Mathematisches
Forschungsinstitut
Oberwolfach

Mitglied der Leibniz Gemeinschaft

Oberwolfach Preprints

OWP 2019 - 06

KARL H. HOFMANN AND LINUS KRAMER

Group Algebras of Compact Groups
A New Way of Producing Group Hopf Algebras over Real and Complex Fields:
Weakly Complete Topological Vector Spaces

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

The MFO publishes a preprint series **Oberwolfach Preprints (OWP)**, ISSN 1864-7596, which mainly contains research results related to a longer stay in Oberwolfach, as a documentation of the research work done at the MFO. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

The full copyright is left to the authors. With the submission of a manuscript, the authors warrant that they are the creators of the work, including all graphics. The authors grant the MFO a perpetual, non-exclusive right to publish it on the MFO's institutional repository.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX – XX). Additionally, each preprint will get a Digital Object Identifier (DOI).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI 10.14760/OWP-2019-06

Karl H. Hofmann • Linus Kramer

Group Algebras of Compact Groups

**A New Way of Producing Group Hopf Algebras over Real and Complex Fields:
Weakly Complete Topological Vector Spaces**

**A Preprint of
Mathematisches Forschungsinstitut Oberwolfach**

**Technische Universität Darmstadt, Germany
and Tulane University, New Orleans, USA**

University of Münster, Germany

Table of Contents 2

Abstract. Weakly complete real or complex associative algebras A are necessarily projective limits of finite dimensional algebras. Their group of units A^{-1} is a pro-Lie group with the associated topological Lie algebra $A_{\rm Lie}$ of A as Lie algebra and the globally defined exponential function $\exp\colon A \to A^{-1}$ as the exponential function of A^{-1}. With each topological group G, a weakly complete group algebra $\mathbb{K}[G]$ is associated functorially so that the functor $G \mapsto \mathbb{K}[G]$ is left adjoint to $A \mapsto A^{-1}$. The group algebra $\mathbb{K}[G]$ is a weakly complete Hopf algebra. If G is compact, then $\mathbb{R}[G]$ contains G as the set of grouplike elements. The category of all real weakly complete Hopf algebras A with a compact group of grouplike elements whose linear span is dense in A is equivalent to the category of compact groups. The group algebra $A = \mathbb{R}[G]$ of a compact group G contains a copy of the Lie algebra $\mathfrak{L}(G)$ in $A_{\rm Lie}$; it also contains all probability measures on G. The dual of the group algebra $\mathbb{R}[G]$ is the Hopf algebra $\mathcal{R}(G,\mathbb{R})$ of representative functions of G. The rather straightforward duality between vector spaces and weakly complete vector spaces thus becomes the basis of a duality $\mathcal{R}(G,\mathbb{R}) \leftrightarrow \mathbb{R}[G]$ and thus yields a new aspect of Tannaka duality. In the case of a compact abelian G, an alternative concrete construction of $\mathbb{K}[G]$ is given both for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$. Because of the presence of $\mathfrak{L}(G)$, the enveloping algebra of weakly complete Lie algebras are introduced and placed into relation with $\mathbb{K}[G]$.

Acknowledgments. Substantial portions of this text, notably Chapters 6, 7, and 8 were written while the authors were partners in the program Research in Pairs at the Mathematisches Forschungsinstitut Oberwolfach MFO in the Blackforest from February 4 through 23, 2019. The authors are grateful for the environment and infrastructure of MFO which made this research possible. The authors also acknowledge the joint work with Rafael Dahmen of the Karlsruher Institut für Technologie (notably on the earlier chapters; a good deal of that work also did take place at the Technische Universität Darmstadt in 2018).

Chapter 1

The Basics on Real and Complex Vector Spaces

We let $\mathbb{K}$ denote the field $\mathbb{R}$ of real numbers, respectively, the field $\mathbb{C}$ of complex numbers and, accordingly, $\mathcal{V}_{\mathbb{R}}$ the category of real, respectively, $\mathcal{V}_{\mathbb{C}}$ the category of complex vectorspaces.

If V is a $\mathbb{K}$-vector space then its *dual* $V^* \stackrel{\text{def}}{=} \mathcal{V}_{\mathbb{K}}(V, \mathbb{K}) \subseteq \mathbb{K}^V$ of $\mathbb{K}$-linear forms inherits a vector space topology from $\mathbb{K}^V$, called the topolgy of pointwise convergence, or the *weak-*-topology* or simply the *weak topology.* If B is any basis of V (which exists by the Axiom of Choice (AC)), then the topological vector space V^* is isomorphic to the product topological vector space V^B. Conversely, if $W = \mathbb{K}^X$ in the category of topological vector spaces, then $W \cong V^{(X)}$. A topological $\mathbb{K}$-vector space is called *weakly complete* iff it is isomorphic to $\mathbb{K}^X$ for some set X.

The full subcategory $\mathcal{W}_{\mathbb{K}}$ of the category of all topological $\mathbb{K}$-vector spaces and continuous linear maps between them (cf. [10], EA3.10, p. 755) is called the category of weakly complete spaces.

We just observed that a weakly complete topological vector space $W = \mathbb{K}^X$ is the dual of the vector space $V \stackrel{\text{def}}{=} \mathbb{K}^{(X)}$, and a close look at the topological dual $W' \stackrel{\text{def}}{=} \mathcal{W}_K(W, \mathbb{K}) \cong \mathcal{W}(\mathbb{K}^X, \mathbb{K}) \cong (\mathcal{W}_K(\mathbb{K}, \mathbb{K})^{(X)} \cong \mathbb{K}^{(X)} = V$ shows that $W \cong W'^*$. It is equally straightforward to oberserve that $V^{*\prime} \cong V$ for each $\mathbb{K}$-vector space V. This is the core of the following theorem.

The Duality of the categories $\mathcal{V}_K$ and $\mathcal{W}_K$

Theorem 1.1. *The categories $\mathcal{V}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces and $\mathcal{W}_K$ of weakly complete topological $\mathbb{K}$-vector spaces are naturally dual to each other.* □

A few comments are in order. For the concept of two contravariant functors $F: \mathcal{A} \to \mathcal{B}$ and $U: \mathcal{B} \to \mathcal{A}$ being adjoint on the right see e.g. [compbook], Definition A3.30, p. 773. If these functors implement an equivalence of the categories $\mathcal{A}$ and the opposite $\mathcal{B}^{\text{op}}$ of $\mathcal{B}$ (see [10], A3.39 on p. 777) then the categories $\mathcal{A}$ and $\mathcal{B}$ are said to be naturally *dual* to each other. A celebrated example is given by the category $\mathbb{AB}$ of abelian groups and the category $\mathbb{CAB}$ of compact abelian groups called *Pontryagin–van Kampen Duality* (cf. [10]). The duality of $\mathcal{V}_{\mathbb{K}}$ and $\mathcal{W}_{\mathbb{K}}$ is discussed in detail in [10], E7.12ff., pp.325–340, and again in [11], Appendix 2, pp. 629–650. It is no accident that weakly complete vector spaces are discussed extensively in a book dedicated to a comprehensive study of pro-Lie groups such as [11]. The Lie algebra $\mathfrak{L}(G)$ of a pro-Lie group is a weakly complete vector space; in particular, this applies to each and every compact group (and indeed to every almost connected locally compact group. It is shown in [10], Proposition 7.5 that the duality between $\mathcal{V}_{\mathbb{K}}$ and $W_{\mathbb{K}}$ is a special case of the Pontryagin formalism. It is also argued that the category $\mathcal{V}_K$ may be considered as a category of locally convex topological spaces since every $\mathbb{K}$ vector space has a unique finest locally convex topology.

Given this wealth of information on the background of the Duality Theorem 1.1 we emphasize that the basic facts on the duality here are very simple and direct, but it is this simple rather systematic aspect that will serve us well in the present project. Some of the basic properties of weakly compact vector spaces are surprisingly simple.

Proposition 1.2. *For a topological $\mathbb{K}$-vector space W the following statements are eqivalent:*

(i) *W is profinite-dimensional, that is, W is the strict projective limit of its finite dimensional quotients.*
(ii) *W is weakly complete.*

Proof. This is a simple consequence of the duality and the elementary fact, that every $\mathbb{K}$-vector space is the directed union of the system of its finite dimensional vector subspaces. □

(Cf. [10], Proposition 7.26, p. 329.)

Proposition 1.3. *Every weakly complete vector space is linearly compact, that is, any filterbasis of closed affine subspaces has a nonempty intersection.*

Proof. See [11], Theorem A2.14, p. 643. □

Let us conclude these remarks with the observation that most of what we discuss here remains true over any locally compact topological field $\mathbb{K}$ in place of $\mathbb{R}$ or $\mathbb{K}$; however we emphasie that we cannot work with the discrete topology of the ground field which was done in similar situations in contexts like [4] or [12] and a considerable body of literature related to these sources.

Chapter 2

Hopf Algebras: The Category Theoretical Background

We work in an environment which generalizes the category of $\mathbb{K}$-vector spaces plus the presence of the $\mathbb{K}$-tensor products. Such environments were traditionally provided by the theory of *Commutative Monoidal Categories* $\mathcal{A}$ which support a functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such as e.g. the category of sets with the cartesian product $(X, Y) \mapsto X \times Y$. The precise definitions were collected in [10], Appendix 3, p. 787ff., see notably Definition A3.62 on p. 789. What is relevant here is that not only the category $\mathcal{V}_{\mathbb{K}}$ of $\mathbb{K}$-vector spacs has the familiar tensor product $\otimes$ but that the category $\mathcal{W}_{\mathbb{K}}$ of weakly complete vector spaces has a tensor product which was first intruduced by R. Dahmen in [1] and was used readily in [3]. The essence of this tensor product is that for two weakly complete vector spaces $W_1 = \mathbb{K}^X$ and $W_2 = \mathbb{K}^Y$ we have $W_1 \otimes W_2 = \mathbb{K}^{X \times Y}$ and that there is a bilinear homeomorphism $(w_1, w_2) \mapsto w_1 \otimes w_2 : W_1 \times W_2 \to W_1 \otimes W_2$ such that any continuous bilinear function $b \colon W_1 \times W_2 \to W_3$ for a weakly complete member W_3 of $\mathcal{W}_{\mathbb{K}}$ factors through a unique continuous linear map (that is, a morphism of $\mathcal{W}_{\mathbb{K}}$) $b' \colon W_1 \otimes W_2 \to W_3$ such that $b(w_1, w_2) = b'(w_1 \otimes w_2)$.

Proposition 2.1. *The category $\mathcal{W}_{\mathbb{K}}$ together with its tensor product $\otimes$ is a commutative monoidal category such that for two $\mathbb{K}$-vector spaces V_1 and V_2 and two weakly complete $\mathbb{K}$-vector spaces W_1 and W_2 we have natural isomorphisms*

$$(V_1 \otimes V_2)^* \cong V_1^* \otimes V_2^* \quad \textit{and} \quad (W_1 \otimes W_2)' \cong W_1' \otimes W_2'.$$

Proof. These assertions are straightforward exercises; see also [1] and [3]. □

This means

Corollary 2.2. *The symmetric monoidal categories $(\mathcal{V}_K, \otimes)$ and $(W_{\mathbb{K}}, \otimes)$ are naturally dual.*

Proof. This is a reinterpretation of Theorem 1.1 and Proposition 2.1. □

In any abstract or topological abelian category, an algebra V with a mutliplication poses the problem that multiplication $(x, y) \mapsto xy; V \times V \to V$ is not a morphism because it is bilinear and not linear. The presence of a tensor product "$\otimes$" that transforms bilinearity into linearity is therefore an ideal tool to deal with algebras in a systematic way.

We call a morphism $m\colon A \otimes A \to A$ *multiplication* and call it *associative* if the following diagram is commutative in which the natural isomorphism $\alpha_{ABC}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ describes the associativity of $\otimes$ (cf. [10], pp. 788–789, preceding Definition A3.62 of a commutative monoidal category):

$$(D1)\qquad \begin{array}{ccc} A \otimes (A \otimes A) & \xrightarrow{\alpha_{AAA}} & (A \otimes A) \otimes A \\ {\scriptstyle \mathrm{id}_A \otimes m}\downarrow & & \downarrow{\scriptstyle m \otimes \mathrm{id}_A} \\ A \otimes A & & A \otimes A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[\mathrm{id}_A]{} & A. \end{array}$$

A commutative monoidal category posseses an *identity object* E (in the case of sets a singleton set, in the case of $\mathcal{V}_{\mathbb{K}}$ the vector space $\mathbb{K}$) and isomorphisms $\iota_A\colon E \otimes A \to A$ and $\iota'_E\colon A] \otimes E \to A$. An identity $u\colon E \to A$ for a multiplication m is characterized by the commutativity of the diagram

$$(D2)\qquad \begin{array}{ccccc} E \otimes A & \xrightarrow{u \otimes \mathrm{id}_A} & A \otimes A & \xleftarrow{\mathrm{id}_A \otimes u} & A \otimes E \\ {\scriptstyle \iota_A}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \iota'_A} \\ A & \xrightarrow[\mathrm{id}_A]{} & A & \xleftarrow[\mathrm{id}_A]{} & A. \end{array}$$

Then a *monoid* in a commutative monoidal category is an object A in the category, an associative multplication m with an identity u. (See [10], p, 791.) In the category of sets with multiplication being the cartesian product, a monoid is exactly the classical concept of a monoid, that is, a semigroup with identity. In the category of topological spaces and continuous maps, a monoid in the categorical sense is a topological semigroup with identity. In the category $\mathcal{V}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces with the classical tensor product, it is a unital associative $\mathbb{K}$-algebra, and finally,

> *a monoid in the commutative monoidal category $\mathcal{W}_K$ with the tensor product of weakly complete $\mathbb{K}$-vector spaces is a weakly complete topological unital associative $\mathbb{K}$-algebra.*

The category theoretical view point regarding commutative monoidal categories $(\mathcal{A}, \otimes)$ is valuable if one needs to reverse arrows and to consider, say, monoids in the opposite category $(\mathcal{A}^{\rm op}, \otimes)$.

Definition 2.3. A *coalgebra* in a commutative monoidal category is an object with a coassociative comultipolication $c\colon A \to A \otimes A$, that is, a morphism in the category satisfying a commutative diagram obtained from (D1) by reversing all arrows, and with a coidentity $k\colon A \to E$, that is, a morphism satisfying a commutative diagram obtained from (D2) by reversing all arrows. □

In a commutative monoidal category $(SET, \times)$ of sets with the cartesian product and the singleton set $\{*\}$ as identity object E, *every* set X gives rise to a coalgebra with comultiplication $x \mapsto (x, x) : X \to X \times X$ and coidentity $x \mapsto * : X \to E$. A comprehensive survey of of coalgebras was provided by Michaelis in [12]. One fundamental Theorem is the following, attributed in [12] to Cartier, formulated here for the category $\mathcal{V}_{\mathbb{K}}$:

The Fundamental Theorem on Coalgebras

Theorem 2.4. *Every coalgebra in the category $\mathcal{V}_K$ of $\mathbb{K}$-vector spaces is the directed union of the set of its finite dimensional subcoalgebras.*

Proof. See [12], Theorem 4.12, p. 742. □

In its structure this result is similar to the elementary remark that every vector space is the directed union of its finite dimensional vector subspaces which led to the characterisation of weakly compact vector spaces in Proposition 1.2. In a similar way we derive from Theorem 2.4 the following important consequence:

The Fundamental Theorem of Weakly Complete Algebras

Corollary 2.5. *Every weakly complete unital topological $\mathbb{K}$-algebra is the strict projective limit of the projective system of quotient morphisms between its finite dimensional unital quotient algebras.*

Proof. The proof is a straightforward application of 2.4 via duality. Cf. e.g. [3], Theorem 3.2. □

In [2] and [3] it was deduced from 2.5 that the multiplicative group A^{-1} of units of any weakly complete algebra A had special properties. Recall that a topological group G is *almost connected* if the component factor group G/G_0 is compact. Pro-Lie groups were extensively studied in [11]. In particular, every pro-Lie group has a weakly complete Lie algebra $\mathfrak{L}(G)$ which is a pro-Lie algebra in the sense that is is the strict projective limit of its finite dimensional quotient algebras, and that there is a canonical exponential function $\exp_G : \mathfrak{L}(G) \to G$.

The Group of Units of a Weakly Complete Algebra

Theorem 2.6. *Let A be a weakly complete associative unital $\mathbb{K}$-algebra and G its group A^{-1} of invertible elements. Let $A_{\rm Lie}$ denote the weakly complete pro-Lie algebra with the Lie bracket $[x,y] = xy - yx$. Then the following conclusions hold:*

(i) $\overline{G} = A$.

(ii) *G is an almost connected pro-Lie group (which is connected if $\mathbb{K} = \mathbb{C}$). The exponential function of A is everywhere defined by $\exp x = \sum_{m=0}^{\infty} \frac{1}{m!}{\cdot}x^m$ and yields the exponential function of G given by $\exp_G : A_{\rm Lie} \to G$.*

Proof. Sie [2] and [3], 3.11, 3.12, and 4.1. □

We return briefly to the category theoretical background with the following definition:

Definition 2.7. (a) A *bimonoid* in a commutative monoidal category is an object together with both a monoid structure (m, u) and comonoid structure (c, k),

$$(\sigma) \qquad A \xrightarrow{\ c\ } A \otimes A \xrightarrow{\ m\ } A \quad \text{and} \quad E \xrightarrow{\ u\ } A \xrightarrow{\ k\ } E,$$

such that c is a monoid morphism.

(b) A *group* (or often *group object* in a commutative monoidal category) is a bimonoid with commutative comultiplication and with an $\mathcal{A}$-morphism $\sigma: A \to A$, called *inversion* or *symmetry* (as the case may be) which makes the following diagram commutative

$$(\sigma) \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\sigma\otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle c}\big\uparrow & & \big\downarrow{\scriptstyle m} \\ A & \xrightarrow[u\circ k]{} & A, \end{array}$$

plus a diagram showing its compatibility with the comultiplication (see [10], Definition A3.64.ii).

(c) In our commutative monoidal categories $(V_{\mathbb{K}}, \otimes)$ and $(\mathcal{W}_{\mathbb{K}}, \otimes)$ of $\mathbb{K}$-vector spaces,respectively,weakly complete $\mathbb{K}$-vectorspaces,a group object (A, m, c, u, k, σ) is called a *Hopf algebra*, respectively, *a weakly complete Hopf algebra.* □

In reality, the definition of a bimonoid is symmetric and the equivalent conditions that c be a monoid morphism, respectively, that m be a comonoid morphism can be expressed in one commutative diagram (see [10], Diagram following Definition A3.64, p. 793). Also it can be shown that in a group the diagram arising from the diagram (σ) by replacing $\sigma \otimes \mathrm{id}$ by $\mathrm{id} \otimes \sigma$ commutes as well.

In any theory of Hopf algebras it is common to single out two types of special elements, and we review them in the case of weakly complete Hopf algebras.

Definition 2.8. Let A be a weakly complete coassociative coalgebra with comultiplication c and coidentity k. Then an element $a \in A$ is called *grouplike* if $k(a) = 1$ and $c(a) = a \otimes a$. The set of grouplike elements of A will be called $\Gamma(A)$.

If A is a bialgebra, $a \in A$ is called *primitive*, if $c(a) = a \otimes 1 + 1 \otimes a$. The set of primitive elements of A will be called $\Pi(A)$. □

For any $a \in A$ with $c(a) = a \otimes a$, the conditions $a \neq 0$ and $k(a) = 1$ are equivalent.

These definitions apply, in particular, to any weakly complete Hopf algebra. The set $\Gamma(A)$ of grouplike elements of a weakly complete bialgebra A is a closed submonoid of $(A, \cdot)$ and the set $\Pi(A)$ of primitive elements of A is a closed Lie subalgebra of A_{Lie}. If A is a Hopf algebra, then $\Gamma(A)$ is a closed subgroup of A^{-1} hence is a pro-Lie group.

For a morphism $f: W_1 \to W_2$ in $\mathcal{W}_{\mathbb{K}}$ let $f': W_2' \to W_1'$ in $\mathcal{V}_{\mathbb{K}}$ denote the dual morphism of vector spaces.

For a weakly complete coalgebra A let $A' = \mathcal{W}_{\mathbb{K}}(A,\mathbb{K})$ be the dual of A. Then A' is an algebra: If $c\colon A \to A\otimes A$ is the comultiplication of A, then $c'\colon A'\otimes A' \to A'$ is the multiplication of A'.

For a unital algebra R and a weakly complete coalgebra A in duality let $(a,g) \mapsto \langle a,g\rangle : R\times A \to \mathbb{K}$ denote the pairing between R and A, where for $f \in R = \mathcal{W}_{\mathbb{K}}(A,\mathbb{K})$ and $a\in A$ we write $\langle f,a\rangle = f(a)$.

Definition 2.9. Let R be a unital algebra over $\mathbb{K}$. Then a *character* of R is a morphism of unital algebras $R \to \mathbb{K}$. The subset of $\mathbb{K}^R$ consisting of all algebra morphisms inherits the topology of pointwise convergence from $\mathbb{K}^R$ and as a topological space is called the *spectrum* of R and is denoted $\operatorname{Spec}(R)$. □

Now let R be a unital algebra and $A \overset{\text{def}}{=} R^*$ its dual weakly complete coalgebra with comultiplication c such that $ab = c'(a\otimes b)$ for all $a,b\in R$. In these circumstances we have:

Proposition 2.10. *Let $g\in A$. Then the following statements are equivalent:*
(i) *$g \in A$ is grouplike in the coalgebra A.*
(ii) *$g\colon R\to\mathbb{K}$ is a character of R, that is, is an element of* $\operatorname{Spec}(R)$*.*
There is a natural bijection between $\operatorname{Spec}(R)$ *and* $\Gamma(A)$*.*

Proof. See e.g. [3], 6.5. □

Let $K(R)$ denote the closed two sided ideal in R of all $x\in R$ such that $f(x)=0$ for all $f\in\operatorname{Spec}(R)$. We set $\mathbb{S}(A) \overset{\text{def}}{=} \overline{\operatorname{span}(\Gamma(A))}$

Proposition 2.11. *In the annihilator mechanism of the duality between R and A we have $K(R) = \mathbb{S}(A)^\perp$ and $K(R)^\perp = \mathbb{S}(A)$.*

Proof. For the annhilator mechanism in the presence of duality (at least in the case of abelian groups) see e.g. [10], 7.12ff. Given that routine, the proof of the proposition is straightforward. □

Definition 2.12. A unital $\mathbb{K}$-algebra R is called *reduced* if $K(A) = \{0\}$. □

(Cf. [5], pp. 29.) Thus R is reduced iff $\mathbb{S}(A) = A$ (for $A = R^*$) iff the linear span of $\Gamma(A)$ is dense. We also note that $\mathbb{S}(A)$ is a subcoalgebra of A.

Chapter 3

The Weakly Complete Group Algebra K[G] and its Hopf Structure

We obtain one significant weakly complete Hopf algebra via the left adjoint existence theorem starting from the functor $A \mapsto A^{-1}$ from weakly complete unital algebras to the category of topological groups: This left adjoint functor $G \mapsto \mathbb{K}[G]$ from the category of topological groups to the category of weakly complete unital algebras is the topic of investigation here, and $\mathbb{K}[G]$ is called the $\mathbb{K}$-group algebra of the topological group G:

Proposition 3.1. *To each topological group G there is attached functorially a weakly complete group algebra $\mathbb{K}[G]$ with a natural morphism $\eta_G : G \to \mathbb{K}[G]^{-1}$ such that the following universal property holds:*
For each weakly complete unital algebra A and each morphism of topological groups $f: G \to A^{-1}$ there exists a unique morphism of weakly complete unital algebras $f': \mathbb{K}[G] \to A$ restricting to a morphism $f'': \mathbb{K}[G]^{-1} \to A^{-1}$ of topological groups such that $f = f' \circ \eta_G$.

Proof. See [3], Theorem 5.1. □

A schematic display of the universal property may be helpful:

$$
\begin{array}{ccc|c}
\text{top groups} & & & \text{wc algebras} \\
\hline
G & \xrightarrow{\eta_G} & \mathbb{K}[G]^{-1} & \mathbb{K}[G] \\
{\scriptstyle \forall f}\big\downarrow & & \big\downarrow{\scriptstyle f''} & \big\downarrow{\scriptstyle \exists! f'} \\
A^{-1} & \xrightarrow[\text{id}]{} & A^{-1} & A
\end{array}
$$

In [3] the following facts were established on the weakly complete group algebra:

The Weakly Complete Group Algebra $\mathbb{K}[G]$

Theorem 3.2. *For any topological group G, the following statements hold:*

(1) *The $\mathbb{K}$-linear span* $\mathrm{span}(\eta_G(G))$ *of the image* $\eta_G\colon G \to \mathbb{K}[G]^{-1}$ *is dense in* $\mathbb{K}[G]$.

(2) *For topological groups G and H there is a natural isomorphism*

$$\alpha_{GH}\colon \mathbb{K}[G \times H] \to \mathbb{K}[G] \otimes \mathbb{K}[H].$$

The continuous algebra morphism

$$\mathbb{K}[G] \xrightarrow{\mathbb{K}[\mathrm{diag}]} \mathbb{K}[G \times G] \xrightarrow{\alpha_{GH}} \mathbb{K}[G] \otimes \mathbb{K}[G]$$

is the natural comultiplication of a Hopf algebra structure on $\mathbb{K}[G]$.

(3) $\eta_G(G)$ *is a subgroup of* $\Gamma(\mathbb{K}[G])$, *the closed subgroup of the pro-Lie group* $\mathbb{K}[G]^{-1}$ *of grouplike elements.*

(4) *The closed $\mathbb{K}$-vector space of primitive elements* $\Pi(\mathbb{K}[G])$ *is the Lie algebra of the pro-Lie group* $\Gamma(\mathbb{K}[G])$, *and the exponential function of* $\mathbb{K}[G]$ *induces the exponential function* $\exp_G\colon \mathfrak{L}(G) = \Pi(\mathbb{K}[g]) \to \Gamma(K[G])$.

(5) *For any compact group G, the function η_G is an embedding algebraically and topologically. In this case we may consider G as a subgroup of* $\mathbb{K}[G]^{-1}$.

(6) *If G is a compact group and G is considered a subgroup of* $\Gamma(\mathbb{K}[G])$ *according to* (3) *and* (5), *then* $G = \Gamma(\mathbb{R}[G])$ *and is a proper subgroup of* $\Gamma(\mathbb{C}[G])$.

Proof. For (1) see [3], 3.5, for (2) see 5.5 and 5.9, for (3) see 5.7, for (4) see Theorem 2.5 above, for (5) see [3], 5.4, and for (6) see [3], 8.7. □

Chapter 4

The Group Algebra K[G] and its Duality

To some extent, we are dealing here with the duality theory of weakly complete Hopf algebras A. For this purpose we let G denote the pro-Lie group $\Gamma(A)$ of primitive elements of A.

The underlying weakly complete vector space of A is a topological left and right G-module A with the module operations

$$\begin{array}{lll}(g,a)\mapsto g{\cdot}a: & G\times A\to A, & g{\cdot}a := ga, \quad \text{and}\\ (a,g)\mapsto a{\cdot}g: & G\times A\to A, & a{\cdot}g := ag.\end{array}$$

We let $\mathbb{I}(A)$ denote the filterbasis of closed two-sided ideals J of A such that A/J is a finite dimensional algebra and that $A\cong \lim_{J\in\mathbb{I}(A)} A/J$. We can reformulate Corollary 2.4 in terms of G-modules as follows:

Lemma 4.1. *For the topological group $G=\Gamma(A)$, the G-module A has a filter basis $\mathbb{I}(A)$ of closed two-sided submodules $J\subseteq A$ such that $\dim(A/J)<\infty$ and that $A=\lim_{J\in\mathbb{I}(A)} A/J$ is a strict projective limit of finite dimensional G-modules. The filter basis $\mathbb{I}(A)$ in A converges to $0\in A$.*

Proof. This is indeed a reformulation of Corollary 2.4 □

For a $J\in\mathbb{I}(A)$ let $J^\perp=\{f\in A' : (\forall a\in J)\,\langle f,a\rangle=0\}$ denote the annihilator of J in the dual V of A. We compare the “Annihilator Mechanism” from [10], Proposition 7.62 and observe the following configuration:

$$\begin{array}{ccc}A & \{0\} & \\ \Big| & \Big| & \Big\}\ \cong\ (A/J)' \\ J & J^\perp & \\ \Big| & \Big| & \Big\}\ \cong\ J' \\ \{0\} & A'. & \end{array}$$

In particular, we recall the fact that $J^\perp\cong(A/J)'$ showing that $J^\perp$ is a finite-dimensional G-module on either side. By simply dualizing Lemma 4.1, we obtain

Lemma 4.2. *For the topological group $G=\Gamma(A)$, the dual G-module $R\stackrel{\text{def}}{=}A'$ of the weakly complete G-module A has an up-directed set $\mathbb{D}(R)$ of finite-dimensional* *two-sided G-submodules (and $\mathbb{K}$-coalgebras!) $F \subseteq R$ such that R is the direct limit*

$$R = \operatorname*{colim}_{F \in \mathbb{D}(R)} F = \bigcup_{F \in \mathbb{D}(R)} F.$$

The colimit is taken in the category of (abstract) G-modules, i.e. modules without any topology.

Proof. This is a consequence of vector space duality 1.1 in view of Lemma 4.1. □

This means that for the topological group $G = \Gamma(A)$, every element ω of the dual of A' is contained in a finite dimensional left- and right-G-module.

We record this in the following form:

Lemma 4.3. *Let $\omega \in A'$. Then the vector subspaces* $\operatorname{span}(G \cdot \omega)$ *and* $\operatorname{span}(\omega \cdot G)$ *of both the left orbit and the right orbit of ω are finite dimensional, and both are contained in a finite dimensional $\mathbb{K}$-subcoalgebra of A'.*

Proof. Straightforward from the preceeding. □

For any $\omega \in A'$ the restriction $f \stackrel{\text{def}}{=} \omega|G : G \to \mathbb{K}$ is a continuous function such that each of the sets of translates f_g, $f_g(h) = f(gh)$, respectively, ${}_gf$, ${}_gf(h) = f(hg)$ forms a finite dimensional vector subspace of the space $C(G, \mathbb{K})$ of the vector space of all continuous $\mathbb{K}$-valued functions f on G.

Definition 4.4. For an arbitrary topological group G we define $\mathcal{R}(G, \mathbb{K}) \subseteq C(G, \mathbb{K})$ to be that set of continuous functions $f: G \to \mathbb{K}$ for which the linear span of the set of translations ${}_gf$, ${}_gf(h) = f(hg)$, is a finite dimensional vector subspace of $C(G, \mathbb{K})$. The functions in $\mathcal{R}(G, \mathbb{K})$ are called *representative functions.* □

In Lemma 4.3 we saw that for a weakly complete Hopf algebra A and its dual A' (consisting of continuous linear forms) we have a natural linear map

$$\tau_A: A' \to \mathcal{R}(\Gamma(A), \mathbb{K}), \ \tau_A(\omega)(g) = (\omega|\Gamma(A))(g).$$

Lemma 4.5. *There is an exact sequence of $\mathbb{K}$-vector spaces*

$$0 \to \mathbb{S}(A)^\perp \xrightarrow{\text{inc}} A' \xrightarrow{\tau_A} \mathcal{R}(\Gamma(A), \mathbb{K}).$$

Proof. An element $\omega \in A'$ is in the kernel of τ_A if and only if $\omega(\Gamma(A)) = \{0\}$ if and only if $\omega\big(\mathbb{S}(A)\big) = \{0\}$ if and only if $\omega \in \mathbb{S}(A)^\perp$. □

We recall from Theorem 3.2(1) that for a weakly complete group algebra $A = \mathbb{K}[G]$ the term $\mathbb{S}(A)^\perp$ in the exact sequence vanishes, and so τ_A is injective in that case. In general, it does not appear to be evident under which circumstances Lemma 4.5 can be improved.

Still in the case of $A = \mathbb{K}[G]$ we are in a much better situation.

Lemma 4.6. *For any topological group G and any $f \in \mathcal{R}(G,\mathbb{K})$ there is an $\omega \in \mathbb{K}[G]'$ such that $\omega \circ \eta_G = f$.*

Proof. See [3], 7.6. □

The Dual of a Weakly Complete Group Algebra $\mathbb{K}[G]$

Theorem 4.7. (i) *For an arbitrary topological group G, the function*

$$F_G\colon \mathbb{K}[G]' \to \mathcal{R}(G,\mathbb{K}), \quad F(\omega) = \omega \circ \eta_G$$

is a natural isomorphism of Hopf algebras.

(ii) *If A is a weakly complete Hopf algebra satisfying $\mathbb{S}(A) = A$, and if G is the group $\Gamma(A)$ of grouplike elements of A, then $\tau_A\colon A' \to \mathcal{R}(G,\mathbb{K})$, $\tau(\omega) = \omega|G$ is an injective morphism of Hopf algebras, embedding A' as Hopf subalgebra of $\mathcal{R}(G,\mathbb{K})$.*

Proof. (i) By Theorem 3.2(i) we have $\overline{\text{span}}(\eta(G)) = \mathbb{K}[G]$ and so the relation $(\omega \circ \eta_G)(G) = \omega(\eta_G(G)) = \{0\}$ implies $\omega = 0$. Hence the natural linear function $F_G\colon \mathbb{K}[G]' \to \mathcal{R}(G,\mathbb{K})$, $F_G(\omega) = \omega_G \circ \eta_G$ is injective. By Lemma 4.6, it is surjective and so it is a natural isomorphism of vector spaces.

We recall that $\mathbb{K}[G] \otimes \mathbb{K}[G] \cong \mathbb{K}[G \times G]$ in the category of weakly complete vector spaces by Theorem 3.2(2). By Proposition 2.1 we derive that $\mathbb{K}[G]' \otimes \mathbb{K}[G]'$ is naturally isomorphic to $\mathbb{K}[G \times G]'$, and so $G \mapsto \mathbb{K}[G]'$ is a functor mapping topological groups to group objects in the category of vector spaces, that is, to Hopf algebras.

Analogously, $\mathcal{R}(G,\mathbb{K}) \otimes \mathcal{R}(H,\mathbb{K}) \cong \mathcal{R}(G \times H,\mathbb{K})$ in the category of vector spaces. Hence the functor $G \mapsto \mathcal{R}(G,\mathbb{K})$ is a functor mapping topological groups into the category of Hopf algebras (in the category of $\mathbb{K}$-vector spaces). The naturality of F_G then implies that it is a morphism of Hopf algebras.

(ii) The relation $\mathbb{S}(A) = A$ is equivalent to $\mathbb{S}(A)^\perp = \{0\}$ in A', and so by Lemma 4.5, the linear function τ_A is injective. By (i) we identify $\mathcal{R}(G,\mathbb{K})$ and $\mathbb{K}[G]'$ as Hopf algebras. Then the injection $\tau_A\colon A' \to \mathcal{R}(G,\mathbb{K}) = \mathbb{K}[G]'$, $G = \Gamma(A)$ is none other than the dual ε'_A of the surjective back adjunction $\varepsilon_A\colon \mathbb{K}[G] = \mathbb{K}[\Gamma(A)] \to A$ of the adjunction of Corollary 6.10 in [3], and so it is a morphism of Hopf algebras and the assertion follows. □

The vector space $\mathcal{R}(G,\mathbb{K})$ is familiar in the literature as the vector space of representative functions on G, where it is most frequently formulated for compact groups G and where it is also considered as a Hopf-algebra. In that case, the isomorphism of Theorem 4.7 is also an isomorphism of Hopf algebras. We are choosing here the covariant group algebra $\mathbb{K}[G]$ to be at the center of attention and obtain $\mathcal{R}(G,\mathbb{K})$ via vector space duality from $\mathbb{K}[G]$. Conversely, if one asks for a "concrete" description of $\mathbb{K}[G]$, then the answer may now be that, in terms of topological vector spaces, as a topological vector space, $\mathbb{K}[G]$ is the algebraic dual (consisting of all linear forms) of the (abstract) vector subspace $\mathcal{R}(G,\mathbb{K})$ of the vector space $C(G,\mathbb{K})$ of continuous functions $G \to \mathbb{K}$. If G is a compact group, $C(G,\mathbb{K})$ is a familiar Banach space.

Chapter 5

The Group Algebra R[G] for Compact Groups

The weakly complete group algebras $\mathbb{K}[G]$ are particularly perfect for $\mathbb{K} = \mathbb{R}$ and compact groups G. Recall the hyperplane ideal $I = \ker k$ for the augmentation $k: \mathbb{R}[G] \to \mathbb{R}$. Let $B(G) \subseteq 1 + I$ denote the closed convex hull of $G \subseteq \mathbb{R}[G]$.

Indeed we have

Real Group Algebras of Compact Groups

Theorem 5.1. *Let G be a compact group and abbreviate the real weakly complete group Hopf algebra $\mathbb{R}[G]$ by A. Then the following statements hold:*

(1) *The natural morphism of topological groups $\eta_G: G \to \Gamma(A)$ is an isomorphism of compact groups.*

(2) *The Lie algebra $\mathfrak{L}(G)$ is isomorphic to the pro-Lie algebra $\Pi(A)$, and the restiction of the exponential function of A is (upon natural identification) equal to $\exp_G: \mathfrak{L}(G) \to G$.*

(3) *$B(G)$ is a compact submonoid with zero of $\mathbb{R}[G]^{-1}$ which is naturlly isomorphic to the compact convex set of probability measures of G containing Haar measure $\gamma \in 1 + I$ as the zero element of $B(G)$.*

(4) *The subspace $J \stackrel{\text{def}}{=} \mathbb{R}{\cdot}\gamma$ is a one-dimensional ideal, and*

$$\mathbb{R}[G] = I \oplus J$$

is the ideal direct sum of I and J. The vector subspace J is a minimal nonzero ideal. In particular, $J \cong \mathbb{R}[G]/I \cong \mathbb{R}$ and $I \cong \mathbb{R}[G]/J$. The Lie algebra of primitive elements $\Pi(\mathbb{R}[G]) \cong \mathfrak{L}(G)$ is contained in $I = \ker k$. Trivially, $\exp I \subseteq 1 + I$.

Proof. See [3], Section 6 and 7. □

We note that item (1) is not correct for $\mathbb{K} = \mathbb{C}$. Out next chapter will shed some light onto how matters relate between $\mathbb{R}$ and $\mathbb{C}$.

In the converse direction we have the following information.

The Hopf Algebra Side of the Issue

Theorem 5.2. *Let A be a real weakly complete Hopf algebra and abbreviate the group $\Gamma(A)$ of its grouplike elements by G. Assume that G is compact and generates the algebra A algebraically and topologically. Then the natural morphism $\varepsilon_A: \mathbb{R}[\Gamma(A)] \to A$ is an isomorphism of weakly complete Hopf algebras.*

Proof. See [3], Theorem 8.12. □

Proposition 5.3. *A real weakly complete Hopf algebra A which satisfies* $A = \mathbb{S}(A)$ *is automatically cocommutative.*

Proof. By Theorem 4.7(ii) we have an injection of Hopf algebras $A' \to \mathcal{R}(\Gamma(A), \mathbb{K})$. Since $\mathcal{R}(\Gamma(A), \mathbb{K})$ is a commutative algebra, A' is a commutative algebra. Hence its dual A is cocommutative. □

Definition 5.4. A real weakly complete Hopf algebra A is called *compactlike* if the multiplicative subgroup $\Gamma(A)$ is compact and its linear span is dense in A, that is $A = \mathbb{S}(A)$. □

Cf. [3] . By Proposition 5.3, a real weakly complete compactlike Hopfalgebra is automatically cocommutative.

The Equivalence Theorem

Theorem 5.5. *The categories of compact groups and of weakly compact compactlike Hopf algebras are equivalent.*

Proof. The result follows from Theorem 5.1(1) and Theorem 5.2. □

In particular this means that the topological $\mathbb{K}$-linear representation theories of compact groups and of the topological algebras underlying wealky complete compactlike Hopf algebras are the same.

In a similar vein,

> *if a category turns out to be dual to the category of weakly complete compactlike Hopf algebras, then it is also dual to the category of compact groups and vice versa.*

Here is a simple example:

A Duality Theorem

Theorem 5.6. *The category of weakly complete compactlike Hopf algebras is dual to the category of real reduced Hopf algebras.*

Proof. A real weakly complete compactlike Hopf algebra is isomorphic to $\mathbb{R}[G]$ for a compact group G. By Theorem 4.7, the dual $\mathbb{R}[G]'$ is isomorphic to the real Hopf algebra $\mathcal{R}(G, \mathbb{R})$ which is reduced (see Definition 2.10 and the subsequent comments) since $\mathbb{S}\big(\mathbb{R}[G]\big) = \mathbb{R}[G]$. □

Remark. A real reduced Hopf algebra is automatically a commutative algebra.

The Tannaka Duality Theorem

Corollary 5.7. [14] *The categories of compact groups and of real reduced Hopf algebras are duals of each other.*

Proof. Now clear. □

Chapter 6

An Alternative View: Compact Abelian Groups

We have seen the usefulness of the concept of a weakly complete group algebra $\mathbb{K}[G]$ over the real or complex numbers. We obtained its existence out the adjoint functor existence theorem. This is rather remote from a concrete construction. It may therefore be helpful to see the whole apparatus in a much more concrete way at least for a substantial subcategory of the category of compact groups, namely, the category of compact abelian groups for which we already have a familiar duality theory due to PONTRYAGIN and VAN KAMPEN (see e.g. [10], Chapter 7).

In this chapter let G be a compact abelian group and $\widehat{G} = \mathrm{Hom}_c(G, \mathbb{R}/\mathbb{Z})$ its discrete character group. These groups are written additively. For the current discussion, we consider a character χ of G as a function $G \to \mathbb{S}^1 \to \mathbb{C}^\times$ and let $A = A(G)$ denote the discrete abelian multiplicative group of these χ. Then clearly $A = A(G) \cong \widehat{G}$. (Since in former chapters the letter A frequently denoted some topological algebra, the reader should perhaps be warned that in the present context it designates a multiplicatively written discrete abelian group.)

The Hopf algebra $\mathcal{R}(G, \mathbb{C})$.

Following [10], Theorem 3.28ff. we have $\mathcal{R}(G, \mathbb{C}) = \bigoplus_{\chi \in A} \mathbb{C}{\cdot}\chi$. Therefore, considering $\mathbb{C}^{(A)} \subseteq \mathbb{C}^A$, we have an isomorphism $\iota_G \colon \mathbb{C}^{(A)} \to \mathcal{R}(G, \mathbb{C})$,

$$\iota_G(f) = \sum_{\chi \in A} f(\chi){\cdot}\chi, \tag{1}$$

as a finite sum. Thus, if $\varepsilon \in A$ is the identity character, we have $\iota(\delta_{\chi\varepsilon}) = \chi$ for $\chi \in A$.

We recall that the comultiplication $c_{\mathcal{R}}$ of $\mathcal{R}(G, \mathbb{C}) \cong \mathbb{C}^{(A)}$ is simply given by $c_{\mathcal{R}}(\varphi)(g, h) = \varphi(g + h)$ which equals $\varphi(g)\varphi(h) = (\varphi \otimes \varphi)(g, h)$, if φ is a character of G, i.e. $\varphi \in A$. Thus on the basis elements $\delta_{\chi,\varepsilon}$ of $\mathbb{C}^{(A)}$, the comultiplication c of $\mathbb{C}^{(A)}$ is simply given by $c(\delta_{\chi\varepsilon}) = \delta_{\chi\varepsilon} \otimes \delta_{\chi\varepsilon}$ and it is linearly extended from there. In this fashion, $\mathbb{C}^{(A)}$ becomes a commutative and cocommutative Hopf algebra isomorphic to the Hopf algebra $\mathcal{R}(G, \mathbb{C})$.

The dual Hopf algebra $\mathbb{C}[G]$.

The vector space dual of $\mathbb{C}^{(A)}$ is the Hopf algebra $\mathbb{C}^A$ in the category $\mathcal{WV}$ of weakly complete $\mathbb{C}$-vector spaces with the dual pairing

$$\langle -, - \rangle \colon \mathbb{C}^A \times \mathbb{C}^{(A)} \to \mathbb{C}, \quad \langle F, f \rangle = \sum_{\chi \in A} F(\chi) f(\chi) \in \mathbb{C}.$$

Definition 6.1. We write $\mathbb{C}[G] \stackrel{\text{def}}{=} \mathbb{C}^A$ as the set of all functions $F\colon A \to \mathbb{C}$ given the topology of pointwise convergence. Its commutative and cocommutative Hopf algebra structure is dual to that of $\mathbb{C}^{(A)}$ which we just recalled. □

Multiplication on $\mathbb{C}[G]$:

Let $F_1, F_2 \in \mathbb{C}^A$. We identify $\mathbb{C}^A \otimes \mathbb{C}^A$ with $\mathbb{C}^{A\times A}$ in the category $\mathcal{VW}$. Let $m\colon \mathbb{C}^{A\times A} \to \mathbb{C}^A$ denote the multiplication of the Hopf algebra $\mathbb{C}[G]$. Then for all $\chi \in A$ we have

$\langle m(F_1, F_2), \chi\rangle = \langle F_1 \otimes F_2, c(\chi)\rangle$
(since m and c are duals of each other)
$= \langle F_1 \otimes F_2, \delta_{\chi\varepsilon} \otimes \delta_{\chi\varepsilon}\rangle = \langle F_1, \delta_{\chi,\varepsilon}\rangle\langle F_2, \delta_{\chi,\varepsilon}\rangle$
(by the definition of the comultiplication c on $\mathbb{C}[A]$)
$= F_1(\chi)F_2(\chi)$
(by the definition of the dual pairing)
$= (F_1F_2)(\chi)$.

So multiplication on $\mathbb{C}[G] = \mathbb{C}^G$ is the pointwise multiplication of functions making the algebra structure of $\mathbb{C}^G$ the one arising from the product of copies of $\mathbb{C}$.

Comultiplication $\gamma\colon \mathbb{C}[G] = \mathbb{C}^G \to \mathbb{C}[G] \otimes \mathbb{C}[G] = \mathbb{C}^{G\times G}$ on $\mathbb{C}[G]$:

Let $F \in \mathbb{C}^A$ and $\chi_1, \chi_2 \in A$. If $\mathbb{C}^{(A)} \otimes \mathbb{C}^{(A)}$ in the category of vector spaces is identified naturally with $\mathbb{C}^{(A\times A)}$, then $\delta_{(\chi_1,\chi_2),(\varepsilon,\varepsilon)}$ is identified with $\delta_{\chi_1,\varepsilon} \otimes \delta_{\chi_1,\varepsilon}$. Then on a basis of $\mathbb{C}^{(A)} \otimes \mathbb{C}^{(A)}$ we have $\langle \gamma(F), \delta_{\chi_1\varepsilon} \otimes \delta_{\chi_2,\varepsilon}\rangle = \langle F, \delta_{\chi_1\chi_2,\varepsilon}\rangle$
(since γ is dual to the multiplication on $\mathbb{C}[G] = \mathbb{C}^{(A)}$)
$= F(\chi_1\chi_2)$
(by definition of the dual pairing).

We summarize:

Lemma 6.2. *For a compact abelian group G, the multiplication of the weakly complete Hopf algebra $\mathbb{C}[G] = \mathbb{C}^{A(G)}$ is given by the natural product multiplication $(F_1F_2)(\chi) = F_1(\chi)F_2(\chi)$, and the comultiplication is given by $\gamma(F)(\chi_1, \chi_2) = F(\chi_1\chi_2)$ (with $\mathbb{C}^{A\times A}$ and $\mathbb{C}^A \otimes \mathbb{C}^A$ identified in the category of weakly complete vector spaces).*

Proof. This is a summary of what we had before. □

What are the grouplike elements $F \in \mathbb{C}[G] = \mathbb{C}^A$?

An element $F \in \mathbb{C}^{\mathbb{A}}$ is grouplike iff it is nonzero and $\gamma(F) = F \otimes F$, that is, for all $\chi_1, \chi_2 \in A$ we have

$\gamma(F)(\chi_1, \chi_2) = (F \otimes F)(\chi_1, \chi_2) = F(\chi_1)F(\chi_2)$.

But from the previous subsection we have $\gamma(F)(\chi_1, \chi_2) = F(\chi_1\chi_2)$. Thus F in the weakly complete Hopf algebra $\mathbb{C}[G] = \mathbb{C}^A$ is grouplike iff $F \in \mathrm{Hom}(A, \mathbb{C}^\times)$.

What are the primitive elements $F \in \mathbb{C}[G] = \mathbb{C}^A$? The identity $\mathbf{1}$ of the algebra $\mathbb{C}^A$ is the constant function with value $1 \in \mathbb{C}$. An element $F \in \mathbb{C}^{\mathbb{A}}$ is primitive iff

$\gamma(F) = (F \otimes \mathbf{1} + \mathbf{1} \otimes F)$ iff

$(\forall \chi_1, \chi_2 \in A)\, F(\chi_1\chi_2) = \gamma(F)(\chi_1, \chi_2) = (F \otimes \mathbf{1} + \mathbf{1} \otimes F)(\chi_1, \chi_2)$
$= F(\chi_1) + F(\chi_2)$ iff F is a morphism of abelian groups iff
$F \in \mathrm{Hom}(A, (\mathbb{C}, +))$.

Again we summarize for the Hopf algebra $\mathbb{C}[G] = \mathbb{C}^{A(G)}$:

Lemma 6.3. *The group $G^\sharp = \Gamma(\mathbb{C}[G])$ is $\mathrm{Hom}(A, \mathbb{C}^\times) \subseteq (\mathbb{C}^\times)^A$ with the topology of pointwise convergence.*

The Lie algebra of primitive elements is the vector space $\mathrm{Hom}(A, \mathbb{C}) \subseteq \mathbb{C}^A$ with the topology of pointwise convergence.

Proof. See above. □

The following is noted in [10]:

Remark 4. $\mathrm{Hom}(A, \mathbb{R})$ is naturally isomorphic to $\mathrm{Hom}(\mathbb{R} \otimes_{\mathbb{Z}} A, \mathbb{R})$, a weakly complete $\mathbb{R}$-vector space.

Proof. See Proposition 7.35, p. 338 of [10]. □

This remains true in the case $\mathbb{K} = \mathbb{C}$. These facts may be shown directly. In [10], Theorem 8.20 it is discussed that G contains totally disconnected compact subgroups Δ such that the annihilator in the character group of G, say, $\Delta^\perp \subseteq A$ is free, and $A/\Delta^\perp$ is a torsion group. This means that G/Δ is a torus. We note that the inclusion $\Delta^\perp \to A$ induces an isomorphism $\mathbb{K} \otimes_{\mathbb{Z}} \Delta^\perp \to \mathbb{K} \otimes_{\mathbb{Z}} A$ and the (torsion free) rank of A is $\mathrm{rank}\,\Delta^\perp$. If $\Delta^\perp \cong \mathbb{Z}^{(X)}$ for a set X of cardinality $\mathrm{rank}\,\Delta^\perp$, then $\mathrm{Hom}(A, \mathbb{K}) \cong \mathbb{K}^X$.

Remark 6. 5. If $\aleph$ is any cardinal and A is an abelian group with torsion free rank $\aleph$, then $\mathrm{Hom}(A, \mathbb{R}) \cong \mathbb{R}^\aleph$.

Proof. See the preceding explanation. □

The exponential function of $\mathbb{C}[G] = \mathbb{C}^A$.

We recall from [3], Theorem 3.12, that every weakly complete associative unital algebra W such as e.g. $\mathbb{C}^A$ has an exponential function, which is immediate in this particular case as it is calculated componentwise. If the weakly complete algebra W is even a Hopf algebra, such as $\mathbb{C}^A$, then the group $G^\sharp \stackrel{\mathrm{def}}{=} \Gamma(W)$ is a pro-Lie group with $\mathbb{C}$-Lie algebra $\mathfrak{L}(G^\sharp)$ being the pro-Lie subalgebra $\Pi(W)$ of W_{Lie} of all primitive elements of W by [3], Theorem 6.15. If $W = \mathbb{C}^A = \mathbb{C}[G]$, then the exponential function $\exp_{G^\sharp}\colon \mathfrak{L}(G^\sharp) \to G^\sharp$ of $G^\sharp$ is the restriction of the (componentwise!) exponential function $\exp\colon \mathbb{C}^A \to (\mathbb{C}^A)^{-1} = (\mathbb{C}^\times)^A$ to $\mathfrak{L}(G^\sharp) = \mathrm{Hom}(A, \mathbb{C})$.

The real case derived from the complex case

If we consider $\mathbb{R}$ as a subfield of $\mathbb{C}$, then it is natural to consider $\mathcal{R}(G,\mathbb{R})$ as a vector subspace of $\mathcal{R}(G,\mathbb{C})$ in the natural way. Let $\kappa: \mathbb{C} \to \mathbb{C}$ be conjugation: $\kappa(z) = \overline{z}$. Then $\kappa^2 = \mathrm{id}_\mathbb{C}$

The involution $\kappa \otimes \mathrm{id}_E$ on the complexification $\mathbb{C} \otimes_\mathbb{R} E$ of a real vector space E is treated at some length in [10] in the last section of Chapter 3, headlined *Complexificatin of Real Representations* (cf. p. 83ff.).

We define the involution $\kappa_\mathcal{R}$ on $\mathcal{R}(G,\mathbb{C})$ by $\kappa(\varphi)(g) = \overline{\varphi(g)}$ for all $g \in G$. Then a function $\varphi \in \mathcal{R}(G,\mathbb{C})$ is in $\mathcal{R}(G,\mathbb{R})$ iff φ is fixed under $\kappa_\mathcal{R}$,

If $\chi: G \to \mathbb{S}^1 \subseteq \mathbb{C}$ is a (multiplicative) character, then its coextension $G \to \mathbb{C}$, which we may also denote χ, is a member of $\mathcal{R}(G,\mathbb{C})$, and in this sense we may write $A \subseteq \mathcal{R}(G,\mathbb{C})$. Now, if $\chi \in A$, then $\kappa_\mathcal{R}(\chi)(g) = \overline{\chi(g)} = \chi(g)^{-1} = \chi^{-1}(g)$, since $\chi(G) \subseteq \mathbb{S}^1$, as G is compact. Thus $\kappa_\mathcal{R}$ leaves $A \subseteq \mathcal{R}(G,\mathbb{C})$ invariant and induces on A an involution. Accordingly, we define on $\mathbb{C}^{(A)}$ an involution ρ_G by $\rho_G(f)(\chi) = \overline{\kappa_\mathcal{R}(\chi)}$. Then $\iota_G(\kappa_\mathcal{R}(\varphi)) = \rho_G(\iota(\varphi))$ for $\varphi \in \mathcal{R}(G,\mathbb{C})$, that is, $\iota_G \circ \kappa_\mathcal{R} = \rho_G \circ \iota_G$. Accordingly, $f \in \mathbb{C}^{(A)}$ is in $\mathbb{R}^{(A)}$ iff it is fixed under ρ_G.

The involution ρ_G of $\mathbb{C}^{(A)}$ extends to an involution σ_G of $\mathbb{C}^A$ via $\sigma_G(F) = \overline{F \circ \kappa_\mathcal{R}}$. Then the duality of $\mathbb{C}^A$ and $\mathbb{C}^{(A)}$, which is implemented by the pairing

$$\langle F, f\rangle = \sum_{\chi \in A} F(\chi) f(\chi),$$

is clearly compatible with σ_G in the sense that

$$\overline{\langle F, f\rangle} = \langle \sigma_G(F), \rho_G(f)\rangle. \tag{2}$$

Definition 6.6. We define $\mathbb{R}[G] \subseteq \mathbb{C}[G] = \mathbb{C}^A$ to be the fixed vector space of the involution σ_G of $\mathbb{C}[G]$. □

For the following we recall that $\Gamma(\mathbb{R}[G]) \subseteq G^\sharp = \mathrm{Hom}(A, \mathbb{C}^\times)$ and that $\Pi(\mathbb{R}[G]) = \mathfrak{L}(G) \subseteq \mathrm{Hom}(A, \mathbb{C})$.

The Case of Compact Abelian Groups Revisited

Theorem 6.7. *For compact abelian groups G the following statements hold*

(i) $\mathbb{R}[G]$ *is a weakly complete real Hopf subalgebra of the weakly complete complex Hopf algebra* $\mathbb{C}[G] = \mathbb{C}^A$.

(ii) $\Gamma(\mathbb{R}[G]) = \mathrm{Hom}(A, \mathbb{S}^1)$, *and therefore* $\Gamma(\mathbb{R}[G])$ *is naturally isomorphic to* G.

(iii) $\Pi(\mathbb{R}[G]) = \mathrm{Hom}(A, i\mathbb{R}) = i\,\mathrm{Hom}(A, \mathbb{R})$, *and therefore* $\Pi(\mathbb{R}[G])$ *is isomorphic to* $\mathbb{R}^{\mathrm{rank}(A)}$ *where rank*(A) *is the torsion free rank of the abelian group* $A \cong \widehat{G}$.

(iv) *The exponential function* $\exp: \mathbb{C}[G] = \mathbb{C}^A \to \mathbb{C}[G]^{-1} = (\mathbb{C}^\times)^A$, $(\exp F)(\chi) = e^{F(\chi)}$ *maps* $\Pi(\mathbb{R}(G)) \cong \mathfrak{L}(G)$ *into* $\Gamma(\mathbb{R}[G]) \cong G$ *and induces the exponential function* $\exp_G: \mathfrak{L}(G) \to G$ *of* G.

Proof. (i) Let $\gamma: G \to H$ be a morphism of compact groups. Then $A(\gamma): A_2 \to A_1$, $A_j = A(G_j)$, $j = 1, 2$, is a morphism of abelian groups, and $\mathbb{C}[G] = \mathbb{C}^{A_1} \to \mathbb{C}^{A_2} =$

$\mathbb{C}[H]$ is a morphism of weakly complete Hopf algebras. Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[G] & \xrightarrow{\mathbb{C}[\gamma]} & \mathbb{C}[H] \\ {\scriptstyle \sigma_G}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_H} \\ \mathbb{C}[G] & \xrightarrow[\mathbb{C}[\gamma]]{} & \mathbb{C}[H]. \end{array}$$

Thus the involution $\sigma\colon \mathbb{C}[-] \to \mathbb{C}[-]$ is a natural transformation. In view of $\mathbb{C}[G \times H] \cong \mathbb{C}[G] \otimes \mathbb{C}[H]$ in the category of weakly complete vector spaces, σ_G is an involution of Hopf algebras on $\mathbb{C}[G]$, that is it respects multiplication and comultiplication as well as identities and coidentities plus symmetries. Hence the fixed point set $\mathbb{R}[G]$ of σ_G on $\mathbb{C}[G]$ is a Hopf subalgebra.

(ii) The naturality of σ and $\mathbb{C}[G \times H] = \mathbb{C}^{G\times H} \cong \mathbb{C}[G] \otimes \mathbb{C}[H]$ in the category of weakly complete vector spaces imply readily that $G^\sharp = \Gamma(\mathbb{C}[G]) = \mathrm{Hom}(A, \mathbb{C}^\times)$ is invariant under σ. Let $F\colon A \to \mathbb{C}^\times$ be in $G^\sharp$ and let $\chi \in A$. Then $\sigma(F)(\chi) = \overline{F(\chi^{-1})} = \big(\overline{F(\chi)}\big)^{-1}$ and $\sigma(F) = F$ imply that $F(\chi)^{-1} = \overline{F}(\chi)$ within $\mathbb{C}$, whence $|F(\chi)| = 1$. Hence $F \in G^\sharp$ is σ-invariant iff $F(A) \subseteq \mathbb{S}^1$. But $\mathrm{Hom}(A, \mathbb{S}^1) \cong \mathrm{Hom}(A, \mathbb{R}/\mathbb{Z}) = \widehat{A} = \widehat{\widehat{G}} \cong G$ by Pontryagin Duality: See [10], Theorem 7.63, p. 358.

(iii) We assume that $F \in \mathrm{Hom}(A, \mathbb{C})$ is σ-invariant. that is $F(\chi) = (\sigma F)(\chi) = \overline{F(\chi^{-1})} = -\overline{F(\chi)}$. If $F(\chi) = x + iy$ with $x, y \in \mathbb{R}$, then $x + iy = -(x - iy)$, implying $x = 0$. Thus $F(A) \subseteq i\mathbb{R}$. Thus $\Pi(\mathbb{R}[G]) \cong \mathrm{Hom}(A, i\mathbb{R}) \cong \mathrm{Hom}(A, \mathbb{R}) \cong \mathbb{R}^{\mathrm{rank}(A)}$ by Remark 5.

(iv) If $F \in \mathrm{Hom}(A, i\mathbb{R}) \subseteq \mathbb{C}^A$, then for any $\chi \in A$ we have $(\exp F)(\chi) = e^{F(\chi)} \in \mathbb{S}^1$. The remainder follows from [dh1], Theorem 6.15. □

We observe that $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ and $\mathbb{C}^\times = \mathbb{R}_0^\times \times \mathbb{S}^1 \cong \mathbb{R} \times \mathbb{T}$ in a canonical fashion. Let the $\mathbb{R}$-vector space $\mathrm{Hom}(A, \mathbb{R}) \cong \mathrm{Hom}(\mathbb{R} \otimes_{\mathbb{Z}} A, \mathbb{R})$ be denoted $G^\flat$. Then we have the following Corollary for which we recall $G^\sharp = \Gamma(\mathbb{C}[G])$ □

Corollary 6.8. *Let G be a compact abelian group. Then*
(i) $G^\sharp \cong G^\flat \times G$,
(ii) $\mathfrak{L}_{\mathbb{C}}(G) \cong G^\flat \oplus \mathfrak{L}(G)$.
(iii) $\exp_{G^\sharp} = \mathrm{id}_{G^\flat} + \exp_G$ *in the sense that* $\exp_{G^\sharp}$ *acts componentwise.*

Proof. (i) By Theorem 7(ii), $G^\sharp = \mathrm{Hom}(A, \mathbb{C}^\times) \cong \mathrm{Hom}(A, \mathbb{R}_0^\times \times \mathbb{S}^1)$ $\cong \mathrm{Hom}(A, \mathbb{R}_0^\times) \times \mathrm{Hom}(A, \mathbb{S}^1) \cong \mathrm{Hom}(A, \mathbb{R}) \times \mathrm{Hom}(\widehat{G}, \mathbb{T}) \cong G^\flat \times \widehat{\widehat{G}} \cong G^\flat \times G$ by duality. We calculate that σ operates on $G^\flat \times G$ via $(\varphi, g) \to (-\varphi, g)$. So the fixed point set of $G^\flat \times G$ is $\{0\} \times G \cong G$.

(ii) By Theorem 7(iii), $\Pi(\mathbb{C}[G]) = \mathrm{Hom}(A, \mathbb{C}) = \mathrm{Hom}(A, \mathbb{R} \oplus i\mathbb{R})$ $\cong \mathrm{Hom}(A, R) \oplus \mathrm{Hom}(A, iR) = G^\flat \oplus i\,\mathrm{Hom}(A, R) = G^\flat \oplus iG^\flat$. Now σ acts on $G^\flat \oplus iG^\flat$ via $\varphi + i\psi \mapsto -\varphi + i\psi$. So the fixed point set of $G^\flat \oplus iG^\flat$ is $iG^\flat = \mathfrak{L}_{\mathbb{R}}(G)$.

Assertion (iii) is immediate. □

Chapter 7

Preservation Properties of the Group Algebra Functor

In our discussion of the functor $G \mapsto \mathbb{K}[G]$ we have mainly concentrated on the objects. But in concrete situations it is just as important to know how this functor treats morphisms. So we insert a chapter on the prepservation of the most basic properties.

Does the functor $\mathbb{K}[-]$ preserve surjectivity?

Let $f: G \to H$ be a surjective morphism of compact groups.

We know from [3] that G may be viewed as the multiplicative subgroup of group like elements of $\mathbb{R}[G]$ and that

$$\overline{\operatorname{span}(G)} = \mathbb{R}[G]. \tag{1}$$

Since $f = \mathbb{R}[f]|G$ has H as image,we know that

$$H \subseteq \operatorname{im}(\mathbb{R}[f]). \tag{2}$$

The function $\mathbb{R}[f]$ is, in particular, a morphism of weakly complete vector spaces. These morphisms always have closed images (see [10], Theorem 7.30(iv)). By (1), H generates $\mathbb{R}[H]$ algebraically and topologically. Thus $\mathbb{R}[H]$ is contained in the image of the morphism $\mathbb{R}[f]$ which therefore is surjective. But then the commuting diagram

$$\begin{array}{ccc} \mathbb{C} \otimes \mathbb{R}[G] & \xrightarrow{\mathbb{C}\otimes\mathbb{R}[f]} & \mathbb{C} \otimes \mathbb{R}[H] \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{C}[G] & \xrightarrow[\mathbb{C}[f]]{} & \mathbb{C}[H] \end{array}$$

shows that $\mathbb{C}[f]$ is also surjective. So we have the following Lemma:

Lemma 7.1. *For every surjective morphism f of compact groups the morphism $\mathbb{K}[f]$ of weakly complete Hopf algebras is surjective.*

Proof. See above. □

This seems natural in so far as left adjoints tend to preserve epimorphisms.

The particular left adjoint $\mathbb{K}[-]$, however, also preserves injectives:

Theorem 7.2. *If G is a closed subgroup of the compact group H, then $\mathbb{K}[G] \subseteq \mathbb{K}[H]$ (up to natural isomorphism).*

Proof. From the injectivity of a morphism of compact groups $j: G \to H$ we derive the surjectivity of $C(j, \mathbb{K}): C(H, \mathbb{K}) \to C(G, \mathbb{K})$. (Cf. the Gelfand-Naimark

Theorem.) Accordingly, the function $L^2(j,\mathbb{K})\colon L^2(H,\mathbb{K}) \to L^2(G,\mathbb{K})$ is surjective as well. Now we set $M \overset{\text{def}}{=} C(f,\mathbb{K})\big(\mathcal{R}(H,\mathbb{K})\big) \subseteq \mathcal{R}(G,\mathbb{K})$. Since $\mathcal{R}(H,\mathbb{K})$ is dense in $C(H,\mathbb{K})$ in the norm topology, M is dense in $\mathcal{R}(G,\mathbb{K})$ in the norm topology. Then it is dense in $L^2(G,\mathbb{K})$ in the L^2-topology, and M is a G-module. In the case of $\mathbb{K} = \mathbb{R}$ we can now apply Lemma 8.11 of [3] and conclude that $M = \mathcal{R}(G,\mathbb{R})$. Thus $\mathcal{R}(G,j)\colon \mathcal{R}(H,\mathbb{R}) \to \mathcal{R}(G,\mathbb{R})$ is surjective. By Theorem 7.7 of [3] this implies that $\mathbb{R}[f]'\colon \mathbb{R}[H]' \to \mathbb{R}[G]'$ is surjective. The duality between $\mathbb{K}$-vector spaces and weakly complete $\mathbb{K}$-vetorspaces shows that $\mathbb{R}[f]\colon \mathbb{R}[G] \to \mathbb{R}[H]$ is injective. This proves the theorem for $\mathbb{K} = \mathbb{R}$. But then the commuting diagram

$$\begin{array}{ccc} \mathbb{C}\otimes\mathbb{R}[G] & \xrightarrow{\mathbb{C}\otimes\mathbb{R}[f]} & \mathbb{C}\otimes\mathbb{R}[H] \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ \mathbb{C}[G] & \xrightarrow[\mathbb{C}[f]]{} & \mathbb{C}[H] \end{array}$$

shows that $\mathbb{C}[f]$ is also injective. In the category of weakly complete vector spaces every injective morphism is an embedding by duality since every surjective morphism of vector spaces is a coretraction. □

Corollary 7.3. (i) *For each compact group G with identity component G_0, the Hopf algebra $\mathbb{K}[G_0]$ is a Hopf subalgebra of $\mathbb{K}[G]$.*

(ii) $\mathbb{R}[G_0]$ *is algebraically and topologically generated by* $\Pi(\mathbb{R}[G])$.

Proof. (i) is a consequence of Theorem 7.2.

(ii) The compact group G_0 is algebraically and topologically generated by $\exp_G(\mathfrak{L}(G))$ (cf. [11], Corollary 4.22, p. 191, and $\overline{\operatorname{span}(G_0)} = \mathbb{R}[G_0]$ bu Theorem 3.2(1). □

Chapter 8

The Impact of Weakly Closed Enveloping Algebras of Lie Algebras

We have observed that at least for compact groups G, the weakly complete group algebra $\mathbb{R}[G]$ contains a substantial volume of materials: G itself, the Lie algebra $\mathfrak{L}(G)$ of G as a pro-Lie group, the exponential function between them and, as was at least indicated in Theorem 5.1(3) and which was described in greater detail in [3], a substantial portion of the Radon measure theory of G. The topological Hopf algebra $\mathbb{K}[G]$ is, in a sense, univerally generated by G. So it seems natural to ask the question whether $\mathfrak{L}(G)$ generates $\mathbb{K}[G]$ in a universal way—perhaps in some fashion that resembles the universal enveloping algebra of a Lie algebra such as it is presented in the famous POINCARÉ-BIRKHOFF-WITT-Theorem. This is not exactly the case, but a few aspects can and proably should be discussed

We let $\mathbb{K}$ denote one of the topological fields $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{WA}$ denote the category of weakly complete associative unital algebras over $\mathbb{K}$ and and $\mathcal{WL}$ the category of weakly complete Lie algebras over $\mathbb{K}$. The functor $A \mapsto A_{\rm Lie}$ which associates with a weakly complete associative algebra A the weakly compliete Lie algebra obtained by considering on the weakly complete vector space A the Lie algebra obtained w.r.t. the Lie bracket $[x, y] = xy - yx$ is called the *underlying Lie algebra functor*. Since A is a strict projective limit of finite dimensional K-algebras by [3], Theorem 3.2, then $A_{\rm Lie}$ is a strict projective limit of finite dimensional $\mathbb{K}$-Lie algebras, briefly called *pro-Lie algebras*. Every pro-Lie algebra is weakly complete. Caution: A comment following Theorem 3.12 of [3] exhibits an example of a weakly complete $\mathbb{K}$-Lie algebra which is not a pro-Lie algebra.

Lemma 8.1. *The "underlying Lie algebra" functor $A \mapsto A_{\rm Lie}$ from $\mathcal{WA}$ to $\mathcal{WL}$ has a left adjoint $\mathbf{U}: \mathcal{WL} \to \mathcal{WA}$.*

Proof. The category $\mathcal{WL}$ is complete. (Exercise. Cf. Theorem A3.48 of [10], p. 781.) The "Solution Set Condition" (of Definition A3.59 in [10], p. 786) holds. (Exercise: Cf. the proof of [3], Section 5.1 "The solution set condition".) Hence $\mathbf{U}$ exists by the Adjoint Functor Existence Theorem (i.e., Theorem A3.60 of [10], p. 786). □

In other words, for each weakly complete Lie algebra L there is a natural morphism $\lambda_L: L \to \mathbf{U}(L)$ such that for each continuous Lie algebra morphism $f: L \to A_{\rm Lie}$ for a weakly complete associative unital algebra A there is a unique $\mathcal{WA}$-morphism $f': \mathbf{U}(L) \to A$ such that $f = f'_{\rm Lie} \circ \lambda_L$.

$$\begin{array}{ccc|c}
 & \mathcal{WL} & & \mathcal{AA} \\
\hline
L & \xrightarrow{\lambda_L} & \mathbf{U}(L)_{\rm Lie} & \mathbf{U}(L) \\
{\scriptstyle\forall f}\big\downarrow & & \big\downarrow{\scriptstyle f'_{\rm Lie}} & \big\uparrow{\scriptstyle\exists! f'} \\
A_{\rm Lie} & \xrightarrow[\rm id]{} & A_{\rm Lie} & A.
\end{array}$$

If necessary we shall write $\mathbf{U}_\mathbb{K}$ instead of $\mathbf{U}$ whenever the ground field should be emphasized. We shall call $\mathbf{U}_\mathbb{K}(L)$ *the weakly complete enveloping algebra* of L (over $\mathbb{K}$).

Example 8.2. Let $L = \mathbb{K}$. Then $\mathbf{U}(L) = \mathbb{K}\langle X\rangle$ (see [3], Definition following Corollary 3.3), and define $\lambda_L: L \to \mathbf{U}(L)_{\rm Lie}$ by $\lambda_L(t) = t{\cdot}X$. Then the universal property is satisfied by [3], Corollary 3.4. Indeed, let $f: K \to A_{\rm Lie}$ a morphism of weakly compliete Lie algebras. Then there is a unique morphism $f': \mathbf{U}(L) \to A$ such that $f'(X) = f(1)$ by [3], Corollary 3.4. Then $f'(t{\cdot}X) = t{\cdot}F'(X) = t{\cdot}f(1) = f(r)$.

Thus by Lemma 3.5 of [3] and the subsequent remarks we have:

> *The weakly complete enveloping algebra $\mathbf{U}_\mathbb{C}$ over $\mathbb{C}$ of the smallest nonzero Lie algebra is isomorphic to the weakly complete commutative algebra $\mathbb{C}[[X]]^\mathbb{C}$ with the complex power series algebra $\mathbb{C}[[X]] \cong \mathbb{C}^{\mathbb{N}_0}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.*

The size of the weakly complete enveloping algebras therefore is considerable.

Proposition 8.3. *The universal enveloping functor* $\mathbf{U}$ *is multiplicative, that is, there is a natural isomorphism* $\alpha_{L_1L_2}: \mathbf{U}(L_1 \times L_2) \to \mathbf{U}(L_1) \otimes \mathbf{U}(L_2)$.

Proof. We have a natural bilinear inclusion morphism of weakly complete vector spaces $j: \mathbf{U}(L_1) \times \mathbf{U}(L_2) \to \mathbf{U}(L_1) \otimes \mathbf{U}(L_2)$ yielding

$$L_1 \times L_2 \xrightarrow{\lambda_1 \times \lambda_2} \mathbf{U}(L_1)_{\rm Lie} \times \mathbf{U}(L_2)_{\rm Lie} \xrightarrow{j} \mathbf{U}(L_1)_{\rm Lie} \otimes \mathbf{U}(L_2)_{\rm Lie}$$

and

$$\mathbf{U}(L_1)_{\rm Lie} \otimes \mathbf{U}(L_2)_{\rm Lie} = (\mathbf{U}(L_1) \otimes \mathbf{U}(L_2))_{\rm Lie},$$

the composition α_0 of which is a morphism of weakly complete Lie algebras. Hence the universal property yiels a morphism of weakly complete associative algebras

$$(1) \qquad \alpha: \mathbf{U}(L_1 \times L_2) \to \mathbf{U}(L_1) \otimes \mathbf{U}(L_2)$$

such that $\alpha_0 = \alpha_{\rm Lie} \circ \lambda_{L_1} \otimes \lambda_{L_2}$.

The functorial property of $\mathbf{U}$ allows us to argue that each of $\mathbf{U}(L_m)$, $m = 1, 2$ is a retract of $\mathbf{U}(L_1 \times L_2)$ so that we may assume $\mathbf{U}(L_m) \subseteq \mathbf{U}(L_1 \times L_2)$, $m = 1, 2$. Now the multiplication in $\mathbf{U}(L_1 \times L_2)$ gives rise to a continuous bilinear map $\mathbf{U}(L_1) \times \mathbf{U}(L_2) \to \mathbf{U}(L_1 \times L_2))$, and then the universal property of the tensor product of weakly complete vector spaces yields the morphism

$$(2) \qquad \beta: \mathbf{U}(L_1) \otimes \mathbf{U}(L_2) \to \mathbf{U}(L_1 \times L_2).$$

Similarly to the proof of [3], Theorem 5.5 (preceding the statement of the theorem) we argue that α and β are inverses of each other, and so α of (1) is the desired isomorphism $\alpha_{L_1L_2}$. □

Corollary 8.4. *For a weakly complete Lie algebra L let $p_L\colon L \to (\mathbf{U}(L) \otimes \mathbf{U}(L))_{\rm Lie}$ denote morphism of weakly complete vector spaces given by $p_L(x) = \gamma_L(x) \otimes 1 + 1 \otimes \gamma_L(x)$. Then p_L is a morphism of Lie algebras. Thus p_L turns out to be a morphism of weakly complete Lie algebras and therefore, by the universal property of $\mathbf{U}$ produces a unique natural morphism of weakly complete associative unital algebras $\gamma_L\colon \mathbf{U}(L) \to \mathbf{U}(L) \otimes \mathbf{U}(L)$ such that $p_L = (\gamma_L)_{\rm Lie} \circ \lambda_L$. Then each weakly complete enveloping algebra $\mathbf{U}(L)$ is a weakly complete Hopf algebra with the comultiplication γ_L and the coidentity $U(k)\colon \mathbf{U}(L) \to \mathbb{K}$, where $k\colon L \to \{0\}$ is the constant morphism.*

Proof. Let $x_j \in \mathfrak{L}$ and $y_j = \lambda_L(x_j)$, $j = 1, 2$. Since λ_L is a morphism of weakly complete Lie algebras, $\lambda_L([x_1, x_2]) = [y_1, y_2] = y_1y_2 - y_2y_1$ in $\mathbf{U}(L)_{\rm Lie}$. Now let $z_j = y_j \otimes 1 + 1 \otimes y_j$ for $j = 1, 2$. Then in $(\mathbf{U}(L) \otimes \mathbf{U}(L))_{\rm Lie}$, just as in the classical case, we calculate

$$[z_1, z_2] = z_1z_2 - z_2z_1 = (y_1\otimes 1+1\otimes y_1)(y_2\otimes 1+1\otimes y_2)-(y_2\otimes 1+1\otimes y_2)(y_1\otimes 1+1\otimes y_1)$$
$$= (y_1y_2 \otimes 1 + y_1 \otimes y_2 + y_2 \otimes y_1 + 1 \otimes y_1y_2) - (y_2y_1 \otimes 1 + y_2 \otimes y_1 + y_1 \otimes y_2 + 1 \otimes y_2y_1)$$
$$= [y_1, y_2] \otimes 1 + 1 \otimes [y_1, y_2],$$

showing that $p_L\colon L \to \mathbf{U}(L)_{\rm Lie}$ is a morphism of weakly complete Lie algebras. Now γ_L is a morphism of weakly complete unital algebras satisfying $\gamma_L(y) = y\otimes 1+1\otimes y$ for $y = \lambda(x)$, $x \in L$ and the associativity of this comultiplication is readily checked as in the case of abstract enveloping algebras. The constant morphism of weakly complete Lie algebras $L \to \{0\}$ yields a morphism of weakly complete unital algebras $\mathbf{U}(L) \to \mathbb{K}$ which is the coidentity of the Hopf algebra. □

Our results from [3] regarding weakly complete associative unital algebras and Hopf algebras over $\mathbb{K}$ apply to the present situation.

The Weakly Complete Enveloping Algebra

Theorem 8.5. *Let L be a weakly complete Lie algebra. Then*

(i) *$\mathbf{U}(L)$ is a strict projective limit of finite-dimensional Lie algebras and the group of units $\mathbf{U}(L)^{-1}$ is dense in $\mathbf{U}(L)$. It is an almost connected pro-Lie group (which is connected in the case of $\mathbb{K} = \mathbb{C}$). The algebra $\mathbf{U}(L)$ has an exponential function $\exp_{UU(L)}\colon \mathbf{U}(L)_{\rm Lie} \to \mathbf{U}(L)^{-1}$,*

(ii) *the pro-Lie algebra $\Pi(\mathbf{U}(L))$ of primitive elements of $\mathbf{U}(L)$ contains $\lambda_L(L)$,*

(iii) *the pro-Lie algebra $\Pi(\mathbf{U}(L))$ is the Lie algebra of the pro-Lie group $G_L \stackrel{\rm def}{=} \Gamma(\mathbf{U}(L))$ of grouplike elements of $\mathbf{U}(L)$, and the exponential function $\exp_{G_L}$:*

$\mathfrak{L}(G_L) \to G_L$ is the restriction and corestriction of $\exp_{U(L)}$. Its image generates algebraically and topologically the identity component $(G_L)_0$.

Proof. (i) See [3] Theorems 3.2, 3.11, 3.12, 4.1.

(ii) The very definition of the comultiplication in Corollary 8.4 shows that for any $y \in \lambda_L(L)$, the image under the comultiplication γ_L is $y \otimes 1 + 1 \otimes y$, which means that y is primitive.

(iii) See [3], Theorem 6.15. □

We note right away that for any weakly complete Lie algebra L which has at least one nonzero finite dimensional $\mathbb{K}$-linear representation, the morphism $\lambda_L : L \to \mathbf{U}(L)_{\mathrm{Lie}}$ is nonzero. By Ado's Theorem, this applies, in particular, to any Lie algebra which has a nontrivial finite dimensional quotient and therefore is true for all pro-Lie groups.

Corollary 8.6. (i) *The weakly complete enveloping algebra $\mathbf{U}(L)$ of a weakly complete Lie algebra L with a nontrivial finite dimensional quotient has nontrivial grouplike elements.*

(ii) *If L is a pro-Lie algebra, then $\lambda_L : L \to \mathbf{U}(L)_{\mathrm{Lie}}$ maps L isomorphically into the set $\Pi(\mathbf{U}(L))$ of primitive elements.*

Proof. (i) By Theorem 8.5 (iii) $\mathbf{U}(L)$ has nontrivial grouplike elements if $\Pi(\mathbf{U}(L))$ is nonzero. By Theorem 8.5 (ii) this is the case if γ_L is nonzero which is the case for all L satisfying the hypothesis of the Corollary by the remark preceding it.

(ii) Since each finite dimensional quotient of L has a faithful representation by the Theorem of Ado, and since the finite dimensional quotients separate the points of L, the morphism γ_L is injective. However, injective morphisms of weakly complete vector spaces are open onto their images. □

It follows that for pro-Lie algebras L we may assume that L is in fact a closed Lie subalgebra of $\mathbf{U}(L)$ which generates $\mathbf{U}(L)$ algebraically and topologically.

One application of the functor $\mathbf{U}$ is of present interest to us. Recall that for a compact group we naturally identify G with the group of grouplike elements of $\mathbb{R}[G]$, and that $\mathfrak{L}(G)$ may be identified with the pro-Lie algebra $\Pi(\mathbb{R}[G])$ of primitive elements. In the case of $\mathbb{K} = \mathbb{C}$ it is still true that $\Pi(\mathbb{C}[G])$ is the Lie algebra of the pro-Lie group $\Gamma(\mathbb{C}[G])$.

Proposition 8.7. (i) *Let G be a compact group. Then there is a natural morphism of weakly complete algebras $\omega_G : \mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)) \to \mathbb{R}[G]$.*

(ii) *The image of ω_G is the closed subalgebra $\mathbb{R}[G_0]$ of $\mathbb{R}[G]$.*

(iii) *The pro-Lie group $\Gamma(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ is mapped onto $G_0 = \Gamma(\mathbb{R}[G_0]) \subseteq \mathbb{R}[G]$.*

Proof. (i) follows at once from the universal property of U.

(ii) As a morphism of weakly complete Hopf algebras, $\omega\colon \mathbf{U}(\mathfrak{L}(G_0)) \to \mathbb{R}[G_0]$ maps grouplike elements into grouplike elements.

But the closed group G_0 is generated algebraically and topologically by $\exp(G)$ Hence G_0 is contained in the algebra algebraically and topologically generated by $\exp\big(\mathfrak{L}(G)\big) = \Pi(\mathbb{R}[G])$. □

Example 8.8. Let $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then $\mathbb{C}[G] = \mathbb{C}^{\mathbb{Z}}$ as was exposed in Chapter 6. Then $\Gamma(\mathbb{C}[G]) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^\times) \cong \mathbb{C}^\times$ and $\Pi(\mathbb{C}[G]) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{C}) \cong \mathbb{C}$. Further, by Example 2.2, $\mathbf{U}(\mathbb{C}) = \mathbb{C}\langle X\rangle \cong \mathbb{C}[[X]]^{\mathbb{C}}$. Then the morphism of Hopf algebras $\omega_{\mathbb{T}}$ of Proposition 10 is a morphism $\mathbb{C}[[X]]^{\mathbb{C}} \to \mathbb{C}^{\mathbb{Z}} \cong \mathbb{C}[[X]]$. Thus the enveloping Hopf algebra of $\mathfrak{L}(G) \subseteq \mathbb{C}[G]$ tends to be vastly larger than the group algebra $\mathbb{C}[G]$.

References

[1] Dahmen, R., "Tensor Products of Weakly Complete Topological Vector Spaces," Diplomarbeit am FB. Mathematik der Technischen Universität Darmstadt **230**, 2007, 66 pp.
[2] Dahmen, R., and K. H. Hofmann, *On the Component Factor Group G/G_0 of a Pro-Lie Group G,* J. of Lie Theory **29** (2019), 221–225
[3] Dahmen, R., and K. H. Hofmann, *The Pro-Lie Group Aspect of Weakly Complete Algebras and Weakly Complete Group Hopf Algebras,* J. of Lie Theory **29** (2019), *–*.
[4] Dieudonné, J., "Introduction to the Theory of Formal Groups," Pure and Applied Math., A Series of Monographs and Textbooks **20**, Marcel Dekker, New York, 1973.
[5] Hochschild, G., "The Structure of Lie Groups," Holden-Day, San Francisco etc., 1965, x+230 pp.
[6] Hochschild, G., and G. D. Mostow, *Representations and representative functions of Lie groups,* Ann. of Math. (2) **66** (1957), 495–542.
[6] Hochschild, G., and G. D. Mostow, *Representations and representative functions of Lie groups. II,* Ann. of Math. (2) **68** (1958), 295–313.
[8] Hochschild, G., and G. D. Mostow, *Representations and representative functions of Lie groups. III,* Ann. of Math. (2) **70** (1959), 85–100.
[10] Hofmann, K. H., and S. A. Morris, "The Structure Theory of Compact Groups–A Primer for the Student–a Handbook for the Expert," 3rd Edition, De Gruyter Studies in Mathematics **25**, Berlin/Boston, xxii+924pp.
[11] Hofmann, K. H., and S. A. Morris, The Lie Theory of Connected Pro-Lie Groups,–A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups, and Connected Locally Compact Groups, European Mathematical Society Publishing House, 2006, xii+663pp.
[12] Michaelis, W., "Coassociative Coalgebras", in: M. Hazewinkel, Ed., Handbook of Algebra, Elsevier, Amsterdam etc. Vol 3, 2003, 587–788
[13] Serre, J.-P., Gèbres, Enseignements Math. **39**b(1993), 33–85.
[14] Tannaka, T., *Über den Dualitätssatz der nichtkommutativen topologischen Gruppen*, Tôhoku Math. J. **45** (1939), 1–12.