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Cataland: Why the Fuß?

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Cataland: Why the Fuß?

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To Anke, Magali and Maria.

Contents

Abstract

The three main objects in noncrossing Catalan combinatorics associated to a finite Coxeter system are noncrossing partitions, clusters, and sortable elements. The first two of these have known Fuß-Catalan generalizations. We provide new viewpoints for both and introduce the missing generalization of sortable elements by lifting the theory from the Coxeter system to the associated positive Artin monoid. We show how this new perspective ties together all three generalizations, providing a uniform framework for noncrossing Fuß-Catalan combinatorics. Having developed the combinatorial theory, we provide an interpretation of our generalizations in the language of the representation theory of hereditary Artin algebras.

Acknowledgements

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We made substantial progress in June 2014 at the workshop “Non-crossing partitions in representation theory” at Bielefeld University. We thank the funding agencies for financial support, and we thank the organizers B. Baumeister, A. Hubery, and H. Krause.

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We thank Drew Armstrong, Christian Krattenthaler, Nathan Reading and Vic Reiner for many helpful discussions.

CHAPTER 1

Introduction

1.1. A brief historical overview

We take the perspective that there are three distinct families of noncrossing Catalan objects—noncrossing partitions, triangulations, and stack-sortable permutations. The objects in each family are counted by the *Catalan number*

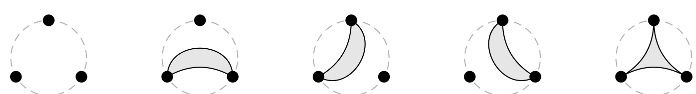
$$\mathsf{Cat}_n := \frac{1}{n+1}\binom{2n}{n}.$$

It could be argued that there are a few other Catalan objects...

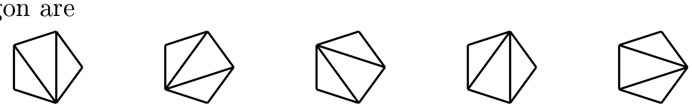
#1

Historically, these three families arose as follows (we follow the organization of the monograph, rather than giving a chronological account).

• In 1972, G. Kreweras introduced and studied *noncrossing set partitions* of the set $\{1,\ldots,n\}$ [**Kre72**]. In addition to many enumerative results, he proved that the noncrossing partitions form a lattice under refinement. This curious property is intimately connected to the $K(\pi,1)$ problem for the braid group. We refer to [**Arm06**, Chapter 1] and [**Bes15**] for details. The five noncrossing partitions for $n=3$ are



• In 1751, L. Euler guessed the enumeration of *triangulations of a convex* $(n+2)$*-gon*. J. Segner discovered the standard combinatorial recurrence in [**DS58**], from which L. Euler was able to prove the enumeration. A complete proof that these triangulations were counted by Cat_n first appeared in 1761—E. C. Catalan was later born in 1814, some 50 years later. We refer to [**Sta15**, Appendix B, History of Catalan Numbers] for a comprehensive historical treatment. The five triangulations of a pentagon are



• In 1968, D. Knuth introduced the notion of *stack-sortability* as those permutations in $\mathfrak{S}_n$ that can be sorted in a single pass through a stack. In [**Knu73**, Exercise 2.2.1.5], he gives the exercise of showing that such permutations are characterized as those avoiding the pattern 231. Of the six permutations in $\mathfrak{S}_3$, the five stack-sortable are

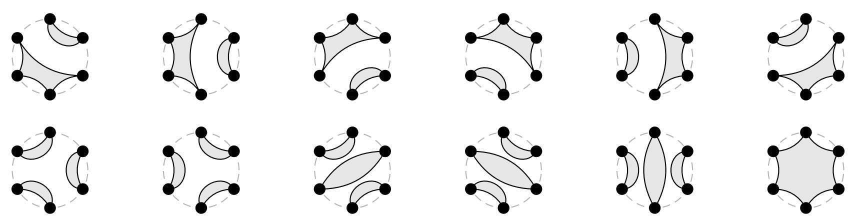
$$123, \quad 213, \quad 132, \quad 321, \quad 312.$$

This monograph is the continuation of a program to simultaneously generalizing these three classical families in two orthogonal directions: a Fuß-Catalan direction, and a Coxeter-theoretic direction. We review these generalizations now.

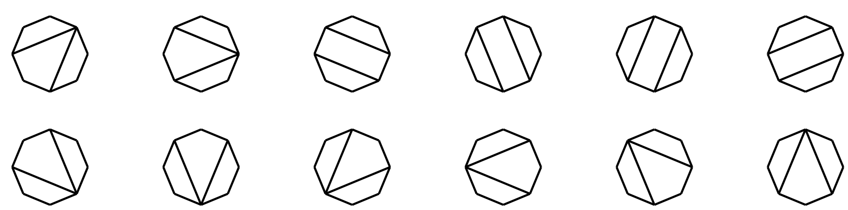
Fuß-Catalan generalizations. This direction seeks *m-eralizations* of Catalan families—generalizations that incorporate an additional nonnegative integral parameter m. Objects should now be counted by the *Fuß-Catalan number*[A]

$$\mathsf{Cat}_n^{(m)} := \frac{1}{mn+1}\binom{(m+1)n}{n}.$$

• G. Kreweras enumerated chains in the noncrossing partition lattice [**Kre72**, Section 6]. In 1980, P. Edelman put a partial order on these chains in [**Ede80**], and interpreted them as those noncrossing partitions on $\{1,\ldots,mn\}$ with block sizes divisible by m. These are illustrated for $n=3$ and $m=2$ as



• The m-eralization of triangulations goes back to N. Fuß in 1791. As detailed in [**Arm06**, Section 3.5], N. Fuß counted the number of dissections of a convex $(mn{+}2)$-gon into $(m{+}2)$-gons [**Fuß91**]. The twelve quadrangulations of an octagon are



• While m-eralizations have been established for the first two families, we are not aware of any previous m-eralization of the stack-sortable permutations. We give one in Chapter 6.

Coxeter-theoretic generalizations. This second direction—championed by R. Simion and V. Reiner—is concerned with a generalization of Catalan objects to finite Coxeter groups W. Although no uniform proof is currently known, Coxeter-Catalan families are counted by the *Catalan number of type W*,

$$\mathsf{Cat}(W) := \prod_{i=1}^{n} \frac{h+d_i}{d_i},$$

where $d_1 \leq d_2 \leq \ldots \leq d_n$ are the degrees of the fundamental invariants of W and $h := d_n$ is the Coxeter number. For the symmetric group $\mathfrak{S}_n$, this definition recovers the classical Catalan numbers. All three classical Catalan objects have such generalizations:

'cause otherwise, calling them Catalan numbers wouldn't make sense.

#2

• In unpublished work from 1993 [**Mon93**], C. Montenegro counted centrally symmetric noncrossing partitions. In 1997, V. Reiner associated these to the root system of type B [**Rei97**] and considered more refined enumerative properties. He posed the problem of generalizing this construction to all reflection groups.

[A] The Fuß-Catalan numbers—and this monograph—are named for N. I. Fuß & E. C. Catalan.

The answer to V. Reiner's question came from the theory of Artin groups. In 1998, J. Birman, K. H. Ko, and S. J. Lee gave a remarkable new presentation of the braid group [**BKL98**]. In the early 2000s—inspired by this and the $K(\pi, 1)$ problem for complex reflection groups—T. Brady and C. Watt [**Bra01, BW02**], and D. Bessis [**Bes03**] gave a uniform definition of noncrossing partition lattices for all finite Coxeter groups.

• Coxeter-theoretic generalizations of triangulations originated in the theory of polytopes. Triangulations of an $(n+2)$-gon naturally index the vertices of an n-dimensional polytope called the *associahedron*, famously studied by D. Tamari after his 1951 thesis (as a poset, which he later showed was a lattice) and in J. Stasheff's 1961 thesis (as a polytope). The centrally symmetric triangulations of a $(2n)$-gon naturally index the vertices of a polytope J. Stasheff called the *cyclohedron* [**Sta97**], which first appeared in 1994 work of R. Bott and C. Taubes [**BT94**]. R. Simion independently asked and solved the problem of finding a type B analogue of the associahedron [**Sim03**].

The construction for general root systems came from the theory of cluster algebras. As a consequence of their classification of finite-type cluster algebras [**FZ03**], S. Fomin and A. Zelevinsky gave uniform definitions of associahedra for all Weyl groups. These were realized as polytopes by many researchers in many ways, as detailed in [**HPS18**]. We refer to the entire collection [**MHPS12**] for further background.

• The Coxeter-theoretic generalization of stack-sortable elements is intimately connected to polytopes, but also to lattice theory. It is well-known that the associahedron can be constructed from the permutahedron by giving a map from permutations to triangulations [**BW97, Ton97**]. Answering a question of R. Simion from [**Sim03**], V. Reiner constructed the cyclohedron from the permutahedron of type B using a theory of equivariant fiber polytopes. In particular, this construction defined a map from signed permutations to centrally symmetric triangulations [**Rei02**].

In 2005, N. Reading uniformly generalized such maps to all finite Coxeter groups using the notion of *Coxeter-sortability*. He used the weak order to define a generalization of the Tamari lattices he called *Cambrian lattices*, giving a combinatorial model for the cluster complex in the corresponding Coxeter group.

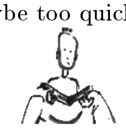
Fuß-Coxeter-Catalan generalizations.

The Coxeter-theoretic generalizations of noncrossing partitions and triangulations have both previously been m-eralized to objects counted by the *Fuß-Catalan numbers of type W*,

$$\mathsf{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i}{d_i}.$$

In his 2006 thesis [**Arm06**], D. Armstrong defined *m-eralized noncrossing partitions* by applying P. Edelman's construction to the uniformly-defined noncrossing partition lattices for finite Coxeter groups. In 2005, S. Fomin and N. Reading found an *m-eralized cluster complex* for bipartite Coxeter elements [**FR05**].

This is the end of the historic treatment. It was a little quick, maybe too quick...



#1

While both m-eralizations have been thoroughly studied, there are several missing pieces:

- an m-eralized cluster complex for general Coxeter elements,
- an m-eralization of sortable elements,
- an m-eralized Cambrian lattice, and
- bijections relating m-eralized noncrossing partitions to the above.

Since the weak order on a finite Coxeter group W is so fundamental to the $m = 1$ versions of the above, it is reasonable to start our search with

- an m-eralization of the weak order.

1.2. Summary of results

This monograph introduces these missing m-eralizations by passing from the Coxeter group to its **positive Artin monoid**. In the remainder of this introduction, we sketch our main results.

tl;dr: we generalized some combinatorics from Coxeter groups to their associated positive Artin monoids.

#3

The weak order. In Chapter 2, we introduce an m-eralization of the weak order using the positive Artin monoid $\boldsymbol{B}^+$ corresponding to W. Let $\boldsymbol{w}_\circ$ be the image of the longest element of W in $\boldsymbol{B}^+$, so that the weak order on W embeds into the weak order on $\boldsymbol{B}^+$ as the interval consisting of those elements that are initial in $\boldsymbol{w}_\circ$. We introduce the Fuß-Catalan parameter m as follows.

Definition 2.11.1. $W^{(m)} := \left\{ \boldsymbol{w} \in \boldsymbol{B}^+ \ : \ \boldsymbol{w}\boldsymbol{u} = \boldsymbol{w}_\circ^m \text{ for some } \boldsymbol{u} \in \boldsymbol{B}^+ \right\}$.

We recall in Proposition 2.11.4 that $W^{(m)}$ contains exactly those elements of $\boldsymbol{B}^+$ with at most m Garside factors, and show in Theorem 2.11.5 that the weak order on $W^{(m)}$ is a rank-symmetric and self-dual lattice.

Subword complexes. In Chapter 3, we recall some background from simplicial complexes and then generalize the notion of subword complexes to the needed generality in Definition 3.2.1 and Proposition 3.2.4. We finally introduce and study Coxeter-initial subword complexes in Sections 3.5 and 3.5. These later turn out to be the generality in which we consider subword complexes in this monograph.

Noncrossing partitions. In Chapter 4, we review background on m-eralized noncrossing partitions, including their description in terms of a dual subword complex in Theorem 4.2.5. We m-eralize the notions of Cambrian rotation and recurrence, and provide a novel perspective—new even for $m = 1$—by defining an m-eralized c-Cambrian poset structure on noncrossing partitions in Definition 4.4.2.

Cluster complexes. In Chapter 5, we consider the missing m-eralization of the c-cluster complex for arbitrary Coxeter elements c. We use an m-eralized c-compatibility relation whose existence and uniqueness are proven by m-eralizing the subword complex approach to c-cluster complexes given in [**CLS14**, **PS15**].

Definition 5.3.1. The simplicial complex $\mathsf{Asso}^{(m)}(W, c)$ is the set of subwords of the c-sorting word $\mathsf{cw}_\circ^m(\mathsf{c})$ (see Definition 2.6.1) whose complements contain a word for $\boldsymbol{w}_\circ^m \in \boldsymbol{B}^+$.

We show in Corollary 5.5.7 that in the special case of bipartite Coxeter elements, this definition recovers the generalized cluster complex of S. Fomin and N. Reading [**FR05**]. This perspective allows us to give simple proofs of many of its known properties. In Theorem 5.6.1 we prove C. Athanasiadis and E. Tzanaki's result from [**AT08**] that $\mathsf{Asso}^{(m)}(W, c)$ is vertex-decomposable (and therefore shellable);

and in Theorem 5.6.2 we show that $\mathsf{Asso}^{(m)}(W,c)$ is a wedge of $\mathsf{Cat}^{(m-1)}(W)$ many spheres of dimension $n-1$ [**FR05**, Proposition 11.1].

Sortable elements. In Chapter 6, we provide the missing m-eralization of Coxeter-sortable elements by lifting N. Reading's definition from $W^{(m)}$ (Definition 6.1.1). We denote the set of all *m-eralized c-sortable elements* by $\mathsf{Sort}^{(m)}(W,c)$, and characterize the individual Garside factors of an m-eralized c-sortable element in Definition 6.4.1 and in Corollary 6.4.4.

Bijections. The following summary of theorems relates D. Armstrong's m-eralization of noncrossing partitions, S. Fomin and N. Reading's m-eralization of cluster complexes, and the new m-eralizations of sortable elements and subword complexes.

THEOREMS 5.7.2, 6.8.5, 6.8.10, 6.9.7. *There are explicit, uniform, and natural bijections between the three families*

- *the m-eralized c-noncrossing partitions $\mathsf{NC}^{(m)}(W,c)$,*
- *the m-eralized c-cluster complexes $\mathsf{Asso}^{(m)}(W,c)$, and*
- *the m-eralized c-sortable elements $\mathsf{Sort}^{(m)}(W,c)$.*

We use the term "explicit" to mean that we provide a bijection (rather than relying on a counting argument), the term "uniform" to indicate that we do not use the classification theorem of finite Coxeter systems, and the term "natural" to mean that the given bijections respect the inductive parabolic structure on each family. In fact, the proofs will very often be based on the inductive structure provided by a modification of Cambrian rotation called the *Cambrian recurrence*, as in Propositions 6.3.1, 4.3.3, and 5.4.5.

Cambrian lattices. Although the flip graph of S. Fomin and N. Reading's m-eralized c-cluster complex can be used to define a Cambrian graph for bipartite Coxeter elements, no corresponding poset has been considered in the literature for $m > 1$. In particular, no orientation of the m-eralized exchange graph was known to be a lattice.

In Sections 4.4, 5.3, and 6.6, we give definitions of the m-eralized c-Cambrian lattice on each of $\mathsf{NC}^{(m)}(W,c)$, $\mathsf{Asso}^{(m)}(W,c)$, and $\mathsf{Sort}^{(m)}(W,c)$. For $\mathsf{NC}^{(m)}(W,c)$ and $\mathsf{Asso}^{(m)}(W,c)$, we construct these posets as the transitive closures of the objects under certain flips, while $\mathsf{Sort}^{(m)}(W,c)$ inherits its poset structure from the weak order on $W^{(m)}$. The present construction m-eralizes N. Reading's Cambrian lattices, which are themselves generalizations of the classical Tamari lattices. As a case of particular interest, this construction provides a new m-eralization of the Tamari lattice, different from the m-Tamari lattice (Section 5.8).

THEOREMS 5.7.3, 6.6.4, 6.8.6. *The restriction of the weak order to $\mathsf{Sort}^{(m)}(W,c)$ is a lattice. It is isomorphic to the increasing flip posets of $\mathsf{NC}^{(m)}(W,c)$ and of $\mathsf{Asso}^{(m)}(W,c)$.*

Positive and rational Catalan combinatorics. In Chapter 7, we study *positive* analogues of the three noncrossing Catalan families, and we explain a special symmetry on positive m-eralized c-noncrossing partitions.

D. Armstrong has proposed that noncrossing Fuß-Catalan combinatorics should have a "rational Catalan" generalization to accommodate a parameter p coprime to the Coxeter number h—the m-eralizations of noncrossing objects should then be recovered for $p = mh + 1$, while the positive analogues should correspond to $p = mh - 1$ [**ARW13, ALW16**]. In Chapter 8, we give conjectural constructions in the classical types.

Sorry, conjectural.

#4

Representation Theory. Chapter 9 discusses the link to the representation theory of hereditary Artin algebras. For W crystallographic, m-eralized noncrossing partitions and m-eralized clusters have been given representation-theoretic interpretations in, respectively, [**BRT12**] and [**Tho07, Zhu08**]. We show that the m-eralized c-sortable elements also fit into this framework, generalizing the work of C. Ingalls and H. Thomas for $m = 1$ [**IT09**]. It turns out that the combinatorial bijections of Theorems 5.7.2 and 6.8.10 coincide in crystallographic types with representation-theoretic bijections [**BRT12, KV88**].

Figure 1 is perfect for use as a desktop background, accent wall decor, or tattoo.

#5

Figure 1 illustrates these objects and their bijections.

Gross omissions and open problems. There are many directions in Coxeter-Catalan combinatorics that are outside the scope of this monograph, and there are also many open problems. We give a short list here, which is by no means meant to be exhaustive:

- We do not give type-by-type combinatorial constructions of our noncrossing Fuß-Catalan objects. Enumerations in the classical types often make use of such constructions [**FR05, Arm06**].
- We do not do any refined enumeration, nor do we address cyclic sieving [**KS18, BR11**].
- We do not talk about *nonnesting* Catalan objects (using root posets or the affine Weyl group) [**Rei97**], or touch on the vast field of q,t-Catalan combinatorics [**Hai94**].
- We do not address recent advances regarding noncrossing parking functions and parking spaces [**Rho14, ARR15**].
- Outside of the historical overview, we do not mention polytopes or discuss realizations of associahedra.
- We do not construct m-eralized cluster *algebras*.
- We do not consider our constructions for infinite Coxeter groups [**RS11**]. This would already be interesting in affine type [**RS18**].

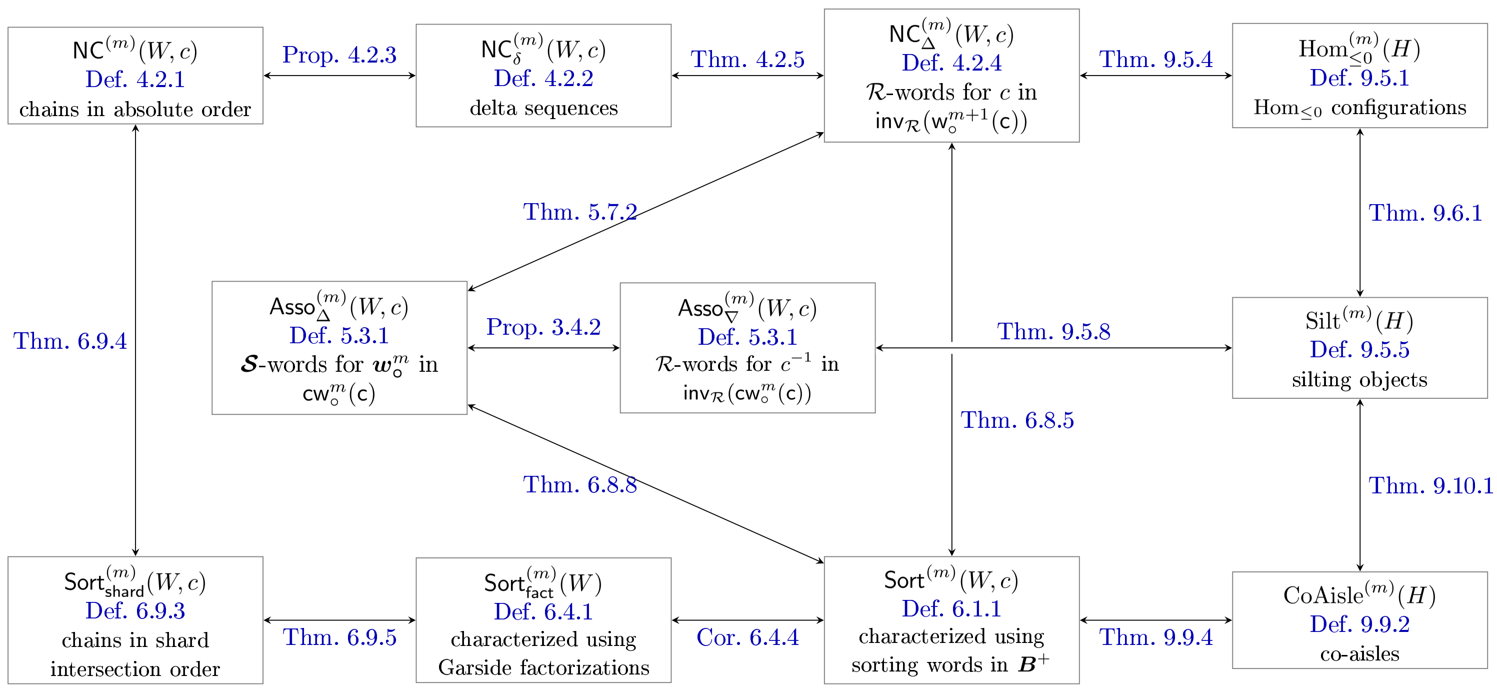


FIGURE 1. A diagram of the Fuß–Catalan objects and their bijections. The top line contains the noncrossing partition-type objects (Chapter 4), the middle line the cluster-type objects (Chapter 5), and the bottom line the sortable-type objects (Chapter 6). The rightmost column contains the representation-theoretic counterparts (Chapter 9).

CHAPTER 2

Background on Coxeter and Artin groups

In this chapter, we review the theory of finite Coxeter and Artin groups. Most of this material is classical and detailed background can be found in [**Hum90**, **BB05**, **DDG$^+$15**]. After recalling this background (Sections 2.1 to 2.5), we extend the theory of sorting words to the positive Artin monoid (Section 2.6) and recall the geometry of Coxeter groups (Sections 2.8 and 2.9). We then discuss the shard intersection order and give a new characterization in the Coxeter group (Section 2.10). We conclude with the definition of the m-eralized weak order inside the positive Artin monoid (Section 2.11).

If you can read Drew Armstrong's monograph, you'll be fine.

#6

2.1. Coxeter and Artin systems

A (finite) *Coxeter system* $(W, \mathcal{S})$ of *rank* $n := |\mathcal{S}|$ is a finite group W together with a distinguished subset $\mathcal{S} \subseteq W$ of generators and a presentation

$$W = \left\langle \mathcal{S} : s^2 = e, \quad [s|t]^{m(s,t)} = [t|s]^{m(t,s)} \text{ for } s, t \in \mathcal{S} \text{ with } s \neq t \right\rangle,$$

for integers $m(s,t) = m(t,s) \geq 2$, where $[s|t]^{m(s,t)}$ consists of $m(s,t)$ alternating factors s and t.[B] The elements in the set $\mathcal{S}$ are called *simple generators* or *simple reflections*. The relations $sts\cdots = tst\cdots$ are called *braid relations*, and invoking one to rewrite a word in $\mathcal{S}$ is called a *braid move*. A braid relation of the form $st = ts$ is called a *commutation relation*.

The set of *reflections* in W is defined to be

$$\mathcal{R} := \left\{ s^w \ : \ s \in \mathcal{S}, w \in W \right\},$$

where we write $u^w := w^{-1}uw$ and ${}^w u := wuw^{-1}$.

The (spherical) *Artin system* $(\boldsymbol{B}, \boldsymbol{\mathcal{S}})$ corresponding to the finite Coxeter system $(W, \mathcal{S})$ is the group $\boldsymbol{B}$ given by a formal copy of the generators $\mathcal{S}$—written $\boldsymbol{\mathcal{S}}$—subject to only the braid relations

$$\boldsymbol{B} = \left\langle \boldsymbol{\mathcal{S}} : [\boldsymbol{s}|\boldsymbol{t}]^{m(s,t)} = [\boldsymbol{t}|\boldsymbol{s}]^{m(t,s)} \text{ for } s, t \in \mathcal{S} \text{ with } s \neq t \right\rangle.$$

We mostly restrict to the submonoid generated by $\boldsymbol{\mathcal{S}}$, called the *positive Artin monoid* $\boldsymbol{B}^+$.

Warning. While we used boldface above to distinguish between the simple generators $s \in \mathcal{S}$ and the corresponding generator $\boldsymbol{s} \in \boldsymbol{\mathcal{S}}$, we usuall do not distinguish between these two generating sets if there is no risk of confusion. When confusion may arise, we still use boldface to distinguish elements $\boldsymbol{w} \in \boldsymbol{B}^+$ and $w \in W$.

This is great notation. It even comes with a 90 day warranty.

#7

[B] We will regularly use the notation $[u|v]^i$ for a positive integer i to mean i alternating copies of u and v—both being elements in the group W, in the positive monoid $\boldsymbol{B}^+$, or words in $\mathcal{S}$ or in $\mathcal{R}$.

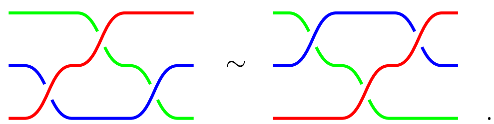
EXAMPLE 2.1.1. When W is the *symmetric group* $\mathfrak{S}_n$, $\boldsymbol{B}$ is the *braid group* $\boldsymbol{\mathfrak{S}}_n$, and $\boldsymbol{B}^+$ is the *positive braid monoid* $\boldsymbol{\mathfrak{S}}_n^+$. We often illustrate definitions and results using the running example of the symmetric group on three letters $\mathfrak{S}_3$, which is the Coxeter group (of type A_2 as given in Section 2.1.2) generated by the simple reflections $\mathcal{S} = \{s, t\}$ corresponding to the transpositions $s = (12)$ and $t = (23)$. We denote the third transposition by $u = (13)$, which completes the set of reflections

$$\mathcal{R} = \{s, u, t\} = \{(12), (13), (23)\}.$$

In addition to these three reflections and the identity element e, $\mathfrak{S}_3$ contains the two long cycles

$$(123) = st = tu = us \quad \text{and} \quad (321) = ts = ut = su.$$

For the symmetric group $\mathfrak{S}_n$ on n letters, we sometimes use $\mathcal{S} = s_1, \ldots, s_{n-1}$ with $s_i = (i\ i{+}1) \in \mathfrak{S}_n$. The positive Artin monoid $\boldsymbol{\mathfrak{S}}_3^+$ contains all words in the letters $\boldsymbol{s}$ and $\boldsymbol{t}$, subject to the relation $\boldsymbol{sts} = \boldsymbol{tst}$, indicated as



2.1.1. Words in generators. We use sans-serif letters to distinguish *words* in generators from *elements* in W or in $\boldsymbol{B}^+$. In this way, $\mathsf{s}_1 \cdots \mathsf{s}_p$ is an $\mathcal{S}$-word, while $s_1 \cdots s_p \in W$ or $\boldsymbol{s}_1 \cdots \boldsymbol{s}_p \in \boldsymbol{B}^+$ are elements. We call two $\mathcal{S}$- or $\mathcal{R}$-words Q and Q' *commutation equivalent* if we may transform one into the other by a sequence of interchanges of consecutive commuting letters, *i.e.*, letters that commute in the group. In this case, we write $\mathsf{Q} \equiv \mathsf{Q}'$. A word u (often a single letter) is *initial* or *final* in Q if u occurs as a prefix or, respectively, as a suffix of some $\mathsf{Q}' \equiv \mathsf{Q}$. We denote the word obtained from a word Q by removing an initial letter s by $\bar{\mathsf{s}}\mathsf{Q}$ and write $\mathsf{Q}\bar{\mathsf{s}}$ for the removal of a final letter s.

We take typography very seriously.

#8

But we don't necessarily respect our own conventions.

#2

We use the notation $\psi(s) := w^{w_\circ} \in \mathcal{S}$ for $s \in \mathcal{S}$ and extend this notation to elements in W and $\boldsymbol{B}^+$ and also to words. We denote the reverse of a word Q by $\mathrm{rev}(\mathsf{Q})$ and extend this notation to elements in $\boldsymbol{B}^+$.

2.1.2. Classifications of Coxeter groups. The *Coxeter diagram* of $(W, \mathcal{S})$ is the graph on $\mathcal{S}$ with an edge $s - t$ if $m(s,t) \geq 3$ labelled by $m(s,t)$. Usually, the label $m(s,t) = 3$ is omitted. The Coxeter system is *simply-laced* if $m(s,t) \in \{2,3\}$ for all $s,t \in \mathcal{S}$ and *crystallographic* if $m(s,t) \in \{2,3,4,6\}$ for all $s,t \in \mathcal{S}$.

A Coxeter system is called *irreducible* if its Coxeter diagram is connected. Irreducible finite Coxeter systems are classified, the classification is shown in Figure 2. See [**BB05**, Appendix A1] for details.

Regardless of whether or not you know this content, you should probably be reading something else.

#9

2.2. The weak order

2.2.1. Words in simple reflections. The (Coxeter) *length* of an element w in W or in $\boldsymbol{B}^+$ is the length $\ell_{\mathcal{S}}(w)$ of a shortest expression for w as a product of the generators in $\mathcal{S}$. Examples are given in Figures 3 and 4. An $\mathcal{S}$-word $\mathsf{s}_1 \cdots \mathsf{s}_p$ is *reduced* if $w = s_1 \cdots s_p$ and $p = \ell_{\mathcal{S}}(w)$. A factorization $w = u \cdot v$ is *reduced* if $\ell_{\mathcal{S}}(w) = \ell_{\mathcal{S}}(u) + \ell_{\mathcal{S}}(v)$. In this case, u is *initial* and v is *final* in w and we set $\overline{u}w := u^{-1}w$, where we emphasize that $\overline{\boldsymbol{u}}\boldsymbol{w} \in \boldsymbol{B}^+$ for $\boldsymbol{w} \in \boldsymbol{B}^+$. A sequence

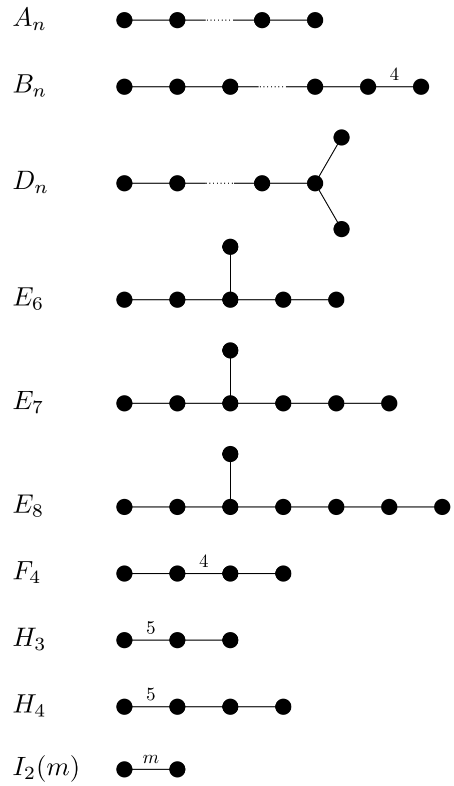


FIGURE 2. Classification of the irreducible Coxeter groups and diagrams.

$s_1, \ldots, s_p \in \mathcal{S}$ is an *initial sequence* for w if s_{i+1} is initial in $\overline{s}_i \ldots \overline{s}_2 \overline{s}_1 w s_1 s_2 \ldots s_i$. We use analogous notation for s final in w and final sequences for w.

E. Brieskorn and K. Saito proved the following essential lemma.

Double induction, oh my God! ⋒

#10

LEMMA 2.2.1 ([**BS72**, Lemma 2.1 & Proposition 2.3]). *Let* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}^+$.

(1) If $\boldsymbol{aub} = \boldsymbol{avb}$ *for* $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{B}^+$*, then* $\boldsymbol{u} = \boldsymbol{v}$.

(2) If $\boldsymbol{su} = \boldsymbol{tv}$ *for* $\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{\mathcal{S}}$*, then there exists* $\boldsymbol{w} \in \boldsymbol{B}^+$ *such that* $\boldsymbol{u} = [\boldsymbol{t}|\boldsymbol{s}]^{m(s,t)-1}\boldsymbol{w}$ *and* $\boldsymbol{v} = [\boldsymbol{s}|\boldsymbol{t}]^{m(s,t)-1}\boldsymbol{w}$. □

Any reduced $\mathcal{S}$-word of an element in W may be transformed to any other by a sequence of braid moves [**BB05**, Theorem 3.3.1]. We define the *support* $\mathrm{supp}(w)$ for $w \in W$ or $\boldsymbol{B}^+$ to be the set $\{s_1, \ldots, s_p\} \subseteq \mathcal{S}$ of simple reflections contained in any reduced word $\mathsf{s}_1 \cdots \mathsf{s}_p$ for w. Support is well-defined, since reduced words are connected under braid moves, and braid moves preserve the set of simple reflections in a reduced word.

Identifying the generating sets for W and $\boldsymbol{B}^+$ gives a natural injection

$$W \hookrightarrow \boldsymbol{B}^+. \tag{2.1}$$

Since $\boldsymbol{B}^+$ is subject to the braid relations, any two reduced $\mathcal{S}$-words for $w \in W$ specify the same element $\boldsymbol{w} \in \boldsymbol{B}^+$. The unique *longest element* in W is denoted by $w_\circ$. The corresponding element $\boldsymbol{w}_\circ \in \boldsymbol{B}^+$ is sometimes called the *fundamental element* in Garside theory. We denote its length by $N := \ell_{\mathcal{S}}(w_\circ) = |\mathcal{R}|$.

2.2.2. The weak order. The (right) *weak order* on W is the partial order $\mathsf{Weak}(W)$ defined by $u \leq w$ if u is initial in w. The weak order on the positive Artin monoid $\mathsf{Weak}(\boldsymbol{B}^+)$ is defined analogously. We record the following fundamental fact about the weak order.

THEOREM 2.2.2. *For W a finite Coxeter group with corresponding Artin monoid $\boldsymbol{B}^+$, $\mathsf{Weak}(W)$ and $\mathsf{Weak}(\boldsymbol{B}^+)$ are lattices.* □

The injection $W \hookrightarrow \boldsymbol{B}^+$ gives a poset isomorphism

$$\mathsf{Weak}(W) \cong [\boldsymbol{e}, \boldsymbol{w}_\circ]_{\mathsf{Weak}(\boldsymbol{B}^+)} \subseteq \mathsf{Weak}(\boldsymbol{B}^+).$$

The *meet* of two elements w, w' in $\mathsf{Weak}(W)$ or in $\mathsf{Weak}(\boldsymbol{B}^+)$ is denoted by $w \wedge w'$, and their *join* is denoted by $w \vee w'$. We recall that w is called *join-irreducible* if it covers exactly one other element. The join-irreducible elements are exactly those elements w such that $w \neq w' \vee w''$ for $w', w'' < w$ and $w \neq e$.

2.3. Descents and cover reflections

We define the *left descent set*, *right descent set*, *left ascent set*, and *right ascent set* of w by

$$\begin{aligned} \mathsf{des}_L(w) &:= \{s \in \mathcal{S} \ : \ s \text{ initial in } w\}, & \mathsf{des}_R(w) &:= \{s \in \mathcal{S} \ : \ s \text{ final in } w\}, \\ \mathsf{asc}_L(w) &:= \mathcal{S} \setminus \mathsf{des}_L(w), & \mathsf{asc}_R(w) &:= \mathcal{S} \setminus \mathsf{des}_R(w). \end{aligned}$$

PROPOSITION 2.3.1. *Let $s \in S$. For an element w of W or $\boldsymbol{B}^+$ such that $\ell_{\mathcal{S}}(sw) > \ell_{\mathcal{S}}(w)$, we have*

$$\mathsf{des}_L(sw) \subseteq \{s\} \cup \mathsf{des}_L(w).$$

PROOF. If $s, t \in \mathsf{des}_L(u)$ for u in W or in $\boldsymbol{B}^+$, then $s \vee t = [s|t]^{m(s,t)} = [t|s]^{m(s,t)}$ is initial in u by Lemma 2.2.1(2). Applying this to the element $u = sw$ yields that $s \vee t$ is initial in sw for any $t \in \mathsf{des}_L(sw)$. If $t \neq s$, we conclude that $[t|s]^{m(s,t)-1}$ is initial in w. □

For $w \in W$, the *covered reflections* and the *covering reflections* are, respectively,

$$\mathsf{cov}_\downarrow(w) := \{{}^w s \ : \ s \in \mathsf{des}_R(w)\}, \mathsf{cov}^\uparrow(w) \quad := \{{}^w s \ : \ s \in \mathsf{asc}_R(w)\}. \tag{2.2}$$

Cue wild applause. We also do not define motives.

#11

We do not define colored versions of covered and covering reflections for elements of $\boldsymbol{B}^+$. Figure 3 illustrates several examples.

2.4. Parabolic subgroups

Protip: if you understand the examples, you understand the theory.

#3

When $J \subseteq \mathcal{S}$, we use the notation W_J for the *standard parabolic subgroup* generated by J, and $\boldsymbol{B}_J^+$ for the corresponding *standard parabolic positive submonoid*. A standard parabolic subgroup is called *maximal* if it is generated by $\langle s \rangle := \mathcal{S} \setminus \{s\}$ for some $s \in \mathcal{S}$. Conjugates of standard parabolic subgroups are called *parabolic subgroups*.

The *parabolic quotient* corresponding to a standard parabolic subgroup is

$$W^J := \{w \in W \ : \ \mathsf{des}_L(w) \cap J = \emptyset\}.$$

w	w(st)	$\ell_{\mathcal{S}}$	$\ell_{\mathcal{R}}$	supp	des_L	des_R	$\mathsf{cov}_\downarrow$	$\mathsf{cov}^\uparrow$
e	st|st|st	0	0	–	–	–	–	s,t
s	st|st|st	1	1	s	s	s	s	u
t	st|st|st	1	1	t	t	t	t	u
st	st|st|st	2	2	s,t	s	t	u	t
ts	st|st|st	2	2	s,t	t	s	u	s
sts	st|st|st	3	1	s,t	s,t	s,t	s,t	–

FIGURE 3. The st-sorting word, length, reflection length, left descents, right descents, covered reflections, and covering reflections of the elements of $\mathfrak{S}_3$.

garside	w(st)	des_L	des_R
$s \cdot s$	st|st|st	s	s
$s \cdot st$	st|st|st	s	t
$t \cdot t$	st|st|st	t	t
$t \cdot ts$	st|st|st	t	s
$st \cdot t$	st|st|st	s	t
$st \cdot ts$	st|st|st	s	s
$ts \cdot s$	st|st|st	t	s

garside	w(st)	des_L	des_R
$ts \cdot st$	st|st|st	t	t
$sts \cdot s$	st|st|st	s,t	s
$sts \cdot t$	st|st|st	s,t	s,t
$sts \cdot st$	st|st|st	s,t	s,t
$sts \cdot ts$	st|st|st	s,t	s,t
$sts \cdot sts$	st|st|st	s,t	s,t

FIGURE 4. The st-sorting word and descent sets of those elements in $\boldsymbol{\mathfrak{S}}_3^+$ having exactly two nontrivial Garside factors.

For $J \subseteq \mathcal{S}$ and an element $w \in W$, we write $w_J := w \wedge w_\circ(J)$, where $w_\circ(J)$ is the longest element of W_J. Every $w \in W$ has a unique *parabolic decomposition*

$$w = w_J w^J \text{ for } w_J \in W_J \text{ and } w^J \in W^J. \tag{2.3}$$

2.5. Garside factorizations

The *Garside factorization* is a certain factorization of an element $\boldsymbol{w} \in \boldsymbol{B}^+$ as a product of elements in $[\boldsymbol{e}, \boldsymbol{w}_\circ]$, given as follows: set $\boldsymbol{w}_1 = \boldsymbol{w}$. For $i = 1, 2, 3, \ldots$, as long as $\boldsymbol{w}_i \neq \boldsymbol{e}$, let

$$\boldsymbol{w}^{(i)} = \boldsymbol{w}_i \wedge \boldsymbol{w}_\circ, \quad \boldsymbol{w}_{i+1} = (\boldsymbol{w}^{(i)})^{-1}\boldsymbol{w}_i$$

and set $k = i$ for the last i with $\boldsymbol{w}_i \neq \boldsymbol{e}$. Then

$$\mathrm{garside}(\boldsymbol{w}) = \boldsymbol{w}^{(1)} \cdot \boldsymbol{w}^{(2)} \cdot \ \cdots \ \cdot \boldsymbol{w}^{(k)},$$

where the *Garside factors* $\boldsymbol{w}^{(i)}$ are separated by a centered dot. The *Garside degree* $\deg(\boldsymbol{w})$ is defined to be k. By construction, every factor $\boldsymbol{w}^{(i)}$ is initial in $\boldsymbol{w}_\circ$ and so can be treated as an element $w^{(i)} \in W$.

The following characterization of Garside factorizations appears in [**Mic99**, Corollary 4.2].

THEOREM 2.5.1. *A factorization $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \cdot \ \cdots \ \cdot \boldsymbol{v}_k$ with $\boldsymbol{v}_i \leq \boldsymbol{w}_\circ$ is the Garside factorization of the element $\boldsymbol{w} = \boldsymbol{v}_1 \cdots \boldsymbol{v}_k \in \boldsymbol{B}^+$ if and only if*

$$\mathsf{des}_R(\boldsymbol{v}_{i-1}) \supseteq \mathsf{des}_L(\boldsymbol{v}_i). \qquad \square$$

2.6. Coxeter elements and sorting words

Coxeter was known to do 50 pushups a day, even into his 90s. Perhaps now's a good time to see how many pushups you can do.

#12

A (standard) *Coxeter element* c for $(W, \mathcal{S})$ is defined to be the product of the elements of $\mathcal{S}$ in any order. All Coxeter elements in W are conjugate, and we denote their common order by the *Coxeter number* h. For $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n$ and $J \subseteq \mathcal{S}$, define the *restriction* $\mathsf{c}\big|_J$ as the subword of c consisting of those simple reflections lying in J. The restriction of the corresponding element c is defined similarly.

Finite Coxeter groups have a *bipartite decomposition* $\mathcal{S} = \mathcal{S}_L \sqcup \mathcal{S}_R$ with the property that all reflections in $\mathcal{S}_L$ pairwise commute, as do all those in $\mathcal{S}_R$. For irreducible finite Coxeter systems, this decomposition is unique up to the interchanging of $\mathcal{S}_L$ and $\mathcal{S}_R$. A Coxeter element is called *bipartite* if it is the product of the reflections in $\mathcal{S}_L$ followed by the product of the reflections in $\mathcal{S}_R$, or vice versa.

We recall N. Reading's definition of c-sorting words [**Rea07b**], and extend it to $\boldsymbol{B}^+$.

This definition is *much* nicer than the notation suggests.

#13

Definition 2.6.1. Let w be an element of W or $\boldsymbol{B}^+$. The *c-sorting word* $\mathsf{w}(\mathsf{c}) = \mathsf{c}\big|_{I_1}\cdots\mathsf{c}\big|_{I_k}$ for w is the lexicographically first (as a sequence of positions) subword of $\mathsf{c}^\infty = (\mathsf{s}_1\cdots\mathsf{s}_n)^\infty$ that is a reduced expression for w. (It is common to separate the different copies of $\mathsf{s}_1\cdots\mathsf{s}_n$ by vertical bars when writing c-sorting words.)

The c-sorting word is attached to a particular $\mathcal{S}$-reduced word for c rather than to c itself. However, since all reduced words for c are commutation equivalent, the different choices of reduced words for a fixed Coxeter element c give commutation equivalent sorting words. Figures 3 and 4 list the st-sorting words of the six elements in $\mathfrak{S}_3$, and the 13 elements in $\mathfrak{S}_3^+$ with exactly two nontrivial Garside factors.

This definition gives the following greedy procedure to compute the c-sorting word of an element w of W or $\boldsymbol{B}^+$:

$$\mathsf{w}(\mathsf{c}) = \begin{cases} \mathsf{s}\ \mathsf{u}(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}) & \text{if } s \in \mathsf{des}_L(w) \text{ with } w = su, \\ \mathsf{w}(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}) & \text{if } s \notin \mathsf{des}_L(w). \end{cases}$$

That is—for s initial in c—the c-sorting word for w begins with s if and only if s is a left descent of w. The remainder of $\mathsf{w}(\mathsf{c})$ then coincides with the $\bar{s}cs$-sorting word of an element of shorter or equal length.

This is *the* way to think of the sorting word.

#4

The following lemmas summarize many previously known properties of sorting words. The first lemma is immediate from the greedy procedure for computing c-sorting words.

Lemma 2.6.2. *Let $w = s_1\cdots s_\ell \in \boldsymbol{B}^+$ and let $w \leq v \in \boldsymbol{B}^+$. If $\mathsf{s}_1\cdots\mathsf{s}_\ell$ is initial in c^∞ for a Coxeter element $c \in W$, then $\mathsf{s}_1\cdots\mathsf{s}_\ell$ is also initial in the c-sorting word $\mathsf{v}(\mathsf{c})$ of v.* □

Even David Speyer needed seven pages.

#14

The following lemma is surprisingly difficult to prove.

Lemma 2.6.3 ([**Spe09**, Corollary 4.1]). *The c-sorting word $\mathsf{w}_\circ(\mathsf{c})$ of the longest element $w_\circ \in W$ is initial in c^∞.* □

Example 2.6.4. We consider two examples in $\mathfrak{S}_5$ for Lemma 2.6.3. First, let $\mathsf{c} = \mathsf{s}_2\mathsf{s}_4\mathsf{s}_1\mathsf{s}_3\mathsf{s}_5$. Underlining the letters in $\mathsf{w}_\circ(\mathsf{c})$ as a subword of c^∞ gives

$$\underline{\mathsf{s}_2}\ \underline{\mathsf{s}_4}\ \underline{\mathsf{s}_1}\ \underline{\mathsf{s}_3}\ \underline{\mathsf{s}_5}\ \Big|\ \underline{\mathsf{s}_2}\ \underline{\mathsf{s}_4}\ \underline{\mathsf{s}_1}\ \underline{\mathsf{s}_3}\ \underline{\mathsf{s}_5}\ \Big|\ \underline{\mathsf{s}_2}\ \underline{\mathsf{s}_4}\ \underline{\mathsf{s}_1}\ \underline{\mathsf{s}_3}\ \underline{\mathsf{s}_5}\ \Big|\ \mathsf{s}_2\ \mathsf{s}_4\ \mathsf{s}_1\ \mathsf{s}_3\ \mathsf{s}_5\ \Big|\ \cdots.$$

The element $\mathsf{w}_\circ(\mathsf{c})$ is a prefix of c^∞. The Coxeter element $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3\mathsf{s}_5\mathsf{s}_4$ gives $\mathsf{w}_\circ(\mathsf{c})$ as the subword

$$\underline{\mathsf{s}_1}\ \underline{\mathsf{s}_2}\ \underline{\mathsf{s}_3}\ \underline{\mathsf{s}_5}\ \underline{\mathsf{s}_4}\ \big|\ \underline{\mathsf{s}_1}\ \underline{\mathsf{s}_2}\ \underline{\mathsf{s}_3}\ \underline{\mathsf{s}_5}\ \underline{\mathsf{s}_4}\ \big|\ \underline{\mathsf{s}_1}\ \underline{\mathsf{s}_2}\ \underline{\mathsf{s}_3}\ \mathsf{s}_5\ \mathsf{s}_4\ \big|\ \underline{\mathsf{s}_1}\ \underline{\mathsf{s}_2}\ \mathsf{s}_3\ \mathsf{s}_5\ \mathsf{s}_4\ \big|\ \cdots\ ,$$

which—though not a prefix—is still initial in c^∞.

We collect the several elementary properties of sorting words for later reference.

Brace yourself! This will keep the machine running smoothly.

#15

LEMMA 2.6.5. *Let c be a Coxeter element and let s be initial in c. Let $\mathsf{c} = \mathsf{c}_1$ and c_2 be two reduced $\mathcal{S}$-words for c such that the first letter of c is s, and let $\bar{\mathsf{s}}\mathsf{c}\mathsf{s}$ be a reduced $\mathcal{S}$-word for $\bar{s}cs$. Let $w \in \boldsymbol{B}^+$. Then*

(1) $\mathsf{w}(\mathsf{c}_1) \equiv \mathsf{w}(\mathsf{c}_2)$;
(2) c *is initial in* $\mathsf{w}_\circ(\mathsf{c})$;
(3) $\psi(\mathsf{c})$ *is final in* $\mathsf{w}_\circ(\mathsf{c})$;
(4) $\psi(\mathsf{w}_\circ(\mathsf{c})) \equiv \mathsf{w}_\circ(\psi(\mathsf{c}))$;
(5) $\mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{w}_\circ(\mathsf{c})) \equiv \mathsf{c}^h$;
(6) $\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c}))) \equiv \mathrm{rev}(\mathsf{w}_\circ(\mathsf{c}))$; *and*
(7) $\mathsf{w}_\circ(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}) \equiv \bar{\mathsf{s}}\mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{s})$.

PROOF. (1) was shown in the discussion after Definition 2.6.1.

(2) follows from the fact that $c \leq w_\circ$ in weak order together with the observation that c is lexicographically minimal in c^∞.

For (4), we note that given two reduced words Q and Q' for $w_\circ$, Q lexicographically precedes Q' inside c^∞ if and only if $\psi(\mathsf{Q})$ precedes $\psi(\mathsf{Q}')$ inside $\psi(\mathsf{c})^\infty$. As $\psi(\mathsf{Q})$ and $\psi(\mathsf{Q}')$ are also reduced words for $w_\circ$ and $\mathsf{w}_\circ(\mathsf{c})$ is lexicographically minimal inside c^∞, we conclude that $\psi(\mathsf{w}_\circ(\mathsf{c}))$ is lexicographically minimal inside $\psi(\mathsf{c})^\infty$.

For the other items, we rely on Lemma 2.6.3.

For (7), we use that $\mathsf{w}_\circ(\mathsf{c}) = \mathsf{s}\mathsf{q}_2 \dots \mathsf{q}_N$ is initial in c^∞, and thus, $\mathsf{q}_2 \dots \mathsf{q}_N$ is initial in $(\bar{\mathsf{s}}\mathsf{c}\mathsf{s})^\infty$. Therefore, Lemma 2.6.2 implies that $\mathsf{q}_2 \dots \mathsf{q}_N$ is initial in $\mathsf{w}_\circ(\bar{\mathsf{s}}\mathsf{c}\mathsf{s})$. Only one letter—$\psi(s)$—is left to obtain the $\bar{\mathsf{s}}\mathsf{c}\mathsf{s}$-sorting word, so that $\mathsf{w}_\circ(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}) \equiv \mathsf{q}_2 \dots \mathsf{q}_N\psi(\mathsf{s})$.

(3) is obtained from (2) by applying (7) n times. Since c is initial in $\mathsf{w}_\circ(\mathsf{c})$ by (2), $\bar{\mathsf{c}}\mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{c}) \equiv \mathsf{w}_\circ(\mathsf{c})$.

(5) follows from (2), (3) and (4): First, we have that $\mathsf{w}_\circ(\mathsf{c})$ is initial in c^∞ and $\mathsf{w}_\circ(\psi(\mathsf{c}))$ is initial in $\psi(\mathsf{c})^\infty$. As $\psi(\mathsf{c})$ is final in $\mathsf{w}_\circ(\mathsf{c})$, this final $\psi(\mathsf{c})$ is directly followed by the initial copy of $\psi(\mathsf{c})$ in $\mathsf{w}_\circ(\psi(\mathsf{c}))$ and we obtain that $\mathsf{w}_\circ(\mathsf{c})\mathsf{w}_\circ(\psi(\mathsf{c}))$ is initial in c^∞. As c is now final in $\mathsf{w}_\circ(\mathsf{c})\mathsf{w}_\circ(\psi(\mathsf{c}))$ and this c can only consist of a consecutive copy of c inside c^∞, this word equals c^k for some k. As its length is $nh = 2N = 2\ell_{\mathcal{S}}(w_\circ)$, we conclude that $k = h$.

Finally, (6) is obtained from (2) and (3). We have that $\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c})))$ is initial in $\big(\mathrm{rev}(\psi(\mathsf{c}))\big)^\infty$ and $\mathsf{w}_\circ(\mathsf{c})$ is initial in c^∞. As $\psi(\mathsf{c})$ is final in $\mathsf{w}_\circ(\mathsf{c})$, we deduce that $\mathrm{rev}(\mathsf{w}_\circ(\mathsf{c}))$ is initial in $\big(\mathrm{rev}(\psi(\mathsf{c}))\big)^\infty$ as well. For $s \in S$, write a_s for the number of occurrences of s in $\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c})))$ and b_s for the number of occurrences of s in $\mathrm{rev}(\mathsf{w}_\circ(\mathsf{c}))$.

By forgetting all simple reflections other than s and t, we note that a subword Q of c^∞ is initial if and only if it has the property that for each pair of noncommuting simple reflections s and t with s preceding t in c, the occurrences of s and t alternate within Q, starting with s.

c	$\psi(\mathsf{c})$	$\mathsf{w}_\circ(\mathsf{c})$	$\mathrm{rev}(\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c}))))$
$\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$	$\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1$	$\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2\mathsf{s}_1$	$\mathsf{s}_1\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1$
$\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2$	$\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2$	$\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2$	$\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2$
$\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2$	$\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2$	$\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2$	$\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2$
$\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3$	$\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1$	$\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3$	$\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1$
$\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1$	$\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3$	$\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1$	$\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3$
$\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1$	$\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$	$\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_3$	$\mathsf{s}_3\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$

FIGURE 5. The $\mathcal{S}$-words for the Coxeter elements in $\mathfrak{S}_4$ and associated sorting words.

This applies to both $\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c})))$ and $\mathrm{rev}(\mathsf{w}_\circ(\mathsf{c}))$. In particular, if s, t are a pair of non-commuting reflections with s appearing before t in $\mathrm{rev}\,\psi(\mathsf{c})$, then $a_t \leq a_s \leq a_t+1$, and similarly $b_t \leq b_s \leq b_t+1$. Whether $a_s = a_t$ or $a_s = a_t+1$ can be determined from the fact that $\mathrm{rev}(\mathsf{c}) = \psi(\mathrm{rev}(\psi(\mathsf{c})))$ is final in $\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c})))$. Since $\mathrm{rev}(\mathsf{c})$ is also final in $\mathrm{rev}(\mathsf{w}_\circ(\mathsf{c}))$, $b_s = b_t$ if and only if $a_s = a_t$, and $b_s = b_t+1$ if and only if $a_s = a_t+1$. It follows that the n-tuples a and b differ only by an overall additive constant. The sum of the entries in a is the length of $\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c})))$, and similarly the sum of the entries in b is the length of $\mathrm{rev}(\mathsf{w}_\circ(\mathsf{c}))$, and these two quantities are therefore equal. Thus $a = b$, and since $\mathrm{rev}(\mathsf{w}_\circ(\mathsf{c}))$ and $\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c})))$ are initial in the same word, they must coincide. □

We leave it to the reader to check Lemma 2.6.5 using Figure 5.

2.7. The absolute order

In addition to studying the group W using the simple reflections $\mathcal{S}$, one may also consider words in general reflections $\mathcal{R}$. The *reflection length* of an element $w \in W$ is the length $\ell_{\mathcal{R}}(w)$ of a shortest expression for w as a product of reflections in $\mathcal{R}$. An $\mathcal{R}$-word $\mathsf{r}_1 \cdots \mathsf{r}_p$ is *reduced* if $w = r_1 \cdots r_p$ and $p = \ell_{\mathcal{R}}(w)$. The *absolute order* $\mathsf{Abs}(W) = (W, \leq_{\mathcal{R}})$ is defined by

$$\begin{aligned} u \leq_{\mathcal{R}} w &\iff \text{there exists } v \in W \text{ with } uv = w \text{ and } \ell_{\mathcal{R}}(u) + \ell_{\mathcal{R}}(v) = \ell_{\mathcal{R}}(w) \\ &\iff \text{there exists } v' \in W \text{ with } v'u = w \text{ and } \ell_{\mathcal{R}}(v') + \ell_{\mathcal{R}}(u) = \ell_{\mathcal{R}}(w). \end{aligned}$$

The equivalence follows from invariance of $\ell_{\mathcal{R}}$ under conjugation.

2.7.1. Dual braid presentations. The idea of using $\mathcal{R}$ as a generating set has been exploited to develop alternative presentations for W, $\boldsymbol{B}$, and $\boldsymbol{B}^+$ [**BKL98, Bra01, BW02, Bes03**].We briefly recall this presentation here, following [**Bes03**].

For a Coxeter element $c \in W$, denote by $\mathrm{Red}_{\mathcal{R}}(c)$ the set of its reduced $\mathcal{R}$-words. There is an action of the classical braid group $\boldsymbol{\mathfrak{S}}_n$ on n strands on $\mathrm{Red}_{\mathcal{R}}(c)$ (not to be confused with the Artin group $\boldsymbol{B}$ of W). To avoid confusion, set $\boldsymbol{\sigma}_i$ be the generator of $\boldsymbol{\mathfrak{S}}_n$ crossing strand $i+1$ over strand i, corresponding to $s_i = (i\ i+1) \in \mathfrak{S}_n$. This action is called the *Hurwitz action*, and is given by

$$\boldsymbol{\sigma}_i(r_1, \ldots, r_n) \mapsto (r_1, \ldots, r_{i-1},\ r_{i+1},\ r_{i+1}^{-1} r_i r_{i+1},\ r_{i+2}, \ldots, r_n) \in \mathrm{Red}_{\mathcal{R}}(c).$$

The Hurwitz action action is transitive on $\mathrm{Red}_{\mathcal{R}}(c)$. For a Coxeter element c, the Artin group $\boldsymbol{B}$ has a *dual braid presentation*

$$\boldsymbol{B} = \Big\langle \boldsymbol{\mathcal{R}} : \boldsymbol{r}_1\boldsymbol{r}_2 = \boldsymbol{r}_2\boldsymbol{r}_3 \text{ for } r_1, r_2, r_3 \in \mathcal{R} \text{ with } r_1 r_2 = r_2 r_3 \leq_{\mathcal{R}} c \Big\rangle.$$

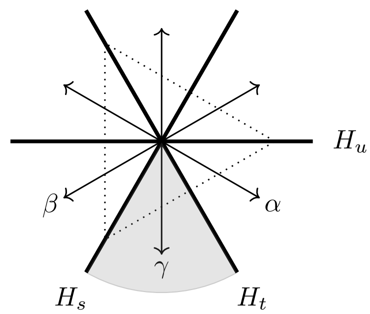


FIGURE 6. The root system of type A_2.

A relation of the form $\boldsymbol{r}_1\boldsymbol{r}_2 = \boldsymbol{r}_2\boldsymbol{r}_3$ is called a *dual braid relation*.

2.8. Root systems

The study of finite Coxeter groups is closely related to the study of finite root systems. We refer to [**Hum90**, Section 5] for a detailed treatment of this relation.

2.8.1. Root systems and reflection arrangements. Let

$$\Delta \subseteq \Phi^+ \subseteq \Phi \subset V$$

be a *root system* inside a Euclidian vector space V associated to the finite Coxeter system $(W, \mathcal{S})$, where $\Delta = \{\alpha_s \ : \ s \in \mathcal{S}\}$ denotes the *simple roots* and Φ^+ denotes the *positive roots*. We write $|\beta|$ for the positive root in $\{\pm\beta\}$.

As there is not necessarily a unique root system associated to a finite reflection group W, we consider a Coxeter system to be given with a fixed associated root system. For $J \subseteq \mathcal{S}$, the *standard parabolic root subsystem* is given by $\Phi_J := \Phi \cap \mathbb{R}\{\alpha_s \ : \ s \in J\}$ with positive roots $\Phi_J^+ := \Phi_J \cap \Phi^+$.

There is a bijection $\mathcal{R} \leftrightarrow \Phi^+$ obtained by sending a reflection $r \in \mathcal{R}$ to the unique positive root β such that $r(\beta) = -\beta$, and write $r = s_\beta$ and $\beta = \alpha_r$ in this case. We define the corresponding *reflecting hyperplane* $H_r := \beta^\perp$ as the orthogonal complement of β. The *reflection arrangement* is the collection of all such reflecting hyperplanes, and the *fundamental chamber* is the connected component of the complement of the reflection arrangement defined by the intersection of the positive half spaces corresponding to the simple roots. This fixes a bijection between the set of connected components $V \setminus \cup_{r\in\mathcal{R}} H_r$ and W; under this bijection, we may refer to a connected component by its corresponding element. A *gallery* is a walk on these connected components, where two components are adjacent if they share a common hyperplane, and any hyperplane is crossed at most once.

EXAMPLE 2.8.1. The Coxeter group $\mathfrak{S}_3$ can be realized as the *dihedral group* of isometries of an equilateral triangle. Its root system contains the simple and positive roots

$$\Delta = \{\alpha, \beta\} \quad \text{and} \quad \Phi^+ = \{\alpha, \gamma, \beta\},$$

where we take $\alpha = e_2 - e_1, \beta = e_3 - e_2$, and $\gamma = e_3 - e_1 = \alpha + \beta$. Here, e_1, e_2, and e_3 denote the standard basis of $\mathbb{R}^3$—but, as usual, we restrict to the hyperplane $x + y + z = 0$. The corresponding reflections are

$$s = s_\alpha, \quad t = s_\beta, \quad \text{and} \quad u = s_\gamma = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta.$$

See Figure 6 for an illustration of this example with shaded fundamental chamber.

2.8.2. The geometry of the absolute order. The *fixed space* of an element $w \in W$ is defined to be

$$\mathrm{Fix}(w) := \ker(w - \mathbb{1}),$$

while its *moved space* is

$$\mathrm{Mov}(w) := \mathrm{im}(w - \mathbb{1}).$$

As W acts by orthogonal transformations, we have $\mathrm{Mov}(w) = \mathrm{Fix}(w)^{\perp}$. The following properties relate reflection length with moved and fixed spaces.

Just think about cycle decomposition in the symmetric group.

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LEMMA 2.8.2 ([**BW02**, Section 2]). *For $w, w_1, w_2, w_3 \in W$ and $r, r_1, r_2 \in \mathcal{R}$, we have*

(i) $\ell_{\mathcal{R}}(w) = \dim \mathrm{Mov}(w)$,
(ii) $w_1 \leq_{\mathcal{R}} w_2 \Rightarrow \mathrm{Mov}(w_1) \subseteq \mathrm{Mov}(w_2)$ *and* $\mathrm{Fix}(w_1) \supseteq \mathrm{Fix}(w_2)$,
(iii) $w_1 \leq_{\mathcal{R}} w_2 \leq_{\mathcal{R}} w_3 \Rightarrow w_2^{-1}w_3 \leq_{\mathcal{R}} w_1^{-1}w_3$,
(iv) if $r_1 \leq_{\mathcal{R}} w$ *and* $r_1 \neq r_2$, *then* $r_1 r_2 \leq_{\mathcal{R}} w \Leftrightarrow r_2 \leq_{\mathcal{R}} r_1 w$, *and*
(v) $\mathrm{Mov}(r) \subseteq \mathrm{Mov}(w) \Rightarrow r \leq_{\mathcal{R}} w$. □

2.9. Inversions and colored inversions

We recall how Coxeter groups act on root systems, and extend this action to the corresponding Artin monoid.

2.9.1. Inversion sets for Coxeter groups. The *inversion set* of an element $w \in W$ is the set $\mathrm{inv}(w) := \Phi^+ \cap w(\Phi^-)$. The Coxeter length of an element $w \in W$ is equal to the cardinality of its inversion set. The choice of a reduced $\mathcal{S}$-word $\mathsf{s}_1 \cdots \mathsf{s}_\ell$ for $w \in W$ induces a total order on the inversion set of w

$$\mathsf{inv}(\mathsf{s}_1 \cdots \mathsf{s}_\ell) := \big(\alpha_{s_1}, s_1(\alpha_{s_2}), \ldots, s_1 \cdots s_{\ell-1}(\alpha_{s_\ell})\big), \tag{2.4}$$

so that

$$\mathrm{inv}(w) = \big\{\alpha_{s_1}, s_1(\alpha_{s_2}), \ldots, s_1 \cdots s_{\ell-1}(\alpha_{s_\ell})\big\} \subseteq \Phi^+.$$

The right weak order in W can be described in terms of inversion sets as

$$w \leq w' \Leftrightarrow \mathrm{inv}(w) \subseteq \mathrm{inv}(w'). \tag{2.5}$$

Subsets of positive roots that are inversion sets of elements were characterized by P. Papi in [**Pap94**]. A subset $A \subseteq \Phi^+$ is called *biclosed* if, for $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = a\alpha + b\beta$ with $a, b \in \mathbb{R}_+$, we have

$$\alpha, \beta \in A \Rightarrow \gamma \in A \text{ and } \gamma \in A \Rightarrow (\alpha \in A \text{ or } \beta \in A). \tag{2.6}$$

The map $w \mapsto \mathrm{inv}(w)$ is a bijection between elements in W and biclosed subsets of Φ^+.

2.9.2. Root orders and reflection orders. The inversion sequence of a reduced $\mathcal{S}$-word for $w_\circ$ specifies a total order on all positive roots which we call a *root order*. Root orders on Φ^+ were shown by P. Papi to be exactly those orderings of Φ^+ with the property that if $\alpha, \beta, \gamma = a\alpha + b\beta \in \Phi^+$ with $a, b \in \mathbb{R}_+$ then γ is in between α and β in the ordering, see [**Pap94**]. Every such root order induces a corresponding reflection order on all reflections. Recall that an order $\prec$ on $\mathcal{R}$ is a *reflection order* as defined by M. Dyer in [**Dye93**, Definition 2.1] if for any two reflections $r_1, r_2 \in \mathcal{R}$ such that the positive roots $\alpha_{r_1}, \alpha_{r_1 r_2 r_1}, \ldots, \alpha_{r_2 r_1 r_2}, \alpha_{r_2}$ are in the positive real span of α_{r_1} and α_{r_2}, either

$$r_1 \prec r_1 r_2 r_1 \prec \ldots \prec r_2 r_1 r_2 \prec r_2$$

holds, or the same statement with r_2 and r_1 swapped holds. The reflection orders on $\mathcal{R}$ are exactly those orders that come from root orders.

Order or partial order? You decide.

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We use the symbol $\leq_{\mathsf{c}}$ for the total order $\mathsf{inv}(\mathsf{w}_\circ(\mathsf{c}))$ on positive roots and for the corresponding reflection order. A reflection order is compatible with a Coxeter element c if it is commutation equivalent to $\mathsf{inv}(\mathsf{w}_\circ(\mathsf{c}))$ for some $\mathcal{S}$-word $\mathsf{s}_1 \cdots \mathsf{s}_n$ for $c = s_1 \cdots s_n$.

We have the following addition to Lemma 2.6.5, see [**RS11**, Lemmas 3.7 and 3.8].

Lemma 2.9.1. *Let c be a Coxeter element and let s be initial in c. If $r_1, r_2 \in \mathcal{R} \cap W_{\langle s \rangle}$ then*

$$r_1 <_c r_2 \Leftrightarrow r_1 <_{\bar{s}cs} r_2 \Leftrightarrow r_1 <_{\bar{s}c} r_2 \text{ in } W_{\langle s \rangle}. \qquad \square$$

2.9.3. Inversion sets for positive Artin monoids. If Q is not a reduced $\mathcal{S}$-word for an element in W, the definition of inversion sequence in (2.4) still makes sense, but may now contain negative or repeated roots. We keep track of this additional information using *colored positive roots*, defined as the set $\Phi^{(\infty)} := \{\beta^{(k)} : \beta \in \Phi^+, k \in \mathbb{N}\}$ consisting of positive roots *colored* by a nonnegative integer superscript. We write $|\beta^{(k)}|$ for the uncolored positive root β.

For a set T of colored positive roots and a parabolic subgroup W_J with $J \subseteq \mathcal{R}$, we write $T_J := \{\beta^{(i)} \in T : \beta \in \Phi^+_J\}$. A simple reflection $s \in \mathcal{S}$ acts on a colored positive root by

$$s(\beta^{(k)}) := \begin{cases} [s(\beta)]^{(k)} & \text{if } \beta \neq \alpha_s \text{ and} \\ \beta^{(k+1)} & \text{if } \beta = \alpha_s \end{cases}. \tag{2.7}$$

The *colored inversion sequence* and *colored reflection sequence* of an $\mathcal{S}$-word $\mathsf{Q} = \mathsf{s}_1 \cdots \mathsf{s}_p$ are defined to be

$$\begin{aligned} \mathsf{inv}(\mathsf{Q}) &:= \left(\beta_1^{(m_1)}, \ldots, \beta_p^{(m_p)}\right), \text{ and} \\ \mathsf{inv}_{\mathcal{R}}(\mathsf{Q}) &:= \left(r_1^{(m_1)}, \ldots, r_p^{(m_p)}\right), \end{aligned} \tag{2.8}$$

where $\beta_i^{(m_i)} = s_1 \cdots s_{i-1}(\alpha_{s_i}^{(0)})$ and $r_i = r_{\beta_i}$.

The *colored inversion set* of an element $\boldsymbol{w} \in \boldsymbol{B}^+$ is the set

$$\mathrm{inv}(\boldsymbol{w}) := \{\beta_1^{(m_1)}, \ldots, \beta_\ell^{(m_\ell)}\}$$

for any reduced $\mathcal{S}$-word $\mathsf{s}_1 \cdots \mathsf{s}_\ell$ for $\boldsymbol{w}$, as in (2.8).

Lemma 2.9.2. *The colored inversion set of $\boldsymbol{w} \in \boldsymbol{B}^+$ is well-defined.*

Well, defined.

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Proof. This follows, as in the case of $w \in W$, from the fact that any two words for $\boldsymbol{w} \in \boldsymbol{B}^+$ are obtained from each other by braid moves. The situation therefore reduces to checking that the colored inversion set stays unchanged under a braid move, which is immediate. $\square$

The definition of colored inversion sequence can be rephrased as follows.

Lemma 2.9.3. *Let $\mathsf{Q} = \mathsf{s}_1 \cdots \mathsf{s}_p$ be an $\mathcal{S}$-word with $\mathsf{inv}(\mathsf{Q}) = \left(\beta_1^{(m_1)}, \ldots, \beta_p^{(m_p)}\right)$. Then the following two properties hold:*

(i) *For any* $1 \leq i \leq p$, *we have* $\beta_i = \left|s_1 \cdots s_{i-1}(\alpha_{s_i})\right| \in \Phi^+$.
(ii) *For any* $\beta \in \Phi^+$ *let* $1 \leq k_0 < \ldots < k_a \leq p$ *be all indices such that* $\beta_{k_i} = \beta$. *Then* $m_{k_i} = i$ *for* $0 \leq i \leq a$.

Proof. Item (i) follows from the observation that the action of $s \in \mathcal{S}$ in (2.7) differs from the usual action of s on roots only in the second case where it changes the color instead of the sign.

We now turn to item (ii). For $w \in W$ and positive roots $\alpha \neq \beta$,

$$|w(\beta)| \neq |w(\alpha)|. \tag{2.9}$$

Fix $\beta \in \Phi^+$ and let $1 \leq k_0 < \ldots < k_a \leq p$ be the indices for which $\beta_{k_i} = \beta$ as in the statement. Fix $0 \leq i \leq a$. By (2.9), we obtain that, when we calculate the color of $\beta_{k_i}^{(m_{k_i})} = s_1 \cdots s_{k_i - 1}(\alpha_{s_{k_i}}^{(0)})$, we add one exactly in the positions $k_0, \ldots, k_{i-1}$, implying that $m_{k_i} = i$, as desired. □

This description of colored inversion sequences has two immediate consequences. First, no colored roots inside $\mathsf{inv}(\mathsf{Q})$ for $\mathsf{Q} = \mathsf{s}_1 \cdots \mathsf{s}_p$ appear multiple times, *i.e.*, $|\operatorname{inv}(\boldsymbol{w})| = p$ for $\boldsymbol{w} = \boldsymbol{s}_1 \cdots \boldsymbol{s}_p \in \boldsymbol{B}^+$. Second, the colored inversion sequence can be computed by first doing the calculation without colors—recording a sequence of positive roots—and then filling in the colors so the colors of each positive root that appears increase consecutively starting from zero. We may therefore suppress the colors in colored inversion sequences and write

$$\mathsf{inv}(\mathsf{s}_1 \cdots \mathsf{s}_p) = (\beta_1, \ldots, \beta_p)$$

with $\beta_i = |s_1 \cdots s_{i-1}(\alpha_{s_i})|$.In the case that $\mathsf{s}_1 \cdots \mathsf{s}_p$ is a reduced word for $w \in W$, this agrees with (2.4). To refer to the words $\left(r_1^{(m_1)}, \ldots, r_p^{(m_p)}\right)$ and $(r_1, \ldots, r_p)$, we also use the notation $\mathsf{r}_1^{(m_1)} \cdots \mathsf{r}_p^{(m_p)}$ and $\mathsf{r}_1 \cdots \mathsf{r}_p$, respectively.

Example 2.9.4. The reduced $\mathcal{S}$-word sts for the element $w_\circ \in \mathfrak{S}_3$ has reflection sequence $\mathsf{inv}_{\mathcal{R}}(\mathsf{sts}) = \mathsf{sut}$ and inversion sequence $\mathsf{inv}(\mathsf{sts}) = (\alpha, \gamma, \beta)$.

The $\mathcal{S}$-word stssts specifies the element $\boldsymbol{w}_\circ^2 \in \boldsymbol{\mathfrak{S}}_3^+$ with colored reflection sequence

$$\mathsf{inv}_{\mathcal{R}}(\mathsf{stssts}) = \mathsf{s}^{(0)}\mathsf{u}^{(0)}\mathsf{t}^{(0)}\mathsf{t}^{(1)}\mathsf{u}^{(1)}\mathsf{s}^{(1)}$$

and colored inversion sequence

$$\mathsf{inv}(\mathsf{stssts}) = \left(\alpha^{(0)}, \gamma^{(0)}, \beta^{(0)}, \beta^{(1)}, \gamma^{(1)}, \alpha^{(1)}\right).$$

In contrast to the situation for the Coxeter group in Section 2.9.1, colored inversion sets do not generally distinguish positive braids. For example, $\boldsymbol{sstt} \neq \boldsymbol{ttss}$ in $\boldsymbol{\mathfrak{S}}_3^+$, even though

$$\operatorname{inv}(\boldsymbol{sstt}) = \operatorname{inv}(\boldsymbol{ttss}) = \{\alpha^{(0)}, \alpha^{(1)}, \beta^{(0)}, \beta^{(1)}\}.$$

Nevertheless, we have the following lemma.

Lemma 2.9.5. *Let* $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{B}^+$ *with* $\operatorname{inv}(\boldsymbol{w}_1) = \operatorname{inv}(\boldsymbol{w}_2)$*, and let* w_1, w_2 *denote their images inside* W*. Then* $w_1 = w_2$*.*

Proof. A reduced $\mathcal{S}$-word for $w \in W$ can be obtained from a (possibly non-reduced $\mathcal{S}$-word) $\mathsf{s}_1 \cdots \mathsf{s}_p$ by applying braid moves and by removing two consecutive equal letters ss. A word Q for $w \in W$ is reduced if and only if its colored inversion set $\operatorname{inv}(\mathsf{Q})$ only has the color 0. By Lemma 2.9.2, braid moves do not change

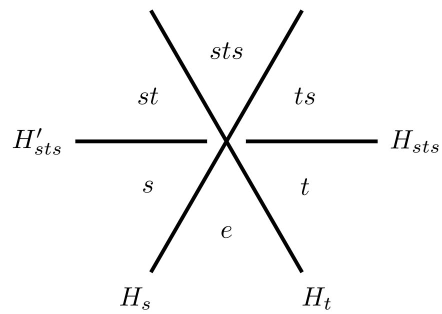

FIGURE 7. The shards of type A_2.

the colored inversion set. On the other hand, removing two consecutive letters ss removes the same positive root from the colored inversion set twice, with two consecutive colors.

Let $\boldsymbol{w} \in \boldsymbol{B}^+$ and write w for its image in W. Then $\operatorname{inv}(w)$ is given by all positive roots in $\operatorname{inv}(\boldsymbol{w})$ that appear an odd number of times. As elements in W are uniquely determined by their inversion sets, this shows that w only depends on $\operatorname{inv}(\boldsymbol{w})$ and not on $\boldsymbol{w}$ itself. □

2.10. Shards and the shard intersection order

The set of all parabolic subgroups, when ordered by inclusion, forms a lattice $\mathcal{P}(W)$. On the other hand, the *intersection lattice* $\mathcal{L}(W)$ is the lattice of flats, *i.e.*, of all intersections of the hyperplanes in the reflection arrangement of W ordered by reverse inclusion. These two lattices are well-known to be related in the following way.

THEOREM 2.10.1 ([**BI99**]). *There is an order-preserving bijection*

$$\mathcal{L}(W) \cong \mathcal{P}(W)$$
$$X \mapsto W_X$$
$$V^U \leftarrow\!\shortmid U,$$

where $W_X := \{w \in W \;:\; X \subseteq \operatorname{Fix}(w)\}$ *and* $V^U := \bigcap_{w\in U} \operatorname{Fix}(w)$. □

2.10.1. Shards. In [**Rea11**], N. Reading defined a delicate slicing procedure on simplicial hyperplane arrangements that cuts hyperplanes into several pieces called *shards*. We are interested in this construction in the case of a reflection arrangement. Recall the identification of chambers and elements in W from Section 2.8.

A hyperplane H in a subarrangement of the reflection arrangement is called *basic* if the connected region containing the fundamental chamber is bounded by H in the subarrangement. For any two hyperplanes H, H', define $\mathcal{A}(H, H')$ to be the subarrangement consisting of all hyperplanes containing $H \cap H'$. One says that H' *cuts* H if H' is a basic hyperplane of $\mathcal{A}(H, H')$ while H is not. In this way, all hyperplanes are cut into *shards*, defined as the closures of the connected pieces

$$H \setminus \bigcup_{H' \text{ cuts } H} H'$$

of hyperplanes. Figure 7 illustrates the slicing of the reflection arrangement of type A_2 into shards.

A *lower shard* for an element $w \in W$ is the shard Σ inside a hyperplane H_r such that $r \in \mathsf{cov}_{\downarrow}(w)$ and the gallery from w to rw crosses Σ. It was shown in [**Rea11**, Proposition 3.3] that shards are in bijection with the join-irreducible elements of the weak order $\mathsf{Weak}(W)$. A shard Σ is associated to the join-irreducible element w that has Σ as its unique lower shard. In type A_2, the join-irreducible elements are s, t, st, ts, which correspond in the obvious way to the four shards shown in Figure 7.

Lemma 2.10.2. *Any gallery from e to w crosses every lower shard of w.*

Proof. Let H be the hyperplane corresponding to a lower shard Σ of w and consider a different shard Σ' in H. Then there is a hyperplane H' which cuts H such that Σ and e are on one side of H' and Σ' is on the other side. Then it is clear that a gallery from e to w will not cross H' and so will not cross Σ'. On the other hand, since e and w are on opposite sides of H, it must cross some shard in H. Therefore it must cross Σ. ☐

2.10.2. The shard intersection order. The *shard intersection order* is the set $\mathsf{Shard}(W)$ of all intersections of shards, ordered by reverse inclusion. N. Reading constructed in [**Rea11**, Proposition 4.7] a bijection between W and the set of shard intersections.

Theorem 2.10.3. *There is a bijection*

$$\begin{aligned} \mathrm{shard} : W &\longrightarrow \mathsf{Shard}(W) \\ w &\longmapsto \bigcap_{\Sigma \text{ a lower shard for } w} \Sigma. \end{aligned}$$ ☐

Figure 7 shows the correspondence between s, t, st, ts and the four shards. To complete the bijection, we observe that e corresponds to the empty intersection, while sts corresponds to the intersection $\{0\} = H_s \cap H_t$.

Since both shards and shard intersections have purely group-theoretic interpretations as elements of W, it is reasonable to ask for a definition of the shard intersection order directly on the group W.

Definition 2.10.4. The *shard intersection order* on W is defined by $u \leq_{\mathrm{Sh}} v$ if

$$\mathrm{inv}(u) \subseteq \mathrm{inv}(v) \text{ and } W_{\mathsf{cov}_{\downarrow}(u)} \subseteq W_{\mathsf{cov}_{\downarrow}(v)}.$$

This definition is justified by the following theorem, which has not previously appeared in the literature.

Theorem 2.10.5. *Let $u, v \in W$. Then*

$$u \leq_{\mathrm{Sh}} v \quad \Leftrightarrow \quad \mathrm{shard}(u) \subseteq \mathrm{shard}(v)$$

Inversion set + cover reflections = shards.

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By (2.5), $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$ if and only if $u \leq v$ in weak order—so this describes the shard intersection order on W explicitly as a weakening of the weak order, as shown in [**Rea11**, Proposition 4.7 (ii)]. Figure 8 shows the shard intersection order of type A_2 with each element labelled by its cover reflections.

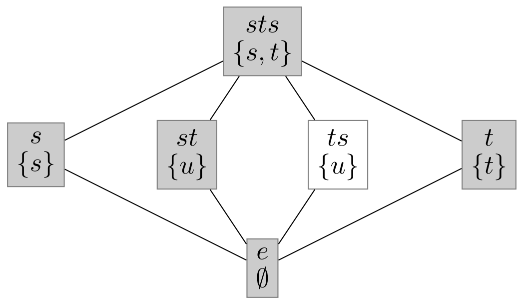

FIGURE 8. The shard intersection order $\mathsf{Shard}(\mathfrak{S}_3)$ with each element labelled by its cover reflections. The st-sortable elements are shaded. These are defined in Chapter 6.

PROOF OF THEOREM 2.10.5. We first show that the two properties $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$ and $W_{\mathsf{cov}_\downarrow(u)} \subseteq W_{\mathsf{cov}_\downarrow(v)}$ imply $u \leq_{\mathrm{Sh}} v$. By the correspondence in Theorem 2.10.1 between the lattice of parabolic subgroups and the intersection lattice of the hyperplane arrangement of W, $W_{\mathsf{cov}_\downarrow(u)} \subseteq W_{\mathsf{cov}_\downarrow(v)}$ implies that

$$\bigcap_{H \in \mathsf{cov}_\downarrow(u)} H \supseteq \bigcap_{H \in \mathsf{cov}_\downarrow(v)} H. \tag{2.10}$$

If we in addition have $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$, then there exists a gallery from e to u to v. Let Σ be the union of the set of shards that this gallery crosses, and note that there is exactly one shard for each hyperplane in $\{H_r \mid r \in \mathrm{inv}(v)\}$ (since each hyperplane is crossed at most once). For any chamber r with lower hyperplane H, the entire facet of r corresponding to H is part of the same shard.

By Lemma 2.10.2, taking the intersection of both sides of (2.10) with Σ implies that the intersection of the lower shards of u contains the intersection of the lower shards of v.

We now show that $u \leq_{\mathrm{Sh}} v$ implies that $W_{\mathsf{cov}_\downarrow(u)} \subseteq W_{\mathsf{cov}_\downarrow(v)}$ and $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$. By [**Rea11**, Proposition 5.5], we know that $W_{\mathsf{cov}_\downarrow(u)} \subseteq W_{\mathsf{cov}_\downarrow(v)}$, since sending shards to their containing hyperplane induces an order-preserving map from the shard intersection order to the intersection lattice of W. Finally, by [**Rea11**, Proposition 4.7 (ii)], sending regions to the intersection of their lower shards is an order-preserving map from the shard intersection order to the weak order. $\square$

We recall the following proposition.

PROPOSITION 2.10.6 ([**Rea11**, Proposition 1.2]).

$$[e, w]_{\mathsf{Shard}(W)} \cong \mathsf{Shard}(W_{\mathsf{cov}_\downarrow(w)}) \cong \mathsf{Shard}(W_{\mathsf{des}_R(w)}). \qquad \square$$

The isomorphism $[e, w]_{\mathsf{Shard}(W)} \cong \mathsf{Shard}(W_{\mathsf{cov}_\downarrow(w)})$ is given by sending $u \leq_{\mathrm{Sh}} w$ to $u_{\mathsf{cov}_\downarrow(w)}$. The bijection $r \mapsto r^w$ from $\mathsf{cov}_\downarrow(w)$ to $\mathsf{des}_R(w)$ induces the poset isomorphism

$$W_{\mathsf{cov}_\downarrow(w)} \cong W_{\mathsf{des}_R(w)}$$
$$u \mapsto u^w,$$

from which the isomorphism $[e, w]_{\mathsf{Shard}(W)} \cong \mathsf{Shard}(W_{\mathsf{des}_R(w)})$ follows.

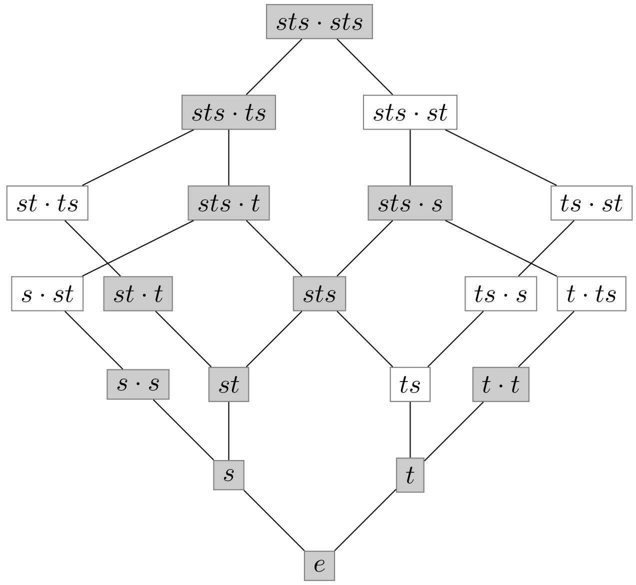

FIGURE 9. The m-eralized weak order $\mathsf{Weak}^{(2)}(\mathfrak{S}_3)$ for $m = 2$. The st-sortable elements are shaded. These are defined in Chapter 6.

2.11. The m-eralized weak order

By (2.1), a finite Coxeter group W injects into its Artin monoid $\boldsymbol{B}^+$ as the interval $\mathsf{Weak}(W) \cong [\boldsymbol{e}, \boldsymbol{w}_\circ]_{\mathsf{Weak}(\boldsymbol{B}^+)}$. This injection suggests the following m-eralization, previously considered by P. Dehornoy in an enumerative context [**Deh07**].

If you like permutations, then you will *love* the m-eralized weak order!

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DEFINITION 2.11.1. The (right) *m-eralized weak order* is defined as the interval

$$\mathsf{Weak}^{(m)}(W) := [\boldsymbol{e}, \boldsymbol{w}_\circ^m]_{\mathsf{Weak}(\boldsymbol{B}^+)}.$$

We denote the elements of $\mathsf{Weak}^{(m)}(W)$ by $W^{(m)}$.

Figure 9 illustrates the Hasse diagram of $\mathsf{Weak}^{(2)}(\mathfrak{S}_3)$.

REMARK 2.11.2. Both $\mathsf{Weak}(W) = \mathsf{Weak}^{(1)}(W)$ and $\mathsf{Weak}(\boldsymbol{B}^+) = \mathsf{Weak}^{(\infty)}(W)$ are known to have beautiful rank-generating functions, which follow respectively from invariant theory [**Hum90**, Section 3.9] and from an inclusion-exclusion argument [**S$^+$09**, Lemma 2.1]. The corresponding rank-generating functions are given by

$$\mathsf{Weak}(W; q) = \prod_{i=1}^{n} [d_i]_q \quad \text{and} \quad \mathsf{Weak}(\boldsymbol{B}^+; q) = \left(\sum_{J \subseteq S} (-1)^{|J|} q^{\ell_S(w_0(J))} \right)^{-1}.$$

The rank generating function $\mathsf{Weak}^{(2)}(\mathfrak{S}_4; q)$ is an irreducible polynomial over $\mathbb{Q}$ of degree 12, suggesting that no nice formula may exist for general m.

2.11.1. The m-eralized weak order and Garside degree. We now show that consideration of the Garside degree of $\boldsymbol{w} \in \boldsymbol{B}^+$ is enough to determine membership in $W^{(m)}$. We first require a technical lemma, extending [**EM94**, Lemma 2.10] to all spherical Artin monoids.

LEMMA 2.11.3. *Let $\boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{B}^+$ such that $\boldsymbol{u}\boldsymbol{w} \geq \boldsymbol{w}_\circ \in \boldsymbol{B}^+$, and let $\boldsymbol{w}^{(1)}$ be the first Garside factor of $\boldsymbol{w}$. Then $\boldsymbol{u}\boldsymbol{w}^{(1)} \geq \boldsymbol{w}_\circ$.*

PROOF. We argue by induction on $\ell_{\mathcal{S}}(\boldsymbol{u})$, the base case when $\boldsymbol{u} = \boldsymbol{e}$ being trivial. Otherwise, let $\boldsymbol{u} = \boldsymbol{u}'\boldsymbol{s}$ and $\boldsymbol{w}' = \boldsymbol{s}\boldsymbol{w}$ for some $\boldsymbol{s} \in \boldsymbol{\mathcal{S}}$, so that $\boldsymbol{u}'\boldsymbol{w}' = \boldsymbol{u}\boldsymbol{w} \geq \boldsymbol{w}_\circ$. By induction, $\boldsymbol{u}'(\boldsymbol{w}')^{(1)} \geq \boldsymbol{w}_\circ$. Since $\boldsymbol{s} \in \mathsf{des}_L(\boldsymbol{w}')$, the first Garside factor $(\boldsymbol{w}')^{(1)} = \boldsymbol{w}' \wedge \boldsymbol{w}_\circ$ has $\boldsymbol{s} \in \mathsf{des}_L((\boldsymbol{w}')^{(1)})$. We may therefore write $(\boldsymbol{w}')^{(1)} = \boldsymbol{s}\boldsymbol{v}$ for some $\boldsymbol{v} \leq \boldsymbol{w}_\circ$ with $\boldsymbol{v} \leq \boldsymbol{w}$. Since $\boldsymbol{w}^{(1)} = \boldsymbol{w} \wedge \boldsymbol{w}_\circ$, $\boldsymbol{v} \leq \boldsymbol{w}_\circ$, and $\boldsymbol{v} \leq \boldsymbol{w}$, we have that $\boldsymbol{v} \leq \boldsymbol{w}^{(1)}$. Then $\boldsymbol{u}\boldsymbol{w}^{(1)} \geq \boldsymbol{u}\boldsymbol{v} = \boldsymbol{u}'(\boldsymbol{w}')^{(1)} \geq \boldsymbol{w}_\circ$. □

We now extend [**EM94**, Theorem 2.11] to all spherical Artin monoids.

So you can think of an element of $W^{(m)}$ as an m-tuple of elements in W satisfying Theorem 2.5.1.

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PROPOSITION 2.11.4. *Let $\boldsymbol{u} \in \boldsymbol{B}^+$. Then $\boldsymbol{u} \in W^{(m)}$ if and only if $\boldsymbol{u}$ has Garside degree at most m.*

PROOF. We show that

(i) $\boldsymbol{w} \leq \boldsymbol{w}_\circ^r \Rightarrow \deg(\boldsymbol{w}) \leq r$ and
(ii) $\deg(\boldsymbol{w}) = r \Rightarrow \boldsymbol{w} \leq \boldsymbol{w}_\circ^r$.

We prove (i) by induction on r. The base case $r = 0$ is trivial, so we assume that $r \geq 1$ and let $\mathrm{garside}(\boldsymbol{w}) = \boldsymbol{w}^{(1)} \cdot \boldsymbol{w}^{(2)} \cdot \cdots \cdot \boldsymbol{w}^{(k)}$. Since $\boldsymbol{w} \leq \boldsymbol{w}_\circ^r$, there exists $\boldsymbol{u} \in \boldsymbol{B}^+$ with $\boldsymbol{u}\boldsymbol{w} = \boldsymbol{w}_\circ^r$. Then $\boldsymbol{u}\boldsymbol{w} \geq \boldsymbol{w}_\circ$, so that $\boldsymbol{u}\boldsymbol{w}^{(1)} = \boldsymbol{w}_\circ\boldsymbol{u}'$ by Lemma 2.11.3. We can therefore write $\boldsymbol{u}\boldsymbol{w} = \boldsymbol{w}_\circ\boldsymbol{u}'\boldsymbol{w}^{(2)}\cdots\boldsymbol{w}^{(k)}$, from which it follows that $\boldsymbol{u}'\boldsymbol{w}^{(2)}\cdots\boldsymbol{w}^{(k)} = \boldsymbol{w}_\circ^{r-1}$, giving $\boldsymbol{w}^{(2)}\cdots\boldsymbol{w}^{(k)} \leq \boldsymbol{w}_\circ^{r-1}$. By induction, we obtain that $k-1 \leq r-1$ implying that $\deg(\boldsymbol{w}) = k \leq r$.

On the other hand, we show (ii) as follows. Let $\mathrm{garside}(\boldsymbol{w}) = \boldsymbol{w}^{(1)} \cdot \boldsymbol{w}^{(2)} \cdot \cdots \cdot \boldsymbol{w}^{(r)}$ and let $\boldsymbol{u}_1 \in \boldsymbol{B}^+$ be the element such that $\boldsymbol{u}_1\boldsymbol{w}^{(1)} = \boldsymbol{w}_\circ \in \boldsymbol{B}^+$. Push this copy of $\boldsymbol{w}_\circ$ to the right to obtain the factorization

$$\boldsymbol{u}_1\boldsymbol{w} = (\boldsymbol{w}^{(2)})^{\boldsymbol{w}_\circ}(\boldsymbol{w}^{(3)})^{\boldsymbol{w}_\circ}\cdots(\boldsymbol{w}^{(k)})^{\boldsymbol{w}_\circ}\boldsymbol{w}_\circ.$$

Iterating, we obtain the desired conclusion. □

2.11.2. Lattice properties of m-eralized weak order. We conclude the discussion of $\mathsf{Weak}^{(m)}(W)$ with some of its poset-theoretic properties.

THEOREM 2.11.5. *$\mathsf{Weak}^{(m)}(W)$ is a rank-symmetric, self-dual lattice.*

Just as the rank-symmetry of $\mathsf{Weak}(W)$ is demonstrated using the anti-automorphism given by acting by the longest element $w_\circ$, we prove Theorem 2.11.5 by showing that $\mathsf{Weak}^{(m)}(W)$ has an anti-automorphism given by acting by $\boldsymbol{w}_\circ^m$.

LEMMA 2.11.6. *The left and the right factors of $\boldsymbol{w}_\circ^m \in \boldsymbol{B}^+$ coincide.*

PROOF. For $\boldsymbol{s} \in \boldsymbol{\mathcal{S}}$, it holds that $\boldsymbol{w}_\circ^m\boldsymbol{s} = \psi^m(\boldsymbol{s})\boldsymbol{w}_\circ^m \in \boldsymbol{B}^+$ Any reduced expression $\boldsymbol{u}\boldsymbol{s}_1\cdots\boldsymbol{s}_k = \boldsymbol{w}_\circ^m \in \boldsymbol{B}^+$, $\psi^m(\boldsymbol{s}_1)\cdots\psi^m(\boldsymbol{s}_k)\boldsymbol{u}$ is again a reduced expression for $\boldsymbol{w}_\circ^m$. □

LEMMA 2.11.7. *The map $\boldsymbol{w} \mapsto \boldsymbol{w}_\circ^m\overline{\boldsymbol{w}}$ on $\boldsymbol{B}^+$ is an anti-isomorphism from right to left weak order when restricted to $\mathsf{Weak}^{(m)}(W)$.*

PROOF. Set $\phi(\boldsymbol{w}) = \boldsymbol{w}_\circ^m \overline{\boldsymbol{w}}$. By Lemma 2.11.6, $\boldsymbol{w} \in W^{(m)}$ is both initial and final in $\boldsymbol{w}_\circ^m$. We also have

$$\ell_{\mathcal{S}}(\boldsymbol{w}_\circ^m \overline{\boldsymbol{w}}) = \ell_{\mathcal{S}}(\overline{\boldsymbol{w}} w_\circ^m) = \ell_{\mathcal{S}}(\boldsymbol{w}_\circ^m) - \ell_{\mathcal{S}}(\boldsymbol{w}) = mN - \ell_{\mathcal{S}}(\boldsymbol{w}).$$

To show that the map reverses the order, it suffices to consider the case for $\boldsymbol{w} \leq \boldsymbol{ws} \in W^{(m)}$. Then $\ell_{\mathcal{S}}(\phi(\boldsymbol{w})) = \ell_{\mathcal{S}}(\boldsymbol{w}_\circ^m) - \ell_{\mathcal{S}}(\boldsymbol{w})$ and

$$\ell_{\mathcal{S}}(\phi(\boldsymbol{ws})) = \ell_{\mathcal{S}}(\boldsymbol{w}_\circ^m) - \ell_{\mathcal{S}}(\boldsymbol{ws}) = \ell_{\mathcal{S}}(\boldsymbol{w}_\circ^m) - \ell_{\mathcal{S}}(\boldsymbol{w}) - 1,$$

so that $\ell_{\mathcal{S}}(\phi(\boldsymbol{ws})) + 1 = \ell_{\mathcal{S}}(\phi(\boldsymbol{w}))$. Furthermore,

$$\phi(\boldsymbol{ws}) = \boldsymbol{w}_\circ^m \overline{\boldsymbol{sw}} = \overline{\psi^m(\boldsymbol{s})} \boldsymbol{w}_\circ^m \overline{\boldsymbol{w}} = \overline{\psi^m(\boldsymbol{s})} \phi(\boldsymbol{w}),$$

so that $\phi(\boldsymbol{w}) = \psi^m(\boldsymbol{s})\phi(\boldsymbol{ws})$. □

Representing an element of $\boldsymbol{B}^+$ as an $\mathcal{S}$-word, the reverse map converts between left weak order and right weak order.

LEMMA 2.11.8. *The map* rev *is an isomorphism from left weak order to right weak order that preserves the interval* $\mathsf{Weak}^{(m)}(W)$*.*

PROOF. Since $\mathrm{rev}(\boldsymbol{sw}) = \mathrm{rev}(\boldsymbol{w})\boldsymbol{s}$, rev is an isomorphism from left weak order to right weak order. If $\boldsymbol{w} \leq \boldsymbol{w}_\circ^m$, then there exists $\boldsymbol{u} \in \boldsymbol{B}^+$ such that $\boldsymbol{uw} = \boldsymbol{w}_\circ^m$ so that $\mathrm{rev}(\boldsymbol{w})\,\mathrm{rev}(\boldsymbol{u}) = \mathrm{rev}(\boldsymbol{uw}) = \mathrm{rev}(\boldsymbol{w}_\circ^m) = \boldsymbol{w}_\circ^m$ and $\mathrm{rev}(\boldsymbol{w}) \leq \boldsymbol{w}_\circ^m$. □

PROPOSITION 2.11.9. *The composition* $(\mathrm{rev} \circ \phi)$ *is an anti-isomorphism of the m-eralized weak order* $\mathsf{Weak}^{(m)}(W)$*.*

PROOF. This follows from Lemmas 2.11.7 and 2.11.8. □

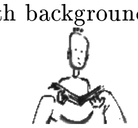
PROOF OF THEOREM 2.11.5. Since it is an interval, $\mathsf{Weak}^{(m)}(W)$ inherits the lattice property of Theorem 2.2.2 from $\mathsf{Weak}(\boldsymbol{B}^+)$. Self-duality and rank-symmetry are a direct consequence of the existence of the anti-isomorphism $(\mathrm{rev} \circ \phi)$ and the fact the $\mathsf{Weak}^{(m)}(W)$ is graded by length. □

CHAPTER 3

Subword complexes

In this chapter, we define and study several variants of subword complexes, generalizing the notion introduced in [**KM05**]. We begin by reviewing properties of simplicial complexes (Section 3.1). Additional background can be found in [**Bjö95**, Section 9] and its references. We then extend the theory of subword complexes to positive Artin monoids (Sections 3.2 and 3.3), and introduce dual subword complexes (Section 3.4) and Coxeter-initial subword complexes (Section 3.5). We conclude with topological properties of Coxeter-initial subword complexes (Section 3.6).

We are mostly done with background.



#5

3.1. Simplicial complexes

A *simplicial complex* with ground set $[m] = \{1, \ldots, m\}$ is a collection $\mathcal{C} \subset 2^{[m]}$ such that

$$A \subseteq B \in \mathcal{C} \Rightarrow A \in \mathcal{C}.$$

The elements of $\mathcal{C}$ are called *faces*, the containment-wise maximal faces are called *facets*. We regularly identify a simplicial complex with its set of facets.

Faces of cardinality one are called *vertices*, and we denote a vertex $\{i\}$ simply by i. Every face $F \in \mathcal{C}$ together with all its subfaces $G \subseteq F$ again forms a simplicial complex 2^F. Simplicial complexes of this form are called *simplices*.

The *dimension* of a face F is given by $\dim(F) := |F| - 1$ and the dimension of $\mathcal{C}$ is the maximal dimension among its faces. The complex $\mathcal{C}$ is *pure* if all its facets have the same dimension, and it is called *flag* if all its containment-wise minimal non-faces have cardinality two. The *one-skeleton* of a simplicial complex is its subcomplex consisting of its 0- and 1-dimensional faces. A simplicial complex is flag if and only if it is the clique complex of its one-skeleton.

3.1.1. Shellability and vertex-decomposability. If $\mathcal{C}$ is pure, than it is said to be *shellable* if there is a linear *shelling order* $F_1, \ldots, F_k$ of its facets such that for $2 \leq i \leq k$, we have

$$F_i \cap \Big(\bigcup_{j<i} F_j \Big)$$

is a nonempty union of facets of the boundary of the simplex 2^{F_i}. A shellable simplicial complex is contractible or has the homotopy type of a wedge of spheres.

Define the *deletion* of a vertex i of $\mathcal{C}$ by

$$\mathrm{del}_i(\mathcal{C}) := \{B \in \mathcal{C} \ : \ i \notin B\},$$

and the *link* of i by

$$\mathrm{lk}_i(\mathcal{C}) := \{B \in \mathcal{C} \ : \ i \notin B, B \cup \{i\} \in \mathcal{C}\}.$$

Finally, a pure complex $\mathcal{C}$ of dimension d is called *vertex-decomposable* if either $\mathcal{C} = \{\emptyset\}$, or if there exists a vertex $i \in \mathcal{C}$ such that

(i) $\mathrm{del}_i(\mathcal{C})$ is pure of dimension d and again vertex-decomposable, and
(ii) $\mathrm{lk}_i(\mathcal{C})$ is pure of dimension $d-1$ and again vertex-decomposable.

The lexicographic order on facets of a vertex-decomposable complex (regarded as increasing tuples of vertices) induced by the vertex ordering arising from the decomposition is a shelling order. We refer to [**BW96, BW97**] and the references therein for a detailed treatment of shellable and vertex-decomposable simplicial complexes.

3.2. Subword complexes

Subword complexes were introduced by A. Knutson and E. Miller in [**KM05, KM04**] in the context of Gröbner geometry. In this monograph, we consider the following generalization.

DEFINITION 3.2.1. Let $\mathsf{Q} = \mathsf{s}_1 \cdots \mathsf{s}_p$ be an $\mathcal{S}$-word, let $w \in W$, and let $a = \ell_{\mathcal{S}}(w) + 2g$ for some $g \geq 0$. The *subword complex* $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, a)$ is the simplicial complex with facets given by subsets of (positions of) letters in Q whose complements form an $\mathcal{S}$-word for w of length a. We call Q the *search word* and we write $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w)$ for $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, \ell_{\mathcal{R}}(w))$.

When $g = 0$, $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w)$ recovers A. Knutson and E. Miller's original definition. Subword complexes for $g = 0$ were studied in connection to Catalan combinatorics in [**SS12, CLS14, PS15**], but related objects have appeared in a number of contexts, including in S. Billey, W. Jockush, and R. Stanley's remarkable formula for Schubert polynomials [**BJS93**] and in J. Morse and A. Schilling's crystal model for $\mathfrak{sl}_n$ [**MS16**].

When two search words are commutation equivalent, they evidently yield isomorphic complexes (see [**CLS14**, Proposition 3.8]).

PROPOSITION 3.2.2. *Let $\mathsf{Q} \equiv \mathsf{Q}'$ be commutation equivalent $\mathcal{S}$-words. Then there is a canonical isomorphism $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, a) \cong \mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}', w, a)$. This is given by the canonical identification of letters in Q and Q'. (To have this canonical identification, we forbid two copies of the same letter from commuting.)* ☐

Check that it is an 8-gon with its 4 diameters.

#6

EXAMPLE 3.2.3. The facets of the subword complex $\mathrm{SUB}_{\mathcal{S}}(\mathsf{ststs}, w_\circ)$ for $\mathfrak{S}_3$ are given by

$$\big\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\big\},$$

as the complements of the subword in positions i and $i+1$ for $i \in \{1,2,3,4\}$ give the reduced $\mathcal{S}$-word sts, while the complement of the subword in positions 1 and 5 gives the reduced word tst. The first column of Figure 13 on page 64 lists the 12 facets of the subword complex $\mathrm{SUB}_{\mathcal{S}}(\mathsf{stststst}, e, 6)$ for $\mathfrak{S}_3$.

3.2.1. Alternative possible definitions for subword complexes. We have seen in Example 2.9.4 that an element of $\boldsymbol{B}^+$ is not uniquely determined by its colored inversion set. Together with Lemma 2.9.5 this suggests two further generalizations of Definition 3.2.1 for a given search word Q:

- For $\boldsymbol{w} \in \boldsymbol{B}^+$, let $\mathrm{SUB}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{w})$ be the *Artin subword complex* with facets given by subsets of (positions of) letters in Q whose complements form an $\mathcal{S}$-word for $\boldsymbol{w}$.

- For a set X of colored positive roots, let $\textsc{Sub}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, X)$ be the *inversion set subword complex* with facets given by (positions of) letters in Q whose complement is a word for some $\boldsymbol{w} \in \boldsymbol{B}^+$ with $\mathrm{inv}(\boldsymbol{w}) = X$.

We choose to work with Definition 3.2.1 because central properties of subword complexes do not generally hold for the other definitions (Lemma 3.3.4, Proposition 3.4.2). As we will see in Section 3.5, all three will coincide in the special circumstances we consider in later chapters.

They want to be the same, but they simply aren't in general.

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Proposition 3.2.4. *Let $\mathsf{Q} = \mathsf{s}_1 \cdots \mathsf{s}_p$ be a search word and let $\boldsymbol{w} \in \boldsymbol{B}^+$. There are injections*

$$\textsc{Sub}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{w}) \hookrightarrow \textsc{Sub}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, \mathrm{inv}(\boldsymbol{w})) \hookrightarrow \textsc{Sub}_{\mathcal{S}}(\mathsf{Q}, w, \ell_{\mathcal{S}}(\boldsymbol{w})),$$

where $\mathrm{inv}(\boldsymbol{w})$ is the colored inversion set of $\boldsymbol{w}$ and w is the projection of $\boldsymbol{w}$ in W.

To repeat, we will discuss in Section 3.5 that these notions of subword complex coincide in the situations we consider in this monograph.

Proof of Proposition 3.2.4. The first injection is clear, as the facets of $\textsc{Sub}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, \mathrm{inv}(\boldsymbol{w}))$ are given by the disjoint union of the facets of all $\textsc{Sub}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{w}')$ such that $\mathrm{inv}(\boldsymbol{w}') = \mathrm{inv}(\boldsymbol{w})$. On the other hand, this argument also shows that this injection is a bijection if and only if there is a unique $\boldsymbol{w} \in \boldsymbol{B}^+$ with $\mathrm{inv}(\boldsymbol{w}) = X$ for a given colored inversion set X.

The second embedding follows from Lemma 2.9.5. In this case, one also has that the facets of $\textsc{Sub}_{\mathcal{S}}(\mathsf{Q}, w, a)$ for $a = \ell_{\mathcal{S}}(\boldsymbol{w})$ are given by the disjoint union of the facets of all $\textsc{Sub}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, X)$ such that the colored inversion set X has cardinality a, contains the roots in $\mathrm{inv}(\boldsymbol{w})$ an odd number of times, and contains all other roots an even number of times (as described at the end of the proof of Lemma 2.9.5). □

Example 3.2.5. For $\mathfrak{S}_3$, for any search word Q

$$\textsc{Sub}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, \{\alpha^{(0)}, \alpha^{(1)}, \beta^{(0)}, \beta^{(1)}\}) = \textsc{Sub}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{sstt}) \sqcup \textsc{Sub}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{ttss}),$$

where the disjoint union should be interpreted on the level of facets.

3.3. Root configurations

Skip this section if you know root configurations for subword complexes. It's still the same story.

#8

Same good taste, but now with 50% less sodium!

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Fix a subword complex $\textsc{Sub}_{\mathcal{S}}(\mathsf{Q}, w)$ with $\mathsf{Q} = \mathsf{s}_1 \cdots \mathsf{s}_p$. Following [**CLS14**, Definition 3.2] and [**PS15**, Definition 3.1], we modify the colored inversion sequence of the $\mathcal{S}$-word Q to account for the choice of a facet $I = \{i_1 < \ldots < i_{p-a}\} \subseteq \{1, 2, \ldots, p\}$. The *root vector* of I is the tuple of colored positive roots $\big(\mathsf{r}_I(1), \ldots, \mathsf{r}_I(p)\big)$ defined by

$$\mathsf{r}_I(i) := s_1 \cdots \widehat{s}_{i_1} \cdots \widehat{s}_{i_k} \cdots s_{i-1}\big(\alpha^{(0)}_{s_i}\big),$$

where the letters $\mathsf{s}_{i_1}, \ldots, \mathsf{s}_{i_k}$ with $I \cap [i-1] = \{i_1 < \ldots < i_k\}$ are omitted. The *root configuration* of the facet I is the set

$$\mathsf{R}(I) = \big\{\mathsf{r}_I(i) \ : \ i \in I\big\}, \tag{3.1}$$

which is totally ordered by I as a subset of positions of Q. The root configurations of the facets of the subword complex from Example 3.2.3 are listed in the fourth column of Figure 13 on page 64.

Lemma 3.3.1. *Let $\mathsf{r}_I(\cdot) = \big(\beta_1^{(m_1)}, \ldots, \beta_p^{(m_p)}\big)$ be a root vector.*

(i) *The map* $\mathsf{r}_I(\cdot) : [p]\backslash I \longrightarrow \mathrm{inv}(\mathsf{s}_1 \cdots \widehat{\mathsf{s}}_{i_1} \cdots \widehat{\mathsf{s}}_{i_{p-a}} \cdots \mathsf{s}_p)$ *is a bijection. Moreover, Lemma 2.9.3(ii) shows how to recover the colors* m_i *for* $i \notin I$.

(ii) *For* $i \in I$, *the color* m_i *equals the number of indices* $j < i$ *with* $j \notin I$ *and* $\beta_j = \beta_i$.

Proof. Item (i) is a direct consequence of the definition. Item (ii) follows from the observation that $\mathsf{r}_I(i)$ only depends on $I \cap [i-1]$. This means that $\mathsf{r}_I(i)$ for $i \in I$ is given by $\beta_i^{(m_i)}$ with $\beta_i = |w(\alpha_{s_i})| \in \Phi^+$ for $w = s_1 \cdots \widehat{s}_{i_1} \cdots \widehat{s}_{i_k} \cdots s_{i-1} \in W$ and the color m_i is equal to the number of occurrences of the root β_i in position j with $j < i$ and $j \notin I$. □

3.3.1. Flips. We say that two facets I and J are *adjacent* if $I \setminus \{i\} = J \setminus \{j\}$ for some positions $1 \leq i, j \leq p$ with $i \neq j$. If I and J are adjacent, the *flip* from I to J is *increasing* if $i < j$ and *decreasing* if $i > j$. The *direction* of the flip is given by the root vector $\mathsf{r}_I(i)$.

The root configuration captures the notions of increasing and decreasing flips in subword complexes, described for $g = 0$ in [**CLS14, PS15**]. Note that when $g = 0$, each vertex can be flipped in only one way [**CLS14, PS15**]. This is no longer true when $g > 0$—it may be possible to find three distinct facets I, J, K with $I \setminus \{i\} = J \setminus \{j\} = K \setminus \{k\}$.

Lemma 3.3.2. *Let* I *and* J *be two adjacent facets of* $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w)$ *with* $I \backslash i = J \backslash j$ *and write* $\beta = |\mathsf{r}_I(i)|$. *Then* $|\mathsf{r}_J(k)| = |s_\beta(\mathsf{r}_I(k))|$ *for* $\min(i,j) \leq k \leq \max(i,j)$ *as uncolored roots, and* $\mathsf{r}_J(k) = \mathsf{r}_I(k)$ *otherwise.*

Proof. This is straightforward from the definition and from the description of colors in Lemma 3.3.1. □

Lemma 3.3.3 (see also [**PS15**, Lemma 3.3(2)]). *Let* I *be a facet of the subword complex* $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, a)$ *with* $i \in I$, *and write* $\beta^{(k)} = \mathsf{r}_I(i)$. *If* J *is adjacent to* I *with* $I \setminus \{i\} = J \setminus \{j\}$, *then* $\mathsf{r}_I(j) = \beta^{(\ell)}$ *for some color* ℓ. *Furthermore,* $i < j$ *if and only if* $k \leq \ell$.

Proof. Assume that $i < j$, the case $i > j$ being completely analogous. Let $\mathsf{s}_1 \ldots \mathsf{s}_p$ be the word obtained from Q by removing all letters in $I \cap J = I \setminus i = J \setminus j$ and let the two remaining letters from I and J in $\mathsf{s}_1 \ldots \mathsf{s}_p$ be $\mathsf{s}_{i'}$ and $\mathsf{s}_{j'}$, respectively. Since $i < j$, we also have $i' < j'$. By definition, we have $p = a + 1$ and

$$w = s_1 \cdots \widehat{s}_{i'} \cdots s_p = s_1 \cdots \widehat{s}_{j'} \cdots s_p$$

and obtain $s_{i'+1} \cdots s_{j'-1} s_{j'} = s_{i'} s_{i'+1} \cdots s_{j'-1}$. This gives $s_{i'}^{\rho} = s_{j'}$ for $\rho = s_{i'+1} \cdots s_{j'-1} \in W$, implying that $\alpha_{s_{i'}} = \rho(\alpha_{s_{j'}})$ and

$$|\mathsf{r}_I(i)| = |s_1 \cdots s_{i'-1}(\alpha_{s_{i'}})| = |s_1 \cdots s_{i'-1} s_{i'+1} \cdots s_{j'-1}(\alpha_{s_{j'}})| = |\mathsf{r}_I(j)|.$$

The statement about the colors follows from Lemma 3.3.1(ii). □

Lemma 3.3.4. *Let* I *be a facet of the subword complex* $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, a)$ *and let* $i \in I$ *and* $j \notin I$ *such that* $|\mathsf{r}_I(i)| = |\mathsf{r}_I(j)|$. *Then* $(I \setminus \{i\}) \cup \{j\}$ *is again a facet of* $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, a)$.

Proof. The proof is the reverse of the argument in the proof of Lemma 3.3.3. Assume again that $i < j$, the case $i > j$ being completely analogous. Let $\mathsf{s}_1 \cdots \mathsf{s}_p$ be the word obtained from Q by removing all letters in $I \setminus i$, let $\mathsf{s}_{i'}$ be the remaining letter from I inside $\mathsf{s}_1 \dots \mathsf{s}_p$, and let $\mathsf{s}_{j'}$ with $i' < j'$ be the letter in Q corresponding to $j \notin I$. We then have $w = s_1 \cdots \widehat{s}_i \cdots s_p$ and

$$|s_1 \cdots s_{i'-1}(\alpha_{s_{i'}})| = |\mathsf{r}_I(i)| = |\mathsf{r}_I(j)| = |s_1 \cdots \widehat{s}_{i'} \cdots s_{j'-1}(\alpha_{s_{j'}})|.$$

Thus, $s_{i'} = \rho s_{j'} \rho^{-1}$ for $\rho = s_{i'+1} \cdots s_{j'-1}$. This gives

$$\begin{aligned} w &= s_1 \cdots \widehat{s}_{i'} \cdots s_p \\ &= s_1 \cdots s_{i'-1} s_{i'} s_{i'} s_{i'+1} \cdots s_p \\ &= s_1 \cdots s_{i'} \rho s_{j'} \rho^{-1} \rho s_{j'} \cdots s_p \\ &= s_1 \cdots \widehat{s}_{j'} \cdots s_p. \end{aligned}$$ □

LEMMA 3.3.5 (see also [**PS15**, Lemma 3.4 and Remark 3.5]). *A facet I of the subword complex* $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, a)$ *is uniquely determined by its root configuration* $\mathsf{R}(I)$.

PROOF. The proof is completely analogous to the proof of [**PS15**, Lemma 3.4]. Let I and J be two distinct facets of $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, a)$, and let i be the smallest index which is in $I \cup J$ but not in $I \cap J$. Assume without loss of generality that $i \in I$. We then have that $\mathsf{r}_I(i) = \mathsf{r}_J(i)$. Also, Lemma 3.3.1(ii) shows that $\mathsf{r}_J(i) = \mathsf{r}_J(j)$ implies $j \leq i$. Consequently, $\mathsf{r}_I(i)$ appears at least once more in $\mathsf{R}(I)$ than in $\mathsf{R}(J)$. □

DEFINITION 3.3.6. The *increasing flip graph* $\mathcal{G}(\mathsf{Q}, w, a)$ of $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, a)$ is the directed graph whose vertices are the facets of $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, a)$ with directed edges given by increasing flips. The *increasing flip poset* is the transitive closure of the increasing flip graph.

Figure 14 on page 66 illustrates the increasing flip poset in $\mathfrak{S}_3$ for the subword complex $\mathrm{SUB}_{\mathcal{S}}(\mathsf{stststst}, e, 6)$. Increasing flips are drawn upwards.

3.4. Dual subword complexes

DEFINITION 3.4.1. Let Q be an $\mathcal{R}$-word, let $w \in W$, and let $a = \ell_{\mathcal{R}}(w) + 2g$ for some $g \geq 0$. The *dual subword complex* $\mathrm{SUB}_{\mathcal{R}}(\mathsf{Q}, w, a)$ is the simplicial complex with facets being all subsets of (positions of) letters in Q that are an $\mathcal{R}$-word for w of length a. We write $\mathrm{SUB}_{\mathcal{R}}(\mathsf{Q}, w)$ for $\mathrm{SUB}_{\mathcal{R}}(\mathsf{Q}, w, \ell_{\mathcal{R}}(w))$.

The analogue of Proposition 3.2.2 evidently holds for dual subword complexes.

These want to be defined in the dual braid group. They usually can't, unless they're of a very special form...

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3.4.1. Duality of subword complexes.

Definitions 3.2.1 and 3.4.1 are related by the following proposition, which associates to any subword complex an isomorphic dual subword complex. In fact, both complexes coincide as abstract complexes on the ground set $\{1, \dots, p\}$.

PROPOSITION 3.4.2. *Let* $\mathsf{Q} = \mathsf{s}_1 \cdots \mathsf{s}_p$ *be an* $\mathcal{S}$*-word, let* $w \in W$*, and let* $a = \ell_{\mathcal{S}}(w) + 2g$ *for some* $g \geq 0$*. Then*

$$\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, a) = \mathrm{SUB}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}), w', b),$$

where $b = p - a$ *and* $w' = w s_p \cdots s_1 \in W$.

PROOF. This proof uses the same argument as the proofs of [**IS10**, Lemma 3.2] and of [**CLS14**, Proposition 2.8]. Let $\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}) = (\mathsf{r}_1, \dots, \mathsf{r}_p)$ and let $I = \{i_1, \cdots, i_b\}$

with $1 \leq i_1 \leq \ldots \leq i_b \leq p$. For each j from 1 to b, replace $\widehat{s}_{i_j}$ by $s_{i_j} s_{i_j}$ and move one copy to the left by conjugation. We obtain

$$s_1 \cdots \widehat{s}_{i_1} \cdots \widehat{s}_{i_b} \cdots s_p = r_{i_1} \cdots r_{i_b} s_1 \cdots s_p.$$

Therefore, the set $\{i_1, \ldots, i_b\}$ is a facet of $\textsc{Sub}_{\mathcal{S}}(\mathsf{Q}, w, a)$ if and only if

$$w = r_{i_1} \cdots r_{i_b} s_1 \cdots s_p$$

or, equivalently, $r_{i_1} \cdots r_{i_b} = w s_p \cdots s_1$. This is the case if and only if $\{i_1, \ldots, i_b\}$ is a facet of $\textsc{Sub}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}), w', b)$. □

Even though subword complexes and their associated dual subword complexes are identical because we are working in the Coxeter group W, they provide two different viewpoints in $\boldsymbol{B}$. This will be essential in the proof of Theorem 3.5.1.

Example 3.4.3. As in Example 3.2.3, let $\mathsf{Q} = \mathsf{stststst}$ be an $\mathcal{S}$-word with $\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}) = \mathsf{sutsutsu}$. The 12 facets of $\textsc{Sub}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}), su)$ are listed in the second column of Figure 13 on page 64.

3.5. Coxeter-initial subword complexes

We now turn to a class of subword complexes that behave particularly nicely, and which play a central role in the m-eralized noncrossing theory.

Theorem 3.5.1. *Let c be a Coxeter element with $\mathcal{S}$-word $\mathsf{c} = \mathsf{s}_1 \ldots \mathsf{s}_n$, and let $\mathsf{Q} = \mathsf{s}_1 \ldots \mathsf{s}_n \mathsf{s}_{n+1} \ldots \mathsf{s}_p$ be initial in c^∞. Set $\boldsymbol{w} = \boldsymbol{s}_{n+1} \ldots \boldsymbol{s}_p \in \boldsymbol{B}^+$ and its projection $w = s_{n+1} \ldots s_p \in W$. Then*

$$\textsc{Sub}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{w}) = \textsc{Sub}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, \mathrm{inv}(\boldsymbol{w})) = \textsc{Sub}_{\mathcal{S}}(\mathsf{Q}, w, p-n).$$

. . . and in that special case it doesn't matter which one you use! (Learn about the dual braid monoid before reading the proof.)

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We call subword complexes of the form given in this theorem *c-initial subword complexes*.

We use the relationship between subword and dual subword complexes given in Proposition 3.4.2 to prove this theorem. The core is a lift of Proposition 3.4.2 to the Artin group $\boldsymbol{B}$. This uses a certain embedding of the reflections $\mathcal{R} \subseteq W$ into $\boldsymbol{B}$ which depends on the choice of a Coxeter element c. This construction is essentially due to D. Bessis [**Bes03**], and we emphasize that this embedding is *not* the injection $\mathcal{R} \subseteq W \hookrightarrow \boldsymbol{B}^+ \subseteq \boldsymbol{B}$ defined in (2.1).

Let $\mathsf{Q} = \mathsf{s}_1 \ldots \mathsf{s}_p$ be initial in c^∞ and set

$$\boldsymbol{r}_i := \boldsymbol{s}_1 \ldots \boldsymbol{s}_{i-1} \boldsymbol{s}_i \boldsymbol{s}_{i-1}^{-1} \ldots \boldsymbol{s}_1^{-1} \in \boldsymbol{B} \tag{3.2}$$

using the identification between $\mathcal{S}$ and $\boldsymbol{\mathcal{S}}$.

Proposition 3.5.2. *Let $c = s_1 \ldots s_n$ be a Coxeter element with $\mathcal{S}$-word c and let $\boldsymbol{c} = \boldsymbol{s}_1 \ldots \boldsymbol{s}_n \in \boldsymbol{B}$. Let $\mathsf{Q} = \mathsf{s}_1 \ldots \mathsf{s}_p$ be a word initial in c^∞ starting with $\mathsf{s}_1 \ldots \mathsf{s}_n$, and let $\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}) = \mathsf{r}_1 \ldots \mathsf{r}_p$ be the inversion sequence of Q. Let $1 \leq i_1 < \cdots < i_n \leq p$ such that $r_{i_n} \ldots r_{i_1} = c \in W$. Then*

$$\boldsymbol{r}_{i_n} \ldots \boldsymbol{r}_{i_1} = \boldsymbol{c} \in \boldsymbol{B}. \tag{3.3}$$

Equivalently,

$$\boldsymbol{s}_1 \ldots \widehat{\boldsymbol{s}}_{i_1} \ldots \widehat{\boldsymbol{s}}_{i_n} \ldots \boldsymbol{s}_p = \boldsymbol{c}^{-1} \boldsymbol{s}_1 \ldots \boldsymbol{s}_p \in \boldsymbol{B}^+.$$

PROOF. This proposition follows for bipartite Coxeter elements from [**Bes03**, Theorem 2.2.5], we only provide a sketch of the argument here. Assume the Coxeter element c to be bipartite. D. Bessis shows that the elements

$$\boldsymbol{\mathcal{R}} = \left\{\boldsymbol{c}^k\boldsymbol{s}\boldsymbol{c}^{-k} \mid k \in \mathbb{Z}, \boldsymbol{s} \in \boldsymbol{\mathcal{S}}\right\} \subset \boldsymbol{B}$$

are in bijection with $\mathcal{R} \subseteq W$ and that these satisfy the dual braid relations as defined in Section 2.7.1. Since c is bipartite and Q is initial in c^∞, we have $\{\boldsymbol{r}_1, \dots, \boldsymbol{r}_p\} \subseteq \boldsymbol{\mathcal{R}}$ for $\boldsymbol{r}_i$ as defined in (3.2), implying that these also satisfy the dual braid relations.

We now deduce (3.3) for bipartite Coxeter elements by recalling that the Hurwitz action is transitive on $\mathrm{Red}_{\mathcal{R}}(c)$. Therefore $r_{i_n} \dots r_{i_1}$ and $s_1 \dots s_n$ are related by dual braid relations, implying that $\boldsymbol{r}_{i_n} \dots \boldsymbol{r}_{i_1}$ and $\boldsymbol{s}_1 \dots \boldsymbol{s}_n$ are also related by dual braid relations. As these relations are satisfied in $\boldsymbol{B}$, we obtain that $\boldsymbol{r}_{i_n} \dots \boldsymbol{r}_{i_1} = \boldsymbol{c}$, as desired. We finally compute

$$\boldsymbol{s}_1 \dots \widehat{\boldsymbol{s}}_{i_1} \dots \widehat{\boldsymbol{s}}_{i_n} \dots \boldsymbol{s}_p = \boldsymbol{r}_{i_1}^{-1} \dots \boldsymbol{r}_{i_n}^{-1} \boldsymbol{s}_1 \dots \boldsymbol{s}_p = \boldsymbol{c}^{-1} \boldsymbol{s}_1 \dots \boldsymbol{s}_p.$$

The proposition for general Coxeter elements now follows from the claim that if the proposition holds for the Coxeter element c than it also holds for the Coxeter element $\bar{s}cs$ because every Coxeter element can be obtained from any other by such a procedure.

We obtain this reduction as follows. Let Q be an $\mathcal{S}$-word and let $\widehat{\mathsf{Q}}$ be the word obtained from Q by removing the first letter. Let c be a Coxeter element with word $\mathsf{c} = \mathsf{s}_1 \dots \mathsf{s}_n$. Lemma 2.6.5(7) then implies that Q is initial in c^∞ if and only if $\widehat{\mathsf{Q}}$ is initial in $\bar{\mathsf{s}}\mathsf{c}\mathsf{s}^\infty$, so we assume both to be initial.

Given indices $2 \leq i_1 < \dots < i_n \leq p$ such that $r_{i_n} \dots r_{i_1} = c \in W$. If the conclusion holds for the Coxeter element c, then $\boldsymbol{r}_{i_n} \dots \boldsymbol{r}_{i_1} = \boldsymbol{c} \in \boldsymbol{B}$. We obtain that

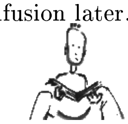
$$\boldsymbol{r}_{i_n}^{\boldsymbol{s}_1} \dots \boldsymbol{r}_{i_1}^{\boldsymbol{s}_1} = \boldsymbol{s}_1^{-1} \boldsymbol{r}_{i_n} \dots \boldsymbol{r}_{i_1} \boldsymbol{s}_1 = \boldsymbol{s}_1^{-1} \boldsymbol{c} \boldsymbol{s}_1 = \boldsymbol{s}_2 \dots \boldsymbol{s}_n \boldsymbol{s}_1 \in \boldsymbol{B}.$$

This implies the conclusion also for the Coxeter element $\bar{s}cs$. □

PROOF OF THEOREM 3.5.1. This is a direct consequence of Proposition 3.4.2 using Proposition 3.5.2. □

EXAMPLE 3.5.3. Let $c = st \in \mathfrak{S}_3$ be a Coxeter element and consider the word $\mathsf{Q} = \mathsf{ststst}$ with reflection sequence $\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}) = \mathsf{sutsut}$. Then

This example is worth going through to avoid confusion later.



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$$\begin{aligned} \mathrm{SUB}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}), c^{-1}) &= \{12, 15, 23, 26, 34, 45, 56\} \\ \mathrm{SUB}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}), c\phantom{^{-1}}) &= \{13, 16, 24, 35, 46\}. \end{aligned}$$

The first is a c-initial subword complex and we have

$$\begin{aligned} \mathrm{SUB}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}), c^{-1}) &= \mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, c^{-1}, 4) \\ &= \mathrm{SUB}_{\mathcal{S}}^{\mathrm{col}}(\mathsf{Q}, \{\alpha^{(0)}, \gamma^{(0)}, \beta^{(0)}, \alpha^{(1)}\}) \\ &= \mathrm{SUB}_{\boldsymbol{\mathcal{S}}}^{\boldsymbol{B}}(\mathsf{Q}, \boldsymbol{stst}). \end{aligned}$$

On the other hand, $\mathrm{SUB}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{Q}), c) = \mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, c, 4)$ is *not* c-initial and does not coincide with the corresponding Artin subword complex.

3.6. Topology of Coxeter-initial subword complexes

When $g = 0$, A. Knutson and E. Miller showed in [**KM04**, Theorem 2.5 & Corollary 3.8] that subword complexes are vertex-decomposable spheres or balls. Subword complexes are generally neither spheres nor balls for $g > 0$, and are not even necessarily vertex-decomposable.

EXAMPLE 3.6.1. Consider the word $\mathsf{Q} = \mathsf{tsstts}$ for $\mathfrak{S}_3$ and the element $\boldsymbol{w} = \boldsymbol{ssts} = \boldsymbol{stst} = \boldsymbol{tstt} \in \mathfrak{S}_3^+$. Then the subword complex is

$$\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, 4) = \{14, 15, 23, 26, 36, 45\},$$

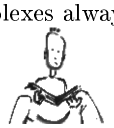
while the Artin subword complex is

$$\mathrm{Sub}_{\mathcal{S}}^{\boldsymbol{B}}(\mathsf{Q}, \boldsymbol{w}) = \{14, 15, 26, 36\}.$$

Since the only words with colored inversion set $\mathrm{inv}(\boldsymbol{w}) = \{\alpha^{(0)}, \alpha^{(1)}, \beta^{(0)}, \gamma^{(0)}\}$ are reduced $\mathcal{S}$-words for $\boldsymbol{w}$, we therefore have $\mathrm{Sub}_{\mathcal{S}}^{\boldsymbol{B}}(\mathsf{Q}, \boldsymbol{w}) = \mathrm{Sub}_{\mathcal{S}}^{\mathrm{col}}(\mathsf{Q}, \mathrm{inv}(\boldsymbol{w}))$. These simplicial complexes are disconnected but of positive dimension, and are therefore not vertex-decomposable.

THEOREM 3.6.2. *Let c be a Coxeter element with $\mathcal{S}$-word $\mathsf{c} = \mathsf{s}_1 \dots \mathsf{s}_n$, and let $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_n \mathsf{s}_{n+1} \dots \mathsf{s}_p$ be initial in c^∞. Set $w = s_{n+1} \dots s_p \in W$. Then the Coxeter-initial subword complexes $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, p-n)$ is vertex-decomposable. Moreover, the lexicographic order on the facets is a shelling order.*

You see—these behave like subword complexes always did.



#10

word.

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PROOF. We show that both the link and the deletion of the first vertex are vertex-decomposable subword complexes by simultaneous induction on the rank of W and on the length of Q. The base case of the empty word Q is clear, so we may assume $p > n$. Without loss of generality, we may also assume that $s_{n+1} = s_1$.

We first address the link. The facets of the link of the first vertex of the complex $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, p-n)$ are given by

$$\big\{ I \setminus 1 \ : \ I \text{ a facet of } \mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, p-n) \text{ with } 1 \in I \big\}.$$

Let $\widehat{\mathsf{Q}}$ be the restriction of Q to those letters s_i with ${}^{s_1 \cdots s_{i-1}} s_i \in W_{\langle s \rangle}$. Then it is clear that

$$\mathrm{Sub}_{\mathcal{R}}(\mathsf{Q}, s_n \cdots s_2) \cong \mathrm{Sub}_{\mathcal{R}}(\widehat{\mathsf{Q}}, s_n \cdots s_2).$$

This complex is vertex-decomposable by induction on the rank of W.

We next consider the deletion. Let I be a facet of $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, p-n)$ with $1 \in I$, so that $I \setminus 1$ is a face of the deletion of the vertex 1. We show that there is a facet J of $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, p-n)$ with $J \setminus j = I \setminus 1$ and $j > 1$. Observe that $\mathsf{r}_I(1) = \alpha_{s_1}^{(0)}$. By Lemma 3.3.3, we thus have to show that α_{s_1} appears in the inversion sequence of Q with the letters in positions I deleted. This follows from Theorem 3.5.1 because the inversion sequence of Q with the letters in positions I deleted is the same as the inversion sequence of $\mathsf{s}_{n+1} \dots \mathsf{s}_p$. By construction, $s_{n+1} = s_1$ and thus $\mathsf{inv}_{\mathcal{R}}(\mathsf{s}_{n+1} \dots \mathsf{s}_p)$ starts with α_{s_1}. □

COROLLARY 3.6.3. *Let c be a Coxeter element with $\mathcal{S}$-word $\mathsf{c} = \mathsf{s}_1 \dots \mathsf{s}_n$, and let $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_n \mathsf{s}_{n+1} \dots \mathsf{s}_p$ be initial in c^∞. Set $w = s_{n+1} \dots s_p \in W$. Let $\widehat{\mathsf{Q}}$ of length q be obtained from Q by removing the longest initial subword that is also initial in $\mathsf{w}_\circ(\mathsf{c})$. Then $\mathrm{Sub}_{\mathcal{S}}(\mathsf{Q}, w, p-n)$ has the homotopy type of a wedge of k spheres where k is the number of facets of $\mathrm{Sub}_{\mathcal{S}}(\widehat{\mathsf{Q}}, w, q-n)$.*

Observe in this corollary, that $\mathsf{w}_\circ(\mathsf{c})$ is initial in c^∞ by Lemma 2.6.5(5). Therefore, $\widehat{\mathsf{Q}}$ is obtained from Q by removing the intersection of $\mathsf{w}_\circ(\mathsf{c})$ and Q as initial segments of c^∞.

PROOF. Because the complex is vertex-decomposable and every facet contains n positions, it has the homotopy type of a wedge of $(n-1)$-dimensional spheres. We count the number of such spheres using the technique of *homology facets*, developed by A. Björner and M. Wachs in [**BW96**] (see also [**Wac07**, Theorem 3.1.3]). The number of spheres is given by the number of facets whose entire boundary is contained in the union of the earlier facets in the lexicographic shelling order of Theorem 3.6.2.

We show that the homology facets of $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, p-n)$ are exactly those facets that do not contain a position corresponding to one of the letters in the initial copy of $\mathsf{w}_\circ(\mathsf{c})$. These facets are in canonical bijection with facets of $\mathrm{SUB}_{\mathcal{S}}(\widehat{\mathsf{Q}}, w, q-n)$.

We first consider the case when I is a facet of $\mathrm{SUB}_{\mathcal{S}}(\mathsf{Q}, w, p-n)$ that does not contain a position corresponding to a letter of the initial $\mathsf{w}_\circ(\mathsf{c})$. Then $k > 0$ for $\beta^{(k)} = \mathsf{r}_I(i)$ for any $i \in I$. Since the root vectors in this initial copy of $\mathsf{w}_\circ(\mathsf{c})$ are the positive zero-colored roots, we can use Lemma 3.3.4 to flip any position $i \in I$ into a letter in this initial copy of $\mathsf{w}_\circ(\mathsf{c})$ to obtain another facet J with $I \setminus i = J \setminus j$ for some position j. Since $j < i$ by construction, the obtained facet J is lexicographically smaller and contains the boundary face $I \setminus i$ of I. The facet I is therefore a homology facet.

Suppose now that I is a facet containing a position i of a letter in this initial $\mathsf{w}_\circ(\mathsf{c})$. The root vector $\beta^{(k)} = \mathsf{r}_I(i)$ is now colored by $k = 0$, and the position $i \in I$ cannot be flipped to a lexicographically smaller facet. The boundary face $I \setminus i$ is therefore not contained in any lexicographically smaller facet, and I is thus not a homology facet. $\square$

CHAPTER 4

Noncrossing partitions

In this chapter, we study m-eralized noncrossing partitions. We begin by reviewing known constructions (Sections 4.1 and 4.2). Detailed background and historical information can be found in [**Arm06**]. We then describe the theory in terms of dual subword complexes and conclude with a definition of the Cambrian recurrence and Cambrian poset on m-eralized noncrossing partitions (Section 4.3).

4.1. Classical noncrossing partitions

A set partition of $\{1, \ldots, n\}$ is called *noncrossing* if the convex hulls of its blocks do not overlap when drawn on a circle. G. Kreweras introduced and studied the lattice of noncrossing partitions ordered by containment in [**Kre72**]—the noncrossing partition lattice of $\{1, 2, 3, 4\}$ is illustrated below.

This sentence is clarified by the picture, until you notice that we haven't drawn convex hulls.

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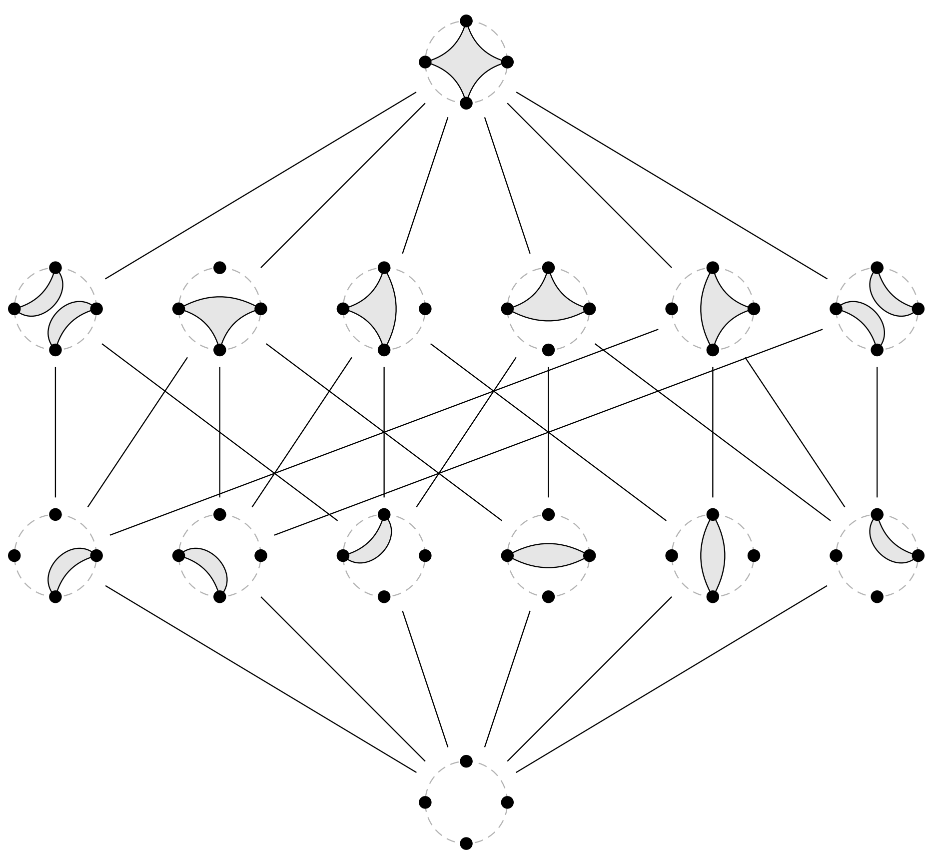

4.1.1. Noncrossing partitions for Coxeter groups. In [**Rei97**], V. Reiner interpreted the noncrossing partition lattice as a type A phenomenon using the identification of the lattice of set partitions with the intersection lattice of the type A_{n-1} reflection arrangement. Together with C. Athanasiadis, he extended this notion to types B_n and D_n in [**Rei97, AR04**]. Noncrossing partitions were subsequently defined in full generality for finite Coxeter systems independently by T. Brady and C. Watt in [**BW02**] and by D. Bessis in [**Bes03**].

Let $c \in W$ be a Coxeter element. An element $w \in W$ is a *c-noncrossing partition* if $w \leq_{\mathcal{R}} c$, and we denote the set of c-noncrossing partitions by $\mathsf{NC}(W, c)$. The following theorem was proven independently by different methods by D. Bessis [**Bes03**, Fact 2.3.1], T. Brady and C. Watt [**BW08**, Theorem 7.8], C. Ingalls and H. Thomas [**IT09**, Theorem 4.2], and N. Reading [**Rea11**, Corollary 8.6].

THEOREM 4.1.1. *The c-noncrossing partitions* $\mathsf{NC}(W, c)$ *partially ordered by the absolute order form a lattice.* □

We denote this *c-noncrossing partition lattice* by

$$\mathsf{NCL}(W, c) := [e, c]_{\mathsf{Abs}(W)}.$$

In type A_{n-1} with $c = (1, 2, \ldots, n) \in \mathfrak{S}_n$, the original lattice of noncrossing set partitions of $\{1, \ldots, n\}$ is recovered by sending a permutation to the set partition given by its cycles. Figure 10 gives an example for $c = (1, 2, 3, 4) \in \mathfrak{S}_4$.

Noncrossing partitions have the following useful characterization, which appeared in a slightly different form in [**BW08**, Lemma 4.8].

PROPOSITION 4.1.2. *Let* $r_1, \ldots, r_k \in \mathcal{R}$. *Then*

$$r_1 \cdots r_k \leq_{\mathcal{R}} c \text{ with } \ell_{\mathcal{R}}(r_1 \cdots r_k) = k$$
$$\Leftrightarrow$$
$$r_1, \ldots, r_k \text{ are pairwise distinct and } r_a r_b \leq_{\mathcal{R}} c \text{ for all } 1 \leq a < b \leq k.$$

Go back and read the lemma about fixed and moved spaces.

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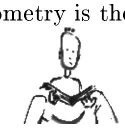
For clarity of the argument, we extract the following lemma before proving Proposition 4.1.2.

LEMMA 4.1.3. *Let* $r_1, r_2 \in \mathcal{R}$ *with* $r_1 \neq r_2$. *Let* $\beta_1, \beta_2 \in \Phi^+$ *such that* $r_1 = s_{\beta_1}$ *and* $r_2 = s_{\beta_2}$ *and let* μ *be a generator of* $\mathrm{Fix}(r_1 c)$ *such that* $\langle \beta_1, \mu \rangle = 1$. *Then*

$$r_1 r_2 \leq_{\mathcal{R}} c \Leftrightarrow \langle \mu, \beta_2 \rangle = 0.$$

Geometry is the key.



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Thanks, Brady and Watt!

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PROOF. By Lemma 2.8.2(i), $\mathrm{Fix}(r_1 c) = \mathrm{Mov}(r_1 c)^{\perp}$ is one-dimensional. By Lemma 2.8.2(v), β_1 is not perpendicular to $\mathrm{Fix}(r_1 c)$, so that μ is well-defined,

$$r_1 \not\leq_{\mathcal{R}} r_1 c \;\Rightarrow\; \mathrm{Mov}(r_1) \not\subseteq \mathrm{Mov}(r_1 c) \;\Rightarrow\; \mathrm{Fix}(r_1 c) \not\subseteq \mathrm{Fix}(r_1) \;\Rightarrow\; \beta_1 \not\perp \mathrm{Fix}(r_1 c).$$

Then

$$r_1 r_2 \leq_{\mathcal{R}} c \;\Leftrightarrow\; r_2 \leq_{\mathcal{R}} r_1 c \;\Leftrightarrow\; \mathrm{Mov}(r_2) \subseteq \mathrm{Mov}(r_1 c) \;\Leftrightarrow\; \langle \beta_2, \mu \rangle = 0,$$

as desired. The first implication follows from Lemma 2.8.2(iv), the second from (ii) and (v), and the third from the definitions of β_2 and μ. □

PROOF OF PROPOSITION 4.1.2. The forward implication is clear: the distinctness of the reflections follows from the fact that $\ell_{\mathcal{R}}(r_1 \ldots r_k) = k$, and for any $1 \leq a < b \leq k$, the pair $r_a r_b$ can be moved to the left by conjugation to reveal that $r_a r_b \leq_{\mathcal{R}} r_1 \cdots r_k \leq_{\mathcal{R}} c$.

The reverse implication is more delicate. As in Lemma 4.1.3, define $\beta_a \in \Phi^+$ such that $r_a = r_{\beta_a}$ and let μ_a be the generator of $\mathrm{Fix}(r_a c)$ such that $\langle \beta_a, \mu_a \rangle = 1$. (As above, μ_a is well-defined because β_a is not perpendicular to $\mathrm{Fix}(r_a c)$.)

Consider the $(k \times k)$-matrix $\langle \alpha_a, \mu_b \rangle_{1 \leq a,b \leq k}$. By Lemma 4.1.3 and the assumption that $r_a r_b \leq_{\mathcal{R}} c$ with $r_a \neq r_b$ for all $1 \leq a < b \leq k$, this matrix is upper-triangular with ones on the main diagonal. Therefore, $\{\mu_1, \ldots, \mu_k\}$ are linearly independent.

We complete the proof by showing by induction that $r_1 \cdots r_{k'} \leq_{\mathcal{R}} c$ for $1 \leq k' \leq k$. The case $k' = 1$ is trivial. Assume that $k' \geq 2$. By induction, for $1 \leq a < k'$, we have

$$\begin{aligned} r_a \leq_{\mathcal{R}} r_1 \cdots r_{k'-1} \leq_{\mathcal{R}} c &\Rightarrow r_a c \geq_{\mathcal{R}} r_{k'-1} \cdots r_1 c \\ &\Rightarrow \mathrm{Fix}(r_a c) \subseteq \mathrm{Fix}(r_{k'-1} \cdots r_i c) \\ &\Rightarrow \mu_a \in \mathrm{Fix}(r_{k'-1} \cdots r_1 c). \end{aligned}$$

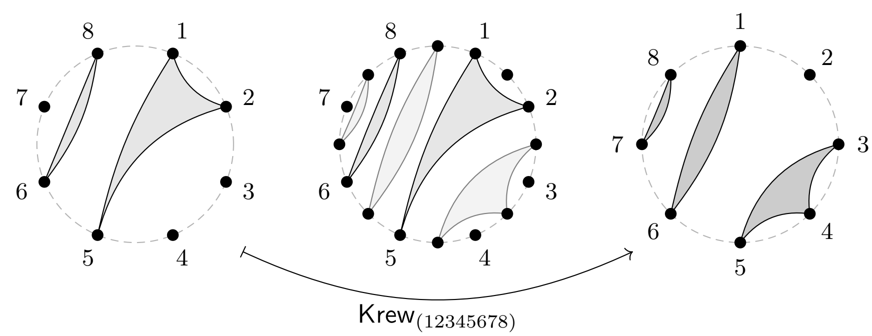
Now $\{\mu_1, \ldots, \mu_{k-1}\}$ are linearly independent by the calculation above and it follows from the inductive hypothesis that $\dim \mathrm{Fix}(r_{k'-1} \cdots r_1 c) = k' - 1$. Therefore, $\{\mu_1, \ldots, \mu_{k'-1}\}$ is a basis of $\mathrm{Fix}(r_{k'-1} \cdots r_1 c)$. As $\beta_{k'}$ is orthogonal to μ_a for $a < k'$, we have that $\beta_{k'} \in \mathrm{Mov}(r_{k'-1} \cdots r_1 c)$. This proves that $r_{k'} \leq_{\mathcal{R}} r_{k'-1} \cdots r_1 c$, so that $r_1 \cdots r_{k'} \leq_{\mathcal{R}} c$. □

4.1.2. The Kreweras complement. The *Kreweras complement* is the bijection on noncrossing partitions

$$\begin{aligned} \mathsf{Krew}_c : \mathsf{NC}(W, c) &\to \mathsf{NC}(W, c) \\ w &\mapsto c w^{-1}. \end{aligned} \tag{4.1}$$

This operation is an anti-automorphism of the lattice $\mathsf{NCL}(W, c)$—if $w \leq_{\mathcal{R}} wr$ for some $r \in \mathcal{R}$, then $\mathsf{Krew}_c(wr) = \mathsf{Krew}_c(w) \cdot {}^w r \leq_{\mathcal{R}} \mathsf{Krew}_c(w)$.

In the graphical description of noncrossing partitions, the Kreweras complement sends a noncrossing partition to the coarsest "complementary" noncrossing partition, as indicated below. Its square $\mathsf{Krew}_c^2(w) = {}^c w$ is conjugation by c, which corresponds to the cyclic rotation $i \mapsto i + 1$.



4.1.3. Noncrossing partitions as dual subword complexes. C. Athanasiadis, T. Brady, and C. Watt used the edge labelling of the noncrossing partition lattice by reflections to prove that $\mathsf{NCL}(W,c)$ is *EL-shellable* [**ABW07**]. This edge labelling gives a unique factorization of each element $w \in \mathsf{NC}(W,c)$ into reflections that increase with respect to the reflection order $\leq_{\mathsf{c}}$, as shown in [**ABW07**, Theorem 3.5]. Such factorizations were also considered by N. Reading in [**Rea07a**, Remark 6.8].

PROPOSITION 4.1.4. *Each $w \in \mathsf{NC}(W,c)$ has a unique reduced $\mathcal{R}$-word $\mathsf{r}_1\mathsf{r}_2\cdots\mathsf{r}_p$ with $r_1 <_{\mathsf{c}} r_2 <_{\mathsf{c}} < \cdots <_{\mathsf{c}} r_p$.* □

Since $\mathsf{Krew}_c(w)\cdot w = c$, by combining a noncrossing partition with its Kreweras complement and invoking the unique factorization of Proposition 4.1.4, we obtain a refined description of noncrossing partitions considered in [**Arm06, Tza08, BRT12**]. We m-eralize this description in Definition 4.2.4 below.

PROPOSITION 4.1.5. *There is a canonical bijection between c-noncrossing partitions $\mathsf{NC}(W,c)$ and facets of the dual subword complex*

$$\mathsf{NC}_\Delta(W,c) := \mathrm{SUB}_{\mathcal{R}}\big(\mathsf{inv}_{\mathcal{R}}(\mathsf{c}^h), c\big),$$

for any reduced $\mathcal{S}$-word c for c.

Having access to individual reflections is going to come in handy.

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The construction of $\mathsf{NC}_\Delta(W,c)$ depends on the chosen reduced word c for c, but the complexes for all possible choices are canonically isomorphic by Lemma 2.6.5(1) and Proposition 3.2.2. We therefore prefer to attach this complex to the Coxeter element c itself rather than to any specific reduced word.

PROOF OF PROPOSITION 4.1.5. The map $w \mapsto (\mathsf{Krew}_c(w), w)$ is a bijection from $\mathsf{NC}(W,c)$ to the set of pairs (δ_0,δ_1) such that $\delta_0,\delta_1 \in \mathsf{NC}(W,c)$, $\delta_0\delta_1 = c$, and $\ell_{\mathcal{R}}(\delta_0)+\ell_{\mathcal{R}}(\delta_1) = \ell_{\mathcal{R}}(c)$. Let $(\delta_0(c),\delta_1(c))$ be the pair obtained by applying the unique factorizations from Proposition 4.1.4 to each of δ_0 and δ_1, each specifying certain positions in $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))$. By construction, this yields a bijection to facets of a dual subword complex

$$\mathsf{NC}(W,c) \longrightarrow \mathrm{SUB}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))\,\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c})), c).$$

Finally, it follows from Lemma 2.6.5(5) that

$$\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))\,\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c})) = \mathsf{inv}_{\mathcal{R}}\left(\mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{w}_\circ(\mathsf{c}))\right) \equiv \mathsf{inv}_{\mathcal{R}}(\mathsf{c}^h).$$ □

EXAMPLE 4.1.6. We illustrate Proposition 4.1.5 for $\mathfrak{S}_4$ in Figure 10, using

$$\begin{aligned}\mathsf{inv}_{\mathcal{R}}(c^h) &\equiv \mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))\,\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c})) \\ &= (12)(13)(14)(23)(24)(34)\cdot(12)(13)(14)(23)(24)(34).\end{aligned}$$

4.1.4. The Cambrian recurrence. For s initial in c, there is an isomorphism

$$\begin{aligned}\mathsf{NCL}(W,c) &\longrightarrow \mathsf{NCL}(W,\bar{s}cs) \\ w &\longmapsto \bar{s}ws.\end{aligned}$$

A modification of this conjugation map produces a simple defining recurrence for $\mathsf{NC}(W,c)$ called the *c-Cambrian recurrence* [**Rea06**, Theorem 6.1].

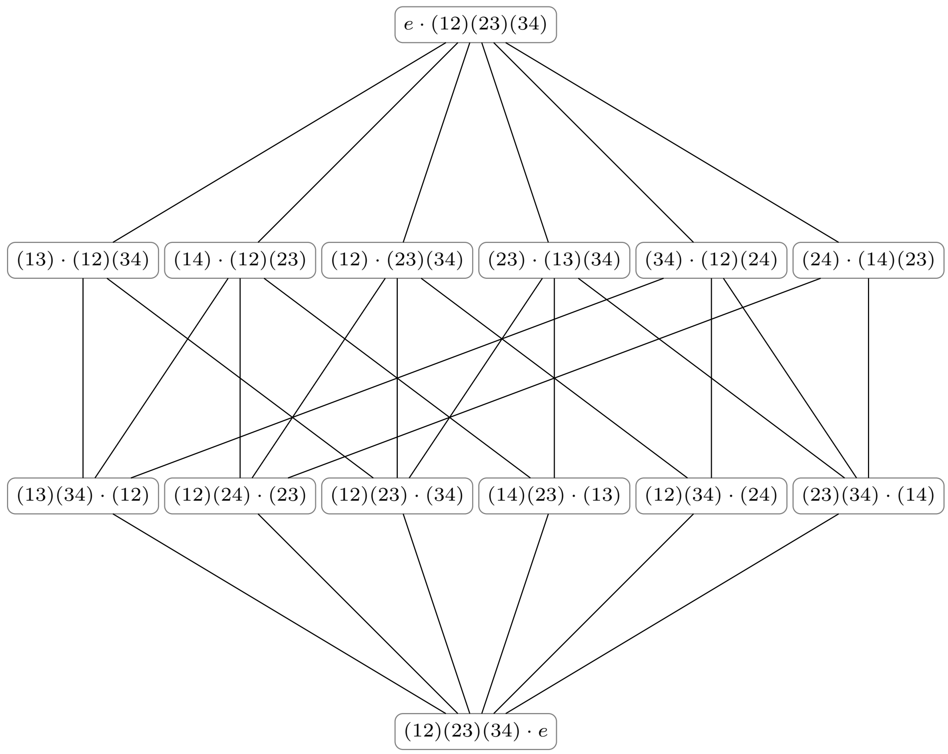

FIGURE 10. The noncrossing partition lattice $\mathsf{NCL}\big(\mathfrak{S}_4, (1234)\big)$. Each noncrossing partition is represented by its corresponding 1-delta sequence.

PROPOSITION 4.1.7. *Let s be initial in c. Then*

$$w \in \mathsf{NC}(W,c) \Leftrightarrow \begin{cases} sw \in \mathsf{NC}(W_{\langle s \rangle}, \bar{s}c) & \text{if } sw \leq_{\mathcal{R}} w \\ \bar{s}ws \in \mathsf{NC}(W, \bar{s}cs) & \text{otherwise} \end{cases}.$$ □

This *natural* recurrence shows up everywhere in noncrossing Catalan combinatorics!

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4.2. m-eralized noncrossing partitions

We now review the m-eralization of noncrossing partitions. Generalizing a construction given by P. Edelman in [**Ede80**] from type A to all finite Coxeter groups, D. Armstrong defined m-eralized c-noncrossing partitions as m-multichains in absolute order.

4.2.1. m-eralized noncrossing partitions as chains.

DEFINITION 4.2.1 ([**Arm06**, Definition 3.2.2(1)]). The *m-eralized c-noncrossing partitions* are the m-multichains

$$\mathsf{NC}^{(m)}(W,c) := \big\{(w_1 \geq_{\mathcal{R}} w_2 \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} w_m) \ : \ w_i \in \mathsf{NC}(W,c)\big\}.$$

The *support* of $(w_1 \geq_{\mathcal{R}} w_2 \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} w_m)$ is $\mathrm{supp}(w_1) \subseteq \mathcal{S}$.

4.2.2. m-eralized noncrossing partitions as delta sequences. D. Armstrong also gave an equivalent construction encoding the intervals between consecutive elements of the m-multichain. This m-eralizes the pairing of a noncrossing partition with its Kreweras complement.

Definition 4.2.2 ([**Arm06**, Definition 3.2.2(2)]). A sequence $\delta = (\delta_0, \delta_1, \ldots, \delta_m)$ with $\delta_i \in \mathsf{NC}(W, c)$ is an *m-delta sequence* if

$$\delta_0\delta_1 \cdots \delta_m = c \quad \text{and} \quad \sum_{i=0}^{m} \ell_{\mathcal{R}}(\delta_i) = \ell_{\mathcal{R}}(c).$$

The *support* of δ is defined to be $\operatorname{supp}(\delta) := \operatorname{supp}(\delta_1 \cdots \delta_m) \subseteq \mathcal{S}$, and we denote the set of all such sequences by $\mathsf{NC}^{(m)}_{\delta}(W, c)$.

The following proposition relates Definitions 4.2.1 and 4.2.2, and is immediate from the definitions.

Proposition 4.2.3 ([**Arm06**, Lemma 3.2.4]). *There is a canonical bijection*

$$\begin{aligned} \mathsf{NC}^{(m)}(W,c) &\overset{\sim}{\longleftrightarrow} \mathsf{NC}^{(m)}_{\delta}(W,c) \\ \big(w_1 \geq_{\mathcal{R}} \ldots \geq_{\mathcal{R}} w_m\big) &\longmapsto \big(cw_1^{-1}, w_1w_2^{-1}, \ldots, w_{m-1}w_m^{-1}, w_m\big) \\ \big(\delta_1 \cdots \delta_m \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} \delta_{m-1}\delta_m \geq_{\mathcal{R}} \delta_m\big) &\longleftarrow \big(\delta_0, \ldots, \delta_m\big). \end{aligned}$$ □

D. Armstrong m-eralized the lattice $\mathsf{NCL}(W, c)$ by considering $\mathsf{NC}^{(m)}_{\delta}(W, c)$ under componentwise absolute order.

$$(\delta_0, \delta_1, \ldots, \delta_m) \leq_{\mathcal{R}} (\delta'_0, \delta'_1, \ldots, \delta'_m) \Leftrightarrow \delta_i \leq_{\mathcal{R}} \delta'_i \text{ for all } 1 \leq i \leq m. \tag{4.2}$$

(Note that we do not compare the zero-th components.) This order on $\mathsf{NC}^{(m)}_{\delta}(W, c)$ is a graded meet semilattice [**Arm06**, Theorem 3.4.4]. For $\mathfrak{S}_n$ it coincides with the refinement order on *m-shuffle noncrossing partitions*—the noncrossing partitions of $m(n+1)$ such that the elements of each block are congruent modulo m [**Arm06**, Section 4.3.1, Figure 4.6].

It is straightforward to m-eralize the *Kreweras complement* by

$$\mathsf{Krew}_c(\delta) := \left(c\delta_m c^{-1}, \delta_0, \delta_1, \ldots, \delta_{m-1}\right),$$

so that $\operatorname{ord}(\mathsf{Krew}_c) = (m+1)h$. On the combinatorial model of m-shuffle noncrossing partitions, Krew_c^{m+1} acts as a rotation by m.

4.2.3. m-eralized noncrossing partitions as dual subword complexes. Using dual subword complexes, we generalize Proposition 4.1.5 and refine m-delta sequences to the level of reflections.

Definition 4.2.4. Define the dual subword complex

$$\mathsf{NC}^{(m)}_{\Delta}(W, c) := \textsc{Sub}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ^{m+1}(\mathsf{c})), c).$$

Deal with it.

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As in Proposition 4.1.5, the construction depends on the chosen reduced $\mathcal{S}$-word but the complexes for all possible choices are canonically isomorphic—so we again prefer to attach this complex to the element c itself. As seen in Example 3.5.3, the dual subword complexes for m-eralized noncrossing partitions are *not* Coxeter-initial subword complexes.

It is convenient to represent a facet I of $\mathsf{NC}^{(m)}_{\Delta}(W, c)$ as a word of colored reflections $\mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_n^{(i_n)}$, where

$\mathsf{NC}^{(2)}(\mathfrak{S}_3, st)$	$\mathsf{NC}^{(2)}_{\delta}(\mathfrak{S}_3, st)$	$\mathsf{NC}^{(2)}_{\Delta}(\mathfrak{S}_3, st)$	$\mathrm{supp}(w)$
$e \geq_{\mathcal{R}} e$	(st, e, e)	**s**u**t**.sut.sut	–
$st \geq_{\mathcal{R}} st$	(e, e, st)	sut.sut.**s**u**t**	s, t
$st \geq_{\mathcal{R}} s$	(e, u, s)	sut.s**u**t.**s**ut	s, t
$u \geq_{\mathcal{R}} e$	(t, u, e)	su**t**.s**u**t.sut	s, t
$s \geq_{\mathcal{R}} e$	(u, s, e)	s**u**t.**s**ut.sut	s
$t \geq_{\mathcal{R}} t$	(s, e, t)	**s**ut.sut.su**t**	t
$st \geq_{\mathcal{R}} u$	(e, t, u)	sut.su**t**.s**u**t	s, t
$st \geq_{\mathcal{R}} e$	(e, st, e)	sut.**s**u**t**.sut	s, t
$t \geq_{\mathcal{R}} e$	(s, t, e)	**s**ut.su**t**.sut	t
$u \geq_{\mathcal{R}} u$	(t, e, u)	su**t**.sut.s**u**t	s, t
$s \geq_{\mathcal{R}} s$	(u, e, s)	s**u**t.sut.**s**ut	s
$st \geq_{\mathcal{R}} t$	(e, s, t)	sut.**s**ut.su**t**	s, t

FIGURE 11. The three variants of the m-eralized st-noncrossing partitions for $\mathfrak{S}_3$ with $m = 2$, together with their supports. They are arranged according to their orbits under Cambrian rotation, defined in Section 4.3.

- a reflection r_j is colored according to the copy of $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))$ to which it belongs,
- $r_a <_{\mathsf{c}} r_b$ if $a < b$ and $i_a = i_b$, and
- $r_1 \cdots r_n = c$.

For notational simplicity, we write $r \in I$ to mean that $r \in \{r_1, \ldots, r_n\}$. The *support* of I is defined to be $\mathrm{supp}(I) := \mathrm{supp}(\prod r_k) \subseteq \mathcal{S}$, where the product is over all roots $r_k \in I$ with color $i_k > 0$.

Each noncrossing partition definition has a slightly different flavor...

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...and is equally delicious in its own unique way.

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THEOREM 4.2.5. *There is a canonical bijection*

$$\mathsf{NC}^{(m)}_{\delta}(W, c) \overset{\sim}{\longleftrightarrow} \mathsf{NC}^{(m)}_{\Delta}(W, c).$$

PROOF. As in the proof of Proposition 4.1.5, apply the factorization of Proposition 4.1.4 separately to each component of the m-delta sequence. □

Proposition 4.2.3 and Theorem 4.2.5 give canonical preserving bijections between the three variants of noncrossing partitions, and we move freely between them. Figure 11 shows all 12 elements of

$$\mathsf{NC}^{(2)}(\mathfrak{S}_3, st) \overset{\sim}{\longleftrightarrow} \mathsf{NC}^{(2)}_{\delta}(\mathfrak{S}_3, st) \overset{\sim}{\longleftrightarrow} \mathsf{NC}^{(2)}_{\Delta}(\mathfrak{S}_3, st)$$

with their supports.

4.3. The Cambrian rotation and recurrence

For s initial in c, define the bijection

$$\begin{aligned} \mathsf{Shift}_s : \mathsf{NC}^{(m)}_{\Delta}(W, c) &\longrightarrow \mathsf{NC}^{(m)}_{\Delta}(W, \overline{s}cs) \\ \mathsf{r}_1^{(i_1)}\mathsf{r}_2^{(i_2)}\cdots\mathsf{r}_n^{(i_n)} &\longmapsto \begin{cases} \mathsf{r}_2^{(i_2)}\cdots\mathsf{r}_n^{(i_n)}\mathsf{s}^{(m)} & \text{if } r_1^{(i_1)} = s^{(0)} \\ \mathsf{t}_1^{(j_1)}\cdots\mathsf{t}_n^{(j_n)} & \text{otherwise} \end{cases}, \end{aligned}$$

where $t_k = r_k^s$ and $j_k = \begin{cases} i_k & \text{if } t_k \neq s \\ i_k - 1 & \text{if } t_k = s \end{cases}$. By Lemma 2.6.5(7) and since s is final in $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\bar{\mathsf{s}}\mathsf{cs}))$, the ordering on the reflections in the word in the image of this map is compatible with $\bar{s}cs$.

EXAMPLE 4.3.1. Alternately applying Shift_s and Shift_t to $\mathsf{s}^{(0)}\mathsf{t}^{(0)} \in \mathsf{NC}_\Delta^{(2)}(\mathfrak{S}_3, st)$ gives the orbit

$$\begin{array}{ccccc} \mathsf{sut.sut.sut} & \xmapsto{\mathsf{Shift}_s} & \mathsf{tus.tus.tus} & \xmapsto{\mathsf{Shift}_t} & \mathsf{sut.sut.sut} \\ & \xmapsto{\mathsf{Shift}_s} & \mathsf{tus.tus.tus} & \xmapsto{\mathsf{Shift}_t} & \mathsf{sut.sut.sut} \\ & \xmapsto{\mathsf{Shift}_s} & \mathsf{tus.tus.tus} & \xmapsto{\mathsf{Shift}_t} & \mathsf{sut.sut.sut} \\ & \xmapsto{\mathsf{Shift}_s} & \mathsf{tus.tus.tus} & \xmapsto{\mathsf{Shift}_t} & \mathsf{sut.sut.sut} \end{array}.$$

DEFINITION 4.3.2. For $\mathsf{c} = \mathsf{s}_1 \cdots \mathsf{s}_n$, the *$m$-eralized c-Cambrian rotation* is

$$\mathsf{Camb}_\mathsf{c} := \mathsf{Shift}_{s_n} \circ \cdots \circ \mathsf{Shift}_{s_1} : \mathsf{NC}_\Delta^{(m)}(W, c) \longrightarrow \mathsf{NC}_\Delta^{(m)}(W, c).$$

As usual, this composition does not depend on the chosen reduced word c. The elements in Figure 11 are arranged according to their orbits under Cambrian rotation.

A modification of the first case in the definition of the shift operator gives an inductive characterization of $\mathsf{NC}_\Delta^{(m)}(W, c)$ called the *m-eralized c-Cambrian recurrence*.

PROPOSITION 4.3.3. *Let s be initial in c and let $I = \mathsf{r}_1^{(i_1)}\mathsf{r}_2^{(i_2)} \cdots \mathsf{r}_n^{(i_n)}$. Then*

$$I \in \mathsf{NC}_\Delta^{(m)}(W, c) \Leftrightarrow \begin{cases} \mathsf{r}_2^{(i_2)} \cdots \mathsf{r}_n^{(i_n)} \in \mathsf{NC}_\Delta^{(m)}(W_{\langle s \rangle}, \bar{s}c) & \text{if } r_1^{(i_1)} = s^{(0)} \\ \mathsf{Shift}_s(I) \in \mathsf{NC}_\Delta^{(m)}(W, \bar{s}cs) & \text{otherwise} \end{cases}.$$

PROOF. This follows from Lemma 2.6.5(7) and Lemma 2.9.1. □

WARNING. Some care must be taken to correctly run this recurrence in reverse. It might seem that the facet $\mathsf{s}^{(0)}\mathsf{r}_2^{(i_2)} \cdots \mathsf{r}_n^{(i_n)} \in \mathsf{NC}_\Delta^{(m)}(W, c)$ is produced by both $\mathsf{r}_2^{(i_2)} \cdots \mathsf{r}_n^{(i_n)} \in \mathsf{NC}_\Delta^{(m)}(W_{\langle s \rangle}, \bar{s}c)$ and by $\mathsf{r}_2^{(i_2)} \cdots \mathsf{r}_n^{(i_n)}\mathsf{s}^{(m)} \in \mathsf{NC}_\Delta^{(m)}(W, \bar{s}cs)$. But facets of $\mathsf{NC}_\Delta^{(m)}(W, \bar{s}cs)$ of the form $\mathsf{r}_2^{(i_2)} \cdots \mathsf{r}_n^{(i_n)}\mathsf{s}^{(m)}$ are not in the image of the recurrence, and so it is crucial not to use them when running the recurrence in reverse.

We emphasize here that when reversing any of the Cambrian recurrences in Proposition 5.4.5 and in Propositions 6.2.1, 6.3.1, 6.4.3, and 6.8.3, the elements on the right hand side are required to have the form specified by the recurrence.

For later: the parabolic subgroup gets Cambrian rotated to the top of the Cambrian poset.

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EXAMPLE 4.3.4. The Cambrian recurrence for the $(e, s, t) \in \mathsf{NC}_\delta^{(2)}(\mathfrak{S}_3, st)$ is computed as

$$\underbrace{(e,s,t)}_{st} \mapsto \underbrace{(s,e,u)}_{ts} \mapsto \underbrace{(u,e,s)}_{st} \mapsto \underbrace{(t,s,e)}_{ts} \mapsto \underbrace{(e,s,e)}_{s} \mapsto \underbrace{(s,e,e)}_{s} \mapsto \underbrace{(e,e,e)}_{e},$$

where the subscript identifies the (parabolic) Coxeter element.

4.4. Cambrian posets

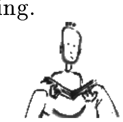
In this section, we define a new poset structure m-eralizing N. Reading's Cambrian lattices. We later prove in Theorems 6.6.4 and 6.8.6 that this poset is a sublattice of the weak order.

4.4.1. The weak order on covering and covered reflections. Starting with $\mathsf{cov}^{\uparrow}(e) = \mathcal{S}$ and $\mathsf{cov}_{\downarrow}(e) = \emptyset$, we build $\mathsf{Weak}(W)$ on the covering and covered reflections of the elements in W by describing cover relations. Supposing that $\mathsf{cov}^{\uparrow}(w)$ and $\mathsf{cov}_{\downarrow}(w)$ are known, there will be one cover of w for each $r \in \mathsf{cov}^{\uparrow}(w)$. Choose any $r \in \mathsf{cov}^{\uparrow}(w)$ and let α_r be the associated positive root. The element wr covering w has $\mathsf{cov}^{\uparrow}(rw)$ and $\mathsf{cov}_{\downarrow}(rw)$ given by

$$\mathsf{cov}^{\uparrow}(rw) = \left\{u^r \ : \ u \in \mathsf{cov}^{\uparrow}(w), r(\alpha_u) > 0\right\} \cup \left\{u^r \ : \ u \in \mathsf{cov}_{\downarrow}(w), r(\alpha_u) < 0\right\},$$
$$\mathsf{cov}_{\downarrow}(rw) = \left\{u^r \ : \ u \in \mathsf{cov}_{\downarrow}(w), r(\alpha_u) > 0\right\} \cup \left\{u^r \ : \ u \in \mathsf{cov}^{\uparrow}(w), r(\alpha_u) < 0\right\}.$$

Finally, an element $w \in W$ can be reconstructed from its covered and covering reflections—these tell us exactly which hyperplanes bound the corresponding chamber, which determines w up to multiplication by $w_\circ$. Since we have distinguished covering and covered reflections, we have therefore uniquely specified w.

This is worth remembering.



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4.4.2. Cambrian posets. We mimic this characterization of $\mathsf{Weak}(W)$ with a small twist. Delta sequences in $\mathsf{NC}^{(1)}_{\delta}(W,c)$ have two components, $\delta_{\underline{w}} = (\delta_0, \delta_1)$. The analogy we wish to draw is that the factorization of δ_0 of Proposition 4.1.4 should be thought of as a variant of $\mathsf{cov}^{\uparrow}(w)$, while the factorization of δ_1 behaves like $\mathsf{cov}_{\downarrow}(w)$. We now define the cover relations, jumping immediately to the definition for general m.

Let $I = \mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_a^{(i_a)} \cdots \mathsf{r}_n^{(i_n)}$ be a facet of $\mathsf{NC}^{(m)}_{\Delta}(W,c)$. If $i_a \neq m$, the *increasing flip* of I in the direction r_a is given by

$$\mathsf{Flip}^{\uparrow}_{r_a}(I) := \mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_{a-1}^{(i_{a-1})} \ \underbrace{\mathsf{t}_{a+1}^{(i'_{a+1})} \cdots \mathsf{t}_b^{(i'_b)} \ \mathsf{r}_a^{(i_a+1)}}_{\text{modified}} \ \mathsf{r}_{b+1}^{(i_{b+1})} \cdots \mathsf{r}_n^{(i_n)}, \tag{4.3}$$

where b is chosen maximally so that $r_b^{(i_b)} <_{\mathsf{c}} r_a^{(i_a+1)}$. We therefore have

$$\underbrace{r_{a+\ell} <_{\mathsf{c}} \cdots <_{\mathsf{c}} r_b}_{\text{color } i_a+1} <_{\mathsf{c}} r_a <_{\mathsf{c}} \underbrace{r_{a+1} <_{\mathsf{c}} \cdots <_{\mathsf{c}} r_{a+\ell-1}}_{\text{color } i_a},$$

and for $a+1 \leq j \leq b$, the modified portion of the colored factorization is given by $t_j = r_j^{r_a}$ and

$$i'_j = \begin{cases} i_j & \text{if } r_a(\alpha_{r_j}) \in \Phi^+ \\ i_j - 1 & \text{if } r_a(\alpha_{r_j}) \in \Phi^- \end{cases}.$$

The *decreasing flip* $\mathsf{Flip}^{\downarrow}_{r_a}(I)$ for $i_a \neq 0$ is defined analogously. Examples of flips are illustrated in Figure 12 on page 57. Before showing that this notion of flips is well-defined on m-eralized noncrossing partitions, we remark that this is *not* the notion of flip we defined for subword complexes in Section 3.3.1.

An innocent composition of Hurwitz moves that somehow mirrors some rather risqué quiver mutations.

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Proposition 4.4.1. *If $I = \mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_a^{(i_a)} \cdots \mathsf{r}_n^{(i_n)}$ is a facet of $\mathsf{NC}^{(m)}_{\Delta}(W,c)$ with $i_a \neq m$, then $\mathsf{Flip}^{\uparrow}_{r_a}(I) \in \mathsf{NC}^{(m)}_{\Delta}(W,c)$.*

PROOF. For $\mathsf{Flip}^{\uparrow}_{r_a}(I)$ to be a subword, we must check that—up to commutation of commuting letters—it respects the reflection order $\leq_{\mathsf{c}}$. The idea is simple. We rotate $\mathsf{w}_\circ(\mathsf{c})$ so that r_a is conjugated to a simple reflection, perform the flip in this simple system where it is straightforward to check the condition, and then rotate back. By assumption, we have that the reflections in the colored factorization

$$\mathsf{r}_a^{(i_a)}\mathsf{r}_{a+1}^{(i_{a+1})}\cdots\mathsf{r}_b^{(i_b)} \tag{4.4}$$

are ordered by $r_{a+\ell}\cdots <_{\mathsf{c}} r_b <_{\mathsf{c}} r_a <_{\mathsf{c}} r_{a+1} <_{\mathsf{c}} \cdots <_{\mathsf{c}} r_{a+\ell-1}$.

Let $\mathsf{w} = \mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_{i-1}$ be the prefix of $\mathsf{w}_\circ(\mathsf{c}) = \mathsf{s}_1\cdots\mathsf{s}_N$ so that ${}^{w}s_i = r_a$. Applying Lemma 2.6.5(7), there is some Coxeter element c' so that the cyclic rotation $\mathsf{s}_i\cdots\mathsf{s}_N\mathsf{s}_1\cdots\mathsf{s}_{i-1} \equiv \mathsf{w}_\circ(\mathsf{c}')$. Conjugating (4.4) by w shifts all reflections into the same color giving the colored factorization

$$\mathsf{s}_i^{(i_a)}(r_{a+1}^{w})^{(i_a)}\cdots(r_b^{w})^{(i_a)}. \tag{4.5}$$

Up to commutation of commuting letters, this factorization respects the reflection order $\leq_{\mathsf{c}'}$. By Lemma 2.9.1, conjugating (4.5) by s_i gives us the colored factorization

$$\underbrace{(r_{a+1}^{ws_i})^{(i_a)}\cdots(r_b^{ws_i})^{(i_a)}}_{\text{in } W_{\langle s_i\rangle}}\mathsf{s}_i^{(i_a+1)} \tag{4.6}$$

that respects the reflection order $\leq_{\bar{\mathsf{s}}\mathsf{c}'\mathsf{s}}$, up to commutations. For any $\beta \in \Phi^+_{\langle s\rangle}$, $s_i(\beta) = \beta + k\alpha_i$ for $k \geq 0$. By Lemma 2.9.1, $s_i(\beta)$ therefore occurs weakly earlier in the orders $\leq_{\mathsf{c}'}$ and $\leq_{\bar{\mathsf{s}}\mathsf{c}'\mathsf{s}}$.

We now undo the applications of Lemma 2.6.5(7) to recover $\mathsf{w}_\circ(\mathsf{c})$, which has the effect of conjugating (4.6) by w. Since ${}^{w}s_i = r_a$, the colored factorization $\mathsf{t}_{a+1}^{(i'_{a+1})}\cdots\mathsf{t}_b^{(i'_b)}\mathsf{r}_a^{(i_a)+1}$ respects the reflection order $\leq_{\mathsf{c}}$. Because reflections have only moved weakly earlier in $\leq_{\mathsf{c}}$ as a result of the sequence of conjugations, it is only possible for colors to remain the same or decrease by one. A color decreases by one exactly when the corresponding root changed sign when reflected by ${}^{w}s_i = r_a$, as desired. □

By analogy to the weak order, flips define the cover relations of an m-eralized c-Cambrian poset.

DEFINITION 4.4.2. The *m-eralized c-Cambrian poset* $\mathsf{Camb}^{(m)}_{\mathsf{NC}}(W,c)$ is the partial order on the elements of $\mathsf{NC}^{(m)}_{\Delta}(W,c)$ with minimal element $\mathsf{s}_1^{(0)}\cdots\mathsf{s}_n^{(0)}$ and with covering relations

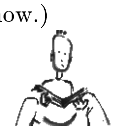
$$\mathsf{r}_1^{(i_1)}\cdots\mathsf{r}_a^{(i_a)}\cdots\mathsf{r}_n^{(i_n)} \lessdot \mathsf{Flip}^{\uparrow}_{r_a}\left(\mathsf{r}_1^{(i_1)}\cdots\mathsf{r}_a^{(i_a)}\cdots\mathsf{r}_n^{(i_n)}\right)$$

for $i_a < m$. Cover relations are labelled by the colored reflection $r_a^{(i_a)}$.

We stick to the name *Cambrian poset* until we prove its lattice property in Section 6.6.

WARNING. Note that this notion of flip does *not* come from the flip on the subword complex corresponding to $\mathsf{NC}^{(m)}_{\Delta}(W,c)$ defined in Proposition 3.4.2.

We never use the subword complex flip for noncrossing partitions, so don't confuse these two notions of flip! (Well, ignore Theorem 7.3.5 for now.)



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It follows from the definitions that Shift_s and $\mathsf{Flip}^{\uparrow}_r$ indeed interact nicely, as the following proposition shows.

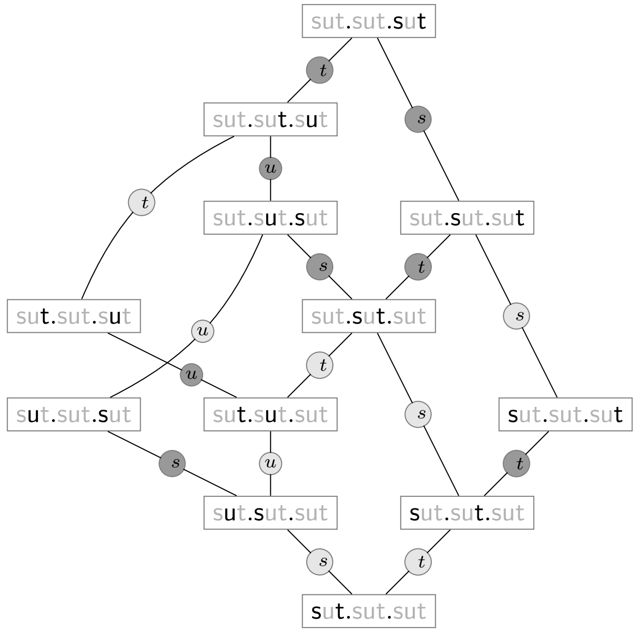

FIGURE 12. The flip poset $\mathsf{Camb}^{(2)}_{\mathsf{NC}}(\mathfrak{S}_3, st)$ on the noncrossing partitions $\mathsf{NC}^{(2)}(\mathfrak{S}_3, st)$. The colored reflections labelling the edges correspond to the direction of the flips, with the color indicated by the shading.

PROPOSITION 4.4.3. *For s initial in c, let $I = \mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_a^{(i_a)} \cdots \mathsf{r}_n^{(i_n)} \in \mathsf{NC}^{(m)}_{\Delta}(W, c)$, and suppose $I \lessdot \mathsf{Flip}^{\uparrow}_{r_a}(I)$. Then for $b + c = m + 1$,*

$$\begin{cases} \left(\mathsf{Flip}^{\downarrow}_{r_a}\right)^b \circ \mathsf{Shift}_s(I) = \mathsf{Shift}_s \circ \left(\mathsf{Flip}^{\uparrow}_{r_a}\right)^c (I) & \text{if } r_1^{(i_1)} = s^{(0)} \text{ and } r_a = s \\ \mathsf{Flip}^{\uparrow}_{r_a} \circ \mathsf{Shift}_s(I) = \mathsf{Shift}_s \circ \mathsf{Flip}^{\uparrow}_{r_a}(I) & \text{if } r_1^{(i_1)} = s^{(0)} \text{ and } r_a \neq s \\ \mathsf{Flip}^{\uparrow}_{r_a^s} \circ \mathsf{Shift}_s(I) = \mathsf{Shift}_s \circ \mathsf{Flip}^{\uparrow}_{r_a}(I) & \text{otherwise.} \end{cases}$$

PROOF. This follows immediately from the definition of Shift_s and the definition of $\mathsf{Flip}^{\uparrow}_r$. ☐

Figure 12 illustrates the m-eralized c-Cambrian poset for $\mathfrak{S}_3$ for $m = 2$. We define a Cambrian graph by allowing longer flips that send a reflection from the i^{th} copy of $w_\circ(c)$ to the j^{th} copy for $i < j$.

DEFINITION 4.4.4. Fix $\mathsf{c} = \mathsf{s}_1 \cdots \mathsf{s}_n$ a reduced $\mathcal{S}$-word for a Coxeter element c. The *m-eralized c-Cambrian graph* $\mathcal{G}\mathsf{Camb}^{(m)}_{\mathsf{NC}}(W, c)$ is the directed graph on the elements of $\mathsf{NC}^{(m)}_{\Delta}(W, c)$ with source $\mathsf{s}_1^{(0)} \cdots \mathsf{s}_n^{(0)}$ and edges

$$\mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_a^{(i_a)} \cdots \mathsf{r}_n^{(i_n)} \to \left(\mathsf{Flip}^{\uparrow}_{r_a}\right)^k \left(\mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_a^{(i_a)} \cdots \mathsf{r}_n^{(i_n)}\right)$$

for $k > 0$ with $i_a + k \leq m$.

COROLLARY 4.4.5. *Let c, c' be two Coxeter elements. Then there is an* undirected *graph isomorphism $\mathcal{G}\mathsf{Camb}^{(m)}_{\mathsf{NC}}(W, c) \longrightarrow \mathcal{G}\mathsf{Camb}^{(m)}_{\mathsf{NC}}(W, c')$.*

The Cambrian poset defined directly on noncrossing partitions!

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PROOF. Since any two Coxeter elements c, c' are conjugate in W, there exists an initial sequence $s_1, \dots, s_p$ for c such that $c' = \overline{s}_p \cdots \overline{s}_1 c s_1 \cdots s_p$. The undirected graph isomorphism is induced by the composition of the shift operations for the sequence $s_1, \dots, s_p$. $\square$

4.5. Some enumerations

We collect the following enumeration results for m-eralized c-noncrossing partitions and their Cambrian posets.

THEOREM 4.5.1. *We have*

$$(1-q)^{n+1} \sum_{m=0}^{\infty} \left|\mathsf{NC}^{(m)}(W,c)\right| q^m = \sum_{\mathsf{r}_1\cdots\mathsf{r}_n \in \mathrm{Red}_{\mathcal{R}}(c)} q^{\mathrm{des}(\mathsf{r}_1\cdots\mathsf{r}_n)}$$

where $\mathrm{des}(\mathsf{r}_1, \dots, \mathsf{r}_n)$ *is the number of descents* $r_i >_c r_{i+1}$ *in the reflection order induced by the Coxeter element* c.

Is this trivial? It feels trivial.

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PROOF. This is a direct consequence of Theorem 4.2.5, after observing that this formula is equivalent to

$$\left|\mathsf{NC}^{(m)}(W,c)\right| = \sum_{\mathsf{r}_1\cdots\mathsf{r}_n \in \mathrm{Red}_{\mathcal{R}}(c)} \binom{m+1+\mathrm{asc}(\mathsf{r}_1\cdots\mathsf{r}_n)}{n}$$

where $\mathrm{asc}(\mathsf{r}_1, \dots, \mathsf{r}_n)$ is the number of ascents $r_i <_c r_{i+1}$. Given a reduced $\mathcal{R}$-word $\mathsf{r}_1\cdots\mathsf{r}_n$ of c, one can place each reflection r_i into one copy of the $m+1$ copies of $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))$ inside the search word $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ^{m+1}(\mathsf{c}))$, and two consecutive reflections r_i and r_{i+1} possibly into the same copy if and only if $r_i <_c r_{i+1}$. This shows that the number of ways to place the $\mathcal{R}$-word $\mathsf{r}_1\cdots\mathsf{r}_n$ into the search word equals the number of tuples $1 \le a_1 \le \ldots \le a_n \le m+1$ with weak inequalities $a_i \le a_{i+1}$ if $r_i <_c r_{i+1}$ and strict inequalities otherwise. The statement then follows by summing over all factorizations as such tuples $1 \le a_1 \le \ldots \le a_n \le m+1$ are clearly counted by the above binomial. $\square$

THEOREM 4.5.2. *We have that*

(i) the posets $\mathsf{Camb}^{(m)}_{\mathsf{NC}}(W,c)$ *and* $\mathsf{Camb}^{(m)}_{\mathsf{NC}}(W,\psi(c))$ *are dual, and*

(ii) the number of elements of $\mathsf{Camb}^{(m)}_{\mathsf{NC}}(W,c)$ *with k upper covers equals the number of elements* $(w_1, \dots, w_m) \in \mathsf{NC}^{(m)}(W,c)$ *such that* $\ell_{\mathcal{R}}(w_1) = k$.

PROOF. The first item (i) follows from Lemma 2.6.5(6). The second item (ii) follows from Definition 4.4.2. $\square$

CHAPTER 5

Cluster complexes

In this chapter, we study m-eralized cluster complexes. We review the theory of cluster complexes as subword complexes (Section 5.1) and define m-eralized compatibility relations and m-eralized cluster complexes (Sections 5.2 and 5.3). We define the Cambrian recurrence and the Cambrian poset on m-eralized cluster complexes (Section 5.4). We use this recurrence to show that our definition recovers known constructions of generalized cluster complexes for bipartite Coxeter elements (Section 5.5). We conclude with topological properties of cluster complexes in our framework (Section 5.6), and give a natural bijection to noncrossing partitions (Section 5.7). We conclude with a comparison of our m-eralized Cambrian posets of type A_n and m-Tamari lattices (Section 5.8).

5.1. Classical cluster complexes

Let $\Phi_{\geq -1} = \Phi^+ \cup -\Delta$ be the set of *almost positive roots*. For $s \in \mathcal{S}$, define a bijection

$$\begin{aligned} \tau_s : \Phi_{\geq -1} &\longrightarrow \Phi_{\geq -1} \\ \beta &\longmapsto \begin{cases} \beta & \text{if } \beta \in -(\Delta \setminus \alpha_s) \\ s(\beta) & \text{otherwise} \end{cases}. \end{aligned} \tag{5.1}$$

In their study of finite type cluster algebras, S. Fomin and A. Zelevinsky used τ_s to define a binary relation on $\Phi_{\geq -1}$ [**FZ02, FZ03**]. R. Marsh, M. Reineke, and A. Zelevinsky [**MRZ03**] and, independently, N. Reading [**Rea07a**] interpreted this as the bipartite case of a more general family of relations, depending on a Coxeter element c (or, equivalently, on an orientation of the Coxeter diagram).

I recommend you look at these references before continuing this chapter!

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Definition 5.1.1. The *c-compatibility relations* are the unique family of relations $\|_c$ on $\Phi_{\geq -1}$ characterized by the following two properties:

(i) for $\alpha \in \Delta$,
$$-\alpha \,\|_c\, \beta \Leftrightarrow \beta \in \Phi_{\langle s_\alpha \rangle},$$
(ii) for s final in c,
$$\beta_1 \,\|_c\, \beta_2 \Leftrightarrow \tau_s(\beta_1) \,\|_{sc\bar{s}}\, \tau_s(\beta_2).$$

The *c-cluster complex* $\mathsf{Asso}(W, c)$ is the simplicial complex given by all collections of pairwise c-compatible almost positive roots.

The two properties overdetermine this binary relation. For a proof that $\|_c$ is well-defined, we refer to [**Rea07a**, Section 7]. By symmetry, (ii) can be equivalently stated for s initial in c.

A *c-cluster* is a facet of the c-cluster complex $\mathsf{Asso}(W, c)$—that is, a maximal subset of almost positive roots which are pairwise c-compatible. In crystallographic type, the c-cluster complexes are isomorphic to the cluster complex defined in [**FZ02**]. We m-eralize Definition 5.1.1 in Definition 5.2.3, and show in Corollary 5.5.7 that this m-eralization of compatibility relations reduces for generalized cluster complexes (see [**FR05**]) and bipartite Coxeter elements to S. Fomin and N. Reading's compatibility relation recalled in Definition 5.5.4.

K. Igusa and R. Schiffler found an explicit rule for the compatibility of two roots under $\|_c$ in [**IS10**, Theorem 2.5], using the connection between representation theory and clusters introduced in [**BMR$^+$06**]; see also [**BW08**, Section 8] when c is bipartite. This combinatorics was studied and extended in a purely combinatorial setting beyond crystallographic type by C. Ceballos, J.-P. Labbé, and C. Stump in [**CLS14**], obtaining isomorphisms

$$\mathsf{Asso}(W,c) \cong \mathrm{S{\scriptstyle UB}}_{\mathcal{S}}(\mathsf{cw}_\circ(\mathsf{c}), w_\circ) = \mathrm{S{\scriptstyle UB}}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ(\mathsf{c})), c^{-1}).$$

The root configuration gives a bijection between the facets of these subword complexes and noncrossing partitions [**PS15**, Theorem 6.23]. These bijections are m-eralized in Section 5.7 and in Section 5.5, respectively.

5.2. m-eralized compatibility relations

We construct in this section an m-eralized compatibility relation simmultaneously for all Coxeter elements. This ties together the compatibility relation from Definition 5.1.1 for all Coxeter elements and the compatibility relation of S. Fomin and N. Reading for bipartite Coxeter elements. After having established the existence of the proposed relation, we show in Section 5.5.2 that our construction recovers the Fomin-Reading relation in the bipartite case.

5.2.1. Colored almost positive roots. Let

$$\Phi^{(m)}_{\geq -1} := \left\{\alpha^{(k)} \;:\; \alpha \in \Phi^+, 0 \leq k < m\right\} \cup \left\{\alpha^{(m)} \;:\; \alpha \in \Delta\right\}$$

be the set of *m-colored almost positive roots*. The color m plays the role of the negative simple roots. In particular, we use the identification

$$\Phi^{(1)}_{\geq -1} \longleftrightarrow \Phi_{\geq -1}, \quad \text{where } \beta^{(0)} \mapsto \beta, \text{ and } \alpha^{(1)} \mapsto -\alpha.$$

Given this notation of colored roots, a similar inductive description yields an m-eralization of Definition 5.1.1 for general Coxeter elements.

For $s \in \mathcal{S}$, define the bijection

$$\tau_s^{(m)} : \Phi^{(m)}_{\geq -1} \longrightarrow \Phi^{(m)}_{\geq -1}$$

$$\beta^{(k)} \longmapsto \begin{cases} \beta^{(k-1)} & \text{if } \beta = \alpha_s \text{ and } k > 0 \\ \beta^{(m)} & \text{if } \beta = \alpha_s \text{ and } k = 0 \\ \beta^{(k)} & \text{if } \beta \neq \alpha_s \text{ and } k = m \\ [s(\beta)]^{(k)} & \text{otherwise} \end{cases} . \tag{5.2}$$

Remark 5.2.1. It follows from the definition that the order of $\tau_s^{(m)}$ is given by $\operatorname{lcm}\{m+1, 2\}$. Thus, $\tau_s^{(m)}$ is an involution if and only if $m = 1$, and $\tau_s^{(m)}$ reduces to τ_s in this case.

For later usage, we record the following elementary property of $\tau_s^{(m)}$.

Lemma 5.2.2. *Let $s, s' \in \mathcal{S}$ with $s \neq s'$. Then $\tau_s^{(m)}$ restricts to a bijection on $(\Phi_{\langle s' \rangle})^{(m)}_{\geq -1}$.*

PROOF. All four definitions in (5.2) map $\left(\Phi_{\langle s'\rangle}\right)^{(m)}_{\geq -1}$ to itself. As $\tau_s^{(m)}$ is a bijection, the property follows. □

5.2.2. The compatibility relation.

DEFINITION 5.2.3. The *m-eralized c-compatibility relation* $\|_c$ is the unique family of relations $\|_c$ on $\Phi^{(m)}_{\geq -1}$ by the following two properties. For s final in c, we have

(i) $\alpha_s^{(m)} \|_c \beta^{(k)} \Leftrightarrow \beta \in \Phi_{\langle s\rangle}$, and
(ii) $\beta_1^{(k)} \|_c \beta_2^{(\ell)} \Leftrightarrow \tau_s^{(m)}(\beta_1^{(k)}) \|_{sc\bar{s}} \tau_s^{(m)}(\beta_2^{(\ell)})$.

In contrast to the case $m = 1$, Definition 5.2.3(ii) is not equivalent to the corresponding statement for s initial in c. We now show that property (i)—which assumes s to be final in c—implies the analogue of Definition 5.1.1(i).

So that s is now final, not initial. There was a reason for this choice, I promise.

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We just don't remember it.

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PROPOSITION 5.2.4. *For $s \in \mathcal{S}$, we have*

$$\alpha_s^{(m)} \|_c \beta^{(k)} \Leftrightarrow \beta \in \Phi_{\langle s\rangle}.$$

For $m = 1$, the m-eralized c-compatibility relation $\|_c$ therefore recovers Definition 5.1.1.

PROOF. Assume that s is not final in c and write $c = s_1 \cdots s_n$ with $s = s_p$ for $1 \leq p < n$. As $s_i \neq s$ for $i > p$, we obtain

$$\begin{aligned} \alpha_s^{(m)} \|_c \beta^{(k)} &\Leftrightarrow \alpha_s^{(m)} \|_{c'} \tau^{(m)}_{s_{p+1}} \circ \cdots \circ \tau^{(m)}_{s_n}\left(\beta^{(k)}\right) \\ &\Leftrightarrow \tau^{(m)}_{s_{p+1}} \circ \cdots \circ \tau^{(m)}_{s_n}\left(\beta^{(k)}\right) = \gamma^{(\ell)} \text{ with } \gamma \in \Phi_{\langle s\rangle} \text{ and some color } \ell \\ &\Leftrightarrow \beta \in \Phi_{\langle s\rangle}, \end{aligned}$$

for $c' = s_{p+1} \cdots s_n c \bar{s}_n \cdots \bar{s}_{p+1}$. The first equivalence follows from the defining property (ii) and the fact that $\tau^{(m)}_{s_{p+1}} \circ \cdots \circ \tau^{(m)}_{s_n}\left(\alpha_s^{(m)}\right) = \alpha_s^{(m)}$. The second equivalence follows from the defining property (i), and the third equivalence follows from Lemma 5.2.2. □

Analogously to [**Rea07a**, Lemma 7.1], we show that a binary relation on $\Phi^{(m)}_{\geq -1}$ satisfying the two defining properties of $\|_c$ is unique, if it exists. Existence is deferred to Theorem 5.5.2.

As $\tau_s^{(m)}$ coincides with τ_s on non-simple roots of a given color, the following can be seen as the m-eralized version of [**Rea07a**, Lemma 7.1].

LEMMA 5.2.5. *Let c be a Coxeter element, and $\beta^{(k)} \in \Phi^{(m)}_{\geq -1}$. Then there exists a final sequence $s_1, \ldots, s_p, s$ for c such that*

$$\tau^{(m)}_{s_p} \circ \cdots \circ \tau^{(m)}_{s_1}(\beta^{(k)}) = \alpha_s^{(k)}.$$

PROOF. We first consider the case $k = m$. In this case, $\beta = \alpha_s$ for some $s \in \mathcal{S}$. Write $c = s_n \ldots s_1$ and set $p < n$ such that $s_{p+1} = s$. As $s_i \neq s$ for $i \leq p$, we obtain the desired property by (5.2).

We next consider the case $k < m$. Let $\mathsf{c} = \mathsf{s}_n \ldots \mathsf{s}_1$ and consider

$$\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c}))) = \mathsf{s}_1 \cdots \mathsf{s}_N.$$

Lemma 2.6.5(7) implies that $s_1,\dots,s_N$ is a final sequence for c. Set $0 \le p < N$ such that $s_1\cdots s_p(\alpha_{s_{p+1}}) = \beta$ and compute

$$\tau^{(m)}_{s_p} \circ \cdots \circ \tau^{(m)}_{s_1}(\beta^{(k)}) = \big[s_p \cdots s_1(\beta)\big]^{(k)} = [\alpha_{s_{p+1}}]^{(k)}$$

by (5.2). □

LEMMA 5.2.6. *Let c be a Coxeter element and let $\beta \in \Phi^+$. There there exist a final sequence $s_1,\dots,s_p,s$ for c such that*

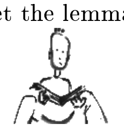
$$\tau^{(m)}_{s_p} \circ \cdots \circ \tau^{(m)}_{s_1}(\beta^{(k)}) = \alpha^{(m)}_s.$$

PROOF. Applying Lemma 5.2.5, we obtain a final sequence $s_1,\dots,s_p,s$ for c such that

$$\tau^{(m)}_{s_p} \circ \cdots \circ \tau^{(m)}_{s_1}(\alpha^{(k)}) = \alpha^{(k)}_s.$$

If $k = m$, we conclude the result. Otherwise, we apply $\tau^{(m)}_s$ to obtain $\alpha^{(k-1)}_s$, and proceed by again applying Lemma 5.2.5. □

Keep this in mind and forget the lemmas.



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PROPOSITION 5.2.7. *The m-eralized c-compatibility relation $\|_c$ is uniquely determined by its two defining properties.*

PROOF. This is an immediate consequence of Lemma 5.2.6. □

5.3. m-eralized cluster complexes

We construct m-eralized cluster complexes as subword complexes and in terms of colored almost positive roots.

5.3.1. Cluster complexes as subword complexes. E. Tzanaki encoded m-eralized clusters as certain colored factorizations for bipartite c [**Tza08**, Theorem 1.1], m-eralizing the result for $m = 1$ in [**IS10**, **BW08**]. We extend this approach to general Coxeter elements.

DEFINITION 5.3.1. Let c be a Coxeter element and let c be a reduced word for c. The *m-eralized c-cluster complex* is the c-initial subword complex defined by either

$$\begin{aligned}\mathsf{Asso}^{(m)}_{\Delta}(W,c) &:= \mathrm{SUB}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{cw}^m_\circ(\mathsf{c}), \boldsymbol{w}^m_\circ),\\ &= \mathrm{SUB}_{\mathcal{S}}(\mathsf{cw}^m_\circ(\mathsf{c}), w^m_\circ, mN), \text{ or}\\ \mathsf{Asso}^{(m)}_{\nabla}(W,c) &:= \mathrm{SUB}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}^m_\circ(\mathsf{c})), c^{-1}).\end{aligned}$$

The *m-eralized c-Cambrian poset* $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(W,c)$ is the increasing flip poset of $\mathsf{Asso}^{(m)}_{\Delta}(W,c)$.

The equality $\mathsf{Asso}^{(m)}_{\Delta}(W,c) = \mathsf{Asso}^{(m)}_{\nabla}(W,c)$ follows from the general equality proven in Proposition 3.4.2, where we again emphasize the different flavours of both. We may represent a face I of $\mathsf{Asso}^{(m)}_{\nabla}(W,c)$ as a word of colored reflections $\mathsf{r}^{(i_1)}_1 \cdots \mathsf{r}^{(i_k)}_k$, where $k \le n$ and

- a reflection r_j is colored by the copy of $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))$ to which it belongs, or by m if it belongs to the final copy of $\mathsf{inv}_{\mathcal{R}}(\mathsf{c})$;
- $r_a <_{\mathsf{c}} r_b$ when $a < b$ and $i_a = i_b$; and
- $r_k \cdots r_1 \le_{\mathcal{R}} c$ with $\ell_{\mathcal{R}}(r_k \cdots r_1) = k$.

By construction, every face is of the form above. We will prove in Proposition 5.4.9 that these three properties characterize the faces of $\mathsf{Asso}_{\nabla}^{(m)}(W, \mathsf{c})$.

5.3.2. Cluster complexes and colored almost positive roots. We now begin to relate the above construction to $\|_c$. Let $\mathsf{c} = \mathsf{s}_1 \cdots \mathsf{s}_n$ for a Coxeter element c. We know from Lemma 2.6.5(2) and (3) that $\mathsf{cw}_\circ(\mathsf{c}) \equiv \mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{c})$ and thus that

$$\mathsf{cw}_\circ^m(\mathsf{c}) \equiv \mathsf{w}_\circ^m(\mathsf{c})\psi^m(\mathsf{c}).$$

We therefore have

$$\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c})) \equiv \underbrace{\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))\,\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))\cdots\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))}_{\text{colors } 0,\ldots,m-1}\underbrace{\mathsf{inv}_{\mathcal{R}}(\mathsf{c})}_{\text{color } m}, \tag{5.3}$$

where the reflections in the i^{th} copy of $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))$ have color $(i-1)$, and the n roots of color m all come from the final copy of $\psi^m(\mathsf{c})$. This defines a bijection

$$\begin{aligned} \phi_c : \mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c})) &\longrightarrow \Phi_{\geq -1}^{(m)} \\ r^{(k)} &\longmapsto \begin{cases} \alpha_r^{(k)} & \text{if } k < m \\ \big[s_{i-1}\cdots s_1(\alpha_r)\big]^{(k)} & \text{if } k = m \end{cases}, \end{aligned} \tag{5.4}$$

where i is chosen so that $\alpha_r = s_1 \cdots s_i(\alpha_{s_{i+1}})$ if $\mathsf{c} = \mathsf{s}_1 \cdots \mathsf{s}_n$.

Definition 5.3.2. Let $\mathsf{Asso}^{(m)}(W, c)$ be the image of the faces of $\mathsf{Asso}_{\nabla}^{(m)}(W, c)$ under the map ϕ_c defined in (5.4). That is,

$$\begin{aligned} \phi_c : \mathsf{Asso}_{\nabla}^{(m)}(W, c) &\longrightarrow \mathsf{Asso}^{(m)}(W, c) \\ \mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_k^{(i_k)} &\longmapsto \{\phi_c(r_1^{(i_1)}), \ldots, \phi_c(r_k^{(i_k)})\}. \end{aligned}$$

We show in Theorem 5.5.2 that two m-colored almost positive roots are compatible under $\|_c$ if and only if they appear together in a face of $\mathsf{Asso}^{(m)}(W, c)$.

Definition 5.3.3. The *support* of a facet I of $\mathsf{Asso}_{\Delta}^{(m)}(W, c)$ is

$$\mathrm{supp}(I) := \Big\{ s \in \mathcal{S} \ :\ \alpha_s^{(0)} \notin \mathsf{R}(I) \Big\}.$$

The support on $\mathsf{Asso}_{\nabla}^{(m)}(W, c)$ and on $\mathsf{Asso}^{(m)}(W, c)$ are defined analogously.

Figure 13 shows all 12 facets of

$$\mathsf{Asso}_{\Delta}^{(2)}(\mathfrak{S}_3, st) \cong \mathsf{Asso}_{\nabla}^{(2)}(\mathfrak{S}_3, st) \cong \mathsf{Asso}^{(2)}(\mathfrak{S}_3, st),$$

with their supports.

5.4. The Cambrian rotation and recurrence

5.4.1. The shift operator. Let c be a reduced word for c with first letter s. It is immediate from Lemma 2.6.5(5) and (7) that for $\mathsf{cw}_\circ^m(\mathsf{c}) = \mathsf{s}_1\mathsf{s}_2 \ldots \mathsf{s}_{n+mN}$,

$$\mathsf{s}_2 \ldots \mathsf{s}_{n+mN}\psi^m(\mathsf{s}_1) \equiv \bar{\mathsf{s}}\mathsf{csw}_\circ^m(\bar{\mathsf{s}}\mathsf{cs}).$$

This identification defines a canonical isomorphism

$$\mathsf{Shift}_s : \mathsf{Asso}_{\Delta}^{(m)}(W, c) \longrightarrow \mathsf{Asso}_{\Delta}^{(m)}(W, \bar{s}cs).$$

$\mathsf{Asso}^{(2)}_{\triangle}(\mathfrak{S}_3, st)$	$\mathsf{Asso}^{(2)}_{\nabla}(\mathfrak{S}_3, st)$	$\mathsf{Asso}^{(2)}(\mathfrak{S}_3, st)$	$\mathsf{R}(I)$	$\mathrm{supp}(I)$
st.sts.tst	sut.sut.su	$\alpha^{(0)}, \gamma^{(0)}$	$\alpha^{(0)}, \beta^{(0)}$	–
st.sts.tst	sut.sut.su	$\alpha^{(2)}, \beta^{(2)}$	$\alpha^{(2)}, \beta^{(2)}$	s, t
st.sts.tst	sut.sut.su	$\gamma^{(1)}, \beta^{(1)}$	$\gamma^{(1)}, \alpha^{(2)}$	s, t
st.sts.tst	sut.sut.su	$\beta^{(0)}, \alpha^{(1)}$	$\beta^{(0)}, \gamma^{(1)}$	s, t
st.sts.tst	sut.sut.su	$\gamma^{(0)}, \beta^{(0)}$	$\gamma^{(0)}, \alpha^{(1)}$	s
st.sts.tst	sut.sut.su	$\alpha^{(0)}, \beta^{(2)}$	$\alpha^{(0)}, \beta^{(2)}$	t
st.sts.tst	sut.sut.su	$\beta^{(1)}, \alpha^{(2)}$	$\beta^{(1)}, \gamma^{(2)}$	s, t
st.sts.tst	sut.sut.su	$\alpha^{(1)}, \gamma^{(1)}$	$\alpha^{(1)}, \beta^{(1)}$	s, t
st.sts.tst	sut.sut.su	$\alpha^{(0)}, \gamma^{(1)}$	$\alpha^{(0)}, \beta^{(1)}$	t
st.sts.tst	sut.sut.su	$\beta^{(0)}, \alpha^{(2)}$	$\beta^{(0)}, \gamma^{(2)}$	s, t
st.sts.tst	sut.sut.su	$\gamma^{(0)}, \beta^{(1)}$	$\gamma^{(0)}, \alpha^{(2)}$	s
st.sts.tst	sut.sut.su	$\alpha^{(1)}, \beta^{(2)}$	$\alpha^{(1)}, \beta^{(2)}$	s, t

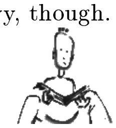
FIGURE 13. The three variants of the m-eralized st-clusters for $\mathfrak{S}_3$ with $m = 2$, together with their root configurations and supports. They are arranged according to their orbits under Cambrian rotation, defined in Section 5.4.

This isomorphism is explicitly described on $\mathsf{Asso}^{(m)}_{\nabla}(W, c) \longrightarrow \mathsf{Asso}^{(m)}_{\nabla}(W, \bar{s}cs)$ by

$$\mathsf{Shift}_s : \mathsf{inv}_{\mathcal{R}}(\mathsf{c}\mathsf{w}_\circ^m(\mathsf{c})) \longrightarrow \mathsf{inv}_{\mathcal{R}}(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}\mathsf{w}_\circ^m(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}))$$

$$r^{(k)} \longmapsto \begin{cases} s^{(k-1)} & \text{if } r = s \text{ and } k > 0 \\ \left[\,{}^{\bar{s}c}s\right]^{(m)} & \text{if } r = s \text{ and } k = 0 \\ [r^s]^{(k)} & \text{if } r \neq s \end{cases} . \tag{5.5}$$

Everything here behaves as expected, no pitfalls anywhere. Notationally a little heavy, though.



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EXAMPLE 5.4.1. We visualize the definition of Shift_s for the initial $s = s_1$ in $c = s_1 s_2 s_3$ with $\bar{s}cs = s_2 s_3 s_1$. In this case, we obtain

s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_1	
$(12)^{(0)}$	$(13)^{(0)}$	$(14)^{(0)}$	$(23)^{(0)}$	$(24)^{(0)}$	$(34)^{(0)}$	$(12)^{(1)}$	$(13)^{(1)}$	$(14)^{(1)}$	
	$(23)^{(0)}$	$(24)^{(0)}$	$(13)^{(0)}$	$(14)^{(0)}$	$(34)^{(0)}$	$(12)^{(0)}$	$(23)^{(1)}$	$(24)^{(1)}$	$(13)^{(1)}$
	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_1	s_3

where the first two rows show $\mathsf{w}_\circ(\mathsf{c})$ and $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))$, the third row shows the map $\mathsf{Shift}_{(12)}$, and the last two rows show $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}))$ and $\mathsf{w}_\circ(\bar{\mathsf{s}}\mathsf{c}\mathsf{s})$. (Observe that in the later, the last two commuting reflections s_1 and s_3 are interchanged.)

5.4.2. The Cambrian rotation. For s initial in c, we extend the definition of Shift_s to words in $\mathsf{inv}_{\mathcal{R}}(\mathsf{c}\mathsf{w}_\circ^m(\mathsf{c}))$—and thus to faces—by

$$\mathsf{Shift}_s : \mathsf{Asso}^{(m)}_{\nabla}(W, c) \longrightarrow \mathsf{Asso}^{(m)}_{\nabla}(W, \bar{s}cs)$$

$$\mathsf{r}_1^{(i_1)}\mathsf{r}_2^{(i_2)}\cdots\mathsf{r}_k^{(i_k)} \longmapsto \begin{cases} \mathsf{Shift}_s(\mathsf{r}_2^{(i_2)}\cdots\mathsf{r}_k^{(i_k)}\mathsf{s}^{(0)}) & \text{if } \mathsf{r}_1^{(i_1)} = \mathsf{s}^{(0)} \\ \mathsf{Shift}_s(\mathsf{r}_1^{(i_1)}\mathsf{r}_2^{(i_2)}\cdots\mathsf{r}_k^{(i_k)}) & \text{otherwise} \end{cases} . \tag{5.6}$$

EXAMPLE 5.4.2. Parallel to Example 4.3.1, alternately applying Shift_s and Shift_t to the facet $\{1,2\} \in \mathsf{Asso}^{(2)}_{\Delta}(\mathfrak{S}_3, st)$ gives the orbit

$$\begin{array}{ccccc} \mathsf{ststst st} & \xrightarrow{\mathsf{Shift}_s} & \mathsf{tststs ts} & \xrightarrow{\mathsf{Shift}_t} & \mathsf{ststst st} \\ & \xrightarrow{\mathsf{Shift}_s} & \mathsf{tststs ts} & \xrightarrow{\mathsf{Shift}_t} & \mathsf{ststst st} \\ & \xrightarrow{\mathsf{Shift}_s} & \mathsf{tststs ts} & \xrightarrow{\mathsf{Shift}_t} & \mathsf{ststst st} \\ & \xrightarrow{\mathsf{Shift}_s} & \mathsf{tststs ts} & \xrightarrow{\mathsf{Shift}_t} & \mathsf{ststst st} \end{array}.$$

The same computation in $\mathsf{Asso}^{(2)}_{\nabla}(\mathfrak{S}_3, st)$ gives

$$\begin{array}{ccccc} \mathsf{s}^{(0)}\mathsf{u}^{(0)} & \xrightarrow{\mathsf{Shift}_s} & \mathsf{t}^{(0)}\mathsf{u}^{(2)} & \xrightarrow{\mathsf{Shift}_t} & \mathsf{s}^{(2)}\mathsf{u}^{(2)} \\ & \xrightarrow{\mathsf{Shift}_s} & \mathsf{s}^{(1)}\mathsf{t}^{(2)} & \xrightarrow{\mathsf{Shift}_t} & \mathsf{u}^{(1)}\mathsf{t}^{(1)} \\ & \xrightarrow{\mathsf{Shift}_s} & \mathsf{t}^{(1)}\mathsf{u}^{(1)} & \xrightarrow{\mathsf{Shift}_t} & \mathsf{t}^{(0)}\mathsf{s}^{(1)} \\ & \xrightarrow{\mathsf{Shift}_s} & \mathsf{u}^{(0)}\mathsf{s}^{(0)} & \xrightarrow{\mathsf{Shift}_t} & \mathsf{s}^{(0)}\mathsf{u}^{(0)} \end{array}.$$

DEFINITION 5.4.3. The *m-eralized c-Cambrian rotation* is

$$\mathsf{Camb}_c := \mathsf{Shift}_{s_n} \circ \cdots \circ \mathsf{Shift}_{s_1} : \mathsf{Asso}^{(m)}(W,c) \longrightarrow \mathsf{Asso}^{(m)}(W,c).$$

This composition evidently does not depend on the chosen reduced word. The elements in Figure 13 are arranged according to their orbits under Cambrian rotation.

PROPOSITION 5.4.4.

$$\mathrm{ord}(\mathsf{Camb}_c) = \begin{cases} mh+2 & \text{if } \psi \not\equiv \mathbb{1} \text{ and } m \text{ odd} \\ (mh+2)/2 & \text{otherwise} \end{cases}.$$

PROOF. The statement follows from the facts that $w_\circ$ is an involution on $\mathcal{S}$, $2N = nh$, and the length of the word $\mathsf{cw}_\circ^m(\mathsf{c})$ is $n + mN$. ☐

5.4.3. The Cambrian recurrence. We modify the first case in the definition of the shift operator to obtain an inductive characterization of $\mathsf{Asso}^{(m)}(W,c)$, the *m-eralized c-Cambrian recurrence*.

PROPOSITION 5.4.5. *Let s be initial in c and let $I = \mathsf{r}_1^{(i_1)}\mathsf{r}_2^{(i_2)}\cdots\mathsf{r}_n^{(i_n)}$. If $r^{(i_1)} = s^{(0)}$, we set $I_{\langle s\rangle} := [\mathsf{r}_2^s]^{(i_2)}\cdots[\mathsf{r}_n^s]^{(i_n)}$. Then*

$$I \in \mathsf{Asso}^{(m)}_{\nabla}(W,c) \Leftrightarrow \begin{cases} \mathsf{Shift}_s(I_{\langle s\rangle}) \in \mathsf{Asso}^{(m)}_{\nabla}(W_{\langle s\rangle}, \overline{s}c) & \text{if } r_1^{(i_1)} = s^{(0)} \\ \mathsf{Shift}_s(I) \in \mathsf{Asso}^{(m)}_{\nabla}(W, \overline{s}cs) & \text{otherwise} \end{cases}.$$

PROOF. If $\mathsf{r}_1^{(i_1)} \neq \mathsf{s}^{(0)}$, the statement is clear, so assume otherwise. We compute that $r_2^s \cdots r_n^s = (r_1 \cdots r_n)s = c^{-1}s = (sc)^{-1}$. Each r_i^s is a reflection inside the parabolic subgroup $W_{\langle s\rangle}$, since $\mathsf{s}\mathsf{r}_n^s \cdots \mathsf{r}_2^s$ is a reduced $\mathcal{R}$-word for c. It is a facet by Lemma 2.6.5(7) and Lemma 2.9.1. ☐

EXAMPLE 5.4.6. Parallel to Example 4.3.4, the Cambrian recurrence for the facet $s^{(1)}u^{(2)} \in \mathsf{Asso}^{(2)}_{\nabla}(\mathfrak{S}_3, st)$ is computed as

$$\underbrace{\mathsf{s}^{(1)}\mathsf{u}^{(2)}}_{st} \mapsto \underbrace{\mathsf{s}^{(0)}\mathsf{t}^{(2)}}_{ts} \mapsto \underbrace{\mathsf{u}^{(0)}\mathsf{t}^{(1)}}_{st} \mapsto \underbrace{\mathsf{t}^{(0)}\mathsf{u}^{(1)}}_{ts} \mapsto \underbrace{\mathsf{s}^{(1)}}_{s} \mapsto \underbrace{\mathsf{s}^{(0)}}_{s} \mapsto \underbrace{-}_{e}.$$

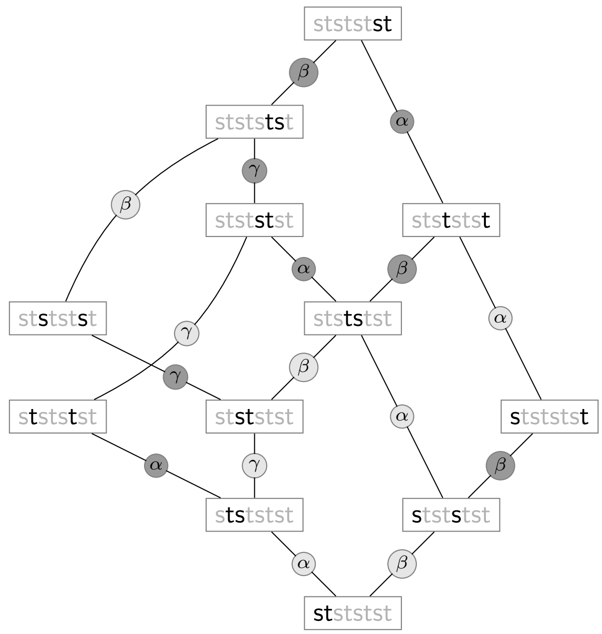

FIGURE 14. The flip poset $\mathsf{Camb}^{(2)}_{\mathsf{Asso}}(\mathfrak{S}_3, st)$ on the facets of $\mathsf{Asso}^{(2)}(\mathfrak{S}_3, st)$. The colored roots labelling the edges correspond to the direction of the flips, with the color indicated by the shading.

The same computation for ststst**st** $\in \mathsf{Asso}^{(2)}_{\Delta}(\mathfrak{S}_3, st)$ is

$$\underbrace{\mathsf{ststst}}_{st} \mapsto \underbrace{\mathsf{tststs}}_{ts} \mapsto \underbrace{\mathsf{ststst}}_{st} \mapsto \underbrace{\mathsf{tststs}}_{ts} \mapsto \underbrace{\mathsf{ss}}_{s} \mapsto \underbrace{\mathsf{sss}}_{s} \mapsto \underbrace{-}_{e}.$$

Figure 14 illustrates $\mathsf{Camb}^{(2)}_{\mathsf{Asso}}(\mathfrak{S}_3, st)$.

PROPOSITION 5.4.7. *For any two Coxeter elements c, c', the two m-eralized cluster complexes $\mathsf{Asso}^{(m)}_{\Delta}(W, c)$ and $\mathsf{Asso}^{(m)}_{\Delta}(W, c')$ are isomorphic.*

PROOF. Let $s_1, \dots, s_p$ be an initial sequence for c. Then,

$$\mathsf{Shift}_{s_p} \circ \cdots \circ \mathsf{Shift}_{s_1} : \mathsf{Asso}^{(m)}_{\Delta}(W, c) \longrightarrow \mathsf{Asso}^{(m)}_{\Delta}(W, \overline{s}_p \cdots \overline{s}_1 c s_1 \cdots s_p)$$

is the desired isomorphism. □

REMARK 5.4.8. For any two Coxeter elements c and c', the underlying unoriented flip graphs of the complexes $\mathsf{Asso}^{(m)}(W, c)$ and $\mathsf{Asso}^{(m)}(W, c')$ are isomorphic. The isomorphism is induced by the shift operation, and is given by the isomorphism in Proposition 5.4.7. In particular, the map Camb_c is a graph automorphism. For $m = 1$, the increasing flip graph coincides with the Hasse diagram of the Cambrian poset, while for $m \geq 2$ the increasing flip graph is no longer transitively reduced. Therefore, the shift operation does not induce a isomorphism between the unoriented Hasse diagrams of $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(W, c)$ and $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(W, c')$.

5.4.4. Flagness of m-eralized cluster complexes. We conclude this section by using the Cambrian recurrence to intrinsically describe the faces of the complex $\mathsf{Asso}^{(m)}_{\nabla}(W,\mathsf{c})$. We then deduce that the m-eralized c-cluster complex is flag, and so can be reconstructed as the clique complex of its edges.[C]

PROPOSITION 5.4.9. *Let $\mathsf{r}_1^{(i_1)}\cdots\mathsf{r}_k^{(i_k)}$ be a subword of $\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c}))$. Then $\mathsf{r}_1^{(i_1)}\cdots\mathsf{r}_k^{(i_k)}$ is a face of $\mathsf{Asso}^{(m)}_{\nabla}(W,c)$ if and only if $r_k\cdots r_1 \leq_{\mathcal{R}} c$ with reflection length $\ell_{\mathcal{R}}(r_k\cdots r_1)=k$.*

PROOF. We have already seen in Section 5.3 that these are necessary conditions for a subword to be a face, so we only need to show that these are also sufficient. For the empty word, the statement is clear, so we can assume $k\geq 1$ and that $r_k\cdots r_1 \leq_{\mathcal{R}} c$ with $\ell_{\mathcal{R}}(r_k\cdots r_1)=k$. We have to show that this subword of $\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c}))$ can be extended to a subword that is a reduced $\mathcal{R}$-word for c^{-1}.

But this is a direct consequence of the Cambrian recurrence. Either the first letter $r_1^{(i_1)}$ is not equal to $s^{(0)}$ in which case we conclude the statement by induction on the second equivalence in Proposition 5.4.5. Otherwise, the first letter $\mathsf{r}_1^{(i_1)}$ is $\mathsf{s}^{(0)}$. In this case, we conclude the statement by induction from the first equivalence in Proposition 5.4.5 as follows. Removing this first letter $\mathsf{s}^{(0)}$ from the facet and conjugating all other letters by s gives the facet $I_{\langle s\rangle}=[\mathsf{r}_2^s]^{(i_2)}\cdots[\mathsf{r}_k^s]^{(i_k)}\in\mathsf{Asso}^{(m)}_{\nabla}(W,\overline{s}c)$. By induction, $I_{\langle s\rangle}$ can be extended to a reduced subword of $\mathsf{inv}_{\mathcal{R}}(\overline{\mathsf{s}}\mathsf{cw}_\circ^m(\overline{\mathsf{s}}\mathsf{c}))$ whose product is $c^{-1}s$. Lifting this subword back to $\mathsf{Asso}^{(m)}_{\nabla}(W,c)$ has the effect of conjugating by s, giving a reduced subword of $\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c}))$ whose product is sc^{-1}. Adding back the first letter $\mathsf{s}^{(0)}$ gives a reduced subword for c^{-1}. ☐

EXAMPLE 5.4.10. The cluster complex $\mathsf{Asso}^{(1)}_{\nabla}(\mathfrak{S}_4,s_1s_2s_3)$ has search word

$$\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ(\mathsf{c}))=(12)^{(0)}(13)^{(0)}(14)^{(0)}(23)^{(0)}(24)^{(0)}(12)^{(1)}(34)^{(0)}(13)^{(1)}(14)^{(1)}.$$

There are 21 subwords of this search word of the form $\mathsf{r}_1^{i_1}\mathsf{r}_2^{i_2}$ with $r_2r_1\leq_{\mathcal{R}} c$ and $r_1\neq r_2$. Two such examples are $(23)^{(0)}(14)^{(1)}$ and $(24)^{(0)}(12)^{(1)}$.

COROLLARY 5.4.11. *$\mathsf{Asso}^{(m)}(W,c)$ is flag—that is, all its minimal nonfaces have cardinality 2.*

Note the dependence on Proposition 4.1.2. Subtle.

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PROOF. We first check that every letter is contained in a facet of $\mathsf{Asso}^{(m)}_{\Delta}(W,c)$. Write $\mathsf{cw}_\circ^m(\mathsf{c})=\mathsf{c}\mathsf{c}\big|_{I_1}\cdots\mathsf{c}\big|_{I_k}$ where $\mathsf{c}\big|_{I_1}\cdots\mathsf{c}\big|_{I_k}$ is the c-sorting word of $w_\circ^m$ decomposed into its subwords of the individual copies of c. By definition, the letters in the first copy of c form a facet. Furthermore, Lemma 2.6.3 implies that $I_1\supseteq I_2\supseteq\cdots\supseteq I_k$—so the word obtained from $\mathsf{cw}_\circ^m(\mathsf{c})$ omitting any $\mathsf{c}\big|_{I_j}$ still contains a copy of $w_\circ^m$. Therefore, every letter is contained in facet.

Now let $\mathsf{r}_1^{(i_1)}\cdots\mathsf{r}_k^{(i_k)}$ be a subword of $\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c}))$ such that $\mathsf{r}_a^{(i_a)}\mathsf{r}_b^{(i_b)}$ for $1\leq a<b\leq k$ is a face of $\mathsf{Asso}^{(m)}_{\nabla}(W,c)$. Therefore $r_br_a\leq_{\mathcal{R}} c$ with $r_b\neq r_a$ by Proposition 5.4.9. By Proposition 4.1.2, $r_k\cdots r_1\leq c$ with $\ell_{\mathcal{R}}(r_k\cdots r_1)=k$. Therefore, $\mathsf{r}_1^{(i_1)}\cdots\mathsf{r}_k^{(i_k)}$ is a face of $\mathsf{Asso}^{(m)}_{\nabla}(W,c)$ by Proposition 5.4.9. ☐

[C] This statement was inadvertently omitted in [**CLS14**].

5.5. Subword complexes and compatibility

5.5.1. Compatibility via the m-eralized cluster complex. We show that the construction of $\mathsf{Asso}^{(m)}(W,c)$ given in Definition 5.3.1 implies the existence of the $\|_c$ relation proposed in Definition 5.2.3. Write $\|'_c$ for the relation on $\Phi^{(m)}_{\geq -1}$ given by

$$r^{(k)} \|'_c\, t^{(\ell)} \text{ if } \{\mathsf{r}^{(k)}, \mathsf{t}^{(\ell)}\} \text{ is a face of } \mathsf{Asso}^{(m)}(W,c). \tag{5.7}$$

LEMMA 5.5.1. *The bijection $\phi : \mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c})) \longrightarrow \Phi^{(m)}_{\geq -1}$ defined in (5.4) sends Shift_s to $\tau_s^{(m)}$. That is, for $r^{(k)} \in \mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c}))$ and s initial in c, we have*

$$\phi_{\bar{s}cs}\big(\mathsf{Shift}_s(r^{(k)})\big) = \tau_s^{(m)}\big(\phi_c(r^{(k)})\big).$$

PROOF. This follows immediately from the definitions (5.2) and (5.5). Note that the third and fourth cases in (5.2) are merged together as the third case in (5.5) using (5.4). □

THEOREM 5.5.2. *The m-eralized c-compatibility relation $\|_c$ exists and is symmetric. Moreover,*

$$r^{(k)} \|'_c\, t^{(\ell)} \Leftrightarrow \phi_c(r^{(k)}) \|_c\, \phi_c(t^{(\ell)}).$$

PROOF. Applying ϕ_c^{-1} and using Lemma 5.5.1, we prove that $\|'_c$ satisfies the defining properties Definition 5.2.3 of $\|_c$ with $\tau_s^{(m)}$ replaced by Shift_s. That is, for s final in c

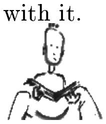
(i) $[^{sc}s]^{(m)} \|'_c\, t^{(k)} \Leftrightarrow \phi_c(t^{(k)}) \in \Phi_{\langle s\rangle}$, and
(ii) $t^{(k)} \|'_c\, r^{(\ell)} \Leftrightarrow \mathsf{Shift}_s(t^{(k)}) \|_{sc\bar{s}}\, \mathsf{Shift}_s(r^{(\ell)})$,

with $t^{(k)}, r^{(\ell)} \in \mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ^m(\mathsf{c}))$. When $k=m$, the first property holds because the colored reflections of color m form a facet of $\mathsf{Asso}^{(m)}(W,c)$ and the roots of color m in $\Phi_{\geq -1}$ are simple. For $k<m$ we compute

$$\begin{aligned}
[^{sc}s]^{(m)} \|'_c\, t^{(k)} &\overset{(5.6)}{\Longleftrightarrow} \mathsf{Shift}_s^{-1}\big([^{sc}s]^{(m)}\big) \|'_{sc\bar{s}}\, \mathsf{Shift}_s^{-1}\big(t^{(k)}\big)\\
&\overset{(5.5)}{\Longleftrightarrow} s^{(0)} \|'_{sc\bar{s}}\, \mathsf{Shift}_s^{-1}\big(t^{(k)}\big)\\
&\overset{5.4.5}{\Longleftrightarrow} \mathsf{Shift}_s\mathsf{Shift}_s^{-1}\big(t^{(k)}\big) = t^{(k)} \in \mathsf{Asso}_\nabla^{(m)}(W_{\langle s\rangle}, c\bar{s})\\
&\overset{5.4.11}{\Longleftrightarrow} t^{(k)} \in \mathsf{inv}_{\mathcal{R}}(\mathsf{c\bar{s}w}_\circ^m(\mathsf{c\bar{s}}))\\
&\overset{(5.4)}{\Longleftrightarrow} t \in W_{\langle s\rangle} \Leftrightarrow \phi_c(t^{(k)}) = \beta_t^{(k)} \in \Phi_{\geq -1},
\end{aligned}$$

implying the first defining property for $\|'_c$. The second defining property directly follows from (5.6) using (5.4).

If we had initial, the arguments here would be even more technical. But we are finally done with it.



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As shown in Proposition 5.2.7, these properties uniquely determine the relation. It follows from Lemma 5.5.1 that

$$t^{(k)} \|'_c\, r^{(\ell)} \Leftrightarrow \phi_c(t^{(k)}) \|_c\, \phi_c(r^{(\ell)}).$$ □

COROLLARY 5.5.3. *The m-eralized c-cluster complex $\mathsf{Asso}^{(m)}(W,c)$ is the simplicial complex given by all collections of pairwise $\|_c$ compatible m-colored almost positive roots.* □

5.5.2. Recovering generalized cluster complexes. In [**FR05**], S. Fomin and N. Reading gave an m-eralization of Definition 5.1.1 for bipartite Coxeter elements. Let $\mathcal{S} = \mathcal{S}_L \sqcup \mathcal{S}_R$ be the bipartition of the simple reflections from Section 2.6. The *Fomin-Reading map*[D] $\mathrm{FR}^{(m)} : \Phi^{(m)}_{\geq -1} \longrightarrow \Phi^{(m)}_{\geq -1}$ is defined by

$$\mathrm{FR}^{(m)}(\beta^{(k)}) = \begin{cases} \beta^{(k+1)} & \text{if } 0 \leq k < m-1 \\ \mathrm{FR}(\beta)^{(0)} & \text{if } k = m-1 \\ \mathrm{FR}(-\beta)^{(0)} & \text{if } k = m \end{cases},$$

where $\mathrm{FR} := \prod_{s \in \mathcal{S}_L} \tau_s \prod_{s \in \mathcal{S}_R} \tau_s$ and $\gamma^{(0)}$ is interpreted as $[-\gamma]^{(m)}$ for $\gamma \in -\Delta$.

DEFINITION 5.5.4. The FR *m-eralized compatibility relation* is the unique relation $\|^{\mathrm{FR}}$ on $\Phi^{(m)}_{\geq -1}$ characterized by the following two properties [**FR05**, Theorem 3.4]:

(i) for $\alpha \in \Delta$ and $\beta^{(k)} \in \Phi^{(m)}_{\geq -1}$, $\alpha^{(m)} \|^{\mathrm{FR}} \beta^{(k)} \Leftrightarrow \beta \in \Phi_{\langle \alpha \rangle}$,

(ii) for $\beta^{(k)}, \gamma^{(\ell)} \in \Phi^{(m)}_{\geq -1}$, $\beta^{(k)} \|^{\mathrm{FR}} \gamma^{(\ell)} \Leftrightarrow \mathrm{FR}^{(m)}(\beta^{(k)}) \|^{\mathrm{FR}} \mathrm{FR}^{(m)}(\gamma^{(\ell)})$.

We show that the compatibility relation $\|_c$ recovers $\|^{\mathrm{FR}}$ for bipartite $c = c_L c_R$.

PROPOSITION 5.5.5. *The Fomin-Reading map* $\mathrm{FR}^{(m)}$ *satisfies*

$$\left(\mathrm{FR}^{(m)}\right)^{-1} = \psi \circ \tau^{(m)}_{s_N} \circ \cdots \circ \tau^{(m)}_{s_1},$$

where $\mathsf{w}_\circ(\mathsf{c_R c_L}) = \mathsf{s}_1 \cdots \mathsf{s}_N$ *is the* $c_R c_L$*-sorting word for* $w_\circ$*, and where* $\psi : \Phi^+ \to \Phi^+$ *is the involution on positive roots given by* $\psi(\beta) := -w_\circ(\beta)$ *and* $\psi(\beta^{(k)}) := \psi(\beta)^{(k)}$.

PROOF. It is a straightforward check that this composition satisfies the two cases in the definition of the Fomin-Reading map $\mathrm{FR}^{(m)}$. □

EXAMPLE 5.5.6. Let $c = c_L c_R = st = (12)(23) \in \mathfrak{S}_3$ and let $m = 2$. In this case, $\mathsf{w}_\circ(\mathsf{c_R c_L}) = \mathsf{tst}$, so we consider the identity

$$\left(\mathrm{FR}^{(2)}\right)^{-1} = \psi \circ \tau^{(2)}_t \circ \tau^{(2)}_s \circ \tau^{(2)}_t.$$

We compute the images of this composition on $\Phi^{(2)}_{\geq -1}$:

$$\begin{array}{rccccccccc}
 & \alpha^{(0)} & \gamma^{(0)} & \beta^{(0)} & \alpha^{(1)} & \gamma^{(1)} & \beta^{(1)} & \alpha^{(2)} & \beta^{(2)} \\
\tau^{(2)}_t \downarrow & & & & & & & & \\
 & \gamma^{(0)} & \alpha^{(0)} & \beta^{(2)} & \gamma^{(1)} & \alpha^{(1)} & \beta^{(0)} & \alpha^{(2)} & \beta^{(1)} \\
\tau^{(2)}_s \downarrow & & & & & & & & \\
 & \beta^{(0)} & \alpha^{(2)} & \beta^{(2)} & \beta^{(1)} & \alpha^{(0)} & \gamma^{(0)} & \alpha^{(1)} & \gamma^{(1)} \\
\tau^{(2)}_t \downarrow & & & & & & & & \\
 & \beta^{(2)} & \alpha^{(2)} & \beta^{(1)} & \beta^{(0)} & \gamma^{(0)} & \alpha^{(0)} & \gamma^{(1)} & \alpha^{(1)} \\
\psi \downarrow & & & & & & & & \\
 & \alpha^{(2)} & \beta^{(2)} & \alpha^{(1)} & \alpha^{(0)} & \gamma^{(0)} & \beta^{(0)} & \gamma^{(1)} & \beta^{(1)} \\
\mathrm{FR}^{(2)} \downarrow & & & & & & & & \\
 & \alpha^{(0)} & \gamma^{(0)} & \beta^{(0)} & \alpha^{(1)} & \gamma^{(1)} & \beta^{(1)} & \alpha^{(2)} & \beta^{(2)}
\end{array}.$$

[D] We have chosen to place the "negative" simple roots in the separate color m, while S. Fomin and N. Reading place the negative simple roots along with the full copy of positive roots in color 0. We have modified their definitions accordingly.

Corollary 5.5.7. *For $c = c_L c_R$ a bipartite Coxeter element, $\beta^{(i)} \parallel_c \gamma^{(j)}$ if and only if $\beta^{(i)} \parallel^{\mathrm{FR}} \gamma^{(j)}$.*

Proof. The defining condition Definition 5.5.4(i) for $\parallel^{\mathrm{FR}}$ holds for $\parallel_c$. We must show that $\parallel_c$ also satisfies the defining condition Definition 5.5.4(ii). By [**FR05**, Lemma 6.2], the m-eralized compatibility relation is invariant under ψ:

$$\beta^{(k)} \parallel^{\mathrm{FR}} \gamma^{(\ell)} \Leftrightarrow \psi(\beta^{(k)}) \parallel^{\mathrm{FR}} \psi(\gamma^{(\ell)}).$$

Proposition 5.5.5 implies that Definition 5.5.4(ii) can be expressed as

$$\beta^{(k)} \parallel^{\mathrm{FR}} \gamma^{(\ell)} \Leftrightarrow \big(\tau_{s_N}^{(m)} \circ \cdots \circ \tau_{s_1}^{(m)}\big)^{-1}(\beta^{(k)}) \parallel^{\mathrm{FR}} \big(\tau_{s_N}^{(m)} \circ \cdots \circ \tau_{s_1}^{(m)}\big)^{-1}(\gamma^{(\ell)}).$$

By applying Definition 5.2.3(ii) N times, we conclude for $c = c_L c_R$ that the same recurrence relation holds for $\parallel_c$. It follows that

$$\beta^{(k)} \parallel^{\mathrm{FR}} \gamma^{(\ell)} \Leftrightarrow \beta^{(k)} \parallel_c \gamma^{(\ell)}. \qquad \square$$

5.6. Topological properties

In this section, we apply the general results from Section 3.6 to m-eralized cluster complexes. In [**AT08**], C. Athanasiadis and E. Tzanaki showed that the generalized cluster complex $\mathsf{Asso}^{(m)}(W, c_L c_R)$ of Fomin-Reading is vertex-decomposable. For $m = 1$, [**CLS14**] gives an elegant proof of vertex-decomposability by realizing the cluster complex as a subword complex and invoking [**KM04**, Theorem 2.5], which states that *all* subword complexes are vertex-decomposable. Based on Theorem 3.6.2, we also obtain the vertex-decomposability for m-eralized c-cluster complexes.

Aren't we lucky to have Coxeter-initial complexes?

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Theorem 5.6.1. *$\mathsf{Asso}^{(m)}(W, c)$ is vertex-decomposable, hence shellable. The lexicographic order of the facets (as sorted lists of positions) of $\mathsf{Asso}^{(m)}(W, c)$ is a shelling order.*

Proof. This is a direct consequence of Theorem 3.6.2 given that $\mathsf{cw}_\circ^m(\mathsf{c})$ is initial in c^∞. $\square$

As a consequence we also obtain the following uniform theorem first proven by S. Fomin and N. Reading in [**FR05**, Proposition 11.1].

Theorem 5.6.2. *The m-eralized c-cluster complex $\mathsf{Asso}^{(m)}(W, c)$ has the homotopy type of a wedge of $\mathsf{Cat}^{(m-1)}(W)$ spheres of dimension $n-1$.*

Proof. This follows from Corollary 3.6.3 by observing that the subcomplex of $\mathsf{Asso}^{(m)}(W, c)$ not containing positions corresponding to a letter in the initial copy of $\mathsf{cw}_\circ^m(\mathsf{c})$ is equal to the cluster complex $\mathsf{Asso}^{(m-1)}(W, \psi(c))$. By Proposition 5.4.7 and Corollary 5.5.7, $\mathsf{Asso}^{(m-1)}(W, \psi(c))$ is isomorphic to $\mathsf{Asso}^{(m-1)}(W, c)$. Since the latter is known to be counted by $\mathsf{Cat}^{(m-1)}(W)$, we conclude the theorem. $\square$

Corollary 5.6.3. *The h-polynomial $h_0 + h_1 q + \cdots + h_d q^d$ of the m-eralized c-cluster complex $\mathsf{Asso}^{(m)}(W, c)$ is given by the upper and lower covers generating function of $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(W, c)$,*

$$h_i = \big\{F \in \mathsf{Camb}^{(m)}_{\mathsf{Asso}}(W, c) \ : \ F \text{ has exactly } i \text{ upper covers } \big\}.$$

The coefficients in this corollary are called *W-Fuß-Narayana numbers.*

PROOF. Since the lexicographic order is a linear extension of $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(W,c)$, the statement follows from Theorem 5.6.1 together with the fact that the reverse of a shelling order is again a shelling order. □

5.7. Clusters and noncrossing partitions

The connection between $\mathsf{Asso}^{(m)}_{\Delta}(W,c)$ and $\mathsf{NC}^{(m)}_{\Delta}(W,c)$ is based on the m-eralization of the $m=1$ results in [**PS15**, Proposition 6.20]. This recovers, in crystallographic types, the representation-theoretic bijections given by A. Buan, I. Reiten, and H. Thomas [**BRT11, BRT09**].

The root configurations of the facets of $\mathsf{Asso}^{(m)}_{\Delta}(W,\mathsf{c})$ satisfy an *m-eralized c-Cambrian recurrence*.

PROPOSITION 5.7.1. *Let s be initial in c and let I be a facet of $\mathsf{Asso}^{(m)}_{\Delta}(W,c)$. Then the root configuration satisfies*

$$\mathsf{R}(I) = \begin{cases} \{\alpha_s^{(0)}\} \cup \mathsf{R}(I_{\langle s\rangle}) & \text{if } s^{(0)} \in \mathsf{R}(I) \\ s\big(\mathsf{R}(\mathsf{Shift}_s(I))\big) & \text{otherwise} \end{cases},$$

with $I_{\langle s\rangle} \in \mathsf{Asso}^{(m)}_{\Delta}(W_{\langle s\rangle}, \overline{s}c)$ and $\mathsf{Shift}_s(I) \in \mathsf{Asso}^{(m)}(W, \overline{s}cs)$ as defined in Proposition 5.4.5.

PROOF. Both cases follow from Proposition 5.4.5 and (3.1). □

As discussed, we use the term *natural* to mean that a bijection respects the Cambrian recurrence.

THEOREM 5.7.2. *The root configuration induces a natural bijection*

$$\mathsf{Asso}^{(m)}_{\Delta}(W,\mathsf{c}) \overset{c}{\longleftrightarrow} \mathsf{NC}^{(m)}_{\Delta}(W,c)$$
$$I \longmapsto \mathsf{t}_1^{(i_1)} \cdots \mathsf{t}_n^{(i_n)},$$

for $\mathsf{R}(I) = \{\beta_1^{(i_1)}, \dots, \beta_n^{(i_n)}\}$ and $t_k = s_{\beta_k}$ for $1 \le k \le n$.

And... scene.

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PROOF. Immediate from the Cambrian recurrence for noncrossing partitions in Proposition 4.3.3 and the recurrence for root configurations in Proposition 5.7.1. □

THEOREM 5.7.3. *The natural bijection in Theorem 5.7.2 induces a poset isomorphism $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(W,c) \cong \mathsf{Camb}^{(m)}_{\mathsf{NC}}(W,c)$.*

PROOF. By Lemma 3.3.2 and Equation (4.3), flips in $\mathsf{Asso}^{(m)}(W,c)$ are sent to flips in $\mathsf{NC}^{(m)}_{\Delta}(W,c)$ under the bijection of Proposition 5.7.1. □

COROLLARY 5.7.4. *The h-polynomial of $\mathsf{Asso}^{(m)}(W,c)$ is given by $\sum q^{\ell_{\mathcal{R}}(w_m)}$, where the sum ranges over all $(w_1,\dots,w_m) \in \mathsf{NC}^{(m)}(W,c)$.*

PROOF. This follows from Corollary 5.6.3, from Theorem 5.7.3, and from Theorem 4.5.2(ii). □

5.8. The linear m-eralized Cambrian poset and the m-Tamari lattice

We conclude this chapter with a brief combinatorial description of the m-eralized c-Cambrian poset $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(\mathfrak{S}_n, c)$ for the linear Coxeter element

$$c = (1, 2, \ldots, n) = s_1 \cdots s_{n-1} \in \mathfrak{S}_n$$

and a comparison of $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(\mathfrak{S}_n, c)$ with the m-Tamari lattice. We refer for example to [**BPR12**] for a detailed treatment of the latter.

5.8.1. m-eralized clusters and dissections of polygons. Using Corollary 5.5.7, the following description of m-eralized clusters in $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(\mathfrak{S}_n, c)$ for the linear Coxeter element c can be deduced from the description for the bipartite Coxeter element given by S. Fomin and N. Reading in [**FR05**, Section 5.1]. Consider a regular $(mn+2)$-gon and associate to the letter s_i inside the k-th copy of $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2 \ldots \mathsf{s}_{n-1}$ inside the search word $\mathsf{cw}_\circ^m(\mathsf{c})$ the diagonal between vertices

$$m(k-1)+1 \quad \text{and} \quad m(k+i-1)+2,$$

where the indices are considered cyclically.

Example 5.8.1. For $n = 3$ and $m = 2$, we obtain the identification between letters and diagonals of an octagon as

s	t	s	t	s	t	s	t
14	16	36	38	58	52	72	74

.

For $n = 4$ and $m = 2$, we obtain the identification between letters and diagonals of an decagon as

1	1	1	2	2	2	3	3	4	3	4	5	4	5	5
s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_1	s_3	s_2	s_1	s_3	s_2	s_3
14	16	18	36	38	30	58	50	70	52	72	92	74	94	96

,

where the first row indicates the copy of c in which the letter sits, and where a 0 in the last row denotes the tenth vertex.

We then have the following characterization of the m-eralized cluster complex $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(\mathfrak{S}_n, c)$.

Theorem 5.8.2. *The m-eralized c-cluster complex $\mathsf{Asso}^{(m)}_{\Delta}(\mathfrak{S}_n, c)$ for the Coxeter element $c = s_1 s_2 \ldots s_{n-1}$ is isomorphic to the complex of dissections of a regular $(mn+2)$-gon into $(m+2)$-gons. The isomorphism is given by the above identification between letters in the search word for $\mathsf{Asso}^{(m)}_{\Delta}(\mathfrak{S}_n, c)$ and diagonals inside the regular polygon.*

Proof. This is a direct consequence of the isomorphism constructed in [**FR05**, Section 5.1]. We leave the details to the reader. □

Observe that this identification also yields a description of the m-eralized Cambrian poset in this case in terms of rotations of diagonals inside the polygon. An example is shown in Figure 15. The reader is invited to check this isomorphism in the two examples above.

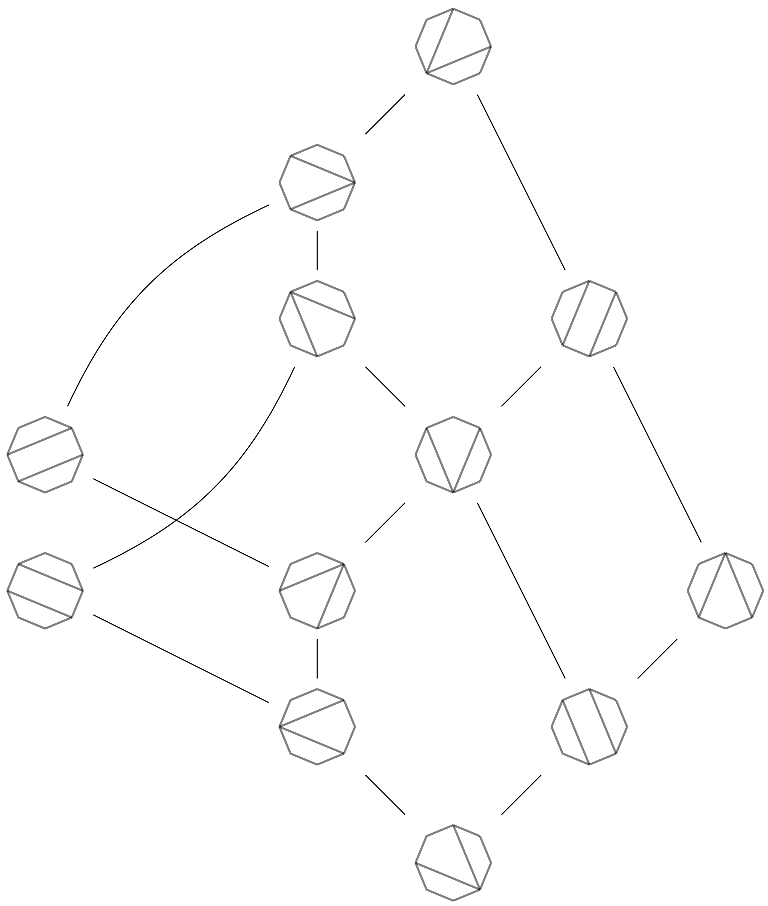

FIGURE 15. The linear Cambrian poset $\mathsf{Camb}^{(2)}_{\mathsf{Asso}}(\mathfrak{S}_3, st)$. Elements are quadrangulations of an octogon, and cover relations are given by increasing rotations of a diagonal.

5.8.2. Comparison to the m-Tamari lattice. An *m-Dyck path* is a lattice path from $(0,0)$ to (mn,n) with east and north steps, never going below the line betwen them. The *m-Tamari lattice* on Dyck paths is defined by cover relations: whenever there is an east step followed immediately by a north step, a line is drawn with slope $1/m$ from their common endpoint upwards until it first touches the Dyck path again. The cover relation consists of swapping the east step with the subpath cut out by the line segment.

The 1-Tamari lattice is isomorphic to the 1-Cambrian poset with linear Coxeter element, but this is no longer the case for $m > 1$ and $n > 2$.

OBSERVATION 5.8.3. *The 2-Cambrian poset $\mathsf{Camb}^{(2)}_{\mathsf{Asso}}(\mathfrak{S}_3, st)$ does not coincide with the 2-Tamari lattice for $n = 3$. The latter is illustrated in Figure 16, while $\mathsf{Camb}^{(2)}_{\mathsf{Asso}}(\mathfrak{S}_3, st)$ is shown in Figure 15.*

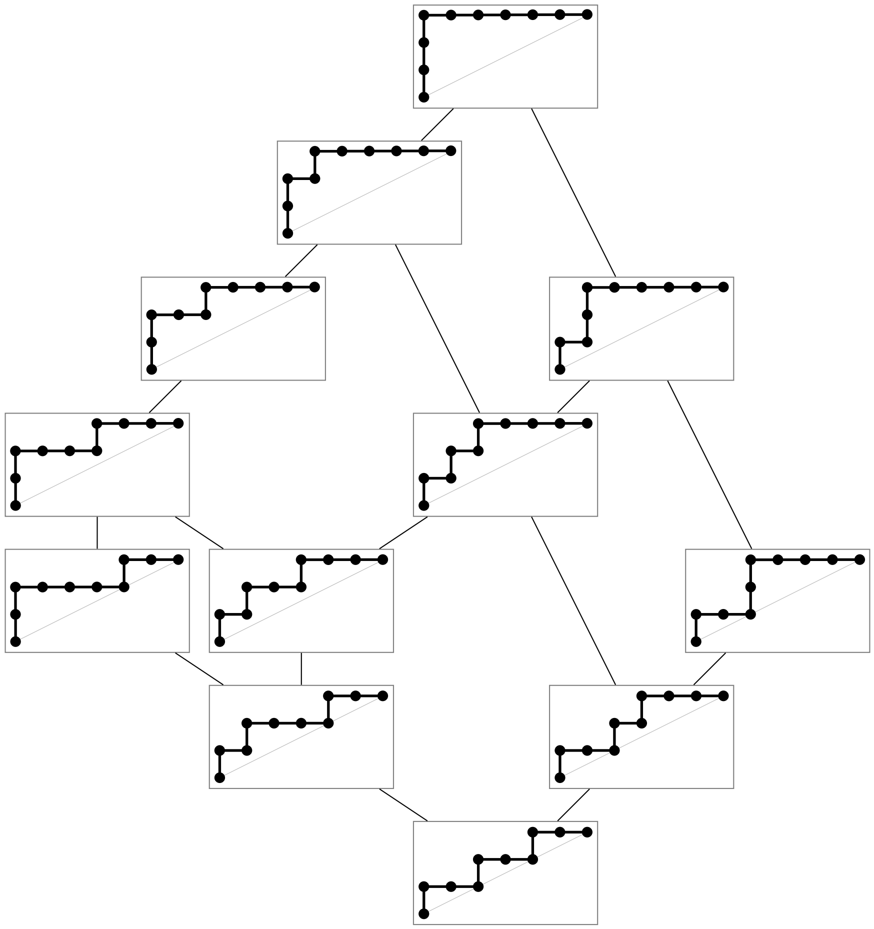

FIGURE 16. The 2-Tamari lattice on the 2-Dyck paths.

While these two posets are generally not isomorphic, there is nevertheless the following surprising connection.

THEOREM 5.8.4. *We have*

$$\sum_I q^{\#upper\ covers\ of\ I} = \sum_D q^{\#upper\ covers\ of\ D},$$

where the first sum ranges over all m*-eralized clusters using the cover relation in* $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(\mathfrak{S}_n, c)$*, and the second sum ranges over all* m*-Dyck paths using the cover relation in the* m*-Tamari lattice of Dyck paths from* $(0,0)$ *to* (mn, n)*.*

PROOF. The number of upper covers of an m-Dyck path in m-Tamari lattice is given by its number of valleys. The generating function of all m-Dyck paths by their number of valleys is well-known to be the m-Narayana polynomial. The m-Narayana polynomial is also equal to the h-polynomial of $\mathsf{Asso}^{(m)}(\mathfrak{S}_n, c)$, see Corollary 5.6.3 and [**FR05**]. And we have seen in Corollary 5.6.3, that this equals the generating function of the number of upper covers in $\mathsf{Camb}^{(m)}_{\mathsf{Asso}}(\mathfrak{S}_n, c)$. (Alternatively, one can deduce this second Narayana count also from Corollary 5.7.4 and [**Arm06**, Section 3.5].) □

We conjecture that there is an even deeper relationships between the two posets. M. Bousquet-Mélou, E. Fusy, and L.-F. Préville-Ratelle [**BMFPR12**] proved that the number of intervals in the m-Tamari lattice is

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}.$$

In a subsequent paper, M. Bousquet-Mélou, G. Chapuy, and L.-F. Préville-Ratelle proved that the number of labelled intervals in the m-Tamari lattice—where the top element of the interval is labelled by a parking function compatible with the corresponding Dyck path—equals $(m+1)^n(mn+1)^{n-2}$ [**BMCPR13**]. Labelling $(w_1 \geq_{\mathcal{R}} w_2 \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} w_m) \in \mathsf{NC}^{(m)}(W, c)$ by cosets of the parabolic subgroup $W_{\mathrm{Fix}(w_m)}$ [**Rho14**], we conjecture the following relationship between the number of intervals and labelled intervals.

Conjecture 5.8.5. Let $c = (1, 2, \ldots, n) \in \mathfrak{S}_n$ be the linear Coxeter element. Then

- the number of intervals in the m-Tamari lattice equals the number of intervals in the m-eralized Cambrian poset, and
- the number of intervals in the lattices also coincide if each interval is weighted by the number of parking functions or, respectively, by the number of cosets of the top element.

Noncrossing q, t-Catalan combinatorics?

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We also conjecture the following refined version of the second interval enumeration.

Conjecture 5.8.6. For any integer partition $\lambda \vdash n$, there are the same number of intervals $\alpha \leq \beta$

- in the m-eralized Cambrian poset for which λ records the sorted block sizes of the last component δ_m of the m-delta sequence $\beta = (\delta_0, \delta_1, \ldots, \delta_m)$, as
- in the m-Tamari lattice for which λ records the sorted sizes of the vertical runs of the m-Dyck path β.

Both conjectures have been verified for $\mathfrak{S}_n$ for $n \in \{3, 4\}$ and $m \in \{2, 3, 4\}$.

CHAPTER 6

Sortable elements

In this chapter, we study m-eralized sortable elements as certain elements in the m-eralized weak order. After giving the general definition (Section 6.1), we review the known theory of sortable elements in Coxeter groups (Section 6.2). Detailed background can be found in [**Rea06, Rea07a, Rea07b**]. We discuss the Cambrian recurrence on m-eralized sortable elements (Section 6.3). We describe sortable elements in terms of their Garside factors (Section 6.4), define the Cambrian poset on sortable elements, and show that this poset is a lattice (Section 6.6). We construct natural bijections between m-eralized sortable elements, m-eralized noncrossing partitions, and m-eralized clusters (Section 6.8). We connect m-eralized sortable elements with chains in the shard intersection order (Section 6.9).

This chapter might get a bit technical, so please buckle up.

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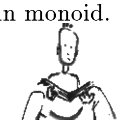
6.1. m-eralized sortable elements

N. Reading introduced c-sortable elements in [**Rea06**] as a subset of elements of the Coxeter group W. His notion extends verbatim to elements of the positive Artin monoid $\boldsymbol{B}^+$. Let c be a Coxeter element with reduced $\mathcal{S}$-word c. For w an element of W or $\boldsymbol{B}^+$, recall from Definition 2.6.1 that the c-sorting word is the lexicographically first subword of c^∞ that is a reduced $\mathcal{S}$-word for w.

Definition 6.1.1. An element w of W or $\boldsymbol{B}^+$ is *c-sortable* if its c-sorting word $\mathsf{w}(\mathsf{c}) = \mathsf{c}\big|_{I_1}\mathsf{c}\big|_{I_2}\cdots\mathsf{c}\big|_{I_k}$ satisfies $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$.

This is the definition you probably know, but now in the positive Artin monoid.



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By Lemma 2.6.3, $\mathsf{w}_\circ(\mathsf{c})$ is initial in c^∞. Hence, $w_\circ$ is c-sortable. Since $\boldsymbol{w}_\circ^2 = \boldsymbol{c}^h \in \boldsymbol{B}^+$, we conclude that that $\boldsymbol{w}_\circ^m$ is c-sortable for any positive integer m and any Coxeter element c.

Example 6.1.2. We illustrate Definition 6.1.1 with the following non-example. In $\mathsf{Weak}^{(2)}(\mathfrak{S}_4)$ with $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$, the element $w = s_1s_2s_3s_1s_2 \cdot s_3s_2s_1$ (where the dot denotes the separation of Garside factors), has c-sorting word

$$\mathsf{w}(\mathsf{c}) = \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ s_1 & s_2 & s_3 & s_1 & s_2 & s_3 & - & s_2 & - & s_1 & - & - \end{array}\right).$$

It is not c-sortable since s_1 occurs in the fourth but not the third copy of c. Note that the vertical bars here serve only to distinguish the copies of c.

Although the word $\mathsf{w}(\mathsf{c})$ depends on a particular choice of a reduced word c for the Coxeter element c, the property of being c-sortable does not by Lemma 2.6.5(1).

We write $\mathsf{Sort}(W, c)$ for the set of c-sortable elements in the Coxeter group, $\mathsf{Sort}(\boldsymbol{B}^+, c)$ for those in the positive Artin monoid, and $\mathsf{Sort}^{(m)}(W, c)$ for the restriction of $\mathsf{Sort}(\boldsymbol{B}^+, c)$ to the interval $W^{(m)} = [\boldsymbol{e}, \boldsymbol{w}_\circ^m]$. We characterize m-eralized sortable elements using Garside factorizations in Definition 6.4.1. Since

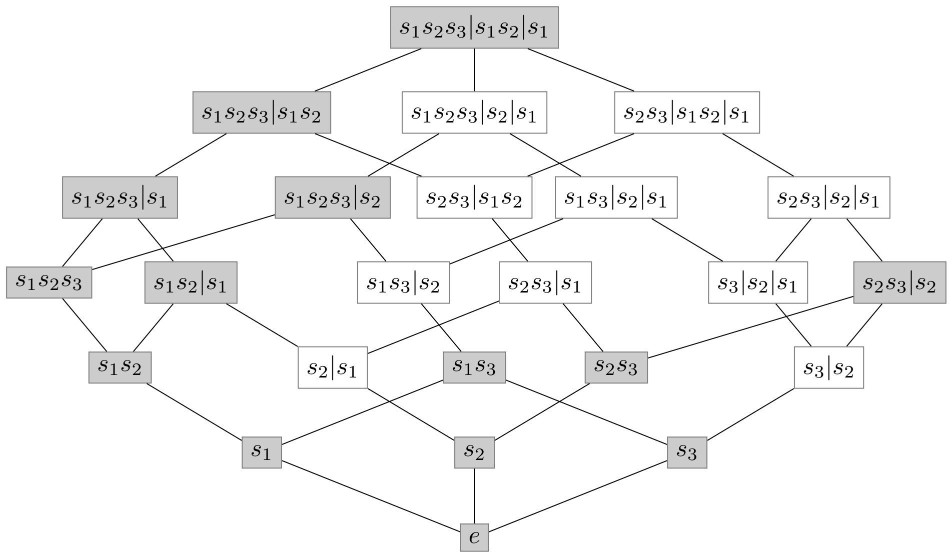

FIGURE 17. The weak order $\mathsf{Weak}(\mathfrak{S}_4)$. The $s_1s_2s_3$-sortable elements are shaded.

$\mathsf{Sort}^{(m)}(W,c) \subseteq \boldsymbol{B}^+$, sortable elements inherit the notion of *support* from Section 2.2.

Figure 17 illustrates the weak order of $\mathfrak{S}_4$, with the $s_1s_2s_3$-sortable elements shaded. $\mathsf{Sort}^{(2)}(\mathfrak{S}_3)$ is illustrated in Figure 9 on page 34, and the first, second, and fifth columns of Figure 18 list the Garside factorizations, sorting words, and supports of the 12 elements in $\mathsf{Sort}^{(2)}(\mathfrak{S}_2, st)$.

6.2. Classical sortable elements

In this section, we review known properties of sortable elements.

6.2.1. Cambrian recurrence. There is an immediate defining recurrence for $\mathsf{Sort}(W,c)$ from Definition 6.1.1, called the *c-Cambrian recurrence*.

PROPOSITION 6.2.1 (N. Reading [**Rea06**, Lemma 2.1 and 2.2]). *Let s be initial in c. Then*

$$w \in \mathsf{Sort}(W,c) \Leftrightarrow \begin{cases} w \in \mathsf{Sort}(W_{\langle s \rangle}, \overline{s}c) & \text{if } s \in \mathsf{asc}_L(w) \\ \overline{s}w \in \mathsf{Sort}(W, \overline{s}cs) & \text{if } s \in \mathsf{des}_L(w) \end{cases}. \qquad \square$$

6.2.2. Lattice properties of sortable elements. The following properties of sortable elements are most easily explained using the following characterization of the sortable property in the language of biclosed sets from (2.6).

THEOREM 6.2.2 ([**RS11**, Theorem 4.3]). *An element $w \in W$ is c-sortable for a Coxeter element $c \in W$ if and only if, for $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = a\alpha + b\beta$ with $a, b \in \mathbb{R}_+$, we have*

$$\gamma \in \mathrm{inv}(w) \Rightarrow \min_{\leq_c}\{\alpha, \beta\} \in \mathrm{inv}(w),$$

where $\leq_c$ *is the root order induced by the* c*-sorting word* $\mathsf{w}_\circ(\mathsf{c})$. *In particular,* $w_\circ \in \mathsf{Sort}(w, c)$. □

The following lemma is an immediate consequence of Theorem 6.2.2.

LEMMA 6.2.3. *For* $w \in \mathsf{Sort}(W, c)$ *and* $J \subseteq \mathcal{S}$,

$$w_J \in \mathsf{Sort}\left(W_J, c\big|_J\right). \qquad \square$$

Under the inclusion $W_{\langle s \rangle} \subset W$, a sortable element in $W_{\langle s \rangle}$ is again sortable in W. The following lift from $W_{\langle s \rangle}$ to W is more subtle.

LEMMA 6.2.4 (N. Reading [**Rea07b**, Lemma 2.8 and 2.9]). *For* s *initial in* c, *if* $w \in \mathsf{Sort}(W_{\langle s \rangle}, \bar{s}c)$, *then* $w \vee s$ *is both* c*-sortable and* $\bar{s}cs$*-sortable with*

$$\mathsf{cov}_\downarrow(w \vee s) = \mathsf{cov}_\downarrow(w) \cup \{s\}. \qquad \square$$

The proofs of the m-eralized statements to Lemmas 6.2.3 and 6.2.4 will occupy many, many pages.

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6.2.3. The Cambrian lattice. The *c-Cambrian lattice* $\mathsf{Camb}_{\mathsf{Sort}}(W, c)$ is the restriction of the weak order $\mathsf{Weak}(W)$ to $\mathsf{Sort}(W, c)$. The inversion set of the meet of two c-sortable elements in the weak order on W is simply the intersection of their inversion sets, which again directly follows from Theorem 6.2.2.

PROPOSITION 6.2.5. *Let* $w, u \in \mathsf{Sort}(W, c)$. *Then* $u \wedge v \in \mathsf{Sort}(W, c)$ *and*

$$\mathrm{inv}(u \wedge v) = \mathrm{inv}(u) \cap \mathrm{inv}(v). \qquad \square$$

Although the join does not enjoy such a simple description in terms of inversion sets, if $w, u \in \mathsf{Sort}(W, c)$, then also $u \vee v \in \mathsf{Sort}(W, c)$. We conclude the following theorem [**RS11**, Theorem 7.3].

THEOREM 6.2.6. $\mathsf{Camb}_{\mathsf{Sort}}(W, c)$ *is a sublattice of* $\mathsf{Weak}(W)$. □

We m-eralize Proposition 6.2.5 in Lemma 6.6.3 and m-eralize Theorem 6.2.6 in Theorem 6.6.4.

6.2.4. Cambrian rotation. The following *c-Cambrian rotation* is well-defined by Proposition 6.2.1 and Lemma 6.2.4:

$$\begin{aligned} \mathsf{Shift}_s : \mathsf{Sort}(W, c) &\longrightarrow \mathsf{Sort}(W, \bar{s}cs) \\ w &\longmapsto \begin{cases} w \vee s & \text{if } s \in \mathsf{asc}_L(w) \\ \bar{s}w & \text{if } s \in \mathsf{des}_L(w) \end{cases}. \end{aligned}$$

6.2.5. Sortable elements, noncrossing partitions, and clusters. By associating a sortable element to its cover reflections, N. Reading gave a natural bijection between c-sortable elements and c-noncrossing partitions [**Rea07a**, Theorem 6.1]. This bijection is given by

$$\begin{aligned} \mathsf{Sort}(W, c) &\overset{c}{\longleftrightarrow} \mathsf{NC}(W, c) \\ w &\longmapsto r_1 \cdots r_k, \end{aligned} \tag{6.1}$$

where $\mathsf{cov}_\downarrow(w) = \{r_1, \ldots, r_k\}$ with $r_1 <_{\mathsf{c}} \cdots <_{\mathsf{c}} r_k$. N. Reading and D. Speyer gave an alternative description of this bijection using the notion of a *skip set* [**RS11**, Section 5], which we m-eralize in Theorem 6.8.5.

N. Reading proved the existence of the compatibility $\|_c$ by showing that it arises naturally from his theory of sortable elements [**Rea07a**], providing a bijection between sortable elements and clusters.

$\mathsf{Sort}^{(2)}(\mathfrak{S}_3, st)$	$\mathsf{Sort}^{(2)}_{\mathsf{fact}}(\mathfrak{S}_3, st)$	$\mathsf{Sort}^{(2)}_{\mathsf{shard}}(\mathfrak{S}_3, st)$	$\mathcal{C}_c(w)$	$\mathrm{supp}(w)$
st|st|st	e	$e{\geq_{\mathrm{Sh}}}e$	$\alpha^{(0)}, \beta^{(0)}$	–
st|st|st	$sts \cdot tst$	$sts{\geq_{\mathrm{Sh}}}sts$	$\alpha^{(2)}, \beta^{(2)}$	s, t
st|st|st	$sts \cdot t$	$sts{\geq_{\mathrm{Sh}}}s$	$\gamma^{(1)}, \alpha^{(2)}$	s, t
st|st|st	st	$st{\geq_{\mathrm{Sh}}}e$	$\beta^{(0)}, \gamma^{(1)}$	s, t
st|st|st	s	$s{\geq_{\mathrm{Sh}}}e$	$\gamma^{(0)}, \alpha^{(1)}$	s
st|st|st	$t \cdot t$	$t{\geq_{\mathrm{Sh}}}t$	$\alpha^{(0)}, \beta^{(2)}$	t
st|st|st	$sts \cdot ts$	$sts{\geq_{\mathrm{Sh}}}st$	$\beta^{(1)}, \gamma^{(2)}$	s, t
st|st|st	sts	$sts{\geq_{\mathrm{Sh}}}e$	$\alpha^{(1)}, \beta^{(1)}$	s, t
st|st|st	t	$t{\geq_{\mathrm{Sh}}}e$	$\alpha^{(0)}, \beta^{(1)}$	t
st|st|st	$st \cdot t$	$st{\geq_{\mathrm{Sh}}}st$	$\beta^{(0)}, \gamma^{(2)}$	s, t
st|st|st	$s \cdot s$	$s{\geq_{\mathrm{Sh}}}s$	$\gamma^{(0)}, \alpha^{(2)}$	s
st|st|st	$sts \cdot s$	$sts{\geq_{\mathrm{Sh}}}t$	$\alpha^{(1)}, \beta^{(2)}$	s, t

FIGURE 18. The three variants of the m-eralized st-sortable elements for $\mathfrak{S}_3$ with $m = 2$, together with their skip sets and their supports. They are arranged according to their orbits under Cambrian rotation, defined in Section 6.7.

6.3. The Cambrian recurrence

There is a simple inductive characterization of $\mathsf{Sort}^{(m)}(W, c)$ called the *m-eralized c-Cambrian recurrence*, which follows directly from Lemma 2.6.5(7) and Lemma 2.9.1.

PROPOSITION 6.3.1. *Let s be initial in c. Then*

$$w \in \mathsf{Sort}^{(m)}(W, c) \Leftrightarrow \begin{cases} w \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle}, \bar{s}c) & \text{if } s \in \mathsf{asc}_L(w) \\ \bar{s}w \in \mathsf{Sort}^{(m)}(W, \bar{s}cs) & \text{if } s \in \mathsf{des}_L(w) \end{cases}. \qquad \square$$

EXAMPLE 6.3.2. Parallel to Examples 4.3.4 and 5.4.6, the Cambrian recurrence for $sts \cdot s \in \mathsf{Sort}^{(2)}(\mathfrak{S}_3, st)$ is

$$\underbrace{sts \cdot s}_{st} \mapsto \underbrace{ts \cdot s}_{ts} \mapsto \underbrace{s \cdot s}_{st} \mapsto \underbrace{s}_{ts} \mapsto \underbrace{s}_{s} \mapsto \underbrace{e}_{s} \mapsto \underbrace{e}_{e},$$

where the subscript identifies the (parabolic) Coxeter element.

6.4. Factorwise sortable elements

Didn't see that one coming, did you?

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We now give a description of m-eralized c-sortable elements using their Garside factorizations. Since each Garside factor may be interpreted as an element of the group W, this relates m-eralized c-sortable elements to tuples of elements of W.

Let c be a Coxeter element with word c, and let $w \in W$ with $\mathsf{des}_R(w) = \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\}$ ordered so that

$${}^{w}s_{i_1} <_{\mathsf{c}} {}^{w}s_{i_2} <_{\mathsf{c}} \cdots <_{\mathsf{c}} {}^{w}s_{i_k}.$$

Define the *twisted restriction* of a Coxeter element c with respect to the element w to be the Coxeter element $c|^{w} := s_{i_1}s_{i_2}\cdots s_{i_k}$ of the parabolic subgroup $W_{\mathsf{des}_R(w)}$.

DEFINITION 6.4.1. An element $w \in W^{(m)}$ with $\mathrm{garside}(w) = w^{(1)} \cdot \cdots \cdot w^{(m)}$ is *factorwise c-sortable* if $w^{(i)}$ is $c^{(i)}$-sortable for all $1 \leq i \leq m$, where we set $\mathsf{c}^{(1)} = \mathsf{c}$ and $\mathsf{c}^{(i)} := \mathsf{c}^{(i-1)}\Big|^{w^{(i-1)}}$ for $1 < i \leq m$. We denote the set of factorwise c-sortable elements by $\mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c)$.

Let this definition gently wash over you.

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EXAMPLE 6.4.2. Let $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3 \in \mathfrak{S}_4$ and let $w \in \mathfrak{S}_4^+$ have the Garside factorization

$$w^{(1)} \cdot w^{(2)} = s_1s_2s_3s_2 \cdot s_3s_2 = s_1s_3s_2s_3 \cdot s_3s_2.$$

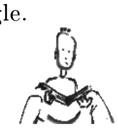
Since $\mathsf{des}(w^{(1)}) = \{s_3, s_2\}$ with $s_3^{w^{(1)}} = (13) <_{\mathsf{c}^{(1)}} (34) = s_2^{w^{(1)}}$, we have

$$\mathsf{c}^{(2)} = \mathsf{c}^{(1)}\Big|^{w^{(1)}} = \mathsf{s}_3\mathsf{s}_2.$$

Then w is factorwise c-sortable, because $w^{(1)}$ is $c^{(1)}$-sortable and $w^{(2)}$ is $c^{(2)}$-sortable.

6.4.1. Sortable and factorwise sortable elements. We show that the factorwise sortable elements $\mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c)$ coincides with $\mathsf{Sort}^{(m)}(W,c)$ by proving that factorwise c-sortable elements satisfy the Cambrian recurrence.

Be prepared: there is a beautiful beach to reach, but the path goes through a deep jungle.



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PROPOSITION 6.4.3. *Let s be initial in c. Then*

$$w \in \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c) \Leftrightarrow \begin{cases} w \in \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W_{\langle s\rangle}, \bar{s}c) & \text{if } s \in \mathsf{asc}_L(w) \\ \bar{s}w \in \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W, \bar{s}cs) & \text{if } s \in \mathsf{des}_L(w) \end{cases}.$$

PROOF. Let $w \in \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c)$ and suppose

$$\mathrm{garside}(w) = w^{(1)} \cdot w^{(2)} \cdot \cdots \cdot w^{(m)},$$

where some of these factors may be the identity.

If $s \in \mathsf{asc}_L(w) = \mathsf{asc}_L(w^{(1)})$, then $w^{(1)}$ is c-sortable by assumption, and Proposition 6.2.1 implies that $s \notin \mathrm{supp}(w^{(1)})$, so that $s \notin \mathsf{des}_R(w^{(1)})$. We conclude that $s \notin \mathrm{supp}(w)$ since all further Garside factors $w^{(2)}, \ldots, w^{(m)}$ are inside $W_{\mathsf{des}_R(w^{(1)})}$, implying that w is factorwise $\bar{s}c$-sortable as an element of $W^{(m)}_{\langle s\rangle}$ by Lemma 2.9.1. The converse direction also follows from Lemma 2.9.1.

Otherwise, $s \in \mathsf{des}_L(w)$ so that $w = su$ is reduced. We set $\mathrm{garside}(u) = u^{(1)} \cdot u^{(2)} \cdot \cdots \cdot u^{(m)}$, and thus need to show that

$$w \in \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c) \Leftrightarrow u \in \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W, \bar{s}cs).$$

If $su^{(1)} \leq w_\circ$, then $\mathrm{garside}(w) = su^{(1)} \cdot u^{(2)} \cdot \cdots \cdot u^{(m)}$ by Theorem 2.5.1, since $\mathsf{des}_R(su^{(1)}) \supseteq \mathsf{des}_R(u^{(1)})$. By Proposition 6.3.1, $w^{(1)}$ is c-sortable if and only if $\bar{s}w^{(1)} = u^{(1)}$ is $\bar{s}cs$-sortable, and we conclude this case.

We consider both implications individually in the final case $su^{(1)} \not\leq w_\circ$, so that $s \in \mathsf{des}_L(u^{(1)})$.

Suppose $u \in \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W, \bar{s}cs)$, and let $\bar{\mathsf{s}}\mathsf{cs}$ be a word for $\bar{s}cs$ ending in the letter s. Since s is final in the reflection order associated to $\bar{\mathsf{s}}\mathsf{cs}$ and $u^{(1)}$ is $\bar{s}cs$-sortable, the simple reflection s is the final reflection in the inversion sequence $\mathsf{inv}(\mathsf{u}^{(1)}(\bar{\mathsf{s}}\mathsf{cs}))$, and so corresponds to the final simple reflection r in $\mathsf{u}^{(1)}(\bar{\mathsf{s}}\mathsf{cs})$. More succinctly, we have the reduced expression $u^{(1)}r = su^{(1)}$, and obtain $u^{(1)} = su^{(1)}r^{-1}$. In this case, set $w^{(1)} = u^{(1)} = su^{(1)}r^{-1}$, which is c-sortable by Proposition 6.3.1.

We drop the first Garside factor and repeat the preceding argument on the element $ru^{(2)}\cdots u^{(m)}$, obtaining an element of $\mathsf{Sort}^{(m-1)}_{\mathsf{fact}}(W,c)$ with Garside factorization $w^{(2)}\cdot\ \cdots\ \cdot w^{(m)}$. We now claim that $w^{(1)}\cdot w^{(2)}\cdot\ \cdots\ \cdot w^{(m)}$ is the Garside factorization of w. We did not change the first Garside factor $w^{(1)}=u^{(1)}$ since $r\in\mathsf{des}_R(u^{(1)})$ and $\mathsf{des}_L(ru^{(2)})\subseteq\{r\}\cup\mathsf{des}_L(u^{(2)})$ (by Proposition 2.3.1). $w^{(2)}$ thus lies in $W\Big|^{u^{(1)}}$. We conclude that $w^{(1)}$ was indeed the first Garside factor of w. The result follows by induction on the number of Garside factors of w.

Now suppose $w\in\mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c)$. Running the argument above in reverse, we obtain the candidate Garside factorization of u, $u^{(1)}=w^{(1)}$ and $u^{(2)}\cdot\ \cdots\ \cdot u^{(m)}=r^{-1}w^{(2)}\cdots w^{(m)}$, where $u^{(1)}r=su^{(1)}$. Since $\mathsf{des}_R(u^{(1)})=\mathsf{des}_R(w^{(1)})$, and since $w^{(2)}$ only uses simple reflections in $\mathsf{des}_R(w^{(1)})$, $u^{(1)}$ is indeed the first Garside factor. The result again follows by induction on the number of Garside factors of w. □

Corollary 6.4.4. $\mathsf{Sort}^{(m)}(W,c)=\mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c)$.

Like, equal as sets of elements of $\boldsymbol{B}^+$.

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Proof. Since both $\mathsf{Sort}^{(m)}(W,c)$ and $\mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c)$ satisfy the same recurrence and initial conditions, and element $w\in W^{(m)}$ is c-sortable if and only if it is factorwise c-sortable. □

Example 6.4.5. By Example 6.4.2, the element $w=w^{(1)}\cdot w^{(2)}=s_1s_2s_3s_2\cdot s_3s_2\in\mathfrak{S}_4$ is factorwise c-sortable for $c=s_1s_2s_3$. Its c-sorting word is

$$\mathsf{w}(\mathsf{c})=\left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3\\ s_1 & s_2 & s_3 & - & s_2 & s_3 & - & s_2 & - & - & - & - \end{array}\right).$$

By Definition 6.1.1, this element is also c-sortable.

6.4.2. Sorting and factorwise sorting words. Not only do the notions of sortable and factorwise sortable agree, it turns out that the c-sorting word of a c-sortable element w is *commutation equivalent* to the concatenation of the individual sorting words of the Garside factors of w.

For $w\in\mathsf{Sort}^{(m)}(W,c)$, define

$$\mathrm{garside}(\mathsf{w})(\mathsf{c}):=\left[\mathsf{w}^{(1)}(\mathsf{c}^{(1)})\right]\cdot\left[\mathsf{w}^{(2)}(\mathsf{c}^{(2)})\right]\cdot\ \cdots\ \cdot\left[\mathsf{w}^{(m)}(\mathsf{c}^{(m)})\right]$$

to be the concatenation of the $\mathsf{c}^{(i)}$-sorting words defined from the Garside factors of w as in Definition 6.4.1.

Proposition 6.4.6. *For $w\in\mathsf{Sort}^{(m)}(W,c)$ with sorting word $\mathsf{w}(\mathsf{c})$, we have*

$$\mathrm{garside}(\mathsf{w})(\mathsf{c})\equiv\mathsf{w}(\mathsf{c}).$$

Proof. Let $w\in\mathsf{Sort}^{(m)}(W,c)$ with s initial in c and $s\in\mathsf{des}_L(w)$. If we let $w=su$, then the proof of Proposition 6.4.3 shows that the Garside factorization of u is obtained by removing the initial s from the Garside factorization of w and performing commutations. The same relationship is trivially true for the sorting words of w and u. The result now follows from the Cambrian recurrence. □

This proposition is a generalization of Lemma 2.6.5(5) to arbitrary sortable elements—applying Proposition 6.4.6 to $\boldsymbol{w}_\circ^2\in\mathsf{Sort}^{(2)}(w,c)$ gives

$$\mathsf{w}_\circ(\mathsf{c})\cdot\mathsf{w}_\circ(\psi(\mathsf{c}))=\mathrm{garside}(\mathsf{w}_\circ^2)(\mathsf{c})\equiv\mathsf{w}_\circ^2(\mathsf{c})=\mathsf{c}^h.$$

EXAMPLE 6.4.7. In $\mathfrak{S}_4$ with $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$, we compute

$$\begin{aligned}\text{garside}(\mathsf{w}_\circ^2)(\mathsf{c}) = \mathsf{w}_\circ(\mathsf{c}) \cdot \mathsf{w}_\circ(\psi(\mathsf{c})) = &\quad (\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3|\mathsf{s}_1\mathsf{s}_2|\mathsf{s}_1) \cdot (\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1|\mathsf{s}_3\mathsf{s}_2|\mathsf{s}_3) &= \mathsf{w}_\circ^2(\mathsf{c})\\ \equiv &\quad \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3|\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3|\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3|\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3 &= \mathsf{c}^4\end{aligned}$$

As a corollary to Proposition 6.4.6, the color of a reflection in $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}(\mathsf{c}))$ for $w \in \mathsf{Sort}^{(m)}(W, c)$ matches the Garside factor containing the corresponding letter in garside $\mathsf{w}(\mathsf{c})$.

COROLLARY 6.4.8. *Let* $w \in \mathsf{Sort}^{(m)}(W, c)$ *with* $\mathsf{w}(\mathsf{c}) = \mathsf{s}_1\mathsf{s}_2 \cdots \mathsf{s}_p$ *and* $\mathsf{inv}(\mathsf{w}(\mathsf{c})) = \left(\beta_1^{(i_1)}, \ldots, \beta_p^{(i_p)}\right)$. *Then the letter* s_a *belongs to* $\mathsf{w}^{(j+1)}(\mathsf{c}^{(j+1)})$ *under the identification* $\mathsf{w}(\mathsf{c}) \equiv \text{garside}(\mathsf{w})(\mathsf{c})$ *if and only if* $i_a = j$. □

6.5. Lattice properties of m-eralized sortable elements

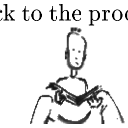
In this section, we m-eralize both Lemma 6.2.3 and Lemma 6.2.4. The proofs of these theorems are somewhat technical.

6.5.1. Projections of sortable elements to parabolic subgroups. We first m-eralize Lemma 6.2.3 to m-eralized sortable elements and their colored inversion sets.

THEOREM 6.5.1. *For* $w \in \mathsf{Sort}^{(m)}(W, c)$, *define* $w_J := w \wedge w_\circ^m(J)$. *Then*

$$w_J \in \mathsf{Sort}^{(m)}(W_J, c\big|_J) \text{ and } \operatorname{inv}(w_J) = \operatorname{inv}(w)\big|_J.$$

You may also come back to the proof later.



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"Later."

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PROOF. For each $m \geq k \geq 0$, define $w_k^m(J) := w_\circ^k w_\circ(\psi^k(J))^{m-k}$ and

$$w(k) := w \wedge w_k^m(J)$$

with Garside factorization $w(k) = w_k^{(1)} \cdot \cdots \cdot w_k^{(m)}$. By construction of Garside factorization, the first k Garside factors of $w(k)$ agree with those of w.

Define J_k to be the set of right descents s_i of $w_k^{(k)}$ such that $w_k^{(1)} \cdots w_k^{(k)}(\alpha_{s_i}) \in \Phi_J$. By convention, we set $w_m^{(m+1)} = e$ and $J_m = \emptyset$. We prove the following statements by decreasing induction on k:

- $w_k^{(k+1)} \cdots w_k^{(m)}$ uses only letters from J_k,
- the inversions in $\operatorname{inv}(w(k))$ of color weakly greater than k are exactly those inversions in $\operatorname{inv}(w)$ of color weakly greater than k that lie in Φ_J, and
- the element $w(k)$ is factorwise c-sortable.

The base case $k = m$ follows from the fact that $w(m) = w$ and w has no inversions of color greater than m. Suppose we have shown the statements for k; we will show them for $k - 1$.

Writing the parabolic decomposition of $w^{(k)}$ with respect to J_{k-1} from Equation (2.3) as $w^{(k)} = (w^{(k)})_{J_{k-1}}(w^{(k)})^{J_{k-1}}$, we have

$$\begin{aligned} w(k) &= w^{(1)} \cdot \cdots \cdot w^{(k-1)} \cdot w^{(k)} \cdot w_k^{(k+1)} \cdot \cdots \cdot w_k^{(m)} \\ &= w^{(1)} \cdot \cdots \cdot w^{(k-1)} \cdot \left[(w^{(k)})_{J_{k-1}}(w^{(k)})^{J_{k-1}}\right] w_k^{(k+1)} \cdot \cdots \cdot w_k^{(m)}. \end{aligned}$$

As $w_{k-1}^m(J)$ is initial in $w_k^m(J)$, we have that $w(k-1) = w(k) \wedge w_{k-1}^m(J)$. Since $w(k-1)$ is an initial segment of $w(k)$, $\operatorname{inv}(w(k-1))\big|_J \subseteq \operatorname{inv}(w(k))\big|_J$. We will conclude equality by showing that there are exactly the right number of inversions in $\mathrm{inv}(w(k-1))$. Since $w(k)$ and $w(k-1)$ share their first $k-1$ Garside factors, we are only concerned with the remaining $m-k+1$ factors.

By induction, every inversion corresponding to a letter in $w_k^{(k+1)} \cdots w_k^{(m)}$ lies in Φ_J. To show that these are again inversions in $w(k-1)$, we must check that every letter in $w_k^{(k+1)} \cdots w_k^{(m)}$ can move past $(w^{(k)})^{J_{k-1}}$ as a simple reflection in $\mathcal{S}_{J_{k-1}}$.

By induction, $w_k^{(k+1)} \cdots w_k^{(m)}$ uses only letters from J_k and $w(k)$ is factorwise c-sortable. Fix s_i a letter used in $w_k^{(k+1)} \cdots w_k^{(m)}$. By definition of factorwise c-sortablility, $s_i \in \mathsf{des}_R(w^{(k)})$. By the inductive hypothesis, when computing $\mathrm{inv}(w(k))$, the uncolored root corresponding to this s_i lies in Φ_J.

Since $s_i \in \mathsf{des}_R(w^{(k)})$, it must be that $(w^{(1)} \cdot \cdots \cdot w^{(k)})(\alpha_{s_i})$ occurs among the roots corresponding to $(w^{(k)})_{J_{k-1}}$ in $\mathrm{inv}(w(k))$. Furthermore, since this root can be removed from $\mathrm{inv}(w^{(k)})$ while preserving the biclosed property, it can also be removed from $\mathrm{inv}((w^{(k)})_{J_{k-1}})$.

We conclude that for any letter $s_i \in J_k$, there exists $s_i' \in \mathcal{S}_{J_{k-1}}$ with $s_i' \in \mathsf{des}_R((w^{(k)})_{J_{k-1}})$ and

$$\left(w^{(1)} \cdots w^{(k-1)} (w^{(k)})_{J_{k-1}}\right)(\alpha_{s_i'}) = \left(w^{(1)} \cdots w^{(k)}\right)(\alpha_{s_i}) \in \Phi_J.$$

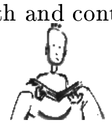
We may therefore move every letter in $w_k^{(k+1)} \cdots w_k^{(m)}$ past $(w^{(k)})^{J_{k-1}}$ as a simple reflection in $\mathcal{S}_{J_{k-1}}$. Finally, moving all letters of $w_k^{(k+1)} \cdots w_k^{(m)}$ past $(w^{(k)})^{J_{k-1}}$ (in order, from left to right) and then dropping the trailing $(w^{(k)})^{J_{k-1}}$ preserves factorwise c-sortability. ☐

Example 6.5.2. Neither Corollary 6.4.8 nor Theorem 6.5.1 hold for general non-sortable elements. For example, let $w = sst \in \mathfrak{S}_3^+$ with Garside factorization $s \cdot st$. Then Corollary 6.4.8 fails for w. Since $\mathsf{inv}(s \cdot st) = \left(\alpha^{(0)}, \alpha^{(1)}, \beta^{(0)}\right)$, we see that $\mathtt{t}$ appears in the second Garside factor, but the corresponding root β has color 0. Theorem 6.5.1 also fails for w with $J = \{t\}$, since $\mathrm{inv}(w)\big|_J = \{\beta^{(0)}\}$ but $\mathrm{inv}(w_J) = \mathrm{inv}(e) = \emptyset$.

6.5.2. Joins of sortable elements with initial simple reflections. The following theorem generalizes part of Lemma 6.2.4, and will be needed for m-eralized Cambrian rotation to be well-defined. Although Theorem 6.5.3 does not describe the change in cover reflections from w to $w \vee s^k$ (because we have not defined cover reflections for elements of $\boldsymbol{B}^+$), the related Lemma 6.8.7 indicates how a related set of colored roots changes.

Theorem 6.5.3. *For s be initial in c, if $w \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle}, \bar{s}c)$, then $w \vee s^k$ is both c-sortable and $\bar{s}cs$-sortable for any $0 \le k \le m$.*

If you enjoyed the proof on the previous page, take a deep breath and continue.



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Set $w(0) = w_0^{(1)} \cdot \cdots \cdot w_0^{(m)}$ to be the Garside factorization of the element $w \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle}, \bar{s}c)$ in the statement of Theorem 6.5.3. For $1 \le k \le m$ inductively set

$$\begin{aligned} w(k) =& w_{k-1}^{(1)} \cdot w_{k-1}^{(2)} \cdot \cdots \cdot w_{k-1}^{(k-1)} \cdot \left(w_{k-1}^{(k)} \vee s_k\right) \cdot (w_{k-1}^{(k+1)})^{v_k} \cdot \cdots \cdot (w_{k-1}^{(m)})^{v_k} \\ =& w_k^{(1)} \cdot w_k^{(2)} \cdot \cdots \cdot w_k^{(m)} \end{aligned}$$

where $s_k, v_k \in W$ are given by

This one is actually a little harder, but the idea is still simple.

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$$s_k = s_{k-1}^{\left(w_{k-1}^{(k-1)}\right)^{-1}} \qquad \text{and} \qquad v_k = \left(w_{k-1}^{(k)}\right)^{-1}\left(w_{k-1}^{(k)} \vee s_k\right).$$

We will show that the decomposition $w(k) = w_k^{(1)} \cdot \; \cdots \; \cdot w_k^{(m)}$ is the Garside factorization of $w \vee s^k$, and that this factorization is factorwise c- and $\bar{s}cs$-sortable.

LEMMA 6.5.4. *The element $w(k)$ is c-sortable and $\bar{s}cs$-sortable with Garside factorization $w_k^{(1)} \cdot \; \cdots \; \cdot w_k^{(m)}$.*

As the proof of Lemma 6.5.4 is a somewhat lengthy induction on k, we extract the base case into a separate lemma for readability.

LEMMA 6.5.5. *The element $w(1)$ is c-sortable and $\bar{s}cs$-sortable with Garside factorization*

$$w(1) = (w_0^{(1)} \vee s) \cdot (w_0^{(2)})^v \cdot \; \cdots \; \cdot (w_0^{(m)})^v,$$

where $v = \left(w_0^{(1)}\right)^{-1}\left(w_0^{(1)} \vee s\right) \in W$. Furthermore, $\mathsf{cov}_\downarrow\left(w_0^{(1)} \vee s\right) = \mathsf{cov}_\downarrow\left(w_0^{(1)}\right) \cup \{s\}$.

PROOF. Since $w(0)$ is c-sortable (and thus factorwise c-sortable) by assumption, its first factor $w_0^{(1)}$ is also c-sortable. This element $w_0^{(1)}$ satisfies the assumption of Lemma 6.2.4, and we obtain that $w_0^{(1)} \vee s$ is both c-sortable and $\bar{s}cs$-sortable, and that

$$\mathsf{cov}_\downarrow(w_0^{(1)} \vee s) = \mathsf{cov}_\downarrow(w_0^{(1)}) \cup \{s\}. \tag{6.2}$$

The factorwise c-sortability also implies that the Garside factors $w_0^{(2)}, \ldots, w_0^{(m)}$ all live in the parabolic subgroup $W_{\mathsf{des}_R(w_0^{(1)})}$. Now, conjugating all these Garside factors by v simply takes those right descents of $w_0^{(1)}$, maps them to the cover reflection $\mathsf{cov}_\downarrow(w_0^{(1)})$ by conjugating with $w_0^{(1)}$, and then turns these cover reflections back to the corresponding right descents of $w_1^{(1)} = w_0^{(1)} \vee s$ using (6.2). Since this rearrangement of the right descents is clearly compatible with the defining property of factorwise c-sortability and factorwise $\bar{s}cs$-sortability, the statements follow. □

PROOF OF LEMMA 6.5.4. We write $c_j^{(i)}$ for the Coxeter element $c^{(i)}$ for $w(j)$, as defined at the beginning of Section 6.4. We will prove the statement of Lemma 6.5.4 along with

- $\mathsf{cov}_\downarrow(w_k^{(k)}) = \mathsf{cov}_\downarrow(w_{k-1}^{(k)} \vee s_k) = \mathsf{cov}_\downarrow(w_{k-1}^{(k)}) \cup \{s_k\}$; and
- s_k is initial in $c_{k-1}^{(k-1)}$,

by induction on k. These are established for $k = 1$ by Lemma 6.5.5; we interpret $c_0^{(0)}$ as c. It remains to conclude the statements for k, assuming that they hold for $k-1$.

The first $k-1$ Garside factors have not changed, and so are still Garside factors, and each of them is sortable in its corresponding parabolic subgroup given by the definition of factorwise sortability.

We next show that $s_k \in \mathsf{des}_R(w_{k-1}^{(k-1)})$ (which, in particular, shows that it is a simple reflection) and $s_k \notin \operatorname{supp}(w_{k-1}^{(k)})$. We can assume by induction that

$$\mathsf{cov}_\downarrow(w_{k-1}^{(k-1)}) = \mathsf{cov}_\downarrow(w_{k-2}^{(k-1)} \vee s_{k-1}) = \mathsf{cov}_\downarrow(w_{k-2}^{(k-1)}) \cup \{s_{k-1}\}.$$

Therefore, $s_k = s_{k-1}^{w_{k-1}^{(k-1)}}$ is the right descent of $w_{k-1}^{(k-1)}$ corresponding to its cover reflection s_{k-1}, implying the first property $s_k \in \mathsf{des}_R(w_{k-1}^{(k-1)})$. Moreover, $w_{k-1}^{(k)} =$

$\big(w_{k-2}^{(k)}\big)^{v_{k-1}}$ sits inside the right descents of $w_{k-1}^{(k-1)}$ in the same way as $w_{k-2}^{(k)}$ sits in the right descents of $w_{k-2}^{(k-1)}$. Since s_{k-1} was not a covered reflection of $w_{k-2}^{(k-1)}$, the right descent s_k of $w_{k-1}^{(k-1)}$ corresponding to this covered reflection cannot be contained in the support of $w_{k-1}^{(k)}$, yielding the second property $s_k \notin \mathrm{supp}(w_{k-1}^{(k)})$.

The induction hypothesis gives us that s_{k-1} is initial in $c_{k-2}^{(k-2)}$. Therefore s_k is initial in $c_{k-1}^{(k-1)}$ by the definition of $c_{k-1}^{(k-1)}$ since $s_{k-1} \in \mathsf{cov}_\downarrow(w_{k-1}^{(k-1)})$ is the cover reflection corresponding to $s_k \in \mathsf{des}_R(w_{k-1}^{(k-1)})$.

We can therefore apply Lemma 6.2.4 to the $c_{k-1}^{(k-1)}$-sortable element $w_{k-1}^{(k)}$ to obtain that $w_k^{(k)} = w_{k-1}^{(k)} \vee s_k$ is again $c_{k-1}^{(k-1)}$- and $(s_k^{-1} c_{k-1}^{(k-1)} s_k)$-sortable with

$$\mathsf{cov}_\downarrow(w_k^{(k)}) = \mathsf{cov}_\downarrow(w_{k-1}^{(k)} \vee s_k) = \mathsf{cov}_\downarrow(w_{k-1}^{(k)}) \cup \{s_k\}.$$

Therefore $w_{k-1}^{(k)}$ lives in the parabolic subgroup generated by $\mathsf{cov}_\downarrow(w_k^{(k-1)}) = \mathsf{cov}_\downarrow(w_{k-1}^{(k-1)})$.

The final part of the proof is to conjugate the remaining Garside factors $w_{k-1}^{(k+1)}$ through $w_m^{(k+1)}$ by v_k. This part is completely analogous to the argument given in the proof of Lemma 6.5.5. □

Proof of Theorem 6.5.3. We show that $w(k) = w \vee s^k$, and again first consider the case $k = 1$.

Clearly, $s \leq w(1)$ since the Garside factorization begins with $w_0^{(1)} \vee s$ which is above s in W and therefore has a reduced $\mathcal{S}$-word starting with s. Also $w \leq w(1)$ since

$$\begin{aligned} w(1) = w_1^{(1)} \cdot w_1^{(2)} \cdot \ \cdots \ \cdot w_1^{(m)} &= \big(w_0^{(1)} v_1\big) \cdot \big(v_1^{-1} w_0^{(2)} v_1\big) \cdot \ \cdots \ \cdot \big(v_1^{-1} w_0^{(m)} v_1\big) \\ &= w_0^{(1)} \cdots w_0^{(m)} v_1 = w v_1, \end{aligned}$$

where we write $v_1 = (w_0^{(1)})^{-1} w_1^{(1)} = (w_0^{(1)})^{-1} w_0^{(1)} \vee s_k$ as before, and write $v_1 = (w_0^{(1)} \vee s_k)^{-1} (w_0^{(1)}) \in B^+$. The first equality is given by the definition of $w(1)$ in terms of $w(0)$. Then Lemma 6.5.5 implies that $w_1^{(1)} \cdot \cdots \cdot w_1^{(m)}$ is indeed the Garside factorization $w(1)$.

It remains to show that $w(1)$ is minimal among all elements above s and w. Although the colored inversion set of an element of B^+ is not necessarily unique to that element, the number of inversions still tells us its length. Any element above w must contain all inversions of w, and any element above $w_0^{(1)}$ and s must contain the inversions of $w_0^{(1)} \vee s$. The inversion set of $w(1)$ contains all these inversions and no others, and therefore has the minimal desired length; we conclude that $w(1) = w \vee s$.

For the case of general k, we first check that $w(k-1) \leq w(k)$ and that $s^k \leq w(k)$. For $w(k-1)$, we have

$$\begin{aligned} w(k) =& w_{k-1}^{(1)} \cdot w_{k-1}^{(2)} \cdot \ \cdots \ \cdot w_{k-1}^{(k-1)} \cdot (w_{k-1}^{(k)} \vee s_k) \cdot (w_{k-1}^{(k+1)})^{v_k} \cdot \ \cdots \ \cdot (w_{k-1}^{(m)})^{v_k} \\ =& w_{k-1}^{(1)} w_{k-1}^{(2)} \cdots w_{k-1}^{(k-1)} w_{k-1}^{(k)} \Big((w_{k-1}^{(k)})^{-1} (w_{k-1}^{(k)} \vee s_k) \Big) (w_{k-1}^{(k+1)})^{v_k} \cdots (w_{k-1}^{(m)})^{v_k} \\ =& w_{k-1}^{(1)} \cdots w_{k-1}^{(m)} \Big((w_{k-1}^{(k)})^{-1} (w_{k-1}^{(k)} \vee s_k) \Big). \end{aligned}$$

For s^k, we have

$$\begin{aligned} w(k) =& w^{(1)}_{k-1} w^{(2)}_{k-1} \cdots w^{(k-1)}_{k-1} (w^{(k)}_{k-1} \vee s_k)(w^{(k+1)}_{k-1})^{v_k} \cdots (w^{(m)}_{k-1})^{v_k} \\ =& w^{(1)}_{k-1} w^{(2)}_{k-1} \cdots w^{(k-1)}_{k-1} s_k \cdots = w^{(1)}_{1} w^{(2)}_{2} \cdots w^{(k-1)}_{k-1} s_k \cdots \\ =& s(w^{(1)}_{1} w^{(2)}_{2} \cdots w^{(k-1)}_{k-1}) \cdots = s s^{k-1} \cdots = s^k \cdots . \end{aligned}$$

We now show that $w(k)$ is the minimal element above s^k and $w(k-1)$. Let $u = (w^{(1)}_{k-1} w^{(2)}_{k-1} \cdots w^{(k-1)}_{k-1})$. We claim that $s^k \vee w(k-1) = (su) \vee w(k-1)$. We first show that $s^k \vee u = su$. The element $s^k \vee u$ is divisible by s^k, and since u is divisible by s^{k-1} but not by s^k, the length of $s^k \vee u$ is at least one more than the length of u. As the element $su = us_k$ is divisible by both s^k and u, we conclude that $su = s^k \vee u$. Therefore, since u is initial in $w(k-1)$, we conclude that

$$su \vee w(k-1) = s^k \vee u \vee w(k-1) = s^k \vee w(k-1).$$

Now $su = us_k$, so that us_k and $w(k-1)$ share their first $k-1$ Garside factors. Therefore,

$$su \vee w(k-1) = us_k \vee w(k-1) = u(s_k \vee (w^{k}_{k-1} \cdot \ \cdots \ \cdot w^{(m)}_{k-1})).$$

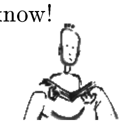
By the inversion set argument used above for $k=1$, we may now conclude that the final $m-k+1$ Garside factors of $s_k \vee (w^{k}_{k-1} \cdot \ \cdots \ \cdot w^{(m)}_{k-1})$ are of the specified form, so that $w(k) = s^k \vee w(k-1)$. $\square$

6.6. Cambrian lattices

Definition 6.6.1. The *m-eralized c-Cambrian poset* $\mathsf{Camb}^{(m)}_{\mathsf{Sort}}(W,c)$ is the restriction of $\mathsf{Weak}^{(m)}(W)$ to $\mathsf{Sort}^{(m)}(W,c)$.

Figure 19 shows all 12 st-sorting elements in $\mathsf{Camb}^{(2)}_{\mathsf{Sort}}(\mathfrak{S}_3, st)$. By Equation (2.5), weak order on W is characterized as containment of inversion sets. Although comparison of colored inversion sets does not recover $\mathsf{Weak}^{(m)}(W)$ for $m \geq 2$, it does capture relations among c-sortable elements.

Inversion sets for sortables behave in the positive Artin monoid in the same way inversion sets for elements in the Coxeter group do—good to know!



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Theorem 6.6.2. *For* $w, u \in \mathsf{Sort}^{(m)}(W,c)$,

$$w \leq u \text{ if and only if } \mathrm{inv}(w) \subseteq \mathrm{inv}(u).$$

Proof. If $w \leq u$ then it is clear that $\mathrm{inv}(w)$ is contained in $\mathrm{inv}(u)$, since w is initial in u. We now argue the converse. Suppose $\mathrm{inv}(w) \subseteq \mathrm{inv}(u)$ and let s be initial in c.

- Suppose $s \in \mathsf{asc}_L(w)$ and $s \in \mathsf{asc}_L(u)$. Then we are done by restriction to $W_{\langle s \rangle}$.
- The case $s \in \mathsf{des}_L(w)$ and $s \in \mathsf{asc}_L(u)$ is not possible as $\mathsf{des}_L(w) \subseteq \mathsf{des}_L(u)$.
- Suppose $s \in \mathsf{asc}_L(w)$ and $s \in \mathsf{des}_L(u)$. It is clear that $u_{\langle s \rangle} \leq u$. Since $w \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle}, \overline{s}c)$, we have that $\mathrm{inv}(w) \subseteq \mathrm{inv}(u_{\langle s \rangle})$ by Theorem 6.5.1, so that by induction on rank (since both $w, u_{\langle s \rangle} \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle}, \overline{s}c)$), $w \leq u_{\langle s \rangle}$. Since $u_{\langle s \rangle} \leq u$, we conclude that $w \leq u$.
- Suppose finally that $s \in \mathsf{des}_L(w)$ and $s \in \mathsf{des}_L(u)$. Then we get the statement for $\overline{s}u = u'$ and $\overline{s}w = w'$ by induction on length. Multiplying by s does not change containment of inversion sets (since multiplication by s just multiplies all inversions by s, and then adds $\alpha_s^{(0)}$). $\square$

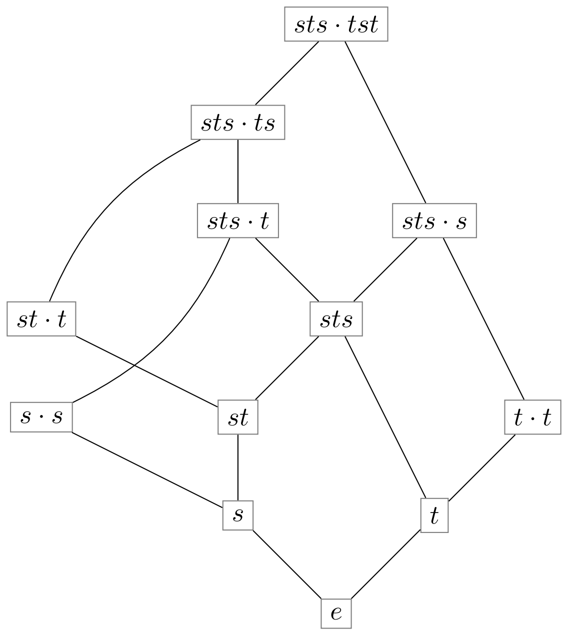

FIGURE 19. The Cambrian lattice $\mathsf{Camb}^{(2)}_{\mathsf{Sort}}(\mathfrak{S}_3, st)$. Each sortable element is represented by its Garside factorization.

In fact, the intersection of colored inversion sets of two c-sortable elements is again the inversion set of a c-sortable element, m-eralizing Proposition 6.2.5.

LEMMA 6.6.3. *Let* $u, v \in \mathsf{Sort}^{(m)}(W, c)$. *Then* $u \wedge v \in \mathsf{Sort}^{(m)}(W, c)$ *and*

$$\mathrm{inv}(u \wedge v) = \mathrm{inv}(u) \cap \mathrm{inv}(v).$$

PROOF. Let s be initial in c. We consider again the four possible cases.

- If $s \in \mathsf{asc}_L(u)$ and $s \in \mathsf{asc}_L(v)$, then we are done by restriction to $W_{\langle s \rangle}$ and induction on rank.
- If $s \in \mathsf{des}_L(u)$ and $s \in \mathsf{asc}_L(v)$, then $v \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle}, \overline{s}c)$. Then $u \wedge v = (u \wedge (w_\circ^m(J) \wedge v)) = ((u \wedge w_\circ^m(J)) \wedge v) = u_{\langle s \rangle} \wedge v$. By Theorem 6.5.1, $u_{\langle s \rangle}$ is $\overline{s}c$-sortable, so that $u \wedge v$ is c-sortable with inversion set given by the previous case.
- The case $s \in \mathsf{des}_L(v)$ and $s \in \mathsf{asc}_L(u)$ follows by symmetry.
- Finally, if $s \in \mathsf{des}_L(u)$ and $s \in \mathsf{des}_L(v)$, then by Proposition 6.3.1, $\overline{s}u$ and $\overline{s}v$ are both $\overline{s}cs$-sortable. By induction on length, $(\overline{s}u) \wedge (\overline{s}v)$ is $\overline{s}cs$-sortable. Then $u \wedge v = s(\overline{s}u \wedge \overline{s}v)$, which is again c-sortable by Proposition 6.3.1. Furthermore, we have
$$\mathrm{inv}(\overline{s}u \wedge \overline{s}v) = \mathrm{inv}(\overline{s}u) \cap \mathrm{inv}(\overline{s}v)$$
by induction on length. This gives
$$\begin{aligned} \mathrm{inv}(w \wedge u) &= \{\alpha_s^{(0)}\} \cup s(\mathrm{inv}(\overline{s}w \wedge \overline{s}u)) \\ &= \{\alpha_s^{(0)}\} \cup s\,(\mathrm{inv}(\overline{s}w) \cap \mathrm{inv}(\overline{s}u)) \\ &= \mathrm{inv}(w) \cap \mathrm{inv}(u). \end{aligned}$$
□

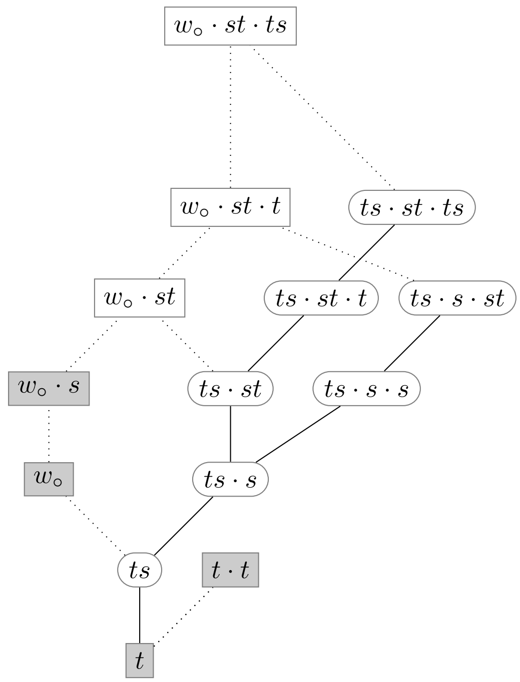

FIGURE 20. A part of $\mathsf{Weak}^{(3)}(\mathfrak{S}_3)$. The st-sortable elements are shaded. The rounded boxes indicate those elements lying above t, but not lying above any larger st-sortable element. There are two such maximal elements.

An element $w' \in W$ is called *c-antisortable* if $w'w_\circ$ is c^{-1}-sortable. N. Reading showed that every element $w \in \mathsf{Weak}(W)$ lies above a unique largest c-sortable element $\pi_\downarrow^c(w)$ in weak order, and lies below a unique smallest c-antisortable element $\pi_c^\uparrow(w)$ [**Rea07b**]. He proved that the fibers of the $\pi_\downarrow^c : \mathsf{Weak}(W) \to \mathsf{Sort}(W, c)$ are given by the intervals $[\pi_\downarrow^c(w), \pi_c^\uparrow(w)]_{\mathsf{Weak}(W)}$, so that $\pi_\downarrow^c$ defines a lattice congruence [**Rea07b**, Proposition 3.1].

In contrast, the m-eralized c-sortable elements no longer form a lattice quotient of $\mathsf{Weak}^{(m)}(W)$ for the reason that the maps $\pi_\downarrow^c$ and $\pi_c^\uparrow$ are not well-defined.

For $\mathfrak{S}_3$ with $m = 3$, the elements of $\mathsf{Weak}^{(3)}(\mathfrak{S}_3)$ lying above t—but not above any larger st-sortable element—do not form an interval. This is illustrated in Figure 20.

Although $\mathsf{Sort}^{(m)}(W, c)$ is no longer a lattice quotient of $\mathsf{Weak}^{(m)}(W)$, the restriction of $\mathsf{Weak}^{(m)}(W)$ to $\mathsf{Sort}^{(m)}(W, c)$ is still a lattice. The proof is analogous to the proof of [**Rea07b**, Theorem 1.2], except that we do not have a projection map $\pi_\downarrow^c$, and so cannot rely on its properties to compute the join.

This is maybe the cleanest way of seeing this.

THEOREM 6.6.4. *$\mathsf{Camb}^{(m)}_{\mathsf{Sort}}(W, c)$ is a sublattice of $\mathsf{Weak}^{(m)}(W)$.*

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Proof. Lemma 6.6.3 shows that $u \wedge v \in W^{(m)}$ is c-sortable for $u, v \in \mathsf{Sort}^{(m)}(W, c)$. A similar argument proves that $u \vee v \in W^{(m)}$ is c-sortable for $u, v \in \mathsf{Sort}^{(m)}(W, c)$. Again, let s be initial in c.

- If $s \in \mathsf{des}_L(u)$ and $s \in \mathsf{des}_L(v)$, then $s \in \mathsf{des}_L(u \vee v)$. By induction on length, $(\bar{s}u) \vee (\bar{s}v)$ is $\bar{s}c$-sortable. Then $u \vee v = s(\bar{s}u \vee \bar{s}v)$ is c-sortable.
- If $s \in \mathsf{des}_L(v)$ and $s \in \mathsf{asc}_L(u)$, then $u \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle}, \bar{s}c)$ and $s \vee u$ is c-sortable by Theorem 6.5.3. We compute that $u \vee v = s \vee (u \vee v) = (s \vee u) \vee v$, so that $u \vee v$ is c-sortable by the previous case.
- The case $s \in \mathsf{des}_L(u)$ and $s \in \mathsf{asc}_L(v)$ follows by symmetry.
- Finally, suppose $s \in \mathsf{asc}_L(u)$ and $s \in \mathsf{asc}_L(v)$. Then $u, v \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle})$ and we conclude the result by induction on rank. □

6.7. The Cambrian rotation

For s initial in c, define the bijection

$$\mathsf{Shift}_s : \mathsf{Sort}^{(m)}(W, c) \longrightarrow \mathsf{Sort}^{(m)}(W, \bar{s}cs)$$

$$w \longmapsto \begin{cases} w \vee s^m & \text{if } s \in \mathsf{asc}_L(w) \\ \bar{s}w & \text{if } s \in \mathsf{des}_L(w) \end{cases}, \tag{6.3}$$

where $w \vee s^m$ denotes the join in $\mathsf{Weak}^{(m)}(W)$. The first case is well-defined by Theorem 6.5.3 from which it also follows that $w \vee s^m \neq u \vee s^m$ for $w \neq u$.

Example 6.7.1. Parallel to Examples 4.3.1 and 5.4.2, alternately applying Shift_s and Shift_t to $e \in \mathsf{Sort}^{(2)}(\mathfrak{S}_3, st)$ gives the orbit

$$\begin{array}{ccccc} e & \xmapsto{\mathsf{Shift}_s} & s \cdot s & \xmapsto{\mathsf{Shift}_t} & sts \cdot sts \\ & \xmapsto{\mathsf{Shift}_s} & tst \cdot st & \xmapsto{\mathsf{Shift}_t} & sts \cdot t \\ & \xmapsto{\mathsf{Shift}_s} & tst & \xmapsto{\mathsf{Shift}_t} & st \\ & \xmapsto{\mathsf{Shift}_s} & t & \xmapsto{\mathsf{Shift}_t} & e \end{array}.$$

Definition 6.7.2. The *m-eralized c-Cambrian rotation* $\mathsf{Camb}_c : \mathsf{Sort}^{(m)}(W, c) \to \mathsf{Sort}^{(m)}(W, c)$ is given by

$$\mathsf{Camb}_c = \mathsf{Shift}_{s_n} \circ \cdots \circ \mathsf{Shift}_{s_1}$$

for any reduced $\mathcal{S}$-word $\mathsf{s}_1\mathsf{s}_2 \cdots \mathsf{s}_n$ for c.

This composition evidently does not depend on the chosen reduced word. The elements in Figure 18 are arranged according to their orbits under Cambrian rotation.

6.8. Sortable elements, noncrossing partitions, and clusters

In this section, we use skip sets to relate sortable elements to noncrossing partitions (Theorem 6.8.5) and to clusters (Theorem 6.8.8), where we recall that the term *natural* means that a bijection respects the Cambrian recurrence. These bijections were introduced by N. Reading and D. Speyer for $m = 1$ in [**RS11**].

Are you ready to see the parts of the theory coming together naturally?

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Thanks, Reading and

6.8.1. Sortable elements and noncrossing partitions. We define a natural bijection between m-eralized c-sortable elements and m-eralized c-noncrossing partitions, m-eralizing the constructions in [**Rea07a, RS11**].

Let the c-sorting word of $w \in \mathsf{Sort}^{(m)}(W, c)$ be $\mathsf{w}(\mathsf{c}) = \mathsf{s}_1 \cdots \mathsf{s}_p$, and let $s \in \mathcal{S}$. We say that w ***skips*** $s \in \mathcal{S}$ in position $k+1$ if the leftmost instance of s in c^∞ not used in $\mathsf{w}(\mathsf{c})$ occurs between s_k and s_{k+1}. The ***skip set*** of colored positive roots is defined as

$$\mathcal{C}_c(w) = \left\{ \beta_s^{(\ell_s)} \ : \ s \in \mathcal{S} \right\},$$

where $\beta_s^{(\ell_s)} = s_1 \cdots s_k(\alpha_s^{(0)})$ when w skips s in position $k+1$, and we say that the skip for s has color ℓ_s. The skip set is ordered by the indices of the skipped positions.

Example 6.8.1. The skip set of $s_1s_2s_3|s_2s_3|s_2 \in \mathsf{Sort}^{(2)}(\mathfrak{S}_4, s_1s_2s_3)$ is

$$\begin{array}{ccc|cccc|cccc|ccc} s_1 & s_2 & s_3 & \underset{(23)^{(0)}}{s_1} & & s_2 & s_3 & - & s_2 & & \underset{(34)^{(1)}}{s_3} & - & \underset{(14)^{(2)}}{s_2} & - \end{array}.$$

The skip for s_1 has color 0, the skip for s_3 has color 1, and the skip for s_2 has color 2. Figure 18 on page 80 shows the skip sets of all elements in $\mathsf{Sort}^{(2)}(\mathfrak{S}_3, st)$.

Remark 6.8.2. Let $w \in \mathsf{Sort}^{(m)}(W, c)$. We recover its c-sorting word $\mathsf{w}(\mathsf{c})$ from its skip set $\mathcal{C}_c(w)$ by reading the word $\mathsf{c}^\infty = \mathsf{s}_1\mathsf{s}_2\ldots$ from left to right, deleting letters as follows: if there is no next letter to read in the copy of c^∞ from which we have deleted some letters, the remaining letters spell $\mathsf{w}(\mathsf{c})$. Otherwise, let s be the letter in the current position, and let u be the product of the undeleted letters strictly to its left. If $u(\alpha_s^{(0)}) \in \mathcal{C}_c(u)$, delete the current letter s and all occurrences of s to the right.

The skip sets satisfy an *m-eralized c-Cambrian recurrence*, proven for $m = 1$ by N. Reading and D. Speyer in [**RS11**].

Proposition 6.8.3. *Let s be initial in c and let $w \in \mathsf{Sort}^{(m)}(W, c)$. Then*

$$\mathcal{C}_c(w) = \begin{cases} \left\{\alpha_s^{(0)}\right\} \cup \mathcal{C}_{\overline{s}c}(w) & \text{if } s \in \mathsf{asc}_L(w) \\ s\big(\mathcal{C}_{\overline{s}cs}(\overline{s}w)\big) & \text{if } s \in \mathsf{des}_L(w) \end{cases}.$$

Proof. If $s \in \mathsf{asc}_L(w)$ then $w \in \mathsf{Sort}^{(m)}(W_{\langle s \rangle}, \overline{s}c)$ with skip set $\mathcal{C}_{\overline{s}c}(w)$. Treating w as an element of $\mathsf{Sort}^{(m)}(W, c)$ does not change the positions of the skips $t \neq s$. But w now skips $s = s_1$ in position 1, which adds $\left\{\alpha_s^{(0)}\right\}$ to its skip set. If $s \in \mathsf{des}_L(w)$, no simple reflection is skipped in position 1 and so each skip $s_1 \cdots s_k(\alpha_t) \in \mathcal{C}_c(w)$ can be identified with $s(s_2 \cdots s_k(\alpha_t^{(0)})) \in s\mathcal{C}_{\overline{s}cs}(\overline{s}w)$. □

Example 6.8.4. Parallel to Example 6.3.2, the sequence of skip sets for the Cambrian recurrence on $sts \cdot s \in \mathsf{Sort}^{(2)}(\mathfrak{S}_3, st)$ is

$$\underbrace{\alpha^{(1)}, \beta^{(2)}}_{st} \mapsto \underbrace{\alpha^{(0)}, \gamma^{(2)}}_{ts} \mapsto \underbrace{\gamma^{(0)}, \alpha^{(2)}}_{st} \mapsto \underbrace{\beta^{(0)}, \alpha^{(1)}}_{ts} \mapsto \underbrace{\alpha^{(1)}}_{s} \mapsto \underbrace{\alpha^{(0)}}_{s} \mapsto \underbrace{-}_{e}.$$

We deduce the following theorem.

THEOREM 6.8.5. *The map $\mathcal{C}_c$ induces a natural bijection*

$$\mathsf{Sort}^{(m)}(W,c) \overset{c}{\longleftrightarrow} \mathsf{NC}^{(m)}_{\Delta}(W,c)$$
$$w \longmapsto \mathsf{t}_1^{(i_1)} \cdots \mathsf{t}_n^{(i_n)},$$

for $\mathcal{C}_c(w) = \{\beta_1^{(i_1)}, \dots, \beta_n^{(i_n)}\}$ *and* $t_k = s_{\beta_k}$.

PROOF. Immediate from Proposition 6.8.3 and Proposition 4.3.3: the inductive structure on $\mathsf{Sort}^{(m)}(W,c)$ is sent to the inductive structure on $\mathsf{NC}^{(m)}_{\Delta}(W,c)$, and both sides have the same initial conditions. ☐

Comparing rows in Figures 11 and 18 illustrates for $\mathfrak{S}_3$ the natural bijection $\mathsf{Sort}^{(2)}(\mathfrak{S}_3, st) \longrightarrow \mathsf{NC}^{(2)}_{\Delta}(\mathfrak{S}_3, st)$. We use the bijection of Theorem 6.8.5 to prove that $\mathsf{Camb}^{(m)}_{\mathsf{Sort}}(W,c)$ and $\mathsf{Camb}^{(m)}_{\mathsf{NC}}(W,c)$ are isomorphic.

THEOREM 6.8.6. $\mathsf{Camb}^{(m)}_{\mathsf{Sort}}(W,c) \cong \mathsf{Camb}^{(m)}_{\mathsf{NC}}(W,c)$.

To prove this theorem, we require control over the skip set of covers of the elements in $\mathsf{Sort}^{(m)}(W_{\langle s\rangle}, \overline{s}c)$.

LEMMA 6.8.7. *Let* $w \in \mathsf{Sort}^{(m)}(W_{\langle s\rangle}, \overline{s}c)$ *with*

$$\mathcal{C}_c(w) = \{\alpha_s^{(0)}, \beta_2^{(\ell_2)}, \dots, \beta_i^{(\ell_i)}, \beta_{i+1}^{(\ell_{i+1})}, \dots, \beta_n^{(\ell_n)}\}$$

for $0 \le \ell_2 \le \cdots \le \ell_i < k \le \ell_{i+1} \le \cdots \le \ell_n$, *then*

$$\mathcal{C}_c(w \vee s^k) = \{[s(\beta_2)]^{(\ell_2)}, \dots, [s(\beta_i)]^{(\ell_i)}, \alpha_s^{(k)}, \beta_{i+1}^{(\ell_{i+1})}, \dots, \beta_n^{(\ell_n)}\}.$$

PROOF. The proof relies on the description of how the Garside factorization of $w \vee s^k$ is obtained from the Garside factorization of $w \vee s^{k-1}$ in Lemma 6.5.4. In particular, the colored inversion sequence of the c-sorting word only changes within the k^{th} Garside factor. [**RS11**, Proposition 5.4] shows that the roots in this Garside factor change as described in the statement of the lemma for $k=1$ and $m=1$, so that the skip set $\mathcal{C}_c(w \vee s^k)$ is obtained from the skip set $\mathcal{C}_c(w \vee s^{k-1})$ by

- replacing $\alpha_s^{(k-1)}$ by $\alpha_s^{(k)}$;
- replacing all other $(k-1)$-colored roots $\beta^{(k-1)}$ by $[s(\beta)]^{(k-1)}$; and
- leaving all other colored roots unchanged.

The lemma follows by applying this procedure k times. ☐

PROOF OF THEOREM 6.8.6. We show that a cover relation

$$u \lessdot v \text{ in } \mathsf{Camb}_{\mathsf{Sort}}(W,c)$$

corresponds under the map $\mathcal{C}_c$ to a cover relation

$$I = \mathsf{r}_1^{(i_1)} \cdots \mathsf{r}_n^{(i_n)} \lessdot \mathsf{t}_1^{(j_1)} \cdots \mathsf{t}_n^{(j_n)} = \mathsf{Flip}_r^{\uparrow}(I) \text{ in } \mathsf{Camb}_{\mathsf{NC}}(W,c).$$

- ∘ If $s \in \mathsf{asc}_L(u)$ and $s \in \mathsf{asc}_L(v)$ (equivalently, $\mathsf{r}_1^{(i_1)} = \mathsf{t}_1^{(j_1)} = \mathsf{s}^{(0)}$), the statement follows by the Cambrian recurrences in Propositions 6.2.1 and 4.3.3 and Theorem 6.8.5.
- ∘ The case $s \in \mathsf{des}_L(u)$ and $s \in \mathsf{asc}_L(v)$ (equivalently, $\mathsf{r}_1^{(i_1)} \neq \mathsf{s}^{(0)}$ and $\mathsf{t}_1^{(j_1)} = s^{(0)}$) is impossible if we are starting from a cover $u \lessdot v$ in $\mathsf{Camb}_{\mathsf{Sort}}(W,c)$ since $u \le v$ in weak order, and it is impossible if we are starting from a cover in $\mathsf{Camb}_{\mathsf{NC}}(W,c)$ because the increasing flip of $\mathsf{r}_j^{(i_j)}$ changes only reflections that appear after it in c^∞, and it changes them into reflections which still appear after it in c^∞.

- If $s \in \mathsf{des}_L(u)$ and $s \in \mathsf{des}_L(v)$ (equivalently, $\mathsf{r}_1^{(i_1)} \neq \mathsf{s}^{(0)}$ and $\mathsf{t}_1^{(j_1)} \neq \mathsf{s}^{(0)}$), the statement follows again by the Cambrian recurrences.
- Then suppose $s \in \mathsf{asc}_L(u), s \in \mathsf{des}_L(v)$ (equivalently, $\mathsf{r}_1^{(i_1)} = \mathsf{s}^{(0)}$ and $\mathsf{t}_1^{(j_1)} \neq \mathsf{s}^{(0)}$). Since $u \leq v$ in weak order and $s \in \mathrm{supp}(v)$, we have that $s \leq v$ in weak order, so that $u \vee s \leq v$. Since $s \leq vs \notin \mathrm{supp}(u)$, Lemma 6.8.7 for $k = 1$ implies that $u \vee s$ is c-sortable. Therefore, $v = u \vee s$ and the skip set of v may be obtained from the skip set of u by Lemma 6.8.7. On the other hand, (4.3) shows that $\mathsf{Flip}_r^{\uparrow}(I)$ behaves in exactly the same way. □

6.8.2. Sortable elements and clusters. Using Theorem 5.7.2, we obtain a bijection between m-eralized c-sortable elements and the m-eralized c-cluster complex by identifying the skip set of a sortable element and root configuration of a facet.

THEOREM 6.8.8. *There is a natural bijection*

$$\begin{aligned} \mathsf{Sort}^{(m)}(W,c) &\overset{c}{\longleftrightarrow} \mathsf{Asso}_{\Delta}^{(m)}(W,c) \\ w &\longmapsto I, \end{aligned}$$

given by $\mathcal{C}_c(w) = \mathsf{R}(I)$.

PROOF. This is a direct consequence of Propositions 5.7.1 and 6.8.3. □

We now m-eralize [**Rea07a**, Theorem 8.1] to give a satisfyingly direct bijection between $\mathsf{Sort}^{(m)}(W,c)$ and $\mathsf{Asso}^{(m)}(W,c)$. Write the c-sorting word of $w \in \mathsf{Sort}^{(m)}(W,c)$ as $\mathsf{w}(\mathsf{c}) = \mathsf{s}_1 \cdots \mathsf{s}_p$.

For $s \in \mathrm{supp}(w)$, let $\beta_s^{(\ell_s)}$ be the colored positive root $\mathsf{s}_1 \cdots \mathsf{s}_{k-1}(\alpha_{s_k}^{(0)})$, where s_k is the *last* occurrence of the letter s in $\mathsf{w}(\mathsf{c})$. Define

$$\mathcal{C}_c^*(w) := \left\{\beta_s^{(\ell_s)} \ : \ s \in \mathrm{supp}(w)\right\} \cup \left\{\alpha_s^{(m)} \ : \ s \notin \mathrm{supp}(w)\right\}.$$

PROPOSITION 6.8.9. *Let s be initial in c and let $w \in \mathsf{Sort}^{(m)}(W,c)$. Then*

$$\mathcal{C}_c^*(w) = \begin{cases} \mathcal{C}_{\overline{s}c}^*(w) \cup \left\{\alpha_s^{(m)}\right\} & \text{if } s \in \mathsf{asc}_L(w) \\ \left(\tau_s^{(m)}\right)^{-1} \mathcal{C}_{\overline{s}cs}^*(\overline{s}w) & \text{if } s \in \mathsf{des}_L(w) \end{cases}.$$

PROOF. The first case $s \in \mathsf{asc}_L(w)$ is clear, since $s \notin \mathrm{supp}(w)$. If $s \in \mathsf{des}_L(w)$, note that $\left(\tau_s^{(m)}\right)^{-1}$ acts as the simple reflection s on $\left\{\beta_s^{(\ell_s)} \ : \ s \in \mathrm{supp}(w)\right\}$, while fixing $\left\{\alpha_s^{(m)} \ : \ s \notin \mathrm{supp}(w)\right\}$. This is exactly how $\mathcal{C}_c^*(\overline{s}w)$ is obtained from $\mathcal{C}_c^*(w)$. □

THEOREM 6.8.10. *The map $\mathcal{C}_c^*$ is a natural bijection*

$$\begin{aligned} \mathsf{Sort}^{(m)}(W,c) &\overset{c}{\longleftrightarrow} \mathsf{Asso}^{(m)}(W,c) \\ w &\longmapsto \mathcal{C}_c^*(w). \end{aligned}$$

PROOF. This follows from the Cambrian recurrences in Proposition 6.2.1 and in Proposition 6.8.9. □

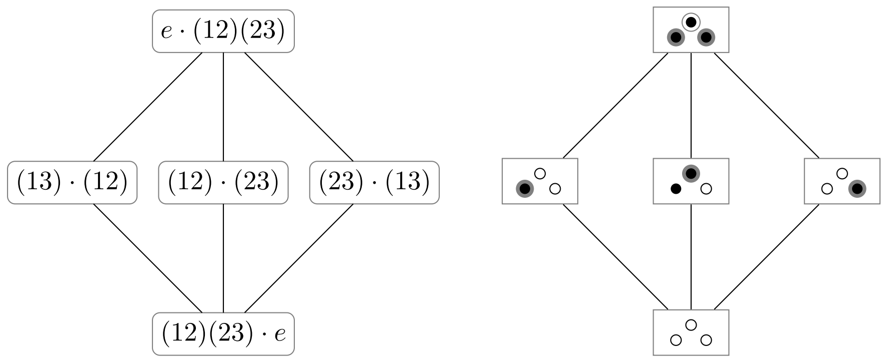

FIGURE 21. The isomorphic lattices $\mathsf{NCL}(\mathfrak{S}_3, st)$ and $\mathsf{Shard}(\mathfrak{S}_3)$ restricted to $\mathsf{Sort}(\mathfrak{S}_3, st)$. Sortable elements are specified by their inversion sets as described in Example 6.9.2.

6.9. Sortable elements and the shard intersection order

In this section, we give an alternative description of the bijection in Theorem 6.8.5. This reconciles the m-eralization of c-sortable elements with D. Armstrong's m-eralization of noncrossing partitions using the shard intersection order.

N. Reading gave a beautiful proof of the lattice property of the noncrossing partition lattice $\mathsf{NCL}(W, c)$, showing that the restriction of $\mathsf{Shard}(W)$ to the sortable elements $\mathsf{Sort}(W, c)$ is isomorphic to $\mathsf{NCL}(W, c)$ ([**Rea11**, Theorem 8.5]).

THEOREM 6.9.1. *The restriction of* $\mathsf{Shard}(W)$ *to* $\mathsf{Sort}(W, c)$ *is isomorphic to* $\mathsf{NCL}(W, c)$ *under the natural bijection of Theorem 6.8.5.* □

EXAMPLE 6.9.2. Figure 21 illustrates $\mathsf{NCL}(\mathfrak{S}_3, (123))$, as well as $\mathsf{Shard}(\mathfrak{S}_3)$ restricted to $\mathsf{Sort}(\mathfrak{S}_3, (123))$. These should be compared with the corresponding shard intersection order, drawn in Figure 8. Sortable elements are specified by their inversion sets—inversions are black circles, non-inversions are white circles, covered reflections are circled in gray, and other inversions in the parabolic subgroup generated by the covered reflections are circled in white. Circles are indexed by reflections in the order $\begin{array}{ccc} & 13 & \\ 12 & & 23 \end{array}$.

A brief glimpse from the shining nonnesting world.

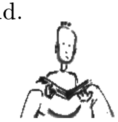

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Figure 22 depicts the restriction of $\mathsf{Shard}(\mathfrak{S}_4)$ to $\mathsf{Sort}(\mathfrak{S}_4, (1234))$, and should be compared with the noncrossing partition lattice in Figure 10. Circles are indexed by reflections in the order $\begin{array}{ccccc} & & 14 & & \\ & 13 & & 24 & \\ 12 & & 23 & & 34 \end{array}$.

But that's not organic.

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Combining Theorem 6.9.1 with Definition 4.2.1 suggests a definition of m-eralized sortable elements as m-multichains of sortable elements in $\mathsf{Shard}(W)$.

DEFINITION 6.9.3. The *shard c-sortable elements* are the m-multichains

$$\mathsf{Sort}^{(m)}_{\mathsf{shard}}(W, c) := \left\{ (w_1 \geq_{\mathrm{Sh}} w_2 \geq_{\mathrm{Sh}} \cdots \geq_{\mathrm{Sh}} w_m) \ : \ w_i \in \mathsf{Sort}(W, c) \right\}.$$

By construction, the shard c-sortable elements are in bijection with the m-eralized c-noncrossing partitions.

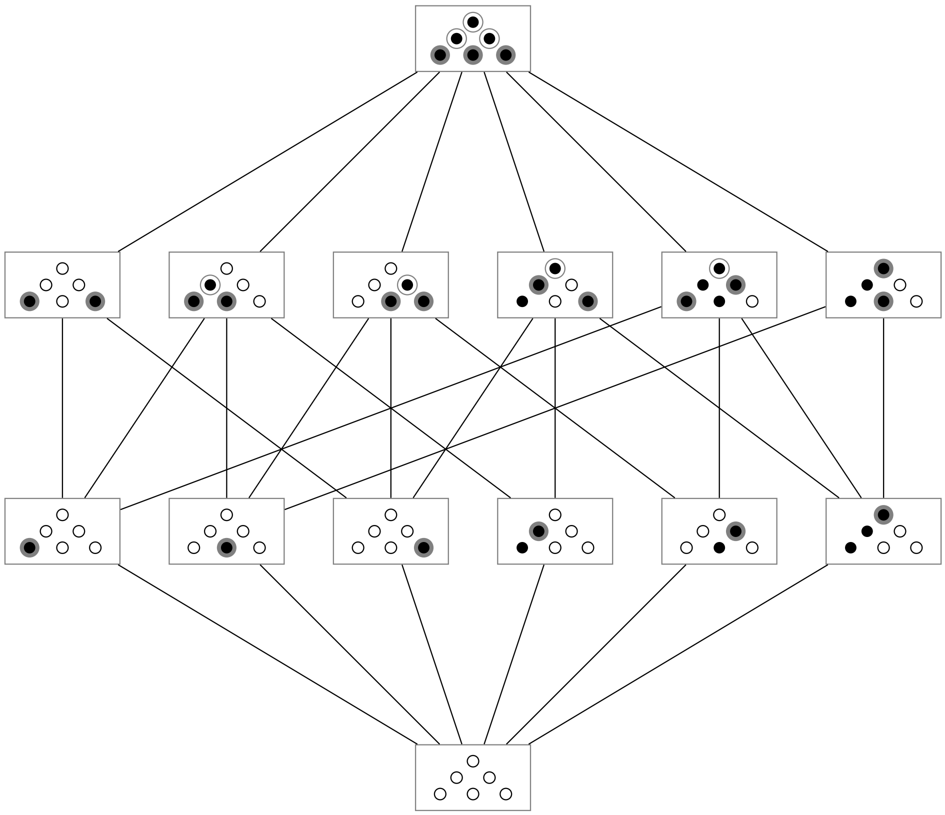


FIGURE 22. $\mathsf{Shard}(\mathfrak{S}_4)$ restricted to $\mathsf{Sort}(\mathfrak{S}_4, s_1 s_2 s_3)$. Sortable elements are specified by their inversion sets as described in Example 6.9.2.

THEOREM 6.9.4. *There is a natural bijection*

$$\mathsf{Sort}^{(m)}_{\mathsf{shard}}(W,c) \overset{c}{\longleftrightarrow} \mathsf{NC}^{(m)}(W,c). \qquad \square$$

The main theorem of this section draws an analogy between the m-eralized Cambrian lattices and D. Amrstrong's m-eralization of the noncrossing partition lattice from Section 4.2—just as multichains of noncrossing partitions should be ordered by componentwise absolute order, multichains of sortable elements should be ordered by componentwise weak order.

THEOREM 6.9.5. *There is a natural bijection*

$$\mathsf{Sort}^{(m)}_{\mathsf{shard}}(W,c) \overset{c}{\longleftrightarrow} \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W,c)$$

that sends $\mathsf{Sort}^{(m)}_{\mathsf{shard}}(W,c)$ *under componentwise weak order to* $\mathsf{Camb}^{(m)}_{\mathsf{Sort}}(W,c)$.

REMARK 6.9.6. A naive guess for a bijection from $\mathsf{Sort}^{(m)}_{\mathsf{shard}}(W,c)$ to $\mathsf{Sort}^{(m)}(W,c)$ would be to simply multiply the individual factors of $\mathsf{Sort}^{(m)}_{\mathsf{shard}}(W,c)$ (as an element of $\boldsymbol{B}^+$). This naive guess is wrong. For example, for $c = s_1 s_2 s_3 \in \mathfrak{S}_4$, this map would send the chain of sortable elements in shard order

$$(s_1s_2s_3s_1 \geq_{\mathrm{Sh}} s_1s_2s_3) \text{ to the element } s_1s_2s_3|s_1\cdot\cdot|s_1s_2s_3 \in \mathfrak{S}_4^+.$$

This element is evidently not c-sortable.

Proof of Theorem 6.9.5. Given a multichain

$$(w_1 \geq_{\mathrm{Sh}} w_2 \geq_{\mathrm{Sh}} \cdots \geq_{\mathrm{Sh}} w_m) \in \mathsf{Sort}^{(m)}_{\mathsf{shard}}(W, c),$$

we produce the Garside factorization of an element in $\mathsf{Sort}^{(m)}(W, c)$ as follows.

Proposition 2.10.6 gives a bijection

$$[e, w]_{\mathsf{Shard}(W)} \cong \mathsf{Shard}(W_{\mathsf{des}_R(w)}).$$

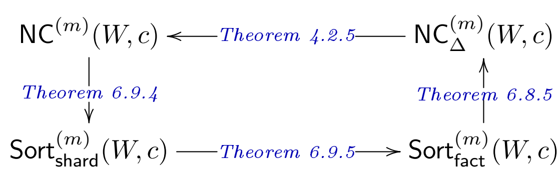
If $u \leq_{\mathrm{Sh}} w$ with $w, u \in \mathsf{Sort}(W, c)$, u is sent to a $c\big|^{w}$-sortable element in $W_{\mathsf{des}_R(w)}$ under this bijection using Theorem 6.9.1.

We iterate this procedure to obtain elements $w^{(1)}, w^{(2)}, \ldots, w^{(m)}$ such that $w^{(1)} := w_1 \in \mathsf{Sort}(W, c)$, and for all $1 < i \leq m$, $w^{(i)} \in \mathsf{Sort}(W_{\mathsf{des}_R(w_{(i-1)})}, c^{(i)})$, where $c^{(i)} := c^{(i-1)}\big|^{w^{(i-1)}}$. Then $w^{(1)} \cdot w^{(2)} \cdot \;\cdots\; \cdot w^{(m)}$ satisfies the condition in Theorem 2.5.1 to be a Garside factorization, it satisfies the factorwise conditions of Definition 6.4.1, and so it is an element of $\mathsf{Sort}^{(m)}(W, c) = \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W, c)$ by Corollary 6.4.4.

The inverse of this bijection is given by explicitly describing the inverse of Proposition 2.10.6 on c-sortable elements: given $u \in W_{\mathsf{des}_R(w)}$, we conjugate it by w to an element of $W_{\mathsf{cov}_\downarrow(w)}$. Since c-sortable elements are uniquely defined by their cover reflections by (6.1), there is a unique way to complete the inversion set in $W_{\mathsf{cov}_\downarrow(w)}$ to the inversion set of a c-sortable element in W.

Theorem 6.6.2 implies that the componentwise weak order on $\mathsf{Sort}^{(m)}_{\mathsf{shard}}(W, c)$ recovers $\mathsf{Camb}^{(m)}_{\mathsf{Sort}}(W, c)$. □

Theorem 6.9.7. *Starting at any corner of the square below and traveling around the square via the given bijections results in the identity map.*



Proof. It suffices to check that the composition of the bijections described in Proposition 4.2.3, Theorem 6.9.4, and Theorem 6.9.5 is the inverse of the bijection $\mathcal{C}_c$ given in Theorem 6.8.5.

Choose $(u_1 \geq_{\mathcal{R}} u_2 \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} u_m) \in \mathsf{NC}^{(m)}(W, c)$, write the corresponding chain of sortable elements as

$$(w_1 \geq_{\mathrm{Sh}} w_2 \geq_{\mathrm{Sh}} \cdots \geq_{\mathrm{Sh}} w_m) \in \mathsf{Sort}^{(m)}_{\mathsf{shard}}(W, c),$$

and denote the corresponding factorwise c-sortable element by

$$w = w^{(1)}w^{(2)} \cdots w^{(m)} \in \mathsf{Sort}^{(m)}_{\mathsf{fact}}(W, c).$$

Let $\mathcal{C}_c(w) = (\delta_0, \delta_1, \delta_2, \ldots, \delta_m) \in \mathsf{NC}^{(m)}_{\delta}(W, c)$. We must show that

$$(\delta_0, \delta_1, \delta_2, \ldots, \delta_m) = (cu_1^{-1}, u_1u_2^{-1}, u_2u_3^{-1}, \cdots, u_m).$$

We argue by induction on m, the base case following from the $m = 1$ theory.

First, any skips of color 0 that could be attributed to the $w^{(2)} \cdots w^{(m)}$ piece of $\mathcal{C}_c(w)$ are already skips of color 0 from the $w^{(1)}$ piece of the product, since the support of $w^{(2)} \cdots w^{(m)}$ is contained in the support of $w^{(1)}$. By Proposition 6.4.6, these skips of color 0 are not affected by the addition of the piece $w^{(2)} \cdots w^{(m)}$, and therefore account for the first piece of the delta sequence, cu_1^{-1} by the $m = 1$ bijection between sortable elements and delta sequences.

Second, since the covering reflections of $w^{(1)}$ correspond to a parabolic Coxeter element, conjugating by $w^{(1)}$ to map $W_{\mathsf{des}_R(w^{(1)})}$ to $W_{\mathsf{cov}_\downarrow(w^{(1)})}$, we have by induction that the skip set of $w^{(2)} \cdots w^{(m)}$ is indeed $(u_1u_2^{-1}, u_2u_3^{-1}, \cdots, u_m)$. $\square$

Example 6.9.8. We compute an extended example of Theorem 6.9.7, starting and ending at $\mathsf{NC}^{(m)}(\mathfrak{S}_4, c)$. Fix $m = 2$ and $c = s_1s_2s_3 \in \mathfrak{S}_4$. The element $(23)^{(0)}(34)^{(1)}(14)^{(2)} \in \mathsf{NC}_\Delta^{(m)}(\mathfrak{S}_4, c)$ corresponds under the map in Proposition 4.2.3 to the chain

$$(134) \geq_{\mathcal{R}} (14) \in \mathsf{NC}^{(m)}(\mathfrak{S}_4, c).$$

By Theorem 6.9.4, this chain corresponds to the shard sortable element

$$(w_1 \geq_{\mathrm{Sh}} w_2) = (s_1s_2s_3s_2 \geq_{\mathrm{Sh}} s_1s_2s_3) \in \mathsf{Sort}^{(m)}_{\mathsf{shard}}(\mathfrak{S}_4, c).$$

We use Theorem 6.9.5 to find the Garside factorization of an element of $\mathsf{Sort}^{(m)}(W, c)$ from this shard sortable element. We compute

$$\mathrm{inv}_{\mathcal{R}}(w_1) = \big\{(12), (13), (14), (34)\big\} \supseteq \big\{(12), (13), (14)\big\} = \mathrm{inv}_{\mathcal{R}}(w_2).$$

The covered reflections $\mathsf{cov}_\downarrow(w_1) = \{(13), (34)\}$ generate the (nonstandard) parabolic subgroup $W_{\mathsf{cov}_\downarrow(w_1)}$ with reflections $\{(13), (14), (34)\}$. The associated descents $\mathsf{des}_R(w_1) = \{s_3, s_2\}$ generate the standard parabolic subgroup $W_{\mathsf{des}_R(w_1)}$.

Under the first isomorphism $[e, w_1]_{\mathsf{Shard}(W)} \cong \mathsf{Shard}(W_{\mathsf{cov}_\downarrow(w_1)})$ of Proposition 2.10.6, the element $(w_2)_{\mathsf{cov}_\downarrow(w_1)} \in W_{\mathsf{cov}_\downarrow(w_1)}$ has inversions $\{(13), (14)\}$. Passing to the standard parabolic $\mathsf{Shard}(W_{\mathsf{des}_R(w_1)})$ by the second isomorphism of Proposition 2.10.6, the corresponding element of $W_{\mathsf{des}_R(w_1)}$ has inversions $\{(34), (24)\}$ and reduced $\mathcal{S}$-word s_3s_2. We conclude that

$$(s_1s_2s_3s_2 \geq_{\mathrm{Sh}} s_1s_2s_3) \mapsto s_1s_2s_3s_2 \cdot s_3s_2 \in \mathsf{Sort}^{(2)}_{\mathsf{fact}}(\mathfrak{S}_4).$$

This element is factorwise c-sortable by Example 6.4.2 and its skip set was computed in Example 6.8.1, recovering the initial colored factorization $(23)^{(0)}(34)^{(1)}(14)^{(2)} \in \mathsf{NC}_\Delta^{(m)}(\mathfrak{S}_4, c)$.

CHAPTER 7

Positive m-eralized structures

In this chapter, we study *positive m-eralized analogues* of the structures we have previously considered. We define and discuss their embeddings (Section 7.1), study their enumeration (Section 7.2) and their symmetries under Kreweras complements and Cambrian rotations (Section 7.3).

Home stretch—two easy chapters, then representation theory!

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7.1. The positive m-eralized structures and their embeddings

Recall that the support $\mathrm{supp}(w)$ of $w \in \boldsymbol{B}^+$ is the set $\{s_1, \ldots, s_p\}$ of simple reflections contained in any reduced $\mathcal{S}$-word $\mathsf{s}_1 \cdots \mathsf{s}_p$ for w. The natural bijections

$$\mathsf{Sort}^{(m)}(W,c) \overset{c}{\longleftrightarrow} \mathsf{Asso}^{(m)}(W,c) \overset{c}{\longleftrightarrow} \mathsf{NC}^{(m)}(W,c)$$

preserve the various notion of support given in Definitions 4.2.1 and 5.3.3.

The uniform underlying idea is to restrict attention from all Fuß-Catalan objects to those with full support. This is a well-worn path for $m = 1$, and has also been developed for m-eralized c-cluster complexes with bipartite c in [**FR05**, Section 12].

We define the following positive analogues together with natural embeddings:

$$\mathsf{Sort}_+^{(m)}(W,c) \hookrightarrow \mathsf{Sort}^{(m)}(W,c)$$
$$\mathsf{NC}_+^{(m)}(W,c) \hookrightarrow \mathsf{NC}^{(m)}(W,c)$$
$$\mathsf{Asso}_+^{(m)}(W,c) \hookrightarrow \mathsf{Asso}^{(m)}(W,c)\ .$$

An element of $\mathsf{NC}^{(m)}(W,c) \overset{c}{\longleftrightarrow} \mathsf{Asso}^{(m)}(W,c) \overset{c}{\longleftrightarrow} \mathsf{Sort}^{(m)}(W,c)$ is said to have *full support* if its support is all of $\mathcal{S}$. Restricting to those elements of full support gives a quick, uniform definition for positive versions of each construction. To better describe the symmetries of these positive structures, we prefer to give slightly different interpretations.

An element $w \in \mathsf{Sort}^{(m)}(W,c)$ has full support if and only if the sorting word $\mathsf{w}(\mathsf{c})$ starts with an initial copy of c. We define the *positive m-eralized c-sortable elements* to be

$$\big\{w \in [\boldsymbol{e}, \bar{\boldsymbol{c}}\boldsymbol{w}_\circ^m]_{\mathsf{Weak}(\boldsymbol{B}^+)}\ :\ w \text{ is } c\text{-sortable}\big\}.$$

We therefore recover the sortable elements of full support using the embedding

Take care, this is *not* the embedding as a subset.

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$$\mathsf{Sort}_+^{(m)}(W,c) \hookrightarrow \mathsf{Sort}^{(m)}(W,c)$$
$$\boldsymbol{w} \mapsto \boldsymbol{c}\boldsymbol{w}.$$

Multiplying an element less than or equal to $\bar{\boldsymbol{c}}\boldsymbol{w}_\circ^m$ by $\boldsymbol{c}$ evidently does not increase its Garside degree beyond m. This embedding has thus the correct image by Proposition 2.11.4. Some caution is warranted—it is already the case that

$\mathsf{Sort}_+^{(2)}(\mathfrak{S}_3, st)$	$\hookrightarrow$	$\mathsf{Sort}^{(2)}(\mathfrak{S}_3, st)$
· · |st|st	$\mapsto$	st|st|st
· · |st|st	$\mapsto$	st|st|st
· · |st|st	$\mapsto$	st|st|st
· · |st|st	$\mapsto$	st|st|st
· · |st|st	$\mapsto$	st|st|st
· · |st|st	$\mapsto$	st|st|st
· · |st|st	$\mapsto$	st|st|st

FIGURE 23. The positive m-eralized st-sortable elements $\mathsf{Sort}_+^{(2)}(\mathfrak{S}_3, st)$, with their embedding into $\mathsf{Sort}^{(2)}(\mathfrak{S}_3, st)$.

$\mathsf{NC}_+^{(2)}(\mathfrak{S}_3, st)$	$\hookrightarrow$	$\mathsf{NC}^{(2)}(\mathfrak{S}_3, st)$
· · t.sut.sut	$\mapsto$	sut.sut.sut
· · t.sut.sut	$\mapsto$	sut.sut.sut
· · t.sut.sut	$\mapsto$	sut.sut.sut
· · t.sut.sut	$\mapsto$	sut.sut.sut
· · t.sut.sut	$\mapsto$	sut.sut.sut
· · t.sut.sut	$\mapsto$	sut.sut.sut
· · t.sut.sut	$\mapsto$	sut.sut.sut

FIGURE 24. The positive m-eralized st-noncrossing partitions $\mathsf{NC}_+^{(2)}(\mathfrak{S}_3, st)$.

$\mathsf{Sort}_+^{(m)}(W, c) \subseteq \mathsf{Sort}^{(m)}(W, c)$, but this is not the intended embedding. Figure 23 illustrates the example in $\mathfrak{S}_3$ with $m = 2$.

A facet I of the dual subword complex $\mathsf{NC}_\Delta^{(m)}(W, c)$ has full support if and only if it does not contain any letter from the first copy of c of the initial copy of $w_\circ$ of the search word $\mathsf{w}_\circ^{m+1}(\mathsf{c})$. That is, I has full support if and only if

$$I \cap \left\{ \mathsf{r}^{(0)} \;:\; \mathsf{r} \in \mathsf{inv}_{\mathcal{R}}(c) \right\} = \emptyset.$$

We define the *positive m-eralized c-noncrossing partitions* to be

$$\mathsf{NC}_+^{(m)}(W, c) := \mathrm{SUB}_{\mathcal{R}}(\bar{\mathsf{c}}\mathsf{w}_\circ^{m+1}(\mathsf{c}), c).$$

On the level of the colored reflection sequences, the inclusion $\bar{\mathsf{c}}\mathsf{w}_\circ^{m+1}(\mathsf{c}) \hookrightarrow \mathsf{w}_\circ^{m+1}(\mathsf{c})$ conjugates every reflection by c. Therefore, reinserting the initial missing copy of c defines an embedding

$$\begin{aligned} \mathsf{NC}_+^{(m)}(W, c) &\hookrightarrow \mathsf{NC}^{(m)}(W, c) \\ I &\mapsto \{i + n \;:\; i \in I\}. \end{aligned}$$

Since the product of the reflections in a facet of $\mathsf{NC}_+^{(m)}(W, c)$ is an $\mathcal{R}$-word for c, this conjugation gives another $\mathcal{R}$-word for c. The image is therefore exactly those facets with full support. Figure 23 illustrates the example in $\mathfrak{S}_3$ with $m = 2$.

As in the case of $\mathsf{NC}_\Delta^{(m)}(W, c)$, a facet I of the subword complex $\mathsf{Asso}_\Delta^{(m)}(W, c)$ has full support if and only if it does not contain any letter from the first copy of c of the initial copy of $w_\circ$ in the search word $\mathsf{c}\mathsf{w}_\circ^{m}(\mathsf{c})$. We define the *positive*

$\mathsf{Asso}_{+}^{(2)}(\mathfrak{S}_3, st)$	$\hookrightarrow$	$\mathsf{Asso}_{\Delta}^{(2)}(\mathfrak{S}_3, st)$
$\cdots$.sts.tst	$\mapsto$	st.sts.tst
$\cdots$.sts.tst	$\mapsto$	st.sts.tst
$\cdots$.sts.tst	$\mapsto$	st.sts.tst
$\cdots$.sts.tst	$\mapsto$	st.sts.tst
$\cdots$.sts.tst	$\mapsto$	st.sts.tst
$\cdots$.sts.tst	$\mapsto$	st.sts.tst
$\cdots$.sts.tst	$\mapsto$	st.sts.tst

FIGURE 25. The positive m-eralized st-clusters $\mathsf{Asso}_{+}^{(2)}(\mathfrak{S}_3, st)$.

m-eralized c-cluster complex to be

$$\mathsf{Asso}_{+}^{(m)}(W,c) := \mathrm{SUB}_{\mathcal{S}}(\mathsf{w}_\circ^m(\mathsf{c}), \bar{c}w_\circ^m, mN-n) = \mathrm{SUB}_{\boldsymbol{\mathcal{S}}}^{\boldsymbol{B}}(\mathsf{w}_\circ^m(\mathsf{c}), \bar{\boldsymbol{c}}\boldsymbol{w}_\circ^m).$$

The inclusion $\mathsf{w}_\circ^m(\mathsf{c}) \hookrightarrow \mathsf{cw}_\circ^m(\mathsf{c})$ gives the embedding

$$\begin{aligned} \mathsf{Asso}_{+}^{(m)}(W,c) &\hookrightarrow \mathsf{Asso}_{\Delta}^{(m)}(W,c) \\ I &\mapsto \{i+n \;:\; i \in I\}\,. \end{aligned} \tag{7.1}$$

By Definition 5.3.3, the image of this embedding recovers those elements of full support. Figure 25 illustrates the example in $\mathfrak{S}_3$ with $m = 2$.

7.2. Enumeration of positive m-eralized structures

Before discussing some structural properties of the positive constructions, we recall that they are uniformly enumerated by the *positive Fuß-Catalan* (or *Fuß-Dogolon*[E]) *numbers of type W*, defined by

I'm almost positive this nomenclature has my full support.

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$$\mathsf{Cat}_{+}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i - 2}{d_i}. \tag{7.2}$$

We refer to [**FR05**, Corollary 12.4] for a proof of this is the counting formula for $\mathsf{Asso}_{+}^{(m)}(W,c)$. Formally extending (1.1) to accommodate negative numbers, we obtain $\mathsf{Cat}_{+}^{(m)}(W) = (-1)^n \mathsf{Cat}^{(-m-1)}(W)$. This purely enumerative identity may be seen as an instance of combinatorial *reciprocity* [**Ath05**, Corollary 1.3].

THEOREM 7.2.1. *We have*

$$\left|\mathsf{Sort}_{+}^{(m)}(W,c)\right| = \left|\mathsf{NC}_{+}^{(m)}(W,c)\right| = \left|\mathsf{Asso}_{+}^{(m)}(W,c)\right| = \mathsf{Cat}_{+}^{(m)}(W).$$

PROOF. By the embedding $\mathsf{Asso}_{+}^{(m)}(W,c) \hookrightarrow \mathsf{Asso}_{\Delta}^{(m)}(W,c) \longrightarrow \mathsf{Asso}^{(m)}(W,c)$, Corollary 5.5.7 gives a bijection between $\mathsf{Asso}_{+}^{(m)}(W,c)$ and facets in the Fomin-Reading generalized cluster complex that do not use m-colored simple roots. The enumeration for $\mathsf{Asso}_{+}^{(m)}(W,c)$ now follows from [**FR05**, Proposition 12.4]. As the bijections

$$\mathsf{NC}^{(m)}(W,c) \longleftrightarrow \mathsf{Asso}^{(m)}(W,c) \longleftrightarrow \mathsf{Sort}^{(m)}(W,c)$$

respect support, we conclude the result for the remaining objects. □

We obtain the following counting formula analogous to Theorem 4.5.1.

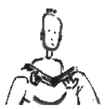
[E]This nomenclature is due to D. Bessis or comes from the fact that these numbers *lay doggo* without being discovered for far longer than the classical Catalan numbers.

THEOREM 7.2.2. *We have*

$$(1-q)^{n+1}\sum_{m=0}^{\infty}\left|\mathsf{NC}_{+}^{(m)}(W,c)\right|q^m = \sum_{\mathsf{r}_1\cdots\mathsf{r}_n\in\mathrm{Red}_{\mathcal{R}}(c)} q^{\mathrm{asc}(\mathsf{r}_1\cdots\mathsf{r}_n)}$$

where $\mathrm{asc}(\mathsf{r}_1,\ldots,\mathsf{r}_n)$ *is the number of ascents* $r_i <_c r_{i+1}$ *in the reflection order induced by the Coxeter element c.*

Look at the two formulas, aren't they cute.



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PROOF. This is a direct consequence of (7.1), after observing that this formula is equivalent to

$$\left|\mathsf{NC}_{+}^{(m)}(W,c)\right| = \sum_{\mathsf{r}_1\cdots\mathsf{r}_n\in\mathrm{Red}_{\mathcal{R}}(c)}\binom{m+\mathrm{des}(\mathsf{r}_1\cdots\mathsf{r}_n)}{n}$$

where $\mathrm{des}(\mathsf{r}_1,\ldots,\mathsf{r}_n)$ is the number of ascents $r_i >_c r_{i+1}$. The only difference to the proof of Theorem 4.5.1 is that we now how only m copies of $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ(\mathsf{c}))$ in the search word, and we search for c^{-1} rather than for c. □

7.3. Symmetries of the positive m-eralized structures

Cambrian rotation realizes a remarkable symmetry on the m-eralized c-cluster complex $\mathsf{Asso}^{(m)}(W,c)$ (Definition 5.4.3). This symmetry is lost when studying the positive m-eralized c-cluster complex $\mathsf{Asso}_{+}^{(m)}(W,c)$—in fact, $\mathsf{Asso}_{+}^{(m)}(W,c)$ and $\mathsf{Asso}_{+}^{(m)}(W,\overline{s}cs)$ are not necessarily isomorphic simplicial complexes.

The situation is reversed for the complexes $\mathsf{NC}^{(m)}(W,c)$ and $\mathsf{NC}_{+}^{(m)}(W,c)$. Although the dual subword complexes $\mathsf{NC}^{(m)}(W,c)$ and $\mathsf{NC}^{(m)}(W,\overline{s}cs)$ are not necessarily isomorphic, it turns out that $\mathsf{NC}_{+}^{(m)}(W,c)$ and $\mathsf{NC}_{+}^{(m)}(W,\overline{s}cs)$ *are* isomorphic Analogously to the situation for $\mathsf{Asso}^{(m)}(W,c)$, composing this isomorphism n times in the order induced by c realizes a symmetry on the complex. This symmetry is illustrated in Figure 26.

We explain this symmetry by defining a shift operation on $\mathsf{NC}_{+}^{(m)}(W,c)$, analogously to (5.6) for $\mathsf{Asso}(W,c)$. For s initial in c, Lemma 2.6.5 implies that the search words $\mathrm{inv}_{\mathcal{R}}(\mathsf{Q})$ for $\mathsf{NC}_{+}^{(m)}(W,c)$ and $\mathrm{inv}_{\mathcal{R}}(\mathsf{Q}')$ for $\mathsf{NC}_{+}^{(m)}(W,\overline{s}cs)$ are related by

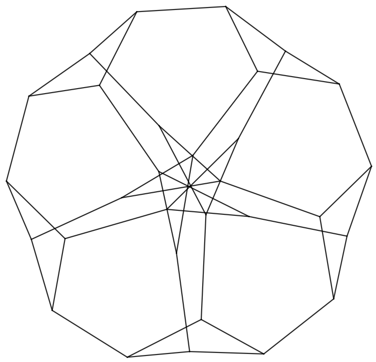
$$\overline{\mathsf{s}}\mathsf{Q}\psi^{m+1}(\mathsf{s}) \equiv \mathsf{Q}'. \tag{7.3}$$

This relationship between the search words Q and Q' induces a bijection on the facets of $\mathsf{NC}_{+}^{(m)}(W,c)$ and $\mathsf{NC}_{+}^{(m)}(W,\overline{s}cs)$. For s initial in c, define

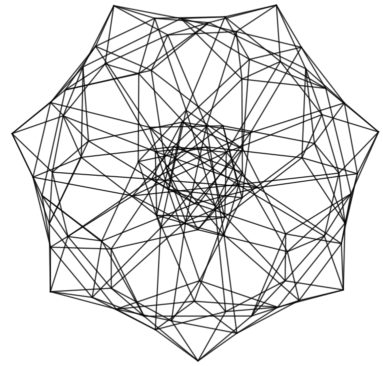
$$\begin{aligned}\mathsf{Shift}_s : \mathsf{NC}_{+}^{(m)}(W,c) &\longrightarrow \mathsf{NC}_{+}^{(m)}(W,\overline{s}cs)\\ I &\longmapsto \mathsf{Shift}_s(I)\end{aligned}$$

using the identification provided by (7.3).

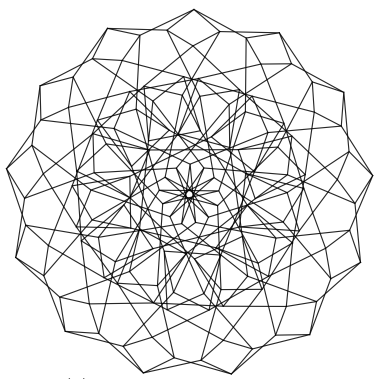
We check that this is a bijection. Recall that every facet I of $\mathsf{NC}_{+}^{(m)}(W,c)$ spells out a reduced $\mathcal{R}$-word for c. Write the reflections corresponding to the positions of I as the reduced $\mathcal{R}$-word $\mathsf{t}_1\mathsf{t}_2\mathsf{t}_3\cdots\mathsf{t}_n$ for c. If the first letter of $\mathrm{inv}_{\mathcal{R}}(\mathsf{Q})$ is not in I, then $\mathsf{Shift}_s(I)$ spells out the reduced $\mathcal{R}$-word $\mathsf{t}_1^s\mathsf{t}_2^s\mathsf{t}_3^s\cdots\mathsf{t}_n^s$ for $\overline{s}cs$, and so specifies a facet in $\mathsf{NC}_{+}^{(m)}(W,\overline{s}cs)$. Otherwise, the first letter of Q—corresponding to the reflection s—is in I. The reflection corresponding to the last letter in $\overline{\mathsf{s}}\mathsf{Q}\psi^{m+1}(\mathsf{s})$ is s^{cs}, so that



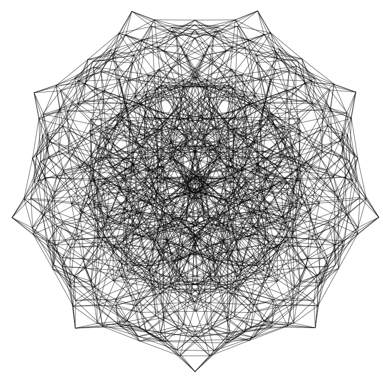
$\mathsf{NC}_+^{(2)}(\mathfrak{S}_4, c)$ with 30 facets



$\mathsf{NC}_+^{(3)}(\mathfrak{S}_4, c)$ with 91 facets



$\mathsf{NC}_+^{(2)}(\mathfrak{S}_4, c)$ with 143 facets

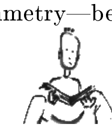


$\mathsf{NC}_+^{(3)}(\mathfrak{S}_4, c)$ with 612 facets

FIGURE 26. Some examples of the symmetries of $\mathsf{NC}_+^{(m)}(W, c)$ under the positive Kreweras complement Krew_c^+.

the map $\mathsf{Shift}_s(I)$ spells out the $\mathcal{R}$-word $\mathsf{t}_2^s \mathsf{t}_3^s \cdots \mathsf{t}_n^s \mathsf{s}^{cs}$. We compute $t_2^s t_3^s \cdots t_n^s s^{cs} = (cs)s^{cs} = \overline{s}cs$, as desired. It is now clear that Shift_s is an isomorphism.

Another source of symmetry—beautiful.



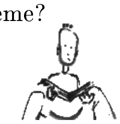
#33

We summarize the preceding discussion with the following theorem.

THEOREM 7.3.1. *For any two Coxeter elements c, c', the simplicial complexes $\mathsf{NC}_+^{(m)}(W, c)$ and $\mathsf{NC}_+^{(m)}(W, c')$ are isomorphic.* □

Composing shifts now defines a map from $\mathsf{NC}_+^{(m)}(W, c)$ to itself.

Am I supposed to understand this naming scheme?



#34

DEFINITION 7.3.2. Let $\mathsf{w}_\circ(\mathsf{c}) = \mathsf{s}_1 \mathsf{s}_2 \cdots \mathsf{s}_N$ be the c-sorting word for $w_\circ$. Define the *positive Kreweras complement* $\mathsf{Krew}_c^+ : \mathsf{NC}_+^{(m)}(W, c) \to \mathsf{NC}_+^{(m)}(W, c)$ by

$$\mathsf{Krew}_c^+ := \mathsf{Shift}_{s_N} \circ \cdots \circ \mathsf{Shift}_{s_2} \circ \mathsf{Shift}_{s_1}.$$

REMARK 7.3.3. Combinatorial interpretations of m-eralized noncrossing partitions in classical types in terms of set partitions have been studied in [**Arm06**].

Although evocative of Cambrian rotation, it is more appropriate to call this composition the *positive Kreweras complement*—it may be shown to act as a rotation of combinatorial models for positive Fuß-Catalan noncrossing partitions in classical types [**KS18**].

We now compute the order of Krew_c^+. Since conjugation by $w_\circ$ is an involution, $h = 2|\mathcal{R}|/|\mathcal{S}|$, and there are exactly $(m+1)|\mathcal{R}| - |\mathcal{S}|$ letters used to construct $\mathsf{NC}_+^{(m)}(W,c)$,

$$\big(\mathsf{Krew}_c^+\big)^{(m+1)h-2} \equiv \mathbb{1}.$$

THEOREM 7.3.4. *The order of the positive Kreweras complement is given by*

$$\operatorname{ord}(\mathsf{Krew}_c^+) = \begin{cases} (m+1)h/2 - 1, & \textit{if } \psi \equiv \mathbb{1} \textit{ or } \big(n = 2 \textit{ and } m = 1\big), \\ (m+1)h - 2, & \textit{otherwise.} \end{cases}$$

This is the same order as the positive Panyushev map on order ideals in root posets!

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PROOF. By the discussion above, $\big(\mathsf{Krew}_c^+\big)^{(m+1)h-2} \equiv \mathbb{1}$. When $\psi \equiv \mathbb{1}$, the symmetry of order 2 gives $\big(\mathsf{Krew}_c^+\big)^{(m+1)h/2-1} \equiv \mathbb{1}$. We conclude by showing that for any two letters in Q, there is a positive m-divisible noncrossing partition containing exactly one. Using the positive Kreweras complement, it is simple to see that this is the case if we are in rank bigger than 2. Thus, the only special cases are the dihedral groups for which $\psi \not\equiv \mathbb{1}$. For those, the situation is fine if $m > 1$, while we also have a symmetry of order 2 otherwise. This completes the proof. □

In the following theorem, we do not mean flips as defined in Section 4.4.2, but honest subword complex flips.

THEOREM 7.3.5. *The flip graph of* $\mathsf{NC}_+^{(m)}(W,c)$ *is regular with degree* $n(m-1)$.

Does this embed into the complexified hyperplane complement?

#50

PROOF. Choose a facet I of the dual subword complex $\mathsf{NC}_+^{(m)}(W,c)$ and a letter in the facet to flip. If the rank of W is one, this letter can flip to any other letter and so has the desired degree. Otherwise the rank is greater than one and we may find another letter in the facet. By the symmetry of the Shift_s maps, we can move this second letter to be the leftmost reflection s in the leftmost copy of $\mathsf{w}_\circ(\mathsf{c})$ in the search word. It follows from Lemma 2.9.1 that the facet I without this new first letter is naturally a facet of $\mathsf{NC}_+^{(m)}(W_{\langle s\rangle}, \bar{s}c)$. We conclude the result by induction on the rank of W. □

CHAPTER 8

Conjectures on rational structures

In this chapter, we give an elementary definition of the rational Catalan numbers (Section 8.1), and then discuss a possible approach and its limitations for constructing noncrossing combinatorial objects counted by the rational Catalan numbers (Section 8.2). For the sake of completeness, we begin the latter considerations with a brief discussion of the ∞-eralized structures.

8.1. Rational Catalan numbers

From a purely enumerative perspective, it is believed that noncrossing Catalan objects ought to have combinatorial generalizations beyond their m-eralization and positive m-eralization. Such generalizations have been studied in type A, and we refer to [**ARW13, ALW16, GM16**] for details.

Definition 8.1.1. Let W be a finite Coxeter group and let p be a positive integral parameter coprime to the Coxeter number h. The *rational Catalan number of type W* is given by

$$\mathsf{Cat}^{[p]}(W) := \prod_{i=1}^{n} \frac{p + (pe_i \bmod h)}{d_i},$$

where $e_1 = d_1 - 1 \leq \ldots \leq e_n = d_n - 1$ are the *exponents* of W.

For Coxeter groups and $p = mh \pm 1$, this formula recovers the Fuß-Catalan and positive Fuß-Catalan numbers from (1.1) and (7.2). Figure 27 lists some rational Catalan numbers in the classical types.

If the Coxeter group W is crystallographic, it is well-known that the two sets $\{pe_1, \ldots, pe_n\}$ and $\{e_1, \ldots, e_n\}$ coincide modulo h (for p coprime to h). In this case, Definition 8.1.1 simplifies to

$$\mathsf{Cat}^{[p]}(W) = \prod_{i=1}^{n} \frac{p + e_i}{d_i}. \tag{8.1}$$

This formula has a combinatorial interpretation—for crystallographic W, M. Haiman showed that $\mathsf{Cat}^{[p]}(W)$ counts W-orbits of Q/pQ (where Q is the root lattice) [**Hai94**, Theorem 7.4.4].

We are getting a bit sidetracked here.

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A general formula for the rational Catalan numbers was first considered by I. Gordon and S. Griffeth in the context of *rational Cherednik algebras* associated to complex reflection groups [**GG12**]. We refer to [**LT09**] for all needed background on complex reflection groups and extend Definition 8.1.1 to all well-generated complex reflection groups. To see that this definition coincides with the definition in [**GG12**], we recall some notions from T. A. Springer's seminal paper on regular

p	1	2	3	4	5	6	7	8	9	10	11	12	13
A_1	1	·	2	·	3	·	4	·	5	·	6	·	7
A_2	1	2	·	5	7	·	12	15	·	22	26	·	35
A_3	1	·	5	·	14	·	30	·	55	·	91	·	140
A_4	1	3	7	14	·	42	66	99	143	·	273	364	476
A_5	1	·	·	·	42	·	132	·	·	·	728	·	1428
A_6	1	4	12	30	66	132	·	429	715	1144	1768	2652	3876
B_2	1	·	3	·	6	·	10	·	15	·	21	·	28
B_3	1	·	·	·	10	·	20	·	·	·	56	·	84
B_4	1	·	5	·	15	·	35	·	70	·	126	·	210
B_5	1	·	6	·	·	·	56	·	126	·	252	·	462
B_6	1	·	·	·	28	·	84	·	·	·	462	·	924
D_4	1	·	·	·	20	·	50	·	·	·	196	·	336
D_5	1	·	7	·	27	·	77	·	182	·	378	·	714
D_6	1	·	8	·	·	·	112	·	294	·	672	·	1386

FIGURE 27. Some rational Catalan numbers of classical types.

elements [**Spr74**]. We refer to that paper for further definitions and background on complex reflection groups.

Let W be a well-generated complex reflection group acting irreducibly on a complex n-dimensional vector space V. It is well-known that W acts on the polynomial ring $\mathbb{C}[V]$ and that the coinvariant algebra $\mathbb{C}[V]/\mathbb{C}[V]^W_+$ carries the regular representation of W. For an irreducible representation $\varphi \in \mathrm{Irred}(W)$, the *$\varphi$-exponents* $e_1(\varphi) \leq \ldots \leq e_{\dim(\varphi)}(\varphi)$ are defined to be the degrees of the graded components of $\mathbb{C}[V]/\mathbb{C}[V]^W_+$ in which the $\dim(\varphi)$ copies of φ live. Its generating function

$$f(\varphi, q) = \sum_{i=1}^{\dim(\varphi)} q^{e_i(\varphi)} = \sum_{k \geq 0} \left[\mathbb{C}[V]/\mathbb{C}[V]^W_+; \varphi\right]_k q^k$$

is called the *fake degree polynomial* of φ, where $\left[\mathbb{C}[V]/\mathbb{C}[V]^W_+; \varphi\right]_k$ denotes the multiplicity of φ inside the k^{th} graded component of the coinvariant algebra.

Let $h = e_n(V) + 1$ be the Coxeter number of W, let ζ be a primitive h^{th} root of unity, and for p coprime to h let $g : \mathbb{C} \to \mathbb{C}$ be the automorphism sending ζ to ζ^p. In [**GG12**], the *rational Catalan number of type W* is defined as

$$\mathsf{Cat}^{[p]}(W) := \prod_{i=1}^{n} \frac{p + e_i(g(V))}{d_i}. \tag{8.2}$$

These rational Catalan numbers are dimensions of certain modules over the rational Cherednik algebra associated to W and the rational parameter p/h. In particular, these numbers are indeed integral—which was not obvious from Definition 8.1.1.

We deduce from Springer theory that Definition 8.1.1 and (8.2) agree.

PROPOSITION 8.1.2. *Let W be a well-generated complex reflection group acting irreducibly on a complex n-dimensional vector space V. The fake degree polynomials of V and of $g(V)$ at primitive h^{th} roots of unity are related by*

$$f(g(V), \zeta) = f(V, \zeta^p).$$

In particular, the two sets $\{e_1(g(V)), \ldots, e_n(g(V))\}$ *and* $\{pe_i, \ldots, pe_n\}$ *coincide modulo* h.

PROOF. Let c be a ζ-regular element of W (which is known to exist in well-generated groups). Then $g(c)$ is ζ^p-regular, and the eigenvalues of $g(c)$ are obtained from the eigenvalues of c by replacing every ζ by ζ^p.

Since both c and c^p satisfy the assumption in [**Spr74**, Section 2.5], we apply [**Spr74**, Proposition 4.5] for both ζ and ζ^p, expressing the respective fake degree polynomials in terms of the eigenvalues. This gives the desired relation between the fake degree polynomials of V and of $g(V)$. □

REMARK 8.1.3. This definition of the rational Catalan numbers has the q-analogue

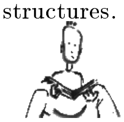
$$\mathsf{Cat}^{[p]}(W;q) := \prod_{i=1}^{n} \frac{\big[p + (pe_i \bmod h)\big]_q}{\big[d_i\big]_q},$$

where $[a]_q = 1 + q + \ldots q^{a-1}$ is the usual q-integer. This q-number appears as a graded Hilbert series in the context of rational Cherednik algebras [**GG12**].

8.2. ∞-eralized structures and possible rational restrictions

Rational Dyck paths and their combinatorial properties have been the subject of considerable recent research [**ARW13, ALW16, GM16, Bod17**]. Such paths are, by construction, counted by the rational Catalan numbers of type A. As such, they belong to the *nonnesting* rational Fuß-Catalan structures and do not play a prominent role here. On the other hand, the search for *noncrossing* rational Fuß-Catalan structures was initiated for type A in [**ARW13**]. In this section, we propose an approach to such structures in classical types.

It's not quite working, even though the definitions seem to be tailored also for the rational structures.



#37

Yeah, it seems that Coxeter-initial complexes ought to be the right thing.

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We attempt to follow the Catalan to Fuß-Catalan to positive Fuß-Catalan progression to its logical conclusion. The central results of this monograph establish that this progression replaces $w_\circ \in W$ by $\boldsymbol{w}_\circ^m \in \boldsymbol{B}^+$ to produce m-eralizations, and then further by $\bar{\boldsymbol{c}}\boldsymbol{w}_\circ^m$ for positive m-eralizations.

For p coprime to h, it therefore seems reasonable to search for initial segments w_p of c^∞, from which we could define *rational c-sortable elements*, *rational c-noncrossing partitions*, and the *rational c-cluster complex* by

$$\begin{aligned}
\mathsf{Sort}^{[p]}(W,c) &:= \big\{\boldsymbol{w} \in [\boldsymbol{e}, \boldsymbol{w}_p]_{\mathsf{Weak}(\boldsymbol{B}^+)} \ : \ \boldsymbol{w} \text{ is } c\text{-sortable}\big\},\\
\mathsf{NC}_{\Delta}^{[p]}(W,c) &:= \mathrm{SUB}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_{p+h}(\mathsf{c})), c), \text{ and}\\
\mathsf{Asso}_{\Delta}^{[p]}(W,c) &:= \mathrm{SUB}_{\boldsymbol{\mathcal{S}}}^{\boldsymbol{B}}(\mathsf{cw}_p(\mathsf{c}), \boldsymbol{w}_p).
\end{aligned}$$

The goal is to find $\boldsymbol{w}_p \in \boldsymbol{B}^+$ having a reduced word $\mathsf{w}_p(\mathsf{c})$ initial in c^∞ so that all three definitions give objects counted by the rational Catalan number $\mathsf{Cat}^{[p]}(W)$.

It turns out to be convenient to first study the situation for $m \to \infty$.

8.2.1. ∞-eralized structures. We may view $\mathsf{w}_\circ^m(\mathsf{c})$ as an initial subsequence of $\mathsf{w}_\circ^{m+1}(\mathsf{c})$ and thereby interpret $\mathsf{Asso}_{\Delta}^{(m)}(W,c)$ as a subset of $\mathsf{Asso}_{\Delta}^{(m+1)}(W,c)$ and similarly for $\mathsf{NC}^{(m)}_{\Delta}(W,c)$ and $\mathsf{Sort}^{(m)}(W,c)$. We then define

$$\mathsf{Asso}^{(\infty)}_{\Delta}(W,c) := \bigcup_{m\geq 1} \mathsf{Asso}^{(m)}_{\Delta}(W,c),$$
$$\mathsf{NC}^{(\infty)}_{\Delta}(W,c) := \bigcup_{m\geq 1} \mathsf{NC}^{(m)}_{\Delta}(W,c),$$
$$\mathsf{Sort}^{(\infty)}(W,c) := \bigcup_{m\geq 1} \mathsf{Sort}^{(m)}(W,c),$$

and consider $\mathsf{Camb}^{(\infty)}_{\mathsf{Sort}}(W,c)$ as the restriction of $\boldsymbol{B}^{+}$ to $\mathsf{Sort}^{(\infty)}(W,c)$. (We could also equivalently define $\mathsf{Camb}^{(\infty)}_{\mathsf{NC}}$ using $\mathsf{Asso}^{(\infty)}_{\Delta}(W,c)$ or $\mathsf{NC}^{(\infty)}_{\Delta}(W,c)$.)

Since the bijection $\mathsf{Asso}^{(m)}_{\Delta}(W,c) \longleftrightarrow \mathsf{Sort}^{(m)}(W,c)$ is given by the skip set, and $\mathsf{Asso}^{(m)}_{\Delta}(W,c) \longleftrightarrow \mathsf{NC}^{(m)}(W,c)$ is given by the root configuration, we also obtain bijections

$$\mathsf{Sort}^{(\infty)}(W,c) \longleftrightarrow \mathsf{NC}^{(\infty)}_{\Delta}(W,c) \longleftrightarrow \mathsf{Asso}^{(\infty)}_{\Delta}(W,c).$$

We moreover obtain for any initial segment w_p of c^{∞}, that these bijections restrict to bijections

$$\mathsf{Sort}^{[p]}(W,c) \longleftrightarrow \mathsf{NC}^{[p]}_{\Delta}(W,c) \longleftrightarrow \mathsf{Asso}^{[p]}_{\Delta}(W,c).$$

8.2.2. A first counterexample. Even though m-eralized and the positive m-eralized structures both fit into this framework, our expectations were somewhat diminished by the following example (checked by computer).

OBSERVATION 8.2.1. *For $\mathfrak{S}_6$ with bipartite $c = (12)(34)(56)\cdot(23)(45)$, there does not exist an initial w_p of c^{∞} such that*

$$|\mathrm{SUB}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{c}\mathsf{w}_p, \boldsymbol{w}_p)| = \mathsf{Cat}^{[5]}(\mathfrak{S}_6) = 66.$$

One still might maintain the following hope.

HOPE 8.2.2. Let $(W,\mathcal{S})$ be a finite Coxeter system with p coprime to h. Then there exists *some* Coxeter element c and a word w_p initial in $\mathsf{w}_\circ(\mathsf{c})$ for which

$$|\mathsf{Sort}^{[p]}(W,c)| = |\mathsf{NC}^{[p]}_{\Delta}(W,c)| = |\mathsf{Asso}^{[p]}_{\Delta}(W,c)| = \mathsf{Cat}^{[p]}(W),$$

with $\mathsf{w}_{p+h}(\mathsf{c}) \equiv \mathsf{w}_p(\mathsf{c})\mathsf{w}_\circ(\mathsf{c})$.

Since such a w_p is initial in c^{∞}, $\mathsf{Asso}^{[p]}(W,c)$ is a c-initial subword complex, and hence is vertex decomposable by Theorem 3.6.2. The relation between $\mathsf{w}_{p+h}(\mathsf{c})$ and $\mathsf{w}_p(\mathsf{c})$ implies that for $p > h$, $\mathsf{Asso}^{[p]}(W,c)$ has the homotopy type of a wedge of $\mathsf{Cat}^{[p-h]}(W)$ spheres of dimension $n-1$.

8.2.3. Conjectural rational constructions. The constructions in this section provide conjectural combinatorial models for rational noncrossing structures in all infinite families of finite Coxeter systems.

Amazing.

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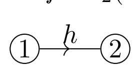
THEOREM 8.2.3. *Hope 8.2.2 holds for $I_2(h)$ with Coxeter element c given by*



and $\boldsymbol{w}_p = \boldsymbol{c}^{(p-1)/2}$.

PROOF. Write s, t for the simple reflections, and fix the Coxeter element $c = st$. We prove the statement by explicitly counting

$$\mathsf{NC}_{\Delta}^{[p]}(W,c) = \mathrm{SUB}_{\mathcal{R}}\big(\,\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_{p+h}(\mathsf{c})), c\big).$$

We immediately obtain that

$$\begin{aligned} |\mathsf{NC}_{\Delta}^{[p]}(W,c)| &= |\mathsf{NC}_{\Delta}^{[p-h]}(W,c)| + \ell_{\mathcal{S}}(\boldsymbol{w}_p) + 1 \\ &= |\mathsf{NC}_{\Delta}^{[p-h]}(W,c)| + p \end{aligned}$$

where $\ell_{\mathcal{S}}(\boldsymbol{w}_p) = p - 1$ and $|\mathsf{NC}_{\Delta}^{[p-h]}(W,c)| = 0$ for $p \leq h$. The first summand in this expression comes from those facets using no reflection from the initial copy of $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_{\circ}(\mathsf{c}))$, the second summand from those using a single reflection in this initial copy, and the final 1 comes from the single facet entirely contained in the initial copy.

We compute for any p that

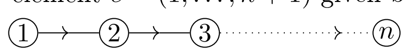
$$\begin{aligned} \mathsf{Cat}^{[p]}(W) &= \frac{1}{2h}\Big[\big(p + (p \bmod h)\big)\big(p + (-p \bmod h)\big)\Big] \\ &= \frac{1}{2h}\Big[p^2 + ph + (p \bmod h)(-p \bmod h)\Big] \\ &= \frac{1}{2h}\Big[p^2 - ph + (p \bmod h)(-p \bmod h)\Big] + p \\ &= \frac{1}{2h}\Big[p^2 - 2ph + h^2 + (p-h)h + (p \bmod h)(-p \bmod h)\Big] + p \\ &= \frac{1}{2h}\Big[(p-h)^2 + (p-h)h + (p \bmod h)(-p \bmod h)\Big] + p \\ &= \frac{1}{2h}\Big[\big((p-h) + (p \bmod h)\big)\big((p-h)(-p \bmod h)\big)\Big] + p \\ &= \mathsf{Cat}^{[p-h]}(W) + p, \end{aligned}$$

with $\mathsf{Cat}^{[p]}(W) = p$ for $1 \leq p \leq h$. □

We conjecture Hope 8.2.2 to hold in types A_n, B_n, D_n, and H_3.

CONJECTURE 8.2.4. Hope 8.2.2 holds for

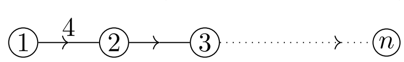
(1) type A_n with Coxeter element $c = (1, \ldots, n+1)$ given by



and for $p = mh + r$ coprime to h with $0 \leq r < h$

$$\boldsymbol{w}_p = \boldsymbol{c}_{a_{r-1}} \cdots \boldsymbol{c}_{a_1} \boldsymbol{w}_{\circ}^m, \quad a_i := \lfloor ih/r \rfloor, \quad \boldsymbol{c}_j := \boldsymbol{s}_1 \boldsymbol{s}_2 \cdots \boldsymbol{s}_j,$$

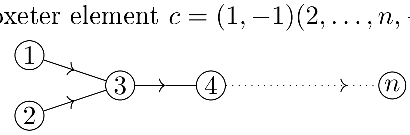
(2) type B_n with Coxeter element $c = (1, \ldots, n, -1, \ldots, -n)$ given by



and

$$\boldsymbol{w}_p = \boldsymbol{c}^{(p-1)/2},$$

(3) type D_n with any Coxeter element $c = (1, -1)(2, \ldots, n, -2, \ldots, -n)$ given by



and

$$\boldsymbol{w}_p = \boldsymbol{c}^{(p-1)/2},$$

We have computed it several times, it seems to *really* fail.

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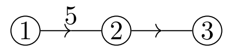
(4) type H_3 with Coxeter element c given by



and

$$\boldsymbol{w}_p = \boldsymbol{c}^{(p-1)/2},$$

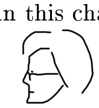
REMARK 8.2.5. Unfortunately, this hope fails to work in at least some other types. For example, in types F_4 and H_4, there do not exist Coxeter elements c and words w_p initial in c^∞ such that Hope 8.2.2 holds.

CHAPTER 9

m-eralized structures in representation theory

In this chapter, we exhibit the connection between the Catalan combinatorics which we have been studying and the representation theory of finite-dimensional hereditary Artin algebras. Our main references for this representation theory are [**DR76, Hap88, ARS97**].

If you were wondering where I was, I've mostly been hanging out in this chapter.



#1

We give a more detailed introduction to the connections to be explored in this chapter (Section 9.1) and present some combinatorial constructions which match directly with the combinatorics built into the representation theory (Section 9.2). We then give a quick introduction to the representation theory of Artin algebras (Section 9.3). We introduce exceptional sequences, and explain their equivalence to $\mathcal{R}$-factorizations of Coxeter elements (Section 9.4). We use exceptional sequences to give representation-theoretic interpretations of m-eralized clusters and m-eralized noncrossing partitions (Section 9.5), and of the natural bijection between them (Section 9.6). We show that these interpretations can be extended to the positive setting (Section 9.7). We show that certain symmetries of $\mathsf{Asso}_{\nabla}^{(m)}(W,c)$ and $\mathsf{NC}_{+}^{(m)}(W,c)$ can be accounted for representation-theoretically (Section 9.8). We give a representation-theoretic interpretation of m-eralized c-sortable elements (Section 9.9), and a representation-theoretic analogue of the bijection to m-eralized clusters (Section 9.10).

9.1. Overview

$\mathsf{Asso}_{\nabla}^{(m)}(W,c)$ and $\mathsf{NC}_{\Delta}^{(m)}(W,c)$ are known to admit interpretations in terms of representation theory of hereditary Artin algebras. See, for example, [**BRT12**], where this perspective was employed to give the first definition of a uniform bijection between these two sets. In view of these existing interpretations, it is satisfying to provide a similar interpretation for m-eralized sortable elements and m-eralized Cambrian lattices. In this chapter, we show that the elements of $\mathsf{Sort}^{(m)}(W,c)$ correspond bijectively to a certain natural class of co-aisles in the bounded derived category of a hereditary Artin algebra defined in terms of W and c. The co-aisles in question all contain a certain "standard" co-aisle, and the inversion sets of the elements of $\mathsf{Sort}^{(m)}(W,c)$ designate the indecomposable objects in the corresponding co-aisle other than those in the standard co-aisle. It follows that $\mathsf{Camb}_{\mathsf{Sort}}^{(m)}(W,c)$ is the inclusion order on this class of co-aisles.

Further, the combinatorial bijections which we have constructed in previous chapters also have corresponding representation-theoretic versions in crystallographic types. We will show that the bijection $\mathsf{Asso}_{\nabla}^{(m)}(W,c) \longleftrightarrow \mathsf{NC}_{\Delta}^{(m)}(W,c)$ in Theorem 5.7.2 agrees (up to a choice of convention) with the bijection from [**BRT12**].

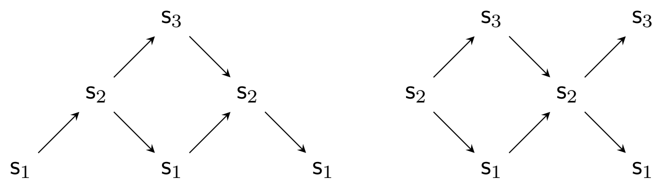


FIGURE 28. Combinatorial AR quivers for $w_\circ(\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3) = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2\mathsf{s}_1$, and for $w_\circ(\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3) = \mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3$.

We will also show that the bijection $\mathsf{Camb}^{(m)}_{\mathsf{Sort}}(W,c) \longleftrightarrow \mathsf{Asso}^{(m)}_{\nabla}(W,c)$ is a combinatorial version of a bijection between (co)-aisles and silting objects which goes back to B. Keller and D. Vossieck [**KV88**].

We omit proofs of most of the purely representation-theoretic statements in this chapter, preferring to refer the reader to the existing literature. On the other hand, we include some proofs either because we consider them to be instructive or because it was not easy to find the desired results in the literature.

I spent a pleasant day reading all the algebra books in the Oberwolfach library so you don't have to.

#2

9.2. Combinatorial constructions

In this section, we give some combinatorial constructions, based on considerations from previous chapters, in order to prepare for our applications to the representation theory of hereditary Artin algebras.

Let $(W,\mathcal{S})$ be a Coxeter system with $\mathcal{S} = \{s_1,\ldots,s_n\}$. For a (possibly infinite or bi-infinite) $\mathcal{S}$-word Q, we define the *combinatorial AR quiver* of Q to be the quiver whose vertex set is the letters of Q, and such that for each letter s_i in Q, and each $s_j \in \mathcal{S}$ such that s_i and s_j do not commute, there is an arrow from that s_i to the next occurrence of s_j, if any. (The poset whose Hasse diagram is the combinatorial AR quiver is known in the literature as the *heap* of Q [**Vie86**].)

This is the best way to convince a combinatorialist to read this chapter.

#39

The *combinatorial AR translation* is a partially defined map from the letters of Q to the letters of Q, sending each s_i to the previous instance of s_i, if any.

If $\mathsf{Q} = \mathsf{s}_1\mathsf{s}_2\ldots$ is a (possibly infinite, but not bi-infinite) $\mathcal{S}$-word, then there is an inversion sequence of colored positive roots $\mathrm{inv}(Q)$, namely $\alpha_1^{(0)}, s_1(\alpha_2^{(0)}),\ldots,$ as defined in Section 2.9.3. We refer to the colored positive root corresponding to a letter in Q as its colored root label.

We are especially interested in three cases of the combinatorial AR quiver construction, depending on a choice of $\mathsf{c} = \mathsf{s}_1\ldots\mathsf{s}_n$.

- $\mathsf{Q} = \mathsf{w}_\circ(\mathsf{c})$. Recall that $\mathsf{w}_\circ(\mathsf{c})$ is the c-sorting word for $w_\circ$; see Definition 2.6.1 for the definition and Lemma 2.6.5 for some of its important properties. Two examples of these quivers, for the Coxeter group of type A_3, with $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$ and $\mathsf{c} = \mathsf{s}_2\mathsf{s}_1\mathsf{s}_3$, are given in Figure 28. (As always, when we give examples in type A_n, we use s_i to represent the adjacent transposition $(i\ i+1)$.)
- $\mathsf{Q} = \mathsf{c}^\infty$. The beginning of the combinatorial AR quiver for c^∞ in the Coxeter group of type A_3 with $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$ is given in Figure 29. Note that, as can be seen in the example, and as shown by Lemma 2.6.5, $\mathsf{w}_\circ(\mathsf{c})$ is initial in c^∞.
- $\mathsf{Q} = {}^\infty\mathsf{c}^\infty$. By definition, ${}^\infty\mathsf{c}^\infty$ is the bi-infinite word consisting of repetitions of c. Part of the combinatorial AR quiver for ${}^\infty\mathsf{c}^\infty$ is given in Figure 30. Note that,

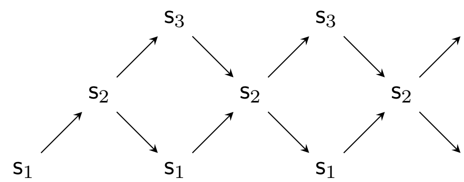


FIGURE 29. Initial subquiver of the combinatorial AR quiver of c^∞ for $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$.

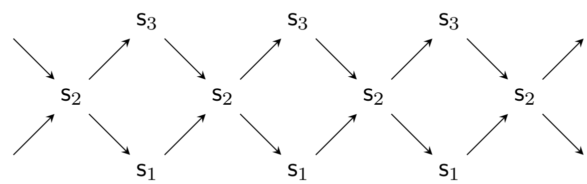


FIGURE 30. Subquiver of the combinatorial AR quiver of ${}^\infty\mathsf{c}^\infty$ for $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$.

up to commutations, the word ${}^\infty\mathsf{c}^\infty$ does not depend on the choice of c, and the corresponding combinatorial AR quiver does not depend on the choice of c at all.

By Lemma 2.6.5, $\mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{w}_\circ(\mathsf{c})) \equiv \mathsf{c}^h$. We can therefore view c^∞ as an infinite alternating repetition of $\mathsf{w}_\circ(\mathsf{c})$ and $\psi(\mathsf{w}_\circ(\mathsf{c}))$. Because $\mathsf{w}_\circ(\mathsf{c})$ is initial in c^∞, the colored root labelling of letters from $\mathsf{w}_\circ(\mathsf{c})$ are the same in $\mathsf{w}_\circ(\mathsf{c})$ and c^∞. The set of these labels are exactly the 0-colored positive roots.

We define the *combinatorial shift* sending a letter of $\mathsf{w}_\circ(\mathsf{c})$ to the corresponding letter in the following $\psi(\mathsf{w}_\circ(\mathsf{c}))$, and similarly sending letters of $\psi(\mathsf{w}_\circ(\mathsf{c}))$ to the corresponding letter of the following copy of $\mathsf{w}_\circ(\mathsf{c})$. This sends the reflection corresponding to the colored almost positive root $\beta^{(i)}$ to the reflection corresponding to $\beta^{(i+1)}$.

Similarly, we can view ${}^\infty\mathsf{c}^\infty$ as a bi-infinite word consisting of alternating copies of $\mathsf{w}_\circ(\mathsf{c})$ and $\psi(\mathsf{w}_\circ(\mathsf{c}))$, although this factorization is not canonical. Using such a factorization, we can also define the combinatorial shift, and inverse combinatorial shift, on the letters of ${}^\infty\mathsf{c}^\infty$. This definition does not depend on the choice of factorization.

Let us define the set of *$\mathbb{Z}$-colored positive roots* to be $\Phi^+ \times \mathbb{Z}$. Once we fix a copy of $\mathsf{w}_\circ(\mathsf{c})$ inside ${}^\infty\mathsf{c}^\infty$, there is a well-defined labelling of the letters of ${}^\infty\mathsf{c}^\infty$ by $\mathbb{Z}$-colored positive roots. The chosen copy of $\mathsf{w}_\circ(\mathsf{c})$ corresponds to the roots labelled with color 0; other reflections are labelled so that the combinatorial shift sends the reflection corresponding to $\beta^{(i)}$ to the reflection corresponding to $\beta^{(i+1)}$.

9.3. A brief introduction to hereditary Artin algebras

We now introduce the representation-theoretic setting in which we shall work. Let $\Bbbk$ be a field, and let H be a finite-dimensional basic hereditary algebra over $\Bbbk$, with n isomorphism classes of simple objects. We consider the category $\operatorname{mod} H$ of finite-dimensional left H-modules. We suppose H to be representation-finite, i.e., having only finitely many isomorphism classes of indecomposable modules. Introductions to the representation theory of finite-dimensional hereditary Artin algebras can be found in [**ARS97**, **ASS06**, **DDPW08**] and [**SY17**, Chapter 7].

9.3.1. The Grothendieck group. The Grothendieck group of H, denoted $K_0(H)$, is the free abelian group on the set of isomorphism classes of H-modules, modulo the subgroup generated by all expressions of the form $[M] - [L] - [N]$ for any short exact sequence $0 \to L \to M \to N \to 0$. The Grothendieck group is isomorphic to $\mathbb{Z}^n$; the classes of the simple H-modules form a $\mathbb{Z}$-basis for $K_0(H)$. The expansion of $[M]$ in the basis of the simple modules records the multiplicity with which each simple module appears in a composition series for M.

$K_0(H)$ is naturally equipped with a symmetric, bilinear form (the symmetrization of the Euler form) which is positive definite. The map sending a module to its class in the Grothendieck group defines a bijection from the isomorphism classes of indecomposable modules to the positive roots of a root system Φ compatible with this symmetric, bilinear form.

The classes in $K_0(H)$ of the simple modules are the simple roots. The Coxeter group corresponding to the root system, W, acts naturally on $K_0(H)$. Any finite crystallographic root system can be realized in this way for some choice of H. If Φ is a simply-laced root system, we can take H to be the path algebra of an orientation of the corresponding Coxeter diagram over an arbitrary ground field $\Bbbk$. For more details on constructing algebras corresponding to non-simply-laced root systems, see for example [**DDPW08**].

For a simple reflection $s_i \in W$, write S_i for the corresponding simple H-module, P_i for the projective cover of S_i, I_i for the injective envelope of S_i, and e_i for the corresponding idempotent in H. Let c_H be the product of the simple reflections taken in an order such that s_i precedes s_j if $\mathrm{Ext}^1(S_j, S_i) \neq 0$. Although there may be different orders which satisfy this condition, the product c_H is well-defined as an element of W. The Coxeter combinatorics associated to the element c_H turns out to encode a great deal of the structure of the category of H-modules. We will write c for c_H if there is no risk of confusion.

9.3.2. The AR translation and the AR quiver. If M is an indecomposable non-projective module, the Auslander-Reiten translation of M, written τM, is the indecomposable module characterized by the fact that $[\tau M] = c[M]$. Similarly, the inverse Auslander-Reiten translation of M is characterized by $[\tau^{-1}M] = c^{-1}[M]$ if M is a non-injective module [**ARS97**, Proposition VIII.2.2].

The isomorphism classes of indecomposable H-modules are naturally organized as the vertices of what is called its *Auslander-Reiten quiver*, or AR quiver for short. By definition, we draw an arrow from X to Y if there is a morphism from X to Y which is not a sum of morphisms factoring through other indecomposable objects (and is not an isomorphism). This is a slightly simplified version of the AR quiver, which suffices for our purposes. We are ignoring the multiplicities of the arrows, which require some additional book-keeping if $\Bbbk$ is not algebraically closed. We are also avoiding the discussion of AR sequences, because we do not actually need them for our purposes

My favorite kind of proof!

#53

The following theorem shows that the structure of the AR quiver of $\operatorname{mod} H$ corresponds to the structure of $\mathsf{w}_\circ(\mathsf{c})$. This theorem is already known; the proof we give consists of a sequence of references to the literature for its different components.

THEOREM 9.3.1. *Let H be a finite-dimensional Artin algebra, and let $c = c_H$ be the corresponding Coxeter element as defined above.*

(1) *The AR quiver of* $\operatorname{mod} H$ *coincides with the combinatorial AR quiver of* $\mathsf{w}_\circ(\mathsf{c})$.
(2) *AR translation in* $\operatorname{mod} H$ *is given by the combinatorial AR translation.*
(3) *The indecomposable module M corresponds to the simple reflection in the combinatorial AR quiver whose corresponding inversion is the 0-colored root $[M]^{(0)}$.*
(4) *The projective indecomposable P_i correponds to the first instance of* s_i *in* $\mathsf{w}_\circ(\mathsf{c})$. *Its class in the Grothendieck group is $[P_i] = s_1 \dots s_{i-1}(\alpha_i)$.*
(5) *The injective indecomposable I_i corresponds to the final instance of* s_i *in* $\mathsf{w}_\circ(\mathsf{c})$. *Its class in the Grothendieck group is $-c^{-1}[P_i]$.*

PROOF. We start by considering the combinatorial AR quiver for c^∞. The AR quiver of H is a subquiver of the combinatorial AR quiver for c^∞ with P_i corresponding to the first occurrence of s_i, by [**ARS97**, Proposition VIII.1.15] and the discussion before it. This also establishes that the AR translation in $\operatorname{mod} H$ agrees with the combinatorial AR translation in c^∞.

The class in the Grothendieck group of P_i is $s_1 \dots s_{i-1}(\alpha_i)$, by [**DR76**, Lemma 1.6]. The subset of c^∞ corresponding to the AR quiver is as follows: take the first k_i copies of s_i, where k_i is maximal such that $[P_i]$, $c[P_i]$, $c^2[P_i]$, $\dots$, $c^{k_i}[P_i]$ are all positive roots, by [**ARS97**, Proposition VIII.1.15]. These are the roots corresponding to successive copies of s_i in c^∞, so k_i can also be described as maximal such that the roots corresponding to the first k_i copies of s in c^∞ are positive. This establishes that the subset of c^∞ corresponding to the AR quiver agrees with the subset corresponding to $\mathsf{w}_\circ(\mathsf{c})$, proving (1) and (2).

We know that the j-th copy of s_i corresponds to the module $\tau^{-j+1}P_i$, whose class in the Grothendieck group is $c^{j-1}s_1 \dots s_{i-1}(\alpha_i)$. This root, with color zero, is the colored positive root labelling the j-th copy of s_i in c^∞. This shows (3).

We have already established (4). Now (5) follows from [**ARS97**, Proposition VIII.2.2]. □

9.3.3. The bounded derived category. The bounded derived category of H, denoted $D^b(H)$, is a triangulated category, with shift functor denoted [1]. Material on derived categories can be found in [**Hap88**], in textbooks on homological algebra, or in [**Kel07**].

We write $\operatorname{ind} D^b(H)$ for the indecomposable objects of $D^b(H)$. Any indecomposable object of $D^b(H)$ is of the form $M[i]$, for $i \in \mathbb{Z}$ and M an indecomposable H-module [**Hap88**, Theorem I.5.2]; the hypothesis in this section of [**Hap88**] that the ground field is algebraically close is not used for this result. The indecomposable objects in $D^b(H)$ are in bijection with the $\mathbb{Z}$-colored positive roots. For M an indecomposable object in $\operatorname{mod} H$ and $i \in \mathbb{Z}$, we define

$$\underline{\dim}\ M[i] := [M]^{(i)}.$$

If X is the direct sum of indecomposable objects $X_1 \oplus \dots \oplus X_r$, we define $\underline{\dim} X$ to be the set $\{\underline{\dim} X_1, \dots, \underline{\dim} X_r\}$.

There is also an AR quiver defined for the bounded derived category. As before, we define the AR quiver by saying that there is a vertex for each isomorphism class of indecomposable modules, and there is an arrow between two vertices when there is an irreducible morphism between the corresponding modules. (This agrees with the usual definition by [**Hap88**, Proposition I.4.3], up to the fact that we are not keeping track of valuation data on the arrows of the AR quiver.) There is also an Auslander-Reiten translation which sends indecomposable objects to indecomposable objects.

The next theorem shows that the AR quiver of $D^b(H)$ is the combinatorial AR quiver of ${}^\infty\mathsf{c}^\infty$. Recall that we showed in Section 9.2 that ${}^\infty\mathsf{c}^\infty$ can also be described as a bi-infinite alternating sequence of $\mathsf{w}_\circ(\mathsf{c})$ and $\psi(\mathsf{w}_\circ(\mathsf{c}))$. Again, we include the known proof for the convenience of the reader.

THEOREM 9.3.2. *The AR quiver of $D^b(H)$ is the combinatorial AR quiver of ${}^\infty\mathsf{c}^\infty$, and the AR translation is given by the combinatorial AR translation. Fixing a copy of $\mathsf{w}_\circ(\mathsf{c})$ inside ${}^\infty\mathsf{c}^\infty$, we can identify it with the AR quiver of* $\operatorname{mod} H$ *inside $D^b(H)$; the following copy of $\psi(\mathsf{w}_\circ(\mathsf{c}))$ is then identified with* $\operatorname{mod} H[1]$*, the following copy of $\mathsf{w}_\circ(\mathsf{c})$ is identified with* $\operatorname{mod} H[2]$*, and so on in both directions.*

PROOF. [**Hap88**, I.4.7] shows that the AR quiver and the AR translation restricted to each shift of the module category, are the same as in the module category. To understand the AR quiver, we need only understand how successive copies of the module category are connected.

The arrows between successive copies of the module category are given by [**Hap88**, Lemma I.5.4]: there is an arrow from $I_i[-1]$ to each (projective) vertex in the module category with an arrow to P_i, and an arrow to P_i from each vertex in the $[-1]$-shift of the module category with an arrow from $I_i[-1]$. (This lemma is proved under the assumption that $\Bbbk$ is algebraically closed; the same proof applies in the general case, though somewhat heavier notation is needed.) This establishes the statements in the theorem about the AR quiver.

The statement about AR translation follows from the corresponding statement about AR translation in the module category, together with the fact that $\tau P_i = I_i[-1]$ from [**Hap88**, Lemma I.5.4] and the fact that $[I_i] = -c^{-1}[P_i]$. ☐

EXAMPLE 9.3.3. We give the representation theory corresponding to Example 2.8.1. Let H be the path algebra of the quiver

$$1 \longleftarrow 2.$$

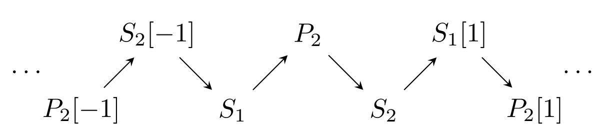
Is there a way to get type H_2, too?

#40

There are three indecomposable H-modules, the simples S_1 and S_2, and P_2, the projective at 2. Their corresponding classes in $K_0(H)$ can be identified as the positive roots of the A_2 root system, respectively α, β, and γ. The Coxeter group is isomorphic to the symmetric group on three letters, and is generated by simple reflections s (corresponding to S_1) and t (corresponding to S_2). The Coxeter element is $c_H = st$, and $\mathsf{w}_\circ(\mathsf{c}) = \mathsf{sts}$, which gives us the shape of the AR quiver of the module category. The AR quiver of the bounded derived category can either be thought of as repeated copies of Q with connecting arrows, or repeated copies of the module category, with connecting arrows. The AR quiver is

Well, technically, not all of the AR quiver.

#3



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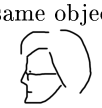
9.4. Factorizations of Coxeter elements and exceptional sequences

A key structure underlying the interplay between the representation theory of finite type Artin algebras and Catalan combinatorics is the correspondence between exceptional sequences and $\mathcal{R}$-factorizations of c^{-1} (i.e., $\mathrm{Red}_{\mathcal{R}}(c^{-1})$, as in Section 2.7.1. We now explain this connection.

Fun fact: exceptional sequences and factorizations of Coxeter elements were developed by disjoint sets of mathematicians who were unaware they were studying the same objects.



#4

9.4.1. Exceptional sequences in the module category.

DEFINITION 9.4.1. A sequence $X_1, \ldots, X_r$ of H-modules is called an *exceptional sequence* if each X_j is indecomposable and $\mathrm{Ext}^i(X_\ell, X_j) = 0$ for $\ell > j$ and $i = 0, 1$. An exceptional sequence is called *complete* if its length is n.

Exceptional sequences of length n are called complete because that is the maximum possible length of an exceptional sequence. Basic references for exceptional sequences are [**CB93**] and (over an arbitrary ground field) [**Rin94**].

Recall that indecomposable H-modules correspond bijectively to positive roots, which correspond bijectively to reflections in W. A sequence of indecomposable modules may therefore be considered as a sequence of reflections. It turns out that there is a very nice way to characterize the sequences of reflections which correspond to exceptional sequences of modules.

The following result was shown in the simply-laced case in [**IT09**]. A proof in a much more general setting, which, in particular, drops the simply-laced assumption, can be found in [**IS10**]. See also [**HK16**] for another presentation.

THEOREM 9.4.2. *If $X_1, \ldots, X_n$ is a sequence of n indecomposable modules, then it is an exceptional sequence if and only if $s_{[X_1]} \ldots s_{[X_n]} = c^{-1}$.* □

EXAMPLE 9.4.3. Continuing the A_2 example which we began above, there are three exceptional sequences in H-mod: (S_1, P_2), (P_2, S_2), and (S_2, S_1). They correspond respectively to the $\mathcal{R}$-factorizations

$$c^{-1} = su = ut = ts.$$

The braid group on n strands acts on $\mathrm{Red}_{\mathcal{R}}(c^{-1})$ by the Hurwitz action, which we have recalled in Section 2.7.1. For $1 \le i \le n-1$, let $\boldsymbol{\sigma}_i$ be the generator taking the $(i+1)^{\text{th}}$ strand over the i^{th}. Given a factorization $c^{-1} = r_1 \ldots r_n$, the result of acting by $\boldsymbol{\sigma}_i$ is to move r_i past r_{i+1}, at the cost of conjugating r_i, i.e., $(r_i, r_{i+1}) \mapsto (r_{i+1}, r_{i+1} r_i r_{i+1})$. The corresponding moves on exceptional sequences are known as *mutations*. (Right) mutation replaces a subsequence (X, Y) by $(Y, R_Y X)$. $R_Y X$ can be defined as the unique module in the abelian, extension-closed subcategory generated by X and Y such that $(Y, R_Y X)$ is an exceptional sequence. By Theorem 9.4.2, $R_Y X$ can also be characterized by the fact that $[R_Y X] = |s_{[Y]}[X]|$.

EXAMPLE 9.4.4. We continue the same example. If we apply right mutation to (P_2, S_2), we obtain (S_2, S_1). If we apply right mutation to (S_2, S_1), we obtain (S_1, P_2). If we apply right mutation to (S_1, P_2), we obtain (P_2, S_2).

9.4.2. Exceptional sequences in the derived category. It turns out that it is useful to consider exceptional sequences in $D^b(H)$, and in fact, from now on, when we refer to exceptional sequences, we mean exceptional sequences in $D^b(H)$. The definition is almost the same: a sequence $X_1, \ldots, X_r$ of indecomposable objects from $D^b(H)$ is exceptional if $\mathrm{Ext}^i(X_\ell, X_j) = 0$ for $\ell > j$ and all $i \in \mathbb{Z}$.

It turns out that exceptional sequences in the derived category are very closely related to exceptional sequences in the module category. To aid the reader, we provide a proof of the following simple lemma.

LEMMA 9.4.5. *Let $M_1, \dots, M_r$ be H-modules, and let $i_1, \dots, i_r \in \mathbb{Z}$. The sequence $(M_1, \dots, M_r)$ is then an exceptional sequence of H-modules if and only if $(M_1[i_1], \dots, M_r[i_r])$ is an exceptional sequence in $D^b(H)$.*

PROOF. The definition of exceptional sequence in $D^b(H)$ is clearly insensitive to applying [1] to any of the terms. It is therefore enough to check that if $(M_1, \dots, M_r)$ is a sequence of modules, then it is an exceptional sequence in $D^b(H)$ if and only if it is an exceptional sequence in $\operatorname{mod} H$. On the face of it, the two definitions look different, because for an exceptional sequence in $D^b(H)$, we check the vanishing of $\operatorname{Ext}^i(M_\ell, M_j)$ for $\ell > j$ and all i, whereas in the definition in $\operatorname{mod} H$, we only check this vanishing for $i = 0$ and $i = 1$. The point is that, since M_ℓ and M_j are H-modules, we know that $\operatorname{Ext}^i(M_\ell, M_j) = 0$ for all $i \neq 0, 1$. □

By Theorem 9.4.2, we can think of complete exceptional sequences in $D^b(H)$ as $\mathcal{R}$-factorizations of c^{-1} where each factor receives an (arbitrary) color in $\mathbb{Z}$.

For exceptional sequences in $D^b(H)$, there is a way to specify precisely how mutations act with respect to shift so that mutations still induce a braid group action. There is an easy way to state the outcome concretely: $R_Y X$ is shifted so that it weakly precedes X in the AR quiver partial order, but it is far to the right as possible subject to that condition.

More algebraically, it can be described as follows. There will be at most one $b \in \mathbb{Z}$ such that $\operatorname{Hom}(X, Y[b]) \neq 0$. Having found this b, take an $\operatorname{End}(Y)$-basis $f_1, \dots, f_a$ of $\operatorname{Hom}(X, Y[b])$. Then define the map f from X to the sum of a copies of $Y[b]$ such that the map into the i-th copy is given by f_i. (This map is known as the left thick $\operatorname{add} Y$ approximation to X.) Then complete this map to a triangle. (This is a routine operation in a triangulated category, akin to taking the kernel or cokernel of a morphism in an abelian category.) The third term of the triangle is defined to be $R_Y X$, which is determined up to isomorphism,

$$R_Y X \to X \to Y^a[b] \to R_Y X[1].$$

(See the proof of [**BRT12**, Lemma 7.1] for more on this.)

EXAMPLE 9.4.6. Let us redo Example 9.4.4 in the derived category. If we apply right mutation to (P_2, S_2), we obtain (S_2, S_1). If we apply right mutation to (S_2, S_1), we obtain (S_1, P_2). (In these two cases, the results were the same as before.) But if we apply right mutation to (S_1, P_2), we obtain $(P_2, S_2[-1])$.

The following useful lemma can be found, for example, in [**BRT12**].

LEMMA 9.4.7. *Given a complete exceptional sequence $(X_1, \dots, X_n)$, if we successively mutate it so as to move X_1 to the other end, the result is a sequence $(X_2, \dots, X_n, \tau^{-1}X_1[-1])$.* □

COROLLARY 9.4.8. *Given a complete exceptional sequence $(X_1, \dots, X_n)$, if we successively mutate it so as to move X_1 part way through the sequence, the result is a sequence $(X_2, \dots, X_i, Y, X_{i+1}, \dots, X_n)$ where Y is located between $\tau^{-1}X_1[-1]$ and X_1 in the AR-quiver.* □

9.5. m-eralized clusters and noncrossing partitions

In this section we define silting objects and $\mathrm{Hom}_{\leq 0}$ configurations in the derived category, and show how they correspond to m-eralized clusters and m-eralized noncrossing partitions, respectively. In order to do this, we recall from Section 8.2.1 the constructions of $\mathsf{Sort}^{(\infty)}(W,c)$, $\mathsf{Asso}_{\nabla}^{(\infty)}(W,c)$ and $\mathsf{NC}_{\Delta}^{(\infty)}(W,c)$, and of the Cambrian poset structure on these denoted by $\mathsf{Camb}_{\mathsf{NC}}^{(\infty)}$.

9.5.1. m-eralized noncrossing partitions.

Definition 9.5.1. An object X in $D^b(H)$ is called a $\mathrm{Hom}_{\leq 0}$ *configuration* if its indecomposable direct summands can be ordered as $X_1,\dots,X_n$ such that the sequence is exceptional and $\mathrm{Hom}(X_i,X_j)=0$ for $i\neq j$.

- ◦ We write $\mathrm{Hom}_{\leq 0}(H)$ for the $\mathrm{Hom}_{\leq 0}$ configurations in $D^b(H)$.
- ◦ We write $\mathrm{Hom}_{\leq 0}^{(\infty)}(H)$ for the $\mathrm{Hom}_{\leq 0}$ configurations all of whose indecomposable summands are of the form $M[i]$ with $i\geq 0$.
- ◦ We write $\mathrm{Hom}_{\leq 0}^{(m)}(H)$ for the $\mathrm{Hom}_{\leq 0}$ configurations all of whose summands are of the form $M[i]$ with $m\geq i\geq 0$.

Proposition 9.5.2. *X is a $\mathrm{Hom}_{\leq 0}$ configuration if and only if the direct summands of X read in some (or equivalently, any) order compatible with the AR quiver of $D^b(H)$, yields a sequence of reflections whose product is c.*

Proof. Suppose X is a $\mathrm{Hom}_{\leq 0}$ configuration. Let the indecomposable summands of X be $X_1,\dots,X_n$, ordered in the reverse of any order compatible with the AR quiver. By definition, then, $\mathrm{Ext}^i(X_\ell,X_j)=0$ for $\ell>j$ and $i>0$ (because X_ℓ will be to the left of $X_j[i]$). On the other hand, because X is a $\mathrm{Hom}_{\leq 0}$ configuration, we know that $\mathrm{Ext}^i(X_\ell,X_j)=0$ for all $\ell\neq j$ and $i\leq 0$. This establishes that $X_1,\dots,X_n$ is an exceptional sequence, and, by Theorem 9.4.2, the corresponding product of reflections is c^{-1}. Thus, if we read the reflection in the AR order, rather than in the reverse order, we obtain c.

Conversely, let $X_n,\dots,X_1$ be the summands of X in some order compatible with the AR quiver, and suppose that the corresponding product of reflections $t_{[X_n]}\dots t_{[X_1]}=c$, so $t_{[X_1]}\dots t_{[X_n]}=c^{-1}$, and thus, by Theorem 9.4.2, we know that $X_1,\dots,X_n$ is an exceptional sequence. We now reverse the previous argument to deduce that X is a $\mathrm{Hom}_{\leq 0}$ configuration. □

Example 9.5.3. We continue the same example. $\mathrm{Hom}_{\leq 0}$-configurations are of one of the three following forms:

- $S_1[i]\oplus P_2[j]$ with $j<i$,
- $P_2[i]\oplus S_2[j]$ with $j<i$,
- $S_2[i]\oplus S_1[j]$ with $j\leq i$.

Theorem 9.5.4. *There is a natural bijection*

$$\begin{array}{ccc}\mathrm{Hom}_{\leq 0}^{(\infty)}(H) & \overset{c}{\longleftrightarrow} & \mathsf{NC}_{\Delta}^{(\infty)}(W,c)\\ X & \longmapsto & \underline{\dim}\ X.\end{array}$$

This bijection restricts to a bijection from $\mathrm{Hom}_{\leq 0}^{(m)}(H)$ to $\mathsf{NC}_{\Delta}^{(m)}(W,c)$.

Proof. Reading $\mathsf{inv}_{\mathcal{R}}(\mathsf{w}_\circ^\infty(\mathsf{c}))$ is equivalent to (a particular way of) reading the classes in the Grothendieck group of the objects in the AR quiver of $D^b(H)$. The theorem now follows from Proposition 9.5.2. Because the bijection is induced from a bijection between indecomposables and letters in c^∞, the fact that the Cambrian recursion is satisfied is clear. The statement about the restrictions is also clear. $\square$

9.5.2. m-eralized clusters.

DEFINITION 9.5.5. An object X in $D^b(H)$ is called *silting* if $\mathrm{Ext}^i(X, X) = 0$ for $i > 0$, and X is the direct sum of n pairwise non-isomorphic indecomposables.

- We write $\mathrm{Silt}(H)$ for the set of silting objects in $D^b(H)$.
- We write $\mathrm{Silt}^{(\infty)}(H)$ for the silting objects in $D^b(H)$ all of whose indecomposable summands are of the form $M[i]$ with M an H-module and $i \geq 0$, or $I[-1]$ with I an indecomposable injective.
- We write $\mathrm{Silt}^{(m)}(H)$ for the silting objects in $D^b(H)$ all of whose indecomposable summands are of the form $M[i]$ with $m > i \geq 0$, or $I[-1]$ for I an indecomposable injective.

Statements similar to the following can be found in [**IS10**] and [**BRT11**]; we omit the proof, which is very similar to that of Proposition 9.5.2.

PROPOSITION 9.5.6. *X is silting if and only if the direct summands of X read in some (or equivalently, any) order compatible with the AR quiver of $D^b(H)$, yields a sequence of reflections whose product is c^{-1}.* $\square$

EXAMPLE 9.5.7. We continue the same example. Silting objects are of one of the three following forms:

- $S_1[i] \oplus P_2[j]$ with $j \geq i$,
- $P_2[i] \oplus S_2[j]$ with $j \geq i$,
- $S_2[i] \oplus S_1[j]$ with $j > i$.

Now, we can state the following theorem.

THEOREM 9.5.8. *There is a natural bijection*

$$\begin{array}{ccc} \mathrm{Silt}^{(\infty)}(H) & \stackrel{c}{\longleftrightarrow} & \mathsf{Asso}_{\nabla}^{(\infty)}(W, c) \\ X & \longmapsto & \underline{\dim}\ \tau^{-1}X \end{array}$$

This bijection restricts to a bijection from $\mathrm{Silt}^{(m)}(H)$ to $\mathsf{Asso}_{\nabla}^{(m)}(W, c)$.

PROOF. The proof is the analogue of the proof of Theorem 9.5.4, using Proposition 9.5.6. $\square$

9.6. The bijection between m-eralized clusters and noncrossing partitions

[**BRT12**] gives a bijection from $\mathrm{Silt}^{(\infty)}(H)$ to $\mathrm{Hom}_{\leq 0}^{(\infty)}(H)$. Define Fund of an exceptional sequence $(E_1, \ldots, E_n)$ to be obtained by applying the fundamental element of the braid group $\mathfrak{S}_n$, and then applying [1]. More explicitly, $\mathrm{Fund}(E_1, \ldots, E_n)$ is obtained by moving E_1 to the end of the sequence (changing it as it moves), moving E_2 to just before the modified E_1, moving E_3 to just before the modified E_2, etc., and then applying [1].

In order to apply Fund to a silting object or a Hom-configuration, we order its indecomposable summands into an exceptional sequence, apply Fund, and then take the direct sum of the resulting terms, forgetting their order. The result is well-defined independent of the choice of ordering.

THEOREM 9.6.1 ([**BRT12**]). *The map* Fund *is a natural bijection from silting objects to* $\mathrm{Hom}_{\leq 0}$ *configurations*

$$\begin{array}{ccc} \mathrm{Silt}(H) & \overset{c}{\longleftrightarrow} & \mathrm{Hom}_{\leq 0}(H) \\ X & \longmapsto & \mathrm{Fund}(X). \end{array}$$

This bijection restricts to a bijection from $\mathrm{Silt}^{(\infty)}(H)$ *to* $\mathrm{Hom}^{(\infty)}_{\leq 0}(H)$*, and further restricts to a bijection from* $\mathrm{Silt}^{(m)}(H)$ *to* $\mathrm{Hom}^{(m)}_{\leq 0}(H)$*.*

PROOF. We refer to [**BRT12**] for the proof of the existence of the bijection. We prove here that it respects the Cambrian recurrence.

Let $\gamma_1^{(k_1)}, \dots, \gamma_n^{(k_n)}$ be in $\mathsf{Asso}^{(\infty)}_{\nabla}(W,c)$. Let the corresponding indecomposable objects be $X_1, \dots, X_n$, with X their direct sum, so $X \in \mathrm{Silt}^{(\infty)}(H)$.

Let s_1 be initial in c, and suppose that $\gamma_1^{(k_1)}$ is not equal to $\alpha_1^{(0)}$. Equivalently, this means that τS_1 is not a summand of X. In this case, it is clear that Fund respects the Cambrian recurrence: Fund is defined in terms of the derived category, so it is not sensitive to the difference between H and the algebra obtained by applying a reflection functor at 1. (See [**DR76**] for reflections functors over general fields.)

Now suppose that $\gamma_1^{(c_1)} = \alpha_1^{(0)}$. In representation-theoretic terms, it means that τS_1 is a summand of X. The summand of $\mathrm{Fund}(X)$ corresponding to τS_1, is S_1, by Lemma 9.4.7. The other summands of X will satisfy that $\mathrm{Ext}^i(X_j, \tau S_s) = 0$ for $i > 0$, because X is silting. By Auslander-Reiten duality, it follows that $\mathrm{Hom}^i(S_1, X_j) = 0$ for $i \geq 0$. This means that the other summands of X_j can be identified with objects in $D^b(H/He_1H)$. The definitions of Fund in $D^b(H)$ and $D^b(H/He_1H)$ agree. This shows that Fund satisfies the Cambrian recurrence. $\square$

THEOREM 9.6.2. *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{Silt}^{(\infty)}(H) & \xleftrightarrow{\text{Theorem 9.6.1}} & \mathrm{Hom}^{(\infty)}_{\leq 0}(H) \\ \text{Theorem 9.5.8} \updownarrow & & \updownarrow \text{Theorem 9.5.4} \\ \mathsf{Asso}^{(\infty)}_{\nabla}(W,c) & \xleftrightarrow[\text{Theorem 5.7.2}]{} & \mathsf{NC}^{(\infty)}_{\Delta}(W,c). \end{array}$$

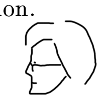
The bijections all restrict to the appropriate m-eralized versions.

PROOF. We have already shown that all of the bijections in this diagram respect the Cambrian recurrence, so the diagram commutes. $\square$

9.7. Positive Fuß-Catalan combinatorics in representation theory

We can proceed in a similar fashion to define representation-theoretic objects corresponding to positive analogues of noncrossing partitions or of clusters.

This section is basically nothing but definitions, but it's put to some use in the next section.



#5

DEFINITION 9.7.1. Define $\mathsf{Hom}^{(\infty)}_{\leq 0,+}(H)$ to consist of $\mathrm{Hom}_{\leq 0}$ configurations in $D^b(H)$ all of whose indecomposable summands are of the form $M[i]$ with M an H-module and $i \geq 0$, and where, if $i = 0$, M is not allowed to be projective. Define

$\mathsf{Hom}^{(m)}_{\leq 0,+}(H)$ to consist of those $\mathrm{Hom}_{\leq 0}$ configurations in $\mathsf{Hom}^{(\infty)}_{\leq 0,+}(H)$ such that, in addition, all of the indecomposable summands are located in shift at most m.

The following lemma is immediate.

LEMMA 9.7.2. *The bijection* $\underline{\dim}$ *from* $\mathrm{Hom}^{(\infty)}_{\leq 0}(H)$ *to* $\mathsf{NC}^{(\infty)}_{\Delta}(W,c)$ *restricts to bijections from* $\mathsf{Hom}^{(\infty)}_{\leq 0,+}(H)$ *to* $\mathsf{NC}^{(\infty)}_{+}(W,c)$ *and from* $\mathsf{Hom}^{(m)}_{\leq 0,+}(H)$ *to* $\mathsf{NC}^{(m)}_{+}(W,c)$. □

DEFINITION 9.7.3. Define $\mathrm{Silt}^{(\infty)}_{+}(H)$ to consist of silting objects in $D^b(H)$ all of whose indecomposable summands are of the form $M[i]$ with M an H-module and $i \geq 0$. Similarly, define $\mathrm{Silt}^{(m)}_{+}(H)$ to consist of silting objects in $D^b(H)$ all of whose indecomposable summands are of the form $M[i]$ with M an H-module and $m > i \geq 0$.

In other words, compared to $\mathrm{Silt}^{(\infty)}(H)$ and $\mathrm{Silt}^{(m)}(H)$ respectively, we are simply forbidding the indecomposable injectives in shift $[-1]$ as possible summands. The following lemma is immediate.

LEMMA 9.7.4. *The bijection* $\underline{\dim} \circ \tau^{-1}$ *from* $\mathrm{Silt}^{(\infty)}(H)$ *to* $\mathsf{Asso}^{(\infty)}_{\nabla}(W,c)$ *restricts to bijections from* $\mathrm{Silt}^{(\infty)}_{+}(H)$ *to* $\mathsf{Asso}^{(\infty)}_{+}(W,c)$ *and from* $\mathrm{Silt}^{(m)}_{+}(H)$ *to* $\mathsf{Asso}^{(m)}_{+}(W,c)$. □

The positive analogue of Theorem 9.6.1 is also shown in [**BRT12**]. We omit the easy proof.

THEOREM 9.7.5. Fund *defines a bijection*

$$\begin{array}{ccc} \mathrm{Silt}^{(\infty)}_{+}(H) & \longleftrightarrow & \mathsf{Hom}^{(\infty)}_{\leq 0,+}(H) \\ X & \longmapsto & \mathrm{Fund}(X). \end{array}$$

This bijection restricts to a bijection from $\mathrm{Silt}^{(m)}_{+}(H)$ *to* $\mathsf{Hom}^{(m)}_{\leq 0,+}(H)$. □

9.8. Representation theory as a source of symmetries

Excitement!

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There are cyclic group actions on $\mathsf{Asso}^{(m)}_{\nabla}(W,c)$, and $\mathsf{NC}^{(m)}_{+}(W,c)$, as discussed in Section 5.4 and Section 7.3 respectively. In this section, we explain how they can be seen as arising out of the representation theory.

9.8.1. Orbit categories. We begin with a quick introduction to orbit categories, which we will make use of for studying both clusters and noncrossing partitions.

DEFINITION 9.8.1. Let G be an autoequivalence of $D^b(H)$. The *orbit category* of $D^b(H)$ with respect to G, which we denote $D^b(H)/G$, is a category whose objects are the objects of $D^b(H)$, and whose morphisms are defined by

$$\mathrm{Hom}_{D^b(H)/G}(X,Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{D^b(H)}(X, G^i Y). \tag{9.1}$$

Define an autoequivalence of $D^b(H)$ by $F^{(m)} = [m]\tau^{-1}$. Let $\mathcal{C}^{(m)}$ be the orbit category with respect to $F^{(m)}$. It is easy to see that $\mathcal{C}^{(m)}$ has finite-dimensional Hom-spaces, because there are only a finite number of non-zero summands in the direct sum (9.1). It is a deeper result that this category is triangulated ([**Kel05**], see also [**Ami07**]).

A triangulated category $\mathcal{C}$ is called *k-Calabi–Yau* if for any X and Y objects of $\mathcal{C}$, we have that $\mathrm{Ext}^i_{\mathcal{C}}(X,Y)$ is naturally isomorphic to the dual of $\mathrm{Ext}^{k-i}_{\mathcal{C}}(Y,X)$.

LEMMA 9.8.2 ([**Kel05**]). *$\mathcal{C}^{(m)}$ is $(m+1)$-Calabi–Yau.*

PROOF. After unwinding the definitions, this follows from Auslander-Reiten duality. □

We now divide into two cases, to treat clusters and noncrossing partitions.

9.8.2. m-eralized clusters. Let $m \geq 0$. The orbit category $\mathcal{C}^{(m)}$ is called the *m-cluster category*. An object X in $\mathcal{C}^{(m)}$ is called a (basic) *m-cluster tilting object* if no two indecomposable direct summands of X are isomorphic, $\mathrm{Ext}^i(X,X) = 0$ for $1 \leq i \leq m$, and X is maximal with respect to this property, i.e., if there is some Y such that $\mathrm{Ext}^i(X \oplus Y, X \oplus Y) = 0$, then Y is a direct sum of summands of X. The maximality property can also be captured by saying that X is the direct sum of n pairwise non-isomorphic indecomposable objects.

Let $\mathcal{F}^{(m)}$ denote the full additive category of $D^b(H)$ whose indecomposable modules are of the form either $M[i]$ for $0 \leq i < m$ or $I[-1]$, with I an indecomposable injective. $\mathcal{F}^{(m)}$ is a fundamental domain for the action of $F^{(m)}$ on ind $D^b(H)$.

LEMMA 9.8.3 ([**BRT11**]). *Let X be an object in $\mathcal{F}^{(m)}$. X is a silting object if and only if the image of X in $\mathcal{C}^{(m)}$ is an m-cluster tilting object.*

PROOF. Suppose that X and Y are indecomposable objects in $\mathcal{F}^{(m)}$. It suffices to show that they can permissibly occur together as summands of an m-cluster tilting object if and only if they can permissibly occur together as summands of a silting object . In other words, we must check that $\mathrm{Ext}^i_{\mathcal{C}^{(m)}}(X,Y) = 0 = \mathrm{Ext}^i_{\mathcal{C}^{(m)}}(Y,X)$ for all $1 \leq i \leq m$ if and only if $\mathrm{Ext}^i_{D^b(H)}(X,Y) = 0 = \mathrm{Ext}^i_{D^b(H)}(Y,X)$ for all $i > 0$. We may assume that X is weakly to the left of Y in the AR quiver for $D^b(H)$.

We first consider the forwards implication. Since X is weakly to the left of Y, it follows that $\mathrm{Ext}^i(X,Y) = 0$ for all $i > 0$. Because X and Y both lie in $\mathcal{F}^{(m)}$, $\mathrm{Ext}^i(Y,X) = 0$ for $i > m$. The fact that $\mathrm{Ext}^i(Y,X) = 0$ for $1 \leq i \leq m$ follows from the corresponding statement for $\mathcal{C}^{(m)}$.

For the reverse implication, we observe that, because X and Y are both in $\mathcal{F}^{(m)}$, and X is weakly to the left of Y, the only summand of $\mathrm{Ext}^i_{\mathcal{C}^{(m)}}(Y,X)$ which could be nonzero is $\mathrm{Ext}^i_{D^b(H)}(Y,X)$, but this is zero by assumption. The vanishing of $\mathrm{Ext}^i_{\mathcal{C}^{(m)}}(X,Y)$ now follows from the $(m+1)$-Calabi-Yau property of $\mathcal{C}^{(m)}$. □

PROOF. Since τ commutes with $F^{(m)}$, it descends to an autoequivalence of the m-cluster category. Acting on $\mathcal{C}^{(m)}$, it is clear that τ preserves the property of being an m-cluster tilting object. Thus, it defines a cyclic action on the set of silting objects in $\mathcal{F}^{(m)}$, and thus on $\mathsf{Asso}^{(m)}_{\nabla}(W,c)$. Since τ corresponds to the combinatorial AR translation, it gives rise to the Cambrian rotation Camb_c. □

9.8.3. Positive m-eralized noncrossing partitions. Let $m \geq 0$. It turns out that the suitable orbit category in which to understand the m-eralized noncrossing partitions is $\mathcal{C}^{(m)}_+ = \mathcal{C}^{(-m-1)}$, that is to say, we use the same construction as above, but with a negative value for the parameter. Write $F^{(m)}_+$ for $F^{(-m-1)}$. This approach was introduced by R. Coelho Simoes for $m = 1$ [**CS12**].

Let $\mathcal{F}_+^{(m)}$ be the fundamental domain for the action of $F_+^{(m)}$ on $D^b(H)$ which consists of all the indecomposable objects in shifts 0 to m of $D^b(H)$, except for the indecomposable projectives in shift zero.

An object X in $\mathcal{C}_+^{(m)}$ is called an $\mathrm{Hom}_{\leq 0}$ configuration if $\mathrm{Ext}^i(X, X) = 0$ for $-1 \geq i \geq -m$, while $\mathrm{Hom}(X, X)$ consists of linear combinations of the identity maps on indecomposable summands of X, and X is maximal with respect to this property.

Lemma 9.8.4. *Let X be an object in $\mathcal{F}_+^{(m)}$. X is $\mathrm{Hom}_{\leq 0}$ configuration in $D^b(H)$ if and only if the image of X in $\mathcal{C}_+^{(m)}$ is a $\mathrm{Hom}_{\leq 0}$ configuration in $\mathcal{C}_+^{(m)}$.* □

We omit the proof, which is identical to the proof of Lemma 9.8.3.

Proposition 9.8.5. *$[-1]$ induces the positive Kreweras complement action Krew_c on $\mathsf{NC}_+^{(m)}(W, c)$.*

Proof. Similarly to the m-cluster case, since $[-1]$ commutes with $F_+^{(m)}$, it descends to an autoequivalence of $\mathcal{C}_+^{(m)}$, where it induces a cyclic action on $\mathrm{Hom}_{\leq 0}$ configurations, and thus also induces a cyclic action of $\mathrm{Hom}_{\leq 0}$ configurations in $\mathcal{F}_+^{(m)}$, and thus on $\mathsf{NC}_+^{(m)}(W, c)$. Since $[-1]$ corresponds to the combinatorial shift, it gives rise to the positive Kreweras complement action Krew_c. □

The Auslander-Reiten translation τ also gives a symmetry of $\mathsf{NC}_+^{(m)}(W, c)$, but the group of symmetries induced by [1] includes the group of symmetries induced by τ, and sometimes the inclusion is strict.

9.9. m-eralized sortable elements and co-aisles in the derived category

When we speak of a subcategory of $D^b(H)$, we always mean a full subcategory closed under direct sums and direct summands. Thus, one way to specify a subcategory of $D^b(H)$ is to specify its indecomposable objects.

Definition 9.9.1. A subcategory $\mathcal{V}$ of $D^b(H)$ is called a *co-aisle* if the following hold [**KV88**, Proposition 1.3].

(1) $X \in \mathcal{V}$ implies $X[-1] \in \mathcal{V}$.
(2) If we have a triangle $X \to Y \to Z \to$ in $D^b(H)$ such that X and Z are in $\mathcal{V}$, then Y is also in $\mathcal{V}$.
(3) For each object Z of $D^b(H)$, there is some $X \in \mathcal{V}$ such that the map $\mathrm{Hom}_{\mathcal{V}}(X, \cdot) \to \mathrm{Hom}(Z, \cdot)|_{\mathcal{V}}$ is an epimorphism.

(Note that [**KV88**] works with aisles; we work with the dual notion of co-aisles for convenience.) Following the terminology of [**KV88**], a co-aisle $\mathcal{V}$ is called *separated* if there is some m such that $\mathcal{V}$ is contained in the additive category generated by $\operatorname{mod} H[i]$ for $i \leq m$.

Note that if $\mathcal{V}$ is a subcategory of $D^b(H)$ satisfying this condition, then, to show that $\mathcal{V}$ is a separated co-aisle, it suffices to check conditions (1) and (2) above. Condition (3) will automatically be satisfied because for any $Z \in D^b(H)$, there will only be a finite number of indecomposable modules of $\mathcal{V}$ admitting a non-zero morphism from Z, so we could take X to be their direct sum.

The standard co-aisle $\mathcal{V}_{\mathrm{st}}(H)$ in $D^b(H)$ is additively generated by $M[i]$ for $M \in \operatorname{mod} H$ and $i < 0$.

DEFINITION 9.9.2. Write $\mathrm{CoAisle}^{(\infty)}(H)$ for the separated co-aisles in $D^b(H)$ which contain $\mathcal{V}_{\mathrm{st}}(H)$. Write $\mathrm{CoAisle}^{(m)}(H)$ for the separated co-aisles in $\mathrm{CoAisle}^{(\infty)}(H)$ which are contained in $\mathcal{V}_{\mathrm{st}}(H)[m]$.

For $\mathcal{V} \in \mathrm{CoAisle}_H^{(\infty)}$, denote the collection of indecomposable objects in $\mathcal{V}$ in degrees ≥ 0 by $\mathcal{V}^\sharp$. Write $\mathrm{CoAisle}^\sharp(H)$ for the collections which arise as $\mathcal{V}^\sharp$ for some $\mathcal{V} \in \mathrm{CoAisle}^{(\infty)}(H)$. For $\mathcal{T} \in \mathrm{CoAisle}^\sharp(H)$, write $\overline{\mathcal{T}}$ for the corresponding separated co-aisle.

For $\mathcal{V}$ a separated co-aisle containing $\mathcal{V}_{\mathrm{st}}(H)$, define

$$\underline{\dim}^\sharp(\mathcal{V}) = \{\underline{\dim}(X) \mid X \in \mathcal{V}^\sharp\}.$$

We show that the collections which arise as $\underline{\dim}^\sharp(\mathcal{V})$ for $\mathcal{V}$ a separated co-aisle containing $\mathcal{V}_{\mathrm{st}}(H)$ are exactly the inversion sets of c-sortable elements of the Artin monoid, by establishing the appropriate Cambrian recursion for the sets $\underline{\dim}^\sharp(\mathcal{V})$.

Let $c = s_1 \dots s_n$. Define $\mu_1(H)$ to be the algebra obtained by applying a reflection functor at vertex 1. As usual, we identify $\mathrm{ind}\,\mathrm{mod}\,\mu_1(H)$ with $\mathrm{ind}\,\mathrm{mod}\,H$, with S_1 removed, and $S_1[1]$ added. We will also need to consider $H' = H/He_1H$, a hereditary Artin algebra of rank $n-1$.

PROPOSITION 9.9.3. *Let s_1 be initial in c_H. Then*

$$\mathcal{T} \in \mathrm{CoAisle}^\sharp(H) \Leftrightarrow \begin{cases} \mathcal{T} \in \mathrm{CoAisle}^\sharp(H') & \text{if } S_1 \notin \mathcal{T} \\ \mathcal{T} \in \mathrm{CoAisle}^\sharp(\mu_1(H)) & \text{if } S_1 \in \mathcal{T} \end{cases}.$$

PROOF. Suppose that $\mathcal{T} \in \mathrm{CoAisle}^\sharp(H)$. Consider first the case that $S_1 \notin \mathcal{T}$. Let M be an H-module such that $\mathrm{Hom}(S_1, M) \neq 0$. We claim that $M[i] \notin \mathcal{V}$ for any $i \geq 0$. Because $\overline{\mathcal{T}}$ is closed under $[-1]$, it suffices to show that $M \notin \mathcal{T}$. Suppose otherwise. We have a short exact sequence of modules $0 \to S_1 \to M \to M' \to 0$, which gives rise to a triangle:

$$M'[-1] \to S_1 \to M \to M'[0]$$

Now $M'[-1] \in \overline{\mathcal{T}}$ because $\overline{\mathcal{T}}$ contains $\mathcal{V}_{\mathrm{st}}(H)$. Then (2) implies that S_1 is in $\mathcal{T}$, contrary to our assumption.

Therefore, all the indecomposable objects of $\mathcal{T}$ are of the form $M[i]$ for M an H-module satisfying $\mathrm{Hom}(S_s, M) = 0$ (and $i \geq 0$); equivalently, they are of the form $M[i]$ for M an H'-module (and $i \geq 0$). Thus, $\mathcal{T} \in \mathrm{CoAisle}^\sharp(H')$.

Next, suppose that $S_1 \in \mathcal{T}$. In this case, $\overline{\mathcal{T}}$ is a co-aisle containing $\mathcal{V}_{\mathrm{st}}(\mu_1(H))$, so $\mathcal{T} \in \mathrm{CoAisle}^{(\infty)}(\mu_1(H))$.

Conversely, suppose that we are given $\mathcal{T}$ in $\mathrm{CoAisle}^\sharp(H')$. Define X to be the additive subcategory of $D^b(H)$ generated by $\mathcal{T}$ and $\mathcal{V}_{\mathrm{st}}(H)$. (1) obviously holds. Condition (2) follows from the long exact sequence for homology. Together these imply that X is a co-aisle in $\mathrm{CoAisle}^{(\infty)}(H)$, so $\mathcal{T} \in \mathrm{CoAisle}^\sharp(H)$.

Finally, suppose we are given $\mathcal{T}$ in $\mathrm{CoAisle}^\sharp(\mu_1(H))$. It is clear that $\mathcal{T} \cup \{S_1\}$ lies in $\mathrm{CoAisle}^\sharp(H)$. □

THEOREM 9.9.4. *There is a natural bijection*

$$\begin{array}{ccc} \mathrm{CoAisle}^{(\infty)}(H) & \overset{c}{\longleftrightarrow} & \mathsf{Sort}^{(\infty)}(W, c) \\ \mathcal{V} & \longmapsto & \underline{\dim}^\sharp(\mathcal{V}) \end{array}$$

This bijection restricts to a bijection from $\mathrm{CoAisle}^{(m)}(H)$ *to inversion sets of* m-c-*sortable elements in the Artin monoid.*

It was clear from the beginning of this project that this bijection would give us sets of colored roots that should be considered the m-eralization of the inversion sets of c-sortable elements, but at that point it wasn't clear there was anything interesting to be said about them.

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PROOF. Both satisfy the same Cambrian recursion. The statement about restriction is straightforward, since a c-sortable elment is m-c-sortable if and only if all its inversions have color at most m. □

An analogous result in the abelian case was shown for kQ-mod in [**IT09**]. Another description of aisles in the hereditary setting was provided by [**SvR16**].

9.10. m-eralized clusters and sortable elements

There is a bijection between silting objects and aisles which goes back to [**KV88**, Section 5]. We follow [**KV88**] except that we use co-aisles instead of aisles. An object Y in a co-aisle $\mathcal{V}$ is called Ext-injective in $\mathcal{V}$ if $\mathrm{Ext}^i(M, Y) = 0$ for all $M \in \mathcal{V}$ and all $i > 0$. We write $\mathrm{ExtInj}(\mathcal{V})$ for the direct sum of the Ext-injective indecomposables of $\mathcal{V}$.

Does that mean, we are reinventing ideas from the 1980s?

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THEOREM 9.10.1 ([**KV88**]). *There is a bijection from silting objects to co-aisles containing the standard co-aisle given as follows*

$$\begin{array}{rcl} \mathrm{Silt}^{(\infty)}(H) & \longleftrightarrow & \mathrm{CoAisle}^{(\infty)}(H) \\ X & \longmapsto & \{M \in D^b(H) \mid \mathrm{Ext}^i(M, X) = 0 \text{ for } i \geq 0\} \\ \mathrm{ExtInj}(\mathcal{V}) & \longleftarrow & \mathcal{V}. \end{array}$$

This bijection restricts to a bijection between $\mathrm{Silt}^{(m)}(H)$ *and* $\mathrm{CoAisle}^{(m)}(H)$.

We can now state the following theorem.

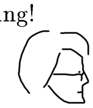
THEOREM 9.10.2. *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{Silt}^{(\infty)}(H) & \xleftrightarrow{\text{Theorem 9.10.1}} & \mathrm{CoAisle}^{(\infty)}(H) \\ \updownarrow \text{Theorem 9.5.8} & & \updownarrow \text{Theorem 9.9.4} \\ \mathsf{Asso}_{\nabla}^{(\infty)}(W, c) & \xleftrightarrow[\text{Theorem 6.8.10.}]{} & \mathsf{Sort}^{(\infty)}(W, c) \end{array}$$

The bijections all restrict to the appropriate m-*eralized versions.*

PROOF. All the bijections in the diagram respect the Cambrian recursion. □

And on that note, we leave you. Thanks for reading!



#7

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