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The mathematics of aquatic locomotion

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Aquatic locomotion is a self-propelled motion through a liquid medium. It can be of biological nature (fish, microorganisms,...) or performed by robotic swimmers. This snapshot aims to introduce the reader to some of the challenges raised by the mathematical modelling of aquatic locomotion, even in seemingly very simple cases.

1 Introduction

Understanding the propelling mechanism of living organisms is a question which fascinated scientists for centuries. The first text devoted in particular to this question is, perhaps, Aristotle's *De Motu Animalium* (*Movement of Animals*) [12], written in the 4th century BC. The subject has been discussed by G. A. Borelli (considered to be the father of biomechanics), in a book borrowing the title of Aristotle's treatise [4], which has been first edited in 1680. Borelli utilized mathematics to prove his theories and firstly considered the possibility of building machines using locomotion strategies inspired by animals; for example, a submarine. Now, several centuries after Borelli's work, the literature on the mathematical modelling of aquatic locomotion has become overwhelming. Excellent introductions to this subject can be found in the monograph of Childress [5] or the collection of papers of Lighthill [9]. The interest in designing robots imitating the swimming mechanism of animals (*biomimetism*) increased after the famous paper of Triantafyllou and Triantafyllou [17] about robotic tuna fish, and it is nowadays a major research subject.

In this snapshot we focus on the situation when forces due to the viscosity[1] are much larger than forces due to acceleration. Consequently, the models described below are not appropriate for understanding macroscopic animal swimming in water. They do, however, give quite an accurate description of the locomotion of microscopic organisms in various biological fluids. As mentioned in a famous paper of Purcell [13], these organisms evolve in a world which is "quite different form the one that we have developed our intuitions in". Locomotion in this world is a difficult task. For a macroscopic swimmer (a human, for instance), equivalent swimming conditions are described in [13]:

> Well you put him in a swimming pool that is full of molasses, and the[n] you forbid him to move any pare [sic.] of his body faster than 1 cm/min. Now imagine yourself in that condition; you're under the swimming pool in molasses, and now you can only move like the hands of a clock.

Before continuing the discussion, it is important (at least from a mathematician's viewpoint) to define what *swimming* means in the context of this snapshot. We adopt here a definition from [13], which states that an object swims if it changes its position using "cyclic deformations of the body on which there are no external torques or forces except those exerted by the surrounding fluid".

2 The scallop theorem and related questions

To translate the above definition of swimming into a mathematical language, we first consider a very simple swimmer which can be seen as a toy model of a scallop. A real scallop swims by opening its shell slowly and closing it fast. Our swimmer exists in a two-dimensional world[2] and is composed of two arms attached at one end (standing for the two halves of the real shell). These arms can be opened and closed (together) up to the angle θ formed by each arm with a given line Ox (see the left side of Figure 1). The fluid is supposed to fill the remainder of the plane. The symmetry of the swimmer's shape with respect to the axis Ox at each instant t, suggests that the only possible displacements of our idealized scallop are in the direction of this axis. In spite of the simplicity of the swimmer described above, the mathematics necessary to understand this system is quite complicated. There is not, as far as I know, a rigorous mathematical proof that the idealized scallop above can indeed swim in water.

[1] Viscosity can be thought of as describing a fluid's resistance to objects moving through it.

[2] By this we mean the swimmer can move only left, right, up or down (not towards or away from the reader), and these are also the only directions that forces can act on it by the fluid (remember, though, we neglect forces due the swimmer's acceleration, such as gravity.)

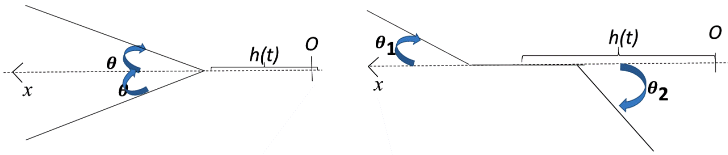

Figure 1: A simplified representation of a scallop (left side) and Purcell's three-link swimmer (right side).

To make this discussion more precise we need some notation. For $t \geqslant 0$ we denote by $h(t)$ the position of the scallop at instant t, as indicated in the left side of Figure 1. According to Newton's second law of motion, the equation describing the evolution of the swimmer's position is

$$m\ddot{h}(t) = f(t), \tag{2.1}$$

where m denotes the mass of the swimmer, $\ddot{h}$ is the second derivative of h[3] (the acceleration of the swimmer) and $f(t)$ is the force exerted by the fluid on the swimmer at instant t. In general, $f(t)$ depends in a very complicated manner on all the history of the function h and of its derivatives up to instant t. More precisely, if we assume that the surrounding liquid is water, then $f(t)$ is determined by the solution of the *incompressible Navier–Stokes equations* in the fluid domain. This is a complex system of equations, and is very difficult to solve. This is why we focus on the case when the surrounding fluid is much more viscous than water and that the maximal rotation velocity and acceleration of the scallop's arms are not too large. In other words, as mentioned above, we assume that the forces due to the acceleration of the swimmer are negligible with respect to viscous forces. Mathematically, this means that the left-hand side of (2.1) vanishes (we can assume the acceleration $\ddot{h}$ equals zero). Also, one of the coefficients, called the *Reynolds number* (see, for instance, [5] for a precise

[3] The notation $\dot{h}$ to represent the derivative of h is called *Newton's notation* or *dot notation*, and is used to denote a derivative with respect to time.

definition of this quantity), which occurs in the nonlinear Navier–Stokes system mentioned above, becomes very small. Therefore, all the terms containing this coefficient are neglected. Consequently, the equations describing the motion of fluid particles become linear and they do not contain any acceleration terms, forming the simpler *Stokes system.* Now the right-hand side of (2.1) can be written as

$$f(t) = \dot{h}(t) f_1(\theta(t)) + \dot{\theta}(t) f_2(\theta(t)), \tag{2.2}$$

where $\dot{h}$, and $\dot{\theta}$ are the derivatives of h and θ, respectively. In other words, $\dot{h}$ is the scallop's velocity and $\dot{\theta}$ is the angular velocity of its arms. The terms f_1 and f_2 are real continuous functions, with $f_1(\theta) \neq 0$ for every $\theta \neq 0$.

Using the above assumptions and writing $g(\theta) = -\frac{f_2(\theta)}{f_1(\theta)}$, Equation (2.1) becomes

$$\dot{h}(t) = g(\theta(t))\dot{\theta}(t). \tag{2.3}$$

Solving (finding $h(t)$) this equation will enable us to describe the movement of the scallop. The simple result below is a particular case of the so-called "scallop theorem" due to Purcell.

Proposition 2.1. *Assume that θ is a function of class C^1 and periodic of period $\tau > 0$. Then any function h verifying* (2.3) *satisfies $h(0) = h(\tau)$.*

Proof. Let $G : \mathbb{R} \to \mathbb{R}$ be such that $G' = g$. Integrating (2.3) on $[0, \tau]$ yields

$$h(\tau) - h(0) = G(\theta(\tau)) - G(\theta(0)) = 0, \tag{2.4}$$

which ends the proof. □

Let us see what the proposition effectively says. The function θ describes the angle of the scallop's arms. The condition that it be of class C^1 means the movement of the arms is smooth. This condition, together with the constraints on f_1 and f_2 (and, hence, g), allows us to invoke the *fundamental theorem of calculus.* This theorem ensures the existence of the function G in the proof and of the leftmost equality in Equation (2.4)[4]. Saying that θ is periodic of period τ means that τ is the time it takes the scallop to complete a cycle of opening and closing its arms. In other words, at time $t = 0$ (the beginning) the arms are closed and so the angle $\theta(0)$ equals zero, but so also at time $t = \tau$, and we get $0 = \theta(0) = \theta(\tau) = \theta(2\tau) = \cdots$. This gives us the rightmost equality in Equation (2.4). Putting all these together we get that the scallop's position at the start $h(0)$ is the same as $h(\tau)$ – its position at time τ. Proposition 2.1 actually states that under all our assumptions, the idealized scallop cannot swim!

[4] Can you spot the usage of the chain rule of differentiation?

To obtain an idealized animal able to swim in a very viscous fluid, it suffices to add one hinge to the idealized scallop described above. We obtain in this way the *3-link swimmer* described in the right side of Figure 1. This swimmer has two arms – one at each end. Each arm can be opened independently – the front one up to angle θ_1 and the back one up to angle θ_2. It can be shown that, not only can this creature move its center of mass $h(t)$ during a stroke (this has been remarked in [13]), but that it can be steered to any final position by appropriately choosing the time periodic functions θ_1 and θ_2 (see, for instance, Dal Maso et al. [7] and references therein).

3 Ciliates and control theory

In this section we will move on to a three-dimensional swimmer, albeit still assuming that gravity pulling the swimmer down can be neglected. We begin by remarking that swimmers are often propelled by deformations which are symmetric with respect to the direction of locomotion (an example is the "scallop" in the left side of Figure 1). Assuming, again, that the fluid is very viscous, that it fills the whole space and adopting some standard approximations, the equations describing the movement of the swimmer are written in a form which can be seen as a generalization of (2.3) (compare the two), which is

$$\dot{h}(t) = \sum_{k=1}^{m} \dot{\theta}_k(t) g_k(\theta_1(t), \ldots \theta_m(t)). \tag{3.1}$$

In the above equation, θ_k $(1 \leqslant k \leqslant m)$ are the functions controlling the shape of the swimmers (as θ controlled the shape of the scallop's arms), and the functions g_k $(1 \leqslant k \leqslant m)$ are not, in general, explicitly known. As in the previous section, we try to find a case where these functions are determined by simple formulas. An example of such a case is a simplified model for the swimming of *ciliates*. Ciliates are swimming microorganisms which exploit the bending of a large number of small and densely-packed protrusions, termed cilia, in order to propel themselves in a viscous fluid. In this model for ciliary locomotion, instead of the dynamics of the individual cilia, we consider time-periodic displacements of the points on the surface of the microorganism; see Taylor [16], Blake [3] or the recent review paper of Lauga and Powers [8] for a detailed description of this model. We assume that, in its initial configuration, the swimmer occupies a ball of radius 1 and that the points on the surface of the ball move in the same way on each meridian. More precisely, for each $t \geqslant 0$, the point x on the surface

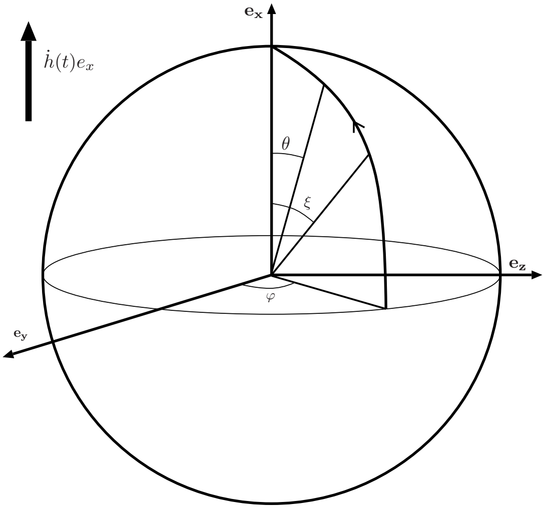

Figure 2: The ball swims in the direction of e_x by moving a generic boundary point of spherical coordinates $(1, \varphi, \xi)$ to a point of spherical coordinates $(1, \varphi, \theta)$ with $\theta = \chi(\xi, t)$.

of the ball of spherical coordinates $(1, \xi, \varphi)$[5], is displaced to a point of whose spherical coordinates are $(1, \theta, \varphi)$, where

$$\theta = \chi(\xi, t) = \xi + \theta_1(t)q_1(\xi) + \theta_2(t)q_2(\xi) \quad (\xi \in [0, \pi],\ t \geqslant 0). \tag{3.2}$$

This means that a point on the surface of the ball (radius 1), stays at the same horizontal angle (φ) and changes only its vertical angle (θ); see Figure 2. The angle θ is dependent on time and the angle ξ. This dependence is encoded by the function χ involving the functions θ_1, θ_2, q_1 and q_2. In Formula (3.2), the functions q_1 and q_2 are supposed to be given and smooth, with $q_i(0) = q_i(\pi) = 0$ for $i \in \{1, 2\}$. It can be shown (see, for instance, [11] or [15]) that under the assumption that our spherical swimmer moves in a very viscous medium, and that the swimmer is centered at the origin at time $t = 0$, the equation describing the displacement $h(t)$ of its center is

$$\dot{h}(t) = \sum_{i=1}^{2} \dot{\theta}_i(t) g_i(\theta_1(t), \theta_2(t)), \quad h(0) = 0, \tag{3.3}$$

[5] Each point in the three-dimensional space can be represented as a triple (r, θ, φ), where r is its distance from the origin, θ its angle above (or below) the e_y–e_z plane, and φ its angle with respect to the e_y axis. Such a representation is called *spherical representation*.

where it has been demonstrated that

$$g_i(\theta_1(t), \theta_2(t)) = \int_0^\pi \chi_i'(\xi)\mathcal{F}(\chi(\xi, t))\, \mathrm{d}\xi \quad (i \in \{1, 2\},\ t \geqslant 0), \tag{3.4}$$

$$\mathcal{F}(x) = \frac{1}{8}\left(\sin(2x) - 2x\right) \quad (x \in \mathbb{R}), \tag{3.5}$$

and χ has been defined in (3.2).

The main result in this section, proved in [15], states as follows:

Theorem 3.1. *Assume that the functions q_1 and q_2 in* (3.2) *are such that*

$$\int_0^\pi [q_1'(\xi)q_2(\xi) - q_1(\xi)q_2'(\xi)] \sin^2 \xi \, d\xi \neq 0. \tag{3.6}$$

Then for every $h_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exist $\tau > 0$ and smooth functions $\theta_1,\ \theta_2 : [0, \tau] \to \mathbb{R}$, with

$$\begin{aligned} \theta_1(0) = \theta_2(0) = \theta_1(\tau) = \theta_2(\tau) = 0, \\ |\theta_k(t)| + |\dot{\theta}_k(t)| \leqslant \varepsilon \quad (t \in [0, \tau]), \end{aligned} \tag{3.7}$$

such that the solution h of (3.3) *satisfies $h(\tau_0) = h_0$.*

Condition (3.6) in the above theorem means, roughly speaking, that the two functions q_1 and q_2 are independent. It is very easy to find such functions, a typical example being $q_1(\xi) = \sin \xi,\ q_2(\xi) = \sin(2\xi)$. The properties in (3.7) mean, basically, that we can find functions θ_1, θ_2 (describing the change of angle of the points on the surface of the swimmer) that are periodic and not varying too much.

Giving the detailed proof of Theorem 3.1 would require some preparation, so we only sketch here the main steps of this proof. Readers not inclined to read the proof now are invited to jump to the conclusions in Section 4. The first one is to remark that, according to (3.7), we also have to control the final value of the functions θ_1 and θ_2. Therefore, it is useful to describe our system in terms of *control theory.* Control theory is the study of the possibility of ensuring a dynamical system will end up in a given state, if we can control it with some *control functions.* In these terms, our system can be written as the three dimensional vector

$$X(t) = \begin{bmatrix} h(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix},$$

and, in our case, the control functions are $\dot{\theta}_1(t)$ and $\dot{\theta}_2(t)$. With this notation, Equation (3.3) becomes

$$\dot{X}(t) = \dot{\theta}_1(t)\eta_1(X(t)) + \dot{\theta}_2(t)\eta_2(X(t)), \tag{3.8}$$

where the functions $\eta_1, \eta_2 : \mathbb{R} \times [0, \pi]^2 \to \mathbb{R}^3$ are defined by

$$\eta_1(X) = \begin{bmatrix} g_1(\theta_1, \theta_2) \\ 1 \\ 0 \end{bmatrix}, \quad \eta_2(X) = \begin{bmatrix} g_2(\theta_1, \theta_2) \\ 0 \\ 1 \end{bmatrix}, \tag{3.9}$$

with g_1 and g_2 given by (3.4). Now that we have written our system in the form (3.8), we can apply a well-known theorem from control theory due to Chow (see, for instance, [6, Section 3.3]) which, together with condition (3.6), gives the result of Theorem 3.1. In less technical terms, this final result means that under suitable conditions, there are periodic movements of the swimmer's surface that will enable it to swim to any given point h_0. We refer to [15] for the detailed proof.

4 Conclusions, comments and open questions

The mathematical models considered above describe locomotion in a fluid when acceleration forces are negligible with respect to viscosity forces. A mathematically interesting feature of these models is that they can be described using only ordinary differential equations[6], as in (3.8). These models are relatively simple in the cases considered in this snapshot, when the swimmer-fluid system fills the whole space, but they become quite complex in the presence of walls; see, for instance, Alouges and Giraldi [1]. The question, tackled in Theorem 3.1, of the existence of controls steering the swimmer to a prescribed final position is important from a theoretical viewpoint but, in view of applications, optimization problems[7] play a central role. We refer to Michelin and Lauga [11] or Lohéac et al. [10] for studies in which various optimization question of this type have been considered.

The main challenge to be raised in the years to come consists of understanding the propelling mechanism of macroscopic swimmers (like real or robotic fish). This requires to consider both inertial and viscosity forces, leading to models based on the *Navier–Stokes evolution partial differential equations*[8]. We refer to San Martin et al. [14] for a description of such models and their mathematical analysis, and to Bergmann and Iollo [2] and references therein

[6] *Ordinary differential equations* are such that describe a relation between a function and its derivative.

[7] Optimization problems are concerned with finding the "best" solution to a problem. For a discussion of such problems, see Snapshot 2/2015 *Minimizing energy* by C. Breiner.

[8] *Partial differential equations* are differential equation relating multivariable functions and their partial derivatives. For a discussion of such equations see, for example, Snapshot 7/2015 *Darcy's law and groundwater flow modelling* by B. Schweizer.

for numerical simulations of this complex system. However, the mathematical research driven by these problems is at an embryonic stage. Obtaining controllability results in the case of fish-like swimming or developing fast numerical methods, efficient for optimization purposes, are fascinating challenges, at the interface of mathematical fluid dynamics and control theory.

Image credits

All figures by the author.

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