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# A McKay Correspondence for Reflection Groups

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# A MCKAY CORRESPONDENCE FOR REFLECTION GROUPS

RAGNAR-OLAF BUCHWEITZ†, ELEONORE FABER, AND COLIN INGALLS

*Dedicated to the memory of our friend and collaborator Ragnar-Olaf Buchweitz*

ABSTRACT. We construct a noncommutative desingularization of the discriminant of a finite reflection group $G$ as a quotient of the skew group ring $A = S * G$. If $G$ is generated by order two reflections, then this quotient identifies with the endomorphism ring of the reflection arrangement $\mathcal{A}(G)$ viewed as a module over the coordinate ring $S^G/(\Delta)$ of the discriminant of $G$. This yields, in particular, a correspondence between the nontrivial irreducible representations of $G$ to certain maximal Cohen–Macaulay modules over the coordinate ring $S^G/(\Delta)$. These maximal Cohen–Macaulay modules are precisely the nonisomorphic direct summands of the coordinate ring of the reflection arrangement $\mathcal{A}(G)$ viewed as a module over $S^G/(\Delta)$. We identify some of the corresponding matrix factorizations, namely the so-called logarithmic co-residues of the discriminant.

## 1. INTRODUCTION

The classical McKay correspondence relates representations of a finite subgroup $G \leqslant \mathrm{SL}(2,\mathbb{C})$ to exceptional curves on the minimal resolution of singularities of the Kleinian singularity $\mathbb{C}^2/G$. By a theorem of Maurice Auslander [Aus86], this correspondence can be extended to maximal Cohen–Macaulay (=CM)-modules over the invariant ring of the $G$-action. In particular, Auslander's version of the correspondence holds more generally for *small* finite subgroups $G \leqslant \mathrm{GL}(n,\mathbb{C})$. It is natural to ask what happens if $G$ is replaced by a group that contains (pseudo-)reflections: The goal of this work is to establish a similar correspondence in the case where $G$ is a pseudo-reflection group, that is, a group that is *generated* by pseudo-reflections.

To this end, let $G \leqslant \mathrm{GL}(n,\mathbb{C})$ be a finite group acting on $\mathbb{C}^n$. By the theorem of Chevalley–Shephard–Todd the quotient $\mathbb{C}^n/G$ is smooth if and only if $G$ is a pseudo-reflection group, that is, it is generated by pseudo-reflections. Thus, if $G$ is a pseudo-reflection group, at first sight there are no singularities to resolve and it is impossible to "see" the irreducible representations as CM-modules over the invariant ring $R$ of the group action: $R$ is a regular ring and it is well-known that in this case all CM-modules are isomorphic to some $R^n$! However, the key idea of this work is to consider the irregular orbits of the group action, on $\mathbb{C}^n$ this is the *reflection arrangement* $\mathcal{A}(G)$ (the set of mirrors of $G$) and in the quotient $\mathbb{C}^n/G$ this is the projection of $\mathcal{A}(G)$, the so-called *discriminant* of $G$.
This is translated into algebra as follows: $G \leqslant \mathrm{GL}(n,\mathbb{C})$ also acts on $S := \mathrm{Sym}_{\mathbb{C}}(\mathbb{C}^n)$, then

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$\mathbb{C}^n = \mathrm{Spec}(S)$, the quotient $\mathbb{C}^n/G = \mathrm{Spec}(R)$, where $R := S^G$ is the invariant ring. If $G$ is a pseudo-reflection group, then $R$ is itself isomorphic to a polynomial ring, and $\mathcal{A}(G)$ is defined by the *Jacobian* $J \in S$, a (not necessarily reduced) product of linear forms in $S$. The discriminant is given by a polynomial $\Delta \in R$ and its coordinate ring is $R/(\Delta)$.

Let us follow this train of thought further: Auslander's theorem states that for a small subgroup $G \leqslant \mathrm{GL}(n, \mathbb{C})$ acting on the polynomial ring $S$ the twisted group ring $A = S * G$ is isomorphic to the endomorphism ring $\mathrm{End}_R(S)$, where $R = S^G$. In particular, $\mathrm{gldim}\, A = \dim R = n$, $A$ is a CM-module over $R$ and the nonisomorphic $R$-direct summands of $S$ correspond to the indecomposable projectives of $A$ and consequently to the irreducible representations of $G$, as these correspond to the simple modules over the group ring $\mathbb{C}G$. For $G$ a pseudo-reflection group, the twisted group ring $A$ still has global dimension $n$ and is a CM-module over the invariant ring $R$. Following our idea, we would like to write $A$ as endomorphism ring over the discriminant, whose coordinate ring is $R/(\Delta)$, but an easy computation shows that the centre of $A$ is in some sense too large: $Z(A) = R$. In order to remedy this, we will consider the quotient $\overline{A} = A/AeA$, where $e = \frac{1}{|G|}\sum_{g \in G} g \in A$ is the idempotent for the trivial representation. This quotient has nice properties:

**Theorem A** (=Thm. 3.12, Cor. 3.13, and Cor. 3.19)**.** *Let $G \leqslant \mathrm{GL}(n, \mathbb{C})$ be a finite group (more generally: $G \leqslant \mathrm{GL}(n, K)$, where $K$ is an algebraically closed field such that $|G|$ is invertible in $K$) and assume that $G$ is generated by pseudo-reflections. Denote $A = S * G$ the twisted group ring and set $\overline{A} = A/AeA$. Then $\overline{A}$ is a CM-module over $S/(J)$, the coordinate ring of the reflection arrangement, as well as over $R/(\Delta)$. Moreover, $\overline{A}$ is Koszul, and* $\mathrm{gldim}\, \overline{A} \leq n$*. If $G \not\leqslant \mu_2$, then* $\mathrm{gldim}\, \overline{A} = n$*.*

In particular, interpreting $A$, $AeA$ and $\overline{A}$ geometrically, we exhibit a matrix factorization $(\varphi, \psi)$ of $J \in S$ whose cokernel is $\overline{A}$ as left $S$-module. Curiously, this matrix factorization comes from the group matrix of $G$ (see Section 3) and it is (skew-)symmetric in that the $S$–dual (or transpose matrix) $\psi^*$ is equivalent to $\varphi$.

The next step is to show that the quotient $\overline{A}$ is isomorphic to an endomorphism ring over $R/(\Delta)$ if $G$ is generated by reflections of order two. First we generalize Auslander's theorem "noncommutatively": For any $G \leqslant \mathrm{GL}(n, \mathbb{C})$ consider the small group $\Gamma := G \cap \mathrm{SL}(n, \mathbb{C})$ and its invariant ring $T := S^\Gamma$. Then $\Gamma \leqslant G$ is a normal subgroup and

$$1 \to \Gamma \to G \to G/\Gamma \to 1$$

is a short exact sequence of groups. Assume that $H := G/\Gamma$ is complementary to $\Gamma$, as will be the case for $H$ cyclic of prime order. From this we obtain the following generalization of Auslander's theorem:

**Theorem B** (see Prop. 4.12 for a more general formulation)**.** *In this situation we have $\mathbb{C}$-algebra isomorphisms*

$$A = S * G \cong S * \Gamma * H \cong \mathrm{End}_{T*H}(S * H)\,,$$

*and $S * H \cong Ae_\Gamma$ as right $T * H \cong e_\Gamma A e_\Gamma$-module, where $e_\Gamma \in A$ is the idempotent $\frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} \gamma$. In particular, if $G = \Gamma$ is in $\mathrm{SL}(n, \mathbb{C})$, then this recovers Auslander's theorem.*

In order to show that $\overline{A}$ is an endomorphism ring, we first view $A$ as a CM-module over the (noncommutative) ring $T * H$ and will use the functor

$$i^* : \mathbf{Mod}(T * H) \to \mathbf{Mod}(R/(\Delta))\,,$$

coming from a standard recollement. For this part we will need that $G$ is a *true reflection group*, that is, generated by reflections of order 2. Then clearly $H \cong \mu_2$. In order to use the recollement, we consider more generally a regular ring $R$, a nonzero $f \in R$, and define the path algebra

$$B := R\left(\begin{tikzcd} e_+ \arrow[rr, bend left, "v"] \arrow[rr, dashed, leftrightarrow, "f"] & & e_- \arrow[ll, bend left, "u"] \end{tikzcd}\right),$$

with relations ,

$$e_\pm^2 = e_\pm,\ e_+ + e_- = 1,\ u = e_+ u e_-,\ v = e_- v e_+,\ uv = fe_+, \text{ and } vu = fe_- .$$

Then matrix factorizations over $B/Be_-B \cong R/(f)$ (Lemma 4.1) can be seen as CM-modules over $B$, which leads to the following reformulation of Eisenbud's theorem on matrix factorizations [Eis80]:

**Theorem C** (=Thm. 4.3)**.** *Let $f \in R$ and $B$ as above and let $i^* : \mathbf{Mod}(B) \to \mathbf{Mod}(B/Be_-B)$ be the functor $i^* = - \otimes_B B/Be_-B$ from the standard recollement. Then $i^*$ induces an equivalence of categories*

$$\mathbf{CM}(B)/\langle e_-B\rangle \simeq \mathbf{CM}(R/(f))\ ,$$

*where $\langle e_-B\rangle$ is the ideal in the category $\mathbf{CM}(B)$ generated by the object $e_-B$. (Here $\mathbf{CM}(\Lambda)$ stands for the category of CM-modules over a ring $\Lambda$).*

In particular, this theorem implies Knörrer's result [Knö87] that $\mathbf{CM}(T * \mu_2) \simeq MF(f)$, where $MF(f)$ stands for the category of matrix factorizations of $f$. The last ingredient comes from Stanley's work on semi-invariants: set $R = S^G$ and $f = \Delta$ and $B = T * H$ in the above theorem, then using that $T \cong R[J]/(J^2 - \Delta)$ as $R$-modules (see Lemma 4.14) one can calculate $i^*(S * H) \cong S/(J)$ as $R/(\Delta)$-module (see Prop. 4.16). This leads directly to the main theorem:

**Theorem D** (=Thm. 4.17 and Corollaries)**.** *Let $G$ be a true reflection group. Then with notation as just introduced, the quotient algebra $\overline{A} = A/AeA$ is isomorphic to the endomorphism ring $\mathrm{End}_{R/(\Delta)}(S/(J))$.*

*In particular, we have established a correspondence between the indecomposable projective $\overline{A}$-modules and the nontrivial irreducible $G$-representations on the one hand and the non-isomorphic $R/(\Delta)$-direct summands of $S/(J)$ on the other hand.*

*Moreover, $\overline{A}$ constitutes a noncommutative resolution of singularities (=NCR) of $R/(\Delta)$ of global dimension $n = \dim R + 1$ for $G \neq \mu_2$.*

For a true reflection group $G \leqslant \mathrm{GL}(2, \mathbb{C})$ this implies that $S/(J)$ is a representation generator of $\mathbf{CM}(R/(\Delta))$, and $R/(\Delta)$ is an ADE-curve singularity (see Thm. 4.22).

The remainder of the paper is dedicated to a more detailed study of $\overline{A}$ and $S/(J)$ as $R/(\Delta)$-modules, for any pseudo-reflection group $G \leqslant \mathrm{GL}(n, \mathbb{C})$: we determine the ranks of the isotypical components of $S/(J)$ over $R/(\Delta)$ using Hilbert–Poincaré series and can give precise formulas in terms of Young diagrams in the case $G = S_n$ (Prop. 5.4). Further, we determine the decomposition of $\overline{A}$ into indecomposable summands over $R/(\Delta)$ and the rank of $\overline{A}$ as $R/(\Delta)$ module, using the codimension 1-structure (Cor. 5.10). We can also deduce that $\overline{A}$ is not an endomorphism ring over the discriminant if $G$ has generating pseudo-reflections of order $\geq 3$ (Cor. 5.11).

Then, using Solomon's theorem and results from Kyoji Saito and Hiroaki Terao we can identify some of the isotypical components of $S/(J)$ (again for any pseudo-reflection group $G$): the isotypical component of the defining representation $V$ of $G$ and its higher exterior powers $\Lambda^l V$ are given by the cokernels of the natural inclusions $\Lambda^l \Theta_R(-\log \Delta) \to \Lambda^l \Theta_R$ of the module of logarithmic derivations into the derivations on $R$, dubbed the *logarithmic co-residues*. In particular, for $l = 1$ one gets that $j_\Delta$, the Jacobian ideal of the discriminant viewed as a module over $R/(\Delta)$, is a direct summand of $S/(J)$, see Thm. 5.24. The other isotypical components have yet to be determined in general.

Finally we are asking about the quiver of the algebra $\overline{A}$ (the McKay algebra): what are the arrows and relations? Using Young diagrams, we can determine the precise shape of the McKay quiver of $G = S_n$ (see Thm. 6.3). The quivers for other reflection groups and the relations remain mysterious.

The paper ends with the example of the discriminant of $G = S_4$ acting on $\mathbb{C}^3$, the well-known *swallowtail*. Here we can explicitly determine all matrix factorizations for the non-isomorphic direct summands of $S/(J)$.

The results in this paper have been announced in [BFI17], where more background on the McKay correspondence and examples may be found.

## CONTENTS

## 2. DRAMATIS PERSONAE

| | |
|---|---|
| $K$ | an algebraically closed field, mostly $\mathbb{C}$ |
| $\operatorname{char} K$ | the characteristic of $K$ |
| $V$ | a finite dimensional vector space over $K$ |
| $n = \dim_K V$ | the dimension of $V$ over $K$ |
| $G \leqslant \mathrm{GL}(V) \cong \mathrm{GL}(n, K)$ | a finite subgroup of $K$–linear automorphisms of $V$ |
| $\Gamma = G \cap \mathrm{SL}(V)$ | the kernel of the determinant homomorphism restricted to $G$ |
| $\lvert G\rvert$ | the order of $G$, assumed not to be divisible by $\operatorname{char} K$ |
| $KG$ | the group algebra on $G$ over $K$. According to our assumption, a semi–simple $K$–algebra, product of matrix algebras over $K$ |
| $S = \mathrm{Sym}_K V$ | the symmetric algebra on $V$ over $K$ |
| $R = S^G$ | the invariant subring of the action of $G$ on $V$ |
| $S^G = K[f_1, ..., f_n]$ | the invariant subring when $G \leqslant \mathrm{GL}(V)$ is a subgroup generated by pseudo-reflections |
| $d_i = \deg f_i$ | the degrees of basic invariants, so that $\lvert G\rvert = d_1 \cdots d_n$ |
| $m = \sum_{i=1}^{n}(d_i - 1)$ | the number of pseudo-reflections in $G$ |
| $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\dots,n}$ | the Jacobian determinant of the basic invariants that is a polynomial in $S$ of degree $m$ |
| $z$ | the squarefree polynomial underlying $J$ |
| $m_1 = \deg z$ | the degree of $z$, that is, the number of mirrors in $G$ |
| $\Delta = zJ \in S^G$ | the discriminant of the reflection group $G$ that is thus of degree $m + m_1$ |
| $V_i, i = 0, ..., r,$ | representatives of the isomorphism classes of irreducible $G$–representations. |
| $V_0 = K_{\mathrm{trv}} = \mathrm{triv}$ | the trivial representation |
| $V_1 = V$ | the defining representation $G \hookrightarrow \mathrm{GL}(V)$ if that is irreducible |
| $V_{\det} = \det V = \lvert V\rvert$ | the linear one-dimensional representation of $G$ afforded by the determinant of the defining representation $V$ |
| $\mathrm{rank}_C M$ | the rank function on the minimal primes in $\operatorname{Spec} C$ for a module $M$ over a reduced commutative ring $C$ |

**Conventions.** Throughout the paper let $K = \mathbb{C}$,[1] if not explicitly otherwise specified. Let $V$ be a finite dimensional vector space over the field $K$ and $\mathrm{GL}(V)$ the group of invertible linear transformations on it. If we choose a basis to identify $V \cong K^n$, we identify, as usual,

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[1] Most of our results also hold if the characteristic of the field $K$ does not divide the order $\lvert G\rvert$ of the group $G$. However, in order to facilitate the presentation, we restrict to $K = \mathbb{C}$.

$\mathrm{GL}(n) = \mathrm{GL}(n, K) \cong \mathrm{GL}(V)$ with the group of invertible $n \times n$ matrices over $K$. Further, let $G$ be a finite subgroup of linear transformations on $V$. The group $G$ acts then linearly and faithfully on the polynomial ring $S = \mathrm{Sym}_K V \cong K[x_1, ..., x_n]$ over $K$, where $x_1, ..., x_n$ constitutes a $K$–basis of $V$. We may consider $S$ as a graded ring with standard grading $|x_i| = 1$ for all $i$. If $s = f(\mathbf{x}) \in S$, then we write $g(s) = f(g\mathbf{x})$ for the action of $g \in G$ on $s$, with $\mathbf{x} = (x_1, ..., x_n)$ and $g\mathbf{x} = (g(x_1), ..., g(x_n))$. Note that if $g = (a_{ij})_{i,j=1,...,n} \in \mathrm{GL}(V)$, then $g\mathbf{x} = (a_{ij})(x_1, ..., x_n)^t$, where $(-)^t$ denotes the transpose[2].

The *invariant ring* of the action of $G$ on $V$ will be denoted by $R := S^G = \{s \in S : g(s) = s$ for all $g \in G\}$.

**Twisted group rings.** Assume that $G \leqslant \mathrm{GL}(V)$ is any finite subgroup. The group ring of $G$ will be denoted by $KG$. We denote by $Q = Q(S)$ the field of fractions of $S$ and note that $G$ acts on $Q$ as well. We consider the following $K$–algebras.

**Definition 2.1.** Assume $G$ acts on a $K$–algebra $S$ through $K$–algebra automorphisms. The *twisted* or *skew group ring* defined by these data is $A = S{*}G = S\tilde{\otimes}_K KG$, where the $\tilde{\otimes}$ is meant to indicate that $A = S \otimes_K KG$ as a left $S$–, right $KG$–module, but the multiplication is twisted by the action of $G$ on $S$.

In more detail, $A$ is the free left $S$–module with basis indexed by $G$, thus, $A = \bigoplus_{g \in G} S\delta_g$, where $\delta_g$ stands for the basis element parametrized by $g \in G$.

The multiplication is defined to be *twisted by the action of* $G$ *on* $S$ in that $\delta_g s = g(s)\delta_g$, for $s \in S, g \in G$. In particular the multiplication of two elements $s'\delta_{g'}, s\delta_g$ is given by

$$(s'\delta_{g'})(s\delta_g) = (s'g'(s))\delta_{g'g} \in S\delta_{g'g} \quad \text{for} \quad g', g \in G, s', s \in S\,.$$

Our notation here follows [KK86a] and is meant to clearly distinguish, say, the element $\delta_g s \in A$ from the element $g(s) \in S$.

However, even if $S$ is commutative, its image is usually not in the centre of $A$, whence the ring homomorphism $S \to A$ only endows $A$ with an $S$–bimodule structure over $K$, with the action from the left simply multiplication in $S$, while the action from the right is determined by $\delta_g s = g(s)\delta_g$ for $g \in G, s \in S$. In particular, each left $S$–module direct summand $S\delta_g \subseteq A$ is already an $S$–bimodule direct summand of $A$.

Similarly $Q * G \cong Q \otimes_S A$ as well as ring homomorphisms $Q \to Q * G$ and $QG \to Q * G$, where $Q = Q(S)$. As noted in [Aus86, p.515] or in [KK86b, Sect.2], [KK86a, 4.1($I_{23}$)] the map

$$\tau\colon Q * G \longrightarrow Q * G \quad , \quad \tau(f\delta_g) = g^{-1}(f)\delta_{g^{-1}} \quad g \in G, f \in Q$$

is an involutive algebra anti-isomorphism that restricts to an anti-isomorphism, denoted by the same symbol, $\tau\colon A \xrightarrow{\cong} A$. In particular, $A \cong A^{op}$ as $K$-algebras.

If $|G|$ is invertible in $S$, we can set $e = \frac{1}{|G|}\sum_{g \in G} \delta_g$. It is an idempotent element of $A$ and $A\left(\sum_{g \in G} \delta_g\right) A = AeA \subseteq A$ is an idempotent ideal in $A$.

**Lemma 2.2.** *Let $e$ be the idempotent just introduced.*

(1) *The left multiplication $e(\ )\colon S \to A, s \mapsto es$, yields an isomorphism of right $A$–modules $S \xrightarrow{\cong} eS = eA$.*

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[2] Let us point out that many authors use $S = \mathrm{Sym}_K(V^*)$ with $g$ acting on $s = f(x)$ as $g(s) = f(g^{-1}(x))$.

(2) *The right multiplication* $(\ )e\colon S \to A, s \mapsto se$, *yields an isomorphism of left* $A$–*modules* $S \xrightarrow{\cong} Se = Ae$.

(3) *The (two–sided) multiplication* $e(\ )e\colon R \to A, r \mapsto ere = er = re$, *yields an isomorphism of rings* $R \xrightarrow{\cong} eAe$, *where* $R = S^G$ *as defined above.*

(4) *In the commutative squares*

$$\begin{tikzcd}
S \times R \arrow[r, "(s{,}r)\mapsto sr"] \arrow[d, "(\ )e\times e(\ )e"', "\cong"] & S \arrow[d, "\cong"', "(\ )e"] & & R \times S \arrow[r, "(r{,}s)\mapsto rs"] \arrow[d, "e(\ )\times e(\ )"', "\cong"] & S \arrow[d, "\cong"', "e(\ )"] \\
Ae \times eAe \arrow[r, "(ae{,}ea'e)\mapsto aea'e"] & Ae & & eAe \times eA \arrow[r, "(ea'e{,}ea)\mapsto ea'ea"] & eA
\end{tikzcd}$$

*the vertical maps are bijections, thereby identifying the right* $eAe$–*module* $Ae$ *with the (right)* $R$–*module* $S$ *and the left* $eAe$–*module* $eA$ *with the (left)* $R$–*module* $S$. *In particular, the induced map*

$$S \otimes_R S \xrightarrow[\cong]{(\ )e\otimes e(\ )} Ae \otimes_{eAe} eA$$

*is an isomorphism of* $A$–*bimodules.* □

Moreover, taking invariants with respect to the above action of $G$ defines a functor $\mathbf{Mod}\, A \to \mathbf{Mod}\, R$ as the $G$–invariants form a (symmetric) $R$–module. In this way, there is, in particular, a natural homomorphism of rings

$$- \otimes_A Ae : A \cong \mathrm{Hom}_A(A, A) \longrightarrow \mathrm{Hom}_{eAe}(Ae, Ae) \cong \mathrm{Hom}_R(S, S)\,. \tag{1}$$

For any left $A$-modules $M, N$, one has $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_S(M, N)^G$, where $g \in G$ acts on an $S$-linear map $f : M \to N$ through $(g \cdot f)(m) = g(f(g^{-1}(m)))$. Taking invariants $(-)^G$ is an exact functor, whence also $\mathrm{Ext}^i_A(M, N) = \mathrm{Ext}^i_S(M, N)^G$ for all $i$. In particular, an $A$-module $M$ is projective if and only if the underlying $S$-module is projective.

**Lemma 2.3** ([Aus86]). *Let* $S$ *be a regular complete local ring or a graded polynomial ring. One has a functor*

$$\alpha : P(A) \to \mathbf{Mod}\, KG,\ P \mapsto S/\mathfrak{m}_S \otimes_S P\,,$$

*where* $P(A)$ *denotes the category of projective* $A$*-modules and* $\mathfrak{m}_S$ *denotes the maximal ideal (in case* $S$ *is local) or the maximal ideal* $(x_1, \dots, x_n)$ *in* $S$ *(in case* $S$ *is a polynomial ring in* $n$ *variables One also has a functor* $\beta$ *in the other direction that sends a* $KG$*-module* $V$ *to* $S \otimes_K V$. *This pair of functors induces inverse bijections on the isomorphism classes of objects.* □

**Remark 2.4.** Auslander proved this result in the case where $S = K[[x, y]]$ the power series ring in two variables, a proof for the $n$-dimensional complete case can be found e.g. in [LW12]. However, the correspondence also holds in the graded case, i.e., for graded modules over $S = K[x_1, \dots, x_n]$ with $\deg x_i = 1$. For this one uses Swan's theorem, see e.g. [Bas68, XIV, Thm. 3.1].

*Quotients of* $A$ *by idempotent ideals.* Let $\chi$ be the character of an irreducible $G$-representation. This defines the central primitive idempotent associated to this representation as $e_\chi = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})g$ in $KG \subset A$. If we want to stress that $e_\chi \in A$, then we write $e_\chi = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})\delta_g$. In particular, denote $e := e_{\mathrm{triv}} = \frac{1}{|G|}\sum_{g\in G}\delta_g \in A$ the idempotent associated to the trivial representation of $G$, $f := e_{\det^{-1}} = \frac{1}{|G|}\sum_{g\in G}\det(g)\delta_g$, the idempotent associated to the inverse determinantal representation.

In the following we will be interested in the quotient algebra $A/Ae_\chi A$, where $e_\chi$ is an idempotent associated to a linear character $\chi$. The next two results show that the choice of the one-dimensional character does not matter and thus we will sometimes switch between $A/AeA$ and $A/AfA$.

With $\mathrm{Hom}_{\mathrm{gps}}(G, K^*)$ the group of linear characters, consider the map $\alpha : \mathrm{Hom}_{\mathrm{gps}}(G, K^*) \to \mathrm{Aut}_{K-\mathrm{Alg}}(A)$,

$$\lambda \mapsto \alpha_\lambda$$

$$\alpha_\lambda\left(\sum_{g\in G} s_g \delta_g\right) = \sum_{g \in G} s_g \lambda(g^{-1})\delta_g \, .$$

**Lemma 2.5.** *The map $\alpha$ is a homomorphism of groups. If $L$ is the one-dimensional representation defined by $\lambda$ and $\chi$ the character of some $G$-representation $W$, then $L \otimes W$ has character $\lambda \cdot \chi$. If $W$ is irreducible and $e_\chi = \frac{1}{|G|}\sum_{g\in G} \chi(g^{-1})g$ the corresponding idempotent in $KG$, then $\alpha_\lambda(e_\chi) = e_{\lambda\cdot\chi}$.* ☐

**Corollary 2.6.** *Let $\lambda, \lambda'$ be one-dimensional characters of $G$ with respective idempotents $e_\lambda, e_{\lambda'}$. Then the quotient algebras $A/Ae_\lambda A$ and $A/Ae_{\lambda'}A$ are isomorphic $K$-algebras.* ☐

In the next lemma we state some useful properties of the quotient $\overline{A}$. For this we recall the following notion: Let $G \leqslant \mathrm{GL}(V)$ be a finite group and let $\chi$ be a linear character. An element $f \in S$ is a *relative invariant for* $\chi$ if $g(f) = \chi(g)f$ for all $g \in G$. The set of relative invariants for $\chi$ is denoted by $S^G_\chi = \{f \in S : g(f) = \chi(g)f \text{ for all } g \in G\}$, cf. [Sta77]. Clearly one has $S^G_{\mathrm{triv}} = S^G = R$.

**Lemma 2.7.** *Let $G \leqslant \mathrm{GL}(V)$ be a finite group and let $\chi$ be a linear character. Assume that $S^G_\chi$ is a free $R$-module of rank 1, that is, there exists a $f_\chi \in S$ such that $S^G_\chi = f_\chi R$. Then*

$$S/(f_\chi) \cong (A/Ae_\chi A)e$$

*as $S$-modules.*

*Proof.* Denote $\overline{A} := A/Ae_\chi A$. Applying $-_A \otimes Ae$ to the exact sequence

$$0 \to Ae_\chi A \to A \to \overline{A} \to 0$$

yields the exact sequence (since $Ae$ is a flat $A$-module)

$$0 \to Ae_\chi Ae \to Ae \to \overline{A}e \to 0. \tag{2}$$

We have seen in Lemma 2.2 (2) that $Ae \cong Se$ (an explicit calculation shows that $Ae = Se$ as sets). Moreover $Ae_\chi Ae = (Sf_\chi)e \cong Sf_\chi$: for this we first use $Ae_\chi Ae = Ae_\chi Se$. Then using that $\delta_g e_\chi = \chi(g)e_\chi$, for an element $(\sum_{g\in G} t_g\delta_g)e_\chi se$ in $Ae_\chi Se$ we get

$$\begin{aligned}\sum_{g\in G} t_g\delta_g e_\chi se &= \left(\sum_{g\in G}\chi(g)t_g\right)e_\chi se = \left(\sum_{g\in G}\chi(g)t_g\right)\frac{1}{|G|}\sum_{h\in G}\chi(h^{-1})h(s)\delta_h e \\ &= \left(\sum_{g\in G}\chi(g)t_g\right)\left(\frac{1}{|G|}\sum_{h\in G}\chi(h^{-1})h(s)\right)e.\end{aligned}$$

The element $\sum_{h\in G}\chi(h^{-1})h(s)$ is a semi-invariant for $\chi$, so it is in the ideal in $R$ generated by $f_\chi$. Thus it follows that $Ae_\chi Ae \subseteq Sf_\chi e$. And the element $f_\chi e = e_\chi f_\chi e$ is in $Ae_\chi Ae$, thus $Ae_\chi Ae \supseteq Sf_\chi e$. This means that the sequence (2) is isomorphic to

$$0 \to Sf_\chi e \to Se \to \overline{A}e \to 0,$$

which implies that $\overline{A}e \cong (S/(f_\chi))e \cong S/(f_\chi)$ as $S$-modules. ☐

**Reflection groups.** Here we recall some useful facts about complex reflection groups; see, for example, [Bou81, LT09, OT92]. We mostly follow the notation in [OT92].

Recall that an element $g$ in $\mathrm{GL}(V)$, is

(a) a *(true) reflection*, if it is conjugate to a diagonal matrix $\mathsf{diag}(-1, 1, ..., 1)$. In other words, as a linear transformation $g$ fixes a unique hyperplane $H \subset V$ pointwise and has additionally $-1 \neq 1$ as an eigenvalue. We call any nonzero eigenvector for the eigenvalue $-1$ a *root* of the reflection and think of it as a vector "perpendicular" to the *mirror* $H$.

(b) a *pseudo-reflection*, if it is conjugate to a diagonal matrix $\mathsf{diag}(\zeta, 1, ..., 1)$, where $\zeta \neq 1$ is a root of unity in $K$. Again we call the eigenspace $H$ to the eigenvalue 1 the *mirror* of $g$.

For a finite subgroup $G \leqslant \mathrm{GL}(V)$, the subgroup $G' \leqslant G$ generated by the pseudo-reflections is normal in $G$ as the conjugate of a pseudo-reflection is again a pseudo-reflection. For the same reason the subgroup $G'' \leqslant G$ generated by (true) reflections is normal in $G$, contained, of course, in $G'$.

One distinguishes now the extreme possibilities.

**Definition 2.8.** Given a finite subgroup $G \leqslant \mathrm{GL}(V)$,

(a) $G$ is *small* if it contains *no pseudo-reflections*, thus, $G' = 1$.
(b) $G$ is a *(true) reflection* group if it is *generated by its (true) reflections*, thus, $G'' = G$.
(c) $G$ is a *complex reflection* or *pseudo-reflection* group if it is *generated by its pseudo-reflections*, thus, $G' = G$.

In this paper we will always distinguish between true reflection groups as the ones generated by order 2 reflections and pseudo-reflection groups as the ones generated by pseudo-reflections.

**Example 2.9.** Any finite subgroup of $\mathrm{SL}(V)$ is small, since it only contains elements with determinant 1, that is, it does not contain any pseudo-reflections.

The ring $S^G$ is a normal Cohen–Macaulay domain by the Hochster–Roberts Theorem [HR74]. If $G \leqslant \mathrm{SL}(V)$, then $S^G$ is Gorenstein and, conversely, if $G$ is small, then $S^G$ is Gorenstein only if $G \leqslant \mathrm{SL}(V)$ according to a theorem by Kei-Ichi Watanabe [Wat74]. Invariant rings of pseudo-reflection groups are distinguished by the following:

**Theorem 2.10** (Chevalley–Shephard–Todd). *Let $G \leqslant \mathrm{GL}(V)$ be a finite group acting on $S$. Then the invariant ring $R = S^G$ is a polynomial ring itself, that is, $R = K[f_1, ..., f_n] \subseteq S$, where the $f_i$ are algebraically independent homogeneous polynomials of degree $d_i \geqslant 1$, if and only if G is a pseudo-reflection group. Note that, equivalently, the $f_i$ form a homogeneous regular sequence in $S$.*
*Moreover, if G is a pseudo-reflection group, then $S$ is free as an $R$-module, more precisely $S \cong R \otimes_K KG$, as G-modules, where KG denotes the group ring of G.*

This was the second theorem in [Che55] and was as well generalized for pseudo-reflections in the separable case, see [OT92, Thm. 6.19].

*Reflection arrangement and discriminant.* Let us now recall some facts regarding pseudo-reflection groups $G \leqslant \mathrm{GL}(V)$:

(a) Finite pseudo-reflection groups over the complex numbers have been classified by Shephard and Todd [ST54]. They contain true reflection groups and thus all finite Coxeter groups, i.e., all finite groups that admit a realization as a reflection group over the real numbers. Coxeter groups are precisely those true reflection groups that have an invariant of degree 2, see [OT92]. Coxeter groups are moreover the pseudo-reflection groups for which $V$ is isomorphic to its dual $V^*$, see e.g. [Ser77, Thm. 31].

(b) The polynomials $f_i$ in Theorem 2.10 are called the *basic invariants* of $G$. They are not unique but their degrees $d_i$ are uniquely determined by $G$ and one has an equality $|G| = d_1 \cdots d_n$.

(c) Let $H \subset V$ be a hyperplane that is fixed by a cyclic subgroup generated by a pseudo-reflection $g_H \in G$ of order $\rho_H > 1$, so, $g_H(v) = v + L_H(v)a_H$, where $a_H \in V$ and $L_H(v)$ is a linear form such that $H = \{v \in V \mid L_H(v) = 0\}$. The *Jacobian*

$$J = Jac(f_1, ..., f_n) = \det \left| \frac{\partial f_i}{\partial x_j} \right|_{i,j=1,...,n} = u \prod_{\text{mirrors } H} L_H^{\rho_H - 1}$$

is, up to a nonzero constant multiple $u \in K^*$. Therefore, each linear form $L_H$ occurs with multiplicity $\rho_H - 1$. The degree of the Jacobian is $m = \sum_{i=1}^n (d_i - 1)$, which equals the number of pseudo-reflections in $G$.

(d) The differential form

$$df_1 \wedge \cdots \wedge d_{f_n} = J dx_1 \wedge \cdots \wedge dx_n$$

is $G$–invariant, whence $J$ transforms according to $gJ = (\det g)^{-1} J$, thus, $JK$ affords the linear, or one dimensional, *inverse determinant representation* of $G$.

(e) The element $z = \prod_H L_H$ is the reduced defining equation of the *reflection arrangement* $\mathcal{A}(G)$ associated to $G$. It is easy to see that $z$ is a relative invariant for the linear character $\chi = \det$, that is, for all $g \in G$ we have $gz = \det(g)z$. The degree of $z$ is $m_1$, the number of mirrors of $G$.

(f) The *discriminant* of the group action is given by

$$\Delta = zJ = \prod_{H \subset \mathcal{A}(G)} L_H^{\rho_H},$$

an element of $S^G$ of degree $\sum_\kappa \rho_\kappa = m + m_1$. The discriminant polynomial $\Delta \in S^G$ is always reduced (this follows e.g. from Saito's criterion and the fact that $\Theta_S^G \cong \Theta_S(-\log \Delta)$, see [OT92, Chapter 6] for statements and notation). In particular, if $G$ is a true reflection group, then $\rho_H = 2$ for all $H$, and thus $J = z$ and $z^2 = \Delta$ represents the discriminant.

(g) The preceding in geometric terms: if $G$ is a pseudo-reflection group, then the quotient $V/G = \mathrm{Spec}(S^G)$ is an affine regular variety isomorphic to $V \cong \mathbb{A}^n(K)$. Under the natural projection

$$\pi : V \cong \mathrm{Spec}(S) \longrightarrow V/G \cong \mathrm{Spec}(S^G)$$

the image of the hyperplane arrangement $\mathcal{A}(G)$ is the discriminant hypersurface $V(\Delta) \subseteq V/G$.

(h) The discriminant $V(\Delta)$ in $V/G$ and the hyperplane arrangement $\mathcal{A}(G)$ in $V$ are both *free divisors*. This means that the module of logarithmic derivations $\Theta_R(-\log \Delta) = \{\theta \in \Theta_R : \theta(\Delta) \in (\Delta)R\}$ is a free $R = S^G$-module and accordingly $\Theta_S(-\log z)$ is a free $S$-module. This was first shown by Kyoji Saito for Coxeter groups, cf. [Sai93] and by Hiroaki Terao for complex reflection groups [Ter80] .

**Example 2.11.** The true reflection groups $G \leqslant \mathrm{GL}(2,\mathbb{C})$ are classified via the ADE-Coxeter-Dynkin diagrams. The discriminant $\Delta$ of such a $G$ is the corresponding ADE-curve singularity, cf. e.g. [Knö84, Section 3]. For example, the $A_2$-curve singularity $K[x,y]/(x^3-y^2)$ is the discriminant of the group $S_3$ acting on $\mathbb{C}^2$, see Fig. 1.

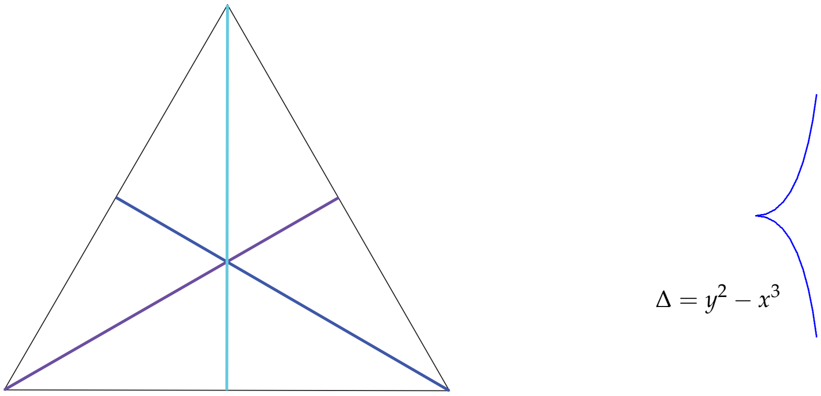

FIGURE 1. The three lines of the hyperplane arrangement of $S_3$ and the discriminant $\Delta$ on the right.

E. Bannai calculated all discriminants for complex reflection groups $G \leqslant \mathrm{GL}(V)$ in [Ban76]. In particular one sees from this list that all discriminants of reflection groups in $\mathrm{GL}(V)$ are curves of type ADE.

**Example 2.12.** The true reflection group $G_{24} \leqslant \mathrm{GL}(3,\mathbb{C})$ is a complex reflection group of order 336 that comes from Klein's simple group, see [OT92] ex. 6.69, 6.118[3] for more details. The reflection arrangement $\mathcal{A}(G_{24})$ consists of 21 hyperplanes. In loc. cit. the basic invariants for this group, and the discriminant matrix are determined. One obtains the equation of the discriminant $\Delta$ as determinant of the discriminant matrix, see Fig. 2. The discriminant $V(\Delta)$ is a non-normal hypersurface in $\mathbb{C}^3$, whose singular locus consists of two singular cubic curves meeting in the origin.

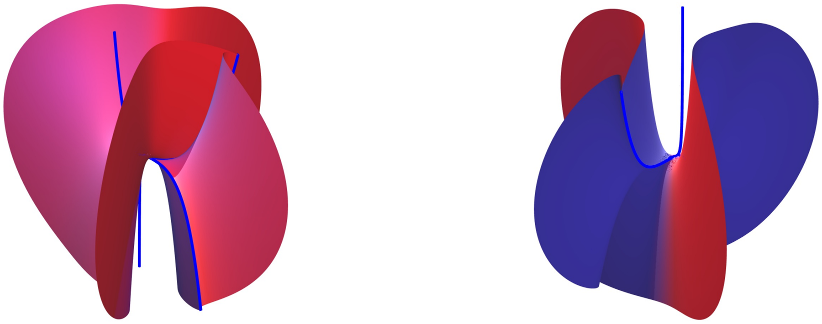

FIGURE 2. Two views of the discriminant of the group $G_{24}$ realized in $\mathbb{R}^3$ with equation $\Delta = -2048x^9y + 22016x^6y^3 - 256x^7z - 60032x^3y^5 + 1088x^4y^2z + 1728y^7 + 1008xy^4z - 88x^2yz^2 + z^3 = 0$.

---

[3] In Ex. 6.118 in [OT92] the sign in front of $256x^7z$ is erroneous.

**Isotypical components.** Let $G \leqslant \mathrm{GL}(V)$ be a pseudo-reflection group, and adopt the notation from the last subsection for $R, S, z, J$, and $\Delta$. Note that $R = S^G = K[f_1, \dots, f_n]$ and $R/(\Delta)$ are graded rings with $\deg f_i = d_i$, the degrees of the basic invariants. The decomposition of $S$ as an $R$-module is given as follows: let $R_+$ be the set of invariants of $G$ with zero constant term. Then $S/(R_+)$ is called the coinvariant algebra (here $(R_+)$ denotes the ideal in $S$ generated by elements in $R_+$) and by the Theorem of Chevalley–Shephard–Todd (Thm. 2.10) one has

$$S \cong R \otimes_K S/(R_+)$$

as graded $R$-modules. As $KG$-modules:

$$S \cong R \otimes_K KG\,.$$

With notation as above, one has the following simple fact.

**Lemma 2.13.** *Let $G$ be a finite group and $M$ a $KG$–module. Suppose that $r$ is the class number of $G$, i.e. the number of conjugacy class of $G$ or equivalently the number of isomorphism classes of irreducible representations of $G$. For $V_i$ an irreducible $G$–representation, the functors $\mathrm{Hom}_{KG}(V_i, -)$ and $(-) \otimes_K V_i$ are adjoint. We write*

$$\mathrm{ev}_{V_i} : \mathrm{Hom}_{KG}(V_i, M) \otimes_K V_i \to M$$

*for the evaluation map, which is the natural transformation of the composition of these functors to the identity functor. The map $\mathrm{ev}_{V_i}$ is a split monomorphism of $KG$–modules, where $G$ acts on $\mathrm{Hom}_{KG}(V_i, M) \otimes_K V_i$ through the second factor. Its image is the isotypical component of $M$ of type $V_i$. The sum of the evaluation maps,*

$$\sum_{i=1}^{r} \mathrm{ev}_{V_i} : \bigoplus_{i=1}^{r} \mathrm{Hom}_{KG}(V_i, M) \otimes_K V_i \xrightarrow{\cong} M\,,$$

*is an isomorphism of $KG$–modules.* □

If $M$ is a module over the skew group ring $S * G$, then each $\mathrm{Hom}_{KG}(V_i, M)$ is a maximal Cohen–Macaulay module over $R = S^G$.
Thus, as $G$-representations we have

$$S \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_{KG}(V_i, S) \otimes_K V_i = \bigoplus_{i=1}^{r} S_i \otimes_K V_i\,,$$

with notation $S_i := \mathrm{Hom}_{KG}(V_i, S)$.

The Jacobian $J \in S$ is an element of the isotypical component of $S$ to the inverse determinantal representation $\det^{-1}$, while $z \in S$ is an element of the isotypical component of $S$ of the determinantal representation det of $G$, and, as $S$ is a free $R$–module, the pair $(J, z)$ constitutes, trivially, a matrix factorization of $\Delta \in R$.

As $J$ and $z$ are relative invariants for $G$, multiplication with these elements on $S$ is $G$–equivariant. More precisely, multiplication with $J, z$, respectively, yields for each $V_i$ a graded $G$–equivariant matrix factorization. For compact notation, set $V_i' = V_i \otimes \det$, which is again an irreducible $G$–representation along with $V_i$. Further recall that the degrees of $J$ and $z$ resp., are $m$ and $m_1$. Now look at the exact sequence

$$0 \to S(-m) \otimes \det^{-1} \xrightarrow{J} S \to S/(J) \to 0.$$

Apply $\mathrm{Hom}_{KG}(V_i, -)$ to get

$$0 \to \mathrm{Hom}_{KG}(V_i, S(-m) \otimes \det^{-1}) \to S_i \to \mathrm{Hom}_{KG}(V_i, S/(J)) \to 0\,.$$

Here $\mathrm{Hom}_{KG}(V_i, S(-m) \otimes \det^{-1}) \cong \mathrm{Hom}_{KG}(V_i \otimes \det, S)(-m)$. If we set as well $S_i' = \mathrm{Hom}_{KG}(V_i', S)$ this is $S_i'(-m)$. Now denoting $\mathrm{Hom}_{KG}(V_i, S/(J)) = M_i$, we have short exact sequences of graded $R$–modules

$$\begin{tikzcd} 0 \arrow[r] & S_i'(-m) \arrow[r, "J"] & S_i \arrow[r] & M_i \arrow[r] & 0 \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} 0 \arrow[r] & S_i(-m-m_1) \arrow[r, "z"] & S_i'(-m) \arrow[r] & N_i \arrow[r] & 0 \end{tikzcd}$$

with $N_i = \mathrm{Hom}_{KG}(V_i, S/(z))(-m)$. Here the second one comes from the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & S(-m-m_1) \arrow[r, "z"] & S \otimes \det^{-1} \arrow[r] & S/(z) \arrow[r] & 0 \end{tikzcd} \tag{4}$$

We also have the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & N_i \arrow[r] & S_i \otimes_R R/(\Delta) \arrow[r] & M_i \arrow[r] & 0 \end{tikzcd} \tag{5}$$

$$\begin{tikzcd} 0 \arrow[r] & M_i(-m-m_1) \arrow[r] & S_i' \otimes_R R/(\Delta)(-m) \arrow[r] & N_i \arrow[r] & 0 \end{tikzcd}$$

which are already short exact sequences of maximal Cohen–Macaulay $R/(\Delta)$–modules. To sum up this discussion, we can state the following

**Lemma 2.14.** *We have the direct sum decompositions:*

$$S/(J) \cong \bigoplus_{i=0}^{r} M_i \otimes_K V_i \quad \textit{and} \quad S/(z) \cong \bigoplus_{i=0}^{r} N_i(m) \otimes_K V_i'$$

*as graded $R/(\Delta)-KG$–bimodules. If $\Delta$ is irreducible it follows that*

$$\mathrm{rank}_{R/(\Delta)} M_i + \mathrm{rank}_{R/(\Delta)} N_i = \dim_K V_i = \mathrm{rank}_R S_i = \mathrm{rank}_R S_i' .$$

□

**Example 2.15.** Consider the representation $V_{\mathrm{triv}}$ (instead of indexing the representations by $V_i$ we index again $V_\rho$ by a specific representation $\rho$) and thus $V_{\mathrm{triv}}' = V_{\det}$. Then the exact sequence (3) looks as follows

$$0 \to Rz(-m) \xrightarrow{J} R \to R/(\Delta) \to 0 ,$$

since $S_{\mathrm{triv}}' \cong Rz$ and $S_{\mathrm{triv}} \cong R$. This means that $M_{\mathrm{triv}} = R/(\Delta)$ and shows that $R/(\Delta)$ is a direct summand of $S/(J)$.

For $V_{\det^{-1}}$ on the other hand we obtain from (3)

$$0 \to R(-m) \xrightarrow{J} RJ \to 0 .$$

Thus $M_{\det^{-1}} = 0$ and the inverse determinantal representation does not contribute a $R$-direct summand of $S/(J)$. Note that if $G$ is a true reflection group, then $\det = \det^{-1}$.

**Endomorphism rings and Auslander's theorem.** One of the key results by M. Auslander in [Aus86, p.515] asserts that the ring homomorphism (1) from $A \to \mathrm{End}_R(S)$ is an isomorphism if $G$ is small, for a detailed proof see [Yos90, Prop. 10.9]:

**Theorem 2.16** (Auslander). *Let $S$ be as above and assume that $G \leqslant \mathrm{GL}(V)$, with $\dim V = n$, is small and set $R = S^G$. Then we have an isomorphism of algebras:*

$$A = S * G \xrightarrow{\cong} \mathrm{End}_R(S)\ , sg \mapsto (x \mapsto sg(x))\ .$$

*Moreover, $S * G$ is a CM-module over $R$ and $\mathrm{gldim}(S * G) = n$.*

**Remark 2.17.** By an obvious calculation, one sees that the centre $Z(A) = R$.

**Noncommutative resolutions of singularities and the McKay correspondence.** A resolution of singularities of an affine scheme $X = \mathrm{Spec}(R)$ is a proper birational map $\pi : \widetilde{X} \to X$ from a smooth scheme $\widetilde{X}$ to $X$ such that $\pi$ is an isomorphism over the smooth points of $X$. Noncommutative resolutions of singularities of a ring $R$ (or of $\mathrm{Spec}(R)$) are certain noncommutative $R$-algebras that should provide an algebraic analog of this geometric notion. For the rationale behind the definition and more background about noncommutative (crepant) resolutions see [Leu12, VdB04, BFI17].

**Definition 2.1.** Let $R$ be a commutative noetherian ring. Let $M$ be a finitely generated $R$-module with $\mathrm{supp}\, M = \mathrm{supp}\, R$. Then $\Lambda = \mathrm{End}_R M$ is called a *noncommutative resolution (NCR)* of $R$ if $\mathrm{gldim}\, \Lambda < \infty$.
If $\Lambda$ is any finitely generated $R$-algebra that is faithful as $R$-module and $\mathrm{gldim}\, \Lambda < \infty$, then we call $\Lambda$ a *weak NCR* of $R$. Note that in the case of a weak NCR we do not require that $\Lambda$ is an Endomorphism ring or even an $R$-order.

**Remark 2.18.** In Van den Bergh's original treatment [VdB04], a *noncommutative crepant resolution (=NCCR)* was defined over a Gorenstein domain. With our definition above, a NCCR over a commutative noetherian ring $R$ is an NCR that is additionally a nonsingular order over $R$. The (weak) NCRs constructed in this paper are (almost) never nonsingular orders: by definition if a finitely generated $R$-algebra $\Lambda$ is a nonsingular $R$-order, then $\mathrm{gldim}(\Lambda)_\mathfrak{p} = \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. This implies in particular that $\mathrm{gldim}\, \Lambda = \dim R$. But our NCRs are of global dimension $\dim R + 1$. For more detail see Remark 3.14 and Cor. 4.19.
NCRs were first defined in [DITV15] over normal rings, we use here the more general definition of [DFI15].

In particular, Auslander's theorem can be reformulated in terms of noncommutative resolutions, cf. [VdB04, IW14]:

**Theorem 2.19.** *Let $G \leqslant \mathrm{GL}(V)$ small. Then $A = S * G$ yields a NCCR over $R = S^G$, that is, $A \cong \mathrm{End}_R\, S$ has global dimension $n$ and is a nonsingular order over $R$.*

*McKay correspondence.* The classical McKay correspondence relates the minimal resolutions of quotients of $\mathbb{C}^2 = K^2$ by finite groups $\Gamma \leqslant \mathrm{SL}(V)$ to the representation theory of $\Gamma$. Here we just shortly state the correspondences and refer to [Buc12, BFI17, GSV83, Rei02] for more details.

Let $\Gamma \leqslant \mathrm{SL}(V)$ be a finite group acting on $V$ with $\dim V = 2$. Denote $R := S^\Gamma$ the invariant ring under this action. Then $X := \mathrm{Spec}(R)$ is a Kleinian singularity, see [Kle93]. Denote by $\pi : \widetilde{X} \to X$ the minimal resolution of $X$ and by $\mathbb{E} = \bigcup_{i=1}^n \mathbb{E}_i$ the exceptional divisor on

$\widetilde{X}$, where the $\mathbb{E}_i$ are the irreducible components. The *dual resolution graph* of $X$ has vertices indexed by the $\mathbb{E}_i$ and there is an edge between $\mathbb{E}_i$ and $\mathbb{E}_j$ if and only if $\mathbb{E}_i \cap \mathbb{E}_j \neq \varnothing$.

On the representation theoretic side, one considers the *McKay quiver* of $\Gamma$. Let $c : \Gamma \hookrightarrow \mathrm{GL}(V)$ be the defining (or canonical) representation of $\Gamma$ and denote the irreducible representations of $\Gamma$ by $\rho_i : \Gamma \to \mathrm{GL}(V_i)$, $i = 0, \ldots, n$ with vector spaces $V_i$ of dimensions $m_i$. Here the trivial representation is $\rho_0 : \Gamma \to K^*$. The McKay quiver of $\Gamma$ has vertices indexed by the $V_i$ and there are $m_{ij}$ arrows from $V_i$ to $V_j$ if and only if $V_i$ is contained with multiplicity $m_{ij}$ in $V_j \otimes V$, so $m_{ij} = \dim_K \mathrm{Hom}_{KG}(V_i, V \otimes V_j)$.

On the algebraic side, look at the *Auslander–Reiten quiver* of $R$. Denote by $\mathbf{CM}(R)$ the category of CM-modules over $R$. The vertices of the Auslander–Reiten quiver are indexed by the indecomposable CM-modules $M_i, i = 0, \ldots, n$ and there are $m_{ij}$ arrows from $M_i$ to $M_j$ if and only if in the AR-sequence $0 \to \tau M_j \to E \to M_j \to 0$ ending in $M_j$, the modules $M_i$ appears with multiplicity $m_{ij}$ in $E$.

By *Herzog's theorem*, [Her78], one has the direct sum decomposition

$$S \cong \bigoplus_{M \in \mathbf{CM}(R)} M^{a_M} , \tag{6}$$

where the direct sum runs over all isomorphism classes of indecomposable CM-modules.

**Theorem 2.20** (Classical McKay correspondence). *For a given $\Gamma \leqslant \mathrm{SL}(V)$, $\dim V = 2$, the following are in $1-1$-correspondence:*

*(1) The irreducible components $\mathbb{E}_i$ of the exceptional divisor of $\widetilde{X}$.*
*(2) The isomorphism classes of irreducible representations $V_i$ of $\Gamma$ (minus the trivial representation).*
*(3) The isomorphism classes of indecomposable CM-modules $M_i$ on $R$ (without the module $R$ itself).*
*(4) The indecomposable $R$-direct summands of $S$ (except $R$ itself).*

*Moreover, the multiplicities $m_i$ of $\mathbb{E}_i$ in the so-called fundamental cycle of $\widetilde{X}$ equals the dimension $m_i$ of $V_i$ equals the multiplicity $a_{M_i}$ of $M_i$ in the decomposition (6) of $S$.*
*The McKay quiver and the Auslander–Reiten quiver are the same and one obtains the dual resolution graph by removing the vertex for the trivial representation and collapsing the 2-cycles on the McKay quiver into edges.*

We are particularly interested in the correspondences (2)–(4): let us note that all three points still hold for any finite $G \leqslant \mathrm{GL}(V)$ not containing any pseudo–reflections and $\dim V = 2$. Herzog's theorem establishes the correspondence between (3) and (4) and Auslander's theorem yields the correspondence between (2) and (4). Moreover, the bijection between the nonisomorphic $R$-direct summands of $S$ and the irreducible representations of $\Gamma \leqslant \mathrm{GL}(V)$ holds for any $\dim V = n \geq 2$ if and only if $G$ does not contain any pseudo-reflections, see [LW12, Cor. 5.20].

## 3. THE GEOMETRY

**Some general facts on group actions.** We begin with the following general results on group actions that we quote from Bourbaki.

**Proposition 3.1** ([Bou64, V.1.9 Cor.]). *Let $G$ be a finite group that acts through ring automorphisms on a commutative integral domain $S$. The group then acts as well through automorphisms* *on the field of fractions $Q(S)$ of $S$ and the fixed field $Q(S)^G$ is the field of fractions of the invariant integral subdomain $R = S^G$, that is, $Q(R) \cong Q(S)^G$.* □

**3.2.** In the setting of the preceding Proposition, a crucial role will be played by the map $\varphi : S \otimes_R S \to \mathsf{Maps}(G, S)$ given by $\varphi\left(\sum_{i=1}^m x_i \otimes y_i\right)(g) = \sum_{i=1}^m x_i g(y_i) \in S$ with $(x_i, y_i) \in S \times S$, for $i = 1, ..., m$, a finite family of pairs from $S$. Both source and target of this map are naturally $R$–modules and $\varphi$ is $R$–linear with respect to these structures.

Moreover, identifying naturally $Q(R) \otimes_R (S \otimes_R S) \cong Q(S) \otimes_{Q(R)} Q(S)$ and $Q(R) \otimes_R \mathsf{Maps}(G, S) \cong \mathsf{Maps}(G, Q(S))$, the induced map $\psi = Q(R) \otimes_R \varphi$ of vector spaces over $Q(R)$ identifies with $\psi\left(\sum_{i=1}^m x_i \otimes y_i\right)(g) = \sum_{i=1}^m x_i g(y_i) \in Q(S)$ for $(x_i, y_i) \in Q(S) \times Q(S)$ a finite family of pairs from $Q(S)$.

Galois descent then yields the following fact[4]:

**Proposition 3.3** ([Bou90, V.§10, no.4, Cor. of Prop. 8])**.** *If $G$ is a finite subgroup of the group of ring automorphisms of a commutative integral domain $S$, then the map*

$$\psi \colon Q(S) \otimes_{Q(R)} Q(S) \longrightarrow \mathsf{Maps}(G, Q(S))$$

*is bijective.* □

**The structure of $\varphi$.** (See also [Wat76] for the material of this subsection.)
To study $\varphi$ further, note next that with respect to the natural $R$–algebra structure on $S \otimes_R S$ and the diagonal $R$–algebra structure on $\mathsf{Maps}(G, S) \cong S^{|G|}$, endowed with the componentwise operations, the map $\varphi$ is a homomorphism of $R$–algebras. Let $\mathrm{ev}_g : \mathsf{Maps}(G, S) \to S$ be the evaluation at $g \in G$, so that

$$\mathrm{ev}_g\, \varphi\left(\sum_{i=1}^m x_i \otimes y_i\right) = \sum_{i=1}^m x_i g(y_i) \in S$$

yields an $R$–algebra homomorphism $\mathrm{ev}_g\, \varphi \colon S \otimes_R S \to S$.

**Lemma 3.4.** *With notation as just introduced and with hypotheses as in Proposition 3.3, one has*

1. *For each $g \in G$, the $R$–algebra homomorphism $\mathrm{ev}_g\, \varphi$ is surjective with kernel the prime ideal*
$$I_g = (1 \otimes s - g(s) \otimes 1; s \in S) \subseteq S \otimes_R S\,.$$
2. *The family of $R$–algebra homomorphisms $(\mathrm{ev}_g)_{g \in G}$ identifies $\mathsf{Maps}(G, S)$ with the $S \otimes_R S$–algebra $\prod_{g \in G} (S \otimes_R S)/I_g$.*
3. *The kernel of $\varphi$ equals $\cap_{g \in G} I_g$. The image of $\varphi$ is isomorphic to the reduced $R$–algebra $\mathrm{Im}\, \varphi \cong (S \otimes_R S)/\cap_{g \in G} I_g$.*
4. *The ideal $\cap_{g \in G} I_g$ is the nilradical of the ring $S \otimes_R S$.*
5. *Kernel and cokernel of $\varphi$ are $R$–torsion modules.*

*Proof.* (1) For any $x \in S$, one has $\mathrm{ev}_g\, \varphi(x \otimes 1) = x \in S$, whence $\mathrm{ev}_g\, \varphi$ is surjective. Its kernel is as claimed due to the following standard argument: Clearly, $I_g \subseteq \mathrm{Ker}(\mathrm{ev}_g\, \varphi)$, and if $\mathrm{ev}_g\, \varphi\left(\sum_{i=1}^m x_i \otimes y_i\right) = \sum_{i=1}^m x_i g(y_i) = 0$ in $S$, then

$$\sum_{i=1}^m x_i \otimes y_i = \sum_{i=1}^m (x_i \otimes y_i - x_i g(y_i) \otimes 1) = (x_i \otimes 1) \sum_{i=1}^m (1 \otimes y_i - g(y_i) \otimes 1) \in I_g\,.$$

Because $S \otimes_R S/I_g \cong S$ is a domain, $I_g \subseteq S \otimes_R S$ is prime.

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[4] This result has also been called "a strong form of Hilbert's Theorem 90"; see `https://math.berkeley.edu/~ogus/Math_250A/Notes/galoisnormal.pdf`

Regarding (2), note that $(\mathrm{ev}_g)_{g\in G} : \mathsf{Maps}(G,S) \to \prod_{g\in G} S$ is bijective by definition and (1) reveals the $S\otimes_R S$–algebra structure induced by $\varphi$ on $\mathsf{Maps}(G,S)$.

The first part of assertion (3) is an immediate consequence of (1) as $\varphi = (\mathrm{ev}_g\,\varphi)_{g\in G}$. The second assertion then follows.

As concerns (4), Proposition 3.3 shows that $S\otimes_R S$ and its image have the same reduction. As the image is reduced, the claim follows — see alternatively [Wat76, Lemma 2.5] for a direct argument.

Likewise, (5) follows from Proposition 3.3. □

**Remark 3.5.** It seems worthwhile to point out the following consequence of (2) above: For $f \in \mathsf{Maps}(G,S)$, the map $sfs' = \varphi(s\otimes s')f \in \mathsf{Maps}(G,S)$ is given by $sfs'(g) = sg(s')f(g)$ for $g\in G$. In particular, even though $\mathsf{Maps}(G,S) \cong S^{|G|}$ as a ring, it is not a symmetric $S$–bimodule when viewed as a $S$–bimodule via $\varphi$.

**The geometric interpretation of $\varphi$.** With $X = \operatorname{Spec} S$, the reduced and irreducible affine scheme defined by the integral domain $S$, the scheme $Y = \operatorname{Spec} R$ identifies with the orbit scheme $Y = X/G$ of $X$ modulo the action of $G$. The canonical map $X \to Y$ corresponds to the inclusion $R\subseteq S$ and $\operatorname{Spec}(S\otimes_R S)\cong X\times_Y X \subseteq X\times X$ is the (schematic) graph of the equivalence relation defined by the action of $G$ on $X$.

For $g\in G$, one may identify $\operatorname{Spec}(S\otimes_R S/I_g) \cong \operatorname{Im}(\operatorname{Spec}(\mathrm{ev}_g\,\varphi) : \{g\}\times X \to X\times_Y X)$, image of the map $(g,x)\mapsto (x, g(x))$ for $x\in X$. The map $\varphi$ corresponds then to $\operatorname{Spec}(\varphi) : G\times X = \coprod_{g\in G}\{g\}\times X \to X\times_Y X$. Proposition 3.3 says that this morphism of schemes is generically an isomorphism, and its image is the graph of the group action, $GX := \bigcup_{g\in G}\operatorname{Im}(\operatorname{Spec}(\mathrm{ev}_g\,\varphi) : \{g\}\times X \subseteq X\times_Y X$ with its reduced structure. Moreover, $GX = (X\times_Y X)_{\mathrm{red}}$ is the reduced underlying scheme of $X\times_Y X$, so that the only difference between $GX$ and $X\times_Y X$ can be embedded components in $X\times_Y X$.

**Interpretation in terms of the twisted group algebra.** The foregoing facts admit the following interpretation in terms of the twisted group algebra $A = S * G$, where $G$ is still a finite subgroup of the group of ring automorphisms of the commutative domain $S$.

**Proposition 3.6.** *With the assumptions just made, the map $x\otimes y \mapsto \sum_{g\in G} x\delta_g y$ defines a surjective homomorphism $\alpha : S\otimes_R S \twoheadrightarrow A\left(\sum_{g\in G}\delta_g\right)A$ of $S$–bimodules, while $\beta : \mathsf{Maps}(G,S)\to A, (s_g)_{g\in G}\mapsto \sum_{g\in G} s_g\delta_g$ is an isomorphism of $S$–bimodules over $R$ when $\mathsf{Maps}(G,S)$ is viewed as an $S$–bimodule via $\varphi$. Note, however, that $\beta$ is clearly not an isomorphism of algebras.*

*There is a commutative diagram of $S$–bimodule homomorphisms*

$$\begin{tikzcd}
 & & A\left(\sum_{g\in G}\delta_g\right)A \arrow[r, hook] & A \\
0 \arrow[r] & \bigcap_{g\in G} I_g \arrow[r] & S\otimes_R S \arrow[r, "\varphi"] \arrow[u, two heads, "\alpha"] & \mathsf{Maps}(G,S)\,, \arrow[u, "\cong"', "\beta"]
\end{tikzcd}$$

*with the bottom row an exact sequence. In particular, as $S$–bimodules*

$$A\left(\sum_{g\in G}\delta_g\right)A \cong (S\otimes_R S)/\cap_{g\in G} I_g\,.$$

*Proof.* It is clear that $\alpha$ is a homomorphism of $S$–bimodules with respect to the natural $S$–bimodule structures on $A$ and its ideal $A\left(\sum_{g\in G}\delta_g\right)A$. It is surjective as for $a = \sum_{h,h'\in G} s_h\delta_h$ and $a' = \sum_{h'\in G}\delta_{h'}s'_{h'}$ in $A$ one has

$$a\left(\sum_{g\in G}\delta_g\right)a' = \sum_{h,h'\in G} s_h\delta_h\left(\sum_{g\in G}\delta_g\right)\delta_{h'}s'_{h'} = \sum_{h,h'\in G} s_h\left(\sum_{g\in G}\delta_g\right)s'_{h'} = \alpha\left(\sum_{h,h'\in G} s_h\otimes s'_{h'}\right),$$

because $\delta_h\left(\sum_{g\in G}\delta_g\right)\delta_{h'} = \sum_{g\in G}\delta_{hgh'} = \sum_{g\in G}\delta_g$ for any $h,h'\in G$.

Note that establishing $\beta$ as an isomorphism of $S\otimes_R S$–modules uses Lemma 3.4(2).

By definition of the various objects and morphisms we have

$$\alpha(x\otimes y) = \sum_{g\in G} x\delta_g y = \sum_{g\in G} xg(y)\delta_g = \beta\varphi(x\otimes y)$$

in $A$, whence the commutativity of the square in the diagram.

What we have established so far shows that the image of $\beta\varphi$ equals $A\left(\sum_{g\in G}\delta_g\right)A\subseteq A$, isomorphic as $S$–bimodule to $S\otimes_R S/\cap_{g\in g} I_g$. ☐

**Corollary 3.7.** *If in the above setting $S\otimes_R S$ is reduced then $\alpha$ is a bijection and the bijections $\alpha,\beta$ identify the map $\varphi$ with the inclusion of the two–sided ideal $A\left(\sum_{g\in G}\delta_g\right)A$ into $A$.* ☐

If $|G|$ is invertible in $S$, let again $e = \frac{1}{|G|}\sum_{g\in G}\delta_g \in A$, cf. Section 2.

**Corollary 3.8.** *If $|G|$ is invertible in $S$, then one can identify $\varphi : S\otimes_R S \to \mathsf{Maps}(G,S)$ with the homomorphism of $A$–bimodules $\mu : Ae\otimes_{eAe} eA \to A$ with $\mu(ae\otimes ea') = aea'$.* ☐

**The structure of $\varphi$ for reflection groups.** Now we return to the situation where $S = \mathrm{Sym}_K V$ and the finite group $G \leqslant \mathrm{GL}(V)$ acts linearly on $S$. In the following key result the equivalence (1)$\Longleftrightarrow$(2) is due to J. Watanabe [Wat76, Cor.2.9, Cor.2.12, Lemma 2.7].

**Theorem 3.9.** *For a finite subgroup $G \leqslant \mathrm{GL}(V)$ with $|G|$ invertible in $K$ the following are equivalent.*

1. *The group $G$ is generated by pseudo–reflections in* $\mathrm{GL}(V)$.
2. *The ring $S\otimes_R S$ is Cohen–Macaulay.*
3. *The ring $S\otimes_R S$ is a complete intersection in the polynomial ring $S\otimes_K S$.*

*If these equivalent conditions are satisfied then $S\otimes_R S$ is reduced and $\varphi\colon S\otimes_R S\to \mathsf{Maps}(G,S)$ is injective. This ring homomorphism is the normalization morphism for $S\otimes_R S$.*

*Proof.* As stated above, (1)$\Longleftrightarrow$(2) is due to Watanabe and clearly (3)$\Longrightarrow$(2). It thus suffices to show (1)$\Longrightarrow$(3). With $S = \mathrm{Sym}_K V \cong K[x_1,\ldots,x_n]$ the polynomial ring, $S\otimes_K S \cong \mathrm{Sym}_K(V\oplus V) \cong K[x'_1,\ldots,x'_n;x''_1,\ldots,x''_n]$ is a polynomial ring in $2n$ variables, where we have set $x'_i = x_i\otimes 1$ and $x''_i = 1\otimes x_i$.

With $f_i\in R\subseteq S$ basic invariants, so that $R = K[f_1,\ldots,f_n]\subseteq S$, one has the presentation

$$S\otimes_R S\cong K[x'_1,\ldots,x'_n;x''_1,\ldots,x''_n]/(f_i(\mathbf{x}'')-f_i(\mathbf{x}');i=1,\ldots,n)\,.$$

Since $S$ is flat (even free) over $R$ by the Chevalley–Shephard–Todd theorem, the $R$–regular sequence $\mathbf{f} = (f_1,\ldots,f_n)$ is also regular on $S$. As $S$ is flat over $K$, the sequence $(f_i\otimes 1)_i$ is regular in $S\otimes_K S$ with quotient $S/(\mathbf{f})\otimes_K S$. As $S/(\mathbf{f})$ is flat over $K$, it follows that $(1\otimes f_i)_i$ forms a regular sequence in $S/(\mathbf{f})\otimes_K S$. Hence $(f_1\otimes 1,\ldots,f_n\otimes 1, 1\otimes f_1,\ldots,1\otimes f_n)$ is a regular sequence in $S \otimes_K S$. This implies that $(1 \otimes f_i - f_i \otimes 1)_i$ is a regular sequence since it is part of the regular sequence $(f_i \otimes 1 - 1 \otimes f_i, 1 \otimes f_i)_i$. Thus, $S \otimes_R S = S \otimes_K S/(f_i \otimes 1 - 1 \otimes f_i)_{i=1,\dots,n}$ is a complete intersection ring as claimed.

Concerning the remaining assertions, if $S \otimes_R S$ is Cohen–Macaulay it cannot contain any nontrivial torsion submodule, whence $\varphi$ is injective by Lemma 3.4(5). As $\varphi$ is injective and generically an isomorphism by Proposition 3.3, it suffices to note that the ring $\mathsf{Maps}(G, S) \cong S^{|G|}$ is normal. □

**Question 3.10.** Can one strengthen Theorem 3.9 by showing that $G$ is a group generated by pseudo–reflections if, and only if, $S \otimes_R S$ is reduced?

**A note on normalization and conductor ideals.** If $\nu \colon C \to \widetilde{C}$ is the normalization homomorphism of a reduced commutative ring $C$, then applying $\mathrm{Hom}_C(-, C)$ to $\nu$ yields an inclusion $\nu^* = \mathrm{Hom}_C(\nu, C) \colon \mathrm{Hom}_C(\widetilde{C}, C) \hookrightarrow C$. The image is the *conductor ideal* $\mathfrak{c} \subseteq C$ (with respect to its normalization.) It is as well an ideal in the larger ring $\widetilde{C}$ and is the largest ideal of $C$ with this property. Alternatively, one may define the conductor ideal as the annihilator $\mathfrak{c} = \mathrm{ann}_C \widetilde{C}/C$.

Below we will use the following facts.

**Lemma 3.11.** *Assume the commutative ring $C$ is noetherian and Gorenstein with its normalization $\widetilde{C}$ a Cohen–Macaulay $C$–module. In this case,*

(1) *The $C$–module $\widetilde{C}/C$ is Cohen–Macaulay of Krull dimension* $\dim C - 1$.
(2) *As $C$–modules* $\mathrm{Ext}^1_C(\widetilde{C}/C, C) \cong C/\mathfrak{c}$.
(3) *As $C$–modules* $\mathrm{Ext}^1_C(C/\mathfrak{c}, C) \cong \widetilde{C}/C$.
(4) *There are isomorphic short exact sequences of $C$–modules*

$$\begin{tikzcd}
0 \arrow[r] & C/\mathfrak{c} \arrow[r] \arrow[d, "\cong"'] & \widetilde{C}/\mathfrak{c} \arrow[r] \arrow[d, "\cong"'] & \widetilde{C}/C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_C(\widetilde{C}/C, C) \arrow[r] & \mathrm{Ext}^1_C(\widetilde{C}/\mathfrak{c}, C) \arrow[r] & \mathrm{Ext}^1_C(C/\mathfrak{c}, C) \arrow[r] & 0\,.
\end{tikzcd}$$

(5) *The conductor ideal $\mathfrak{c}$ is a maximal Cohen–Macaulay $C$–module.*
(6) *If $\widetilde{C}$ is a regular ring, then the (reduced) vanishing locus $V(\mathfrak{c}) \subseteq \mathrm{Spec}\, C$ is the (reduced) singular locus of $C$*

*Proof.* Because the normalization homomorphism is generically an isomorphism, the Krull dimension of $\widetilde{C}/C$ is at most $\dim C - 1$. Because both $\widetilde{C}$ and $C$ are Cohen–Macaulay of Krull dimension $\dim C$ by assumption, the short exact sequence

$$(\dagger) \qquad \begin{tikzcd} 0 \arrow[r] & C \arrow[r, "\nu"] & \widetilde{C} \arrow[r] & \widetilde{C}/C \arrow[r] & 0 \end{tikzcd}$$

shows that the depth of $\widetilde{C}/C$ is at least $\dim C - 1$, whence the dimension and depth coincide and are equal to $\dim C - 1$, thus establishing (1).

For (2) note that $\mathrm{Ext}^i_C(\widetilde{C}, C) = 0$ for $i \neq 0$ as $\widetilde{C}$ is a (necessarily maximal) Cohen–Macaulay $C$–module. Applying $\mathrm{Hom}_C(\ , C)$ to the short exact sequence (†) and noting that $\mathrm{Hom}_C(\widetilde{C}/C, C) = 0$ one obtains the short exact sequence of $C$–modules

$$\begin{tikzcd} 0 & \mathrm{Ext}^1_C(\widetilde{C}/C, C) \arrow[l] & C \arrow[l] & \mathfrak{c} \arrow[l] & 0\,, \arrow[l] \end{tikzcd}$$

and so (2) follows.

As to (3), just apply $\mathrm{Hom}_C(\ ,C)$ to the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{c} \arrow[r, "\nu"] & C \arrow[r] & C/\mathfrak{c} \arrow[r] & 0 \end{tikzcd}$$

and observe that $\mathrm{Hom}_C(\mathfrak{c},C) \cong \widetilde{C}$ as $\widetilde{C}$, being maximal Cohen–Macaulay over $C$, is a reflexive $C$–module.

Finally, apply $\mathrm{Hom}_C(\ ,C)$ to the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{c} \arrow[r, "\nu"] & \widetilde{C} \arrow[r] & \widetilde{C}/\mathfrak{c} \arrow[r] & 0 \end{tikzcd}$$

to obtain first $\mathrm{Ext}^1_C(\widetilde{C}/\mathfrak{c},C) \cong \widetilde{C}/\mathfrak{c}$ and then (4).

Item (5) follows from (2) as $\mathrm{Ext}^1_C(\widetilde{C}/C,C)$ is a Cohen–Macaulay $C$–module (of Krull dimension $\dim C - 1$) along with $\widetilde{C}/C$. Now use the short exact sequence $0 \to \mathfrak{c} \to C \to C/\mathfrak{c} \cong \mathrm{Ext}^1_C(\widetilde{C}/C,C) \to 0$ to conclude.

Item (6) follows from the fact that $V(\mathfrak{c})$ describes the non–normal locus of $\mathrm{Spec}\, C$, thus, $V(\mathfrak{c}) \subseteq \mathrm{Sing}(C)$. Outside of $V(\mathfrak{c})$, the normalization homomorphism is an isomorphism, thus, $\mathrm{Spec}\, C$ is regular there as this holds for $\widetilde{C}$ by assumption. □

Translating this into a statement for the twisted group algebra, we obtain the following structure theorem for the algebra $\overline{A} = A/AeA$.

**Theorem 3.12.** *Assume the finite subgroup $G \leqslant \mathrm{GL}(V)$ with $|G|$ invertible in $K$ is generated by pseudo–reflections*[5] *and let $A = S * G$ be the twisted group algebra.*

(1) *The ideal $AeA$ of $A$ is projective both as left or as right $A$–module.*

(2) *For any (left or right) $\overline{A}$–modules $M, N$, restriction of scalars along $A \twoheadrightarrow \overline{A}$ yields isomorphisms of $R$–modules $\mathrm{Ext}^i_{\overline{A}}(M,N) \xrightarrow{\cong} \mathrm{Ext}^i_A(M,N)$ for all integers $i$. In other words, the ring homomorphism $A \twoheadrightarrow \overline{A}$ is a* homological epimorphism*; see* [GL91, Thm. 4.4.(5), (5′)].

*In particular, $\overline{A}$ is of finite global dimension at most $n = \dim V$. Moreover, $\overline{A}$ is a Cohen–Macaulay $R$–module of Krull dimension $\dim R - 1$.*

*Proof.* (1) In view of Corollary 3.8 and Theorem 3.9, the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is an isomorphism of $A$–bimodules. It thus suffices to prove that $Ae \otimes_{eAe} eA$ is projective as (one–sided) $A$–module. Using again Corollary 3.8, we have also the identification $S \otimes_R S \cong Ae \otimes_{eAe} eA$ as $A$–bimodules. Moreover, by the Chevalley–Shephard–Todd theorem, $S$ is a free $R$–module, whence $S \otimes_R S$ is free as left or right $S$–module. Now $S \cong Ae$ is a projective left $A$–module and $eA \cong S$ is a projective left $eAe \cong R$–module. Thus, $Ae \otimes_{eAe} eA$ is projective as left $A$–module. The statement for the right module structure follows by symmetry.

It is well known that (1) implies (2). This is shown in [APT92] for Artin algebras, but their arguments apply to any rings. For a reference that makes no such restrictive assumption, see [GL91, Thm. 4.4] and [Kra15, Lemma 2.7].

As $\mathrm{gldim}\, A = n$, property (2) implies $\mathrm{gldim}\, \overline{A} \leqslant \mathrm{gldim}\, A$. With $S \otimes_R S \cong Ae \otimes_{eAe} eA$ and $A \cong \mathsf{Maps}(G,S)$, as $S \otimes_R S$–module $\overline{A}$ identifies with the cokernel of the normalization homomorphism $\varphi : S \otimes_R S \to \mathsf{Maps}(G,S)$. As $S \otimes_R S$ is a complete intersection, thus,

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[5] We allow $G$ to be the trivial group.

Gorenstein, and $\mathrm{Maps}(G,S) \cong S^{|G|}$ is Cohen–Macaulay, Lemma 3.11 (1) applies to show that $\overline{A}$ is a Cohen–Macaulay module of Krull dimension $n-1$, equivalently as $S \otimes_R S$ or $R$–module, as claimed. ☐

**Corollary 3.13.** *Let $S$ be as above and $G \leqslant \mathrm{GL}(V)$ be a finite group generated by pseudo-reflections. Let $A = S * G$ and $e_\chi = e_\chi^2 \in A$ an idempotent associated to a linear character $\chi$. Then:*
*(1) The quotient algebra $\overline{A} = A/Ae_\chi A$ is Koszul.*
*(2) If $G \neq \mu_2$, then $\mathrm{gldim}\,\overline{A} = n$.*

*Proof.* By Cor. 2.6 we may assume that $\chi$ is the trivial character and thus $e_\chi = e = \frac{1}{|G|}\sum_{g\in G}\delta_g$. Denote by $V$ the defining representation of $G$. Following [Aus86], the Koszul complex of $K$

$$0 \longrightarrow S(-n)\otimes_K \det V \longrightarrow S(-n+1)\otimes_K \bigwedge^{n-1} V \longrightarrow \cdots \longrightarrow S \longrightarrow K \to 0$$

yields an $A$-projective resolution of $K$. A minimal projective $A$-resolution for any simple $A$-module $W$ is obtained by tensoring this complex with $W$ over $K$, with $G$ acting diagonally:

$$\begin{aligned} 0 \longrightarrow S(-n)\otimes_K \det V \otimes_K W \longrightarrow S(-n+1)\otimes_K \bigwedge^{n-1} V \otimes_K W \longrightarrow \cdots \\ \cdots \longrightarrow S\otimes_K W \longrightarrow W \longrightarrow 0 \end{aligned}$$

Since $e$ is the idempotent for the trivial representation $K$, any irreducible representation $W \ncong K$ gives rise to a nonzero projective $\overline{A}$-module $S \otimes_K W$.
By the theorem, part (1), tensoring the above $A$-projective resolution of $W$ with $-\otimes_A \overline{A}$ yields an $\overline{A}$-projective resolution of $W$ (cf. Thm. 1.6. and Ex. 1 of [APT92]). From this follows that $\overline{A}$ is Koszul and $\mathrm{gldim}\,\overline{A} \leq n$.
For the equality we show that $\mathrm{Ext}^n_{\overline{A}}(W, \det V \otimes W) \neq 0$. Since $G \neq \mu_2$ in statement (2), there exists $W \neq K$ and $W \neq \det(V)^{-1}$ such that $W$ and $\det V \otimes W$ yield nonzero $\overline{A}$-modules. For any $A$- modules $M, N$ it holds that $\mathrm{Ex}^i_A(M,N) = \mathrm{Ext}^i_S(M,N)^G$. Moreover, by the theorem, one has $\mathrm{Ext}^i_{\overline{A}}(M,N) \xrightarrow{\cong} \mathrm{Ext}^i_A(M,N)$ for all $M, N \in \mathbf{Mod}(\overline{A})$. The projection $S\otimes_K \det V \otimes_K W \to \det V \otimes_K W$ represents a nonzero element of $\mathrm{Ext}^n_S(W, \det V \otimes_K W)$ that is $G$-invariant. Thus $\mathrm{Ext}^n_S(W, \det V \otimes_K W)^G \neq 0$. ☐

**Remark 3.14.** If $G = \mu_2$, then since $G$ is generated by a reflection we can choose a basis of $V$ so that $G$ is generated by $\mathrm{diag}(-1,1,\ldots,1)$. Let $C = K[x_2,\ldots,x_n]$. Then an explicit calculation shows that the global dimension of $\overline{A}$ drops indeed: one may realize $A = (K[x_1,\ldots,x_n] * \mu_2)$ as the order

$$\begin{pmatrix} K[x_1^2] & K[x_1^2] \\ x_1^2K[x_1^2] & K[x_1^2] \end{pmatrix} \otimes C.$$

In this description, $e$ is the idempotent matrix $e_{11}$ and $AeA$ is of the form

$$\begin{pmatrix} K[x_1^2] & K[x_1^2] \\ x_1^2K[x_1^2] & x_1^2K[x_1^2] \end{pmatrix} \otimes C.$$

Thus $\overline{A} \cong C$, of global dimension $n-1 < n = \mathrm{gldim}\,A$. Note here that $R/(\Delta) \cong C$ is regular and moreover $\overline{A} \cong R/(\Delta)$.

Until the end of this section we assume the hypotheses of theorem 3.12 that $G \leqslant \mathrm{GL}(V)$ is generated by pseudo–reflections and that $|G|$ is invertible in $K$.

**3.15.** Our next goal is to determine the annihilator of $\overline{A}$ as $S \otimes_R S$–module, equivalently, in view of the preceding Theorem 3.12 and Lemma 3.11, the conductor ideal (of the normalization) of $S \otimes_R S$.

To this end consider the elements $f_i(\mathbf{x}'') - f_i(\mathbf{x}')$ in $S \otimes_K S$ and write

$$(*) \qquad f_i(\mathbf{x}'') - f_i(\mathbf{x}') = \sum_{j=1}^{n} \nabla_i^j(\mathbf{x}', \mathbf{x}'')(x_j'' - x_j')$$

for suitable elements $\nabla_i^j(\mathbf{x}', \mathbf{x}'') \in S \otimes_K S$. Note that $\nabla_i^j(\mathbf{x}, \mathbf{x}) = \partial f_i / \partial x_j \in S$ and recall that

$$J = \det(\partial f_i / \partial x_j) \in S\,,$$

is the Jacobian of the basic invariants $f_i$.

**Lemma 3.16.** *For $g \in G$, one has $\varphi(\det(\nabla_i^j(\mathbf{x}', \mathbf{x}''))) = J\delta_1 \in \mathsf{Maps}(G, S)$.*

*Proof.* By definition of $\varphi$, and because it is a ring homomorphism, one has

$$\varphi(\det(\nabla_i^j(\mathbf{x}', \mathbf{x}'')))(g) = \det(\nabla_i^j(\mathbf{x}, g\mathbf{x})) \in S\,.$$

For $g = 1 \in G$, this evaluates to $J$. For $g \neq 1$, as the $f_i$ are $G$–invariant, one has in $S \otimes_K S$

$$\begin{aligned} f_i(\mathbf{x}'') - f_i(\mathbf{x}') &= f_i(g(\mathbf{x}'')) - f_i(\mathbf{x}') \\ &= \sum_{j=1}^{n} \nabla_i^j(\mathbf{x}', g(\mathbf{x}''))(g(x_j'') - x_j')\,, \end{aligned}$$

and specializing $x'', x' \mapsto x$, this becomes $0 = \sum_{j=1}^{n} \nabla_i^j(\mathbf{x}, g(\mathbf{x}))(g(x_j) - x_j)$ in $S$, whence the linear system $\left(\nabla_i^j(\mathbf{x}, g(\mathbf{x}))\right)(v_1, \ldots, v_n)^T = \mathbf{0}$ has the nontrivial solution $(v_j)_{j=1,\ldots,n} = ((g(x_j) - x_j))_{j=1,\ldots,n} \neq \mathbf{0}$ over the domain $S$. This forces the determinant $\det(\nabla_i^j(\mathbf{x}, g\mathbf{x}))$ to vanish. ☐

**Corollary 3.17.** *In $A$, one has the containment of ideals $A(J\delta_1)A \subseteq AeA$. In particular, $\overline{A}$ is annihilated by $J$ both as left or right $S$–module.*

*Proof.* By Lemma 3.16, $\beta\varphi\left(\det\left(\nabla_i^j(\mathbf{x}', \mathbf{x}'')\right)\right) = J\delta_1$ in $A$ and $J\delta_1 = \alpha\left(\det\left(\nabla_i^j(\mathbf{x}', \mathbf{x}'')\right)\right) \in AeA$. ☐

In fact we have the following precise description of the annihilator of $\overline{A}$ as $S \otimes_R S$–module.

**Proposition 3.18.** *The annihilator ideal of $\overline{A}$ in $S \otimes_R S$ is the conductor ideal $\mathfrak{c}$ of the normalization of $S \otimes_R S$ and*

$$\mathfrak{c} = \mathrm{ann}_{S \otimes_R S} \overline{A} = \left(\det(\nabla_i^j(g(\mathbf{x}'), \mathbf{x}'')), g \in G\right)\,.$$

*The image of this ideal under $\varphi$ is $\varphi(\mathfrak{c}) = \mathsf{Maps}(G, JS) = J\,\mathsf{Maps}(G, S) \subseteq \mathsf{Maps}(G, S)$, the principal ideal generated by the Jacobian $J\cdot 1$ in $\mathsf{Maps}(G, S)$.*

*Proof.* The first statement has been explained already. We next use that for $C = S \otimes_R S$ and $\widetilde{C} = \mathsf{Maps}(G, S)$ the $C$–module $\overline{A}$ identifies with $\widetilde{C}/C$. To determine the conductor, it suffices to compute $\operatorname{Ext}^1_C(\overline{A}, C)$ by Lemma 3.11(2).

To this end observe first that $C = S \otimes_R S \cong S \otimes_K S/(\text{regular sequence})$, where the regular sequence is of length $n = \dim C$. This implies that naturally

$$\operatorname{Ext}^1_C(\overline{A}, C) \cong \operatorname{Ext}^{n+1}_{S\otimes_K S}(\overline{A}, S \otimes_K S)\,.$$

To determine the latter extension module, we make explicit the free $S \otimes_K S$–resolution of $\overline{A}$ as mapping cone of the $S \otimes_K S$–resolutions of $C = S \otimes_R S$ and of $\widetilde{C} = \bigoplus_{g\in G} S \otimes_R S/I_g$, respectively.

As $S \otimes_R S$ is the complete intersection $S \otimes_K S$ modulo the regular sequence $(f_i(\mathbf{x}'') - f_i(\mathbf{x}'))_i$, a free $S \otimes_K S$–resolution is given by the Koszul complex on that regular sequence,

$$\mathbb{K}_{\mathbf{f}} = \mathbb{K}((f_i(\mathbf{x}'') - f_i(\mathbf{x}'))_i, S \otimes_K S) \xrightarrow{\simeq} S \otimes_R S\,,$$

where $\simeq$ is to indicate that that complex is a resolution.

Now for $g \in G$ one has $I_g = (x''_i - g(x'_i); i = 1, ..., n) \subseteq S \otimes_R S$ by Lemma 3.4(1). Applying $g$ to the first tensor factor in equation $(*)$ shows

$$f_i(\mathbf{x}'') - f_i(g(\mathbf{x}')) = \sum_{j=1}^{n} \nabla^j_i(g(\mathbf{x}'), \mathbf{x}'')(x''_j - g(x'_j))\,.$$

As $f_i$ is $G$–invariant, $f_i(g(\mathbf{x}')) = f_i(\mathbf{x}')$ and so there is a containment

$$(f_i(\mathbf{x}'') - f_i(\mathbf{x}'); i = 1, ..., n) \subseteq (x''_j - g(x'_j); j = 1, ..., n) \subset S \otimes_K S$$

of ideals in $S \otimes_K S$. In particular, $S \otimes_R S/I_g \cong S \otimes_K S/(x''_j - g(x'_j); j = 1, ..., n)$ as $S \otimes_K S$–modules.

The sequence $(x''_j - g(x'_j))_j$ is regular in $S \otimes_K S$ as it consists of linearly independent linear forms. Thus, $S \otimes_R S/I_g$ is as well a complete intersection in $S \otimes_K S$ with free resolution the Koszul complex on that regular sequence,

$$\mathbb{K}_g = \mathbb{K}((x''_j - g(x'_j))_j, S \otimes_K S) \xrightarrow{\simeq} S \otimes_R S/I_g\,.$$

With $M_g = \left(\nabla^j_i(g(\mathbf{x}'), \mathbf{x}'')\right)$ the indicated $n \times n$ matrix over $S \otimes_K S$, its exterior powers provide for a lift of the evaluation homomorphism $ev_g\varphi : S \otimes_R S \to S \otimes_R S/I_g$ to a morphism between the resolutions. Putting all these facts together, we obtain

$$\begin{tikzcd}
\mathbb{K}_{\mathbf{f}} \arrow[r, "\simeq"] \arrow[d, "(\Lambda^\bullet M_g)_{g\in G}"'] & S \otimes_R S \arrow[d, "\varphi"] \\
\prod_{g\in G} \mathbb{K}_g \arrow[r, "\simeq"] & \mathsf{Maps}(G, S)\,.
\end{tikzcd}$$

With $\Phi = (\Lambda^\bullet M_g)_{g\in G}$ the indicated morphism between resolutions, the mapping cone on $\Phi$ yields a resolution of $\overline{A} \cong \mathsf{Maps}(G, S)/\operatorname{Im}(\varphi)$ as $S \otimes_K S$–module. This mapping cone is a complex of free $S \otimes_K S$–modules of length $n + 1$, whence we can calculate $\operatorname{Ext}^{n+1}_{S\otimes_K S}(\overline{A}, S \otimes_K S)$ simply as the cokernel of the last differential in the $S \otimes_K S$–dual of that mapping cone. The result is easily seen to be

$$\begin{aligned}\operatorname{Ext}^{n+1}_{S\otimes_K S}(\overline{A}, S\otimes_K S) &= S\otimes_R S/\left(\det M_g; g\in G\right)\\ &= S\otimes_R S/\left(\det\left(\nabla_i^j(g(\mathbf{x}'),\mathbf{x}'')\right); g\in G\right).\end{aligned}$$

Therefore, $\mathfrak{c} = \left(\det\left(\nabla_i^j(g(\mathbf{x}'),\mathbf{x}'')\right); g\in G\right) \subseteq S\otimes_R S$ as claimed.

By the same reasoning as in Lemma 3.16, it follows that $\varphi\left(\det\left(\nabla_i^j(g(\mathbf{x}'),\mathbf{x}'')\right)\right) = J\delta_g \in \mathsf{Maps}(G,S)$. ☐

**Corollary 3.19.** *If $G$ is generated by (pseudo–)reflections of order 2, then $J$ is a squarefree product of linear forms and so $S/\mathfrak{c}\subseteq (S/(J))^{|G|}$ is reduced, $V(\mathfrak{c}) = \operatorname{Sing}(S\otimes_R S)\subseteq \operatorname{Spec}(S\otimes_R S)$.* ☐

**Corollary 3.20.** *For $G$ generated by pseudo–reflections, consider the map $\psi : \mathsf{Maps}(G,S)\to S\otimes_R S$ given by*

$$\psi\Big(\sum_{g\in G} s_g\delta_g\Big) = \sum_{g\in G} s_g \det\left(\nabla_i^j(g(\mathbf{x}'),\mathbf{x}'')\right).$$

*This is $S$–linear on the left and $\varphi\psi = J\,\mathrm{id}_{\mathsf{Maps}(G,S)}$.*

*As both $\mathsf{Maps}(G,S)$ and $S\otimes_R S$ are free (left) $S$–modules, the pair $(\varphi,\psi)$ constitutes a matrix factorization of $J\in S$ whose cokernel is $\overline{A}$ as left $S$–module. In particular, $\overline{A}$ is a maximal Cohen–Macaulay module over the hypersurface ring $S/(J)$.* ☐

**The map $\varphi$ and the group matrix.** Moreover, Lemma 3.11 has the following interesting consequence: In the matrix factorization $(\varphi,\psi)$ of $J$ with coker $\varphi = \overline{A}$, the morphism $\psi$ is the transpose of $\varphi$ up to base change. Indeed, the $S/(J)$–dual of $\overline{A}$, say, with respect to the left $S$–module structure, is the first syzygy module of $\overline{A}$ as $S/(J)$–module.

The equation $\det(\varphi\psi) = J^{|G|}$ that follows from $\varphi\psi = J\,\mathrm{id}$ then entails that

$$J^{|G|} = \det(\varphi)\det(\psi) = \det(\varphi)^2,$$

thus $\det(\varphi) = J^{|G|/2}$. Now $J = \prod_{i=1}^{m_1} L_i^{r_i-1}$, where the $L_i$ are the linear forms defining the mirrors of $G$ and $r_i$ is the order of the cyclic group that leaves the mirror invariant. The hyperplanes $\{L_i = 0\}$ are the irreducible components of the hyperplane arrangement and on such a component the rank of $\overline{A}$ is accordingly $(r_i-1)|G|/2$. Note that this is an integer, as $|G|$ odd implies that each $r_i$ is odd too.

Next note that $\Delta = zJ = \prod_i L_i^{r_i}$. Grouping the hyperplanes into orbits under the action of $G$, we get $\Delta = \prod_{j=1}^q \Delta_j$, where $\Delta_j = \prod_{L_k\in O_j} L_k^{r_k}$ with $O_j$ an orbit and $q$ the number of such orbits. These $\Delta_j$ are the irreducible factors of $\Delta$ in $R$. Note that the exponents $r_k$ are the same for each linear form in an orbit. We abuse notation and denote this common value for $O_j$ also by $r_j$, giving $\Delta_j = (\prod_{L_k\in O_j} L_k)^{r_j}$. Since the stabilizer of a hyperplane in $O_j$ has order $r_j$ we have that $|O_j|\cdot r_j = |G|$. Hence we obtain the following result.

**Corollary 3.21.** *The rank of $\overline{A}$ along the component $\Delta_j$ of the discriminant is $(r_j-1)|G||O_j|/2 = \frac{(r_j-1)|G|^2}{2r_j} = \binom{r_j}{2}\frac{|G|^2}{r_j^2}$.*

We give an alternate proof of this fact that additionally describes the codimension one structure of $\overline{A}$ in Corollary 5.10.

## 4. NONCOMMUTATIVE RESOLUTIONS OF DISCRIMINANTS

**Matrix factorizations as quiver representations, Knörrer's functors.** In order to compare modules over the discriminant and the skew group ring, we will interpret Knörrer's functors from [Knö84] and [Knö87] more generally for path algebras via recollements. This yields a reformulation of Eisenbud's theorem [Eis80] and a generalization of Knörrer's result ([Knö87, Prop. 2.1]) in Theorem 4.3 and Remark 4.6.

*Modules over path algebras:* Let $R$ be a commutative regular ring, $f \in R$ and let

$$B = R\left(\begin{tikzcd} e_+ \arrow[r, bend left, "v"] \arrow[r, dashed, leftrightarrow, "f"] & e_- \arrow[l, bend left, "u"] \end{tikzcd}\right) . \tag{7}$$

This stands for the associative $R$-algebra generated by $e_+, e_-, u, v$, modulo the relations

$$e_+^2 = e_+ \,, \quad e_-^2 = e_- \,, \quad e_- + e_+ = 1 \,,$$
$$u = e_+ u e_- \,, \quad v = e_- v e_+ \,,$$
$$uv = f e_+ \,, \quad vu = f e_- \,.$$

Note that $B$ is free as a $R$-module with basis the four elements $e_-, e_+, u, v$.
A right $B$-module $M$ corresponds to a quiver representation of the form

$$\begin{tikzcd} M_+ \arrow[r, bend left, "v_M"] \arrow[r, dashed, leftrightarrow, "f"] & M_- \arrow[l, bend left, "u_M"] \end{tikzcd} ,$$

where $M_+ = Me_+$ and $M_- = Me_-$ are $R$-modules and $u_M$ and $v_M$ are $R$-linear and must satisfy $u_M v_M = f\mathrm{Id}_{M_+}$ and $v_M u_M = f\mathrm{Id}_{M_-}$. Note here that $M$ is isomorphic to $M_+ \oplus M_-$ as $R$-modules via restriction of scalars. In the following we use the shorthand notation

$$M := (M_+ \underset{u_M}{\overset{v_M}{\rightleftarrows}} M_-) \,.$$

A morphism between $B$-modules $M = (M_+ \rightleftarrows^{v_M}_{u_M} M_-)$ and $M' = (M'_+ \rightleftarrows^{v'_M}_{u'_M} M'_-)$ corresponds to a pair $(\alpha_-, \alpha_+)$ of $R$-module homomorphisms such that the diagram

$$\begin{tikzcd} M_+ \arrow[r, "v_M"] \arrow[d, "\alpha_+"] & M_- \arrow[r, "u_M"] \arrow[d, "\alpha_-"] & M_+ \arrow[d, "\alpha_+"] \\ M'_+ \arrow[r, "v'_M"'] & M'_- \arrow[r, "u'_M"'] & M'_+ \end{tikzcd} \tag{8}$$

commutes.

Conversely, if we start with a quiver representation $(M_+ \rightleftarrows^{v_M}_{u_M} M_-)$, then $M := M_+ \oplus M_-$ is naturally a right $B$-module. If $M$ is finitely generated projective as a $R$-module, then the pair $(u_M, v_M)$ is called a *matrix factorization of $f$* over $R$ and the $B$-module $(M_+ \rightleftarrows^{v_M}_{u_M} M_-)$ is called a *(maximal) Cohen–Macaulay module* over $B$. The category of such modules is denoted $\mathbf{CM}(B)$.

**Lemma 4.1.** *Let $B$ be an algebra of the form (7). Then:*

(a) $B = e_+B \oplus e_-B$ *is the sum of two projective $B$-modules.*
(b) $B/Be_+B \cong B/Be_-B \cong R/(f)$. *In particular, there are natural algebra surjections* $B \twoheadrightarrow R/(f)$.
(c) $e_+Be_+ \cong e_-Be_- \cong R$ .
(d) *The centre of $B$ is $R$.*

*Proof.* The first assertion is clear. We show the other claims for $e_+$, for $e_-$ they follow similarly. For (b) first compute $e_+B = e_+(e_+R \oplus uR \oplus vR \oplus e_-R) = e_+R \oplus uR$ and from this

$$Be_+B = B(e_+R \oplus uR) = e_+R \oplus uR \oplus vR \oplus vuR\,.$$

Now using $vu = e_-f$ it follows that

$$B/Be_+B \cong e_-R/e_-fR \cong e_-(R/(f)) \cong R/(f)\,.$$

For (c) multiply $e_+B = e_+R \oplus uR$ with $e_+$ from the right and get $e_+R \cong R$. For (d) note first that $R \subseteq Z(B)$, as $B$ is a $R$-algebra. Take $x \in Z(B)$, then from $e_+x = xe_+$ and $ux = xu$ it follows that $x = r(e_+ + e_-)$ for some $r \in R$. Thus $Z(B) = R$. ☐

In order to relate modules over $B$ and over $R/(f)$ we look at the standard recollement induced by $e_-$. It is given by

$$\begin{tikzcd}[column sep=huge]
\mathbf{Mod}\, B/Be_-B \arrow[r, "i_*"] & \mathbf{Mod}\, B \arrow[l, bend right=50, "i^* = -\otimes_B B/Be_-B"'] \arrow[l, bend left=50, "i^! = \mathrm{Hom}_B(B/Be_-B, -)"] \arrow[r, "j^* = \mathrm{Hom}_B(e_-B, -)"] & \mathbf{Mod}\, e_-Be_- \arrow[l, bend right=50, "j_! = -\otimes_{e_-Be_-} e_-B"'] \arrow[l, bend left=50, "j_* = \mathrm{Hom}_{e_-Be_-}(Be_-, -)"]
\end{tikzcd} ,$$

where

- $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples,
- the functors $i_*$, $j_*$, and $j_!$ are fully faithful,
- $\mathrm{Im}(i_*) = \ker(j^*)$.

From this it follows that there exist exact sequences

$$\begin{array}{rl} & j_!j^* \to \mathrm{id}_{\mathbf{Mod}\, B} \to i_*i^* \to 0, \\ 0 \to & i_*i^! \to \mathrm{id}_{\mathbf{Mod}\, B} \to j_*j^* \quad , \end{array} \tag{9}$$

see e.g. [FP04, Prop. 4.2]. Note that with Lemma 4.1, one can write $i^* = -\otimes_B R/(f)$, $i^! = \mathrm{Hom}_B(R/(f), -)$. Moreover, $j^* = \mathrm{Hom}_B(e_-B, -) \cong -\otimes_B Be_-$. One easily verifies the following statements:

**Lemma 4.2.** *Let $M$ be a $B$-module, $C$ be a $R/(f) = B/Be_-B$-module and $N$ be a $R = e_-Be_-$-module. The functors in the standard recollement are determined by*

- $i_*C = (0 \rightleftarrows C)$,
- $i^!(M) = \ker v_M$ ,
- $i^*(M) = \mathrm{coker}\, u_M$ ,
- $j^*(M) = M_- = e_-M$ ,
- $j_!(N) = N \otimes_R e_-B = N \otimes_R (R \overset{f}{\underset{\mathrm{id}}{\rightleftarrows}} R) = (N \overset{f}{\underset{\mathrm{id}}{\rightleftarrows}} N)$ ,

- $j_*(N) = \mathrm{Hom}_R(Be_-, N) \cong (N \underset{f}{\overset{\mathrm{id}}{\rightleftarrows}} N)\,.$ $\square$

Recall the definition of an Iwanaga–Gorenstein ring:

**Definition 4.1.** An associative ring $\Lambda$ is called *Iwanaga–Gorenstein* if it is noetherian on both sides and the injective dimension of $\Lambda$ as a left and right $\Lambda$-module is finite.

**Theorem 4.3.** *Let $R$ be a commutative regular ring with $f \in R$, $f \neq 0$ and $B$ as before,*

$$B = R\left(\begin{tikzcd} e_+ \arrow[rr, bend left, "v"] \arrow[rr, dashed, leftrightarrow, "f"] && e_- \arrow[ll, bend left, "u"] \end{tikzcd}\right).$$

*The ring $B$ is Iwanaga–Gorenstein and $M = (M_+ \underset{u_M}{\overset{v_M}{\rightleftarrows}} M_-) \in \mathbf{CM}(B)$ if and only if $i^*M$ is in $\mathbf{CM}(R/(f))$, where $i^*$ is coming from the recollement as described above. The functor $i^*$ induces an equivalence of categories*

$$\mathbf{CM}(B)/\langle e_-B\rangle \simeq \mathbf{CM}(R/(f))\,,$$

*where $e_-B$ is the ideal in the category $\mathbf{CM}(B)$ generated by the object $e_-B$.*

*Proof.* To show that $B$ is Iwanaga–Gorenstein, first note that $B$ is noetherian, since it is finitely generated over $R$. Set $M^* = \mathrm{Hom}_R(M, R)$ for a $B$-bimodule $M$. Since $R$ is Gorenstein, it has a finite injective $R$-module resolution

$$0 \to R \to I_0 \to \cdots \to I_m \to 0.$$

Then apply $\mathrm{Hom}_R(B^*, -)$ to get a complex of $B$-modules

$$0 \to B^{**} \to J_0 \to \cdots \to J_m \to 0.$$

Because $B$ is finite free over $R$, we have $B^{**} \cong B$ as a $B$-module and the $J_i$ are injective $B$-modules. Moreover, the sequence stays exact, thus represents an injective $B$-module resolution of $B$. The argument works for either the left or right module structure on $B$.

Now assume that $M = (M_+ \underset{u_M}{\overset{v_M}{\rightleftarrows}} M_-)$ is in $\mathbf{CM}(B)$. Recall that this means that $M_+$ and $M_-$ are projective over $R$. Set $C := \mathrm{coker}(u_M) = i^*(M)$. Because $f$ is a non-zero-divisor in $R$, multiplication by $f$ on $M_-$ is injective and so is $u_M$ as $v_M u_M = f\,\mathrm{id}_{M_-}$. Therefore

$$0 \to M_- \xrightarrow{u_M} M_+ \to C \to 0 \tag{10}$$

is a projective resolution of $C$ over $R$. This implies $C = i^*(M) \in \mathbf{CM}(R/(f))$ by the Auslander–Buchsbaum formula, and so $i^*$ defines a functor $\mathbf{CM}(B)$ to $\mathbf{CM}(R/(f))$.

Conversely, take any Cohen–Macaulay module $C$ over $R/(f)$ and let (10) be a projective resolution of $C$ over $R$ with $M_+$ and $M_-$ free $R$-modules. One can find $u_M$ and $v_M$ such that

$$\begin{tikzcd} 0 \arrow[r] & M_- \arrow[r, "\mathrm{id}"] \arrow[d, bend left, "\mathrm{id}"] & M_- \arrow[r] \arrow[d, bend left, "u_M"] & 0 \arrow[r] \arrow[d, bend left, "0"] & 0 \\ 0 \arrow[r] & M_- \arrow[r, "u_M"] \arrow[u, bend left, "f"] & M_+ \arrow[r] \arrow[u, bend left, "v_M"] & C \arrow[r] \arrow[u, bend left, "0"] & 0 \end{tikzcd} \tag{11}$$

is a short exact sequence of $B$-modules, that is, $u_M v_M = v_M u_M = f$. The leftmost column of this diagram corresponds to the $B$-module $(M_- \underset{\mathrm{id}}{\overset{f}{\rightleftarrows}} M_-) = j_! j^* M$, which is isomorphic to a direct summand of $(e_-B)^m$ for some $m \geq 0$, since $M_-$ is projective over $R$. In particular, that $B$-module is projective and the $B$-module $M = (M_+ \overset{v}{\underset{u}{\rightleftarrows}} M_-)$ is in $\mathbf{CM}(B)$ and $\operatorname{coker}(u) = i^*(M) = C$. This shows that $i^*$ is a dense functor from $\mathbf{CM}(B)$ to $\mathbf{CM}(R/(f))$. Note that we just established that there is a short exact sequence of functors

$$0 \to j_!j^* \to \mathrm{id}_{\mathbf{CM}(B)} \to i_*i^* \to 0 \tag{12}$$

from $\mathbf{CM}(B)$ to $\mathbf{mod}(B)$, where $\mathbf{mod}(B)$ stands for the category of finitely generated $B$-modules. Here $\mathrm{id}_{\mathbf{CM}(B)} \to i_*i^*$ is the restriction of the unit of the adjunction $(i^*, i_*)$ to $\mathbf{CM}(B)$. Further, $i^*(e_-B) = 0$, whence $i^*$ factors through the quotient $\mathbf{CM}(B)/\langle e_-B\rangle$. From the exact sequence (12) one easily sees that the functor $\mathbf{CM}(B)/\langle e_-B\rangle \to \mathbf{CM}(R/(f))$ induced by $i^*$ is fully faithful. For fullness, note that any morphism $\alpha : i_*i^*M \to i_*i^*M'$ in $\mathbf{CM}(R/(f))$ lifts to a morphism in $\mathbf{CM}(B)$ through the projective resolutions of $i_*i^*M$ and $i_*i^*M'$, and for faithfulness note that if $\alpha = 0$, then any morphism representing $\alpha$ in $\mathbf{CM}(B)$ factors through $e_-B$. ☐

**Example 4.4.** The matrix factorizations to the two indecomposable projective $B$-modules $e_-B$ and $e_+B$ are $(R \overset{f}{\underset{\mathrm{id}}{\rightleftarrows}} R)$ and $(R \overset{\mathrm{id}}{\underset{f}{\rightleftarrows}} R)$, respectively. In particular, one sees that $i^*(e_-B) = 0$ and $i^*(e_+B) = R/(f)$.

Interpreting Theorem 4.3 in terms of matrix factorizations, note that $\mathbf{CM}(B) \simeq MF(f)$, the category of matrix factorizations of $f$. Let $\mathcal{I}$ be the ideal in the category $MF(f)$ generated by the matrix factorization $R \overset{f}{\underset{\mathrm{id}}{\rightleftarrows}} R$. If $f$ is a nonzerodivisor in $R$ then by the above result, the functor $\operatorname{coker}(u_-) : MF(f) \longrightarrow \mathbf{CM}(R/(f))$ induces an equivalence of categories

$$MF(f)/\mathcal{I} \simeq \mathbf{CM}(R/(f)), \tag{13}$$

which is a reformulation of [Eis80, Section 6].

Let $T := R[z]/(z^2 - f)$, so that $\operatorname{Spec}(T)$ is the double cover of $\operatorname{Spec}(R)$ ramified over $V(f) = \{f = 0\}$. The canonical $R$-involution on $T$ that sends $z$ to $-z$ defines a group action of $\mu_2 = \langle \sigma \mid \sigma^2 = 1\rangle$ on $T$. Let $B' = T * \mu_2$ be the corresponding twisted group algebra.

**Proposition 4.5.** *With notation as just introduced*

(1) $B' \cong R\langle z, \delta_\sigma\rangle / \langle z^2 - f, \delta_\sigma z + z\delta_\sigma = 0, \delta_\sigma^2 = 1\rangle$.
(2) *Sending* $\delta_\sigma \mapsto e_+ - e_-$ *and* $z \mapsto u + v$ *defines an injective* $R$*-algebra homomorphism* $B'$ *to* $B$.
(3) *If* 2 *is a unit in* $R$*, then this* $R$*-algebra homomorphism is bijective, with the inverse sending* $e_\pm \mapsto \frac{1}{2}(1 \pm \delta_\sigma)$*, while* $u \mapsto \frac{1}{2}(1 + \delta_\sigma)z$ *and* $v \mapsto \frac{1}{2}(1 - \delta_\sigma)z$.

*Proof.* All statements follow from the fact that $B'$ is a free $R$-module with basis $1, z, \delta_\sigma, z\delta_\sigma$ and $B$ is a free $R$-module with basis $e_+, e_-, u, v$ as we observed at the beginning of this subsection. ☐

**Remark 4.6.** Assuming that 2 is a unit in $R$ and expressing the recollement in terms of $B'$, one regains the functors in [Knö87]. In particular, Theorem 4.3 implies Knörrer's result

$$\mathbf{CM}(T * \mu_2) \simeq MF(f)$$

as established in [Knö87, Prop. 2.1].
Combining this with the equivalence (13) one has the equivalence of categories

$$\mathbf{CM}(T * \mu_2)/(e_{-}B) \simeq MF(f)/\mathcal{I} \simeq \mathbf{CM}(R/(f))\,.$$

**The skew group ring and Bilodeau's isomorphisms.** The results here were inspired by work of Bilodeau [Bil05]. In the following, $K$ is a commutative ring and $G$ a finite group such that the order $|G|$ of $G$ is invertible in $K$. Set $e_G = \frac{1}{|G|}\sum_{g\in G} g \in KG$, the idempotent in the group algebra that belongs to the trivial representation of $G$. Similarly, for a subgroup $H \leqslant G$ we set $e_H = \frac{1}{|H|}\sum_{h\in H} h \in KG$ and say that this idempotent element in $KG$ is defined by $H$.

If $\Gamma, H \leqslant G$ are complementary subgroups in that $\Gamma \cap H = \{1\}$, where $1 \in G$ is the identity element, and $H\Gamma = G$, then every element $g \in G$ can be written uniquely as $g = h\gamma$ with $h \in H, \gamma \in \Gamma$, and also uniquely as $g = \gamma' h'$ with $\gamma' \in \Gamma, h' \in H$.

One has $e_G = e_H e_\Gamma = e_\Gamma e_H$ in $KG$, and for $\Gamma$ normal in $G$ it holds that $g e_\Gamma = e_\Gamma g$ for all $g \in G$, thus, $e_\Gamma$ is then a central idempotent.

**Lemma 4.7.** *Let $M$ be a left $K\Gamma$–module. The $K$–submodule $M^\Gamma = \{m \in M \mid \gamma m = m \text{ for each } \gamma \in \Gamma\}$ equals $e_\Gamma M$.*

*Proof.* If $\gamma m = m$ for each $\gamma \in \Gamma$, then $\left(\sum_{\gamma\in\Gamma}\gamma\right) m = |\Gamma| m$, that is, $e_\Gamma m = m$, and so $M^\Gamma \subseteq e_\Gamma M$. On the other hand, $\gamma e_\Gamma = e_\Gamma$ for each $\gamma \in \Gamma$, thus, $e_\Gamma M \subseteq M^\Gamma$. ☐

**Corollary 4.8.** *If $\Gamma$ acts through $K$–algebra automorphisms on a $K$–algebra $S$, then $T := S^\Gamma = e_\Gamma S$ is a $K$–subalgebra of $S$.*

*Proof.* This is obvious from the description $T = S^\Gamma = \{s \in S \mid \gamma s = s \text{ for each } \gamma \in \Gamma\}$. ☐

**Lemma 4.9.** *With notation as before, let $\Gamma \leqslant G$ be a normal subgroup and set $T = S^\Gamma$ and $H = G/\Gamma$. The quotient group $H$ acts naturally on $T$ through $K$–algebra automorphisms and one can form $T * H$ accordingly. There are a natural isomorphism $S \otimes_K KH \cong Ae_\Gamma$ as right $T * H$–modules and a $K$–algebra isomorphism $T*H \cong e_\Gamma A e_\Gamma$, where $A = S * G$, as before.*

*Proof.* As stated just before Lemma 4.7, for $\Gamma$ normal in $G$, one has $g e_\Gamma = e_\Gamma g$ for each $g \in G$. Further, $\gamma e_\Gamma = e_\Gamma = e_\Gamma \gamma$, whence the element $g e_\Gamma = e_\Gamma g$ depends solely on the coset $g\Gamma$. In that way, $h e_\Gamma = e_\Gamma h$ is a well–defined element of $KG$ for any $h \in H$.

Accordingly, the map $S \otimes_K KH \to Ae_\Gamma$ that sends $s \otimes h \mapsto s(he_\Gamma) \in Ae_\Gamma$ is well defined. It is bijective as for $a = \sum_{g\in G} s_g \delta_g \in A$ one has

$$\begin{aligned} ae_\Gamma &= \sum_{g\in G} s_g \delta_g e_\Gamma \\ &= \sum_{g\Gamma\in H}\sum_{\gamma\in\Gamma} s_{g\gamma}\delta_g\delta_\gamma e_\Gamma \\ &= \sum_{h=g\Gamma\in H}\Big(\sum_{\gamma\in\Gamma} s_{g\gamma}\Big)(he_\Gamma) \end{aligned}$$

whence $ae_\Gamma \mapsto \sum_{h=g\Gamma\in H}(\sum_{\gamma\in\Gamma} s_{g\gamma}) \otimes h$ yields the inverse map. It also follows from this calculation that $\sum_{h'\in H} t_{h'}\delta_{h'} \in T * H$ acts from the right on $Ae_\Gamma$ by

$$\begin{aligned} ae_\Gamma \left(\sum_{h'\in H} t_{h'}\delta_{h'}\right) &= \left(\sum_{h=g\Gamma\in H} s_h h e_\Gamma\right)\left(\sum_{h'\in H} t_{h'}\delta_{h'}\right) \\ &= \sum_{h,h'\in H} s_h h(t_{h'})(hh'e_\Gamma) \\ &= \sum_{h''\in H}\left(\sum_{hh'=h''} s_h h(t_{h'})\right) h''e_\Gamma\,, \end{aligned}$$

where we have used that $e_\Gamma t = te_\Gamma$ and $e_\Gamma h = he_\Gamma$ for $t \in T$ and $h \in H$.

Transporting this structure to $S \otimes_K KH$ under the bijection onto $Ae_\Gamma$, we obtain that $(s \otimes h)\sum_{h'\in H} t_{h'}\delta_{h'} = \sum_{h'} sh(t_{h'}) \otimes hh'$ defines the right $T * H$–module structure on $S \otimes_K KH$ that makes the bijection above $T * H$–linear.

Furthermore, that bijection is $\Gamma$–equivariant with respect to the left $\Gamma$–actions $\gamma(s \otimes h) = \gamma(s) \otimes h$ and $\gamma(ae_\Gamma) = \delta_\gamma ae_\Gamma \in Ae_\Gamma \subseteq A$. Taking $\Gamma$–invariants returns the isomorphism of right $T * H$–modules

$$(S \otimes H)^\Gamma \cong S^\Gamma \otimes H = T \otimes H$$

and

$$(Ae_\Gamma)^\Gamma = e_\Gamma A e_\Gamma\,,$$

whence

$$T \otimes H \cong e_\Gamma A e_\Gamma\,.$$

Transporting the algebra structure on the right to the left via this isomorphism, one obtains $T * H \cong e_\Gamma A e_\Gamma$ as $K$–algebras. □

**Remark 4.10.** If $\Gamma \leqslant G$ admits a complement, necessarily isomorphic to $H$, then the natural $K$–algebra homomorphism $T * H \to S * H$ induces the $T * H$–module structure on $S * H \cong S \otimes_K KH$.

Now we come to the key result.

**Proposition 4.11.** *Let $\Gamma, H \leqslant G$ be complementary subgroups with $\Gamma$ normal in $G$. With $G$ acting through $K$–algebra automorphisms on some $K$–algebra $S$ and with $T = S^\Gamma$, the group $H$ acts naturally on $\mathrm{End}_T(S)$ through algebra automorphisms and there is an isomorphism of $K$–algebras $\Phi\colon \mathrm{End}_T(S) * H \xrightarrow{\cong} \mathrm{End}_{T*H}(S * H)$, where $S * H$ is considered a right $T * H$–module.*

*Proof.* If $h \in H$ and $\alpha \in \mathrm{End}_T(S)$, then $(h\alpha)(s) = h(\alpha(h^{-1}(s)))$ defines the action of $H$ on $\mathrm{End}_T(S)$ through algebra automorphisms. Namely, $h\alpha$ is $T$–linear because

$$\begin{aligned} (h\alpha)(st) &= h(\alpha(h^{-1}(st))) \\ &= h(\alpha(h^{-1}(s)h^{-1}(t))) \end{aligned}$$

as $H$ acts through algebra automorphisms on $S$,

$$= h(\alpha(h^{-1}(s))h^{-1}(t))$$

as $\alpha$ is $T$–linear and $h^{-1}(t) \in T$,

$$\begin{aligned} &= h(\alpha(h^{-1}(s)))h(h^{-1}(t)) \\ &= (h\alpha)(s)t\,. \end{aligned}$$

That $H$ acts through algebra automorphisms on $\operatorname{End}_T(S)$ follows from

$$\begin{aligned} (h(\alpha\beta))(s) &= h(\alpha\beta(h^{-1}(s))) \\ &= h(\alpha(h^{-1}(h\beta h^{-1}(s)) \\ &= (h\alpha)((h\beta)(s))\,. \end{aligned}$$

Accordingly one can form the twisted group algebra $\operatorname{End}_T(S) * H$ as in Definition 2.1. The map $\Phi$ sends $\alpha = \sum_{h\in H} \alpha_h\delta_h$, with $\alpha_h \in \operatorname{End}_T(S)$, to the map $\Phi(\alpha)\colon S * H \to S * H$ defined by

$$\begin{aligned} \Phi(\alpha)\left(\sum_{h'\in H} s_{h'}\delta_{h'}\right) &= \left(\sum_{h\in H} \alpha_h\delta_h\right)\left(\sum_{h'\in H} s_{h'}\delta_{h'}\right) \\ &= \sum_{h,h'\in H} \alpha_h(h(s_{h'}))\delta_h\delta_{h'} \\ &= \sum_{h''\in H}\left(\sum_{hh'=h''} \alpha_h(h(s_{h'}))\right)\delta_{h''}\,. \end{aligned}$$

To show that $\Phi$ is a homomorphism of $K$–algebras, with $\beta = \sum_{h'\in H} \beta_{h'}\delta_{h'} \in \operatorname{End}_T(S) * H$ one finds first

$$\alpha\beta = \sum_{h,h'\in H} \alpha_h h(\beta_{h'})\delta_{hh'}$$

and then

$$\begin{aligned} \Phi(\alpha\beta)\left(\sum_{h''\in H} s_{h''}\delta_{h''}\right) &= \Phi\left(\sum_{h,h'\in H} \alpha_h h(\beta_{h'})\delta_{hh'}\right)\left(\sum_{h''\in H} s_{h''}\delta_{h''}\right) \\ &= \sum_{h,h',h''\in H} (\alpha_h h(\beta_{h'}))((hh')(s_{h''}))\delta_{hh'h''}\,, \end{aligned}$$

whereas

$$\begin{aligned} \Phi(\alpha)\Phi(\beta)\left(\sum_{h''\in H} s_{h''}\delta_{h''}\right) &= \Phi(\alpha)\left(\sum_{h',h''\in H} \beta_{h'}(h'(s_{h''}))\delta_{h'h''}\right) \\ &= \sum_{h,h',h''\in H} \alpha_h(h(\beta_{h'}(h'(s_{h''}))))\delta_{hh'h''} \\ &= \sum_{h,h',h''\in H} \alpha_h(h(\beta_{h'})(h(h'(s_{h''}))))\delta_{hh'h''} \\ &= \sum_{h,h',h''\in H} (\alpha_h h(\beta_{h'})((hh')(s_{h''}))\delta_{hh'h''}\,. \end{aligned}$$

Thus, $\Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta)$ as claimed.

To check that $\Phi(\alpha)$ constitutes an $T{*}H$–linear endomorphism of $S * H$ it suffices to note that there is a commutative diagram of homomorphisms of $K$–algebras

$$\begin{tikzcd}
\operatorname{End}_T(S) * H \arrow[rr, "\Phi"] & & \operatorname{End}_{T*H}(S * H) \\
& T * H \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

where $\varphi$ is induced by the $K$–algebra homomorphism $T \to \operatorname{End}_T(S)$ that sends $t \in T$ to $\lambda_t$, the (left) multiplication by $t$ on $S$, and $\psi$ represents left multiplication by $T * H$ on $S * H$. Indeed,

$$\begin{aligned}
\Phi\varphi(t\delta_h)\left(\sum_{h' \in H} s_{h'}\delta_{h'}\right) &= \Phi(\lambda_t\delta_h)\left(\sum_{h' \in H} s_{h'}\delta_{h'}\right) \\
&= \sum_{h' \in H} th(s_{h'})\delta_{hh'} \\
&= (t\delta_h)\left(\sum_{h' \in H} s_{h'}\delta_{h'}\right) \\
&= \psi(t\delta)\left(\sum_{h' \in H} s_{h'}\delta_{h'}\right) .
\end{aligned}$$

Finally, we show that $\Phi$ is an isomorphism by exhibiting the inverse. Let $f : S * H \to S * H$ be a right $T * H$–linear map. Then

$$f\left(\sum_{h \in H} s_h\delta_h\right) = \sum_{h \in H} f(s_h\delta_1)\delta_h$$

as $f$ is $T * H$–linear. Therefore, $f$ is uniquely determined by $f(s_h\delta_1) = \sum_{h \in H} f_h(s)\delta_h$, where in turn $f_h(s) \in S$ is uniquely determined as the $\delta_h$ form a basis of the (right) $S$–module $S * H$. Now $f$ is $T$–linear on the right, whence necessarily for any $s \in S, t \in T$ the expression

$$f(st\delta_1) = \sum_{h \in H} f_h(st)\delta_h$$

equals

$$\begin{aligned}
f(s\delta_1)t &= \left(\sum_{h \in H} f_h(s)\delta_h\right) t \\
&= \sum_{h \in H} f_h(s)h(t)\delta_h
\end{aligned}$$

Comparing coefficients of $\delta_h$ it follows that $f_h(st) = f_h(s)h(t)$ for each $h \in H$. This implies that the map $\alpha_h(s) = f_h(h^{-1}(s))$ is in $\operatorname{End}_T(S)$ and $\Psi(f) = \sum_{h \in H} \alpha_h\delta_h$ yields the inverse of $\Phi$. Indeed,

$$\begin{aligned}
\Phi\Psi(f)\left(\sum_{h'\in H} s_{h'}\delta_{h'}\right) &= \Phi(\sum_{h\in H}\alpha_h\delta_h)(\sum_{h'\in H} s_{h'}\delta_{h'})\\
&= \sum_{h''\in H}\left(\sum_{hh'=h''}\alpha_h(h(s_{h'}))\right)\delta_{h''}\\
&= \sum_{h''\in H}\left(\sum_{hh'=h''}(f_h(h^{-1}(h(s_{h'}))))\right)\delta_{h''}\\
&= \sum_{h,h'\in H} f_h(s_{h'})\delta_h\delta_{h'}\\
&= f\left(\sum_{h'\in H} s_{h'}\delta_{h'}\right)\,.
\end{aligned}$$

One checks analogously that $\Psi\Phi(\alpha) = \alpha$ for any $\alpha \in \mathrm{End}_T(S) * H$. □

To sum up, let us interpret the preceding result in terms of $A = S * G$:

**Proposition 4.12.** *As noted before, we have*

- *$T{*}H \cong e_\Gamma A e_\Gamma$, an isomorphism of $K$–algebras.*
- *$S{*}H \cong Ae_\Gamma$, an isomorphism of right $e_\Gamma A e_\Gamma$–modules.*

*Furthermore, left multiplication by elements of $A$ defines a $K$–algebra homomorphism*

$$A \to \mathrm{End}_{e_\Gamma A e_\Gamma}(Ae_\Gamma) \cong \mathrm{End}_{T*H}(S*H) \xrightarrow[\cong]{\Psi} \mathrm{End}_T(S) * H\,.$$

*Moreover, as $\Gamma, H$ are complementary subgroups in $G$, one has $A = S * G \cong (S*\Gamma)*H$ as $K$–algebras and, as any skew group ring is isomorphic to its opposite, the sequence of ring homomorphisms becomes*

$$(S*\Gamma)*H \cong A \to \mathrm{End}_{e_\Gamma A e_\Gamma}(Ae_\Gamma) \cong \mathrm{End}_{T*H}(S*H) \xrightarrow[\cong]{\Psi} \mathrm{End}_T(S) * H\,.$$

*Now there is always the natural $K$–algebra homomorphism $f : S*\Gamma \to \mathrm{End}_T(S)$ and the composition of this sequence of $K$–algebra homomorphisms is just $f * H$.*

**Remark 4.13.** By Auslander's theorem, if $G \leqslant \mathrm{GL}(V)$ and $S = K[V]$, or $S = K[[V]]$, is the polynomial ring, respectively the power series ring on the finite dimensional vector space $V$ over $K$, then $f$, and as a consequence also $f * H$, are isomorphisms if $\Gamma$ contains no pseudo–reflections in its linear action on $S$. Thus the above result generalizes Auslander's theorem to the case where $G \leqslant \mathrm{GL}(V)$ is a pseudo-reflection group. Here $\Gamma = G \cap \mathrm{SL}(V)$ is small and $H$ is the quotient $G/\Gamma$ in the exact sequence $1 \to \Gamma \to G \to G/\Gamma \to 1$.

### Intermezzo: Specializing to reflection groups.

*Invariant ring $T = S^\Gamma$ in terms of $R = S^G$.* In the following let $V$ be a finite dimensional vector space over $K$ and $G \leqslant \mathrm{GL}(V)$ be a true reflection group. Set $\Gamma := G \cap SL(V)$ and $H := \det G \cong \mu_2(K) = \langle\sigma\rangle$. This means that we have an exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{\det|_G} H \longrightarrow 1.$$

This sequence splits (by definition $G$ is generated by pseudo-reflections). Let $S = K[x_1, \ldots, x_n]$, $T = S^\Gamma$, $R = S^G = K[f_1, \ldots, f_n] \subseteq S$ and $J = \det\left( (\frac{\partial f_i}{\partial x_j})_{ij} \right)$ the Jacobian of $G$. Note that, since $G$ is generated by order 2 reflections, $J$ is equal to $z$, the polynomial defining the hyperplane arrangement of $G$ and the discriminant of $G$ is $\Delta = z^2 \in R$.

**Lemma 4.14.** *With notation as just introduced, the invariant ring $T$ satisfies $T \cong R \oplus JR$ as an $R$-module and $T \cong R[J]/(J^2 - \Delta)$ as rings.*

*Proof.* This follows from Stanley [Sta77]: let $S^G_\chi$ be the set of invariants relative to the linear character $\chi$, i.e., $S^G_\chi = \{f \in S : g(f) = \chi(g)f \text{ for all } g \in G\}$. In Lemma 4.1 loc. cit. it is shown that

$$S^\Gamma = S^G_{\mathrm{triv}} \oplus S^G_{\det^{-1}}$$

as $S^G = R$-modules, where triv denotes the trivial character and $\det^{-1}$ denotes the inverse of the determinantal character. Since $S^G_{\mathrm{triv}} = S^G = R$ and $S^G_{\det^{-1}}$ is generated by $J = z$ over $R$ (see either [Sta77] or [OT92, Chapter 6]), it follows that

$$T \cong R \oplus JR.$$

From Stanley's description of $T$ as $R$-module, we also see how $H = G/\Gamma \cong \mu_2 = \langle \sigma \rangle$ acts on $R[J]$: $\sigma$ is the identity on $R$ and $\sigma(J) = \det^{-1}(\sigma)(J) = -J$, since the Jacobian is a semi-invariant for $\det^{-1}$ of the reflection group. □

**Remark 4.15.** If $G$ is a pseudo-reflection group, then by [Sta77], the module of relative invariants $S^G_{\det}$ is generated by $z$, the reduced equation for the hyperplane arrangement and $S^G_{\det^{-1}}$ is generated by the Jacobian $J$. Then the relation for the discriminant is $zJ = \Delta$ (see [OT92], Examples 6.39, 6.40 and Def. 6.44).

*The hyperplane arrangement $S/(J)$.* Keeping the notation from above, we further write $A = S * G$, $B = T * H$ and set $e = \frac{1}{|G|}\sum_{g \in G} \delta_g$, $e_\Gamma = \frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} \delta_\gamma$, $e_- = \frac{1}{2}(1 - \delta_\sigma)$ and the (inverse) determinantal idempotent $f = \frac{1}{|G|}\sum_{g \in G} \det^{-1}(g)\delta_g$. Here we show how the module $S/(J)$ over the discriminant $R/(\Delta)$ can be seen as the image of the $B$-module $Ae_\Gamma$.

**Proposition 4.16.** *Denote by $i^* = - \otimes_B B/Be_-B : \mathbf{Mod}(B) \to \mathbf{Mod}(B/Be_-B)$ the standard recollement functor. Then $i^*Ae_\Gamma \cong S/(J)$ as $B/Be_-B \cong R/(\Delta)$-module.*

*Proof.* First compute

$$i^*Ae_\Gamma = Ae_\Gamma \otimes_B B/Be_-B \cong Ae_\Gamma/Ae_\Gamma e_-B.$$

Since $e_-e_\Gamma = e_\Gamma e_- = f$ and $B \cong e_\Gamma A e_\Gamma$ (see Proposition 4.12), this is

$$Ae_\Gamma/Ae_\Gamma e_-B \cong Ae_\Gamma/Ae_\Gamma e_-e_\Gamma Ae_\Gamma \cong Ae_\Gamma/A(e_\Gamma e_-)(e_-e_\Gamma)Ae_\Gamma \cong Ae_\Gamma/AfAe_\Gamma \cong (A/AfA)e_\Gamma.$$

Consider the trivial idempotent $e$ in $A$. Since $\Gamma$ is of index 2 in $G$ and $H$ is the cokernel of $\Gamma \to G$, it follows that $e + f = e_\Gamma$ and with $\overline{A} = A/AfA$ one sees that $\overline{A}e_\Gamma \cong \overline{A}e$. From Lemma 2.7 it follows that $\overline{A}e \cong S/(J)$, since $J$ generates the $R$-module of relative invariants for $\chi = \det^{-1}$. □

**The main theorem.**

**Theorem 4.17.** *Let $G \leqslant \mathrm{GL}(V)$ be a finite true reflection group with $H := \det G \cong \mu_2 = \langle \sigma \rangle$ and set $\Gamma = G \cap SL(V)$. Let $T = S^{\Gamma}$, $R = S^G \subseteq S$, $J$ the Jacobian of $G$ and the discriminant $\Delta = J^2 \in R$. Further denote by $A = S * G$ the skew group ring, $\overline{A} = A/Ae_{\chi}A$, with $e_{\chi} \in A$ an idempotent for a linear representation $\chi$, and $B = T * H$. Then:*

(i) *Then there is an equivalence of categories*
$$\mathbf{CM}(R/\Delta) \simeq \mathbf{CM}(B)/\langle e_{-}B \rangle,$$
*where $e_-$ is the idempotent $e_- = \frac{1}{2}(1 - \delta_{\sigma})$ in $B$.*

(ii) *The skew group ring $A$ is isomorphic to $\mathrm{End}_B(Ae_{\Gamma}) = \mathrm{End}_B(S * H)$ , where $e_{\Gamma} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \delta_{\gamma}$.*

(iii) *The quotient algebra $\overline{A} = A/Ae_{\chi}A$ is isomorphic to $\mathrm{End}_{R/\Delta}(i^*(Ae_{\Gamma}))$, where $i^*$ comes from the standard recollement of* **mod** *$B$,* **mod** *$Be_-B$ and* **mod** *$B/Be_-B$.*

(iv) *The $R/(\Delta)$-module $i^*(Ae_{\Gamma})$ is isomorphic to $S/(J)$, which implies that*
$$\overline{A} \cong \mathrm{End}_{R/(\Delta)}(S/(J)).$$

*Proof.* Without loss of generality we may assume that $e_{\chi} = f = \frac{1}{|G|} \sum_{g \in G} \det^{-1}(g) \delta_g$, cf. Cor. 2.6. From Theorem 4.3 it follows that
$$\mathbf{CM}(B)/\langle e_{-}B \rangle \simeq \mathbf{CM}(B/Be_{-}B).$$
The equivalence is induced by $i^* : \mathbf{Mod}(B) \xrightarrow{-_B \otimes B/Be_-B} \mathbf{Mod}(B/Be_-B)$, as seen in the proof of Theorem 4.3. Since $B/Be_-B$ is isomorphic to $R/(\Delta)$ (see Cor. 4.1, b), it follows that
$$i^* : \mathbf{CM}(B)/\langle e_- B \rangle \xrightarrow{\simeq} \mathbf{CM}(R/(\Delta)).$$
By Prop. 4.12, $A = S * G$ is isomorphic to $\mathrm{Hom}_B(S \otimes H, S \otimes H)$. Since $i^*$ is an equivalence, it follows that
$$i^*(\mathrm{Hom}_B(S \otimes H, S \otimes H)) \cong \mathrm{Hom}_{R/\Delta}(i^*(S \otimes H), i^*(S \otimes H)).$$
Now using $S \otimes H \cong Ae_{\Gamma}$ (as right $B$-module) from Lemma 4.9 yields that $i^*(S \otimes H) = i^* Ae_{\Gamma} = S/(J)$ by Prop. 4.16. Thus in total we get
$$i^*(A) \cong \mathrm{End}_{R/(\Delta)}(S/(J))$$
in $\mathbf{CM}(R/(\Delta))$.
On the other hand, compute the image of $A = \mathrm{Hom}_B(Ae_{\Gamma}, Ae_{\Gamma})$ in $\mathbf{CM}(B)/\langle e_- B \rangle$: we have to identify all morphisms $Ae_{\Gamma} \to Ae_{\Gamma}$ that factor through copies of $e_-B$. These are sums of elements of the form $\alpha \circ \beta$ with $\alpha \in \mathrm{Hom}_B(e_-B, Ae_{\Gamma})$ and $\beta \in \mathrm{Hom}_B(Ae_{\Gamma}, e_-B)$. Since $e_-$ is an idempotent, it follows e.g. from Lemma 4.2 [ASS06] that the first Hom is isomorphic (as right $e_-Be_- = R$-modules)
$$\mathrm{Hom}_B(e_-B, Ae_{\Gamma}) \cong \mathrm{Hom}_B(B, Ae_{\Gamma})e_- \cong Ae_{\Gamma}e_- = Af,$$
since $e_-e_{\Gamma} = e_{\Gamma}e_- = f$. For the other Hom, note that $e_-B = e_-e_{\Gamma}Ae_{\Gamma} = fAe_{\Gamma}$ and thus $\mathrm{Hom}_B(Ae_{\Gamma}, e_-B) = \mathrm{Hom}_B(Ae_{\Gamma}, fAe_{\Gamma})$. For each $f\beta \in \mathrm{Hom}_B(Ae_{\Gamma}, Ae_{\Gamma})$ one sees that the natural map $\Phi : f\,\mathrm{Hom}_B(Ae_{\Gamma}, Ae_{\Gamma}) \to \mathrm{Hom}_B(Ae_{\Gamma}, fAe_{\Gamma})$ sending $f\beta$ to $(ae_{\Gamma} \mapsto f\beta(ae_{\Gamma}))$ is surjective and moreover injective. Thus $\Phi$ is an isomorphism. It follows that
$$\mathrm{Hom}_B(Ae_{\Gamma}, e_-B) \cong f\,\mathrm{Hom}_B(Ae_{\Gamma}, Ae_{\Gamma}) \cong fA$$
as rings. In total we get
$$\mathrm{Hom}_B(Ae_{\Gamma}, Ae_{\Gamma})/\langle e_-B \rangle \cong A/\left((Af)(fA)\right) \cong A/AfA\,.$$

□

This theorem immediately yields that $A/AfA$ is a noncommutative resolution of the discriminant $R/(\Delta)$.

**Remark 4.18.** By Example 2.15 $R/(\Delta)$ is a direct summand of $S/(J)$. Using Thm. 5.3 of [DFI16] it follows that the centre of $\overline{A}$ is equal to $Z(\mathrm{End}_{R/(\Delta)}(S/(J)) = R/(\Delta)$.

**Corollary 4.19.** *Notation as in the theorem. If $G \ncong \mu_2$, then $A/Ae_\chi A \cong \mathrm{End}_{R/(\Delta)}(S/(J))$ yields a NCR of $R/(\Delta)$ of global dimension $n$. If $G \cong \mu_2$, then $A/Ae_\chi A \cong R/(\Delta)$ is a NCCR of $R/(\Delta)$.*

*Proof.* By the theorem $A/Ae_\chi A \cong \mathrm{End}_{R/(\Delta)}(S/(J))$. By Cor. 2.6 $A/Ae_\chi A \cong A/AeA$. By Corollary 3.13 the global dimension of $A/AeA$ is $n$ if $G \ncong \mu_2$. For the remaining case, cf. Rmk. 3.14 and note that $R/(\Delta)$ is regular. □

**Corollary 4.20** (McKay correspondence)**.** *The nontrivial irreducible $G$-representations are in $1-1$-correspondence to the indecomposable projective $\overline{A}$-modules, that are in $1-1$-correspondence to the isomorphism classes of $R/(\Delta)$-direct summands of $S/(J)$.*

*Proof.* Take $\overline{A} = A/AeA$. Similar as in Lemma 2.3 one has functors $\alpha, \beta$ between $P(\overline{A})$ and $\mathbf{Mod}(KG)$. This yields a bijection between the irreducible representations of $KG$ (except the trivial one) and indecomposable projective $\overline{A}$ modules. On the other hand, decompose $S/(J) = \bigoplus_i M_i^{a_i}$ as a finite direct sum of CM-modules over $R/(\Delta)$. Then the indecomposable projective $\mathrm{End}_{R/(\Delta)}(S/(J))$-modules are of the form $\mathrm{Hom}_{R/(\Delta)}(S/(J), M_i)$, which yields the second bijection. □

**Example 4.21.** (The normal crossings divisor as discriminant and its skew group ring) This example was our main motivation for investigating the relationship between $A/AeA$ and $\mathrm{End}_{R/(\Delta)}(S/(J))$: The reflection group $G = (\mu_2)^n$ acts on $V = K^n$ via the reflections $\sigma_1, \dots, \sigma_n$ with

$$\sigma_i(x_j) = \begin{cases} x_j \text{ if } & i \neq j \\ -x_j \text{ if } & i = j\,. \end{cases}$$

So $G$ can be realized as the subgroup of $GL(V)$ generated by the diagonal matrices

$$s_i = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that the invariant ring $R = S^G = K[x_1^2, \dots, x_n^2] = K[f_1, \dots, f_n]$. Then the Jacobian determinant $J = z$ of the basic invariants $(f_1(x), \dots, f_n(x))$ is $J = 2^n x_1 \cdots x_n$. We may omit the constant factor $2^n$ for the remaining considerations. The hyperplane arrangement corresponding to $G$ is the normal crossing divisor $S/(J) = K[x_1, \dots, x_n]/(x_1 \cdots x_n)$. The discriminant $\Delta$ is given by $\Delta = J^2 = f_1 \cdots f_n$. So the coordinate ring of the discriminant is $R/(\Delta) = K[f_1, \dots, f_n]/(f_1 \cdots f_n)$.
By Theorem 4.17, the ring $\overline{A} = A/AeA \cong \mathrm{End}_{R/(\Delta)}(S/(J))$ yields a NCR of $R/(\Delta)$. Here

we can explicitly compute the decomposition of $S/(J)$ as $R/(\Delta)$-module:

$$S/(J) \cong \bigoplus_{I \subsetneq [n]} x^I \cdot \left( R/(f^{[n]\setminus I}) \right),$$

where $[n]$ denotes the set $\{1, \dots, n\}$ and $f^L = \prod_{l \in L} f_l$ for a subset $L \subseteq [n]$. This holds because $S \cong \bigoplus_{I \subsetneq [n]} Rx^I$ as $R$-module and $\mathrm{Ann}_R(S/(J)) = J^2 = \Delta$. Thus $(Rx^I)/(f^{[n]}) \cong (R/(f^{[n]\setminus I})) \cdot x^I$ for any $I \subsetneq [n]$, and it follows that $S/(J)$ is a faithful $R/(\Delta)$-module.
In [DFI15, Thm. 5.5] it was shown that the module $M = \bigoplus_{I \subsetneq [n]} R/(\prod_{i \in I} f_i)$ gives a noncommutative resolution of global dimension $n$ of the normal crossing divisor $R/(\Delta)$. This was proven by showing that $\mathrm{End}_{R/(\Delta)}(M)$ is isomorphic to the order

$$\left( x^{J \setminus I} K[x_1, \dots, x_n] \right)_{I,J} \subset K[x_1, \dots, x_n]^{2^n \times 2^n}, \text{ where } I, J \subseteq [n]. \tag{14}$$

On the other hand, one can also compute that the skew group ring $A$ is in this case is $A = (K[x_1, \dots, x_n] * (\mu_2)^n) \cong \bigotimes_{i=1}^n \Lambda_1$, where $\Lambda_1$ is the skew group ring $K[x] * \mu_2$. Forming the quotient by $AeA$ yields the order (14).

*Coda: Results in dimension* 2. If $G \leqslant \mathrm{GL}(V)$, $\dim V = 2$, is a true reflection group, then the relation between $R = S^G$, $T = S^\Gamma$ and $R/(\Delta)$ can be interpreted in context of the classical McKay correspondence, cf. Section 2: in this case $T$ is isomorphic to $K[x,y,z]/(z^2 + \Delta(x,y))$, where $\{z^2 + \Delta = 0\}$ is an Kleinian surface singularity. Moreover, $T$ is of finite CM-type, that is, there are only finitely many isomorphism classes of indecomposable CM-modules. By Herzog's Theorem [Her78], $S$ is a representation generator for $T$, that is, $\mathbf{add}_T(S) = \mathbf{CM}(T)$.

In the following we show that with Theorem 4.17 one obtains that $R/(\Delta)$ is an ADE-curve and the hyperplane arrangement $S/(J)$ yields a natural representation generator for $R/(\Delta)$:

**Theorem 4.22.** *Let $G \leqslant \mathrm{GL}(V)$, $\dim V = 2$, be a true reflection group, with invariant ring $S^G = R$ and discriminant $R/(\Delta)$. Then $R/(\Delta)$ is of finite CM-type and consequently* $\mathrm{Spec}(R/(\Delta))$ *is an ADE curve singularity. Moreover, $\mathbf{add}_{R/(\Delta)}(S/(J)) = \mathbf{CM}(R/(\Delta))$.*

*Proof.* By Theorem 4.17 $\overline{A} = A/AeA \cong \mathrm{End}_{R/(\Delta)}(S/(J))$ has global dimension 2. Moreover, $R/(\Delta)$ is a direct summand of $S/(J)$. This we deduce from $\overline{A}e \cong S/(J)e$ and multiplying by $e$ from the left yields a direct summand of $\overline{A}$, which can be calculated as $e\overline{A}e \cong R/(\Delta)$. It follows that $S/(J)$ is a generator-cogenerator in the sense of Iyama [Iya03]: since $R/(\Delta)$ is Gorenstein, this just means that $R/(\Delta) \in \mathbf{add}_{R/(\Delta)}(S/(J))$. Now one can use the Auslander lemma, cf. [Iya03, DFI15] to see that $R/(\Delta)$ is of finite CM type, and thus $\mathbf{add}(S/(J)) = \mathbf{CM}(R/(\Delta))$. The only Gorenstein curves of finite CM-type are the ADE-curves, see [GK85]. □

## 5. ISOTYPICAL COMPONENTS AND MATRIX FACTORIZATIONS

Let $G \leqslant \mathrm{GL}(V)$ be any finite pseudo-reflection group. In this section we study direct sum decompositions of $S/(J)$ and $\overline{A}$. Moreover, the Hilbert–Poincaré series of the direct summands of $S/(J)$ as a $R/(\Delta) = S^G/(\Delta)$-modules are computed. Thus we also able to compute the ranks of these direct summands over $R/(\Delta)$ in case $\Delta$ is irreducible. In the case of $G = S_n$ we can even give a more explicit description using Young diagrams. We also compute the rank of $\overline{A}$ for any finite pseudo-reflection group in two ways: using the codimension 1 structure and with Hilbert–Poincaré series (in case $\Delta$ is irreducible).

**Hilbert–Poincaré series of isotypical components of $S/(J)$.** Here we look at the Hilbert–Poincaré series of the direct summands $M_i$ of $S/(J)$: recall from Section 2 that $M_i$ was defined to be the $R/(\Delta)$-module $\mathrm{Hom}_{KG}(V_i, S/(J))$, where $V_i$ is an irreducible $G$-representation. Further we have $S_i = \mathrm{Hom}_{KG}(V_i, S)$ and $S'_i = \mathrm{Hom}_{KG}(V'_i, S) = \mathrm{Hom}_{KG}(V_i \otimes \det, S)$. From the exact sequence (3) it follows that that

$$H_{M_i}(t) = H_{S_i}(t) - t^m H_{S'_i}(t) \ .$$

Let $K_{S_i}(t)$ and $K_{S'_i}(t)$ be the numerator polynomials of the Hilbert-Poincaré series of $S_i$ and $S'_i$ respectively and $H_R(t) = \frac{1}{\prod_{i=1}^n (1-t^{d_i})}$ and $H_{R/(\Delta)}(t) = \frac{1-t^{m+m_1}}{\prod_{i=1}^n (1-t^{d_i})}$ the Hilbert–Poincaré series of $R$ and $R/(\Delta)$ respectively. Then $H_{M_i}(t)$ can be written as

$$H_{M_i}(t) = H_R(t)\left(K_{S_i}(t) - t^m K_{S'_i}(t)\right) = H_{R/(\Delta)}(t)\frac{\left(K_{S_i}(t) - t^m K_{S'_i}(t)\right)}{1 - t^{m+m_1}} \ . \tag{15}$$

**Remark 5.1.** The numerator polynomials $K_{S_i}$ of the $H_{M_i}$ are called *fake degree polynomials*, see e.g. [Car93], or *generalized Kostka polynomials*, see [GP92].

**Example 5.2.** In the case of $G = S_n$, the irreducible representations of $G$ correspond to partitions $\lambda$ of $n$ and each partition $\lambda$ is given by a Young diagram, see e.g. [FH91]. Then the corresponding Hilbert–Poincaré series for the $\lambda$-isotypical component $S_\lambda$ of $S$ is given as

$$H_{S_\lambda}(t) = \prod_{k=1}^{n} \frac{t^{f_k}}{1 - t^{h_k}} \ , \tag{16}$$

where $f_k$ denotes the length of the leg of the hook of the $k$-cell and $h_k$ denotes the length of the hook of the $k$-cell (see Kirillov [Kir84, Thm. 1]) [Note here: for $f_k$ the $k$-cell itself is not counted and for the hooklength it is counted once, cf. [FH91]].

*Ranks of the isotypical components of $S/(J)$.* The ranks of the $M_i = \mathrm{Hom}_{KG}(V_i, S/(J))$ over $R/(\Delta)$ can be computed by evaluating $H_{M_i}(t)$ in $t = 1$, at least when $\Delta$ is irreducible:

**Lemma 5.3.** *Let $R = K[x_1, \dots, x_n]$ be graded by $\deg x_i = d_i \in \mathbb{N}$, let $\Delta \in R$ be a quasi-homogeneous polynomial, $R/(\Delta)$ be a domain, and let $M$ be a finitely generated CM module over $R/(\Delta)$. Then*

$$\mathrm{rank}_{R/(\Delta)}(M) = \lim_{t \to 1} \frac{H_M(t)}{H_{R/(\Delta)}(t)}.$$

*Proof.* Let $S' = K[y_1, \dots, y_{n-1}]$, where the $y_i$ form a system of parameters of $R/(\Delta)$, then $M$ is a finitely generated module over $S'$ Note that the Hilbert–Poincaré series of $M$ (and of $R/(\Delta)$) does not change if we consider both modules over $S'$. If $M$ is a graded CM-module over the graded CM ring $R/(\Delta)$, then $\mathrm{rank}(M) = \frac{e_{S'}(M)}{e_{S'}(R/(\Delta))}$, see [Nor68, Theorem 18] (cf. also Thm. 4.7.9 in [BH93]). Here $e_{S'}(-)$ denotes the multiplicity of a module over $S'$. By work of Smoke [Smo72], one can interpret $e_{S'}(M)$ as $\lim_{t \to 1}(\chi_{S'}(K) H_M(t))$, where $\chi_{S'}(M)$ is the so-called generalized multiplicity of $M$, also cf. [Sta78], and $\chi_{S'}(K)$ is equal to $\prod_{i=1}^{n-1}(1-t^{d'_i})$, where $d'_i = \deg y_i$. Since both $M$ and $R/(\Delta)$ have ranks, both limits $\lim_{t\to 1}(\chi_{S'}(K)H_M(t))$ and $\lim_{t\to 1}(\chi_{S'}(K)H_{R/(\Delta)}(t))$ exist and thus

$$\operatorname{rank}_{R/(\Delta)}(M) = \frac{\lim_{t\to 1}(\chi_{S'}(K)H_M(t))}{\lim_{t\to 1}(\chi_{S'}(K)H_{R/(\Delta)}(t))} = \lim_{t\to 1}\frac{H_M(t)}{H_{R/(\Delta)}(t)}\,.$$

□

**Proposition 5.4.** *With notation as above, let $V_i$ be an irreducible representation of $G$. Then the rank over $R/(\Delta)$ of the $V_i$-isotypical component of $S/(J)$, $M_i$, is given by*

$$\operatorname{rank}_{R/(\Delta)} M_i = \frac{1}{m+m_1}\left(m\dim V'_i + \frac{dK_{S'_i}}{dt}(1) - \frac{dK_{S_i}}{dt}(1)\right),$$

*where $V'_i$ stands again for the twisted representation $V_i \otimes \det$. If $G$ is a true reflection group, this simplifies to*

$$\operatorname{rank}_{R/(\Delta)} M_i = \frac{1}{2}\left(\dim V'_i + \frac{\frac{dK_{S'_i}}{dt}(1) - \frac{dK_{S_i}}{dt}(1)}{m}\right).$$

*Proof.* Using expression (15) and Lemma 5.3 for $H_{M_i}(t)$ we get

$$\operatorname{rank}_{R/(\Delta)} M_i = \lim_{t\to 1}\left(\frac{H_{R/(\Delta)}(t)\frac{K_{S_i}(t)-t^m K_{S'_i}(t)}{1-t^{m+m_1}}}{H_{R/(\Delta)}(t)}\right) = \lim_{t\to 1}\left(\frac{K_{S_i}(t)-t^m K_{S'_i}(t)}{1-t^{m+m_1}}\right).$$

By the rule of l'Hospital this limit is equal to

$$\lim_{t\to 1}\left(\frac{\frac{dK_{S_i}}{dt}(t) - mt^{m-1}K_{S'_i}(t) - t^m\frac{dK_{S'_i}}{dt}(t)}{-(m+m_1)t^{m+m_1-1}}\right).$$

Evaluating this expression in $t=1$ yields the above expression. If $G$ is generated by order 2 reflections, then $m = m_1$ and also $\det = \det^{-1}$, so one obtains the second formula. □

**Proposition 5.5.** *In case of $G = S_n$ and an irreducible representation $\lambda$ the rank of the $\lambda$-isotypical component $M_\lambda$ of $S/(J)$ is given by*

$$\operatorname{rank}_{R/(\Delta)}(M_\lambda) = \dim(V_\lambda)\left(\frac{1}{2} + \frac{A-F}{2m}\right), \tag{17}$$

*where $F = \sum_k f_k$ is the total footlength and $A = \sum_k a_k$ is the total armlength of the Young diagram corresponding to $\lambda$.*

*Proof.* The rank of $M_\lambda$ over $R/(\Delta)$ is given as

$$\operatorname{rank}_{R/(\Delta)} M_\lambda = \lim_{t\to 1}\frac{H_{M_\lambda}(t)}{H_{R/(\Delta)}(t)} = \lim_{t\to 1}\frac{H_{S_\lambda}(t) - t^m H_{S_{\lambda'}}(t)}{H_{R/(\Delta)}(t)},$$

where $\lambda'$ is the conjugate partition to $\lambda$. Note that the hooklengths of the conjugate partitions are the same, that is, the hooklength $h_k$ of the $k$-cell in $\lambda$ is the same as the hooklength $h'_k$ of the corresponding $k$-cell in $\lambda'$. On the other hand, one has that the footlength $f_k$ in $\lambda$ is equal to the armlength $a'_k$ in $\lambda'$ and vice versa. Moreover, these are connected to the hooklength via $h_k = f_k + a_k + 1$. Now substitute Kirillov's formula (16) in the above equation:

$$\operatorname{rank}_{R/(\Delta)} M_\lambda = \lim_{t\to 1}\left( \frac{\frac{1}{\prod_{k=1}^n (1-t^{h_k})}(t^F - t^{m+A})}{\frac{1-t^{2m}}{\prod_{k=1}^n (1-t^{d_k})}} \right) = \lim_{t\to 1}\left[ \left( \frac{t^F - t^{m+A}}{1-t^{2m}} \right) \cdot \left( \frac{\prod_{k=1}^n (1-t^{d_k})}{\prod_{k=1}^n (1-t^{h_k})} \right) \right] ,$$

where $d_k$ are the degrees of the basic invariants of $S_n$. Now using l'Hospital's rule for the two factors in the product yields:

$$\operatorname{rank}_{R/(\Delta)} M_\lambda = \frac{m+A-F}{2m} \cdot \prod_{k=1}^n \frac{d_k}{h_k} .$$

Since $d_k = k$ for all $k = 1, \dots, n$, the product $\prod_{k=1}^n \frac{d_k}{h_k} = \frac{n!}{\prod_{k=1}^n h_k} = \dim(V_\lambda)$ by the hooklength formula, see e.g. [FH91]. This yields the formula in (17). $\square$

**Indecomposable summands and rank of $\overline{A}$.** Let $G \leqslant \mathrm{GL}(V)$ be a finite group. Let $\Gamma$ be a normal subgroup in $G$ containing no pseudo-reflections and $H$ a complementary subgroup such that $\Gamma \cap H = \{1\}$ and $\Gamma H = G$. Here we will determine the indecomposable summands of $\overline{A}$ and then, in the case when $G$ is a true reflection group, of $S/(J)$ over $R/(\Delta)$.

Recall from Section 4 that $A \cong \operatorname{End}_B(Ae_\Gamma)$ with $B = T * H = e_\Gamma A e_\Gamma$. Let $\{e_j^{(i)}\}$ be a complete set of orthogonal primitive idempotents of $KG$ with $e_j^{(i)}$ belonging to the irreducible $G$-representation $V_i$, $j = 1, \dots, \dim V_i$.

**Lemma 5.6.** (1) *The right $A$-module $e_j^{(i)}A$ is projective and indecomposable and $A \cong \bigoplus_{i,j} e_j^{(i)}A$.*
(2) *The modules $e_j^{(i)}Ae_\Gamma$ are indecomposable (right) $B = e_\Gamma A e_\Gamma$-modules.*

*Proof.* For (1) we have to show that $E = \operatorname{End}_A(e_j^{(i)}A)$ has no nontrivial idempotents (see [HGK04, Lemma 2.4.3.]). Now $E$ is positively graded and idempotents necessarily reside in degree 0. The ring $E$ can be written as $\operatorname{End}_A(e_j^{(i)}A) = e_j^{(i)} \operatorname{End}_A(A) e_j^{(i)} = e_j^{(i)} A e_j^{(i)}$. Now look at the degree 0 part of this module:

$$(e_j^{(i)} A e_j^{(i)})_0 \cong e_j^{(i)} KG e_j^{(i)} \cong e_j^{(i)} \operatorname{End}_K(V_i) e_j^{(i)} \cong K .$$

For (2) note that $\operatorname{End}_B(e_j^{(i)} A e_\Gamma) = e_j^{(i)} \operatorname{End}_B(Ae_\Gamma) e_j^{(i)} = e_j^{(i)} A e_j^{(i)}$ by Prop. 4.12. $\square$

Now we get back to true reflection groups $G$. Here $\Gamma = G \cap \mathrm{SL}(V)$ and $H \cong \mu_2$ generated by some reflection $\sigma$. Let $e_- = \frac{1}{2}(1 - \delta_\sigma)$ be the idempotent in $B$. Then $e_-B = e_- e_\Gamma A e_\Gamma = fAe_\Gamma$ with $f = e_{\det^{-1}}$.

**Lemma 5.7.** *When $G$ is a finite true reflection group, then the $e_j^{(i)}Ae_\Gamma$ are contained in $\mathbf{CM}(B)/\langle e_-B \rangle$ and are indecomposable. Moreover,*

$$\bigoplus_{i,j} i^*(e_j^{(i)} A e_\Gamma) \cong S/(J) ,$$

*where $i^* : \mathbf{Mod}(B) \to \mathbf{Mod}(B/Be_-B)$ is the functor from the standard recollement. Moreover, $i^*(e_j^{(i)}Ae_\Gamma)$ is indecomposable as $R/(\Delta) \cong B/Be_-B$-module.*

*Proof.* For the first claim, we have to show that $e_j^{(i)} Ae_\Gamma$ are free $R$-modules. For this, note that $Ae_\Gamma$ is left projective over $A$ and hence projective over $S$ and then also over $R$. Since multiplication with $e_j^{(i)}$ on $Ae_\Gamma$ is $R$-linear, $e_j^{(i)} Ae_\Gamma$ is a direct summand of a projective $R$-module. The indecomposability follows from general considerations: Suppose that $X \in \mathbf{Mod}(B)$ is indecomposable but decomposes as $Y \oplus Z$ with $Y, Z \neq 0$ in $\mathbf{Mod}(B)/\langle e_-B\rangle$. This implies that there exist $U, V \in \langle e_-B\rangle$ such that $X \oplus U \cong Y \oplus Z \oplus V$ in $\mathbf{Mod}(B)$. Since $\mathbf{Mod}\, B$ is Krull–Schmidt, this is a contradiction.
Now using the equivalence $i^* : \mathbf{CM}(B)/\langle e_-B\rangle \simeq \mathbf{CM}(R/(\Delta))$ from Theorem 4.3 the remaining assertions follow. ☐

**Remark 5.8.** Since $f = e_1^{(\nu)}$ for the 1-dimensional representation $V_\nu = \det$, one sees that $Ae_\Gamma \cong \bigoplus_{i,j} e_j^{(i)} Ae_\Gamma \cong \bigoplus_{i \neq \nu, j} e_j^{(i)} Ae_\Gamma$ in $\mathbf{CM}(B)/\langle e_-B\rangle$. No other direct summand can be in $\langle e_-B\rangle$ and thus stays non-zero in the quotient.

*The structure of $\overline{A}$ in codimension* 1. Let $R$ be a DVR with parameter $t$ and quotient field $Q$. Let $\Lambda$ be a standard hereditary order over $R$. So

$$\Lambda = \begin{pmatrix} R & \cdots\cdots & R \\ tR & \ddots \;\; \cdots & \vdots \\ \vdots & \ddots \;\; \ddots & \vdots \\ tR & \cdots \;\; tR & R \end{pmatrix} \subseteq R^{n\times n} \subseteq Q^{n\times n}$$

Let $e$ be a rank one idempotent in $\Lambda$. By this we mean that $e^2 = e$ and $e$ has rank one as a matrix in $Q^{n\times n}$.

**Lemma 5.9.** *Let $e \in \Lambda$ be a rank one idempotent. Then there is a ring isomorphism $\phi : \Lambda \to \Lambda$ such that $\phi(e) = e_{11}$ the matrix unit.*

*Proof.* We first note that $e$ has rank one and so $e = uv^t$, the outer product of vectors $u, v$ in $R^n$ since $R$ is a domain. So

$$e = \begin{pmatrix} u_1v_1 & u_1v_2 & \cdots \\ u_2v_1 & u_2v_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that we have two expressions for the trace of $e$:

$$u_1v_1 + \cdots + u_nv_n = 1.$$

Now if $u_iv_i \in tR$ for all $i$, this would contradict the above equation, so for some $j$ we must have $u_jv_j \notin tR$, so $u_j, v_j \in R^*$ the units of $R$.

Now let

$$y = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \\ t & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

By conjugating $\Lambda$ by $y$ we can assume that $j = 1$.

So now $u_1, v_1 \in R^*$ but $e \in \Lambda$ so $u_iv_j \in tR$ for $i > j$. So in particular $u_2v_1, \ldots, u_nv_1 \in tR$. But $v_1 \in R^*$. So we must have $u_2, \ldots, u_n \in tR$.

Let

$$M = \begin{pmatrix} u_1 & -v_2 & -v_3 & \cdots & -v_n \\ u_2 & v_1 & 0 & \cdots & 0 \\ u_3 & 0 & v_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ u_n & 0 & \cdots & 0 & v_1 \end{pmatrix}$$

We note that $M$ has the following properties:

(1) $\det M \in R^*$.
(2) $M^{-1}\Lambda M = \Lambda$.
(3) $M^{-1}eM = e_{11}$.

and so our isomorphism $\phi$ is simply conjugation by $M$.

To verify the first property note that if we reduce modulo $t$ the $\det M = u_1 v_1^{n-1} \neq 0$. Checking the next properties is made easier by verifying that $RM = MR$ and $eM = Me_{11}$. ☐

Let $A = S * G$ be a skew group ring where $G$ is any finite subgroup of $\mathrm{GL}(V)$. Let $\mathfrak{p}$ be a height one prime ideal of the centre $R = S^G$. Since $A$ is homologically homogeneous in the sense of [BH84] (also see [VdB04]) we see that the global dimension of $A_\mathfrak{p} = A \otimes_R R_\mathfrak{p}$ is one. So $A_\mathfrak{p}$ is a hereditary order. By the structure theory of hereditary orders, as in Reiner [Rei75], there is an étale extension $R'$ of the DVR $R_\mathfrak{p}$ so that $A_{R'} = A \otimes_R R'$ is Morita equivalent to a standard hereditary order:

$$A_{R'} = \begin{pmatrix} R' & \cdots & \cdots & R' \\ tR' & R' & \cdots & R' \\ \vdots & \ddots & \ddots & R' \\ tR' & \cdots & tR' & R' \end{pmatrix}^{[g_1,\dots,g_N]}$$

where the exponent means that we replace the $i, j$ entry $(-)$ by $(-)^{g_i \times g_j}$. We know that if we extend the prime ideal $\mathfrak{p}$ to $\mathfrak{p}R'$ we get the usual formula for ramification $|G| = \sum e_i f_i g_i$. Our étale extension can be chosen to split all residue field extensions. So we have that all $f_i = 1$ and further that $|G| = e_1 g_1 + \cdots + e_N g_N$. Since $S$ is Galois over $S^G$, the ramification indices $e_i$ at the point $\mathfrak{p}$ are all the same. So we obtain:

$$A_{R'} = \begin{pmatrix} R' & \cdots & \cdots & R' \\ tR' & R' & \cdots & R' \\ \vdots & \ddots & \ddots & R' \\ tR' & \cdots & tR' & R' \end{pmatrix}^{g \times g} \subseteq (R')^{rg \times rg}$$

where $r$ is the ramification of the cover $S$ over $S^G$ at the codimension one point $\mathfrak{p}$. Now if we take a rank one idempotent of $A$, we obtain a rank one idempotent $e$ of $A_{R'}$. By Lemma 5.9, we can suppose that $e = e_{11} \in (R')^{eg \times eg}$.

So now we note that

$$A_{R'}/(A_{R'}eA_{R'}) \simeq \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & L & \cdots & L \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & L \end{pmatrix}^{g \times g} \subseteq L^{rg \times rg} ,$$

where $L = Q(R')$. We record the above observations in the following statement.

**Corollary 5.10.** *Let $A = S * G$ be a skew group ring with $G$ any finite subgroup of $\mathrm{GL}(V)$. Then let $\mathfrak{p}$ be a height one prime of the discriminant of the map $S^G \to S$. Let $r$ be the ramification index of this extension at $\mathfrak{p}$ and let $L$ be the algebraic closure of the residue field at $\mathfrak{p}$. Let $\overline{A} = A/(AeA)$. Then*

$$\overline{A} \otimes L \simeq \begin{pmatrix} L & L & \cdots & L \\ 0 & L & \cdots & L \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & L \end{pmatrix}^{g \times g} \subseteq L^{(r-1)g \times (r-1)g}$$

*where $rg = |G|$. Then the rank of $\overline{A}$ over the generic point $\mathfrak{p}$ is $\binom{r}{2}(|G|/r)^2$.*

Note that if $m_1$ is the number of mirrors over our generic point $\mathfrak{p}$ in the discriminant and $m$ is the number of pseudo-reflections with those mirrors, then

$$m_1(r-1) = m$$

where $r$ is the order of the pseudo-reflections with those mirrors, which is the ramification index. This gives

$$\begin{aligned} \binom{r}{2}\frac{|G|^2}{r^2} &= \frac{r(r-1)|G|^2}{2r^2} \\ &= \frac{(r-1)|G|^2}{2r} \\ &= \frac{|G|^2}{2}\frac{m_1(r-1)}{m_1 + m_1(r-1)} \\ &= \frac{|G|^2}{2}\frac{m}{m_1+m}\,. \end{aligned}$$

Note that the above corollary also allows us to conclude the following.

**Corollary 5.11.** *If $G \leqslant \mathrm{GL}(V)$ is generated by pseudo-reflections, some of which have order $\geq 3$, then $\overline{A}$ is not an endomorphism ring over $R/(\Delta)$.*

*Proof.* Suppose there is an $R/(\Delta)$-module $M$ such that $\overline{A} \cong \mathrm{End}_{R/\Delta}(M)$. Let $\mathfrak{p}$ be an associated prime ideal of $R/(\Delta)$ and let $L$ be the algebraic closure of its residue field. Then since $R/(\Delta)$ is reduced, we see that

$$\mathrm{End}_{R/\Delta}(M) \otimes L \cong \mathrm{End}_L(L^n) \cong L^{n\times n}$$

where $n$ is the rank of $M$ on the component corresponding to $\mathfrak{p}$. On the other hand, Corollary 5.10 shows that $A \otimes L$ has a nontrivial Jacobson radical unless the ramification index $r = 2$. Therefore $G$ must be a true reflection group. ☐

**Remark 5.12.** (Rank of $\overline{A}$ via Hilbert–Poincaré series) Here we sketch an alternative way to determine the rank of $\overline{A}$ as $R/(\Delta)$–module, at least when $\Delta$ is irreducible: Again assume that $G \leqslant \mathrm{GL}(V)$ is any finite pseudo-reflection group. If we consider the Hilbert–Poincaré series of $t$, where $S$ is graded in the standard way with $|x_i| = 1$ and $|g| = 0$ for

$g \in G$, one has

$$H_{AeA}(t) = \frac{\prod_{i=1}^{n}(1-t^{d_i})}{(1-t)^{2n}} = \frac{\prod_{i=1}^{n}\left(\sum_{j=0}^{d_i-1} t^j\right)}{(1-t)^n},$$
$$H_A(t) = \frac{|G|}{(1-t)^n},$$

whence

$$H_{\overline{A}}(t) = \frac{|G| - \prod_{i=1}^{n}\left(\sum_{j=0}^{d_i-1} t^j\right)}{(1-t)^n}.$$

Using Lemma 5.3 and the expression for $H_{R/(\Delta)}(t)$, the rank is given by

$$\lim_{t\to 1} H_{\overline{A}}(t)/H_{R/(\Delta)}(t) = |G| \lim_{t\to 1} \frac{\left(|G| - \prod_{i=1}^{n}\left(\sum_{j=0}^{d_i-1} t^j\right)\right)}{1-t^{m+m_1}}, \tag{18}$$

where we have used that

$$\lim_{t\to 1}\prod_{i=1}^{n}\big(\sum_{j=0}^{d_i-1} t^j\big) = \prod_{i=1}^{n} d_i = |G|.$$

For the remaining limit one applies the rule of l'Hospital and obtains

$$\lim_{t\to 1} H_{\overline{A}}(t)/H_{R/(\Delta)}(t) = |G| \frac{-\frac{|G|}{2}m}{-(m+m_1)} = \frac{|G|^2}{2}\frac{m}{m+m_1}.$$

**Example 5.13.** If we take $G = \mu_d$, acting on $K[x]$ as above, then the formula becomes

$$\lim_{t\to 1} H_{\overline{A}}(t)/H_{R/(\Delta)}(t) = \frac{d^2}{2}\frac{d-1}{d} = \binom{d}{2}.$$

In the Shephard–Todd classification these are the groups denoted by $G_3$ or $G(d,1,1)$.

**Example 5.14.** One can explicitly compute the rank of $\overline{A}$ for any finite unitary reflection group with irreducible discriminant in the Shephard–Todd list with the above:

$$\text{rank}_{R/(\Delta)}(\overline{A}) = \frac{|G|^2}{2}\frac{m}{m+m_1}.$$

The number $m$ of reflections is given as $\sum_{i=1}^{n}(d_i-1)$, where $n$ is the rank of $G$, and $d_i$ are the degrees of the basic invariants. The number $m_1$, that is, the number of different mirrors is given by the sum of the *co-exponents* of $G$. These are the degrees of the homogeneous generators minus 1 of the logarithmic derivation module of the reflection arrangement corresponding to $G$ [OT92, Cor. 6.63]. All these numbers can be found in the literature, see e.g. [ST54, Table VII] for the orders and degrees and [OT92, Table B.1] for the co-exponents.

Note that the groups labeled $G_1$ in the Shephard–Todd list are the symmetric groups, which are true reflection groups, so

$$\text{rank}_{R/(\Delta)}(\overline{A}) = \left(\frac{|G|}{2}\right)^2.$$

For the remaining groups one can determine in which cases the discriminant is irreducible, cf. Appendix C in [OT92].

**Identifying isotypical components.** The main result of this section is to identify the module of logarithmic vector fields $\Theta_R(-\log\Delta) \cong \Theta_S^G$ and its exterior powers $\Theta_R^m(-\log\Delta) = \bigwedge^m(\Theta_R(-\log\Delta)) \cong (\bigwedge^m \Theta_S)^G$ as isotypical components of the natural representation $V$ and its exterior powers $\bigwedge^m V$ and their corresponding matrix factorizations. The modules of logarithmic differential forms and logarithmic residues were first defined and studied by Kyoji Saito in [Sai80].

We start with recalling some facts from linear algebra, and introducing the notation for logarithmic vector fields, where we follow [OT92].

**5.15.** Recall the following result from linear algebra: Let $\varphi : P \to Q$ be a linear map between finite projective modules of same rank $m$ over some commutative ring $C$. With $\Lambda^i$ the $i^{th}$ exterior power over $C$ and $|P| = \det P = \Lambda^m P$, $|Q| = \det Q = \Lambda^m Q$ the invertible $C$–modules given by the top exterior powers of $P$ and $Q$, respectively, one has isomorphisms of $C$–modules $\Lambda^i P \cong |P| \otimes_C \Lambda^{m-i}P^*$ and $\Lambda^i Q \cong |Q| \otimes_C \Lambda^{m-i}Q^*$ induced from the nondegenerate pairing $\Lambda^i \otimes_C \Lambda^{m-i} \to \Lambda^m$ . Consider the composition

$$\varphi^{\mathrm{adj}}\varphi\colon \Lambda^i P \xrightarrow{\Lambda^i\varphi} \Lambda^i Q \cong |Q| \otimes_C \Lambda^{m-i}Q^* \xrightarrow{|Q|\otimes_C\Lambda^{m-i}\varphi^*} |Q| \otimes_C \Lambda^{m-i}P^* \cong |Q|/|P| \otimes_C \Lambda^i P\,,$$

where the adjugate morphism $\varphi^{\mathrm{adj}}$ is the composition of the maps to the right of $\varphi$, while $|Q|/|P|$ is shorthand for the invertible $C$–module $|Q| \otimes_C |P|^{-1}$. The top exterior power of $\varphi$ defines the $C$–linear map $\Lambda^m\varphi : |P| \to |Q|$ and the associated $C$–linear section $\det\varphi = \Lambda^m\varphi \otimes_C |P|^{-1} : C \to |Q|/|P|$ of the invertible line bundle $|Q|/|P|$ is the determinant of $\varphi$. The Laplace expansion of the determinant then translates into

$$\varphi^{\mathrm{adj}}\varphi = (\det\varphi)\,\mathrm{id}_P\colon\ \wedge^i P \longrightarrow (|Q|/|P|) \otimes_C \wedge^i P\,.$$

**5.16.** We maintain our usual set-up: $G \leqslant \mathrm{GL}(V)$ is a finite group generated by pseudo-reflections as subgroup of $\mathrm{GL}(V)$, and $S = \mathrm{Sym}_K(V)$ denotes the polynomial ring defined by $V$ over $K$, with $R = S^G$ the invariant subring. Recall that $R \cong \mathrm{Sym}_K W$ is a polynomial ring in its own right, with $W \cong R_+/R_+^2$ the graded $K$–vector space generated by the classes of the basic invariants $f_i \in R_+$.

We denote by $\Omega_S^1$ the Kähler differential forms on $S$ over $K$ and by $\Theta_S = \mathrm{Hom}_S(\Omega_S^1, S)$ its $S$–dual, isomorphic to the $S$–module of $K$-linear derivations, or vector fields, on $S = \mathrm{Sym}_K(V)$. We define $\Omega_R^1$ and $\Theta_R$ similarly by replacing $V$ with $W$.

Restricting a derivation on $S$ to $V = \mathrm{Sym}_K^1 V \subset S$ yields canonical isomorphisms

$$\Theta_S = \mathrm{Hom}_S(\Omega_S^1, S) \xrightarrow{\cong} \mathrm{Hom}_K(V, S) \cong S \otimes_K V^*\,, \quad D \mapsto D|_V\,.$$

Similarly,

$$\Theta_S^i = \mathrm{Hom}_K(\Lambda^i V, S) = S \otimes_K \Lambda^i V^*$$

$$\Omega_S^i = \mathrm{Hom}_K(\Lambda^i V^*, S) = S \otimes_K \Lambda^i V.$$

and these hold if we replace $S$ with $R$ and $V$ with $W$.

**5.17.** If a group $G$ acts on $S$ through $K$–algebra automorphisms then $G$ also acts naturally on $\Omega_S^1$ and $\Theta_S$, respectively.

Let again denote $R = S^G$, then $\Theta_S^G$ is the $R$-module of $G$-invariant derivations and $(\Omega_S^1)^G$ the $R$-module of $G$-invariant differential forms. Employing the isomorphisms above, it follows that

$$\Theta_S^G \xrightarrow{\cong} \operatorname{Hom}_K(V,S)^G \cong \operatorname{Hom}_{KG}(V,S)\,,$$

or, in other words, that the $V$-isotypical component of $S$ is $\Theta_S^G \otimes V$.

**Lemma 5.18.** *If the defining representation $V$ is an irreducible $G$–representation then the evaluation map*

$$\mathrm{ev} : \Theta_S^G \otimes_K V \to S$$

*identifies $\Theta_S^G$ with the isotypical component of $S$ that belongs to $V$. In particular, the evaluation map is a split $R$–monomorphism.* ☐

**5.19.** We have the Jacobian map of $S$-modules $\Omega_R^1 \otimes_R S \xrightarrow{\mathrm{jac}} \Omega_S^1$ defined by the inclusion of $K$–algebras $R \hookrightarrow S$. This gives the Zariski–Jacobi sequence $0 \to \Omega_R^1 \otimes_R S \xrightarrow{\mathrm{jac}} \Omega_S^1 \to \Omega_{S/R}^1 \to 0$, see e.g. [Mat86, Thm 25.1]. Note that jac is injective because $\Omega_R^1 \otimes_R S$ is a free $S$–module as $R$ is smooth over $K$, while the potential kernel is supported on the critical locus of the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$, thus must be zero as the morphism is generically smooth.

Applying $(\ )^* = \operatorname{Hom}_S(-,S)$ yields the map $\mathrm{jac}^* : \Theta_R \otimes S \to \Theta_S$.

$$0 \leftarrow T_{S/R}^1 \leftarrow \Theta_R \otimes S \xleftarrow{\mathrm{jac}^*} \Theta_S \leftarrow 0\,, \tag{†}$$

where $T_{S/R}^1 \cong \operatorname{Ext}_S^1(\Omega_{S/R}^1, S)$ is the first tangent cohomology of $S$ over $R$. In particular, the determinant of the (transposed) Jacobian matrix is given by the $S$–linear co-section

$$\det(\mathrm{jac}^*) : S \longrightarrow \Theta_R^n \otimes_S (\Theta_S^n)^* \cong S \otimes |V|/|W|\,,$$

where $|V|/|W|$ is again shorthand for $\det V \otimes_K (\det W)^{-1}$.

**5.20.** Taking $G$–invariants is exact and applied to the short exact sequence (†) above it returns

$$0 \leftarrow j_\Delta \leftarrow \Theta_R \xleftarrow{(\mathrm{jac}^*)^G} \Theta_S^G \leftarrow 0\,.$$

In [OT92, Cor. 6.57] it is shown that the $R$–linear inclusion $(\mathrm{jac}^*)^G$ identifies $\Theta_S^G$ with $\Theta_R(-\log\Delta) = \{\theta \in \Theta_R : \theta(\Delta) \in \Delta R\}$, the $R$-module of logarithmic vector fields along the discriminant $\Delta$. We have the natural inclusions

$$\mu^* : \Theta_R(-\log\Delta) \cong \Theta_S^G \xrightarrow{(\mathrm{jac}^*)^G} \Theta_R$$

$$\zeta^* : \Theta_R(-\log\Delta) \otimes S \cong \Theta_S^G \otimes S \longrightarrow \Theta_S\,.$$

Accordingly, $j_\Delta = \operatorname{Coker}(\mu^*)$ can be identified with the Jacobian ideal of the discriminant,

$$j_\Delta \cong \{D(\Delta) + (\Delta) \mid D \in \Theta_R\} \subseteq R/(\Delta)\,,$$

and the determinant of $\mu^*$ is the discriminant of $G$. It yields the $R$–linear co-section

$$\det(\mu^*) : R \longrightarrow \Theta_R^n(-\log\Delta) \otimes_R (\Theta_R^n)^* \cong R \otimes |W|/|W'|\,,$$

where $W'$ is a graded $K$–vector space so that $\Theta_R(-\log\Delta) \cong R \otimes (W')^*$. In particular, $R \otimes |W'|$ is a free $R$–module of rank 1 generated in degree $-c$, where $c = \sum_{i=1}^n c_i$ is the sum of the *co–degrees* $0 = c_1 \leqslant \cdots \leqslant c_n$, so that $\Theta_R(-\log\Delta) \cong \oplus_{i=1}^n R(-c_i)$.

As $\Theta_R \cong \oplus_{i=1}^n R(d_i)$, the degree of the discriminant is $|\Delta| = \sum_{i=1}^n (d_i + c_i)$.

**Example 5.21.** If $G$ is a *duality group*, then $d_i - c_i = d_1$, while $d_i + c_{n-i+1} = d_n$, the *Coxeter number* of $G$. Thus for such a group, $|\Delta| = h \cdot n = \sum_{i=1}^{n}(2d_i - d_1)$. Coxeter and Shephard groups are duality groups, and for Coxeter groups $d_1 = 2$ so that for these groups $|\Delta| = 2\sum_{i=1}^{n}(d_i - 1)$, twice the number of reflections in that group, as it should be.

**5.22.** By [OT92, Thm. 6.59] the map $\Theta_R(-\log\Delta) \otimes_R S \to \Theta_S$ is an inclusion as well and identifies in this way the $S$–modules $\Theta_R(-\log\Delta) \otimes_R S \cong \Theta_S(-\log z)$, where $\Theta_S(-\log z) \subseteq \Theta_S$ is the $S$–module of logarithmic vector fields along the hyperplane arrangement given by $\{z = 0\} \subseteq \operatorname{Spec} S$.

Using the same analysis as before, it follows that $z$ has degree $|z| = \sum_{i=1}^{n}(c_i + 1)$, equal, by definition, to the sum of the *co-exponents* of $G$, equal as well to the number of mirrors or reflecting hyperplanes defined by $G$ (this number has been denoted earlier as $m_1$).

We note the following facts.

**Proposition 5.23.** *(a) If the defining representation $V$ of the pseudo-reflection group $G \leqslant \mathrm{GL}(V)$ is irreducible, then $\Lambda^i V$ are irreducible for $1 \leq i \leq \dim(V)$.*
*(b)* $\Omega_R^i \cong (\Omega_S^i)^G \cong (S \otimes \Lambda^i V)^G$
*(c)* $\Theta_R^i(-\log\Delta) \cong (\Theta_S^i)^G \cong (S \otimes \Lambda^i V^*)^G$

*Proof.* The first statement is well known, e.g., [GM06, Thm. 4.6]. The second is [OT92, Theorem. 6.49], and the third statement follows immediately from [OT92, Prop. 6.70], which are both special cases of Solomon's theorem [OT92, Prop. 6.47]. ☐

Now we come to main goal of this section to identify some of the $R$–direct summands of $S/(J)$. By the above, we have the following pair of dual commutative diagrams:

$$\begin{tikzcd}
\Theta_S \arrow[r, equal] & \Theta_S \arrow[d, "\mathrm{jac}^*"] \\
\Theta_R(-\log\Delta) \otimes_R S \arrow[u, "\zeta^*"] \arrow[r, "\mu^* \otimes S"] & \Theta_R \otimes_R S \\
\Theta_R(-\log\Delta) \arrow[u] \arrow[r, "\mu^*"] & \Theta_R \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega_S \arrow[r, equal] \arrow[d, "\zeta"'] & \Omega_S \\
\Omega_R(\log\Delta) \otimes_R S & \Omega_R \otimes_R S \arrow[l, "\mu \otimes S"'] \arrow[u, "\mathrm{jac}"'] \\
\Omega_R(\log\Delta) \arrow[u] & \Omega_R \arrow[u] \arrow[l, "\mu"']
\end{tikzcd}$$

Here the top squares are commutative diagrams of $S$-modules and the bottom squares are commutative diagrams of $R$-modules. Let $\iota : \Lambda^i V \to \Omega_S^i$ and $\iota^* : \Lambda^{n-i} V^* \otimes \Theta_S^{n-i}$ be the natural inclusions. The above maps give us the following pair of commutative diagrams where the vertical maps of the top two squares are the multiplication in the exterior algebra, $a \otimes b \mapsto a \wedge b$.

(19)
$$\begin{tikzcd}
\Theta_S^n \arrow[r, "\Lambda^n \mathrm{jac}^*"] & \Theta_R^n \otimes_R S \\
\Theta_S^i \otimes \Theta_S^{n-i} \arrow[u, "\wedge"] \arrow[r, "\Lambda^i \mathrm{jac}^* \otimes \Lambda^{n-i} \mathrm{jac}^*"] & \Theta_R^i \otimes \Theta_R^{n-i} \otimes_R S \arrow[u, "\wedge \otimes S"'] \\
\Theta_R^i(-\log\Delta) \otimes_R \Lambda^{n-i} V^* \arrow[u, "\Lambda^i \zeta^* \otimes \iota^*"] \arrow[r, "\Lambda^i \mu^* \otimes \Lambda^{n-i} V^*"] & \Theta_R^i \otimes_R \Lambda^{n-i} V^* \arrow[u, "\Theta_R^i \otimes (\Lambda^{n-i} \mathrm{jac}^* \circ \iota^*)"']
\end{tikzcd}$$

(20)
$$\begin{tikzcd}
\Omega_S^n \arrow[rr, "\Lambda^n\zeta"] & & \Omega_R^n(\log\Delta)\otimes_R S \\
\Omega_S^{n-i}\otimes\Omega_S^i \arrow[u, "\wedge"] \arrow[rr, "\Lambda^{n-i}\zeta\otimes\Lambda^i\zeta"] & & \Omega_R^{n-i}(\log\Delta)\otimes\Omega_R^i(\log\Delta)\otimes_R S \arrow[u, "\wedge\otimes S"'] \\
\Omega_R^{n-i}\otimes\Lambda^i V \arrow[u, "\Lambda^{n-i}\mathrm{jac}\otimes\iota"] \arrow[rr, "\Lambda^{n-i}\mu\otimes\Lambda^i V"] & & \Omega_R^{n-i}(\log\Delta)\otimes\Lambda^i V \arrow[u, "\Omega_R^{n-i}(\log\Delta)\otimes(\Lambda^i\zeta\circ\iota)"']
\end{tikzcd}$$

The top squares of these diagrams commute since if $\phi : P \to Q$ is a map of free $S$-modules, then $\Lambda^\bullet\phi : \Lambda^\bullet P \to \Lambda^\bullet Q$ is an homomorphism of $S$-algebras. We know that $\Lambda^n\Omega_R^1(\log\Delta) \cong \Omega_R^n(\log\Delta) \cong \Omega_R^n(|\Delta|)$, since $\Delta$ is a free divisor, and $zJ = \Delta$ so we obtain the following maps

$$\Omega_R^n\otimes_R S \xrightarrow{J} \Omega_S^n \xrightarrow{z} \Omega_R^n(|\Delta|)\otimes_R S.$$

Now we apply the functor $\mathrm{Hom}_{KG}(\Lambda^i V, -)\otimes_K \Lambda^i V$ to this sequence. We first simplify the terms

$$\begin{aligned}
\mathrm{Hom}_{KG}(\Lambda^i V, \Omega_S^n) \cong (\Omega_S^n\otimes\Lambda^i V^*)^G \cong (\Omega_S^{n-i})^G \cong \Omega_R^{n-i} \cong \Omega_R^n\otimes\Theta_R^i\,, \\
\mathrm{Hom}_{KG}(\Lambda^i V, \Omega_R^n\otimes S) &\cong (\Omega_R^n\otimes S\otimes\Lambda^i V^*)^G \\
&\cong \Omega_R^n\otimes(\Theta_S^i)^G \\
&\cong \Omega_R^n\otimes\Theta_R^i(-\log\Delta) \\
&\cong \Omega_R^n(\log\Delta)(-|\Delta|)\otimes\Theta_R^i(-\log\Delta) \\
&\cong \Omega_R^{n-i}(\log\Delta)\,.
\end{aligned}$$

Here we have used the fact that $\Omega^n(\log\Delta)\cong\Omega^n(|\Delta|)$. Now applying the functor with its natural transformation to the identity functor yields the following commutative diagram:

$$\begin{tikzcd}
\Omega_R^n\otimes S \arrow[r, "\Lambda^n\mathrm{jac}^*\otimes\Omega_R^n\otimes\Lambda^n V"] & \Omega_S^n \arrow[r, "\Lambda^n\zeta"] & \Omega_R^n(|\Delta|)\otimes S \\
\Omega_R^{n-i}(\log\Delta)(-|\Delta|)\otimes\Lambda^i V \arrow[u] \arrow[r] & \Omega_R^{n-i}\otimes\Lambda^i V \arrow[u] \arrow[r, "\Lambda^{n-i}\mu\otimes\Lambda^i V"] & \Omega_R^{n-i}(\log\Delta)\otimes\Lambda^i V \arrow[u] \\
\Omega_R^n\otimes\Theta_R^i(-\log\Delta)\otimes\Lambda^i V \arrow[u, "\sim"] \arrow[r, "\Omega_R^n\otimes\Lambda^i\mu^*\otimes\Lambda^i V"] & \Omega_R^n\otimes\Theta_R^i\otimes\Lambda^i V \arrow[u, "\sim"] \arrow[r] & \Omega_R^n\otimes\Theta_R^i(-\log\Delta)(|\Delta|)\otimes\Lambda^i V \arrow[u, "\sim"]
\end{tikzcd}$$

where we have presented two isomorphic interpretations of the bottom row. It is clear that this diagram commutes since the left square with the bottom row is the outer square of the diagram (19) tensored with $\Omega_R\otimes\Lambda^n V$ after applying the isomorphisms $\Lambda^{n-i}V^*\otimes V^n \cong \Lambda^i V$ and $\Theta_R^i\otimes\Omega_R^n \cong \Omega_R^{n-i}$, and the upper right square is the diagram (20). Note that the cokernel of $\Lambda^{n-i}\mu : \Omega_R^{n-i} \to \Omega_R^{n-i}(\log\Delta)$ is the $(n-i)$*-th logarithmic residue*, see [Sai80]. We call the cokernel $\Lambda^i\mu^* : \Theta_R^i(-\log\Delta) \to \Theta_R^i$ the *i-th logarithmic co-residue of* $\Delta$. It is clear that the vertical maps are the evaluations of the natural transformation. Lastly, since the maps on the bottom row are uniquely determined by commuting with the diagram, we obtain the following result.

**Theorem 5.24.** *For all $i$ with $1 \leq i \leq \dim V$, there is a matrix factorization of $\Delta$ given by the pair of maps $\Lambda^{n-i}\mu$ and $\Lambda^i\mu^* \otimes \Omega^n_R$. The cokernels of these maps are the logarithmic residues and co-residues with a degree shift $|\Omega^n_R|$, which occur as $R/(\Delta)$-direct summands of $S/(z)$ and $S/(J)$ respectively, with multiplicity $\binom{n}{i} = \dim \Lambda^i V$. In particular, the first logarithmic residue $\operatorname{coker}(\mu)$ and $\operatorname{coker}(\mu^*) = j_\Delta$ are summands of $S/(z)$ and $S/(J)$ respectively, of multiplicity $n$.*

**Example 5.25.** Let $G = G(r,1,n) \cong \mu_r \wr S_n$ be the full monomial group acting in the usual way on $S = K[x_1, \dots, x_n]$. Let $p_i = \frac{1}{ir}\sum_j x_j^{ri}$ be the $i^{th}$ power sum function of the $x_i^m$, for $i \geq 1$. One choice of generators for the invariants is $p_1, \dots, p_n$. So $R = K[p_1, \dots, p_n]$ as in [ST54, Section 6]. It is now easy to compute that $\mathrm{jac} = (x_i^{jr-1})_{ij}$ in terms of the bases $dp_i$ and $dx_j$ of $\Omega_R$ and $\Omega_S$ respectively. A basis of $\Theta_R(-\log\Delta) \cong \Theta_S^G$ is given by $\theta_i = \sum_j x_j^{(i-1)r+1}\frac{\partial}{\partial x_j}$ as seen in [OT92, Appendix B.1], and so $\zeta^* = (x_j^{(i-1)r+1})_{ij}$ in terms of the bases $\theta_i$ and $\frac{\partial}{\partial x_j}$ of $\Theta_R(-\log\Delta)$ and $\Theta_S$ respectively. Now we can compute $\mu = \zeta\,\mathrm{jac} = r((i+j-1)p_{i+j-1})_{ij}$ in terms of the bases $dp_i$ and $\theta_i$. From this it is easy to compute

$$J = \det(\mathrm{jac}) = (x_1 \cdots x_n)^{r-1}\prod_{i<j}(x_i^r - x_j^r)$$

$$z = \det(\zeta) = x_1 \cdots x_n \prod_{i<j}(x_i^r - x_j^r).$$

Lastly, the maps $\Lambda^i\mu^*$ and $\Lambda^{n-i}\mu$ will determine a matrix factorization for $\Delta$ for each $i$.

## 6. MCKAY QUIVERS

Recall the McKay quiver $\mathrm{Mc}(V,G)$, which was defined in Section 2: let $G \leqslant \mathrm{GL}(V)$ be finite with irreducible representations $V_0 = \mathrm{triv}$, $V_1 = V = V_{\mathrm{stn}}$ the defining representation, $V_2, \dots, V_d$. The McKay quiver consists of vertices corresponding to the $V_i$ and there are $m_{ij}$ arrows from $V_i$ to $V_j$ if $V_j$, where $m_{ij} = \dim_K \operatorname{Hom}_{KG}(V_i, V \otimes V_j)$.

For $G \leqslant \mathrm{SL}(V)$, $\dim V = 2$, the McKay quiver is an extended Coxeter–Dynkin diagram $\overline{\Delta}$ with arrows in both directions. One can show that the skew group algebra $A = S * G$ is Morita equivalent to $\Pi_{\overline{\Delta}}$, the preprojective algebra of the corresponding extended Coxeter–Dynkin diagram $\overline{\Delta}$, see [RVdB89]. The preprojective algebra of $\overline{\Delta}$ is defined to be $K\overline{\Delta}$ modulo the preprojective relations.

A natural question is now to determine the McKay quivers $\mathrm{Mc}(V,G)$ for pseudo-reflection groups $G$ and to to find whether $A$ resp. $\overline{A}$ are Morita equivalent to the path algebra of $\mathrm{McKay}(G)$ modulo suitable relations.

Here we can describe at least the McKay quiver for $G = S_n$, the relations remain mysterious.

**McKay quiver for $S_n$.** We describe the McKay quiver of the group $S_n$ with its standard irreducible representation as a reflection group $V_{\mathrm{stn}}$. Recall that the irreducible representations $V_\lambda$ of $S_n$ are indexed by partitions $\lambda$ of $n$, which we shall label by Young diagrams. Hence the vertices of the McKay quiver are given by partitions of $n$.

Consider $S_{n-1}$ as a subgroup of $S_n$ by taking the permutations that fix $n$, and let

$$\mathrm{Res} : \mathbf{Mod}\, KS_n \to \mathbf{Mod}\, KS_{n-1}$$
$$\mathrm{Ind} : \mathbf{Mod}\, KS_{n-1} \to \mathbf{Mod}\, KS_n$$

be the restriction and induction functors. Frobenius reciprocity states that these functors are adjoint.

We also need the following lemmas.

**Lemma 6.1** ([FH91],Ex. 4.43)**.** *Let $\lambda$ be a partition of $n$. Then*

$$\operatorname{Res} V_\lambda = \bigoplus_\tau V_\tau \,,$$

*where the sum is over all partitions $\tau$ of $n-1$ that are obtained by removing a single block from $\lambda$.*

**Lemma 6.2** ([FH91],Ex. 3.16,Ex. 3.13)**.** *Write $K_{\text{trv}}$ for the trivial representation. If we induce and restrict in general we get*

$$\operatorname{Ind}\operatorname{Res} V \cong V \otimes \operatorname{Ind} K_{\text{trv}}$$

*and for the particular case of $S_{n-1} \subset S_n$ we have*

$$\operatorname{Ind} K_{\text{trv}} \cong V_{\text{stn}} \oplus K_{\text{trv}} \,.$$

We write the number of distinct parts of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ as $p(\lambda)$.

**Theorem 6.3.** *Consider the McKay quiver of $S_n$ with vertices indexed by partitions of $n$. Let $\tau \neq \lambda$ be partitions of $n$. Then there is an arrow from $\lambda$ to $\tau$ if and only if $\lambda$ can be formed from $\tau$ by moving a single block. The number of loops on $\lambda$ is $p(\lambda)-1$, one less than the number of distinct parts of $\lambda$.*

*Proof.* Consider two representations $V_\lambda, V_\tau$ of $S_n$. We have

$$\begin{aligned}
\operatorname{Hom}_{KS_{n-1}}(\operatorname{Res} V_\lambda, \operatorname{Res} V_\tau) &\cong \operatorname{Hom}_{KS_n}(\operatorname{Ind}\operatorname{Res} V_\lambda, V_\tau)\\
&\cong \operatorname{Hom}_{KS_n}(V_\lambda \otimes \operatorname{Ind} K_{\text{trv}}, V_\tau)\\
&\cong \operatorname{Hom}_{KS_n}(V_\lambda \otimes (V_{\text{stn}} \oplus K_{\text{trv}}), V_\tau)\\
&\cong \operatorname{Hom}_{KS_n}((V_\lambda \otimes V_{\text{stn}}) \oplus V_\lambda, V_\tau)
\end{aligned}$$

Now if $\lambda \neq \tau$ we have that $\operatorname{Hom}_{KS_n}(V_\lambda, V_\tau) = 0$. So since $V_{\text{stn}} \cong V_{\text{stn}}^*$ we obtain

$$\operatorname{Hom}_{KS_n}(V_\lambda \otimes V_{\text{stn}}, V_\tau) \cong \operatorname{Hom}_{KS_n}(V_\lambda, V_{\text{stn}} \otimes V_\tau)$$

which is the number of arrows from $\lambda$ to $\tau$ in the McKay quiver. So we only need to note that $\dim \operatorname{Hom}_{KS_{n-1}}(\operatorname{Res} V_\lambda, \operatorname{Res} V_\tau)$ is also the number of isomorphic irreducible summands of $\operatorname{Res} V_\lambda$ and $\operatorname{Res} V_\tau$. Since we obtain these from removing single blocks from $\lambda$ and $\tau$ there can be at most one in common and we obtain the description as stated in the theorem. Now suppose that $\lambda = \tau$ so we obtain

$$\begin{aligned}
\operatorname{Hom}_{KS_{n-1}}(\operatorname{Res} V_\lambda, \operatorname{Res} V_\lambda) &= \operatorname{Hom}_{KS_n}((V_\lambda \otimes V_{\text{stn}}) \oplus V_\lambda, V_\lambda)\\
&= \operatorname{Hom}_{KS_n}(V_\lambda \otimes V_{\text{stn}}, V_\lambda) \oplus K\\
&\cong \operatorname{Hom}_{KS_n}(V_\lambda, V_{\text{stn}} \otimes V_\lambda) \oplus K
\end{aligned}$$

Now note that $\dim \operatorname{Hom}_{KS_{n-1}}(\operatorname{Res} V_\lambda, \operatorname{Res} V_\lambda) - 1$ is the number of loops on $\lambda$ and this is also the number of partitions $\tau$ of $n-1$ obtained from $\lambda$ by removing one block. Note that this number is $p(\lambda)-1$. $\square$

## 7. EXTENDED EXAMPLE: $S_4$ AND THE SWALLOWTAIL

Consider the case of $G = S_4$ acting on $K^3$. We will give an explicit description of the direct summands of $S/(J)$ over the discriminant.

For this example, $S = k[x, y, z]$, $R = k[u, v, w]$ and

$$J = (x - y)(x - z)(y - z)(2x + y + z)(2y + x + z)(2z + x + y).$$

A generator of the discriminant ideal $(\Delta)$ can be computed as the determinant of the matrix $(Jac)^T(Jac)$, where $Jac = \left(\frac{\partial f_i}{\partial x_j}\right)$ is the Jacobian matrix, cf. [Sai93, OT92]. An explicit equation is:

$$\Delta = -v^4 - 2u^3v^2 + 9u^4w + 6uv^2w - 6u^2w^2 + w^3.$$

Spec$(R/(\Delta))$ is called the *swallowtail*. Its singular locus consist of two curves: a parabola (the "self-intersection locus") and a cusp, meeting at the origin, see Fig. 3.

Now let us sketch the computation of the matrix for multiplication by $J$. Consider the map induced by multiplication by $J$ on $S$

$$S \xrightarrow{J} S$$

We know that $S$ is a free $R$-module and that $J^2 = \Delta \in R$, so

$$S \xrightarrow{J} S \xrightarrow{J} S$$

is a matrix factorization of $\Delta$ over $R$ by definition. We wish to decompose $S/(J)$ into indecomposable CM-modules over $R/\Delta$. We can use the grading and the $G$-action to provide information about the decomposition.

First recall that $R = K[f_1, \dots, f_n]$ and let $(R_+)$ be the ideal in $S$ generated by $f_1, \dots, f_n$. Recall that (see Section 2) $S/(R_+) \cong KG$ as $G$-representations, and $S/(R_+) \otimes_K R \cong S$ as graded $RG$-modules.

In this example $G = S_4$. Let us call the irreducible representations

$$K, V, W, V', \det$$

corresponding to the partitions

$$4 = \ydiagram{4}\,,\ 3 + 1 = \ydiagram{3,1}\,,\ 2 + 2 = \ydiagram{2,2}\,,\ 2 + 1 + 1 = \ydiagram{2,1,1}\,,\ 1 + 1 + 1 + 1 = \ydiagram{1,1,1,1}\,.$$

We have that

$$S/(R_+) \simeq K(0) \oplus V(-1) \oplus V(-2) \oplus W(-2) \oplus V(-3) \oplus V'(-3) \oplus V'(-4) \oplus W(-4) \oplus V'(-5) \oplus \det(-6)\,,$$

where the number in $(-)$ indicates the degree shift.

By Section 5, $S$ will decompose into isotypical components via the isomorphism

$$S \cong \bigoplus_{V_i \text{ irreps of } G} \mathrm{Hom}_{KG}(V_i, S) \otimes_K V_i$$

which gives us that the map $S \xrightarrow{J} S$ decomposes into components of the form

$$\mathrm{Hom}_{KG}(U, S) \otimes_K U \xrightarrow{J} \mathrm{Hom}_{KG}(U \otimes \det, S) \otimes_K U \otimes \det$$

for each irreducible representation $U$ of $G$.

So for our example $S_4$ we have the following components

$$K(0) \otimes R \to \det(-6) \otimes R \tag{21}$$

$$\det(-6) \otimes R \to K(0) \otimes R \tag{22}$$

$$(V(-1) \oplus V(-2) \oplus V(-3)) \otimes R \to (V'(-3) \oplus V'(-4) \oplus V'(-5)) \otimes R \tag{23}$$

$$(V'(-3) \oplus V'(-4) \oplus V'(-5)) \otimes R \to (V(-1) \oplus V(-2) \oplus V(-3)) \otimes R \tag{24}$$

$$(W(-2) \oplus W(-4)) \otimes R \to (W(-2) \oplus W(-4)) \otimes R \tag{25}$$

where the maps are the $R$-linear maps given by multiplication by $J$ restricted to each component. Combining the first two components of lines (21) and (22) we obtain the matrix factorization

$$R \to JR \to R$$

where both maps are multiplication by $J$. The cokernels of the two maps are 0 and $R/\Delta$ respectively. By choosing bases of $V(-1) \oplus V(-2) \oplus V(-3)$ and $V'(-3) \oplus V'(-4) \oplus V'(-5)$ we can express multiplication by $J$ in the other components as matrices with entries in $R$. From (23) and (24) we get a pair of $9 \times 9$ matrices

$$M_1 : (V(-1) \oplus V(-2) \oplus V(-3)) \otimes R \to (V'(-3) \oplus V'(-4) \oplus V'(-5)) \otimes R$$

$$M_2 : (V'(-3) \oplus V'(-4) \oplus V'(-5)) \otimes R \to (V(-1) \oplus V(-2) \oplus V(-3)) \otimes R$$

By choosing bases appropriately one can show that both matrices are Kronecker products with the $3 \times 3$ identity matrix $I_3$ so $M_1 = A \otimes I_3$ and $M_2 = B \otimes I_3$. Similarly, for (25) we can compute a matrix

$$M_3 : (W(-2) \oplus W(-4)) \otimes R \to (W(-2) \oplus W(-4)) \otimes R$$

and $M_3 = C \otimes I_2$ for some choice of basis.

We can identify the matrices $A, B, C$ involved in the matrix factorizations of $\Delta$ by using Hovinen's thesis [Hov09, Thm. 4.4.7], where the graded rank one CM-modules over $R/\Delta$ are classified (via matrix factorizations):

**Theorem 7.1** (Hovinen)**.** *Every graded one rank 1 CM-module over the swallowtail $\Delta$ is isomorphic to one of the following list (up to degree shift):*

1. *the free module $R/(\Delta)$,*
2. *the ideal $(v, w)$ and its $R/(\Delta)$-dual,*
3. *the restriction of the so-called open swallowtail and its $R/(\Delta)$-dual, i.e., the ideal defining the self-intersection locus of $\Delta$,*
4. *the modules of the family $\{M_{2,t} : t \in \mathbb{C}\}$ of $2 \times 2$-matrices:*

$$M_{2,t} = \operatorname{coker} \begin{pmatrix} w - (t^2-1)u^2 & v^2 + (t-2)^2(t+1)u^3 \\ v^2 - (t-1)(t+2)^2u^3 & w^2 + 6uv^2 + (t^2-7)u^2w + (t^2-4)^2u^4 \end{pmatrix},$$

5. *the modules of the family $\{M_{4,0,t} : t \in \mathbb{C}^*\}$ of $3 \times 3$-matrices:*

$$M_{4,0,t} = \operatorname{coker} \begin{pmatrix} -w + \frac{1}{t^2}(1-2t)u^2 & 0 & v^2 + \frac{1}{t}(t^2-4t+1)uw + 2v^3 \\ v & -w & 0 \\ -\frac{(t+1)^2}{t}u & v & -w + t(t-2)u^2 \end{pmatrix}$$

6. *the modules of the family $\{M_{4,-3,t} : t \in \mathbb{C}^*\}$ of $3 \times 3$-matrices:*

$$M_{4,-3,t} = \operatorname{coker} \begin{pmatrix} -w + \frac{1}{t^2}(t+4)(3t+4)u^2 & 0 & v^2 + \frac{1}{t}(t^2-t+4)uw - \frac{1}{t}(t+3)(3t+4)u^2 \\ v & -w + 3u^2 & 0 \\ \frac{1}{t}(t+1)(t+4)u & v & -w + (t+1)(t+3)u^2 \end{pmatrix}.$$

*Note here that* $M_{4,-3,-2}$ *is the matrix factorization of the normalization* $\widetilde{R/(\Delta)}$.

*Moreover, the modules are characterized by the first fitting ideals of their corresponding matrix factorizations. This means that the modules in (1)–(6) are pairwise nonisomorphic.*

The result is that $S/(J)$ is a direct sum of 4 nonisomorphic CM-modules corresponding to the nontrivial irreducible representations of $S_4$. One can calculate the ranks explicitly or use the formulas in Section 2:

**Theorem 7.2.** *As a* $R/\Delta$*-module,*

$$S/(J) \cong M_{\yng(4)} \oplus M^3_{\yng(3,1)} \oplus M^3_{\yng(2,1,1)} \oplus M^2_{\yng(2,2)},$$

*where* $M_{\yng(4)} \cong R/(\Delta)$, $M_{\yng(2,1,1)} \cong M_{4,-3,-2}$, *the Jacobian ideal of* $R/(\Delta)$ *(also isomorphic to the normalization of* $R/\Delta$*),* $M_{\yng(3,1)}$ *is the syzygy of* $M_{\yng(2,1,1)}$*, i.e., the module of logarithmic derivations along* $\Delta$*, and* $M_{\yng(2,2)} \cong M_{2,0}$*, which is isomorphic to the ideal defining the singular cusp in* $\Delta = 0$*. The ranks of the modules over* $R/(\Delta)$ *are* $\operatorname{rank}(M_{\yng(4)}) = \operatorname{rank}(M_{\yng(2,1,1)}) = \operatorname{rank}(M_{\yng(2,2)}) = 1$ *and* $\operatorname{rank}(M_{\yng(3,1)}) = 2$.

In particular, this shows that $\operatorname{rank}_{R/(\Delta)}(S/(J)) = 12$, and thus $\operatorname{rank}_{R/(\Delta)}(\operatorname{End}_{R/(\Delta)}(S/(J)) = \operatorname{rank}(\overline{A}) = 144$, as we computed earlier. In Fig. 3 below the curves corresponding to the modules $M_{\yng(2,1,1)}$ and $M_{\yng(2,2)}$ are sketched on the swallowtail from two different perspectives.

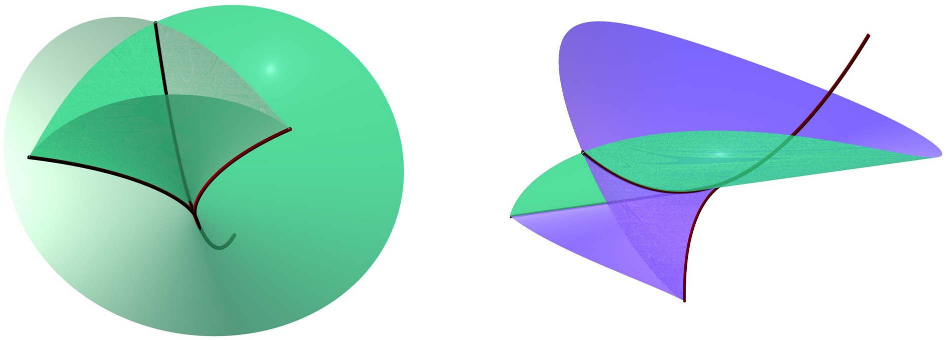
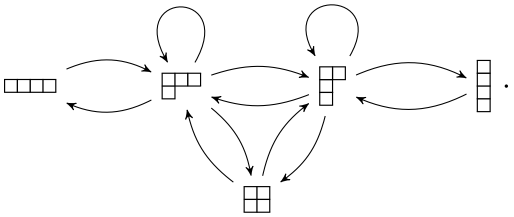


FIGURE 3. The swallowtail.

For this example, one can also draw the McKay quiver, see (26). The quiver of $\overline{A}$ is obtained from (26) by deleting the vertex and incident arrows corresponding to the determinantal representation $\yng(1,1,1,1)$.

(26)



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