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Some Results Related to Schiffer's Problem

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Some results related to Schiffer's Problem

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Abstract

We consider the following semilinear overdetermined problem on a two dimensional bounded or unbounded domain Ω with analytic boundary $\partial\Omega$ having at least one bounded connected component

$$\begin{cases} -\Delta u = g(u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ and } \ u = c \text{ on } \partial\Omega, \end{cases}$$

where c is a constant. When $g(c) = 0$ the constant solution $u \equiv c$ is the unique solution. For $g(c) \neq 0$, we show that the boundary is a circle if and only if the problem admits a solution that has constant third or fourth normal derivative along the boundary. A similar result involving the fifth normal derivative is proved.

Mathematics Subject Classification (2000). 35J25, 35N25, 35J61, 35P99

Keywords. Schiffer problem, Pompeiu problem, overdetermined boundary value problem

1 Introduction

In the present paper we investigate the following overdetermined semilinear problem

$$\left.\begin{array}{rll} \Delta u + g(u) & = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} & = 0 & \text{on } \partial\Omega, \\ u & = c & \text{on } \partial\Omega. \end{array}\right\} \tag{1.1}$$

The study of overdetermined problems has mainly been motivated by several papers of Schiffer ([16], [17]) where he showed how the Hadamard variational method can be extended to study the Dirichlet eigenvalues as a function of the domain. For instance, at a minimum point Ω^* of the first Dirichlet eigenvalue functional $\Omega \mapsto \lambda_1(\Omega)$ defined in the class of bounded domain of prescribed volume, the following overdetermined problem admits a solution

$$-\Delta u = \lambda u, \quad u|_{\partial\Omega} = 0, \quad \partial_\nu u|_{\partial\Omega} = c > 0. \tag{1.2}$$

However this variational characterization of Ω^* was slightly unsatisfactory since on one hand it was well known by the works of Faber and Krahn that the minimum was achieved by a ball, but on the other hand there were no tools to prove that the ball is the unique domain on which (1.2) admits a solution. In 1971, the contribution of Serrin [18] brought a partial answer when in (1.2), $\lambda := \lambda_1(\Omega)$. Indeed in this case, the associated solution u can be chosen to be positive, and therefore the moving plane method as described by Serrin was leading to the conclusion that Ω can only be a ball. Proving that this conclusion holds without positivity assumption is an open question.

Likewise, by considering the critical points of the Neumann eigenvalue functional [7], one is led to the question if the following overdetermined problem

$$\left.\begin{array}{rll} \Delta u + \mu u & = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} & = 0 & \text{on } \partial\Omega, \\ u & = c & \text{on } \partial\Omega, \end{array}\right\} \tag{1.3}$$

with c denoting a nonzero constant, admits a solution $(\mu, u) \in (0,\infty) \times C^2(\bar{\Omega})$. The constant c is assumed to be non-zero, since otherwise an easy application of the Rellich identity (Pohozaev identity) implies that $u \equiv 0$ is the unique function solving (1.3). Note that the constant c can always be renormalized to be $c = -1$, since one can replace a solution u to (1.3) by $-u/c$. Furthermore, we work with a real analytic connected domain, since it is proved in [21] that if Ω is only assumed to be a Lipschitz domain, and if a solution to (1.3) exists, then $\partial\Omega$ has to be real analytic. For $c \neq 0$, the problem (1.3) admits in a ball infinitely many solutions (μ, u) which are given by the radial eigenfunctions associated to the (positive) Neumann eigenvalues of the (negative) Laplacian.

In 1976, Williams showed [20] that the overdetermined problem (1.3) is equivalent to an older problem that originates in a paper by Pompeiu (see Zalcman [24] for a detailed survey on this problem). More specifically, a set Ω is said to have the Pompeiu property if $f \equiv 0$ is the only continuous function on $\mathbb{R}^n$ for which $\int_{\sigma(\Omega)} f(x)\, dx = 0$ for every rigid motion σ. Tchakaloff [19] pointed out that balls in $\mathbb{R}^n$ do not have such a property, which leads to the so-called "Pompeiu problem": *Is the ball the unique bounded smooth simply connected domain that fails to have the Pompeiu property ?* The main result in [20] shows in fact that a simply-connected domain fails to have the Pompeiu property if and only if Problem (1.3) admits a solution.

Note that the solution to Problem (1.3) must satisfy $\nabla u = 0$ on $\partial\Omega$, so that $\partial_i u$ $(i = 1, \cdots, n)$ are Dirichlet eigenfunctions associated to the eigenvalue μ. Therefore Ω has multiple Dirichlet eigenvalues. Since generically the Dirichlet-Laplace spectrum has only simple eigenvalues, Ω must be a special domain. If $u > c$ in Ω, then by using the moving plane method one can show that Ω must be a ball, and the same conclusion can be reached if u has no saddle points [22]. It is also known that if the Problem (1.3) admits infinitely many eigenvalues μ then the domain must necessarily be a ball (see [4], [5]). On the other hand, if Ω is a plane domain different from a disc, it is known that there are no solutions to the overdetermined problem (1.3) when one of the following conditions holds:

(i) Ω is convex and the ratio of inradius over outer radius is smaller than 0.5 [6];

(ii) the boundary can be parametrized by a finite Fourier-series $\sum_{\ell=-m}^{n} a_\ell e^{i\ell s}$, with $a_{-m}a_n \neq 0$ and m, n positive integers [9];

(iii) Ω is convex and $\mu \leq \mu_7$, the seventh Neumann-Laplace eigenvalue with $\mu_1 = 0$ being counted as the first [3], and the same result has been obtained without any convexity assumption in [8];

(iv) Ω is strictly convex and point symmetric and $\mu < \mu_{13}$ [8].

Proving that Problem (1.3) can only admit a solution in a ball is one of the problems that S.T. Yau has added in his list of open problems in [23] (see Problem 80) where this is referred to as "Pompeiu problem, Schiffer Problem". Since then it is commonly called "Schiffer conjecture".

One may also wonder what happens if in the problem (1.3) some higher order normal derivatives are additionally assumed to be constant on the boundary. For instance, Liu proved in [11] that Schiffer's conjecture holds if the third-order interior normal derivative of the corresponding Neumann eigenfunction is constant on the boundary. In [12, Theorem 1.2] this result has been extended to cover overdetermined elliptic fully nonlinear problems, where the linear operator in (1.3) has been replaced by an elliptic operator $F(u, |\nabla u|^2, D^2 u)$.

In the present paper we will focus on dimension two, and show that a similar result holds if we prescribe either the fourth or the fifth-order normal derivative to be constant along the boundary. Throughout the paper our assumption on the domain will be the following

$$\begin{cases} \Omega \text{ is a connected domain in } \mathbb{R}^2, \quad \partial\Omega \text{ is non-empty and analytic,} \\ \qquad \partial\Omega \text{ has at least one } \textit{bounded} \text{ connected component,} \end{cases} \tag{1.4}$$

which allows also exterior domains. Furthermore, the nonlinearity g will be assumed to be real analytic (though for some results g can be less regular), and we will assume that Problem (1.1) admits a real analytic solution.

As we shall see, a solution u to Problem (1.1) satisfies the following two properties (See Proposition 3.2):

(i) The second order normal derivative $\partial^2_{\nu\nu} u$ is constant along the boundary and it is given by $-g(c)$.

(ii) If, moreover, $g(c) = 0$, then $u \equiv c$ is the unique solution which is real analytic up to the boundary. Hence, in order to derive a rigidity result for the domain, we have to assume $g(c) \neq 0$.

To derive statements involving higher order derivatives on the boundary, we parametrize a neighborhood of the boundary, using the arc-length parameter s along the boundary and the distance function d to the boundary (sometimes called "Fermi coordinates"). By rewriting Problem (1.1) in these coordinates (s, d), and taking higher derivatives with respect to the coordinate d, we will derive several useful identities. They lead to our first result:

Theorem 1.1. *Let u be a solution to* (1.1) *with c such that $g(c) \neq 0$. Then the following three statements are equivalent:*

(i) *The boundary $\partial\Omega$ is a circle.*

(ii) *$\partial_d^{(3)} u$ is constant on the boundary $\partial\Omega$.*

(iii) *$\partial_d^{(4)} u$ is constant on the boundary $\partial\Omega$.*

Hence, if one of these conditions holds then the domain is either a disc or the complement of a disc.

Concerning the fifth derivative, we will prove that $\partial_d^{(5)} u$ and the signed curvature κ of the boundary $\partial\Omega$ are related as follows

$$-\kappa'' + 12\kappa^3 - 2g'(c)\kappa + \frac{\partial_d^{(5)} u}{g(c)} = 0. \tag{1.5}$$

This identity shows that if κ is a positive constant, then the fifth order normal derivative of u is constant. In order to prove the converse, the discussion will depend on the signs of $g'(c)$ and of the function $\alpha := \frac{\partial_d^{(5)} u}{g(c)}$. For the case $g'(c) \leq 0$, we will show that (1.5) admits only one periodic solution of prescribed period. This will lead to our next result for the overdetermined problem (1.1):

Theorem 1.2. (The case $g'(c) \leq 0$)
Assume Problem (1.1) *admits a solution u with c such that $g(c) \neq 0$ and $g'(c) \leq 0$. Then $\partial\Omega$ has constant curvature if and only if α is a non-zero constant, and in this case we have*

(i) *Ω is a disc if $\alpha < 0$.*

(ii) *Ω is a complement of a disc if $\alpha > 0$.*

For the case $g'(c) > 0$, we can show the following result.

Theorem 1.3. *Assume Problem* (1.1) *admits a solution u in a domain Ω, with c such that $g(c) \neq 0$, $g'(c) > 0$ and α constant. Then, the following hold:*

(i) *If Ω is bounded and $\alpha < 0$, then Ω is a disc;*
If Ω is unbounded and $\alpha > 0$, then Ω is the complement of a disc.

(ii) *If the curvature of a bounded connected component of $\partial\Omega$ does not change sign, then $\partial\Omega$ is a circle.*

(iii) *If a bounded connected component Γ of $\partial\Omega$ satisfies $g'(c)|\Gamma|^2 \leq 36(1+\sqrt{3})^2\pi^2$ ($|\Gamma|$ the length of Γ), then $\partial\Omega$ is a circle.*

Applied to Schiffer's problem, Theorem 1.3 (iii) amounts to an upper bound on μ in the spirit of [3] and [8], but without convexity assumption on Ω.

This paper is structured as follows. In Section 2, we write the overdetermined Problem (1.1) in Fermi coordinates and derive identities connecting the n-th normal derivative with the curvature of the boundary of the domain. In Section 3, we first show that the constant solution $u \equiv c$ is the unique solution to Problem (1.1) whenever $g(c) = 0$, and then prove our Theorem 1.1. In Section 4, we show that the curvature of the domain Ω is related to the fifth normal derivative via the cubic nonlinear ODE (1.5), and give the associated variational framework. The proof of Theorem 1.2 will follow by noting that the associated functional is convex. The case that the fifth normal derivative is constant on the boundary is treated in Section 5 where our Theorem 1.3 is proved. In the final section we see how similar conclusions can be obtained for the Serrin type overdetermined problem.

2 Some identities for the normal derivatives

In this section, we first introduce some notations and known facts on the signed curvature of a curve. In a second part, we will derive some identities along the boundary of the domain.

2.1 Curvature of a curve

Given a curve $\gamma(s)$ ($s \in (0,L)$) parametrized by the arc-length, we introduce the following unit tangent vector and unit normal vector

$$\mathbf{T} := \gamma'(s) \quad \mathbf{N} := R_{\pi/2}(\mathbf{T}) \tag{2.1}$$

where $R_{\pi/2}$ stands for a left-rotation of angle $\pi/2$. We recall the Serret-Frenet formula

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T},$$

where $\kappa(s)$ is "the signed curvature" at the point $\gamma(s)$.

Given a function $\kappa \in C^0(0,L)$, it is well known that there is a curve parametrized by the arc-length with curvature κ. This curve is unique up to a rigid motion and is explicitly given by

$$\gamma_\kappa(s) = \Big(\int_0^s \cos(K(\tau))d\tau, \int_0^s \sin(K(\tau))d\tau \Big), \quad s \in (0,L), \tag{2.2}$$

where $K(s) := \int_0^s \kappa(\tau)d\tau$. Furthermore, if the function $\kappa \in C^0(\mathbb{R})$ is L-periodic, then the associated curve γ_κ is *closed* and *simple* if and only if the following three conditions are satisfied:

$$\int_0^L \kappa = \pm 2\pi, \tag{2.3}$$

$$\int_0^L e^{iK(\tau)} = 0, \tag{2.4}$$

$$\int_a^b e^{iK(\tau)} \neq 0, \quad \forall 0 \le a < b < L. \tag{2.5}$$

The condition (2.3) is the "Hopf winding number Theorem" which holds for any closed simple curve, and the integral is positive (respectively negative) if the curve is positively oriented (respectively negatively oriented). This Hopf Theorem together with statement (2.4) imply that the functions $\gamma_\kappa, \gamma'_\kappa$ are L-periodic, and the last condition (2.5) ensures that the curve γ_κ does not self-intersect.

The following lemma provides sufficient conditions on a periodic function κ under which the curve γ_κ is closed and simple.

Lemma 2.1. *Let $\kappa \in C^0(\mathbb{R})$ be a periodic function of minimal period $p > 0$ satisfying for some integer $m \geq 2$*

$$\int_0^p \kappa(s)ds = \frac{2\pi}{m}. \tag{2.6}$$

Then the corresponding unit-speed curve γ_κ is closed and has length $P = mp$.
Furthermore, if $\kappa \geq 0$ then the curve γ_κ is simple.

The proof that condition (2.6) ensures that the curve γ_κ is closed can be found in [2]. The fact that this curve is simple under the additional assumption that $\kappa > 0$ is proved in [13, Cor. 5] (and the same proof holds for $\kappa \geq 0$).

Remark 2.2. **(i)** If κ changes sign, then under condition (2.6) the curve γ_κ is closed, but may self-intersect. Several examples can be found in [2].

(ii) When $m = 1$, some additional condition must be added. Indeed, we know by the four-vertex Theorem that the curvature must have at least two local minima and two local maxima.

2.2 Some identities in Fermi Coordinates

Since our domain satisfies (1.4), given a connected component $\Gamma \subset \partial\Omega$, there exists an open set U_Γ such that

$$\overline{U}_\Gamma \cap \partial\Omega = \Gamma.$$

Furthermore, there are simply connected analytic domains Ω_0, Ω_j^h ($j \in J$ with J countable or finite) which allow to write the domain Ω as follows

$$\Omega = \Omega_0 \setminus \bigcup_{j \in J} \Omega_j^h, \tag{2.7}$$

where the sets Ω_j^h are the holes, which are all empty when Ω is simply connected.

- If the domain Ω_0 is bounded, then J is finite and each component of $\partial\Omega$ is bounded,
- If the domain Ω_0 is unbounded, then at least one of $\partial\Omega_j^h$ is compact due to our hypothesis (1.4).

Given a connected component Γ of the boundary, we consider a smooth unit-speed parametrization $\gamma := \gamma_\Gamma$, $\gamma : (0, L) \to \Gamma$ where L is the length of Γ (infinite if Γ is not bounded). The curve γ is oriented in order to have $\mathbf{N}$ (as defined in (2.1)) to coincide with the inner normal

vector field along the boundary of Ω. Hence, γ is positively oriented if $\Gamma = \partial\Omega_0$, whereas it is negatively oriented when Γ is one of the component $\partial\Omega_j^h$.
We parametrize a neighborhood of Γ by using the following "Fermi coordinates" (s,d):

$$\mathbf{x}(s,d) = \gamma(s) + d\,\mathbf{N}, \quad (s,d) \in I \times [0,\varepsilon),$$

for some open interval $I \subset (0,L)$. In these coordinates, the first fundamental form is explicitly given by

$$g_{ij} = \begin{pmatrix} (1-\kappa d)^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where κ stands for the curvature of γ. Therefore, a straight computation shows that the expression of the Laplacian in the Fermi coordinates (s,d) is the following

$$\Delta u = \frac{1}{f}\left\{ \partial_s\Big(\frac{\partial_s u}{f}\Big) + \partial_d\big(f\partial_d u\big) \right\}, \tag{2.8}$$

where

$$f(s,d) = 1 - \kappa(s)d. \tag{2.9}$$

Expanding the relation (2.8) and multiplying by f^3, we see that the overdetermined problem (1.1) satisfies

$$\begin{cases} f\partial_{ss}u + f^3\partial_{dd}u + f^2(\partial_d f)(\partial_d u) - (\partial_s f)(\partial_s u) = -f^3\, g(u) \\ u = c, \quad \partial_s u = \partial_d u = 0 \quad \text{on } (0,L)\times[0,\varepsilon). \end{cases} \tag{2.10}$$

Differentiating the PDE n-times with respect to the variable d, we obtain

Lemma 2.3. *Along $\partial\Omega$ we have in the Fermi coordinates:*

$$\begin{aligned} &\partial_d^{(n+2)}u - (3n+1)\,\kappa\,\big(\partial_d^{(n+1)}u\big) + n[3n-1]\,\kappa^2\,\big(\partial_d^{(n)}u\big) - \; n[n-1]^2\,\kappa^3\,\big(\partial_d^{(n-1)}u\big) \\ &+ \partial_d^{(n)}\partial_{ss}u - n\kappa\,\big(\partial_d^{(n-1)}\partial_{ss}u\big) + n\kappa'(s)\,\big(\partial_d^{(n-1)}\partial_s u\big) \\ &= -\partial_d^{(n)}[g(u)] + 3n\,\kappa\,\partial_d^{(n-1)}[g(u)] - 3n(n-1)\,\kappa^2\,\partial_d^{(n-2)}[g(u)] + n(n-1)(n-2)\,\kappa^3\,\partial_d^{(n-3)}[g(u)] \end{aligned}$$

Proof. Note first that due to (2.9) f^m is a polynomial of degree m in the variable d (m an integer), and therefore $\partial_d^{m+k}f = 0$ for each $k \geq 1$.

We start by differentiating the first term in (2.10)

$$\begin{aligned} \partial_d^n\big(f\partial_{ss}u\big) &= \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} \big(\partial_d^j f\big)\,\big(\partial_d^{(n-j)}\partial_{ss}u\big) \\ &= f\,\big(\partial_d^{(n)}\partial_{ss}u\big) + n\big(\partial_d f\big)\,\big(\partial_d^{(n-1)}\partial_{ss}u\big). \end{aligned} \tag{2.11}$$

We compute the n-th partial derivative with respect to d of $f^3\partial_{dd}u+f^2(\partial_d f)(\partial_d u)$ as follows:

$$
\begin{aligned}
\partial_d^n\big(f^3\partial_{dd}u+f^2(\partial_d f)(\partial_d u)\big) &= \partial_d^n\big(f^3\partial_{dd}u+\frac{1}{3}(\partial_d f^3)(\partial_d u)\big)\\
&= \sum_{j=0}^{3}\left(\begin{array}{c} n\\ j\end{array}\right)\big(\partial_d^{(j)}f^3\big)\big(\partial_d^{(n-j+2)}u\big) +\frac{1}{3}\sum_{j=0}^{2}\left(\begin{array}{c} n\\ j\end{array}\right)\big(\partial_d^{(j+1)}f^3\big)\big(\partial_d^{(n-j+1)}u\big)\\
&= f^3\big(\partial_d^{(n+2)}u\big)+(n+\frac{1}{3})\big(\partial_d f^3\big)\big(\partial_d^{(n+1)}u\big)+\frac{n}{6}(3n-1)\big(\partial_d^{(2)}f^3\big)\big(\partial_d^{(n)}u\big)\\
&\qquad +\frac{n(n-1)^2}{6}\big(\partial_d^{(3)}f^3\big)\big(\partial_d^{(n-1)}u\big). \qquad (2.12)
\end{aligned}
$$

We now differentiate the term $(\partial_s f)(\partial_s u)$ in (2.10)

$$
\begin{aligned}
\partial_d^n\big((\partial_s f)(\partial_s u)\big) &= \sum_{j=0}^{n}\left(\begin{array}{c} n\\ j\end{array}\right)\big(\partial_d^{(j)}\partial_s f\big)\big(\partial_d^{(n-j)}\partial_s u\big)\\
&= \big(\partial_s f\big)\big(\partial_d^{(n)}\partial_s u\big)+n\big(\partial_d\partial_s f\big)\big(\partial_d^{(n-1)}\partial_s u\big). \qquad (2.13)
\end{aligned}
$$

Finally, we consider the term $f^3g(u)$ in the right hand-side of (2.10)

$$
\begin{aligned}
\partial_d^n\big(f^3\,g(u)\big) &= f^3\partial_d^{(n)}[g(u)]+n\big(\partial_d f^3\big)\big(\partial_d^{(n-1)}[g(u)]\big)+\frac{n(n-1)}{2}\big(\partial_d^{(2)}f^3\big)\big(\partial_d^{(n-2)}[g(u)]\big)\\
&\qquad +\frac{n(n-1)(n-2)}{6}\big(\partial_d^{(3)}f^3\big)\big(\partial_d^{(n-3)}[g(u)]\big). \qquad (2.14)
\end{aligned}
$$

Using (2.9), the partial derivatives of f at $d=0$ are given by

$$
f(s,0)=1,\quad \partial_d f^m(s,0)=-m\kappa(s),\quad \partial_d^{(2)}f^3(s,0)=6\kappa^2(s),\quad \partial_d^{(3)}f^3(s,0)=-6\kappa^3(s),
$$

$$
\partial_s f(s,0)=0,\quad \partial_{sd}f(s,0)=-\kappa'(s),
$$

in each (2.11)-(2.14), and adding those relations together, we conclude the proof of the Lemma. □

Using the boundary condition in (2.10), we deduce (by iteration)

1. $n=0$ gives

$$
\partial_{dd}u=-g(c). \qquad (2.15)
$$

2. $n = 1$ gives
$$\partial_d^{(3)} u + 4\kappa g(c) = 3\kappa g(c),$$
that is
$$\partial_d^{(3)} u = -\kappa g(c). \tag{2.16}$$

3. $n = 2$ gives
$$\partial_d^{(4)} u - 3\kappa^2 g(c) = g(c)g'(c) - 6\kappa^2 g(c),$$
or
$$\partial_d^{(4)} u = g(c)(-3\kappa^2 + g'(c)). \tag{2.17}$$

4. $n = 3$ gives
$$\partial_d^{(5)} u + 18\kappa^3 g(c) - 10\kappa(gg')(c) - \kappa'' g(c) \;=\; -8\kappa(gg')(c) + 6\,\kappa^3\, g(c),$$
or rearranged
$$\partial_d^{(5)} u = g(c)\big(\kappa'' - 12\kappa^3 + 2g'(c)\kappa\big). \tag{2.18}$$

Remark 2.4. The linear overdetermined Problem (1.3) with $\mu = 1$ admits in a disc radial solutions given by the Bessel function $J_0(r)$. In this case, we can check that when $J_0'(R) = 0$ then the following relations hold:
$$\frac{J_0''(R)}{J_0(R)} = -1, \quad \frac{J_0^{(3)}(R)}{J_0(R)} = \frac{1}{R}, \quad \frac{J_0^{(4)}(R)}{J_0(R)} = 1 - \frac{3}{R^2}, \quad \frac{J_0^{(5)}(R)}{J_0(R)} = \frac{12}{R^3} - \frac{2}{R}.$$
Note that the derivatives with respect to the Fermi coordinate d and the polar coordinate r are related by $\partial_d u = -\partial_r u$ along the circle of radius R. This is coherent with the identities (2.15) to (2.18). Furthermore, the smallest R_1 for which $J_0'(R_1) = 0$ is given by $R_1 = 3.8317$, for which we have
$$\frac{12}{R_1^3} - \frac{2}{R_1} < 0.$$
Consequently, whenever $J_0'(R) = 0$ (with $R \geq R_1$), we deduce that $\frac{12}{R^3} - \frac{2}{R} < 0$.

The identities (2.15) to (2.18) will be applied in the next section to show that Schiffer's conjecture holds if some higher normal derivative is assumed to be constant along the boundary. In that respect, we note that along the boundary the partial derivatives with respect to the distance variable d, and normal unit outward vector field ν are related to each other by the identity $\partial_d^{(n)} u \equiv (-1)^n \partial_\nu^{(n)} u$ (on $\partial\Omega$).

3 Preliminary results

We start with a general well known result (that holds in a domain $\Omega \subset \mathbb{R}^n$).

Lemma 3.1. *Given two constants $c, \tilde{c} \in \mathbb{R}$, consider the overdetermined problem*

$$-\Delta u = g(u), \quad u|_{\partial\Omega} = c, \quad \partial_\nu u|_{\partial\Omega} = \tilde{c}. \tag{3.1}$$

(i) *On an annulus $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$, the problem* (3.1) *has no radial solution when $g(c) \neq 0$ or $\tilde{c} \neq 0$.*

(ii) *If Ω and g are real analytic, then the problem* (3.1) *admits at most one solution which is real analytic up to the boundary.*

(iii) *If Problem* (3.1) *admits a solution in a real analytic domain such that $\partial\Omega$ admits a connected component which is a sphere, then the domain Ω is either a ball, or the complement of a ball (and the solution is radial).*

Proof. **(i)** If $u(r)$ is a radial solution, then multiplying the equation by $u'(r)$ and integrating we obtain

$$-\int_{R_1}^{R_2} \left\{ (\frac{u'^2}{2})' + (N-1)\frac{u'^2}{r} \right\} = \int_{R_1}^{R_2} [G(u)]',$$

where $G(s) := \int_0^s g(\tau)d\tau$. Since $G\big(u(R_1)\big) = G\big(u(R_2)\big)$ and $u'(R_1) = u'(R_2)$, the first and last term vanish and we obtain $\int_{R_1}^{R_2} \frac{u'^2}{r} = 0$. Hence, $u \equiv c$ which cannot be a solution when $g(c) \neq 0$ or $\tilde{c} \neq 0$.

(ii) This follows from Holmgren's Uniqueness Theorem.

(iii) Let Γ be a connected component of $\partial\Omega$ which is a sphere of radius R. Then, in a neighborhood U_Γ of Γ the Problem admits a radial solution U that is obtained by solving the Cauchy Problem

$$-U'' - (N-1)\frac{U'}{r} = g(U), \quad U(R) = c, \quad U'(R) = \tilde{c}.$$

Hence, $u \equiv U$ in U_Γ and by the unique continuation principle we must have $u \equiv U$ in Ω. Since $\partial\Omega$ must be a level set of U, we deduce that the domain is either a ball, an annulus or the complement of a ball. However the case of an annulus is excluded by part (i). $\square$

We now prove our first result concerning the overdetermined problem (1.1), namely that the second normal derivative is always constant, and that when $g(c) = 0$ the problem admits only the constant solution $u \equiv c$.

We will now focus on Problem (1.1) for which we make the following preliminary observation.

Proposition 3.2. *Let u be a solution to* (1.1).

- **(i)** *Then the second order normal derivative $\partial^2_{dd}u$ is constant along the boundary and it is given by $-g(c)$.*
- **(ii)** *If, moreover, $g(c) = 0$, then on the boundary $\partial\Omega$ any derivative (normal or not) of n-th order vanishes, i.e.*
$$\partial_s^m \partial_d^n u \equiv 0, \qquad \text{on } \partial\Omega, \tag{3.2}$$
for all integers $m, n \in \mathbb{N}$, $(m, n) \neq (0, 0)$. In particular, $u \equiv c$ whenever u is analytical up to the boundary.

Proof. **(i)** The identity (2.15) immediately implies that the second derivative $\partial^2_{dd}u$ is constant at each points with coordinates $(s, 0)$, and is given by $-g(c)$.

(ii) One can apply the uniqueness result in Lemma 3.1, part (ii). Or otherwise, one can also argue directly as follows. If $g(c) = 0$, then the identities (2.15) to (2.18) imply that $\partial_d^{(n)} u \equiv 0$ on $\partial\Omega$ for $n = 2, \cdots, 5$. Let us now argue by induction. Given an integer $n \in \mathbb{N}$ with $n \geq 3$, and assuming that the statement holds for each $k \leq n + 1$, let us first prove that we have $\partial_d^{(n+2)} u = 0$. Applying Lemma 2.3 and the induction assumption, we obtain

$$\partial_d^{(n+2)} u = \sum_{i=0}^{3} \gamma_i \partial^{(n-i)}[g(u)], \quad \text{on } \partial\Omega, \tag{3.3}$$

for some function γ_i. Now by using the chain rule for higher derivatives (Faà di Bruno formula) we get

$$\partial_d^{(m)}[g(u)] = \sum \left\{ \frac{m!}{k_1! \cdots k_m!} g^{(k)}(u) \prod_{i=1}^{n} \frac{1}{k_i!} \left(\frac{\partial_d^{(k-i)} u}{i!} \right)^{k_i} \right\}$$

where the sum is over all nonnegative integers satisfying $\sum_{j=1}^{m} j k_j = m$ and $k := \sum_{j=1}^{m} k_j$. Therefore, by the induction hypothesis, the right hand-side of (3.3) is zero. Therefore, we have $\partial_d^{(n+2)} u = 0$. So the induction argument implies that $\partial_d^{(n)} u \equiv 0$ on $\partial\Omega$ for all $n \in \mathbb{N}$.

In particular at each point of the boundary, we deduce that $\partial_d^{(\beta)} \partial_s^{(\alpha)} u = \partial_s^{(\alpha)} \partial_d^{(\beta)} u = 0$ Hence, we conclude that (3.2) holds. If the function u is furthermore analytic we deduce that u must be constant in $\bar{\Omega}$ and therefore $u \equiv c$. $\square$

Above result gives a full answer to the overdetermined Problem (1.1) when $g(c) = 0$. So, from now on we will consider the case when $g(c) \neq 0$.

Using the identities (2.16) and (2.17) we first prove that $\partial_d^{(3)} u$ or $\partial_d^{(4)} u$ is constant on the boundary if and only if the boundary of the domain is a circle:

Proof of Theorem 1.1: Let Γ be a bounded connected component of Ω.
(iii) $\Longleftrightarrow$ **(i)**: The identity (2.17) shows that $\partial_d^{(4)} u$ is constant on the boundary if and only if the curvature on Γ is constant and equals to $3\kappa^2 = g'(c) - \frac{\partial_d^{(4)} u}{g(c)}$. Hence, by Lemma 3.1 the domain is a disc (if Ω is bounded) or the complement of a disc (if Ω is unbounded).

The equivalence between **(i)** and **(ii)** can be proved in the same way using (2.16) and is already contained in [12] for a bounded domain. $\square$

4 Fifth normal derivative and curvature

Under the condition that Problem (1.1) admits a solution u, the identity (2.18) provides a relation between the fifth normal derivative of u and the curvature of $\partial\Omega$. By introducing the local Fermi coordinates in a neighborhood of each connected component Γ of $\partial\Omega$, and after defining

$$L := |\Gamma|, \quad \alpha(s) := \frac{\partial_d^{(5)} u}{g(c)}(s,0) \quad \text{and} \quad h(s,x) := -12x^3 + 2g'(c)x - \alpha(s) \tag{4.1}$$

along Γ, the identity (2.18) shows that the curvature $\kappa(s)$ along Γ solves

$$\begin{cases} -\kappa'' = h(s,\kappa), & \kappa \in C^2(\mathbb{R}), \\ \kappa, \kappa' \text{ are } L\text{-periodic}. \end{cases} \tag{4.2}$$

Furthermore, we are interested in solutions to (4.2) that represent the signed curvature of Γ which is a closed and simple curve, and consequently each of the conditions (2.3) to (2.5) must be satisfied. Therefore, by writing the domain as in (2.7) we look for solutions to (4.2) that satisfy the additional requirement

$$\int_0^L \kappa(s)ds = \begin{cases} 2\pi & \text{if } \Gamma = \partial\Omega_0, \\ -2\pi & \text{if } \Gamma \in \{\partial\Omega_j^h : j \in J\}. \end{cases} \tag{4.3}$$

Remark 4.1. **(i)** Note that (κ, α) satisfies (4.2) if and only if $(-\kappa, -\alpha)$ does.

(ii) If κ is a solution to Problem (4.2) with minimal period $0 < p < L$, then $p = \frac{L}{m}$ for some integer $m \geq 2$. In particular, if κ is a non-negative solution to (4.2) satisfying (4.3), then by Lemma 2.1 the associated curve γ_κ is closed and simple.

4.1 A variational formulation

To recast the ODE in (4.2) in a variational framework, we introduce the Sobolev space H^1_L defined as the subspace of $H^1_{loc}(\mathbb{R})$ consisting of L-periodic functions on $\mathbb{R}$ endowed with the usual inner product $\langle f, g\rangle := \int_0^L \{f'g' + fg\} ds$. Given a smooth L-periodic function α, we easily check that L-periodic solutions solving the ODE in (4.2) are critical points of the functional

$$J(\kappa) := J^L(\kappa) = \int_0^L \{\frac{\kappa'^2}{2} - H(s,\kappa)\}\, ds, \quad \kappa \in H^1_L \tag{4.4}$$

where

$$H(s,\kappa) := \int_0^\kappa h(s,\tau) d\tau = -3\kappa^4 + g'(c)\kappa^2 - \alpha(s)\kappa. \tag{4.5}$$

We first note that existence of periodic solutions, without any constraints on $\int_0^L \kappa(s)ds$, can be obtained by minimizing the functional (4.4).

Proposition 4.2. *Let $\alpha \in C^\infty(\mathbb{R})$ be a L-periodic function. Then the following holds.*

(i) *The functional J admits a minimizer $\kappa_0 \in H^1_L$. Furthermore,*

$$\alpha\kappa_0^+ \equiv 0 \;\; \text{if}\; \alpha \geq 0 \qquad \text{and} \qquad \alpha\kappa_0^- \equiv 0 \;\; \text{if}\; \alpha \leq 0. \tag{4.6}$$

(ii) *If $g'(c) \leq 0$, then J admits at most one critical point.*

(iii) *If $g'(c)L^2 \leq 2\pi^2$, then for each $M \in \mathbb{R}$ the functional J admits at most one critical point satisfying $\int_0^L \kappa = M$.*

Proof. **(i)** We can check that this functional is coercive and lower semi-continuous in the Hilbert space H^1_L. Hence, standard arguments show that J admits a minimizer κ_0.
If $\alpha \geq 0$ and $\alpha\kappa_0^+ \not\equiv 0$ then $J(-|\kappa_0|) < J(|\kappa_0|)$ in contradiction to the minimizing property of κ_0. Therefore, we conclude that $\int_0^L \alpha\kappa_0^+ = 0$ and thus $\alpha\kappa_0^+ \equiv 0$. The same argument shows that $\alpha\kappa_0^- \equiv 0$ whenever $\alpha \leq 0$.

(ii) Assume the existence of two distinct critical points f and g in H^1_L, and consider the function $\varphi(t) := J\big(tf + (1-t)g\big)$ for $t \in [0,1]$ which satisfies

$$\varphi''(t) = \int_0^L \Big\{|(f' - g')|^2 + \big(36\,(tf + [1-t]g)^2 - 2g'(c)\big)\big(f-g\big)^2\Big\}. \tag{4.7}$$

Since $g'(c) \le 0$, we have

$$\varphi''(t) \ge 0, \quad \varphi'(0) = \varphi'(1) = 0. \tag{4.8}$$

Thus, φ is constant on the interval $[0,1]$ and in particular $\varphi'' \equiv 0$ on $[0,1]$. Since $f \not\equiv g$, and $g'(c) \le 0$, equality (4.7) implies $(tf + [1-t]g)^2 \equiv 0$. Therefore, $f \equiv g \equiv 0$, a contradiction.

(iii) Assume there are two distinct critical points f and g in H^1_L satisfying $\int_0^L f = \int_0^L g = M$, and consider as in (ii) the function $\varphi(t) := J\big(tf + (1-t)g\big)$ for $t \in [0,1]$. Then

$$\varphi''(t) \ge \int_0^L \Big\{|(f' - g')|^2 - 2g'(c)\big(f-g\big)^2\Big\} \ge \Big(\big(\frac{2\pi}{L}\big)^2 - 2g'(c)\Big)\int_0^L \big(f-g\big)^2,$$

where in the last inequality we have used the Wirtinger inequality with the L-periodic $f-g$ which has average zero. Therefore, since $\big(\frac{2\pi}{L}\big)^2 \ge 2g'(c)$ by assumption, the function φ satisfies (4.8). As in part (ii), we deduce $\big(\frac{2\pi}{L}\big)^2 = 2g'(c)$ and $(tf + [1-t]g)^2 \equiv 0$ for all $t \in [0,1]$. Hence $f \equiv g \equiv 0$, which contradicts the fact that f, g are assumed to be distinct. $\square$

The above result has the following implication on Problem (1.1) and provides a proof to our Theorem 1.2.

Proposition 4.3. (The case $g'(c) \le 0$)
Assume that Problem (1.1) *admits a solution u with $g(c) \neq 0$ and $g'(c) \le 0$ in a connected domain that we write as in* (2.7). *Then, the following holds.*

(i) *If $\partial\Omega_0$ is bounded and the set $\alpha^{-1}(0)$ has one-dimensional Hausdorff measure zero in $\partial\Omega_0$, then $\alpha^- \not\equiv 0$ in $\partial\Omega_0$.*

(ii) *If $\partial\Omega_j^h$ is bounded and the set $\alpha^{-1}(0)$ has one-dimensional Hausdorff measure zero in $\partial\Omega_j^h$, then $\alpha^+ \not\equiv 0$ in $\partial\Omega_j^h$.*

(ii) *When α is constant, then $\alpha \neq 0$. Furthermore, if $\alpha < 0$ then Ω is a disc, whereas if $\alpha > 0$ then Ω is the complement of a disc. In both cases the curvature of this disc is the unique solution to $\alpha = -12\kappa^3 + g'(c)\kappa$.*

Proof. When $g'(c) \leq 0$, then the signed curvature κ of each bounded connected component of $\partial\Omega$ solves Problem (4.2) and is given by the unique minimizer of the functional J (by part (ii) of Proposition 4.2).

(i) and **(ii)** If $\alpha \geq 0$ and has a zero set of measure zero in $\partial\Omega_0$ assumed to be bounded, then we deduce from (4.6) that $\kappa_0 \leq 0$ a.e.. Therefore, $\int_0^L \kappa_0 \leq 0$ in contradiction to the first requirement in (4.3). Hence, $\alpha^- \not\equiv 0$. The same arguments shows that $\alpha^+ \not\equiv 0$ on each bounded connected component $\partial\Omega_j^h$.

(iii) When $\alpha \equiv 0$, then the unique solution to the ODE in (4.2) is given by $\kappa_0 \equiv 0$, which does not satisfy the first constraint of (4.3) either. Hence, $\alpha \not\equiv 0$.

If $\alpha < 0$, then the unique critical point of J is given by $\kappa_0 \equiv$ *const* given by the unique (positive) zero of h. If there is a non-empty hole Ω_j^h , then $\alpha < 0$ on $\partial_j^h\Omega$, in contradiction to (i). Hence, Ω is a disc.

The same arguments show that when $\alpha > 0$ then $\Omega = \mathbb{R}^2 \setminus D$ where D is a disc whose curvature is given by the unique (negative) zero of h. $\square$

4.2 Some estimates on the curvature

Since Problem (1.1) is assumed to have a solution u, the curvature κ of each bounded connected component $\Gamma \subset \partial\Omega$ must satisfy (4.2) with $\alpha := \frac{\partial_d^{(5)} u}{g(c)}$ and $L := |\Gamma|$. Hence, some estimates on the curvature of the domain Ω can be obtained by studying the solutions to the ODE in (4.2).

Proposition 4.4. *Let κ be a solution to (4.2) with $\int_0^L \kappa = 2\pi$. Then, the following hold.*

(i) *If $\alpha \geq 0$ in $[0, L]$, then $g'(c)$ must be positive,*

$$\frac{2\pi}{L} \leq \max_{(0,L)} \kappa \leq \sqrt{\frac{g'(c)}{6}} \qquad \text{and} \qquad g'(c)L^2 \geq 24\pi^2, \tag{4.9}$$

and those inequalities are strict unless $\kappa \equiv \frac{2\pi}{L}$.

(ii) *If $\alpha < 0$ in $[0, L]$, then $g'(c)\kappa_{min} < 6\kappa_{min}^3$, and if κ changes sign, we have $g'(c) > 6\kappa_{min}^2 > 0$.*

(iii) *If $\kappa \geq 0$, then we have*

$$\int_0^L \alpha \leq \frac{-96\pi^3}{L^2} + 4\pi g'(c),$$

and equality holds if and only if $\kappa \equiv \frac{2\pi}{L}$.

Proof. **(i)** By the maximum principle, setting $M := \max_{s\in(0,L)} \kappa(s)$ we deduce that

$$-12M^3 + 2g'(c)M \geq \frac{\partial_d^{(5)} u}{g(c)} \geq 0. \tag{4.10}$$

Since $\kappa^+ \not\equiv 0$ on $\partial\Omega$ by assumption, we have $M > 0$ and $0 < 6M^2 \leq g'(c)$ by (4.10). This shows that $\kappa \leq \sqrt{\frac{g'(c)}{6}}$. To discuss the equality case, we set $w := \kappa - \sqrt{\frac{g'(c)}{6}}$, and note that $\sqrt{\frac{g'(c)}{6}}$ is a zero of $P(x) := 12x^3 - 2g'(c)x$. Therefore, we have

$$w'' - 12\kappa\left(\kappa + \sqrt{\frac{g'(c)}{6}}\right) w = \alpha \geq 0 \quad \text{and} \quad w \leq 0 \quad \text{on } [0, L].$$

Hence, the strong maximum principle implies $w \equiv 0$ in an open neigborhood of a point where $w = 0$. Therefore, either $\kappa < \sqrt{\frac{g'(c)}{6}}$ on $[0, L]$, or $\kappa \equiv \sqrt{\frac{g'(c)}{6}}$ and in this latter case $\kappa \equiv \frac{2\pi}{L} = \sqrt{\frac{g'(c)}{6}}$.

Finally, the lower bound on $\max\kappa$ in (4.9) follows from (2.3), and we note that equality holds if and only if $\kappa \equiv \frac{2\pi}{L}$.

(ii) This follows from (2.18) analogous to case (i).

(iii) Integrating (4.2) on $(0, L)$, and using (2.3) we get (by setting $\alpha := \frac{\partial_d^{(5)} u}{g(c)}$)

$$12 \int_0^L \kappa^3 = 4\pi g'(c) - \int_0^L \alpha(s)ds. \tag{4.11}$$

If $\kappa \geq 0$, and Jensen's inequality and (2.3) imply

$$\int_0^L \kappa^3 \geq \frac{1}{L^2}\Big(\int_0^L \kappa\Big)^3 = \frac{8\pi^3}{L^2}, \tag{4.12}$$

with equality holding everywhere in (4.12) if and only if κ is constant. Hence, (4.11) and (4.12) give

$$\int_0^L \alpha(s)ds \leq \frac{-96\pi^3}{L^2} + 4\pi g'(c),$$

with strict inequality unless κ is constant, and in this case $\kappa \equiv \frac{2\pi}{L}$ (since $\int_0^L \kappa = 2\pi$). □

In order to further exploit the property (2.3), we will derive an identity that involves test function of the type $\varphi(K(s))$.

Lemma 4.5. *Let κ be a solution to (4.2). Defining $M := \int_0^L \kappa$ and for $a \in [0, L]$ setting $K(s) := \int_a^s \kappa(\tau)d\tau$ implies*

$$\boxed{\begin{aligned} k'(a)\big(\varphi(M) - \varphi(0)\big) - \tfrac{k^2(a)}{2}\big(\varphi'(M) - \varphi'(0)\big) + 2g'(c)\big(\Phi(M) - \Phi(0)\big) \\ + \textstyle\int_a^{a+L} \kappa^3\big(\tfrac{\varphi''(K)}{2} - 12\varphi(K)\big) = \int_a^{a+L} \alpha(s)\varphi(K). \end{aligned}} \tag{4.13}$$

Proof. Use $\varphi \circ K$ as a test function in the weak formulation of the ODE (4.2), with $\varphi \in C^2(\mathbb{R})$. Integrating by parts on the interval $(a, a+L)$, and noting that $K(a) = 0$ and $K(a+L) = M$, we obtain

$$k'(a)\big(\varphi(M) - \varphi(0)\big) - \frac{k^2(a)}{2}\big(\varphi'(M) - \varphi'(0)\big) + \int_a^{a+L} \frac{\kappa^3}{2}\varphi''(K) + \int_a^{a+L} h(\kappa)\varphi(K) = 0.$$

Therefore, using the explicit definition of h (see (4.1)), we obtain

$$\begin{aligned} k'(a)\big(\varphi(M) - \varphi(0)\big) - \frac{k^2(a)}{2}\big(\varphi'(M) - \varphi'(0)\big) \\ + \int_a^{a+L} \kappa^3\Big(\frac{\varphi''(K)}{2} - 12\varphi(K)\Big) + 2g'(c)\int_a^{a+L} \kappa\varphi(K) = \int_a^{a+L} \alpha(s)\varphi(K). \end{aligned}$$

By setting $\Phi(x) := \int_0^x \varphi(\tau)d\tau$, the last term on the left hand side can be written as

$$\int_a^{a+L} \kappa\varphi(K) = \int_a^{a+L} \frac{d}{ds}[\Phi(K(s))]ds = \Phi(M) - \Phi(0).$$

Therefore, the identity (4.13) follows. $\square$

Example 4.6. Assume Problem (1.1) admits a solution u with $g(c) \neq 0$, and such that α is constant. Then, the curvature κ of each connected component of $\partial\Omega$ satisfies (4.2) with (4.3). Thus by applying (4.13) with the functions $\varphi(s) = \cos s$ and $\varphi(s) = \sin s$ and $M = \pm 2\pi$, we obtain

$$\begin{cases} -\frac{25}{2}\int_a^{a+L} \kappa^3 \cos(K(s)) = \alpha \int_a^{a+L} \cos(K(s))ds, \\ -\frac{25}{2}\int_a^{a+L} \kappa^3 \sin(K(s)) = \alpha \int_a^{a+L} \sin(K(s))ds. \end{cases} \tag{4.14}$$

Since the functions $s \mapsto \cos(\int_a^s \kappa)$ and $s \mapsto \sin(\int_a^s \kappa)$ are L-periodic (follows from (2.3)), we have

$$\int_a^{a+L} \cos(K(s)) = \int_0^L \cos(\int_a^s \kappa) \overset{(2.4)}{=} 0, \quad \int_a^{a+L} \sin(K(s)) = \int_0^L \sin(\int_a^s \kappa) \overset{(2.4)}{=} 0.$$

Thus the right hand-sides in (4.14) are zero, and therefore we obtain

$$\int_a^{a+L} \kappa^3 \sin(K(s))ds = 0, \qquad \int_a^{a+L} \kappa^3 \cos(K(s))ds = 0. \tag{4.15}$$

Instead of choosing trigonometric functions as in Example 4.6, by considering test functions of the type $\varphi(s) := e^{as}$ we obtain the following:

Proposition 4.7. *Let κ be a solution to (4.2), and set $M := \int_0^L \kappa$. Then, if $M \neq 0$ we have*

$$\begin{cases} \kappa^2(a) = \dfrac{g'(c)}{6} - C_M \displaystyle\int_a^{a+L} \alpha(s)\Big(e^{\sqrt{24}\int_a^s \kappa} + e^{\sqrt{24}\int_s^{a+L} \kappa}\Big)ds \\ k'(a) = \sqrt{6}\, C_M \displaystyle\int_a^{a+L} \alpha(s)\Big(e^{\sqrt{24}\int_a^s \kappa} - e^{\sqrt{24}\int_s^{a+L} \kappa}\Big)ds \end{cases} \tag{4.16}$$

where $C_M := \frac{1}{\sqrt{24}(e^{M\sqrt{24}}-1)}$.

Proof. By applying (4.13) with the two functions $\varphi(s) = e^{\sqrt{24}s}$ and $\varphi(s) = e^{-\sqrt{24}s}$ (which satisfy $\frac{\varphi''}{2} - 12\varphi = 0$), we obtain the following system of two equations

$$(e^{M\sqrt{24}} - 1)\begin{pmatrix} -\sqrt{24} & 1 \\ -\sqrt{24}e^{-M\sqrt{24}} & -e^{-M\sqrt{24}} \end{pmatrix}\begin{pmatrix} \frac{\kappa^2(a)}{2} \\ \kappa'(a) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \tag{4.17}$$

where we have set

$$\begin{aligned} A &:= -\frac{2g'(c)}{\sqrt{24}}(e^{M\sqrt{24}} - 1) + \int_a^{a+L} \alpha(s)e^{\sqrt{24}K(s)}ds, \\ B &:= -\frac{2g'(c)}{\sqrt{24}}(1 - e^{-M\sqrt{24}}) + \int_a^{a+L} \alpha(s)e^{-\sqrt{24}K(s)}ds. \end{aligned}$$

By solving the system (4.17), we derive the identity (4.16). □

Proposition 4.8. *Assume Problem* (1.1) *admits a solution u with $g(c) \neq 0$, $g'(c) > 0$ and let κ be the curvature along the boundary of the domain Ω.*

- **(i)** *If α does not change sign on a bounded connected component Γ of $\partial\Omega$, then $\alpha \equiv 0$ on Γ if and only if $\kappa^2(p) = \frac{g'(c)}{6}$ at one point $p \in \Gamma$, and in this case $\kappa^2 \equiv \frac{g'(c)}{6}$ on Γ.*
- **(ii)** *If Ω is bounded and $\alpha \leq 0$ with $\alpha \not\equiv 0$, then Ω is simply connected and the curvature of $\partial\Omega$ satisfies $\kappa > \sqrt{\frac{g'(c)}{6}}$.*

(iii) *Assume $\alpha \geq 0$ with $\alpha \not\equiv 0$. If $\partial\Omega_j^h$ (see (2.7)) is not empty and compact, then the curvature satisfies $|\kappa| > \sqrt{\frac{g'(c)}{6}}$ along $\partial\Omega_j^h$ and in particular Ω_j^h is convex.*

Proof. If Problem (1.1) admits a solution u, then the curvature κ satisfies (4.2) with (4.3).

(i) The statement follows by applying the first identity in (4.16) (with $M = \pm 2\pi$).

(ii) Write the domain Ω as in (2.7) and assume $\Omega_j^h \neq \emptyset$. We apply (4.16) with $M = \pm 2\pi$, and we note that $MC_M > 0$. Hence, (4.16) gives

$$\kappa^2 > \frac{g'(c)}{6} \text{ on } \partial\Omega_0 \quad \text{ and } \quad \kappa^2 < \frac{g'(c)}{6} \text{ on } \partial\Omega \setminus \partial\Omega_0.$$

This is only possible if the collection of holes $\bigcup_{j=1}^{n} \Omega_j^h$ is empty. In fact it follows from [10, Lemma 2.2] that a disc of radius $\sqrt{\frac{6}{g'(c)}}$ contains Ω_0, while Ω_j^h contains a disc of the same radius if it were not empty. Hence, Ω is simply connected.

Furthermore, since $\kappa^+ \not\equiv 0$ (which follows from (4.3)), we deduce $\kappa > 0$ on $\partial\Omega$ (Note that Prop. 4.4 part **(ii)** provides this lower bound only when the curvature is known to be positive).

(iii) Assume $\alpha \geq 0$. Since $C_M < 0$ when $M = -2\pi$ the conclusion follows from (4.16). $\square$

5 Constant fifth normal derivative

When $g'(c) \leq 0$, we have already proved that there is only one L-periodic solution to the ODE (4.2). As a consequence, we proved that Problem (1.1) admits a solution with constant fifth normal derivative on the boundary, then $\partial\Omega$ must be a circle (see Proposition 4.3).

Remark 5.1. Proposition 4.3 was obtained by showing that the functional associated to the ODE (4.2) is convex and admits only one critical point. When α is constant, one can derive this uniqueness result by arguing as follows. Evaluate (4.2) in points where κ attains its maximum κ_M and its minimum k_m and subtract the resulting inequalities for κ'' to arrive at

$$\frac{g'(c)}{6}(\kappa_M - \kappa_m) \geq \kappa_M^3 - \kappa_m^3 = (\kappa_M - \kappa_m)(\kappa_M^2 + \kappa_M\kappa_m + \kappa_m^2). \tag{5.1}$$

So either κ is constant and $\kappa_M = \kappa_m$ or $g'(c) \geq 3(\kappa_M^2 + \kappa_m^2) > 0$.

So, from now on we assume that $g'(c) > 0$, and we can treat the cases $\alpha \leq 0$ and $\alpha > 0$ separately. We first note that when α is constant, Proposition 4.7 implies the following.

Lemma 5.2. *Let κ be a solution to* (4.2) *with $g'(c) > 0$ and α constant. If $\alpha \int_0^L \kappa < 0$, then κ is constant.*

Proof. Set $M := \int_0^L \kappa$. Noting that $\displaystyle\int_a^{a+L} \varphi(K) = \int_0^L \varphi\Big(\int_a^{a+\tau} \kappa(\xi) d\xi \Big) d\tau$, by differentiating the second identity in (4.16) with respect to the variable a, we obtain

$$\kappa''(a) = \sqrt{24}\, C_M\, \alpha \int_0^L \Big(e^{\sqrt{24}\int_a^{a+s}\kappa} + e^{\sqrt{24}\int_{a+s}^{a+L}\kappa} \Big) (\kappa(a+s) - \kappa(a)) ds. \tag{5.2}$$

At a minimum point $a \in [0, L]$ of κ, we have $\kappa''(a) \geq 0$, and $\kappa(a+s) - \kappa(a) \geq 0$ for any $s \in [0, L]$. Since $\alpha M < 0$, we have $\alpha C_M < 0$ (see the definition of C_M in Proposition 4.7). So the nonnegative integrand in (5.2) is identically zero, i.e. $\kappa(a+s) - \kappa(a) = 0$ for all $s \in [0, L]$. Therefore κ is constant. □

Proposition 5.3. *Assume Problem* (1.1) *admits a solution u with $g(c) \neq 0$ and $g'(c) > 0$.*

- **(i)** *If $\alpha \leq 0$ is constant, and Ω is bounded, then Ω is a disc.*
- **(ii)** *If $\alpha > 0$ is constant, and Ω is not simply-connected then Ω is the complement of a disc.*

Proof. **(i)** We already know from Prop. 4.8 that Ω is simply connected (and so $\partial\Omega$ has only one connected component). Since the curvature κ of $\partial\Omega$ satisfies (4.2) and (4.3), Lemma 5.2 implies that κ is constant, and so Ω is a disc.
The same argument allows to prove part **(ii)**. Indeed writing the domain as in (2.7) we see that the signed curvature κ of $\partial\Omega_j^h$ satisfies $\alpha \int_0^L \kappa < 0$ (when $\alpha > 0$). Hence Lemma 5.2 gives that $\partial\Omega_j^h$ is a disc, and by Lemma 3.1 we conclude that Ω is the complement of a disc. □

To discuss the remaining cases $\alpha > 0$ with Ω bounded, and $\alpha < 0$ with Ω unbounded, we note the following:

Lemma 5.4. *Assume Problem* (1.1) *admits a solution u on a domain Ω. Let Γ be a bounded connected component of $\partial\Omega$ and set L to be the length of Γ. Then the rescaled function, $\tilde{u}(\cdot) := u(\frac{\cdot}{\eta})$ (with $\eta > 0$ fixed) is a solution to the overdetermined problem* (1.1) *in the domain $\tilde{\Omega} := \eta\Omega$ and the curvature $\tilde{\kappa}$ of $\tilde{\Gamma} := \eta\Gamma$ of perimeter $\tilde{L} := \eta L$ satisfies*

$$\begin{cases} -\tilde{\kappa}'' + 12\tilde{\kappa}^3 - \frac{2g'(c)}{\eta^2}\tilde{\kappa} + \frac{\alpha}{\eta^5} = 0, \quad \kappa \in C^2(\mathbb{R}), \\ \tilde{\kappa}, \tilde{\kappa}' \text{ are } \tilde{L}\text{-periodic}, \\ \int_0^{\tilde{L}} \tilde{\kappa}(s) ds = \pm 2\pi, \end{cases} \tag{5.3}$$

where the sign in the last requirement is chosen accordingly to (4.3)*.*

Proof. Writing $u(x) = \tilde{u}(\eta x)$ in the Problem (1.1), we obtain

$$-\Delta \tilde{u} = \frac{g(\tilde{u})}{\eta^2} \text{ in } \tilde{\Omega}, \qquad \tilde{u} = c \text{ on } \partial\tilde{\Omega}, \qquad \partial_\nu \tilde{u} = 0 \text{ on } \partial\tilde{\Omega},$$

and its fifth normal derivative on the boundary is given by $\partial_d^{(5)} \tilde{u} = \frac{1}{\eta^5} \partial_d^{(5)} u = \frac{1}{\eta^5}\alpha$. Hence, the same arguments that lead to the problem (4.2) apply to $\tilde{u}$ and show that (5.3) hold. □

Since $(\tilde{\kappa}, \alpha)$ solves (5.3) if and only if $(-\tilde{\kappa}, -\alpha)$ does, it is enough to study the structure of solutions to (5.3) when

$$\alpha > 0 \quad \text{and} \quad \int_0^{\tilde{L}} \tilde{\kappa} = 2\pi.$$

Now, we observe that we can choose $\eta_0 > 0$ for which $\frac{2\pi}{\tilde{L}} (= \frac{2\pi}{\eta_0 L})$ is a solution to the Problem (5.3). Requiring that $\kappa := \frac{2\pi}{\eta_0 L}$ is a solution to the ODE (5.3) gives

$$\eta_0^2 = \frac{1}{4\pi} \frac{\alpha L^3}{g'(c)L^2 - 24\pi^2}$$

which is well defined since (4.9) holds whenever $\alpha > 0$. By setting $\gamma := \frac{g'(c)}{\eta_0^2}$ we note that

$$g'(c)L^2 = \gamma \tilde{L}^2. \tag{5.4}$$

Hence, given $\gamma > 0$ it is enough for our purpose to study the problem

$$\begin{cases} -\tilde{\kappa}'' + 12\tilde{\kappa}^3 - 2\gamma\tilde{\kappa} + \tilde{\alpha} = 0, \quad \tilde{\kappa} \in C^2(\mathbb{R}), \\ \tilde{\alpha} := 2\gamma(\frac{2\pi}{\tilde{L}}) - 12(\frac{2\pi}{\tilde{L}})^3 > 0, \\ \tilde{\kappa}, \tilde{\kappa}' \text{ are } \tilde{L}\text{-periodic}, \\ \int_0^{\tilde{L}} \tilde{\kappa}(s)ds = 2\pi. \end{cases} \tag{5.5}$$

For this reformulated problem and in view of (4.9) we are led to the following question: *Given $\gamma\tilde{L}^2 \geq 24\pi^2$, is the constant solution $\tilde{k} = \frac{2\pi}{\tilde{L}}$ the unique $\tilde{L}$-periodic solution to the Problem* (5.5) *?*

We provide a first answer to that question under the additional assumption $\tilde{\kappa} \geq 0$.

Proposition 5.5. **(i)** *Let $\tilde{\kappa} \geq 0$ be a solution to* (5.5)*. Then, $\tilde{\kappa} \equiv \frac{2\pi}{\tilde{L}}$.*

(ii) *As a consequence, assume Problem* (1.1) *admits a solution u with $g(c) \neq 0$, $g'(c) > 0$ and constant α, in a domain Ω which has a bounded connected component $\Gamma \subset \partial\Omega$ whose curvature does not change sign. Then $\partial\Omega$ is a circle.*

Proof. **(i)** Integrating the ODE in (5.5) on the interval $[0, \tilde{L}]$ we obtain

$$12 \int_0^{\tilde{L}} \tilde{\kappa}^3 = 4\pi\gamma - \tilde{\alpha}\tilde{L} = \frac{96\pi^3}{\tilde{L}^2}. \tag{5.6}$$

If $\tilde{\kappa} \geq 0$ the left hand-side can be bounded from below by Jensen's inequality

$$12 \int_0^{\tilde{L}} \tilde{\kappa}^3 \geq \frac{12}{\tilde{L}} \Big(\int_0^{\tilde{L}} \tilde{\kappa} \Big)^3 = \frac{96\pi^3}{\tilde{L}^2}. \tag{5.7}$$

Hence, by comparing (5.7) with (5.6), we deduce that $\tilde{\kappa}$ must realize the equality in the Jensen's inequality, so that $\tilde{\kappa}$ must be constant.

(ii) Write the domain as in (2.7).
Assume first that the domain is bounded. Then, it is enough to consider the case where $\alpha > 0$ and the domain Ω is simply connected (the other cases are covered by Proposition 5.3). By assumption the curvature of $\partial\Omega$ satisfies $\kappa \geq 0$. Then, referring to Lemma 5.4, after scaling the curvature of $\partial\tilde{\Omega}$ solves the problem (5.5). Hence, by previous argument, $\tilde{\kappa} \equiv \frac{2\pi}{\tilde{L}}$, which implies that $\tilde{\Omega}$ is a disc (and therefore also Ω).

If the domain is unbounded, then by assumption the curvature of one bounded connected component $\partial\Omega_j^h$ satisfies $\kappa \leq 0$. It is enough to consider the case $\alpha < 0$ (the case $\alpha \geq 0$ is covered by Proposition 5.3). Applying above arguments with $(-\kappa, -\alpha)$ we deduce that $\partial\Omega_j^h$ is a circle, and Lemma 3.1 shows that the domain Ω is the complement of a circle. $\square$

Note that the problem (5.5) seems to depend on two parameters $(\gamma, \tilde{L})$. But by doing a proper scaling we can reduce it to a problem depending only on the parameter

$$T := \sqrt{2\gamma}\tilde{L}. \tag{5.8}$$

More specifically, consider the periodic function κ_0 defined through

$$\kappa_0(t) := \frac{1}{\sqrt{2\gamma}} \kappa\Big(\frac{t}{\sqrt{2\gamma}}\Big)$$

Then, substituting $\kappa(s) = \sqrt{2\gamma}\kappa_0(\sqrt{2\gamma}s)$ in (5.5) and by setting $T := \sqrt{2\gamma}\tilde{L}$, we see that κ_0

is a T-periodic satisfying

$$\begin{cases} -\kappa_0'' + 12\kappa_0^3 - \kappa_0 + \alpha_0 = 0, \quad \tilde{\kappa} \in C^2(\mathbb{R}), \\ \alpha_0 := (\frac{2\pi}{T}) - 12(\frac{2\pi}{T})^3 > 0, \\ \kappa_0, \kappa_0' \text{ are } T\text{-periodic}, \\ \int_0^T \kappa_0(t)dt = 2\pi. \end{cases} \tag{5.9}$$

For this rephrased problem, our main question is the following: *Is* $\frac{2\pi}{T}$ *the unique* T*-periodic solution satisfying* (5.9)*?*

In order to give (for the remaining case $g'(c) > 0$, $\alpha > 0$) a uniqueness result that does not assume any sign on the solution κ_0, we undertake a phase plane analysis of the ODE in (5.9).

5.1 A phase plane analysis

In this last subsection, we still assume $\alpha > 0$ and do a preliminary phase plane analysis of the problem

$$y'' = 12y^3 - y + \alpha. \tag{5.10}$$

We first consider a general parameter $\alpha > 0$, and then restrict our attention to the case where α takes the value α_0 defined in (5.9).

By setting $(q(s), p(s)) := (\kappa(s), \kappa'(s))$ the problem (5.10) is equivalent to the following first order ODE in the phase space

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \partial_p \mathcal{H} \\ -\partial_q \mathcal{H} \end{pmatrix} \tag{5.11}$$

where the Hamiltonian $\mathcal{H}$ is explicitly given by

$$\mathcal{H}(q,p) := \frac{p^2}{2} + \underbrace{\Big(-3q^4 + \frac{q^2}{2} - \alpha q \Big)}_{:=H(q)}.$$

The system (5.11) always admits constant solutions $(q, 0)$, where q is a zero of the polynomial $h(q) := -12q^3 + q - \alpha$. Since $\mathcal{H}$ is a constant of the motion, finding a periodic solution to (5.11) which are non-constant is equivalent to finding a closed curve contained in some level set $\{\mathcal{H} = E\}$, which can occur if and only if the function

$$\Psi_E : q \mapsto -H + E, \tag{5.12}$$

satisfies for some $q_1, q_2 \in \mathbb{R}$

$$\Psi_E(q_1) = \Psi_E(q_2) = 0 \quad \text{and} \quad \Psi_E(q) > 0 \quad \forall q \in (q_1, q_2). \tag{5.13}$$

Remark 5.6. The function H is a fourth order polynomial with $\lim_{q\to\pm\infty}(-H(q)) = +\infty$. So the condition (5.13) is satisfied for some $E \in \mathbb{R}$ if and only if H has three distinct critical points, namely if the cubic polynomial $h := H'$ has three distinct zeros.

The number of critical points of the function H depends on the parameter α, as one can see by looking at the graphs of $-H$ plotted in FIgure 1 for $\alpha \in \{\frac{1}{9}, \frac{1}{11}, \frac{1}{15}, \frac{1}{200}\}$.

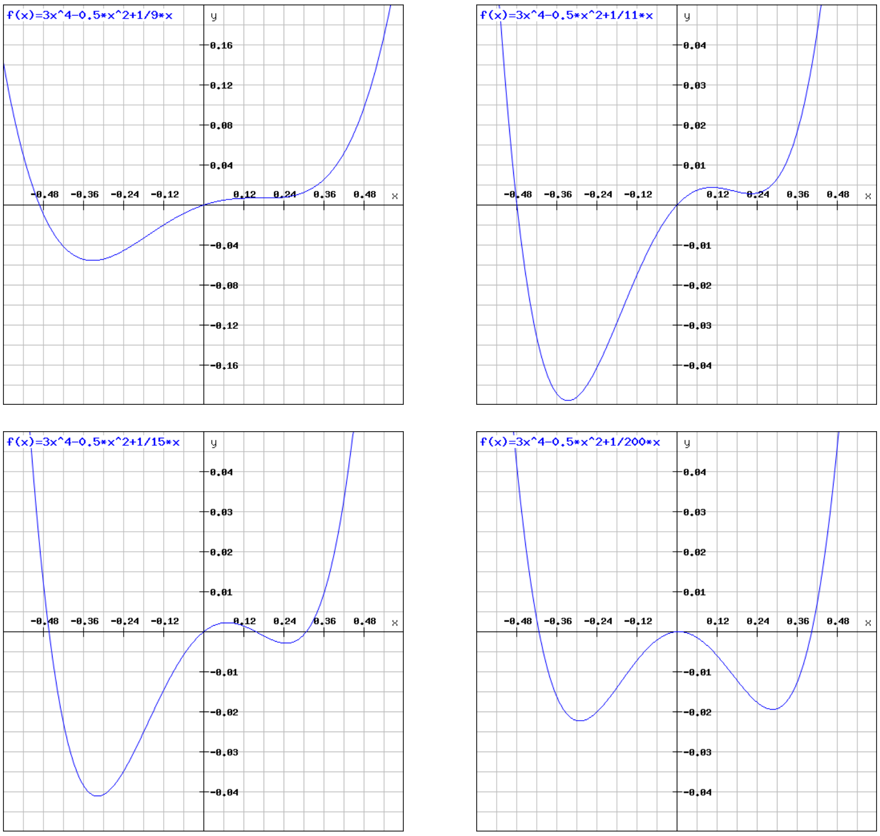

Figure 1: Graph of $-H$ for $\alpha = 1/9,\ 1/11,\ 1/15$ and $1/200$

In the following lemma (stated only for $\alpha > 0$ since this is the range of interest for our goals), we collect some properties of the function H.

Lemma 5.7. *Let $\alpha > 0$. Then, the following hold.*

(i) *The polynomial h has three distinct zeros if and only if $\alpha \in (0, \frac{1}{9})$. Furthermore, in this case the zeros $q_0(\alpha), q_\pm(\alpha)$ satisfy $q_-(\alpha) < 0 < \min\{q_0(\alpha), q_+(\alpha)\}$.*

(ii) *The ODE (5.10) admits non-constant periodic solutions if and only if $\alpha \in (0, \frac{1}{9})$.*

(iii) *When $\alpha \in [\frac{1}{9\sqrt{2}}, \frac{1}{9})$, the non-constant periodic solutions to the ODE* (5.10) *are all non-negative.*

Proof. **(i)** We have $h'\big(\pm\frac{1}{6}\big) = 0$, and the cubic polynomial has three different zeros if and only if

$$h\Big(-\frac{1}{6}\Big) > 0 \quad \text{and} \quad h\Big(\frac{1}{6}\Big) < 0.$$

These two conditions are are satisfied if and only if $|\alpha| < \frac{1}{9}$. Since we assume $\alpha > 0$, (i) follows.

(ii) This follows from part **(i)** and Remark 5.6.

(iii) The pair $(q_c, \alpha_c) := (\frac{1}{3\sqrt{2}}, \frac{1}{9\sqrt{2}})$ is the unique pair for which

$$H(q_c) = 0, \qquad h(q_c) = 0,$$

which follows by explicitly solving the system

$$3q^4 - \frac{q^2}{2} + \alpha q = 0, \qquad 12q^3 - q + \alpha = 0.$$

The phase plane portrait shows that for $\alpha \geq \alpha_c$, the closed curve $(\kappa(s), \kappa'(s))$ satisfies $\kappa \geq 0$. $\square$

In Figure 2 the phase portrait of the cubic ODE looks like the one on the left when $\alpha \in (0, \frac{1}{9\sqrt{2}})$ and like the one on the right when $\alpha \in [\frac{1}{9\sqrt{2}}, \frac{1}{9})$. In the second case each closed curve is on the half-plane $\kappa \geq 0$.

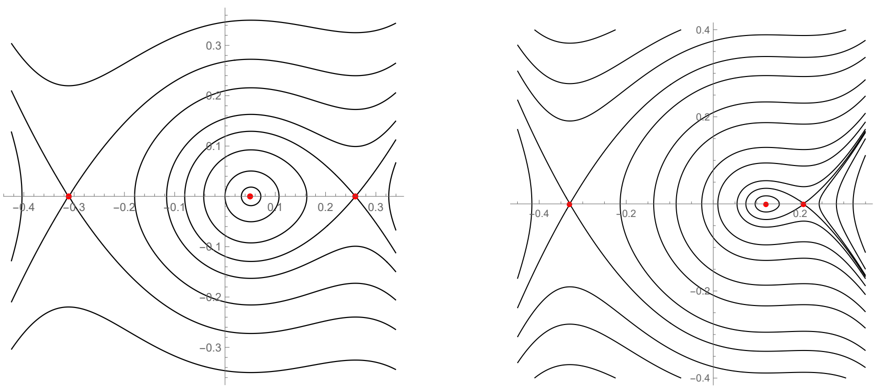

Figure 2: Phase portrait for $\alpha = 1/20$ and $1/10$

The above results will now be applied when α takes the value $\alpha_0 := \frac{2\pi}{T}\big(1 - 12(\frac{2\pi}{T})^2\big)$. Note that we assume $\alpha_0 > 0$ and that the graph of $\alpha_0(T)$ plotted in Figure 3, admits a maximum at $T = 12\pi$.

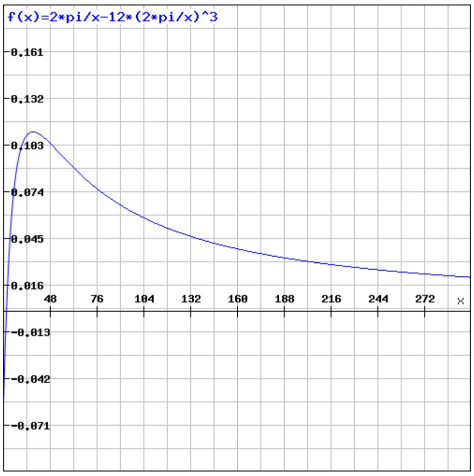

Figure 3: Graph of $\alpha_0(T)$

A straight algebraic computation shows the following.

Lemma 5.8. *Let α_0 be defined in* (5.9)*.*

(i) *Then the polynomial h with $\alpha = \alpha_0$ has three distinct zeros when $T \neq 12\pi$, and they are given by*

$$q_0(\alpha_0) = \frac{2\pi}{T}, \quad q_\pm(\alpha_0) = \frac{1}{2}\Big(-\frac{2\pi}{T} \pm \sqrt{\frac{1}{3} - 3\big(\frac{2\pi}{T}\big)^2}\Big).$$

(ii) *We have $\alpha_0 > 0$ and $q_0(\alpha_0) > q_+(\alpha_0)$ if and only if $T \in (\sqrt{48}\pi, 12\pi)$.*

(iii) *For $T \in \big(12\pi, \sqrt{72}[1+\sqrt{3}]\pi\big]$ we have $\alpha_0(T) \in [\frac{1}{9\sqrt{2}}, \frac{1}{9})$.*

Proof. For (iii), note that $T_1 = 6\pi\sqrt{2}$, $T_2 := 6\pi\sqrt{2}(1+\sqrt{3})$ and $T_3 := 6\pi\sqrt{2}(1-\sqrt{3})$ solve $\alpha_0(T_i) = \frac{1}{9\sqrt{2}}$. $\square$

Based on these results, the phase plane portrait of (κ, κ') looks for each $\alpha_0(T)$ like Figure 2. There are three equilibrium points, and one of them is $(\frac{2\pi}{T}, 0)$. We now deduce a uniqueness result for the Problem (5.9).

Proposition 5.9. *For $T \leq \sqrt{72}[1+\sqrt{3}]\pi$, the constant function $\kappa_0 \equiv \frac{2\pi}{T}$ is the unique solution to Problem* (5.9).

Proof. We consider two different ranges of the parameter T

Case 1: $T \in (\sqrt{48}\pi, 12\pi]$.
By Lemma 5.8 the dynamical system (5.11) with the parameter $\alpha_0(T)$ has three distinct equilibria $(q_0, 0), (q_\pm, 0)$ with $q_0(\alpha_0) := \frac{2\pi}{T}, q_\pm(\alpha_0)$ satisfying $q_0 > q_+ > 0 > q_-$. Furthermore, the phase plane portrait shows that $(q_+, 0)$ is a center and the periodic solutions "move around" $(q_+, 0)$.
Hence, all non-constant periodic solutions κ_0 satisfy $\kappa_0(t) < q_0 = \frac{2\pi}{T}$. In particular, we have $\int_0^T \kappa_0 < 2\pi$ and therefore the last condition in (5.9) cannot be satisfied.

If $T = 12\pi$, the dynamical system (5.11) has only two stationary points, and those (see Remark 5.6) are the only periodic solutions.

Case 2: $T \in \left(12\pi, \sqrt{72}[1+\sqrt{3}]\pi\right]$.
By Lemma 5.8 and Lemma 5.7 the periodic solution to Problem (5.9) are non-negative. Hence, as in Proposition 5.5 (using the convexity of the function $x \mapsto x^3$ on the interval $[0, \infty)$), we deduce that $\kappa_0 \equiv \frac{2\pi}{T}$. □

We can now prove our last result stated in the introduction.

Proof of Theorem 1.3:
Let Γ be a bounded connected component of $\partial\Omega$ and set $L := |\Gamma|$. We note that the parameters $(g'(c), L)$ in the Problem (1.1), and the variable T in the Problem (5.9) are related as follows (follows from (5.4) and (5.8)):

$$2g'(c)L^2 = T^2. \tag{5.14}$$

Claims **(i)** and **(ii)** follow from Proposition 5.3 and Proposition 5.5.

Proof of **(iii):** By Proposition 5.9 and taking into consideration (5.14) the curvature of the domain is constant whenever $g'(c)L^2 \leq 36(1+\sqrt{3})^2\pi^2$. Hence Γ is a circle, and Lemma 3.1 implies that $\partial\Omega$ is a circle. This concludes the proof of our theorem. □

6 Some remarks on Serrin type problems

In this final section we give some remarks on the two-dimensional Serrin type overdetermined problem

$$\left.\begin{array}{rll} \Delta u + g(u) & = 0 & \text{in } \Omega, \\ u & = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} & = \tilde{c} & \text{on } \partial\Omega. \end{array}\right\} \qquad (6.1)$$

Serrin's proof [18] that the domain is a ball holds under the additional assumption that the problem (6.1) admits a *positive* solution on a *bounded* and connected domain. If we replace "positivity" by the condition that some higher normal derivative is constant along the boundary, then our results stated for Schiffer's problem can be easily rephrased here to show that the boundary of the domain is a circle.

We note that by applying Lemma 2.3 we obtain (by iteration starting with $u(s,0) = 0$ and $\partial_d u(s,0) = -\tilde{c}$)

1. $n = 0$, $\partial_{dd} u = -\tilde{c}\kappa - g(0)$;
2. $n = 1$, $\partial_d^{(3)} u = -2\tilde{c}\kappa^2 - g(0)\kappa + \tilde{c}g'(0)$;
3. $n = 2$, $\partial_d^{(4)} u = \tilde{c}\kappa'' - 6\tilde{c}\kappa^3 + 3g(0)\kappa^2 + 2g'(0)\tilde{c}\kappa + \big(g(0)g'(0) - \tilde{c}^2 g''(0)\big)$.

We make first some observation when $\tilde{c} = 0$:

(i) If $(\tilde{c}, g(0)) = (0,0)$, then Lemma 3.1 (part (ii)) shows that $u \equiv 0$ is the unique function solving Problem (6.1).

(ii) If $\tilde{c} = 0$ and $g(0) \neq 0$, then Problem (6.1) is equivalent to Problem (1.1).

Hence, for our discussion it is enough to consider the case when $\tilde{c} \neq 0$.

So, under the assumption that Problem (6.1) admits a solution u in a domain that satisfies (2.7), then the above identities obtained for the normal derivative along the boundary $\partial\Omega$ show the following analogue of Theorem 1.1:

$$\partial\Omega \text{ is a disc} \iff \partial_\nu^{(2)} u \text{ is constant on } \partial\Omega \iff \partial_\nu^{(3)} u \text{ is constant on } \partial\Omega.$$

Concerning the fourth normal derivative, when $g(0) = 0$, by setting $\alpha := \frac{1}{\tilde{c}}\big(\partial_d^{(4)} u + \tilde{c}^2 g''(0)\big)$ we obtain

$$\kappa'' = 6\kappa^3 - 2g'(0)\kappa + \alpha,$$

which is similar to the ODE (4.2). Therefore both Theorem 1.2 and Theorem 1.3 obtained for the Schiffer Problem have an analogue here. We leave the details to the reader. In particular, we derive:

Proposition 6.1. *Assume Problem* (6.1) *admits a solution with* $\tilde{c} \neq 0$*. If* $g(0) = 0$*,* $g'(0) \leq 0$ *and the fourth normal derivative of* u *along* $\partial\Omega$ *is constant, then* $\partial\Omega$ *is a circle.*

Let us give an example where this can be applied.

Example 6.2. Consider the overdetermined problem (6.1) with $g(s) = s^p - s$ (for $p > 1$). In this case, we have $g(0) = 0$ and $g'(0) = -1$ and our result implies that $\partial\Omega$ is a disc if one of the normal derivatives $\partial_\nu^{(n)} u$ for $n \in \{2, 3, 4\}$ is constant on $\partial\Omega$. In particular, when Ω is unbounded, the domain must be the complement of a disc.

Note that for this nonlinearity, Ros, Ruiz, Sicbaldi [15] have shown that Problem (6.1) admits a positive solution on an exterior domain which is not the complement of a disc. This illustrates that extra assumptions are needed to derive symmetry. In [1] and [14] one can find other symmetry results with different assumptions on Serrin's Problem for exterior domains.

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