Omega^a} w(v - u^a) dx, \quad \forall v \in \mathcal{K}.$$

will appears in limit problem (1.13).

Problem (1.13) can be seen as the weak formulation of the following problem.

$$\begin{cases} a'\left(x, (d', D_{x_N} u^a)\right) = 0, \ \text{a.e. in } \Omega^a, \\[2ex] -D_{x_N} a_N\left(x, (d', D_{x_N} u^a)\right) + a_0(x, u^a) + \delta_{\lambda,1} \dfrac{|\partial\omega'|}{|\omega'|} h(x, u^a) \le f, \ \text{in } \Omega^a, \\[2ex] u^a \le g \text{ in } \Omega^a, \\[2ex] (u^a - g)\left(-D_{x_N} a_N\left(x, (d', D_{x_N} u^a)\right) + a_0(x, u^a) + \delta_{\lambda,1} \dfrac{|\partial\omega'|}{|\omega'|} h(x, u^a) - f\right) = 0, \ \text{in } \Omega^a, \\[2ex] -\text{div}(a(x, Du^b)) + a_0(x, u^b) = f, \ \text{in } \Omega^b, \\[2ex] u^a = 0, \ \text{on } \Sigma^a, \\[2ex] u^a = u^b, \quad |\omega'| a_N\left(x, (d', D_{x_N} u^a)\right) = a_N(x, Du^b), \ \text{on } \Sigma^0, \\[2ex] a(x, Du^b) \cdot \nu = 0 \text{ on } \partial\Omega^b \setminus \Sigma^0. \end{cases}$$

The geometry of Ω_ε and problem (1.2) are rigorously introduced in Section 2 and in Section 3, respectively. Section 4 is devoted to introducing an auxiliary problem which transforms the surface integral appearing in problem (1.2) into a volume integral (see also [15], [27], and [36]). The existence, the uniqueness, and *a priori* $H^1(\Omega_\varepsilon)$-estimates (independent of ε) of the solution u_ε to problem (1.2) are proved in Section 5. The proof is based on showing the equi-coercivity, the hemicontinuity, the equi-boundedness, and the strict monotonicity of operator $\mathcal{A}_\varepsilon$ associated with the left-hand side of the variational inequality in (1.2) (see definition (3.16)-(3.17)), and on applying a general result given by J.L. Lions in [34]. Consequently, a convergence result is obtained in Theorem 6.1 in Section 6. In the same section, the method of oscillating test functions introduced by L. Tartar in [45] allows us to prove that the first $N-1$ components of the $L^{\frac{p}{p-1}}(\Omega)$-weak limit of $a\widetilde{(x, Du_\varepsilon^a)}$ are zero (Proposition 6.2). The identification of the other weak limits (Proposition 6.4) is based on a monotone inequality proved in Proposition 6.3. Differently from [5], we explicitly note that the proof of this monotone relation does not make use of the convergence of the energies, which we can not prove in this variational inequality. Finally, the limit variational inequality is identified in Section 7, where also the uniqueness of its solution is proved.

The Signorini boundary conditions were introduced in [43] (see also [26]). About variational inequality, we refer to [11], [24], [31], and [34]. The homogenization of Signorini's type-like problems in perforated domains was studied in [16]. The homogenization of highly rough boundary is largely studied in literature. The pioneering work was the Ph.D. thesis of R. Brizzi and J.-P. Chalot [12] (partially published in [13]) where the Laplace equation with homogeneous Neumann boundary condition was studied, while non-homogeneous Neumann boundary conditions were considered in [27]. Later, the problem has been revisited in [17] by the the unfolding technique (see also [1] for a semi-linear problem). An asymptotic expansion giving an error estimate was built in [35] (see also [39] for the spectral problem). Robin conditions were considered in [19]. Strongly contrasting diffusivity in linear problems was studied in [30]. Monotone operators always with homogeneous Neumann boundary condition were studied in [5]. Integral energies with convex integrands defined on one-dimensional networks were considered in [10]. Linear Signorini boundary conditions were introduced in [38] for the Laplace equation and in [37] for a semi-linear equation. Linear elasticity was investigated in [6], [7], and [8], while Stokes problems in [2] and in [18]. Nonconvex energies were treated in [3]. Problems with more general right-hand side were recently examined in [29] and [28]. See [20], [21], [22], [23], [41], and [42] for control problems. See [4] and [40] for transmission problems in domains separated by an oscillating interface, and [25] for an Helmotz equation posed in two half-planes communicating through a random set of channels. The best constant for the Sobolev trace embedding in Ω_ε was studied in [9]. A large bibliography on the homogenization of boundaries with teeth having vanishing height is also present in literature, but this argument is beyond the scope of this paper and a reader interested in this subject can see the references quoted in [29].

2 The geometry of the domain with highly rough boundary

Let $N \in \mathbb{N}$, $N \geq 2$. A generic element of $\mathbb{R}^{N-1}$ will be denoted by x' and a generic element of $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$ will be denoted by $x = (x', x_N)$. If A is a Lebesgue-measurable subset of $\mathbb{R}^N$ (resp. $\mathbb{R}^{N-1}$), the characteristic function of A in $\mathbb{R}^N$ (resp. $\mathbb{R}^{N-1}$) will be denoted by χ_A and the N-dimensional (resp. $(N-1)$-dimensional) measure of A by $|A|$. The Kronecker delta will be denoted by δ_{ij}.

Let $\Omega' \subset \mathbb{R}^{N-1}$ be a bounded open connected set with Lipschitz boundary. Moreover, let $l^a \in]0, +\infty[$ and $\psi^b \in C(\mathbb{R}^{N-1})$ be such that

$$l^a > 0 > \psi^b(x'), \quad \forall\, x' \in \mathbb{R}^{N-1}.$$

We set

$$\Sigma^a = \left\{x \in \mathbb{R}^N \,;\, x' \in \Omega',\ x_N = l^a\right\},$$

$$\Omega^a = \left\{x \in \mathbb{R}^N \,:\, x' \in \Omega',\ 0 < x_N < l^a\right\},$$

$$\Sigma^0 = \left\{x \in \mathbb{R}^N \,:\, x' \in \Omega',\ x_N = 0\right\},$$

$$\Omega^b = \left\{x \in \mathbb{R}^N \,:\, x' \in \Omega',\ \psi^b(x') < x_N < 0\right\},$$

$$\Omega = \left\{x \in \mathbb{R}^N \,:\, x' \in \Omega',\ \psi^b(x') < x_N < l^a\right\} = \Omega^a \cup \Sigma^0 \cup \Omega^b.$$

Let $\varepsilon \in]0,1[$ be a parameter taking values in a vanishing sequence of real positive numbers, let $\omega' \subset\subset]0,1[^{N-1}$ be a bounded open connected set with C^{m+2}-regularity, $m > \frac{N-1}{2}$, and let Q_ε be the "forest of cylinders" defined by

$$Q_\varepsilon = \bigcup_{k \in \mathbb{Z}^{N-1} :\, \varepsilon\omega' + \varepsilon k \Subset \Omega'} (\varepsilon\omega' + \varepsilon k) \times [0, l^a].$$

Then, we set

$$\Omega^a_\varepsilon = \Omega^a \cap Q_\varepsilon, \quad \Sigma^a_\varepsilon = \Sigma^a \cap Q_\varepsilon, \quad \Sigma^0_\varepsilon = \Sigma^0 \cap Q_\varepsilon, \quad \Sigma^{a,lat}_\varepsilon = \partial\Omega^a_\varepsilon \setminus (\Sigma^a_\varepsilon \cup \Sigma^0_\varepsilon), \quad \Omega_\varepsilon = \Omega^a_\varepsilon \cup \Sigma^0_\varepsilon \cup \Omega^b,$$

(Ω_ε is a "comb" in dimension $N = 2$ or a "brush" in dimension $N = 3$, with basis Ω^b and teeth $Q_\varepsilon \cap \Omega^a$).

Note that

$$\chi_{\Omega^a_\varepsilon} \rightharpoonup |\omega'| \text{ weakly-star in } L^\infty(\Omega^a), \text{ as } \varepsilon \to 0, \tag{2.1}$$

and

$$\chi_{\Sigma^a_\varepsilon} \rightharpoonup |\omega'| \text{ weakly-star in } L^\infty(\Sigma^a), \quad \chi_{\Sigma^0_\varepsilon} \rightharpoonup |\omega'| \text{ weakly-star in } L^\infty(\Sigma^0), \text{ as } \varepsilon \to 0. \tag{2.2}$$

For every ε and for every function $v \in (L^1(\Omega^a_\varepsilon))^M$, with $M \in \mathbb{N}$, $M \geq 1$, we set

$$\widetilde{v}(x) = \begin{cases} v(x), & \text{if } x \in \Omega^a_\varepsilon, \\ 0, & \text{if } x \in \Omega^a \setminus \Omega^a_\varepsilon. \end{cases} \tag{2.3}$$

Note that this extension operator actually depends on ε, although this dependence does not appear in the notation. Moreover, for every function $v \in (L^p(\Omega_\varepsilon))^N$, we set

$$\|v\|_{L^p(\Omega_\varepsilon)} = \|\,|v|\,\|_{L^p(\Omega_\varepsilon)}.$$

3 The problem

Let $p \in [2, +\infty[$, let

$$a : (x,\xi) \in \Omega \times \mathbb{R}^N \to a(x,\xi) = (a_1(x,\xi), \cdots, a_{N-1}(x,\xi), a_N(x,\xi)) = (a'(x,\xi), a_N(x,\xi)) \in \mathbb{R}^N$$

be a function such that

$$a \text{ is a Carathéodory function,} \tag{3.1}$$

$$a(x,\cdot) \text{ is strictly monotone for a.e. } x \in \Omega, \tag{3.2}$$

$$\exists \alpha \in]0,+\infty[,\ \alpha_1 \in L^1(\Omega) \ : \ \alpha|\xi|^p + \alpha_1(x) \leq a(x,\xi)\xi, \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \tag{3.3}$$

$$\exists \beta \in]0,+\infty[,\ \beta_1 \in L^{\frac{p}{p-1}}(\Omega) \ : \ |a(x,\xi)| \leq \beta|\xi|^{p-1} + \beta_1(x), \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N; \tag{3.4}$$

let

$$a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$$

be a function such that

$$a_0 \text{ is a Carathéodory function,} \tag{3.5}$$

$$a_0(x,\cdot) \text{ is monotone for a.e. } x \in \Omega, \tag{3.6}$$

$$\exists \gamma \in]0,+\infty[,\ \gamma_1 \in L^{\frac{p}{p-1}}(\Omega) \ : \ |a_0(x,t)| \leq \gamma|t|^{p-1} + \gamma_1(x), \quad \text{a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}; \tag{3.7}$$

let

$$h : \Omega^a \times \mathbb{R} \to \mathbb{R}$$

be a function such that

$$h \text{ is a continuous function,} \tag{3.8}$$

$$h(x,\cdot) \text{ is monotone for a.e. } x \in \Omega^a, \tag{3.9}$$

$$\exists \eta \in]0,+\infty[,\ \eta_1 \in W^{1,\frac{p}{p-1}}(\Omega^a) \ : \ |h(x,t)| \leq \eta|t|^{p-1} + \eta_1(x), \quad \text{a.e. } x \in \Omega^a, \ \forall t \in \mathbb{R}; \tag{3.10}$$

$$\exists Dh \text{ and it is a Carathéodory valued-function,} \tag{3.11}$$

$$\begin{cases} \exists \theta \in]0,+\infty[,\ \theta_1 \in L^{\frac{p}{p-1}}(\Omega^a) \ : \ |D_t h(x,t)| \leq \theta|t|^{p-2}, \\ \\ |D_{x_i} h(x,t)| \leq \theta|t|^{p-1} + \theta_1(x), \quad \text{a.e. } x \in \Omega^a, \quad \forall t \in \mathbb{R}, \quad \forall i \in \{1,\cdots,N\}; \end{cases} \tag{3.12}$$

let

$$f \in L^{\frac{p}{p-1}}(\Omega), \tag{3.13}$$

$$g \in W^{1,p}(\Omega^a), \quad g \geq 0 \text{ a.e. in } \Omega^a, \quad g_{|_{\Sigma^a \cup \Sigma^0}} = 0, \tag{3.14}$$

and

$$\lambda \in [1,+\infty[. \tag{3.15}$$

Consider the operator

$$\mathcal{A}_\varepsilon : W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon) = \left\{ v \in W^{1,p}(\Omega_\varepsilon) : v_{|_{\Sigma^a_\varepsilon}} = 0 \right\} \longmapsto \left(W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon)\right)' \tag{3.16}$$

that corresponds to the problem (1.1) through the relation

$$\begin{aligned} \langle \mathcal{A}_\varepsilon(u), v \rangle_\varepsilon = & \int_{\Omega_\varepsilon} (a(x,Du(x))Dv + a_0(x,u(x))v)\, dx \\ & + \varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x,u(x)) v d\sigma, \quad \forall u,v \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \end{aligned} \tag{3.17}$$

where $\langle \cdot,\cdot \rangle_\varepsilon$ denotes the duality pairing of $(W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon))'$ and $W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon)$.

The following proposition clarifies the meaning of the last integral in (3.17).

Proposition 3.1. *Assume* (3.8)-(3.12)*. Fix* ε *and let* $u \in W^{1,p}(\Omega_\varepsilon^a)$*. Then,*

$$h(\cdot, u_{|\Sigma_\varepsilon^{a,lat}}(\cdot)) \in L^{\frac{p}{p-1}}(\Sigma_\varepsilon^{a,lat}), \tag{3.18}$$

$$h(\cdot, u(\cdot)) \in W^{1,1}(\Omega_\varepsilon^a), \tag{3.19}$$

$$(h(\cdot, u(\cdot)))_{|\Sigma_\varepsilon^{a,lat}} \in L^1(\Sigma_\varepsilon^{a,lat}), \tag{3.20}$$

and

$$h(\cdot, u_{|\Sigma_\varepsilon^{a,lat}}(\cdot)) = (h(\cdot, u(\cdot)))_{|\Sigma_\varepsilon^{a,lat}}, \text{ a.e. on } \Sigma_\varepsilon^{a,lat}, \tag{3.21}$$

where $u_{|\Sigma_\varepsilon^{a,lat}}(\cdot)$ *and* $(h(\cdot, u(\cdot)))_{|\Sigma_\varepsilon^{a,lat}}$ *denote the trace of* u_ε *and the trace of* $h(\cdot, u(\cdot))$ *on* $\Sigma_\varepsilon^{a,lat}$*, respectively.*

Proof. Assumption (3.10) implies (3.18). Assumptions (3.10) and (3.12) imply (3.19) and consequently (3.20).

To prove (3.21), let $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega_\varepsilon^a})$ be a sequence such that

$$\varphi_n \to u \text{ strongly in } W^{1,p}(\Omega_\varepsilon^a), \text{ as } n \to +\infty. \tag{3.22}$$

Consequently,

$$\varphi_{n|\Sigma_\varepsilon^{a,lat}} \to u_{|\Sigma_\varepsilon^{a,lat}} \text{ strongly in } L^p(\Sigma_\varepsilon^{a,lat}), \text{ as } n \to +\infty. \tag{3.23}$$

Limits (3.22) and (3.23) provide the existence of a subsequence of $\mathbb{N}$, still denoted by $\mathbb{N}$, $(w_0, w_1, \cdots, w_N) \in (L^p(\Omega_\varepsilon^a))^{N+1}$, and $z \in L^p(\Sigma_\varepsilon^{a,lat})$ such that

$$\begin{cases} \varphi_n \to u, \quad D\varphi_n \to Du, \text{ a.e. in } \Omega_\varepsilon^a, \text{ as } n \to +\infty, \\ \\ |\varphi_n| \le w_0, \quad |D_{x_i}\varphi_n| \le w_i, \text{ a.e. in } \Omega_\varepsilon^a, \quad \forall i \in \{1, \cdots, N\}, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.24}$$

and

$$\begin{cases} \varphi_{n|\Sigma_\varepsilon^{a,lat}} \to u_{|\Sigma_\varepsilon^{a,lat}}, \text{ a.e. in } \Sigma_\varepsilon^{a,lat}, \text{ as } n \to +\infty \\ \\ \left|\varphi_{n|\Sigma_\varepsilon^{a,lat}}\right| \le z, \text{ a.e. in } \Sigma_\varepsilon^{a,lat}. \end{cases} \tag{3.25}$$

Using Lebesgue's dominated convergence theorem and assumptions (3.8), (3.10)-(3.12), from (3.24) one deduces

$$h(\cdot, \varphi_n(\cdot)) \to h(\cdot, u(\cdot)) \text{ strongly in } W^{1,1}(\Omega_\varepsilon^a), \text{ as } n \to +\infty, \tag{3.26}$$

and consequently,

$$(h(\cdot, \varphi_n(\cdot)))_{|\Sigma_\varepsilon^{a,lat}} \to (h(\cdot, u(\cdot)))_{|\Sigma_\varepsilon^{a,lat}} \text{ strongly in } L^1(\Sigma_\varepsilon^{a,lat}), \text{ as } n \to +\infty. \tag{3.27}$$

Using the Lebesgue dominated convergence Theorem and assumptions (3.8) and(3.10), from (3.25) one deduces

$$h(\cdot, \varphi_{n|\Sigma_\varepsilon^{a,lat}}(\cdot)) \to h(\cdot, u_{|\Sigma_\varepsilon^{a,lat}}(\cdot)) \text{ strongly in } L^{\frac{p}{p-1}}(\Sigma_\varepsilon^{a,lat}), \text{ as } n \to +\infty. \tag{3.28}$$

On the other side, thanks to (3.8), one has

$$h(\cdot, \varphi_{n|\Sigma_\varepsilon^{a,lat}}(\cdot)) = (h(\cdot, \varphi_n(\cdot)))_{|\Sigma_\varepsilon^{a,lat}}, \text{ on } \Sigma_\varepsilon^{a,lat}. \tag{3.29}$$

Funally, (3.21) is obtained passing to the limit, as n diverges, in (3.29) and using (3.27) and (3.28). □

Remark 3.2. *By virtue of Proposition 3.1, it is not necessary to distinguish between the functions* $h(\cdot, u_{|\Sigma_\varepsilon^{a,lat}}(\cdot))$ *and* $(h(\cdot, u(\cdot)))_{|\Sigma_\varepsilon^{a,lat}}$*, and from now on, by abuse of notation, both functions will be denoted by* $h(\cdot, u)$ *on* $\Sigma_\varepsilon^{a,lat}$*. Moreover from now on, some dependances on* x *will be omitted when it is clear. So, for instance,* $a(x, Du(x))$ *and* $a_0(x, u(x))$ *will be denoted by* $a(x, Du)$ *and* $a_0(x, u)$*, respectively.*

The weak formulation of (1.1) is (for instance, see [37] or our Appendix)

$$\begin{cases} u_\varepsilon \in \mathcal{K}_\varepsilon = \left\{ v \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a) : v \leq g \text{ on } \Sigma_\varepsilon^{a,lat} \right\}, \\ \\ \langle \mathcal{A}_\varepsilon(u_\varepsilon), v - u_\varepsilon \rangle_\varepsilon \geq \displaystyle\int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx, \quad \forall v \in \mathcal{K}_\varepsilon. \end{cases} \tag{3.30}$$

In section 5, the existence, the uniqueness, and *a priori* estimates of the solution u_ε of problem (3.30) will be proved. In what follows, we set

$$u_\varepsilon^a = u_{\varepsilon}{}_{|\Omega_\varepsilon^a} \text{ and } u_\varepsilon^b = u_{\varepsilon}{}_{|\Omega^b}.$$

The goal of this paper is to study the asymptotic behaviour of the solution u_ε to problem (3.30), as ε vanishes.

4 An auxiliary problem

This section is devoted to introduce an auxiliary problem which transforms a $(N-1)$-dimensional integral into a N-dimensional integral (see also [15], [27], and [36]).

Let Ξ be the unique weak solution to the following problem

$$\begin{cases} \Delta \Xi = \dfrac{|\partial \omega'|}{|\omega'|}, \text{ in } \omega', \\ \\ D\Xi \cdot \nu = 1, \text{ on } \partial\omega', \\ \\ \displaystyle\int_{\omega'} \Xi dy' = 0, \end{cases} \tag{4.1}$$

where ν denotes the unit outer normal on $\partial\omega'$ and $|\partial\omega'|$ the $(N-2)$-Hausdorff measure of $\partial\omega'$. Note that Ξ belongs to $C^2(\overline{\omega'})$, since ω' has C^{m+2}-regularity with $m > \frac{N-1}{2}$. Consequently Ξ is also a classical solution to problem (4.1). In what follows, we set

$$C_\Xi = \sup_{\overline{\omega'}} |D\Xi|. \tag{4.2}$$

Lemma 4.1. *Let* Ξ *be denoting also the* $]0,1[^{N-1}$*- periodic extension to* $\displaystyle\bigcup_{k \in \mathbb{Z}^{N-1}} (\overline{\omega'} + k)$ *of the solution to problem* (4.1)*. Then, one has*

$$\varepsilon \int_{\Sigma_\varepsilon^{a,lat}} v d\sigma = \frac{|\partial\omega'|}{|\omega'|} \int_{\Omega_\varepsilon^a} v dx + \varepsilon \int_{\Omega_\varepsilon^a} (D\Xi) \left(\frac{x'}{\varepsilon} \right) D_{x'} v dx, \quad \forall v \in W^{1,1}(\Omega_\varepsilon^a). \tag{4.3}$$

Proof. Problem (4.1) implies

$$\begin{cases} \Delta\left(\Xi\left(\dfrac{x'}{\varepsilon}\right)\right) = \dfrac{1}{\varepsilon^2}\dfrac{|\partial\omega'|}{|\omega'|}, \text{ in } \varepsilon\omega' + \varepsilon k, \\ \\ D\left(\Xi\left(\dfrac{x'}{\varepsilon}\right)\right)\nu = \dfrac{1}{\varepsilon}, \text{ on } \varepsilon\partial\omega' + \varepsilon k, \end{cases} \qquad \forall k \in \mathbb{Z}^{N-1}, \tag{4.4}$$

where ν denotes the unit outer normal on $\bigcup\limits_{k\in\mathbb{Z}^{N-1}} (\varepsilon\partial\omega' + \varepsilon k)$.

Let $v \in W^{1,1}(\Omega_\varepsilon^a)$. Multiplying equations in (4.4) by $\varepsilon^2 v$, integrating by parts, summing up on k such that $\varepsilon\omega' + \varepsilon k \subset \Omega'$, and then integrating in x_N on $]0, l_a[$ give (4.3). □

5 The existence, the uniqueness, and *a priori* estimate of the solution to problem (3.30)

This section is devoted to proving the existence, the uniqueness, and *a priori* estimates of the solution to problem (3.30). To this aim, the Poincaré inequality in $W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a)$ with a constant independent of ε is needed. It was proved in [35] (see Lemma 1.1) in the case $p = 2$. For the convenience of the reader, here with a few of changes the proof in the case $p \in [2, +\infty[$ is provided.

Lemma 5.1. *The following inequality holds.*

$$\begin{aligned} &\exists C_{Poi} \in]0, +\infty[, \quad \exists \varepsilon_0 \in]0, 1[\,:\, \|u\|_{L^p(\Omega_\varepsilon)} \le C_{Poi}\|Du\|_{L^p(\Omega_\varepsilon)}, \\ &\forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a), \quad \forall \varepsilon \in]0, \varepsilon_0[. \end{aligned} \tag{5.1}$$

Proof. Suppose the contrary. Then, there exist a subsequence of $\{\varepsilon\}$, $\{\varepsilon_m\}_{m\in\mathbb{N}}$, and a sequence of functions $\{v_m\}_{m\in\mathbb{N}}$ with $v_m \in W^{1,p}(\Omega_{\varepsilon_m}, \Sigma_{\varepsilon_m}^a)$ such that

$$\lim_{m\to+\infty} \varepsilon_m = 0,$$

$$\|v_m\|_{L^p(\Omega_{\varepsilon_m})} = 1\,, \tag{5.2}$$

$$\|Dv_m\|_{L^p(\Omega_{\varepsilon_m})} < \frac{1}{m}. \tag{5.3}$$

Obviously, the sequence $\{v_m\}$ is bounded in $W^{1,p}(\Omega^b)$ and without loss of generality one can assume that it is a Cauchy sequence in $L^p(\Omega^b)$. Consequently, thanks to (5.3), $\{v_m\}$ is a Cauchy sequence also in $W^{1,p}(\Omega^b)$. Indeed,

$$\|v_m - v_n\|^p_{W^{1,p}(\Omega^b)} \le \|v_m - v_n\|^p_{L^p(\Omega^b)} + 2^{p-1}\left(\frac{1}{m} + \frac{1}{n}\right)^p, \quad \forall m, n \in \mathbb{N}.$$

Hence,

$$\exists v \in W^{1,p}(\Omega^b) \;:\; v_m \to v \text{ strongly in } W^{1,p}(\Omega^b), \text{ as } m \to +\infty. \tag{5.4}$$

On the other side, due to the uniform Dirichlet condition on $\Sigma^a_{\varepsilon_m}$ and (5.3), one get

$$\int_{\Omega^a_{\varepsilon_m}} |v_m|^p \, dx \le l^p \int_{\Omega^a_{\varepsilon_m}} |D_{x_3} v_m|^p \, dx < \frac{l^p}{m^p}, \quad \forall m \in \mathbb{N}. \tag{5.5}$$

Then, (5.2), (5.3), (5.4), and (5.5) imply

$$\begin{cases} 1 = \|v_m\|^p_{L^p(\Omega_{\varepsilon_m})} \longrightarrow \displaystyle\int_{\Omega^b} |v|^p dx, \quad \text{as} \quad m \to +\infty, \\ \displaystyle\int_{\Omega^b} |\nabla v|^p dx = 0. \end{cases}$$

This get

$$v = \frac{1}{\sqrt[p]{|\Omega^b|}}, \text{ a.e. in } \Omega^b. \tag{5.6}$$

Now note that, on the one hand, one has

$$\int_{\Sigma^0_{\varepsilon_m}} |v_m|^p \, dx' \le l^{p-1} \int_{\Omega^a_{\varepsilon_m}} |D_{x_3} v_m|^p \; dx < \frac{l^{p-1}}{m^p} \longrightarrow 0, \text{ as } m \to +\infty.$$

which implies

$$\left| \int_{\Sigma^0_{\varepsilon_m}} v_m \, dx' \right| \le |\Sigma^0_{\varepsilon_m}|^{\frac{p-1}{p}} \|v_m\|_{L^p(\Sigma^0_{\varepsilon_m})} \le |\Omega'|^{\frac{p-1}{p}} \|v_m\|_{L^p(\Sigma^0_{\varepsilon_m})} \longrightarrow 0 \quad \text{as} \quad m \to +\infty. \tag{5.7}$$

On the other hand, (5.4), (5.6), and (2.2) provide

$$\int_{\Sigma^0_{\varepsilon_m}} v_m \, dx' = \int_{\Sigma^0} \chi_{\Sigma^0_{\varepsilon_m}} v_m \, dx' \longrightarrow |\omega'| \frac{1}{\sqrt[p]{|\Omega^b|}} |\Omega'| \neq 0, \text{ as } m \to +\infty,$$

which contradicts (5.7). The lemma is so proved. □

Remark 5.2. *In what follows, all constants in inequalities are independent of the parameter ε. Moreover, the symbol "$\forall \varepsilon$" will mean that "$\exists \varepsilon_0 \in]0,1[\, : \, \forall \varepsilon \in]0, \varepsilon_0[$".*

Theorem 5.3. *Assume* (3.1)-(3.15). *Let $\mathcal{A}_\varepsilon$ be defined by* (3.16)-(3.17). *Then, for every ε, problem* (3.30) *admits a solution u_ε and it is unique. Moreover,*

$$\exists \, c > 0 \; : \; \|Du_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le c, \quad \forall \varepsilon. \tag{5.8}$$

Proof. The proof will be divided into five steps.

Step 1. This step is devoted to proving the equi-coercivity of $\mathcal{A}_\varepsilon$, i.e.

$$\exists c_1, \, c_2 \in]0, +\infty[\, : \, \langle \mathcal{A}_\varepsilon(u), u \rangle_\varepsilon \ge -c_1 + c_2 \|Du\|^p_{L^p(\Omega_\varepsilon)}, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall \varepsilon. \tag{5.9}$$

Assumption (3.3) implies

$$\langle \mathcal{A}_\varepsilon(u), u\rangle_\varepsilon \geq \alpha \|Du\|^p_{L^p(\Omega_\varepsilon)} - \|\alpha_1\|_{L^1(\Omega)} + \int_{\Omega_\varepsilon} a_0(x,u)u dx + \varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x,u)u d\sigma, \tag{5.10}$$
$$\forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall \varepsilon.$$

As far as the second term in the right-hand side of (5.10) is concerned, assumption (3.6) implies

$$\int_{\Omega_\varepsilon} a_0(x,u)u dx = \int_{\Omega_\varepsilon} (a_0(x,u) - a_0(x,0))(u-0)dx + \int_{\Omega_\varepsilon} a_0(x,0)u dx$$
$$\geq \int_{\Omega_\varepsilon} a_0(x,0)u dx, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall \varepsilon. \tag{5.11}$$

Moreover, the Young inequality, (3.7), and the Poincaré inequality provide the following estimate for the last term in (5.11)

$$\left|\int_{\Omega_\varepsilon} a_0(x,0)u dx\right| \leq \frac{\|\gamma_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Omega)}}{\delta^{\frac{p}{p-1}} \frac{p}{p-1}} + \frac{\delta^p}{p}\|u\|^p_{L^p(\Omega_\varepsilon)}$$
$$\leq \frac{p-1}{p}\frac{1}{\delta^{\frac{p}{p-1}}}\|\gamma_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Omega)} + \frac{\delta^p}{p} C^p_{Poi} \|Du\|^p_{L^p(\Omega_\varepsilon)}, \tag{5.12}$$
$$\forall \delta \in]0, +\infty[, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall \varepsilon.$$

By combining (5.11) and (5.12) one obtains

$$\int_{\Omega_\varepsilon} a_0(x,u)u dx \geq -\frac{p-1}{p}\frac{1}{\delta^{\frac{p}{p-1}}}\|\gamma_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Omega)} - \delta^p \frac{C^p_{Poi}}{p} \|Du\|^p_{L^p(\Omega_\varepsilon)}, \tag{5.13}$$
$$\forall \delta \in]0, +\infty[, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall \varepsilon.$$

As far as the last term in the right-hand side of (5.10) is concerned, assumption (3.9) implies

$$\varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x,u)u d\sigma = \varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} (h(x,u) - h(x,0))(u-0)d\sigma + \varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x,0)u d\sigma$$
$$\geq \varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x,0)u d\sigma, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall \varepsilon. \tag{5.14}$$

The Young inequality and (3.10) provide the following estimate for the last term in (5.14)

$$\left|\varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x,0)u d\sigma\right| \leq \varepsilon^\lambda \left(\frac{\|\eta_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Sigma^{a,lat}_\varepsilon)}}{\delta^{\frac{p}{p-1}} \frac{p}{p-1}} + \frac{\delta^p}{p}\|u\|^p_{L^p(\Sigma^{a,lat}_\varepsilon)} \right), \tag{5.15}$$
$$\forall \delta \in]0, +\infty[, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall \varepsilon.$$

On the other side, since $|\eta_1|^{\frac{p}{p-1}} \in W^{1,1}(\Omega^a)$, Lemma 4.1 gives

$$\begin{aligned}
&\varepsilon^\lambda \|\eta_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Sigma^{a,lat}_\varepsilon)} = \varepsilon^{\lambda-1}\left(\frac{|\partial\omega'|}{|\omega'|}\|\eta_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Omega^a_\varepsilon)} + \varepsilon\int_{\Omega^a_\varepsilon}(D\Xi)\left(\frac{x'}{\varepsilon}\right)D_{x'}(|\eta_1|^{\frac{p}{p-1}})dx\right) \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|}\|\eta_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Omega^a)} + C_\Xi\int_{\Omega^a}|D_{x'}(|\eta_1|^{\frac{p}{p-1}})|dx\right) \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|} + C_\Xi\right)\||\eta_1|^{\frac{p}{p-1}}\|_{W^{1,1}(\Omega^a)}, \quad \forall\varepsilon,
\end{aligned} \tag{5.16}$$

where C_Ξ is defined in (4.2). Moreover, since $|u|^p \in W^{1,1}(\Omega^a_\varepsilon)$, Lemma 4.1, (3.15), the Poincaré inequality, and the Young inequality give

$$\begin{aligned}
&\varepsilon^\lambda \|u\|^p_{L^p(\Sigma^{a,lat}_\varepsilon)} = \varepsilon^{\lambda-1}\left(\frac{|\partial\omega'|}{|\omega'|}\|u\|^p_{L^p(\Omega^a_\varepsilon)} + \varepsilon\int_{\Omega^a_\varepsilon}(D\Xi)\left(\frac{x'}{\varepsilon}\right)D_{x'}(|u|^p)dx\right) \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|}C^p_{Poi}\|Du\|^p_{L^p(\Omega^a_\varepsilon)} + C_\Xi\, p\int_{\Omega^a_\varepsilon}|u|^{p-1}|Du|dx\right) \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|}C^p_{Poi}\|Du\|^p_{L^p(\Omega^a_\varepsilon)} + C_\Xi(p-1)\|u\|^p_{L^p(\Omega^a_\varepsilon)} + C_\Xi\|Du\|^p_{L^p(\Omega^a_\varepsilon)}\right) \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|}C^p_{Poi} + C_\Xi(p-1)C^p_{Poi} + C_\Xi\right)\|Du\|^p_{L^p(\Omega^a_\varepsilon)}, \\
&\forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall\varepsilon.
\end{aligned} \tag{5.17}$$

By combining (5.14)-(5.17) one obtains

$$\begin{aligned}
&\varepsilon^\lambda\int_{\Sigma^{a,lat}_\varepsilon}h(x,u)ud\sigma \geq -\frac{p-1}{p}\frac{1}{\delta^{\frac{p}{p-1}}}\left(\frac{|\partial\omega'|}{|\omega'|} + C_\Xi\right)\||\eta_1|^{\frac{p}{p-1}}\|_{W^{1,1}(\Omega^a)} \\
&-\frac{\delta^p}{p}\left(\frac{|\partial\omega'|}{|\omega'|}C^p_{Poi} + C_\Xi(p-1)C^p_{Poi} + C_\Xi\right)\|Du\|^p_{L^p(\Omega^a_\varepsilon)}, \\
&\forall\delta \in]0,+\infty[, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall\varepsilon.
\end{aligned} \tag{5.18}$$

Inequalities (5.10), (5.13), and (5.18) provide

$$\begin{aligned}
&\langle\mathcal{A}_\varepsilon(u), u\rangle_\varepsilon \\
&\geq -\|\alpha_1\|_{L^1(\Omega)} - \frac{p-1}{p}\frac{1}{\delta^{\frac{p}{p-1}}}\left[\|\gamma_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Omega)} + \left(\frac{|\partial\omega'|}{|\omega'|} + C_\Xi\right)\||\eta_1|^{\frac{p}{p-1}}\|_{W^{1,1}(\Omega^a)}\right] \\
&+\left[\alpha - \frac{\delta^p}{p}\left(\left(1 + \frac{|\partial\omega'|}{|\omega'|} + C_\Xi(p-1)\right)C^p_{Poi} + C_\Xi\right)\right]\|Du\|^p_{L^p(\Omega^a_\varepsilon)}, \\
&\forall\delta \in]0,+\infty[, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon), \quad \forall\varepsilon.
\end{aligned} \tag{5.19}$$

Finally, (5.9) follows from (5.19) with a suitable choice of δ.

Step 2. This step is devoted to proving the hemicontinuity of $\mathcal{A}_\varepsilon$, i.e. the continuity of the function

$$t \in [0,1] \to \langle \mathcal{A}_\varepsilon(u_1 + tv), u_2 \rangle_\varepsilon,$$

for every $u_1, u_2, v \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a)$ and for every ε.

Fix $\underline{t} \in [0,1]$, ε, and u_1, u_2, $v \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a)$. By (3.17) one has

$$\begin{aligned} &\langle \mathcal{A}_\varepsilon(u_1 + tv), u_2 \rangle_\varepsilon \\ &= \int_{\Omega_\varepsilon} (a(x, D(u_1 + tv))Du_2 + a_0(x, u_1 + tv)u_2)\, dx + \varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} h(x, u_1 + tv)u_2 d\sigma. \end{aligned} \tag{5.20}$$

Assumptions (3.1), (3.4), (3.5), (3.7), (3.8), (3.10), and Lebesgue's dominated convergence theorem provide that

$$\begin{cases} \lim\limits_{t \to \underline{t}} \displaystyle\int_{\Omega_\varepsilon} a(x, D(u_1 + tv))Du_2 dx = \int_{\Omega_\varepsilon} a(x, D(u_1 + \underline{t}v))Du_2 dx, \\ \lim\limits_{t \to \underline{t}} \displaystyle\int_{\Omega_\varepsilon} a_0(x, u_1 + tv)u_2 dx = \int_{\Omega_\varepsilon} a_0(x, u_1 + \underline{t}v)u_2 dx, \\ \lim\limits_{t \to \underline{t}} \displaystyle\int_{\Sigma_\varepsilon^{a,lat}} h(x, u_1 + tv)u_2 d\sigma = \int_{\Sigma_\varepsilon^{a,lat}} h(x, u_1 + \underline{t}v)u_2 d\sigma. \end{cases} \tag{5.21}$$

By combining (5.20) and (5.21) one obtains

$$\lim_{t \to \underline{t}} \langle \mathcal{A}_\varepsilon(u_1 + tv), u_2 \rangle_\varepsilon = \langle \mathcal{A}_\varepsilon(u_1 + \underline{t}v), u_2 \rangle_\varepsilon.$$

Step 3. This step is devoted to proving the equi-boundedness of $\mathcal{A}_\varepsilon$, i.e.

$$\begin{aligned} &\exists C \in]0, +\infty[\, : \ |\langle \mathcal{A}_\varepsilon(u), v \rangle_\varepsilon| \\ &\le C \left(1 + \|u\|^{p-1}_{W^{1,p}(\Omega_\varepsilon)}\right) \|v\|_{W^{1,p}(\Omega_\varepsilon)}, \quad \forall u, v \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a), \quad \forall \varepsilon. \end{aligned} \tag{5.22}$$

As far as the first integral in the right-hand side of (3.17) is concerned, assumptions (3.4) and (3.7), and Hölder inequality imply

$$\begin{cases} \left| \displaystyle\int_{\Omega_\varepsilon} a(x, Du)Dv dx \right| \\ \le \displaystyle\int_{\Omega_\varepsilon} (\beta |Du|^{p-1} + \beta_1)|Dv| dx \le \left(\beta \|u\|^{p-1}_{W^{1,p}(\Omega_\varepsilon)} + \|\beta_1\|_{L^{\frac{p}{p-1}}(\Omega)}\right) \|v\|_{W^{1,p}(\Omega_\varepsilon)}, \\ \left| \displaystyle\int_{\Omega_\varepsilon} a_0(x, u)v dx \right| \\ \le \displaystyle\int_{\Omega_\varepsilon} (\gamma |u|^{p-1} + \gamma_1)|v| dx \le \left(\gamma \|u\|^{p-1}_{W^{1,p}(\Omega_\varepsilon)} + \|\gamma_1\|_{L^{\frac{p}{p-1}}(\Omega)}\right) \|v\|_{W^{1,p}(\Omega_\varepsilon)}. \\ \forall u, v \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a), \quad \forall \varepsilon. \end{cases} \tag{5.23}$$

As far as the last integral in the right-hand side of (3.17) is concerned, assumption (3.10) and Hölder inequality imply

$$
\begin{aligned}
&\left|\varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} h(x,u)v d\sigma\right| \\
&\leq \varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} (\eta|u|^{p-1}+\eta_1)|v|d\sigma \leq \varepsilon^\lambda \left(\eta\|u\|^{p-1}_{L^p(\Sigma_\varepsilon^{a,lat})} + \|\eta_1\|_{L^{\frac{p}{p-1}}(\Sigma_\varepsilon^{a,lat})}\right)\|v\|_{L^p(\Sigma_\varepsilon^{a,lat})} \\
&= \left(\eta\varepsilon^{\lambda\frac{p-1}{p}}\|u\|^{p-1}_{L^p(\Sigma_\varepsilon^{a,lat})} + \varepsilon^{\lambda\frac{p-1}{p}}\|\eta_1\|_{L^{\frac{p}{p-1}}(\Sigma_\varepsilon^{a,lat})}\right)\varepsilon^{\frac{\lambda}{p}}\|v\|_{L^p(\Sigma_\varepsilon^{a,lat})}, \quad \forall u,v \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a), \quad \forall \varepsilon.
\end{aligned} \tag{5.24}
$$

Moreover, (5.16) gives

$$
\begin{aligned}
&\varepsilon^{\lambda\frac{p-1}{p}}\|\eta_1\|_{L^{\frac{p}{p-1}}(\Sigma_\varepsilon^{a,lat})} = \left(\varepsilon^\lambda\|\eta_1\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Sigma_\varepsilon^{a,lat})}\right)^{\frac{p-1}{p}} \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|} + C_\Xi\right)^{\frac{p-1}{p}} \||\eta_1|^{\frac{p}{p-1}}\|^{\frac{p-1}{p}}_{W^{1,1}(\Omega^a)}, \quad \forall\varepsilon,
\end{aligned} \tag{5.25}
$$

and (5.17) gives

$$
\begin{aligned}
&\varepsilon^{\lambda\frac{p-1}{p}}\|u\|^{p-1}_{L^p(\Sigma_\varepsilon^{a,lat})} = \left(\varepsilon^\lambda\|u\|^p_{L^p(\Sigma_\varepsilon^{a,lat})}\right)^{\frac{p-1}{p}} \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|}C^p_{Poi} + C_\Xi(p-1)C^p_{Poi} + C_\Xi\right)^{\frac{p-1}{p}} \|Du\|^{p-1}_{L^p(\Omega_\varepsilon^a)}, \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|}C^p_{Poi} + C_\Xi(p-1)C^p_{Poi} + C_\Xi\right)^{\frac{p-1}{p}} \|u\|^{p-1}_{(W^{1,p}(\Omega_\varepsilon^a))}, \\
&\forall u \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a), \quad \forall\varepsilon,
\end{aligned} \tag{5.26}
$$

and

$$
\begin{aligned}
&\varepsilon^{\frac{\lambda}{p}}\|v\|_{L^p(\Sigma_\varepsilon^{a,lat})} = \left(\varepsilon^\lambda\|v\|^p_{L^p(\Sigma_\varepsilon^{a,lat})}\right)^{\frac{1}{p}} \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|}C^p_{Poi} + C_\Xi(p-1)C^p_{Poi} + C_\Xi\right)^{\frac{1}{p}} \|Du\|_{L^p(\Omega_\varepsilon^a)}, \\
&\leq \left(\frac{|\partial\omega'|}{|\omega'|}C^p_{Poi} + C_\Xi(p-1)C^p_{Poi} + C_\Xi\right)^{\frac{1}{p}} \|u\|_{W^{1,p}(\Omega_\varepsilon^a)}, \\
&\forall v \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a), \quad \forall\varepsilon.
\end{aligned} \tag{5.27}
$$

Finally, (5.22) follows from (3.17) and (5.23)-(5.27).

Step 4. This step is devoted to proving the strict monotonicity of $\mathcal{A}_\varepsilon$, i.e.

$$\begin{cases} \langle \mathcal{A}_\varepsilon(u_1) - \mathcal{A}_\varepsilon(u_2), u_1 - u_2 \rangle_\varepsilon \geq 0 \quad \forall u_1, u_2 \in W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a), \\ \\ \langle \mathcal{A}_\varepsilon(u_1) - \mathcal{A}_\varepsilon(u_2), u_1 - u_2 \rangle_\varepsilon = 0 \iff u_1 = u_2, \end{cases}$$

for every ε.

This property follows from assumptions (3.2), (3.6), (3.9), and the Poincaré inequality in $W^{1,p}(\Omega_\varepsilon, \Sigma_\varepsilon^a)$.

Thus, the existence and uniqueness of the solution to problem (3.30) for every fixed value ε now follow directly from Theorems 8.2 and 8.3 in [34].

Step 5. This step is devoted to the proof of a priori estimate.

We can choose $v = 0$ as test-function in (3.30), since 0 belongs to $\mathcal{K}_\varepsilon$. Then, one has

$$\langle \mathcal{A}_\varepsilon(u_\varepsilon), u_\varepsilon \rangle_\varepsilon \leq \int_{\Omega_\varepsilon} f u_\varepsilon dx, \quad \forall \varepsilon. \tag{5.28}$$

By virtue of (5.9), the Young inequality, and the Poincaré inequality, inequality (5.28) provides

$$\begin{aligned} -c_1 + c_2 \|Du_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} &\leq \frac{p-1}{p\delta^{\frac{p}{p-1}}} \int_{\Omega_\varepsilon} |f|^{\frac{p}{p-1}} dx + \frac{\delta^p}{p} \int_{\Omega_\varepsilon} |u_\varepsilon|^p dx \\ &\leq \frac{p-1}{p\delta^{\frac{p}{p-1}}} \|f\|^{\frac{p-1}{p}}_{L^{\frac{p-1}{p}}(\Omega)} + \frac{\delta^p}{p} C^p_{Poi} \|Du_\varepsilon\|^p_{L^p(\Omega_\varepsilon)}, \quad \forall \delta \in]0, +\infty[, \quad \forall \varepsilon. \end{aligned} \tag{5.29}$$

Finally, *a priori* estimate (5.8) follows from (5.29) and (3.13), with a suitable choice of δ. $\square$

Remark 5.4. *Of course, the previous proof of the existence and uniqueness of the solution of problem* (3.30) *works also if $\lambda < 1$, but in this last case the previous proof of a priori estimate* (5.8) *does not work. Indeed, if $\lambda < 1$, one can not expect to obtain estimates* (5.8) *without additional assumptions, as showed by the following example proved in [27]. Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the unique weak solution of*

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f, \ \textit{in } \Omega_\varepsilon, \\ \\ Du_\varepsilon \nu_\varepsilon = \varepsilon^\lambda, \ \textit{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.30}$$

where $f \in L^2(\Omega)$ and $\lambda \in [0, 1[$, Then,

$$\exists \mu_1, \, \mu_2 \in]0, +\infty[\, : \, \frac{\mu_1}{\varepsilon^{1-\lambda}} \leq \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \frac{\mu_2}{\varepsilon^{1-\lambda}}, \quad \forall \varepsilon.$$

The following result is a consequence of *a priori* estimate (5.8).

Corollary 5.5. *Assume* (3.1)-(3.15). *Let $\mathcal{A}_\varepsilon$ be defined by* (3.16)-(3.17). *For every ε, let u_ε be the solution to problem* (3.30) *and set $u_\varepsilon^a = u_{\varepsilon|_{\Omega_\varepsilon^a}}$. Then,*

$$\exists C \in]0, +\infty[\, : \, \|h(x, u_\varepsilon^a)\|_{W^{1,1}(\Omega_\varepsilon^a)} \leq C, \quad \forall \varepsilon. \tag{5.31}$$

Proof. Assumptions (3.10) and (3.12), and Hölder inequality provide that

$$\|h(x,u_\varepsilon^a)\|_{L^p(\Omega_\varepsilon^a)} \le \eta \int_{\Omega_\varepsilon^a} |u_\varepsilon^a|^{p-1}dx + \int_{\Omega_\varepsilon^a} |\eta_1|dx$$

$$\le |\Omega|^{\frac{1}{p}} \left(\eta \|u_\varepsilon^a\|^{p-1}_{L^p(\Omega_\varepsilon^a)} + \|\eta_1\|_{L^{\frac{p}{p-1}}(\Omega^a)} \right), \quad \forall \varepsilon.$$

and

$$\|\partial_{x_i}(h(x,u_\varepsilon^a))\|_{L^1(\Omega_\varepsilon^a)} \le \|(\partial_{x_i}h)(x,u_\varepsilon^a)\|_{L^1(\Omega_\varepsilon^a)} + \|(\partial_t h)(x,u_\varepsilon^a)\partial_{x_i}u_\varepsilon^a\|_{L^1(\Omega_\varepsilon^a)}$$

$$\le \theta \int_{\Omega_\varepsilon^a} |u_\varepsilon^a|^{p-1}dx + \int_{\Omega_\varepsilon^a} |\theta_1|dx + \theta \int_{\Omega_\varepsilon^a} |u_\varepsilon^a|^{p-2}||\partial_{x_i}u_\varepsilon^a|dx$$

$$\le |\Omega|^{\frac{1}{p}} \left(\theta \|u_\varepsilon^a\|^{p-1}_{L^p(\Omega_\varepsilon^a)} + \|\theta_1\|_{L^{\frac{p}{p-1}}(\Omega^a)} \right) + \theta \|u_\varepsilon^a\|^{p-2}_{L^p(\Omega_\varepsilon^a)} \|\partial_{x_i}u_\varepsilon^a\|_{L^{\frac{p}{2}}(\Omega_\varepsilon^a)}$$

$$\le |\Omega|^{\frac{1}{p}} \left(\theta \|u_\varepsilon^a\|^{p-1}_{L^p(\Omega_\varepsilon^a)} + \|\theta_1\|_{L^{\frac{p}{p-1}}(\Omega^a)} + \theta \|u_\varepsilon^a\|^{p-2}_{L^p(\Omega_\varepsilon^a)} \|\partial_{x_i}u_\varepsilon^a\|_{L^p(\Omega_\varepsilon^a)} \right), \quad \forall \varepsilon, \quad \forall i \in \{1, \cdots, N\},$$

which imply (5.31), thanks to *a priori* estimate (5.8) and the Poincaré inequality. □

6 Convergence results

As a consequence of (5.3), we have the following result.

Proposition 6.1. *Assume* (3.1)-(3.15). *Let $\mathcal{A}_\varepsilon$ be defined by* (3.16)-(3.17). *For every ε, let u_ε be the solution to problem* (3.30) *and set $u_\varepsilon^a = u_{\varepsilon|_{\Omega_\varepsilon^a}}$, $u_\varepsilon^b = u_{\varepsilon|_{\Omega^b}}$. Let $\mathcal{K}$ be defined in* (1.3). *Then, there exist a subsequence of $\{\varepsilon\}$, still denoted with $\{\varepsilon\}$, and (depending on the subsequence) $(u^a,u^b) \in \mathcal{K}$, $d' \in (L^p(\Omega^a))^{N-1}$, $(A^a,A^b) \in ((L^{\frac{p}{p-1}}(\Omega^a))^N \times (L^{\frac{p}{p-1}}(\Omega^b))^N$, $(A_0^a,A_0^b) \in L^{\frac{p}{p-1}}(\Omega^a) \times L^{\frac{p}{p-1}}(\Omega^b)$, and $H^a \in L^{\frac{p}{p-1}}(\Omega^a)$ such that*

$$\widetilde{u_\varepsilon^a} \rightharpoonup |\omega'|u^a \text{ weakly in } L^p(\Omega^a), \tag{6.1}$$

$$\widetilde{D_{x_N}u_\varepsilon^a} = D_{x_N}\widetilde{u_\varepsilon^a} \rightharpoonup |\omega'|D_{x_N}u^a \text{ weakly in } L^p(\Omega^a), \tag{6.2}$$

$$\widetilde{D_{x'}u_\varepsilon^a} \rightharpoonup |\omega'|d' \text{ weakly in } (L^p(\Omega^a))^{N-1}, \tag{6.3}$$

$$u_\varepsilon^b \rightharpoonup u^b \text{ weakly in } W^{1,p}(\Omega^b), \tag{6.4}$$

$$\widetilde{a(x,Du_\varepsilon^a)} \rightharpoonup |\omega'|A^a \text{ weakly in } ((L^{\frac{p}{p-1}}(\Omega^a))^N, \tag{6.5}$$

$$a(x,Du_\varepsilon^b) \rightharpoonup A^b \text{ weakly in } ((L^{\frac{p}{p-1}}(\Omega^b))^N, \tag{6.6}$$

$$\widetilde{a_0(x,u_\varepsilon^a)} \rightharpoonup |\omega'|A_0^a \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^a), \tag{6.7}$$

$$a_0(x,u_\varepsilon^b) \rightharpoonup A_0^b \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^b), \tag{6.8}$$

$$\widetilde{h(x,u_\varepsilon^a)} \rightharpoonup |\omega'|H^a \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^a), \tag{6.9}$$

as ε tends to zero.

Proof. This proposition is an immediate consequences of *a priori* estimate (5.8), the Poincaré inequality (5.1), and assumptions (3.4), (3.7), (3.10), and (3.12) (for instance, see [5]). For sake of completeness, we just recall the proof of (for instance, see [37])

$$u^a \leq g, \text{ a.e. in } \Omega^a. \tag{6.10}$$

Since $u_\varepsilon \in \mathcal{K}_\varepsilon$, one has

$$\varepsilon \int_{\Sigma_\varepsilon^{a,lat}} u_\varepsilon^a \varphi d\sigma \leq \varepsilon \int_{\Sigma_\varepsilon^{a,lat}} g\varphi d\sigma, \quad \forall \varphi \in C_0^\infty(\Omega^a) : \varphi \geq 0 \text{ in } \Omega^a, \quad \forall \varepsilon. \tag{6.11}$$

Thanks to Lemma 4.1, inequality (6.11) is equivalent to

$$\frac{|\partial\omega|}{|\omega|} \int_{\Omega^a} \widetilde{u_\varepsilon^a} \varphi dx + \varepsilon \int_{\Omega_\varepsilon^a} (D\Xi) \left(\frac{x'}{\varepsilon}\right) D_{x'}(u_\varepsilon^a \varphi) dx$$

$$\leq \frac{|\partial\omega|}{|\omega|} \int_{\Omega^a} \chi_{\Omega_\varepsilon^a} g\varphi dx + \varepsilon \int_{\Omega_\varepsilon^a} (D\Xi) \left(\frac{x'}{\varepsilon}\right) D_{x'}(g\varphi) dx, \quad \forall \varphi \in C_0^\infty(\Omega^a) : \varphi \geq 0 \text{ in } \Omega^a, \quad \forall \varepsilon.$$

By passing to the limit, as ε tends to zero, in this inequality, limits (2.1), (6.1), and *a priori* estimate (5.8) provide

$$\int_{\Omega^a} u^a \varphi dx \leq \int_{\Omega^a} g\varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega^a) : \varphi \geq 0 \text{ in } \Omega^a,$$

which gives (6.10). □

The first $N-1$ components of A^a are identified by the following proposition (compare [5]).

Proposition 6.2. *Assume* (3.1)-(3.15). *Let $\mathcal{A}_\varepsilon$ be defined by* (3.16)-(3.17). *For every ε, let u_ε be the solution to problem* (3.30). *Let $A^a \in \left(L^{\frac{p}{p-1}}(\Omega^a)\right)^N$ be satisfying* (6.5), *up to a subsequence. Then*

$$(A_1^a, \cdots, A_{N-1}^a) = 0, \text{ a.e. in } \Omega^a. \tag{6.12}$$

Proof. Fix $i \in \{1, \cdots, N-1\}$ and let us prove that

$$A_i^a = 0, \text{ a.e. in } \Omega^a. \tag{6.13}$$

Let $w_\pm^i \in W_{per}^{1,\infty}\left(]0,1[^{N-1}\right)$ be a function such that (compare [12])

$$\begin{cases} w_\pm^i(y') = \pm y_i, \text{ for a.e. } y' = (y_1, \cdots, y_{N-1}) \in \overline{\omega'}, \\ |w_\pm^i| \leq 1, \text{ a.e. in }]0,1[^{N-1}. \end{cases}$$

Extend this function by periodicity on $\mathbb{R}^{N-1}$ and for every ε set

$$w_{\varepsilon\pm}^i : x = (x', x_N) \in \mathbb{R}^N \to \varepsilon w_\pm^i\left(\frac{x'}{\varepsilon}\right) - \varepsilon.$$

Note that

$$\|w_{\varepsilon\pm}^i\|_{L^\infty(\mathbb{R}^N)} \leq 2\varepsilon, \quad \forall \varepsilon, \tag{6.14}$$

$$D_{x_j} w^i_{\varepsilon\pm} = \pm\delta_{ij} \text{ in } \Omega^a_\varepsilon, \quad \forall j \in \{1, \cdots, N\}, \quad \forall\varepsilon, \tag{6.15}$$

and, since $w^i_{\varepsilon\pm} \leq 0$, in $\overline{\Omega^a_\varepsilon}$,

$$w^i_{\varepsilon\pm}\psi + u_\varepsilon \in \mathcal{K}_\varepsilon, \quad \forall\psi \in C^\infty_0(\Omega^a) \,:\, \psi \geq 0 \text{ in } \Omega^a, \quad \forall\varepsilon.$$

Choosing these last functions as test functions in the inequality in (3.30) gives

$$\begin{gathered}\int_{\Omega^a_\varepsilon} \left(a(x, Du^a_\varepsilon) D\left(w^i_{\varepsilon\pm}\psi\right) + a_0(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi\right) dx + \varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi d\sigma \\ \geq \int_{\Omega^a_\varepsilon} f w^i_{\varepsilon\pm}\psi dx, \quad \forall\psi \in C^\infty_0(\Omega^a) \,:\, \psi \geq 0 \text{ in } \Omega^a, \quad \forall\varepsilon,\end{gathered} \tag{6.16}$$

that is, thanks to (6.15),

$$\begin{gathered}\int_{\Omega^a_\varepsilon} \left(\pm a_i(x, Du^a_\varepsilon)\psi + w^i_{\varepsilon\pm} a(x, Du^a_\varepsilon) D\psi + a_0(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi\right) dx \\ +\varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi d\sigma \geq \int_{\Omega^a_\varepsilon} f w^i_{\varepsilon\pm}\psi dx, \quad \forall\psi \in C^\infty_0(\Omega^a) \,:\, \psi \geq 0 \text{ in } \Omega^a, \quad \forall\varepsilon.\end{gathered} \tag{6.17}$$

As far as the first integral in (6.17) is concerned, (6.5), (6.7), and (6.14) provide that

$$\begin{gathered}\int_{\Omega^a_\varepsilon} \left(\pm a_i(x, Du^a_\varepsilon)\psi + w^i_{\varepsilon\pm} a(x, Du^a_\varepsilon) D\psi + a_0(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi\right) dx \\ = \int_{\Omega^a} \left(\pm \widetilde{a_i(x, Du^a_\varepsilon)}\psi + w^i_{\varepsilon\pm} \widetilde{a(x, Du^a_\varepsilon)} D\psi + \widetilde{a_0(x, u^a_\varepsilon)} w^i_{\varepsilon\pm}\psi\right) dx \\ \longrightarrow \pm|\omega'| \int_{\Omega^a} A_i \psi dx, \text{ as } \varepsilon \to 0, \ \forall\psi \in C^\infty_0(\Omega^a) \,:\, \psi \geq 0 \text{ in } \Omega^a.\end{gathered} \tag{6.18}$$

As far as the second integral in (6.17) is concerned, estimate (5.31) and Lemma 4.1 give

$$\begin{aligned}&\left|\varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi dx\right| \\ &= \left|\varepsilon^{\lambda-1}\frac{|\partial\omega'|}{|\omega'|} \int_{\Omega^a_\varepsilon} h(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi dx + \varepsilon^\lambda \int_{\Omega^a_\varepsilon} (D\Xi)\left(\frac{x'}{\varepsilon}\right) D_{x'}\left(h(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi\right) dx\right| \\ &= \left|\varepsilon^{\lambda-1}\frac{|\partial\omega'|}{|\omega'|} \int_{\Omega^a_\varepsilon} h(x, u^a_\varepsilon) w^i_{\varepsilon\pm}\psi dx + \varepsilon^\lambda \int_{\Omega^a_\varepsilon} (D\Xi)\left(\frac{x'}{\varepsilon}\right) D_{x'}(h(x, u^a_\varepsilon)) w^i_{\varepsilon\pm}\psi dx + \right. \\ &\left. \varepsilon^\lambda \int_{\Omega^a_\varepsilon} \pm(D_{x_i}\Xi)\left(\frac{x'}{\varepsilon}\right) h(x, u^a_\varepsilon)\psi dx + \varepsilon^\lambda \int_{\Omega^a_\varepsilon} (D\Xi)\left(\frac{x'}{\varepsilon}\right) D_{x'}\psi h(x, u^a_\varepsilon) w^i_{\varepsilon\pm} dx\right| \\ &\leq C\varepsilon^\lambda \left[2\frac{|\partial\omega'|}{|\omega'|}\|\psi\|_{L^\infty(\Omega^a)} + C_\Xi \|\psi\|_{C^1(\Omega^a)}(4\varepsilon + 1)\right] \longrightarrow 0, \\ &\text{as } \varepsilon \to 0, \ \forall\psi \in C^\infty_0(\Omega^a) \,:\, \psi \geq 0 \text{ in } \Omega^a,\end{aligned} \tag{6.19}$$

where C_Ξ is defined in (4.2).

As far as the last integral in (6.17) is concerned, (3.13) and (6.14) provides that

$$\left|\int_{\Omega^a_\varepsilon} f w^i_{\varepsilon\pm}\psi dx\right| \le \varepsilon 2\|\psi\|_{L^\infty(\Omega^a)}\|f\|_{L^1(\Omega^a)} \longrightarrow 0,$$
$$\text{as } \varepsilon\to 0,\ \ \forall\psi\in C_0^\infty(\Omega^a)\ :\ \psi\ge 0 \text{ in } \Omega^a. \tag{6.20}$$

Finally, passing to the limit, as ε vanishes, in (6.17), thanks to (6.18)-(6.20), one obtains

$$\pm\int_{\Omega^a} A_i\psi dx \ge 0, \quad \forall\psi\in C_0^\infty(\Omega^a)\ :\ \psi\ge 0 \text{ in } \Omega^a,$$

i.e.

$$\int_{\Omega^a} A_i\psi dx = 0, \quad \forall\psi\in C_0^\infty(\Omega^a)\ :\ \psi\ge 0 \text{ in } \Omega^a,$$

which implies (6.13). □

Let

$$\mathcal{V} = \Big\{ v=(v^a,v^b)\in C^\infty(\overline{\Omega^a})\times C^\infty(\overline{\Omega^b})\ :$$
$$v^a\le g \text{ a.e. in } \Omega^a,\quad v^a_{|\Sigma^a}=0,\quad v^a_{|\Sigma^0}=v^b_{|\Sigma^0}\Big\}. \tag{6.21}$$

To identify other limits of Proposition 6.1, the following monotone relation is needed.

Proposition 6.3. *Assume* (3.1)-(3.15). *Let $\mathcal{A}_\varepsilon$ be defined by* (3.16)-(3.17). *For every ε, let u_ε be the solution to problem* (3.30). *Let $(u^a,u^b)\in\mathcal{K}$, $d'\in(L^p(\Omega^a))^{N-1}$, $(A^a,A^b)\in((L^{\frac{p}{p-1}}(\Omega^a))^N\times(L^{\frac{p}{p-1}}(\Omega^b))^N$, $(A^a_0,A^b_0)\in L^{\frac{p}{p-1}}(\Omega^a)\times L^{\frac{p}{p-1}}(\Omega^b)$, and $H^a\in L^{\frac{p}{p-1}}(\Omega^a)$ be satisfying* (6.1)-(6.9), *up to a subsequence. Then*

$$\begin{aligned}
&|\omega'|\int_{\Omega^a}\left(A^a_N(D_{x_N}u^a-\tau_N)+a(x,\tau)(\tau-(d',D_{x_N}u^a))+(A^a_0-a_0(x,\phi))\,(u^a-\phi)\right)dx\\
&+\int_{\Omega^b}\left(\left(A^b-a(x,\tau)\right)(Du^b-\tau)+\left(A^b_0-a_0(x,\phi)\right)(u^b-\phi)\right)dx\\
&+\delta_{\lambda,1}|\partial\omega'|\int_{\Omega^a}\left(H^a-h(x,\psi)\right)(u^a-\psi)dx\ge 0,\\
&\forall\tau\in(L^p(\Omega))^3,\quad\forall\phi\in L^p(\Omega),\quad\forall\psi\in L^p(\Omega^a),
\end{aligned}\tag{6.22}$$

where $\delta_{\lambda,1}$ is the Kronecker delta.

Proof. Fix $\tau\in(L^p(\Omega))^3$ and $\phi\in L^p(\Omega)$.

Assumptions (3.2), (3.6), and (3.9) provide

$$\begin{aligned}
&\int_{\Omega_\varepsilon}\left((a(x,\tau)-a(x,Du_\varepsilon))\,(\tau-Du_\varepsilon)+(a_0(x,\phi)-a_0(x,u_\varepsilon))\,(\phi-u_\varepsilon)\right)dx\\
&+\varepsilon^\lambda\int_{\Sigma^{a,lat}_\varepsilon}(h(x,\psi)-h(x,u_\varepsilon)(\psi-u_\varepsilon)d\sigma\ge 0,\quad\forall\psi\in C^\infty(\overline{\Omega^a}),\quad\forall\varepsilon.
\end{aligned}\tag{6.23}$$

On the other side, by virtue of (3.30), one has

$$\int_{\Omega_\varepsilon} (a(x, Du_\varepsilon)(Dv - Du_\varepsilon) + a_0(x, u_\varepsilon)(v - u_\varepsilon))\, dx + \varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} h(x, u_\varepsilon)(v - u_\varepsilon) d\sigma$$
$$\geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx, \quad \forall v \in \mathcal{V}, \quad \forall \varepsilon. \tag{6.24}$$

Adding (6.23) to (6.24) gives

$$\int_{\Omega_\varepsilon} (a(x, Du_\varepsilon)(Dv - \tau) + a(x, \tau)(\tau - Du_\varepsilon) + a_0(x, u_\varepsilon)(v - \phi) + a_0(x, \phi)(\phi - u_\varepsilon))\, dx$$

$$+\varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} (h(x, u_\varepsilon)(v - \psi) + h(x, \psi)(\psi - u_\varepsilon))\, d\sigma$$

$$\geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx, \quad \forall v \in \mathcal{V}, \quad \forall \psi \in C^\infty(\overline{\Omega^a}), \quad \forall \varepsilon.$$

Since

$$h(x, u_\varepsilon)(v - \psi) + h(x, \psi)(\psi - u_\varepsilon) \in W^{1,1}(\Omega_\varepsilon^a),$$

thanks to 4.1 the last inequatily becomes

$$\int_{\Omega_\varepsilon} (a(x, Du_\varepsilon)(Dv - \tau) + a(x, \tau)(\tau - Du_\varepsilon) + a_0(x, u_\varepsilon)(v - \phi) + a_0(x, \phi)(\phi - u_\varepsilon))\, dx$$

$$+\varepsilon^{\lambda-1} \Bigg(\frac{|\partial\omega'|}{|\omega'|} \int_{\Omega_\varepsilon^a} (h(x, u_\varepsilon)(v - \psi) + h(x, \psi)(\psi - u_\varepsilon))\, dx$$

$$+\varepsilon \int_{\Omega_\varepsilon^a} (D\Xi) \left(\frac{x'}{\varepsilon}\right) D_{x'} \left(h(x, u_\varepsilon)(v - \psi) + h(x, \psi)(\psi - u_\varepsilon)\right) dx \Bigg)$$

$$\geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx, \quad \forall v \in \mathcal{V}, \quad \forall \psi \in C^\infty(\overline{\Omega^a}), \quad \forall \varepsilon,$$

i.e.

$$\int_{\Omega^a}\Big(\widetilde{a(x,Du^a_\varepsilon)}(Dv-\tau)+a(x,\tau)(\tau\chi_{\Omega^a_\varepsilon}-\widetilde{Du^a_\varepsilon})+\widetilde{a_0(x,u^a_\varepsilon)}(v-\phi)+a_0(x,\phi)(\phi\chi_{\Omega^a_\varepsilon}-\widetilde{u^a_\varepsilon})\Big)\,dx$$

$$+\int_{\Omega^b}\big(a(x,Du^b_\varepsilon)(Dv-\tau)+a(x,\tau)(\tau-Du^b_\varepsilon)+a_0(x,u^b_\varepsilon)(v-\phi)+a_0(x,\phi)(\phi-u^b_\varepsilon)\big)\,dx$$

$$+\varepsilon^{\lambda-1}\Bigg(\frac{|\partial\omega'|}{|\omega'|}\int_{\Omega^a}\Big(\widetilde{h(x,u^a_\varepsilon)}(v-\psi)+h(x,\psi)(\psi\chi_{\Omega^a_\varepsilon}-\widetilde{u^a_\varepsilon})\Big)\,dx$$

$$+\varepsilon\int_{\Omega^a_\varepsilon}(D\Xi)\left(\frac{x'}{\varepsilon}\right)D_{x'}\left(h(x,u_\varepsilon)(v-\psi)+h(x,\psi)(\psi-u_\varepsilon)\right)dx\Bigg)$$

$$\geq\int_{\Omega^a}f(v\chi_{\Omega^a_\varepsilon}-\widetilde{u^a_\varepsilon})dx+\int_{\Omega^b}f(v-u^b_\varepsilon)dx,\quad\forall v\in\mathcal{V},\quad\forall\psi\in C^\infty(\overline{\Omega^a}),\quad\forall\varepsilon,$$

Now passing to the limit, as ε vanishes, in this inequality, thanks to (2.1), (5.31), (6.1)-(6.9), and (6.12), and taking into account that $h(\cdot,\psi(\cdot))\in W^{1,\frac{p}{p-1}}(\Omega^a)$, one obtains

$$|\omega'|\int_{\Omega^a}\left(A^a_N(D_{x_N}v-\tau_N)+a(x,\tau)(\tau-(d',D_{x_N}u^a))+A^a_0(v-\phi)+a_0(x,\phi)(\phi-u^a)\right)dx$$

$$+\int_{\Omega^b}\big(A^b(Dv-\tau)+a(x,\tau)(\tau-Du^b)+A^b_0(v-\phi)+a_0(x,\phi)(\phi-u^b)\big)\,dx$$

$$+\delta_{\lambda,1}|\partial\omega'|\int_{\Omega^a}\left(H^a(v-\psi)+h(x,\psi)(\psi-u^a)\right)dx$$

$$\geq|\omega'|\int_{\Omega^a}f(v-u^a)dx+\int_{\Omega^b}f(v-u^b)dx,\quad\forall v\in\mathcal{V},\quad\forall\psi\in C^\infty(\overline{\Omega^a}).\tag{6.25}$$

Finally, since $\mathcal{V}$ is dense in $\mathcal{K}$ and $C^\infty(\overline{\Omega^a})$ is dense in $L^p(\Omega^a)$, inequality (6.25) holds true with $v=(u^a,u^b)$ and $\psi\in L^p(\Omega^a)$, too. So (6.22) is proved. □

The limits of Proposition 6.1 will be identified in the next proposition, by using monotone inequality (6.22).

Proposition 6.4. *Assume* (3.1)-(3.15). *Let* $\mathcal{A}_\varepsilon$ *be defined by* (3.16)-(3.17). *For every* ε, *let* u_ε *be the solution to problem* (3.30). *Let* $(u^a,u^b)\in\mathcal{K}$, $d'\in(L^p(\Omega^a))^{N-1}$, $(A^a,A^b)\in((L^{\frac{p}{p-1}}(\Omega^a))^N\times(L^{\frac{p}{p-1}}(\Omega^b))^N$, $(A^a_0,A^b_0)\in L^{\frac{p}{p-1}}(\Omega^a)\times L^{\frac{p}{p-1}}(\Omega^b)$, *and* $H^a\in L^{\frac{p}{p-1}}(\Omega^a)$ *be satisfying* (6.1)-(6.9), *up to a subsequence. Then*

$$a'\left(x,(d',D_{x_N}u^a)\right)=0,\ \ a.e.\ \ in\ \Omega^a,\tag{6.26}$$

$$A^a_N=a_N\left(x,(d',D_{x_N}u^a)\right),\ \ a.e.\ \ in\ \Omega^a,\tag{6.27}$$

$$A^b=a(x,Du^b),\ \ a.e.\ \ in\ \Omega^b,\tag{6.28}$$

$$A^a_0=a_0(x,u^a),\ \ a.e.\ \ in\ \Omega^a,\tag{6.29}$$

$$A_0^b = a_0(x, u^b), \ \ a.e. \ \ in \ \Omega^b, \tag{6.30}$$

$$H^a = h(x, u^a), \ \ a.e. \ \ in \ \Omega^a, \tag{6.31}$$

where $a' = (a_1, \cdots, a_{N-1})$.

Proof. In (6.22) choosing

$$\begin{cases} \tau = (d' + t\varphi, D_{x_N} u^a), \ \text{a.e. in } \Omega^a, \ \text{with } \varphi \in (C_0^\infty(\Omega^a))^{N-1} \text{ and } t \in]0, +\infty[, \\ \tau = Du^b, \ \text{a.e. in } \Omega^b, \\ \phi = u^a, \ \text{a.e. in } \Omega^a, \\ \phi = u^b, \ \text{a.e. in } \Omega^b, \\ \psi = u^a, \ \text{a.e. in } \Omega^a, \end{cases}$$

one derives

$$\int_{\Omega^a} a'\left(x, (d' + t\varphi, D_{x_N} u^a)\right) t\varphi dx \geq 0, \quad \forall \varphi \in (C_0^\infty(\Omega^a))^{N-1}, \quad \forall t \in]0, +\infty[. \tag{6.32}$$

By multiplying (6.32) by $\frac{1}{t}$ and letting t tend to zero, thanks to assumptions (3.1) and (3.4), one obtains

$$\int_{\Omega^a} a'\left(x, (d', D_{x_N} u^a)\right) \varphi dx \geq 0, \quad \forall \varphi \in (C_0^\infty(\Omega^a))^{N-1},$$

which implies (6.26).

In (6.22) choosing

$$\begin{cases} \tau = (d', D_{x_N} u^a + t\varphi), \ \text{a.e. in } \Omega^a, \ \text{with } \varphi \in C_0^\infty(\Omega^a) \text{ and } t \in]0, +\infty[, \\ \tau = Du^b, \ \text{a.e. in } \Omega^b, \\ \phi = u^a, \ \text{a.e. in } \Omega^a, \\ \phi = u^b, \ \text{a.e. in } \Omega^b, \\ \psi = u^a, \ \text{a.e. in } \Omega^a, \end{cases}$$

one derives

$$\int_{\Omega^a} \left(-A_N^a + a_N\left(x, (d', D_{x_N} u^a + t\varphi)\right)\right) t\varphi dx \geq 0, \quad \forall \varphi \in C_0^\infty(\Omega^a), \quad \forall t \in]0, +\infty[. \tag{6.33}$$

By multiplying (6.33) by $\frac{1}{t}$ and letting t tend to zero, thanks to assumptions (3.1) and (3.4), one obtains

$$\int_{\Omega^a} \left(-A_N^a + a_N\left(x, (d', D_{x_N} u^a)\right)\right) \varphi dx \geq 0, \quad \forall \varphi \in C_0^\infty(\Omega^a),$$

which implies (6.27).

By arguing in the same way, one identifies the other limits. More precisely, in (6.22) choosing

$$\begin{cases} \tau = (d', D_{x_N} u^a), \text{ a.e. in } \Omega^a, \\ \\ \tau = Du^b + t\varphi, \text{ a.e. in } \Omega^b, \text{ with } \varphi \in (C_0^\infty(\Omega^b))^N \text{ and } t \in]0, +\infty[, \\ \\ \phi = u^a, \text{ a.e. in } \Omega^a, \\ \\ \phi = u^b, \text{ a.e. in } \Omega^b, \\ \\ \psi = u^a, \text{ a.e. in } \Omega^a, \end{cases}$$

one derives (6.28).

In (6.22) choosing

$$\begin{cases} \tau = (d', D_{x_N} u^a), \text{ a.e. in } \Omega^a, \\ \\ \tau = Du^b, \text{ a.e. in } \Omega^b, \\ \\ \phi = u^a + t\varphi, \text{ a.e. in } \Omega^a, \text{ with } \varphi \in C_0^\infty(\Omega^a) \text{ and } t \in]0, +\infty[, \\ \\ \phi = u^b, \text{ a.e. in } \Omega^b, \\ \\ \psi = u^a, \text{ a.e. in } \Omega^a, \end{cases}$$

one derives (6.29).

In (6.22) choosing

$$\begin{cases} \tau = (d', D_{x_N} u^a), \text{ a.e. in } \Omega^a, \\ \\ \tau = Du^b, \text{ a.e. in } \Omega^b, \\ \\ \phi = u^a, \text{ a.e. in } \Omega^a, \\ \\ \phi = u^b + t\varphi, \text{ a.e. in } \Omega^b, \text{ with } \varphi \in C_0^\infty(\Omega^b) \text{ and } t \in]0, +\infty[, \\ \\ \psi = u^a, \text{ a.e. in } \Omega^a, \end{cases}$$

one derives (6.30).

In (6.22) choosing

$$\begin{cases} \tau = (d', D_{x_N} u^a), \text{ a.e. in } \Omega^a, \\ \\ \tau = Du^b, \text{ a.e. in } \Omega^b, \\ \\ \phi = u^a, \text{ a.e. in } \Omega^a, \\ \\ \phi = u^b, \text{ a.e. in } \Omega^b, \\ \\ \psi = u^a + t\varphi, \text{ a.e. in } \Omega^a, \text{ with } \varphi \in C_0^\infty(\Omega^a) \text{ and } t \in]0, +\infty[, \end{cases}$$

one derives (6.31).

□

7 Proof of Theorem 1.1

Proof. Proposition 6.1, Proposition 6.2, and Proposition 6.4 provide the existence of a subsequence of $\{\varepsilon\}$, still denoted with $\{\varepsilon\}$, and (depending on the subsequence) $(u^a, u^b) \in \mathcal{K}$, $d' \in (L^p(\Omega^a))^{N-1}$ such that (1.4)-(1.12), and (6.26) hold true. Moreover, choosing in (6.25)

$$\begin{cases} \tau = (d', D_{x_N} u^a), \text{ a.e. in } \Omega^a, \\ \tau = Du^b, \text{ a.e. in } \Omega^b, \\ \phi = u^a, \text{ a.e. in } \Omega^a, \\ \phi = u^b, \text{ a.e. in } \Omega^b, \\ \psi = u^a, \text{ a.e. in } \Omega^a, \end{cases}$$

(note that this choice is admissible since $C^\infty(\overline{\Omega^a})$ is dense in $L^p(\Omega^a)$), and taking into account (6.27)-(6.31), one derives

$$\begin{aligned} &|\omega'| \int_{\Omega^a} \left(a_N\left(x, (d', D_{x_N} u^a)\right)\left(D_{x_N} v - D_{x_N} u^a\right) + a_0(x, u^a)(v - u^a)\right) dx \\ &+ \int_{\Omega^b} \left(a(x, Du^b)(Dv - Du^b) + a_0(x, u^b)(v - u^b)\right) dx \\ &+ \delta_{\lambda,1} |\partial\omega'| \int_{\Omega^a} h(x, u^a)(v - u^a) dx \\ &\geq |\omega'| \int_{\Omega^a} f(v - u^a) dx + \int_{\Omega^b} f(v - u^b) dx, \quad \forall v \in \mathcal{V}, \end{aligned}$$

which implies the variational inequality in (1.13), since $\mathcal{V}$ is dense in $\mathcal{K}$.

To prove the uniqueness of the solution to problem (1.13), at first note that this problem is equivalent to

$$
\begin{cases}
(u^a, u^b) \in \mathcal{K}, \quad d' \in (L^p(\Omega^a))^{N-1}, \\[2ex]
|\omega'| \displaystyle\int_{\Omega^a} \left(a\left(x, (d', D_{x_N} u^a)\right)\left((\xi, D_{x_N} v) - (d', D_{x_N} u^a)\right) + a_0(x, u^a)(v - u^a)\right) dx \\[2ex]
+ \displaystyle\int_{\Omega^b} \Big(a(x, Du^b)(Dv - Du^b) + a_0(x, u^b)(v - u^b)\Big)\, dx \\[2ex]
+\delta_{\lambda,1} |\partial\omega'| \displaystyle\int_{\Omega^a} h(x, u^a)(v - u^a) dx \\[2ex]
\geq |\omega'| \displaystyle\int_{\Omega^a} f(v - u^a) dx + \int_{\Omega^b} f(v - u^b) dx, \quad \forall v \in \mathcal{K}, \quad \forall \xi \in (L^p(\Omega^a))^{N-1}.
\end{cases}
\tag{7.1}
$$

Assume that also $(\overline{u}^a, \overline{u}^b) \in \mathcal{K}$ and $\overline{d}' \in (L^p(\Omega^a))^{N-1}$ is a solution to problem (7.1), i.e.

$$
\begin{cases}
(\overline{u}^a, \overline{u}^b) \in \mathcal{K}, \quad \overline{d}' \in (L^p(\Omega^a))^{N-1}, \\[2ex]
|\omega'| \displaystyle\int_{\Omega^a} \left(a\left(x, (\overline{d}', D_{x_N} \overline{u}^a)\right)\left((\xi, D_{x_N} v) - (\overline{d}', D_{x_N} \overline{u}^a)\right) + a_0(x, \overline{u}^a)(v - \overline{u}^a)\right) dx \\[2ex]
+ \displaystyle\int_{\Omega^b} \Big(a(x, D\overline{u}^b)(Dv - D\overline{u}^b) + a_0(x, \overline{u}^b)(v - \overline{u}^b)\Big)\, dx \\[2ex]
+\delta_{\lambda,1} |\partial\omega'| \displaystyle\int_{\Omega^a} h(x, \overline{u}^a)(v - \overline{u}^a) dx \\[2ex]
\geq |\omega'| \displaystyle\int_{\Omega^a} f(v - \overline{u}^a) dx + \int_{\Omega^b} f(v - \overline{u}^b) dx, \quad \forall v \in \mathcal{K}, \quad \forall \xi \in (L^p(\Omega^a))^{N-1}.
\end{cases}
\tag{7.2}
$$

By choosing $v = (\overline{u}^a, \overline{u}^b)$ and $\xi = \overline{d}'$ as test functions in (7.1), $v = (u^a, u^b)$ and $\xi = d'$ as test functions in (7.2), and adding the obtained inequalities, one obtains

$$
\begin{aligned}
&|\omega'| \int_{\Omega^a} \left(a\left(x, (d', D_{x_N} u^a)\right) - a\left(x, (\overline{d}', D_{x_N} \overline{u}^a)\right)\right)\left((\overline{d}', D_{x_N} \overline{u}^a) - (d', D_{x_N} u^a)\right) dx \\
&+|\omega'| \int_{\Omega^a} \left(a_0(x, u^a) - a_0(x, \overline{u}^a)\right)(\overline{u}^a - u^a) dx \\
&+ \int_{\Omega^b} \Big(\big(a(x, Du^b) - a(x, D\overline{u}^b)\big)(D\overline{u}^b - Du^b) + \big(a_0(x, u^b) - a_0(x, \overline{u}^b)\big)(\overline{u}^b - u^b)\Big)\, dx \\
&+\delta_{\lambda,1} |\partial\omega'| \int_{\Omega^a} \left(h(x, u^a) - h(x, \overline{u}^a)\right)(\overline{u}^a - u^a) dx \geq 0.
\end{aligned}
\tag{7.3}
$$

On the other side, by virtue of assumptions (3.2), (3.6), and (3.9), one has

$$
\begin{aligned}
&|\omega'| \int_{\Omega^a} \left(a\left(x, (d', D_{x_N} u^a)\right) - a\left(x, (\overline{d}', D_{x_N} \overline{u}^a)\right)\right) ((\overline{d}', D_{x_N} \overline{u}^a) - (d', D_{x_N} u^a))dx \\
&+|\omega'| \int_{\Omega^a} \left(a_0(x, u^a) - a_0(x, \overline{u}^a)\right) (\overline{u}^a - u^a)dx \\
&+\int_{\Omega^b} \left(\left(a(x, Du^b) - a(x, D\overline{u}^b)\right) (D\overline{u}^b - Du^b) + \left(a_0(x, u^b) - a_0(x, \overline{u}^b)\right) (\overline{u}^b - u^b)\right) dx \\
&+\delta_{\lambda,1} |\partial\omega'| \int_{\Omega^a} \left(h(x, u^a) - h(x, \overline{u}^a)\right) (\overline{u}^a - u^a)dx \leq 0.
\end{aligned}
\tag{7.4}
$$

Comparing (7.3) and (7.4) gives

$$
\begin{aligned}
&|\omega'| \int_{\Omega^a} \left(a\left(x, (d', D_{x_N} u^a)\right) - a\left(x, (\overline{d}', D_{x_N} \overline{u}^a)\right)\right) ((\overline{d}', D_{x_N} \overline{u}^a) - (d', D_{x_N} u^a))dx \\
&+|\omega'| \int_{\Omega^a} \left(a_0(x, u^a) - a_0(x, \overline{u}^a)\right) (\overline{u}^a - u^a)dx \\
&+\int_{\Omega^b} \left(\left(a(x, Du^b) - a(x, D\overline{u}^b)\right) (D\overline{u}^b - Du^b) + \left(a_0(x, u^b) - a_0(x, \overline{u}^b)\right) (\overline{u}^b - u^b)\right) dx \\
&+\delta_{\lambda,1} |\partial\omega'| \int_{\Omega^a} \left(h(x, u^a) - h(x, \overline{u}^a)\right) (\overline{u}^a - u^a)dx = 0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&|\omega'| \int_{\Omega^a} \left(a\left(x, (d', D_{x_N} u^a)\right) - a\left(x, (\overline{d}', D_{x_N} \overline{u}^a)\right)\right) ((d', D_{x_N} u^a) - (\overline{d}', D_{x_N} \overline{u}^a))dx \\
&+|\omega'| \int_{\Omega^a} \left(a_0(x, u^a) - a_0(x, \overline{u}^a)\right) (u^a - \overline{u}^a)dx \\
&+\int_{\Omega^b} \left(\left(a(x, Du^b) - a(x, D\overline{u}^b)\right) (Du^b - D\overline{u}^b) + \left(a_0(x, u^b) - a_0(x, \overline{u}^b)\right) (u^b - \overline{u}^b)\right) dx \\
&+\delta_{\lambda,1} |\partial\omega'| \int_{\Omega^a} \left(h(x, u^a) - h(x, \overline{u}^a)\right) (u^a - \overline{u}^a)dx = 0,
\end{aligned}
$$

which implies, by virtue of (3.2), (3.6), and (3.9),

$$
\begin{cases}
\displaystyle\int_{\Omega^a} \left(a\left(x, (d', D_{x_N} u^a)\right) - a\left(x, (\overline{d}', D_{x_N} \overline{u}^a)\right)\right) ((d', D_{x_N} u^a) - (\overline{d}', D_{x_N} \overline{u}^a))dx = 0, \\
\displaystyle\int_{\Omega^b} \left(a(x, Du^b) - a(x, D\overline{u}^b)\right) (Du^b - D\overline{u}^b)dx = 0.
\end{cases}
\tag{7.5}
$$

Thanks to (3.2), from (7.5) it follows that

$$d' = \overline{d}', \text{ a.e. in } \Omega^a,$$

$$D_{x_N} u^a = D_{x_N} \overline{u}^a, \text{ a.e. in } \Omega^a, \tag{7.6}$$

$$Du^b = D\overline{u}^b, \text{ a.e. in } \Omega^b. \tag{7.7}$$

Due to

$$u^a_{|\Sigma^a} = 0 = \overline{u}^a_{|\Sigma^a},$$

equality (7.6) implies

$$u^a = \overline{u}^a, \text{ a.e. in } \Omega^a. \tag{7.8}$$

Moreover, since

$$u^a_{|\Sigma^0} = u^b_{|\Sigma^0}, \quad \overline{u}^a_{|\Sigma^0} = \overline{u}^b_{|\Sigma^0},$$

equality (7.8) implies

$$u^b_{|\Sigma^0} = \overline{u}^b_{|\Sigma^0},$$

which combined with (7.7) gives

$$u^b = \overline{u}^b, \text{ a.e. in } \Omega^b.$$

The uniqueness of the solution to problem (1.13) is so proved. It implies that limits in (1.4)-(1.12) hold true for all the sequence $\{\varepsilon\}$. The proof of Theorem 1.1 is so completed. □

Remark 7.1. *If in definition* (3.17) *of $\mathcal{A}_\varepsilon$, one adds the term*

$$\varepsilon^\beta \int_{\Sigma^{a,lat}_\varepsilon} wv d\sigma, \quad \forall v \in W^{1,p}(\Omega_\varepsilon, \Sigma^a_\varepsilon),$$

with $\beta \in [1, +\infty[$ and $w \in W^{1,\frac{p}{p-1}}(\Omega^a_\varepsilon)$, then it is easy to prove that the following term

$$\delta_{\beta,1} |\partial\omega'| \int_{\Omega^a} w(v - u^a) dx.$$

will appears in limit problem (1.13).

Remark 7.2. *Let*

$$a(x, \xi) = A(x)\xi, \text{ a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N,$$

where A is a $N \times N$ matrix-valued function such that

$$\begin{cases} A = (A_{ij})_{i,j \in \{1,\cdots,N\}} \in (L^\infty(\Omega))^{N\times N}, \\ \\ \exists \alpha \in]0, +\infty[: \; A(x)\,\xi\,\xi \geq \alpha\,|\xi|^2, \text{ a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N. \end{cases}$$

Set

$$A' = (A_{ij})_{i,j=1,\dots,N-1}, \quad V' = (A_{iN})_{i=1,\cdots,N-1}, \quad H' = (A_{Nj})_{j=1,\cdots,N-1},$$

so that the matrix A can be written by blocks as

$$A = \begin{pmatrix} A' & V' \\ H' & A_{NN} \end{pmatrix}.$$

Moreover, define

$$a_0 = A_{NN} - H'\,(A')^{-1}\,V', \ \ a.e. \ in \ \Omega.$$

Then, it is easily seen that in Theorem 1.1 (for instance, compare [29])

$$d' = -A'^{-1}V'D_{x_N}u^a, \ \ a.e. \ in \ \Omega^a,$$

and

$$a_N\,(x,(d',D_{x_N}u^a)) = a_0 D_{x_N}u^a, \ \ a.e. \ in \ \Omega^a.$$

In particular,

$$d' = (0,\cdots,0) \ and \ a_0 = 1, \ \ a.e. \ in \ \Omega^a,$$

if $A = I$.

In the case of the p-laplacian, i.e.

$$a(x,\xi) = |\xi|^{p-2}\xi,$$

the algebraic system (1.14) *implies* $d' = 0$ *and consequently*

$$a_N\,(x,(d',D_{x_N}u^a)) = |D_{x_N}u^a|^{p-2}D_{x_N}u^a.$$

8 Appendix

This section is devoted to recalling how to obtain the weak formulation (3.30) of the problem (1.1).

Since the trace of g on Σ^0 is equal to zero, extending $g = 0$ on Ω^b, one obtains that $g \in \mathcal{K}_\varepsilon$ (see (3.30) for the definition of $\mathcal{K}_\varepsilon$).

Assume that problem (1.1) admits a classical solution. Multiplying the equation in problem (1.1) by $u_\varepsilon - g$, integrating by parts on Ω_ε, and taking into account the boundary conditions for u_ε give

$$\begin{aligned}
&\int_{\Omega^b} a(x,Du_\varepsilon)Du_\varepsilon dx + \int_{\Omega^a_\varepsilon} a(x,Du_\varepsilon)D(u_\varepsilon - g)dx \\
&+ \int_{\Omega^b} a_0(x,u_\varepsilon)u_\varepsilon dx + \int_{\Omega^a_\varepsilon} a_0(x,u_\varepsilon)(u_\varepsilon - g)dx + \varepsilon^\lambda \int_{\Sigma^{a,lat}_\varepsilon} h(x,u_\varepsilon)(u_\varepsilon - g)d\sigma \\
&= \int_{\Omega^b} f u_\varepsilon dx + \int_{\Omega^a_\varepsilon} f\,(u_\varepsilon - g)\,dx.
\end{aligned} \tag{8.1}$$

Now, multiplying the equation in problem (1.1) by $v - g$ with $v \in \mathcal{K}_\varepsilon$ and integrating by parts on Ω_ε give

$$\begin{aligned}
&\int\limits_{\Omega^b} a(x, Du_\varepsilon)Dv dx + \int\limits_{\Omega_\varepsilon^a} a(x, Du_\varepsilon)D(v-g)dx \\
&+ \int\limits_{\Omega^b} a_0(x, u_\varepsilon)v dx + \int\limits_{\Omega_\varepsilon^a} a_0(x, u_\varepsilon)(v-g)dx + \varepsilon^\lambda \int\limits_{\Sigma_\varepsilon^{a,lat}} h(x, u_\varepsilon)(v-g)d\sigma \\
&= \int\limits_{\Omega^b} f v dx + \int\limits_{\Omega_\varepsilon^a} f(v-g)dx \\
&+ \int\limits_{\Sigma_\varepsilon^{a,lat}} \big(a(x, Du_\varepsilon))\nu_\varepsilon(x) + \varepsilon^\lambda h(x, u_\varepsilon)\big)(v-g)d\sigma, \quad \forall v \in \mathcal{K}_\varepsilon.
\end{aligned} \tag{8.2}$$

Since

$$v \leq g \text{ and } a(x, Du_\varepsilon))\nu_\varepsilon(x) + \varepsilon^\lambda h(x, u_\varepsilon) \leq 0, \text{ a.e. on } \Sigma_\varepsilon^{a,lat},$$

one has

$$\int_{\Sigma_\varepsilon^{a,lat}} \big(a(x, Du_\varepsilon)\nu_\varepsilon(x) + \varepsilon^\lambda h(x, u_\varepsilon)\big)(v-g)\, d\sigma \geq 0. \tag{8.3}$$

Then, combining (8.2) and (8.3) gives

$$\begin{aligned}
&\int\limits_{\Omega^b} a(x, Du_\varepsilon)Dv dx + \int\limits_{\Omega_\varepsilon^a} a(x, Du_\varepsilon)D(v-g)dx \\
&+ \int\limits_{\Omega^b} a_0(x, u_\varepsilon)v dx + \int\limits_{\Omega_\varepsilon^a} a_0(x, u_\varepsilon)(v-g)dx + \varepsilon^\lambda \int\limits_{\Sigma_\varepsilon^{a,lat}} h(x, u_\varepsilon)(v-g)d\sigma \\
&\geq \int\limits_{\Omega^b} f v dx + \int\limits_{\Omega_\varepsilon^a} f(v-g)dx, \quad \forall v \in \mathcal{K}_\varepsilon.
\end{aligned} \tag{8.4}$$

Finally, subtracting (8.1) from (8.4), one obtains

$$\begin{aligned}
&\int_{\Omega_\varepsilon} a(x, Du_\varepsilon)D(v-u_\varepsilon)dx + \int_{\Omega_\varepsilon} a_0(x, u_\varepsilon)(v-u_\varepsilon)dx + \varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} h(x, u_\varepsilon)(v-u_\varepsilon)d\sigma \\
&\geq \int_{\Omega_\varepsilon} f(v-u_\varepsilon)dx, \quad \forall v \in \mathcal{K}_\varepsilon.
\end{aligned}$$

By taking into account definition (3.17) of $\mathcal{A}_\varepsilon$, the weak formulation (3.30) of the problem (1.1) is so obtained.

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