Snapshots of modern mathematics
from Oberwolfach

№ 14/2018

Mixed volumes and mixed integrals

Liran Rotem

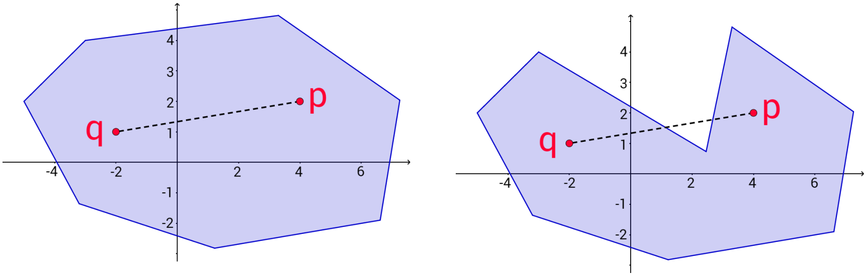
In recent years, mathematicians have developed new approaches to study convex sets: instead of considering convex sets themselves, they explore certain functions or measures that are related to them. Problems from convex geometry become thereby accessible to analytic and probabilistic tools, and we can use these tools to make progress on very difficult open problems.

We discuss in this Snapshot such a functional extension of some "volumes" which measure how "big" a set is. We recall the construction of "intrinsic volumes", discuss the fundamental inequalities between them, and explain the functional extensions of these results.

1 Mixed volumes

A *convex set* is a set A such that for any two points $p, q \in A$ (meaning p and q belong to A), the entire line segment connecting p and q lies inside A. For example, the set in Figure 1a is convex, while the set in Figure 1b is not.

The sets depicted in Figure 1 "live" in the two-dimensional plane. Every point p in the plane can be identified with a pair of real numbers, $p = (x, y)$.



(a) convex

(b) non-convex

Figure 1: A convex set and a nonconvex set.

For example, in Figures 1a and 1b we have $p = (4, 2)$ and $q = (-2, 1)$. We denote the set of all planar points by $\mathbb{R}^2$. Two points $p = (x, y)$ and $q = (z, w)$ in $\mathbb{R}^2$ can be added to create a new point $p + q = (x + z, y + w)$. Moreover, we can multiply the point p by a real number λ to obtain $\lambda p = (\lambda x, \lambda y)$. With these operations, we can give a formal definition of convexity: A set A is convex if for every $p, q \in A$ and every $0 < \lambda < 1$ we also have $\lambda p + (1 - \lambda)q \in A$. The two operations can also be extended to convex sets: If A and B are convex sets, the sum $A + B$ is comprised of all possible sums $p + q$ where $p \in A$ and $q \in B$. Similarly, λA is comprised of all points of the form λp where $p \in A$.

It is easy to understand the operation $A \mapsto \lambda A$ geometrically: We take the set A and either blow it up (if $\lambda > 1$) or shrink it down (if $\lambda < 1$). For example, if A is a disk of radius 4, then $3A$ is a disk of radius 12 and $\frac{1}{4}A$ is a disk of radius 1. As for the sum $A + B$, Figure 2 illustrates the sum of a quadrilateral and a disk.

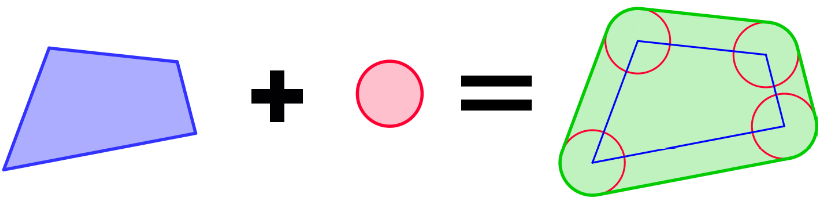


Figure 2: The sum of a quadrilateral and a disk.

For a convex set A in $\mathbb{R}^2$, we denote its area by $|A|$. Given two convex sets A and B in $\mathbb{R}^2$ and a number $t > 0$, what can be said about the area $|A + tB|$? Figure 3 depicts again the case where A is a quadrilateral and B is a disk of radius 1 (so tB is a disk of radius t).

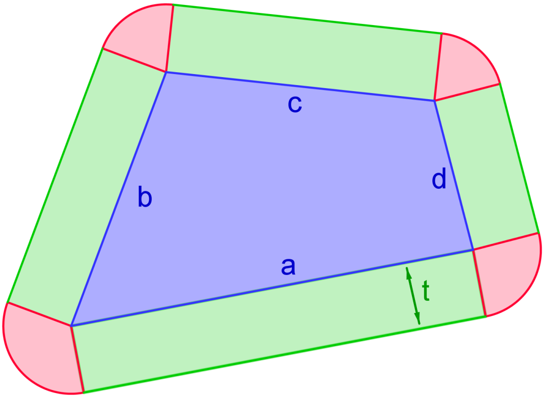


Figure 3: The set $A + tB$, decomposed into three parts.

As we see in this figure, the set $A + tB$ can be decomposed into three parts. The blue part is congruent to A, so its area is simply $|A|$. The green part is comprised of four rectangles. If the lengths of the sides of A are a, b, c, d then the areas of these rectangles are $a \cdot t$, $b \cdot t$, $c \cdot t$, $d \cdot t$. It follows that the total area of the green part is

$$a \cdot t + b \cdot t + c \cdot t + d \cdot t = (a + b + c + d) \cdot t = \mathrm{Per}(A) \cdot t,$$

where $\mathrm{Per}(A)$ is the *perimeter* of A. Finally, the red part is comprised of four sectors of a disk of radius t. Since the sum of the external angles of any polygon is $360°$, the four sectors can be glued together to create exactly one complete disk. It follows that the area of the red part is πt^2, the area of a disk of radius t.

Adding the three areas and using the fact that $|B| = \pi$, we conclude that

$$|A + tB| = |A| + \mathrm{Per}(A) \cdot t + |B| \cdot t^2.$$

It turns out that this formula holds for any convex set A in the plane, as long as B is the disk of radius 1. This is the 2-dimensional case of the important *Steiner formula.*

Even though we can only really visualize convex sets in 2 or 3 dimensions, we can also discuss n-dimensional convex sets for any natural number n. Just like a point in a 2-dimensional space is a pair of real numbers, a point p in an n-dimensional space is just an n-tuple $p = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ of n real numbers, and such points are multiplied with real numbers and added just like points in the plane:

$$\lambda \cdot (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (\lambda x_1 + y_1, \lambda x_2 + y_2, \ldots, \lambda x_n + y_n).$$

We define convex sets and their addition $A + B$ in exactly the same way as in the 2-dimensional case. For an n-dimensional convex set A, we denote by $|A|$ its volume[1].

Again, we would like to know what the volume $|A + tB|$ is, where A is any convex set in $\mathbb{R}^n$ and B is a ball of radius 1. The *n-dimensional Steiner formula* (whose meaning we will explain in a moment) reads as

$$|A + tB| = V_n(A) + V_{n-1}(A) \cdot t + V_{n-2}(A) \cdot t^2 + \cdots + V_0(A) \cdot t^n.$$

Like in the 2-dimensional case, the first coefficient $V_n(A)$ is just $|A|$, and the second coefficient $V_{n-1}(A)$ is the *surface area* of A. The last coefficient, $V_0(A)$, is again just $|B|$, so it doesn't really depend on A. However, the remaining coefficients $V_1(A), V_2(A), \ldots, V_{n-2}(A)$ don't have such a straightforward interpretation. The numbers $V_0(A), V_1(A), \ldots, V_n(A)$ (multiplied with some constants, which we may conveniently ignore here) are known as the *intrinsic volumes* of A. In a very informal sense, these numbers measure how "big" a convex set is, just like the usual volume and the usual surface area.

A natural question which now arises is what happens if we take both A and B to be arbitrary convex sets instead of taking B to be the ball. It turns out that we still have a formula of the form

$$|A + tB| = c_n(A, B) + c_{n-1}(A, B) \cdot t + c_{n-2}(A, B) \cdot t^2 + \cdots + c_0(A, B) \cdot t^n$$

for some coefficients $c_k(A, B)$ that depend on both A and B. In fact, the same is true not just for two sets but for any number of convex sets: the volume $|t_1A_1 + t_2A_2 + \cdots + t_mA_m|$ is some polynomial[2], with coefficients that depend on the convex sets $A_1, A_2, \ldots, A_m$. This result is know as *Minkowski's theorem*, and the coeffcients are known as *mixed volumes*.

Mixed volumes are very important in convex geometry, and much of the theory of convexity is built around inequalities between mixed volumes. For the sake of simplicity we will only discuss intrinsic volumes in the remainder of this note.

2 Inequalities

The *isoperimetric inequality* is one of the most fundamental results in convex geometry. It states that among all sets A in $\mathbb{R}^n$ with a given volume, the ball

[1] We have not said what we mean by the volume of an n-dimensional set, but you can just think of it as something analogous to the area of a 2-dimensional set or the volume of a 3-dimensional set.

[2] Maybe so far you've only seen polynomials that depend on one variable. The volume $|t_1A_1+t_2A_2+\cdots+t_mA_m|$ depends on n variables $t_1, t_2, \ldots, t_n$; such a *multivariate polynomial* is an expression that is built from the variables $t_1, t_2, \ldots, t_n$ by multiplying and adding.

has the minimal surface area. While this result was known to the ancient Greeks, the first formal proof was given by Schwartz in the 19th century, building on the work of Steiner (see [7] and [3] for surveys regarding the isoperimetric inequality and the intimately related Brunn–Minkowski inequality).

Let us write the isoperimetric inequality as an inequality between intrinsic volumes. To do so we will need to remember that the n-dimensional volume is *homogeneous of degree* n, that is, $V_n(rA) = r^n V_n(A)$. More generally, the k-th intrinsic volume V_k is homogeneous of degree k – it satisfies $V_k(rA) = r^k V_k(A)$. A proof of the latter fact is not difficult, but we will not give it here.

To proceed, let A be any convex set and let B be the ball of radius 1. Choose a number r such that the ball $C = rB$ has the same volume as A. Then

$$V_n(A) = V_n(C) = V_n(rB) = r^n V_n(B),$$

and solving for r we obtain $r = \left(\frac{V_n(A)}{V_n(B)}\right)^{1/n}$. By the isoperimetric inequality, the surface area of the ball C is not larger than the surface area of A, so

$$V_{n-1}(A) \geq V_{n-1}(C) = V_{n-1}(rB) = r^{n-1} V_{n-1}(B).$$

Recalling the value of r, and reorganizing the inequality, we conclude that

$$\left(\frac{V_{n-1}(A)}{V_{n-1}(B)}\right)^{\frac{1}{n-1}} \geq \left(\frac{V_n(A)}{V_n(B)}\right)^{\frac{1}{n}}.$$

It turns out that a similar inequality holds for any two intrinsic volumes. The *Alexandrov inequalities* state that if A is any n-dimensional convex set and B is the ball of radius 1, then for every $0 < i \leq j \leq n$ one has

$$\left(\frac{V_i(A)}{V_i(B)}\right)^{\frac{1}{i}} \geq \left(\frac{V_j(A)}{V_j(B)}\right)^{\frac{1}{j}}. \qquad (\star)$$

The Alexandrov inequalities for intrinsic volumes follow easily from the more general *Alexandrov-Fenchel inequality*, which is a very deep inequality between mixed volumes. Some history of the problem, and references to various proofs, appear in Section 1.7 of [1].

3 Functional extensions

In this section we will shift our attention from sets A in $\mathbb{R}^n$ to functions $f : \mathbb{R}^n \to [0, \infty)$, which will turn out to be good tool for understanding convexity. More concretely, we will be interested in log-concave functions. A function $f : \mathbb{R}^n \to [0, \infty)$ is called *log-concave* if for every $x, y \in \mathbb{R}^n$ and every $0 < \lambda < 1$ one has

$$f(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}. \qquad (\star\star)$$

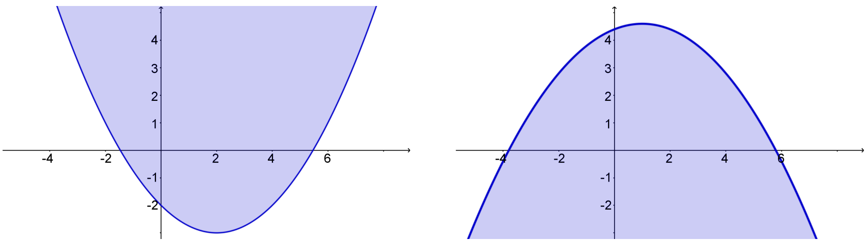
In order to understand the name "log-concave", let us briefly discuss convex and concave functions. A function $\phi : \mathbb{R}^n \to (-\infty, \infty]$ is called *convex* if for every $x, y \in \mathbb{R}^n$ and every $0 < \lambda < 1$ one has

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda \phi(x) + (1-\lambda)\phi(y).$$

Similarly, a function $\phi : \mathbb{R}^n \to [-\infty, \infty)$ is called *concave* if for every $x, y \in \mathbb{R}^n$ and every $0 < \lambda < 1$ one has

$$\phi(\lambda x + (1-\lambda)y) \geq \lambda \phi(x) + (1-\lambda)\phi(y).$$

More geometrically, a function ϕ is convex if the area "above" the graph of ϕ is a convex set. A function ϕ is concave if the area "below" the graph of ϕ is a convex set. You may try to check yourself that these geometric conditions on the graph are indeed captured by the formulas above. Figure 4 depicts such functions in the simplest case $n = 1$. It can be seen from the definition that ϕ is convex if and only if $-\phi$ is concave.



(a) convex (b) concave

Figure 4: A convex function and a concave function.

A function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if and only if $\log f$ is a concave function, which explains the name "log-concave". The reason we are especially interested in log-concave functions is that for every convex set A in $\mathbb{R}^n$, its *indicator function*, which is defined as

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

is log-concave. Let us prove this simple fact: If $x \notin A$ or $y \notin A$ then the inequality

$$\mathbf{1}_A(\lambda x + (1-\lambda)y) \geq \mathbf{1}_A(x)^\lambda \mathbf{1}_A(y)^{1-\lambda}$$

definitely holds, since the right-hand side is 0 and the left-hand side is 0 or 1. If on the other hand $x, y \in A$ then also $\lambda x + (1-\lambda)y \in A$ since A is convex, and the inequality ($\star\star$) becomes

$$1 \geq 1^\lambda \cdot 1^{1-\lambda},$$

which is obviously true.

Not every log-concave function f is an indicator function of a convex set. As an example, the reader may want to check that the function $f : \mathbb{R}^2 \to [0, \infty)$ defined by $f(x, y) = e^{-x^2-y^2}$ is log-concave. We would like to think of log-concave functions as "generalized convex sets" and try to prove geometric statements about them. Over the last decade this approach proved itself to be extremely useful. In fact, even if we are ultimately interested only in convex sets and not in functions, the new functional point of view allows us to attack problems that were completely intractable in the past. A survey of such functional techniques and their success can be found in [4] (which is already slightly outdated).

In the papers [5] and [6], which are joint with V. Milman, we define "mixed volumes" for log-concave functions and prove many inequalities concerning them. In order to present the definition, we should first understand how to add log-concave functions. The obvious guess would be to add the functions pointwise, that is, to consider

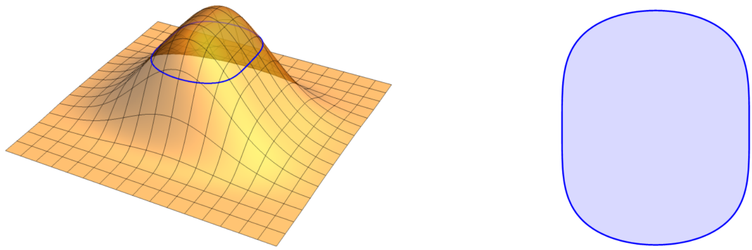
$$(f + g)(x) = f(x) + g(x).$$

However, this definition doesn't make much sense for us, since the pointwise sum of two log-concave functions is not necessarily log-concave: you may try to check yourself that while the indicator functions $\mathbf{1}_{[-2,-1]}$ and $\mathbf{1}_{[1,2]}$ are log-concave, their pointwise sum $\mathbf{1}_{[-2,-1]} + \mathbf{1}_{[1,2]} = \mathbf{1}_{[-2,-1]\cup[1,2]}$ is not.

Hence, instead we make the following definition:

$$(f \oplus g)(x) = \max_{y \in \mathbb{R}^n} \min\{f(y), g(x-y)\}.$$

Instead of explaining this strange-looking definition, let us expound a better way to think about it. For every function $f : \mathbb{R}^n \to [0, \infty)$ and every number $t > 0$, we define $K_t(f)$ to be the set of all points $x \in \mathbb{R}^n$ such that $f(x) \geq t$. These sets are known as the *upper level sets* of f. It is easy to check that if f is log-concave then the sets $K_t(f)$ are convex sets. The sum $f \oplus g$ is then characterized by the simple formula $K_t(f \oplus g) = K_t(f) + K_t(g)$, where the addition on the right hand side is the standard addition of convex sets we already discussed. Figures 5a and 5b illustrate the concept of an upper level set.



(a) graph of f with a level set (in blue) (b) corresponding upper level set

Figure 5: A log-concave function $f : \mathbb{R}^2 \to [0, \infty)$ and an upper level set.

Similarly, if f is a log-concave function and $\lambda > 0$, we define

$$(\lambda \odot f)(x) = f\left(\frac{x}{\lambda}\right).$$

Again, this is a simple operation in terms of the upper level sets, since $K_t(\lambda \odot f) = \lambda K_t(f)$.

In order to to define intrinsic volumes, we considered the volume of $A + tB$. What is the functional equivalent of the volume? In other words, what is the "volume" of a log-concave function f? We claim the answer is the integral $\int f$. One good reason for defining it in this way is the relation $\int \mathbf{1}_A = |A|$, which the reader may be familiar with.

Consider an arbitrary log-concave function f, and define $g = \mathbf{1}_B$, the indicator function of the ball. Like in the classical case, which we considered before, one has

$$\int (f \oplus (t \odot g)) = V_n(f) + V_{n-1}(f) \cdot t + V_{n-2}(f) \cdot t^2 + \cdots + V_0(f) \cdot t^n.$$

Again we have $V_n(f) = \int f$ and $V_0(f) = \int g = |B|$, but the remaining coefficients $V_1(f), V_2(f), \ldots, V_{n-1}(f)$ are new. We call them the *intrinsic integrals* of the function.

Again, one can consider not only the function $g = \mathbf{1}_B$ but any log-concave function g, and even more generally one may consider m different log-concave functions $f_1, f_2, \ldots, f_m$. This will lead to the construction of *mixed integrals*, which generalize the classical mixed volumes. We will not discuss further this more general case.

Now that the intrinsic integrals are defined, we turn our attention to a functional extension of the Alexandrov inequalities ($\star$). In the case of convex sets, we always compared a general set A with the ball B. Therefore, it makes sense to assume that we will now compare a general log-concave function f with the function $g = \mathbf{1}_B$. Surprisingly it turns out that this is not a good approach, and the correct function to compare with, $h : \mathbb{R}^n \to [0, \infty)$, is defined by

$$h(x_1, x_2, \ldots, x_n) = e^{-\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}.$$

Figure 6 shows the graph of this function for the easy-to-draw case $n = 2$.

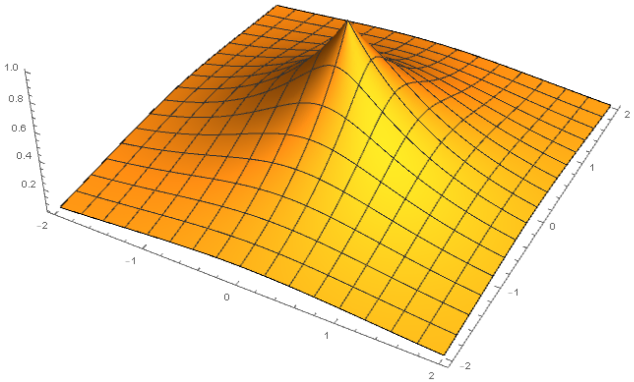


Figure 6: The graph of the function $h : \mathbb{R}^2 \to [0, \infty)$.

We can now formulate the main theorem proved in [6]: Let $f : \mathbb{R}^n \to [0, \infty)$ be any log-concave function with $f(0) = 1$. Then for every $0 < i \leq j \leq n$ one has

$$\left(\frac{V_i(f)}{V_i(h)}\right)^{\frac{1}{i}} \geq \left(\frac{V_j(f)}{V_j(h)}\right)^{\frac{1}{j}}.$$

In particular, by taking $i = n-1$ and $j = n$, we obtain a functional isoperimetric inequality: if f is a log-concave function such that $\int f = \int h$, then f has a larger "surface area" than h. Here the surface area of a log-concave function f should be understood as the intrinsic integral $V_{n-1}(f)$.

Many more inequalities were proved in [6], but we will not discuss them here. Instead, we will conclude this snapshot by mentioning the paper [2] of Bobkov, Colesanti and Fragalà. In this paper the authors independently defined the same intrinsic integrals (but not the more general mixed integrals), and proved a completely different set of inequalities concerning them. For example, it follows from their results that for any log-concave function f, $g = \mathbf{1}_B$, and

$0 < i \leq j \leq n$, one has

$$\left(\frac{V_i(f^i)}{V_i(g^i)}\right)^{\frac{1}{i}} \geq \left(\frac{V_j(f^j)}{V_j(g^j)}\right)^{\frac{1}{j}},$$

which is an analogue of the Alexandrov inequalities ($\star$). Inequalities concerning convex sets have thus inspired discoveries of new inequalities that deal with certain interesting and useful classes of functions.

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Mathematical subjects
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DOI
10.14760/SNAP-2018-014-EN

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