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Real analyticity is concentrated in dimension 2

Jacek Bochnak and Wojciech Kucharz

Abstract. We prove that a real-valued function on a real analytic manifold is analytic whenever all its restrictions to 2-dimensional analytic submanifolds are analytic functions. We also obtain analogous results in the framework of Nash manifolds and nonsingular real algebraic sets. These results can be regarded as substitutes in the real case for the classical theorem of Hartogs, asserting that a complex-valued function defined on an open subset of $\mathbb{C}^n$ is holomorphic if it is holomorphic with respect to each variable separately. In the proofs we use methods of real algebraic geometry even though the initial problem is purely analytic.

Key words. Real analytic manifold, analytic function, Nash manifold, Nash function, real algebraic set, regular function.

Mathematics subject classification (2010). 32C05, 32C25, 58A07, 14P20, 14P05.

1 Introduction

One of the main goals of this paper is to prove the following.

Theorem 1.1. *Let M be a compact real analytic manifold of dimension at least* 3 *and let $f\colon U \to \mathbb{R}$ be a function defined on an open subset U of M. Assume that for every* 2*-dimensional analytic submanifold S of M the restriction $f|_{U\cap S}$ is an analytic function. Then f is an analytic function.*

We always assume submanifolds to be *closed* subsets of the ambient manifold.

Theorem 1.1 can be viewed as a good substitute for the classical Hartogs theorem which asserts that a separately holomorphic function of n complex variables is in fact holomorphic in all the variables. It is well known that the analogous statement does not hold in the real case.

The hypotheses in Theorem 1.1 are global in their nature. We assume analyticity of f on compact analytic surfaces, which are global objects. Proving analyticity of f on U is a local matter, we must prove that f is analytic at each point of U. As indicated below, the global information at our disposal is difficult to translate into local data.

Let $g\colon W \to \mathbb{R}$ be a function defined in an open neighborhood W of $0 \in \mathbb{R}^n$, $n \geq 3$. The local theorem proved in [3, 14] asserts that if the restriction $g|_{W\cap Q}$ is analytic for every 2-dimensional vector subspace $Q \subset \mathbb{R}^n$, then g is analytic at $0 \in \mathbb{R}^n$.

The first idea of how to approach the proof of Theorem 1.1 is to take a local analytic coordinate system $\varphi\colon (V, a) \to (\mathbb{R}^n, 0)$, $n = \dim M$, around a fixed point $a \in V \subset U$

and apply the local result just mentioned above to the function $f \circ \varphi^{-1}$ defined in the open neighborhood $\varphi(V)$ of $0 \in \mathbb{R}^n$. Unfortunately, the global hypotheses in Theorem 1.1 do not directly imply that $f \circ \varphi^{-1}$ is analytic on the intersection of $\varphi(V)$ with each 2-dimensional vector subspace of $\mathbb{R}^n$. We must therefore modify the initial idea and apply more sophisticated methods.

In Section 2 we recall several auxiliary results, including one on series of real homogeneous polynomials in n variables. It should be mentioned that the convergence of series of homogeneous polynomials is more complicated to study than the convergence of conceptually close power series. Our main local result is Theorem 3.3, proved in Section 3. In Section 4, which heavily depends on Theorem 3.3, we investigate global problems and prove Theorem 1.1.

Although Theorem 1.1 is a statement involving only analytic manifolds and analytic functions, in its proof we use, quite unexpectedly, methods and notions from real algebraic geometry (real algebraic sets, semialgebraic sets, real regular functions, the theorems of Tarski–Seidenberg and of Nash–Tognoli, etc.). As an additional bonus we also obtain in Section 4 counterparts of Theorem 1.1, stated below, for Nash functions and regular functions. For background material on real algebraic geometry we refer the reader to the book [1].

By a *Nash manifold* we mean a semialgebraic subset N of $\mathbb{R}^m$ that is an analytic submanifold of an open subset of $\mathbb{R}^m$, for some m. A function $f\colon U \to \mathbb{R}$, defined on an open subset $U \subset N$, is called a *Nash function* if every point $a \in U$ has a semialgebraic open neighborhood $U_a \subset U$ such that the restriction $f|_{U_a}$ is analytic with semialgebraic graph (in that case, one can take $U_a = U$, provided U is a semialgebraic set).

Theorem 1.2. *Let N be a Nash manifold of dimension at least 3 and let $f\colon N \to \mathbb{R}$ be a function whose restriction to every 2-dimensional Nash submanifold of N is a Nash function. Then f is a Nash function.*

By a *real algebraic set* we mean an algebraic subset of $\mathbb{R}^m$ for some m.

Theorem 1.3. *Let X be a nonsingular real algebraic set of pure dimension at least 3 and let $f\colon X \to \mathbb{R}$ be a function whose restriction to every 2-dimensional nonsingular algebraic subset of X is a regular function. Then f is a regular function.*

Theorem 1.3 is a significant refinement of [9, Theorem 6.2]. Section 4 contains also Theorems 4.7 and 4.8 which are more general than Theorems 1.2 and 1.3, respectively.

It is plausible that the compactness of M in Theorem 1.1 is superfluous. We do not require N (resp. X) in Theorem 1.2 (resp. Theorem 1.3) to be compact.

Replacing in Theorems 1.1, 1.2 and 1.3 the nonsingular surfaces by nonsingular curves (in the appropriate category) would lead to a false statement.

Counterexample 1.4. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$f(x,y,z) = \frac{x^8 + y(x^2-y^3)^2 + z^4}{x^{10} + (x^2-y^3)^2 + z^2} \quad \text{for } (x,y,z) \neq (0,0,0) \quad \text{and} \quad f(0,0,0) = 0.$$

Then f is analytic (resp. Nash or regular) on every nonsingular analytic (resp. Nash or algebraic) curve in $\mathbb{R}^3$, but f is not even continuous at $(0,0,0)$.

To establish the first part of the assertion it suffices to prove that for any nonsingular analytic curve $C \subset \mathbb{R}^3$ passing through $(0,0,0)$ the function $f|_C$ is analytic at $(0,0,0)$. Since C is nonsingular, it has near $(0,0,0)$ a local analytic parametrization

$$x = x(t),\ y = y(t),\ z = z(t) \quad \text{for } t \text{ near } 0 \in \mathbb{R},$$

where $x(0) = y(0) = z(0) = 0$, and at least one of the analytic functions $x(t)$, $y(t)$, $z(t)$ has zero of order 1 at $t = 0$. It is not hard to check that the function $f(x(t), y(t), z(t))$ is analytic for t near $0 \in \mathbb{R}$. Thus $f|_C$ is analytic at $(0, 0, 0)$, as required.

Clearly, the function f is not continuous at $(0, 0, 0)$ since on the curve $x^2 - y^3 = 0$, $z = 0$ it is equal to $\frac{1}{x^2}$ away from $(0, 0, 0)$.

2 Auxiliary results

We collect here some results and observations needed in the proofs of the main theorems. The notations and remarks introduced in this section will be used throughout the rest of this paper.

Lemma 2.1. *Let $C = I_1 \times \cdots \times I_n$ be a rectangle in $\mathbb{R}^n$, where the $I_k \subset \mathbb{R}$ are open intervals. Assume that $h \colon C \to \mathbb{R}$ is a polynomial function with respect to each variable separately. Then there exists a polynomial function $H \colon \mathbb{R}^n \to \mathbb{R}$ such that $H|_C = h$.*

Proof. If $I_k = \mathbb{R}$ for $k = 1, \dots, n$, the proof is given in [2, Lemma 1]. The general case can be established using the same method. □

The next lemma is proved in [13], and a simpler proof is given in [3].

Lemma 2.2. *Let $\sum_{k=0}^{\infty} P_k$ be a series of real homogeneous polynomials in n variables, $\deg P_k = k$. Let $\Sigma = \{a \in \mathbb{R}^n \colon \|a\| = 1\}$ and assume that there exists an open nonempty subset Ω of Σ such that for every $a \in \Omega$ one can find a constant $\rho > 0$ (depending on a) such that the series $\sum_{k=0}^{\infty} P_k(x)$ converges at $x = \rho a$. Then there exist constants $c > 0$, $r > 0$ such that*

$$|P_k(z)| \leq \frac{c}{2^k} \quad \text{for } z \in \mathbb{C}^n,\ \|z\| \leq r,\ k \geq 0.$$

In particular, the function $z \mapsto \sum_{k=0}^{\infty} P_k(z)$ is holomorphic in the ball $\|z\| < r$, $z \in \mathbb{C}^n$. □

Let $\mathbb{G}(k, n)$ denote the Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^n$. As in [1, p. 72], we regard $\mathbb{G}(k, n)$ as a nonsingular algebraic subset of $\mathbb{R}^{n^2}$.

Given a vector subspace $V \subset \mathbb{R}^n$ of dimension $n - 2$, we define the subset $O(V)$ of $\mathbb{G}(n - 1, n)$ by

$$O(V) = \{H \in \mathbb{G}(n - 1, n) \colon V \subset H\}.$$

The map

$$\eta \colon \mathbb{G}(n - 1, n) \to G(1, n) = \mathbb{RP}^{n-1}, \quad H \mapsto H^{\perp},$$

sending H to its orthogonal complement $H^{\perp}$ in $\mathbb{R}^n$ (with respect to the standard inner product), is a biregular isomorphism of real algebraic sets, see [1, p. 72]. When convenient we will identify $\mathbb{G}(n-1, n)$ with $\mathbb{RP}^{n-1}$ via η. The image of $O(V)$ by η is a projective line in $\mathbb{RP}^{n-1}$. Indeed, if $H \in O(V)$, then the vector line $H^{\perp}$ is contained in the orthogonal complement $U = V^{\perp}$ of V in $\mathbb{R}^n$. Conversely, to each vector line $L \subset U$ there corresponds $H = \eta^{-1}(L) = L^{\perp} \in O(V)$. Hence $\eta(O(V))$ is the projective line in $\mathbb{RP}^{n-1}$ comprised of all vector lines lying in the vector plane U.

For $n \geq 3$ we define a collection of algebraic (reducible) curves $\mathcal{T} \subset \mathbb{G}(n - 1, n)$ as follows:

- if $n = 3$, then
$$\mathcal{T} = \mathcal{T}(L_1, L_2, L_3) = O(L_1) \cup O(L_2) \cup O(L_3),$$
where $L_1, L_2, L_3 \subset \mathbb{R}^3$ are linearly independent vector lines;

- if $n \geq 4$, then
$$\mathcal{T} = \mathcal{T}(V_1, V_2) = O(V_1) \cup O(V_2),$$
where $V_1, V_2 \subset \mathbb{R}^n$ are vector subspaces of dimension $n-2$ with $\dim(V_1 \cap V_2) = n-4$.

We call each set $\mathcal{T}$ in this collection a *test curve*. The image $\eta(\mathcal{T})$ of $\mathcal{T}$ in $\mathbb{RP}^{n-1}$ can easily be described. If $n = 3$, then $\eta(\mathcal{T}) \subset \mathbb{RP}^2$ is the union of three projective lines in general position (that is, three lines with no common point). If $n \geq 4$, then $\eta(\mathcal{T}) \subset \mathbb{RP}^{n-1}$ is the union of two disjoint projective lines in $\mathbb{RP}^{n-1}$. In either case, $\eta(\mathcal{T})$ is an algebraic curve in $\mathbb{RP}^{n-1}$. Clearly, the union of any three projective lines in general position in $\mathbb{RP}^2$ corresponds to a test curve in $\mathbb{G}(2, 3)$. Similarly, for $n \geq 4$, the union of any two disjoint projective lines in $\mathbb{RP}^{n-1}$ corresponds to a test curve in $\mathbb{G}(n-1, n)$.

Let $e_1, \ldots, e_n$ be the standard vector basis for $\mathbb{R}^n$, $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the ith position. We define the *standard test curve* $\mathcal{T}^* \subset \mathbb{G}(n-1, n)$ as follows:

- if $n = 3$, then
$$\mathcal{T}^* = \mathcal{T}(\mathbb{R}e_1, \mathbb{R}e_2, \mathbb{R}e_3);$$

- if $n \geq 4$, then
$$\mathcal{T}^* = \mathcal{T}(V_1, V_2),$$
where V_1 (resp. V_2) is the vector subspace of $\mathbb{R}^n$ spanned by $e_1, \ldots, e_{n-2}$ (resp. $e_3, \ldots, e_n$).

The following fact, which is a simple exercise in linear algebra, is crucial for our purposes: any *vector* line $L \subset \mathbb{R}^n$ is contained in some hyperplane $Q \in \mathcal{T}^*$, and any *affine* line $l \subset \mathbb{R}^n$ that is parallel to one of the coordinate axes is also contained in some $Q \in \mathcal{T}^*$.

For any two test curves $\mathcal{T}, \mathcal{T}'$ in $\mathbb{G}(n-1, n)$ there exists a linear automorphism $\lambda \in \mathrm{GL}_n(\mathbb{R})$ such that $\mathcal{T}_\lambda = \mathcal{T}'$, where
$$\mathcal{T}_\lambda = \{\lambda(H) \in \mathbb{G}(n-1, n) \colon H \in \mathcal{T}\}.$$

Finally, for any finite subset $\mathcal{H} \subset \mathbb{G}(n-1, n)$ there exists a test curve $\mathcal{T} \subset \mathbb{G}(n-1, n)$ such that $\mathcal{H} \cap \mathcal{T} = \varnothing$.

3 Local results

The following technical result will play a key role in this section.

Proposition 3.1. *Let $\mathcal{T} \subset \mathbb{G}(n-1, n)$ be a test curve, $n \geq 3$. Let $\mathcal{H}$ be a finite (possibly empty) subset of $\mathcal{T}$, H the union of all hyperplanes in $\mathcal{H}$, and $S_1, \ldots, S_p$ the connected components of $\mathbb{R}^n \setminus H$. Let $f \colon U \to \mathbb{R}$ be a function, defined in an open neighborhood U of $0 \in \mathbb{R}^n$, such that for every hyperplane $Q \in \mathcal{T} \setminus \mathcal{H}$ the restriction $f|_{U \cap Q}$ is an analytic function. Then there exist analytic functions $f_i \colon W \to \mathbb{R}$, defined in an open neighborhood $W \subset U$ of $0 \in \mathbb{R}^n$, such that*
$$f_i(0) = f(0) \quad \textit{and} \quad f_i = f \quad \textit{on } W \cap S_i \quad \textit{for } i = 1, \ldots, p.$$

Proof. First we prove the proposition for the standard test curve $\mathcal{T} = \mathcal{T}^*$. By assumption and the properties of $\mathcal{T}^*$ recorded in Section 2, for every vector line $L \subset \mathbb{R}^n \setminus (H \setminus \{0\})$ the function $f|_{U \cap L}$ is analytic. This allows us to define, for each integer $k \geq 0$, the function

$$h_k \colon \mathbb{R}^n \setminus (H \setminus \{0\}) \to \mathbb{R}, \quad x \mapsto \frac{d^k f}{dt^k}(tx)\bigg|_{t=0}.$$

We claim that for each connected component S_i there is a homogeneous polynomial $H_{k,i} \colon \mathbb{R}^n \to \mathbb{R}$, of degree k, such that

$$h_k(x) = H_{k,i}(x) \quad \text{for all } x \in S_i.$$

The claim can be established as follows. Fix S_i and choose an open rectangle $C \subset S_i$ with sides parallel to the coordinate axes. Let $l \subset \mathbb{R}^n$ be an affine line parallel to one of the coordinate axes, with $l \cap C \neq \varnothing$. Since $C \cap H = \varnothing$, the line l is contained in some hyperplane $Q \in \mathcal{T} \setminus \mathcal{H}$. Since $f|_{U \cap Q}$ is analytic, it follows from the definition of h_k that $h_k|_Q$ is a homogeneous polynomial of degree k on Q. In particular, $h_k|_l$ is a polynomial function. By Lemma 2.1, the function $h_k|_C$ is the restriction of a polynomial function $H_{k,C} \colon \mathbb{R}^n \to \mathbb{R}$, which is necessarily homogeneous of degree k. Observe that for any two overlapping open rectangles C and D contained in S_i we have $H_{k,C} = H_{k,D}$ because on $C \cap D$ both polynomials are equal to $h_k|_{C \cap D}$. Furthermore, given any two points a, b in S_i, we can find a sequence of open rectangles $C_1, \ldots, C_q$ contained in S_i such that $a \in C_1$, $b \in C_q$, and $C_j \cap C_{j+1} \neq \varnothing$ for $j = 1, \ldots, q-1$. It follows that $H_{k,C}$ is independent of the choice of C. Hence the claim holds with $H_{k,i} = H_{k,C}$.

We are now ready to construct the required analytic functions f_i. Let Σ be the unit $(n-1)$-sphere in $\mathbb{R}^n$ and let $\Omega_i = S_i \cap \Sigma$. The component S_i has the conic structure: if $x \in S_i$ then $tx \in S_i$ for all $t > 0$. It follows that Ω_i is a nonempty open subset of Σ. For each point $a \in \Omega_i$ the function $t \mapsto f(ta)$ is analytic at $0 \in \mathbb{R}$, so there exists a constant $\rho_a > 0$ such that

$$f(ta) = \sum_{k=0}^{\infty} \frac{1}{k!} h_k(a) t^k = \sum_{k=0}^{\infty} \frac{1}{k!} H_{k,i}(ta) \quad \text{for } |t| < \rho_a.$$

By Lemma 2.2, the series $\sum \frac{1}{k!} H_{k,i}$ of homogeneous polynomials is uniformly convergent in a ball $\|z\| < r_i$, $z \in \mathbb{C}^n$, and its sum f_i is holomorphic there. Clearly, $f_i(ta) = f(ta)$ for $|t| < \min\{r_i, \rho_a\}$. By the identity property for analytic functions,

$$f_i(ta) = f(ta) \quad \text{for } |t| < \rho_i = \min\{r_i, \operatorname{dist}(0, \partial U)\}.$$

It follows that

$$f_i(x) = f(x) \quad \text{for all } x \in S_i,\ \|x\| < \rho_i.$$

If W is the open ball in $\mathbb{R}^n$ with center at $0 \in \mathbb{R}^n$ and radius $\rho = \min\{\rho_1, \ldots, \rho_p\}$, then

$$f_i(0) = f(0) \quad \text{and} \quad f_i = f \quad \text{on } W \cap S_i \quad \text{for } i = 1, \ldots, p,$$

which completes the proof in the case where $\mathcal{T} = \mathcal{T}^*$ is the standard test curve.

For an arbitrary test curve $\mathcal{T}$ in $\mathbb{G}(n-1, n)$, we choose a linear automorphism $\lambda \in \mathrm{GL}_n(\mathbb{R})$ for which $\mathcal{T}_\lambda = \mathcal{T}^*$. Set $\mathcal{H}' = \{\lambda(Q) \colon Q \in \mathcal{H}\}$ and let H' be the union of all hyperplanes in $\mathcal{H}'$. Then $S_1' = \lambda(S_1), \ldots, S_p' = \lambda(S_p)$ are the connected components of $\mathbb{R}^n \setminus H'$. Furthermore, let $U' = \lambda(U)$ and $f' = f \circ \lambda^{-1}|_{U'}$. By applying the first part of the proof to $\mathcal{T}^*$, $\mathcal{H}'$, f', we obtain analytic functions $f'_i\colon W' \to \mathbb{R}$, defined in an open neighborhood $W' \subset U'$ of $\mathbb{R}^n$, such that

$$f'_i(0) = f'(0) \quad \text{and} \quad f'_i = f' \quad \text{on } S'_i \cap W' \quad \text{for } i = 1, \ldots, p.$$

Now we complete the proof for the test curve $\mathcal{T}$ by setting $W = \lambda^{-1}(W')$ and $f_i = f'_i \circ \lambda|_W$. □

Corollary 3.2. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two test curves in $\mathbb{G}(n-1, n)$, $n \geq 3$. Let $\mathcal{H}$ and $\mathcal{H}'$ be finite subsets of $\mathcal{T}$ and $\mathcal{T}'$, respectively, with $\mathcal{H} \cap \mathcal{H}' = \varnothing$. Let H and H' be the unions of all hyperplanes in $\mathcal{H}$ and $\mathcal{H}'$, respectively. Let $f\colon U \to \mathbb{R}$ be a function, defined in an open neighborhood U of $0 \in \mathbb{R}^n$, such that for every hyperplane $Q \in (\mathcal{T} \setminus \mathcal{H}) \cup (\mathcal{T}' \setminus \mathcal{H}')$ the restriction $f|_{U \cap Q}$ is an analytic function. Then there exists an analytic function $g\colon W \to \mathbb{R}$, defined in an open neighborhood $W \subset U$ of $0 \in \mathbb{R}^n$, such that*

$$g(0) = f(0) \quad \textit{and} \quad g = f \ \textit{ on } W \setminus (H \cap H').$$

Proof. Let $S_1, \ldots, S_p$ (resp. $S'_1, \ldots, S'_q$) be the connected components of $\mathbb{R}^n \setminus H$ (resp. $\mathbb{R}^n \setminus H'$). By Proposition 3.1, we can find an open ball $W \subset U$, centered at $0 \in \mathbb{R}^n$, together with analytic functions $f_1, \ldots, f_p$ (resp. $f'_1, \ldots, f'_q$) defined on W such that

$$f_i(0) = f(0), \quad f'_j(0) = f(0) \quad \text{and}$$
$$f_i = f \quad \text{on } S_i \cap W, \quad f'_j = f \quad \text{on } S'_j \cap W \quad \text{for } 1 \leq i \leq p,\ 1 \leq j \leq q.$$

We claim that

$$f_i = f_k \quad \text{for } 1 \leq i \leq p,\ 1 \leq k \leq p.$$

For the proof of the claim we first choose two indices i and k for which the components S_i and S_k are *adjacent*, that it, the intersection $\overline{S_i} \cap \overline{S_k}$ of their closures in $\mathbb{R}^n$ has dimension $n-1$. This means that $\overline{S_i} \cap \overline{S_k}$ is contained in a unique hyperplane $Q \in \mathcal{H}$, and the interior Y of $\overline{S_i} \cap \overline{S_k}$ in Q is nonempty. Choose a point $x_0 \in Y \cap (W \setminus H')$, and let S'_j be a connected component of $\mathbb{R}^n \setminus H'$ containing x_0. Such a component S'_j exists because $\mathcal{H} \cap \mathcal{H}' = \varnothing$ and hence $Q \notin \mathcal{H}'$. Let N be an open neighborhood of x_0 in $S'_j \cap W$. Then $N_i = N \cap S_i$ and $N_k = N \cap S_k$ are nonempty open subsets of S_i and S_k, respectively. By construction,

$$f'_j = f = f_i \quad \text{on } N_i \quad \text{and} \quad f'_j = f = f_k \quad \text{on } N_k.$$

Since f'_j, f_i, f_k are analytic on W, we get

$$f'_j = f_i \quad \text{and} \quad f'_j = f_k \quad \text{on } W.$$

Consequently,

$$f_i = f_k \quad \text{on } W \quad \text{for the chosen } i, k.$$

The claim follows since any two distinct connected components of $\mathbb{R}^n \setminus H$ appear in some sequence $S_{i_1}, \ldots, S_{i_l}$ in which any two consecutive terms are adjacent components.

Setting $g = f_1$, the claim implies that $g = f$ on $W \setminus H$. By interchanging the role of the f_i and the f'_j, we get $g = f$ on $W \setminus H'$. Thus $g = f$ on $W \setminus (H \cap H')$, as required. □

We say that a subset C of $\mathbb{R}^n$, $n \geq 2$, is a *circle centered at* $0 \in \mathbb{R}^n$ if it can be written as

$$C = \{x \in \mathbb{R}^n \colon x \in V,\ \|x\| = r\},$$

where $V \subset \mathbb{R}^n$ is a vector subspace of dimension 2, and r is a positive constant.

The main result of this section is the following.

Theorem 3.3. *Let G be a nonempty Zariski open subset of $\mathbb{G}(n-1,n)$, $n \geq 3$. Let $f\colon U \to \mathbb{R}$ be a function, defined in an open neighborhood U of $0 \in \mathbb{R}^n$, such that for every hyperplane $Q \in G$ the restriction $f|_{U\cap Q}$ is an analytic function. Furthermore, assume that the restriction $f|_C$ is a continuous function for every circle $C \subset U$ centered at $0 \in \mathbb{R}^n$. Then the function f is analytic in a neighborhood of $0 \in \mathbb{R}^n$.*

Proof. For the proper algebraic subset $A = \mathbb{G}(n-1,n) \setminus G$ of $\mathbb{G}(n-1,n)$, we choose two test curves $\mathcal{T}$ and $\mathcal{T}'$ in $\mathbb{G}(n-1,n)$ such that the intersections

$$\mathcal{H} = \mathcal{T} \cap A \quad \text{and} \quad \mathcal{H}' = \mathcal{T}' \cap A$$

are finite sets, and $\mathcal{H} \cap \mathcal{H}' = \varnothing$. This is possible since the test curves are algebraic curves with properties described in Section 2. Clearly, for every hyperplane $Q \in (\mathcal{T}\setminus\mathcal{H})\cup(\mathcal{T}'\setminus\mathcal{H}')$ the restriction $f|_{U\cap Q}$ is an analytic function.

By Corollary 3.2, there exists an analytic function $g\colon W \to \mathbb{R}$, defined in a neighborhood $W \subset U$ of $0 \in \mathbb{R}^n$, such that

$$g(0) = f(0) \quad \text{and} \quad g = f \text{ on } W \setminus H,$$

where H is the union of all hyperplanes in $\mathcal{H}$ (note that a little more is asserted in Corollary 3.2). We claim that $g = f$ on W. It remains to prove that

$$g(y) = f(y) \quad \text{for all } y \in W \cap (H \setminus \{0\}).$$

To this end, fix a point $y \in W \cap (H \setminus \{0\})$ and pick a circle $C \subset W$ centered at $0 \in \mathbb{R}^n$ such that $y \in C$ and $C \cap (W \setminus H) \neq \varnothing$. We can find a sequence $\{x_k\}$ in $C \setminus H$ that converges to y. By construction, $g(x_k) = f(x_k)$ for all k. Since the functions $g|_C$ and $f|_C$ are continuous, we get $g(y) = f(y)$, as required. □

Next we deal with Nash functions and show how to obtain for them suitable variants of the last three results. For the sake of clarity we begin with a short review.

By [1, Chapter 8], a function $f\colon U \to \mathbb{R}$, defined on a connected open subset $U \subset \mathbb{R}^n$, is a Nash function if and only if it is *algebraic*. The latter notion means that f is analytic and satisfies

$$P(x, f(x)) = 0 \quad \text{for all } x \in U,$$

where P is a nonzero real polynomial in $n+1$ variables. Therefore, in view of [4, p. 202, Theorem 6], an analytic function on U is Nash on U as long as it is Nash with respect to each variable separately.

Remark 3.4. With notation as in Proposition 3.1, suppose in addition that for every hyperplane $Q \in \mathcal{T} \setminus \mathcal{H}$ the restriction $f|_{U\cap Q}$ is a Nash function. If W is connected, then the f_i are Nash functions.

Indeed, assume that W is connected. Then each function $f_i\colon W \to \mathbb{R}$ is uniquely determined by the condition $f_i = f$ on $S_i \cap W$. This is the case since f_i is analytic and the intersection $S_i \cap W$ is an open nonempty set. We may therefore assume that $\mathcal{T} = \mathcal{T}^*$ is the standard test curve (see the last step in the proof of Proposition 3.1). For any affine line $l \subset \mathbb{R}^n$ that is parallel to one of the coordinate axes, pick a hyperplane $Q \in \mathcal{T}^*$ with $l \subset Q$. Since

$$f_i = f \quad \text{on } S_i \cap W \cap l \subset S_i \cap W \cap Q,$$

the restriction of f_i to $S_i \cap W \cap l$ is a Nash function. In other words, $f_i|_{S_i\cap W}$ is an analytic function which is Nash with respect to each variable separately. Hence $f_i|_{S_i\cap W}$ is a Nash function. Consequently, f_i is a Nash function on W since it is analytic and W is connected.

Remark 3.5. With notation as in Corollary 3.2, suppose in addition that for every hyperplane $Q \in (\mathcal{T} \setminus \mathcal{H}) \cup (\mathcal{T}' \setminus \mathcal{H}')$ the restriction $f|_{U \cap Q}$ is a Nash function. If W is connected, then g is a Nash function.

Indeed, we can argue as in the proof of Corollary 3.2, making use of Remark 3.4.

A counterpart of Theorem 3.3 for Nash functions takes the following form.

Theorem 3.6. *Let G be a nonempty Zariski open subset of $\mathbb{G}(n-1, n)$, $n \geq 3$. Let $f\colon U \to \mathbb{R}$ be a function, defined in an open neighborhood U of $0 \in \mathbb{R}^n$, such that for every hyperplane $Q \in G$ the restriction $f|_{U \cap Q}$ is a Nash function. Furthermore, assume that the restriction $f|_C$ is a continuous function for every circle $C \subset U$ centered at $0 \in \mathbb{R}^n$. Then f is a Nash function in a neighborhood of $0 \in \mathbb{R}^n$.*

Proof. One can repeat the proof of Theorem 3.3, substituting Remark 3.5 for Corollary 3.2. □

The observations discussed in the rest of this section will not be used in the proofs of Theorems 1.1, 1.2 and 1.3, however, they are of interest in their own right.

Suppose that $n \geq 3$. We say that a set $Y \subset \mathbb{G}(n-1, n)$ is *sufficient* if the following holds: For any function $f\colon U \to \mathbb{R}$, defined in some neighborhood U of $0 \in \mathbb{R}^n$, such that the restriction $f|_{U \cap Q}$ is analytic for every hyperplane $Q \in Y$, the function f is analytic in a neighborhood of $0 \in \mathbb{R}^n$. We say that a set $Z \subset \mathbb{G}(n-1, n)$ is *negligible* if its complement $\mathbb{G}(n-1, n) \setminus Z$ is sufficient.

In order to make the structure of sufficient and negligible sets more transparent, we can transfer the definition from $\mathbb{G}(n-1, n)$ to $\mathbb{RP}^{n-1}$ and call a set $X \subset \mathbb{RP}^{n-1}$ sufficient (resp. negligible) if its inverse image $\eta^{-1}(X) \subset \mathbb{G}(n-1, n)$ is sufficient (resp. negligible), where

$$\eta\colon \mathbb{G}(n-1, n) \to \mathbb{RP}^{n-1}$$

is the biregular isomorphism described in Section 2.

Proposition 3.7. *For any $n \geq 3$, the following hold:*

(1) *Let $T \subset \mathbb{RP}^{n-1}$ be the union of either three projective lines in general position if $n = 3$, or two disjoint projective lines if $n \geq 4$. Then the set T is sufficient.*

(2) *For $T \subset \mathbb{RP}^{n-1}$ as in (1), let $A \subset T$ be a subset comprised of either three points if $n = 3$, or two points if $n \geq 4$. Assume that each point of A belongs to exactly one irreducible component of T. Then the set $T \setminus A$ is not sufficient.*

(3) *Let $B \subset \mathbb{RP}^{n-1}$ be a compact subset that is disjoint from some projective line in $\mathbb{RP}^{n-1}$. Then B is negligible.*

(4) *Let $X \subset \mathbb{RP}^{n-1}$ be an algebraic set. If $\dim X \leq n-3$, then X is negligible (in particular, the curve T in (1) is negligible for $n \geq 4$). If $\dim X = n-2$, then, for some finite subset $F \subset X$, the set $X \setminus F$ is negligible.*

(5) *The complement of a hyperplane in $\mathbb{RP}^{n-1}$ is never sufficient.*

Proof. (1) As explained in Section 2, $\mathcal{T} = \eta^{-1}(T)$ is a test curve in $\mathbb{G}(n-1, n)$. Hence it suffices to apply Proposition 3.1 (with $\mathcal{H} = \varnothing$).

(2) For any vector line $L \subset \mathbb{R}^n$, we define the function $f_L\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f_L(x) = 0 \quad \text{for } x \in \mathbb{R}^n \setminus (L \setminus \{0\}) \quad \text{and} \quad f_L(x) = 1 \quad \text{for } x \in L \setminus \{0\}.$$

Clearly, the restriction $f_L|Q$ is identically 0 (hence analytic) for every vector hyperplane $Q \subset \mathbb{R}^n$ with $Q \cap L = \{0\}$, but $f|_L$ is not analytic at $0 \in \mathbb{R}^n$.

Now we will work with $\mathcal{T} = \eta^{-1}(T)$ and $\mathcal{A} = \eta^{-1}(A)$.

If $n = 3$, then $\mathcal{T} = O(L_1) \cup O(L_2) \cup O(L_3)$, where L_1, L_2, L_3 are linearly independent vector lines in $\mathbb{R}^3$. Furthermore, $\mathcal{A} = \{Q_1, Q_2, Q_3\}$ with $Q_i \in O(L_j)$ whenever $i = j$. Choose a vector line $L \subset \mathbb{R}^3$, $L \neq L_i$ for $i = 1, 2, 3$. Then, for any $Q \in O(L_i)$ with $L \subset Q$, we have

$$Q = L_i + L = Q_i.$$

It follows that $f_L|_Q$ is identically 0 for all $Q \in \mathcal{T} \setminus \mathcal{A}$. Hence (2) holds because f_L is not analytic at $0 \in \mathbb{R}^n$.

If $n \geq 4$, then $\mathcal{T} = O(V_1) \cup O(V_2)$, where V_1, V_2 are $(n-2)$-dimensional vector subspaces of $\mathbb{R}^n$ with $\dim(V_1 \cap V_2) = n-4$. Furthermore, $\mathcal{A} = \{Q_1, Q_2\}$ with $Q_i \in O(V_i)$ for $i = 1, 2$. Choose a vector line $L \subset \mathbb{R}^n$ that is contained in $Q_1 \cap Q_2$, but is not contained in $V_1 \cup V_2$. Then, for any $Q \in O(V_i)$ with $L \subset Q$, we have

$$Q = V_i + L = Q_i.$$

It follows that $f_L|_Q$ is identically 0 for all $Q \in \mathcal{T} \setminus \mathcal{A}$. Hence (2) holds because f_L is not analytic at $0 \in \mathbb{R}^n$.

(3) Suppose that B is disjoint from a projective line $l \subset \mathbb{RP}^{n-1}$. Then we can move l in $\mathbb{RP}^{n-1}$ to produce either three lines in general position if $n = 3$, or two disjoint lines if $n \geq 4$. These lines can be chosen so that their union T is disjoint from B. Hence the set B is negligible since T is sufficient by (1).

(4) If $\dim X \leq n - 3$ (resp. $\dim X = n - 2$), then there exists a set $T \subset \mathbb{RP}^{n-1}$, as in (1), with $T \cap X = \varnothing$ (resp. $F = T \cap X$ finite). Thus (4) holds since X (resp. $X \setminus F$) is contained in $\mathbb{RP}^{n-1} \setminus T$.

(5) Any hyperplane in $\mathbb{RP}^{n-1}$ corresponds via η to a set of the form

$$\tilde{L} = \{Q \in \mathbb{G}(n-1, n) \colon L \subset Q\},$$

where L is a vector line in $\mathbb{R}^n$. The function f_L defined in (2) is not analytic at $0 \in \mathbb{R}^n$, but its restriction $f_L|_Q$ is identically 0 for every Q in $\mathbb{G}(n-1, n) \setminus \tilde{L}$. This completes the proof of (5). □

Proposition 3.7 is counter-intuitive: some "small" sets like the curves in (1) are sufficient, while some "large" sets like the complements of hyperplanes in (5) are not. It does not seem that the size of a set in $\mathbb{RP}^{n-1}$ is essential for sufficiency, but rather its geometrical configuration and, in particular, its position with respect to projective lines in $\mathbb{RP}^{n-1}$.

It would be interesting to characterize *minimal* sufficient subsets of $\mathbb{RP}^{n-1}$, that is, the ones which do not contain proper sufficient subsets. We do not know any explicit example of a minimal sufficient set, but conjecture that the curves T in Proposition 3.7 (1) are minimal. This conjecture is supported by Proposition 3.7 (2).

4 Global results

In this section we prove, after some preparation, the theorems stated in the introduction.

Lemma 4.1. *Let X be a nonsingular real algebraic set of pure dimension m, and let x_0 be a point in X. Then, for any integer n with $m \geq n \geq 0$, there exists a regular map $\varphi\colon X \to \mathbb{R}^n$ for which $0 \in \mathbb{R}^n$ is a regular value and $\varphi(x_0) = 0$.*

Proof. We can assume that $X \subset \mathbb{R}^{m+k} = \mathbb{R}^m \times \mathbb{R}^k$, $x_0 = 0$, and the tangent space to X at 0 is $\mathbb{R}^m \times \{0\}$. Let $\pi\colon X \to \mathbb{R}^n$ be the restriction of the projection $\mathbb{R}^n \times \mathbb{R}^{m-n+k} \to \mathbb{R}^n$. Clearly, π is a submersion at $0 \in X$. Now let $r = (m+k)n$ and let $\lambda_1, \ldots, \lambda_r$ be a basis of the vector space of linear maps from $\mathbb{R}^{m+k}$ into $\mathbb{R}^n$. Then $0 \in \mathbb{R}^n$ is a regular value of the map $\Phi\colon X \times \mathbb{R}^r \to \mathbb{R}^n$ defined by

$$\Phi(x,t) = \pi(x) + t_1\|x\|^2\lambda_1(x) + \cdots + t_r\|x\|^2\lambda_r(x),$$

where $t = (t_1, \ldots t_r)$. By applying the Sard theorem (see [7, p. 79, Theorem 2.7] for details), we can find a point $t \in \mathbb{R}^r$ such that $0 \in \mathbb{R}^n$ is a regular value of the map $\Phi_t\colon X \to \mathbb{R}^n$, $\Phi_t(x) = \Phi(x,t)$. Obviously, the map $\varphi = \Phi_t$ has the required properties. $\square$

The next lemma is inspired by [8].

Lemma 4.2. *Let X be a nonsingular real algebraic set of dimension m. Let $\varphi\colon X \to \mathbb{R}^n$ be a regular map for which $0 \in \mathbb{R}^n$ is a regular value. Then, for any integer k with $0 \leq k \leq n$, there exists a nonempty Zariski open subset $G \subset \mathbb{G}(k,n)$ such that φ is transverse to every k-dimensional vector space H in G.*

Proof. We set $\Gamma = \mathrm{GL}_n(\mathbb{R})$ and work with a fixed $H \in \mathbb{G}(k,n)$.

First observe that if U is a nonempty Zariski open subset of Γ, then its image under the regular map

$$\alpha\colon \Gamma \to \mathbb{G}(k,n), \quad \sigma \mapsto \sigma(H)$$

contains a nonempty Zariski open subset of $\mathbb{G}(k,n)$. This can be seen as follows. The set $\alpha(U)$ is dense in $\mathbb{G}(k,n)$ in the Euclidean topology. Furthermore, by the Tarski–Seidenberg theorem [1], $\alpha(U)$ is a semialgebraic subset of $\mathbb{G}(k,n)$. Thus, in view of [1, Section 2.8], the Zariski closure A of $\mathbb{G}(k,n) \setminus \alpha(U)$ is a proper algebraic subset of $\mathbb{G}(k,n)$. The Zariski open set $\mathbb{G}(k,n) \setminus A$ is nonempty and contained in $\alpha(U)$.

In view of this observation, it suffices to prove that the set

$$\Gamma_0 = \{\sigma \in \Gamma\colon \text{the map } \sigma \circ \varphi\colon X \to \mathbb{R}^n \text{ is transverse to } H\}$$

contains a nonempty Zariski open subset of Γ. To this end, consider the regular map

$$\psi\colon \Gamma \times X \to \mathbb{R}^n, \quad (\sigma, x) \mapsto \sigma(\varphi(x)).$$

By construction, ψ is a submersion at every point $(\sigma, x) \in \Gamma \times X$; this is the case if $x \in \varphi^{-1}(0)$ since $0 \in \mathbb{R}^n$ is a regular value of φ, and is obvious otherwise. In particular, ψ is transverse to H. Consequently, the inverse image $V = \psi^{-1}(H)$ is a nonsingular Zariski closed subset of $\Gamma \times X$. Furthermore, for $\sigma \in \Gamma$, the map

$$\psi_\sigma\colon X \to \mathbb{R}, \quad x \mapsto \psi(\sigma, x) = \sigma(\varphi(x))$$

is transverse to H if and only if σ is a regular value of the regular map

$$\pi\colon V \to \Gamma, \quad (\sigma, x) \mapsto \sigma.$$

Hence the set Γ_0 can be written as

$$\Gamma_0 = \{\sigma \in \Gamma \colon \sigma \text{ is a regular value of } \pi\}.$$

Now we can easily complete the proof. The set $Z \subset V$ of critical points of π is Zariski closed, the map π being regular. By the Tarski–Seidenberg theorem, the set $\pi(Z)$ of critical values of π is semialgebraic. Hence the dimension of $\pi(Z)$ is the same as the dimension of its Zariski closure $B \subset \Gamma$ [1, Proposition 2.8.2]. It remains to prove that $B \neq \Gamma$ since then $\Gamma \setminus B$ is a nonempty Zariski open set contained in Γ_0. The equality $B = \Gamma$ would imply that $\pi(Z)$ contains a nonempty open (in the Euclidean topology) subset of Γ, see [1, Section 2.8], which in turn would violate Sard's theorem. $\square$

Proposition 4.3. *Let X be a nonsingular real algebraic set of pure dimension $n \geq 3$ and let $f \colon U \to \mathbb{R}$ be a function defined on an open (in the Euclidean topology) subset $U \subset X$. Assume that for every $(n-1)$-dimensional nonsingular algebraic subset Y of X the restriction $f|_{U\cap Y}$ is an analytic function. Then f is an analytic function.*

Proof. By Hironaka's theorem on resolution of singularities [6], X is biregularly isomorphic to a Zariski open subset of a compact nonsingular real algebraic set. Thus the proof is reduced to the case in which X is compact.

Let x_0 be a point in U. By Lemma 4.1, there exists a regular map $\varphi \colon X \to \mathbb{R}^n$ for which $0 \in \mathbb{R}^n$ is a regular value and $\varphi(x_0) = 0$. In view of Lemma 4.2, we can find a nonempty Zariski open subset $G \subset \mathbb{G}(n-1, n)$ such that φ is transverse to every hyperplane $Q \in G$. Now we choose a suitable open neighborhood $V' \subset U$ of x_0 so that

$$\Phi = \varphi|_{V'} \colon V' \to \varphi(V') = V$$

is an analytic diffeomorphism and $V \subset \mathbb{R}^n$ is an open ball with center at $0 \in \mathbb{R}^n$. Since X is compact, by replacing V' with a smaller set, we can assume that each point in V is a regular value of φ. We claim that the function

$$g = f \circ \Phi^{-1} \colon V \to \mathbb{R}$$

is analytic in a neighborhood of $0 \in \mathbb{R}^n$. We prove the claim by showing that g satisfies the assumptions of Theorem 3.3.

Let Q be a hyperplane in G. Since the map φ is transverse to Q, the set $Y = \varphi^{-1}(Q)$ is a nonsingular algebraic hypersurface in X. Hence, by assumption, $f|_{U\cap Y}$ is an analytic function. It follows that $g|_{V\cap Q}$ is an analytic function.

Let $C \subset V$ be a circle centered at $0 \in \mathbb{R}^n$. Since V is a ball, we can find an $(n-1)$-dimensional sphere $\Sigma \subset V$ that contains C. The inverse image $\Sigma' = \varphi^{-1}(\Sigma)$ is a nonsingular algebraic hypersurface in X because each point in V is a regular value of φ. By assumption, the function $f|_{U\cap\Sigma'}$ is analytic, hence the function $g|_{\Sigma}$ is also analytic. Consequently, the restriction $g|_C$ is analytic as well.

In view of Theorem 3.3, the function g is analytic in a neighborhood of $0 \in \mathbb{R}^n$, which means that f is analytic in a neighborhood of x_0. Since x_0 is an arbitrary point in U, the function f is analytic on U. $\square$

Corollary 4.4. *Let X be a nonsingular real algebraic set of pure dimension at least 3 and let $f \colon U \to \mathbb{R}$ be a function defined on an open (in the Euclidean topology) subset $U \subset X$. Assume that for every 2-dimensional nonsingular algebraic subset S of X the restriction $f|_{U\cap S}$ is an analytic function. Then f is an analytic function.*

Proof. We proceed by induction on $n = \dim X$. The case $n = 3$ is contained in Proposition 4.3. Suppose that $n \geq 4$. By Proposition 4.3, it suffices to prove that for every algebraic hypersurface $Y \subset X$ the restriction $f|_{U \cap Y}$ is an analytic function. We fix Y, set $V = U \cap Y$, and consider the function $g = f|_V$. If $S \subset Y$ is a nonsingular algebraic surface, then the restriction $g|_{V \cap S} = f|_{U \cap S}$ is analytic by assumption. Thus, by the induction hypothesis, the function g is analytic, which completes the proof. $\square$

Theorem 1.1 can be derived from Corollary 4.4 as follows.

Proof of Theorem 1.1. By the theorem of Nash–Tognoli [1, Theorem 14.1.10], we can find a nonsingular algebraic subset $X \subset \mathbb{R}^m$ and a $\mathcal{C}^\infty$ diffeomorphism $\varphi\colon M \to X$. In view of Grauert's theorem [5], M can be analytically embedded in some $\mathbb{R}^k$. Since X has an analytic tubular neighborhood in $\mathbb{R}^m$, it follows from the Weierstrass approximation theorem that φ can be approximated by analytic diffeomorphisms. We can therefore assume from the beginning that M is a compact nonsingular real algebraic set. Then it suffices to apply Corollary 4.4. $\square$

Our next goal are variants of Proposition 4.3 and Corollary 4.4 for Nash functions.

Proposition 4.5. *Let X be a nonsingular real algebraic set of pure dimension $n \geq 3$ and let $f\colon U \to \mathbb{R}$ be a function defined on an open (in the Euclidean topology) subset $U \subset X$. Assume that for every $(n-1)$-dimensional nonsingular algebraic subset Y of X the restriction $f|_{U \cap Y}$ is a Nash function. Then f is a Nash function.*

Proof. One can argue as in the proof of Proposition 4.3, substituting Theorem 3.6 for Theorem 3.3. $\square$

Corollary 4.6. *Let X be a nonsingular real algebraic set of pure dimension at least 3 and let $f\colon U \to \mathbb{R}$ be a function defined on an open (in the Euclidean topology) subset $U \subset X$. Assume that for every 2-dimensional nonsingular algebraic subset S of X the restriction $f|_{U \cap S}$ is a Nash function. Then f is a Nash function.*

Proof. One can copy the proof of Corollary 4.4, substituting Proposition 4.5 for Proposition 4.3. $\square$

Now we can prove a result somewhat more general than Theorem 1.2.

Theorem 4.7. *Let N be a Nash manifold of dimension at least 3 and let $f\colon U \to \mathbb{R}$ be a function defined on an open subset U of N. Assume that for every 2-dimensional Nash submanifold S of N the restriction $f|_{U \cap S}$ is a Nash function. Then f is a Nash function.*

Proof. The manifold N has finitely many connected components, being a semialgebraic set. Hence, in view of the Artin–Mazur theorem [1], N can be regarded as an open (in the Euclidean topology) subset of a nonsingular real algebraic set. We complete the proof by applying Corollary 4.6. $\square$

Proof of Theorem 1.2. It suffices to take $U = N$ in Theorem 4.7. $\square$

The proof of Theorem 1.3 requires additional preparation. To begin with, we introduce some terminology.

Let X be a nonsingular real algebraic set and let $f\colon U \to \mathbb{R}$ be a function defined on an open (in the Euclidean topology) subset $U \subset X$.

We say that a pair (φ, ψ), where φ and ψ are regular functions on X, is a *rational representation* of f if

$$\psi f = \varphi \quad \text{on } U,$$

and ψ is not identically 0 on any irreducible component of X.

We say that f is a *regular function* if, for every point $x \in U$, it has a rational representation (φ_x, ψ_x) with $\psi_x(x) \neq 0$. In that case, we can find a finite collection $(\varphi_1, \psi_1), \ldots, (\varphi_k, \psi_k)$ of rational representations of f such that

$$U \cap \bigcap_{i=1}^{k} \psi_k^{-1}(0) = \varnothing$$

(this is possible since the Zariski topology on X is Noetherian). Setting

$$\Phi = \sum_{i=1}^{k} \varphi_i \psi_i, \quad \Psi = \sum_{k=0}^{k} \psi_i^2,$$

we obtain a rational representation (Φ, Ψ) of f with $\Psi(x) \neq 0$ for all $x \in U$. The notion of regular function introduced here coincides with the standard one familiar from real algebraic geometry, provided that the set U is Zariski open in X.

We will need the following simple observation, which is explicitly recorded in [10, Proposition 2.1]: If f is analytic and admits a rational representation, then f is actually a regular function. This can also be seen directly since, for any point $x \in U$, the ring of germs of analytic functions at x is faithfully flat over the ring of germs of regular functions at x (the algebraic assertion here is a consequence of [12, Theorem 8.14]).

In what follows certain constructions will depend on the classical theorem of Bertini [11]. For this an appropriate setup is necessary.

One can always express X as $X = \mathbb{X}(\mathbb{R})$, where $\mathbb{X}$ is a nonsingular affine complex algebraic variety defined over $\mathbb{R}$, and $\mathbb{X}(\mathbb{R})$ stands for its set of real points. Moreover, one can choose $\mathbb{X}$ irreducible if X is such.

We denote by $\mathbb{A}^r$ (resp. $\mathbb{P}^r$) complex affine (resp. projective) r-space, regarded as a variety over $\mathbb{R}$.

Theorem 4.8. *Let X be a nonsingular real algebraic set of pure dimension at least 3 and let $f\colon U \to \mathbb{R}$ be a function defined on an open (in the Euclidean topology) subset U of X. Assume that for every 2-dimensional nonsingular algebraic subset S of X the restriction $f|_{U\cap S}$ is a regular function. Then f is a regular function.*

Proof. We can assume without loss of generality that X is irreducible and of the form $X = \mathbb{X}(\mathbb{R})$, where $\mathbb{X}$ is an irreducible nonsingular complex algebraic variety defined over $\mathbb{R}$. We regard $\mathbb{X}$ as a Zariski locally closed subset of $\mathbb{P}^r$ for some r. Set $n = \dim X = \dim \mathbb{X}$. By Corollary 4.6, f is a Nash function, hence it remains to prove that f admits a rational representation. We do it in two steps.

Step 1. Suppose that the set U is connected. Since f is a Nash function, its graph is contained in an irreducible algebraic hypersurface $\mathbb{Y} \subset \mathbb{X} \times \mathbb{A}^1$, defined over $\mathbb{R}$. Note that f has a rational representation if and only if $\pi\colon \mathbb{Y} \to \mathbb{X}$, the restriction of the projection $\mathbb{X} \times \mathbb{A}^1 \to \mathbb{X}$, is a birational morphism. Suppose that π is not birational, that is, it has degree $m > 1$. By Bertini's theorem, for a general linear subspace $\mathbb{L} \subset \mathbb{P}^r$ of dimension

$r-n+2$, both $\mathbb{X}\cap\mathbb{L}$ and $\pi^{-1}(\mathbb{X}\cap\mathbb{L})$ are irreducible algebraic surfaces. We can choose such an $\mathbb{L}$ so that $\mathbb{X}\cap\mathbb{L}$ is nonsingular and $U\cap S\neq\varnothing$, where

$$S=(\mathbb{X}\cap\mathbb{L})(\mathbb{R})=X\cap\mathbb{L}(\mathbb{R}).$$

Note that S is a nonsingular algebraic subset of X of dimension 2. Furthermore, the restriction $\pi_0\colon \pi^{-1}(\mathbb{X}\cap\mathbb{L})\to\mathbb{X}\cap\mathbb{L}$ of π has degree m, and therefore it is not birational. Since the graph of $f|_{U\cap S}$ lies on $\pi^{-1}(\mathbb{X}\cap\mathbb{L})$, the function $f|_{U\cap S}$ does not admit a rational representation. Hence $f|_{U\cap S}$ is not regular, contrary to the assumption. This completes the proof of Step 1.

Step 2. Consider the case of an arbitrary U. Pick a point $a\in U$, and let U_a be the connected component containing a. By Step 1, the restriction $f|_{U_a}$ is a regular function, and hence there exists a pair (φ,ψ) of regular functions on X such that

$$\psi(x)\neq 0 \quad\text{and}\quad \psi(x)f(x)=\varphi(x) \quad\text{for all } x\in U_a.$$

Now the proof is reduced to showing

$$\psi(x)f(x)=\varphi(x) \quad\text{for all } x\in U.$$

Suppose this is not the case. Then the set

$$W=\{x\in U\colon \psi(x)\neq 0,\ \psi(x)f(x)\neq\varphi(x)\}$$

is open and nonempty.

By Bertini's theorem, there exists an irreducible nonsingular algebraic subset $R\subset X$, of dimension 2, such that

$$U_a\cap R\neq\varnothing \quad\text{and}\quad W\cap R\neq\varnothing;$$

we obtain such an R of the form $R=(\mathbb{X}\cap\mathbb{M})(\mathbb{R})$, where $\mathbb{M}\subset\mathbb{P}^r$ is a suitable linear subspace of dimension $r-n+2$. Since the restriction $f|_{U\cap R}$ is a regular function, we can find a pair (α,β) of regular functions on R such that

$$\beta(x)\neq 0 \quad\text{and}\quad \beta(x)f(x)=\alpha(x) \quad\text{for all } x\in U\cap R.$$

By construction,

$$f(x)=\frac{\varphi(x)}{\psi(x)} \quad\text{and}\quad f(x)=\frac{\alpha(x)}{\beta(x)} \quad\text{for all } x\in U_a\cap R.$$

Consequently, we get

$$\beta(x)\varphi(x)=\psi(x)\alpha(x) \quad\text{for all } x\in R$$

because R is irreducible, and the functions $\varphi|_R$, $\psi|_R$, α, β are regular on R. Thus, chosing a point $y\in W\cap R$, we get

$$\varphi(y)=\psi(y)\frac{\alpha(y)}{\beta(y)}=\psi(y)f(y),$$

a contradiction. The proof is complete.

□

Proof of Theorem 1.3. It suffices to take $U=X$ in Theorem 4.8. □

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