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WEIGHTED SURFACE ALGEBRAS: GENERAL VERSION

KARIN ERDMANN AND ANDRZEJ SKOWROŃSKI

ABSTRACT. We introduce general weighted surface algebras of triangulated surfaces with arbitrarily oriented triangles and describe their basic properties. In particular, we prove that all these algebras, except the singular disc, triangle, tetrahedral and spherical algebras, are symmetric tame periodic algebras of period 4.

Keywords: Syzygy, Periodic algebra, Self-injective algebra, Symmetric algebra, Surface algebra, Tame algebra
2010 MSC: 16D50, 16E30, 16G20, 16G60, 16G70

Dedicated to Helmut Lenzing on the occasion of his 80th birthday

1. INTRODUCTION AND MAIN RESULTS

We are interested in the representation theory of tame self-injective algebras. In this paper, all algebras are finite-dimensional associative, indecomposable as algebras, and basic, over an algebraically closed field K of arbitrary characteristic.

Tame self-injective algebras of polynomial growth are currently well understood (see [34, 35]). For non-polynomial growth, much less is known. It would be interesting to describe the basic algebras of arbitrary tame self-injective algebras of non-polynomial growth. Our present project is a step in this direction.

In the modular representation theory of finite groups representation-infinite tame blocks occur only over fields of characteristic 2, and their defect groups are dihedral, semidihedral, or (generalized) quaternion 2-groups. In order to study such blocks, algebras of dihedral, semidihedral and quaternion type were introduced and investigated, over algebraically closed fields of arbitrary characteristic (see [11]). In particular, it was shown in [13, 25] that every algebra of quaternion type is a tame periodic algebra of period 4.

Recently cluster theory has led to new directions. Inspired by this, we study in [15] a class of symmetric algebras defined in terms of surface triangulations, which we call *weighted surface algebras*. They are tame and we show that they are periodic as algebras, of period 4 (with one exception, which we call the singular tetrahedral algebra). We observe that many algebras of quaternion type as described in [11] occur in this setting but the construction in [15] only produces algebras whose Gabriel quiver is 2-regular (that is, at each vertex, two arrows start and two arrows end).

In this paper we extend and improve the results of [15]. We generalize the previous definition slightly, and obtain a larger class of algebras. This new version also includes algebras whose Gabriel quiver is not 2-regular, in particular we obtain now

almost all algebras of quaternion type. As well, we obtain the endomorphism algebras of cluster tilting objects in the stable categories of maximal Cohen-Macaulay modules over minimally elliptic curve singularities, as discussed in [3].

An important further motivation for the generalisation is the study of idempotent algebras. In [18] we show that any Brauer graph algebra occurs as an idempotent algebra of some weighted surface algebra. Analysing an arbitrary idempotent algebra of a weighted surface algebra, we discovered that it is natural to extend the original definition. In a subsequent paper we will give a complete description of all idempotent algebras of weighted surface algebras.

The main result in this paper shows that, with four exceptions, a general weighted surface algebra is periodic as an algebra, of period 4. The exceptions are the singular tetrahedral algebra which already occured in [15], and three others, which we call singular disc algebra, singular triangle algebra, and singular spherical algebra.

Let A be an algebra. Given a module M in mod A, its *syzygy* is defined to be the kernel $\Omega_A(M)$ of a minimal projective cover of M in mod A. The syzygy operator Ω_A is a very important tool to construct modules in mod A and relate them. For A self-injective, it induces an equivalence of the stable module category $\underline{\mathrm{mod}}\,A$, and its inverse is the shift of a triangulated structure on $\underline{\mathrm{mod}}\,A$ [22]. A module M in mod A is said to be *periodic* if $\Omega_A^n(M) \cong M$ for some $n \geq 1$, and if so the minimal such n is called the *period* of M. The action of Ω_A on mod A can effect the algebra structure of A. For example, if all simple modules in mod A are periodic, then A is a self-injective algebra. Sometimes one can even recover the algebra A and its module category from the action of Ω_A. For example, the self-injective Nakayama algebras are precisely the algebras A for which Ω_A^2 permutes the isomorphism classes of simple modules in mod A. An algebra A is defined to be *periodic* if it is periodic viewed as a module over the enveloping algebra $A^e = A^{\mathrm{op}} \otimes_K A$, or equivalently, as an A-A-bimodule. It is known that if A is a periodic algebra of period n then for any indecomposable non-projective module M in mod A the syzygy $\Omega_A^n(M)$ is isomorphic to M.

Finding or possibly classifying periodic algebras is an important problem, because of interesting connections with group theory, topology, singularity theory, cluster algebras, cluster tilting theory (we refer to [14, 27] and the introduction of [15] for some details).

The following three theorems describe basic properties of the general weighted surface algebras.

Theorem 1.1. *Let $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet)$ be a weighted surface algebra over an algebraically closed field K, which is not isomorphic to a singular triangle or spherical algebra. Then Λ is a symmetric algebra.*

Theorem 1.2. *Let $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet)$ be a weighted surface algebra over an algebraically closed field K, which is not isomorphic to a disc algebra, triangle algebra, tetrahedral algebra, spherical algebra. Then the following statements hold:*

(i) Λ degenerates to the biserial weighted surface algebra $B(S, \vec{T}, m_\bullet, c_\bullet)$.
(ii) Λ is a tame algebra of non-polynomial growth.

Theorem 1.3. *Let $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet)$ be a weighted surface algebra over an algebraically closed field K. Then the following statements are equivalent:*

(i) *All simple modules in* $\mathrm{mod}\,\Lambda$ *are periodic of period* 4.

(ii) Λ *is a periodic algebra of period* 4.

(iii) Λ *is a weighted surface algebra other than a singular disc, triangle, tetrahedral or spherical algebra.*

This paper is organized as follows. In Section 2 we introduce the algebras. This is slightly more general as needed for weighted surface algebras, in order to show how they fit into a more general context of tame symmetric algebras; as well it will be needed for the study of idempotent algebras. We review much as needed from [15], this is done in Section 2, and discuss the modifications of the definition needed. Section 3 introduces the algebras which play a special role: the disc algebras, the tetrahedral algebras, the triangle algebras, and the spherical algebras. In Section 4 we prove some general results, in particular we show that weighted surface algebras are symmetric (except for a few small cases which we identify). Section 5 proves the periodicity result. The final section proves tameness, and also classifies polynomial growth. For general background on the relevant representation theory we refer to the books [1, 11, 32, 36].

2. Weighted surface algebras, and the general context

Recall that a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 is a finite set of vertices, Q_1 is a finite set of arrows, and where s, t are maps $Q_1 \to Q_0$ associating to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its target $t(\alpha)$. We say that α starts at $s(\alpha)$ and ends at $t(\alpha)$. We assume throughout that any quiver is connected.

Denote by KQ the path algebra of Q over K. The underlying space has basis the set of all paths in Q. Let R_Q be the ideal of KQ generated by all paths of length ≥ 1. For each vertex i, let e_i be the path of length zero at i, then the e_i are pairwise orthogonal, and their sum is the identity of KQ. We will consider algebras of the form $A = KQ/I$ where I is an ideal of KQ which contains R_Q^m for some $m \geq 2$, so that the algebra is finite-dimensional and basic. The Gabriel quiver Q_A of A is then the full subquiver of Q obtained from Q by removing all arrows α which belong to the ideal I.

The setting for weighted surface algebras and the algebras occuring in [17] and [18] (and also in future work) has unified description, which we will now present.

A quiver Q is *2-regular* if for each vertex $i \in Q_0$ there are precisely two arrows starting at i and two arrows ending at i. All quivers we consider will be 2-regular. Such a quiver has an involution on the arrows, $\alpha \mapsto \bar{\alpha}$, such that for each arrow α, the arrow $\bar{\alpha}$ is the arrow $\neq \alpha$ such that $s(\alpha) = s(\bar{\alpha})$.

A *biserial quiver* is a pair (Q, f) where Q is a (finite) connected 2-regular quiver, with at least two vertices, and where f is a fixed permutation of the arrows such that $t(\alpha) = s(f(\alpha))$ for each arrow α. The permutation f uniquely determines a permutation g of the arrows, defined by $g(\alpha) := \overline{f(\alpha)}$ for any arrow α. Let (Q, f) be a biserial quiver. We say that (Q, f) is a *triangulation quiver* if f^3 is the identity. That is, all cycles of f have length 3 or 1.

In this paper we will focus on triangulation quivers. As we have proved in [15], these are precisely the quivers $(Q(S, \vec{T}), f)$ constructed from a triangulation T of a compact connected (real) surface, with or without boundary, and where the orientation $\vec{T}$ in each triangle can be chosen arbitrarily. For details we refer to [15],

we will not repeat this since we will not use the geometric version in any essential way.

We fix an algebraically closed field K, and we introduce some notation. This will be used throughout. For each arrow α, we fix

$$m_\alpha \in \mathbb{N}^* \quad \text{a multiplicity, constant on } g\text{-cycles, and}$$
$$c_\alpha \in K^* \quad \text{a weight, constant on } g\text{-cycles, and define}$$
$$n_\alpha := \quad \text{the length of the } g\text{-cycle of } \alpha,$$
$$B_\alpha := \alpha g(\alpha) \ldots g^{m_\alpha n_\alpha - 1}(\alpha) \quad \text{the path along the } g\text{-cycle of } \alpha \text{ of length } m_\alpha n_\alpha$$
$$A_\alpha := \alpha g(\alpha) \ldots g^{m_\alpha n_\alpha - 2}(\alpha) \quad \text{the path along the } g\text{-cycle of } \alpha \text{ of length } m_\alpha n_\alpha - 1.$$

For the algebras of the form $A = KQ/I$, we will fix relations such that:
(1) Each paths $\alpha f(\alpha)$ of length 2 occurs in some distinguished element in I.
(2) We will ensure that in the algebra, $c_\alpha B_\alpha = c_{\bar\alpha} B_{\bar\alpha}$, and that the elements B_α span the socle of A.
(3) We will ensure that A has a basis consisting of initial subwords of elements B_α and $B_{\bar\alpha}$.

That is, the cycles of f describe minimal relations, and the cycles of g describe a basis for the algebra. There are two types of distinguished relations,

(Q) $\alpha f(\alpha) = c_{\bar\alpha} A_{\bar\alpha}$ in Λ, only when $f^3(\alpha) = \alpha$ ('quaternion' relations);
(B) $\alpha f(\alpha) = 0$ in Λ ('biserial' relations).

In addition, one needs zero relations so that (2) is satisfied.

This includes Brauer graph algebras: Take an algebra $R = KQ/I$ where (Q, f) is a biserial quiver and where I is generated by biserial relations, for all arrows α, together with relations $B_\alpha = B_{\bar\alpha}$, for all arrows α, taking as the weight function has $c_\alpha = 1$ for all α. For details we refer to [18]. This includes Brauer tree algebra, and motivated by this we think of the cycles of f as 'Green walks'.

In [15] and [17] we have studied biserial weighted surface algebras, these are the Brauer graph algebras where in addition $f^3 = 1$, that is, (Q, f) is a triangulation quiver. These occur for blocks with dihedral defect groups. In this case, the Green walks are in bijection with tubes of rank 3 in the stable Auslander-Reiten quiver. In fact, this suggests that the condition $f^3 = 1$ should play a special role.

On the other extreme, if all distinguished relations are quaternion relations, we get weighted surface algebras, which we will study in detail in this paper.

To deal with tameness, we use special biserial algebras, and we only need those which are symmetric, for the general definition we refer to the literature. It is known that special biserial symmetric algebras are precisely the Brauer graph algebras as described above, for a detailed discussion see [18]. We have the following (proved in this generality in [37], see also [4, 8] for alternative proofs).

Proposition 2.1. *Every special biserial algebra is tame.*

For a positive integer d, we denote by $\mathrm{alg}_d(K)$ the affine variety of associative K-algebra structures with identity on the affine space K^d. Then the general linear group $\mathrm{GL}_d(K)$ acts on $\mathrm{alg}_d(K)$ by transport of the structures, and the $\mathrm{GL}_d(K)$-orbits in $\mathrm{alg}_d(K)$ correspond to the isomorphism classes of d-dimensional algebras (see [26] for details). We identify a d-dimensional algebra A with the point of

$\mathrm{alg}_d(K)$ corresponding to it. For two d-dimensional algebras A and B, we say that B is a *degeneration* of A (A is a *deformation* of B) if B belongs to the closure of the $\mathrm{GL}_d(K)$-orbit of A in the Zariski topology of $\mathrm{alg}_d(K)$.

Geiss' Theorem [20] shows that if A and B are two d-dimensional algebras, A degenerates to B and B is a tame algebra, then A is also a tame algebra (see also [5]). We will apply this theorem in the following special situation.

Proposition 2.2. *Let d be a positive integer, and $A(t)$, $t \in K$, be an algebraic family in $\mathrm{alg}_d(K)$ such that $A(t) \cong A(1)$ for all $t \in K \setminus \{0\}$. Then $A(1)$ degenerates to $A(0)$. In particular, if $A(0)$ is tame, then $A(1)$ is tame.*

A family of algebras $A(t)$, $t \in K$, in $\mathrm{alg}_d(K)$ is said to be *algebraic* if the induced map $A(-) : K \to \mathrm{alg}_d(K)$ is a regular map of affine varieties.

An important combinatorial and homological invariant of the module category $\mathrm{mod}\, A$ of an algebra A is its Auslander-Reiten quiver Γ_A. Recall that Γ_A is the translation quiver whose vertices are the isomorphism classes of indecomposable modules in $\mathrm{mod}\, A$, the arrows correspond to irreducible homomorphisms, and the translation is the Auslander-Reiten translation $\tau_A = D\,\mathrm{Tr}$. For A self-injective, we denote by Γ_A^s the stable Auslander-Reiten quiver of A, obtained from Γ_A by removing the isomorphism classes of projective modules and the arrows attached to them. By a stable tube we mean a translation quiver Γ of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$, and we call r the rank of Γ. We note that, for a symmetric algebra A, we have $\tau_A = \Omega_A^2$ (see [36, Corollary IV.8.6]). In particular, we have the following equivalence.

Proposition 2.3. *Let A be an indecomposable, representation-infinite symmetric algebra. The following statements are equivalent:*

  (i) *Γ_A^s consists of stable tubes.*
  (ii) *All indecomposable non-projective modules in $\mathrm{mod}\, A$ are periodic.*

Therefore, we conclude that, if A is an indecomposable, representation-infinite, symmetric, periodic algebra (of period 4) then Γ_A^s consists of stable tubes (of ranks 1 and 2). We also note that, if A is a representation-infinite special biserial symmetric algebra, then Γ_A^s admits an acyclic component (see [12]), and consequently A is not a periodic algebra.

Let A be an algebra over K and σ a K-algebra automorphism of A. Then for any A-A-bimodule M we denote by ${}_1 M_\sigma$ the A-A-bimodule with the underlying K-vector space M and action defined as $amb = am\sigma(b)$ for all $a, b \in A$ and $m \in M$.

The following has been proved in [21, Theorem 1.4].

Theorem 2.4. *Let A be an algebra over K and d a positive integer. Then the following statements are equivalent:*

  (i) *$\Omega_A^d(S) \cong S$ in $\mathrm{mod}\, A$ for every simple module S in $\mathrm{mod}\, A$.*
  (ii) *$\Omega_{A^e}^d(S) \cong {}_1 A_\sigma$ in $\mathrm{mod}\, A^e$ for some K-algebra automorphism σ of A such that $\sigma(e)A \cong eA$ for any primitive idempotent e of A.*

Moreover, if A satisfies these conditions, then A is self-injective.

The *Cartan matrix* C_A of an algebra A is the matrix $(\dim_K \mathrm{Hom}_A(P_i, P_j))_{1 \leq i,j \leq n}$ for a complete family $P_1, \ldots, P_n$ of a pairwise non-isomorphic indecomposable projective modules in $\mathrm{mod}\, A$. The following main result from [10] shows why the original class of algebras of quaternion type is very restricted compared with the algebras which we will study in this paper.

Theorem 2.5. *Let A be an indecomposable, representation-infinite tame symmetric algebra with non-singular Cartan matrix such that every non-projective indecomposable module in* $\operatorname{mod} A$ *is periodic of period dividing 4. Then* $\operatorname{mod} A$ *has at most three pairwise non-isomorphic simple modules.*

In [15] we define a weighted surface algebra, where the quiver is constructed in terms of a triangulation T of a surface S, with arbitrarily oriented $\vec{T}$ triangles, and such a quiver is denoted by $Q(S, \vec{T})$. Such a quiver is a triangulation quiver (Q, f) as we have defined above. Moreover, it was proved that triangulation quivers are the same as quivers of the form $Q(S, \vec{T})$. We have also at that stage distinguished between weighted surface algebras (which use $Q(S, \vec{T})$) and weighted triangulation algebra (which use (Q, f)).

In the present paper we will almost entirely use triangulation quivers, but we will refer to weighted surface algebras for the algebras constructed. We will now give the general definition, and we use the notation which we have introduced earlier.

Roughly speaking, the modification of the definition consists of

(i) allowing quivers with ≥ 2 vertices (previously we excluded the case of two vertices),
(ii) allowing $m_\alpha n_\alpha \geq 2$ (previously we assumed $m_\alpha n_\alpha \geq 3$).

We require socle conditions as described in part (3) of the notation, as well since $m_\alpha n_\alpha = 2$ we have to modify the zero relations, and exclude a few degenerate cases.

Definition 2.6. *We say that an arrow α of Q is virtual if $m_\alpha n_\alpha = 2$. Note that this condition is preserved under the permutation g, and that virtual arrows form g-orbits of sizes 1 or 2.*

Assumption 2.7. *For the general weighted surface algebra we assume that the following conditions are satisfied:*

(1) $m_\alpha n_\alpha \geq 2$ for all arrows α,
(2) $m_\alpha n_\alpha \geq 3$ for all arrows α such that $\bar{\alpha}$ is virtual and $\bar{\alpha}$ is not a loop,
(3) $m_\alpha n_\alpha \geq 4$ for all arrows α such that $\bar{\alpha}$ is virtual and $\bar{\alpha}$ is a loop.

Condition (1) is a general assumption, and (2) and (3) are needed to eliminate two small algebras, see below. In particular we exclude the possibility that both arrows starting at a vertex are virtual, and also that both arrows ending at a vertex are virtual.

The definition of a weighted surface algebra is now as follows.

Definition 2.8. *The algebra $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet) = KQ/I$ is a weighted surface algebra if (Q, f) is a triangulation quiver, with $|Q_0| \geq 2$, and $I = I(Q, f, m_\bullet, c_\bullet)$ is the ideal of KQ generated by:*

(1) $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}}$ for all arrows α of Q,
(2) $\alpha f(\alpha) g(f(\alpha))$ for all arrows α of Q such that $f^2(\alpha)$ is not virtual,
(3) $\alpha g(\alpha) f(g(\alpha))$ for all arrows α of Q such that $f(\alpha)$ is not virtual.

Note that the ideal is not admissible in general. Namely if an arrow $\bar{\alpha}$ (say) is virtual then by (1) it lies in the square of the radical. In fact, we can see from the relations that the Gabriel quiver Q_Λ of Λ is obtained from Q by removing all virtual arrows.

As long as we do not have any special conditions on other scalars, we can assume that for a virtual arrow α, the weight c_α is equal to 1, namely we may replace α (and $g(\alpha)$) by $c_\alpha^{-1}\alpha$ (and $c_\alpha^{-1}g(\alpha)$), the g-orbit of α has length two, and this scalar does not occur anywhere else. In the first part of this paper we will keep c_α since it will clarify proofs. On the other hand, in the last part we will take $c_\alpha = 1$ for a virtual arrow α since it will simplify the formulae.

We recall some elementary consequences of the definition, which we will use freely throughout.

Lemma 2.9. *Let α be an arrow in Q. We have in Λ the identities:*

(i) $f^2(\alpha) = g^{n_{\bar\alpha}-1}(\bar\alpha)$ *so that* $g(f^2(\alpha) = \bar\alpha$.
(ii) $A_{\bar\alpha} f^2(\alpha) = B_{\bar\alpha}$.
(iii) $\alpha A_{g(\alpha)} = B_\alpha$.
(iv) $c_\alpha B_\alpha = \alpha f(\alpha)f^2(\alpha) = \bar\alpha f(\bar\alpha)f^2(\bar\alpha) = c_{\bar\alpha}B_{\bar\alpha}$.
(v) $A'_\alpha f^2(\bar\alpha) = A_{g(\alpha)}$.

This is the same as Lemma 5.3 in [15].

Definition 2.10. *The algebra $B = B(Q, f, m_\bullet, c_\bullet) = KQ/J$ is a biserial weighted triangulation algebra if (Q, f) is a triangulation quiver, with $|Q_0| \geq 2$, and $J = J(Q, f, m_\bullet, c_\bullet)$ is the ideal of KQ generated by:*

(1) $\alpha f(\alpha)$ for all arrows α of Q,
(2) $c_\alpha B_\alpha - c_{\bar\alpha}B_{\bar\alpha}$ for all arrows α of Q.

We note that $B(Q, f, m_\bullet, c_\bullet) \cong B(Q, f, m_\bullet, \mathbb{1})$, where $\mathbb{1}$ is the constant parameter function taking only value 1. We have the following consequence of [17, Proposition 5.2] (see also [18, Proposition 2.3]).

Proposition 2.11. *Let (Q, f) be a triangulation quiver, $m_\bullet$ and $c_\bullet$ weight and parameter functions of (Q, f), and $B = B(Q, f, m_\bullet, c_\bullet)$. The following statements hold:*

(i) B is finite-dimensional with $\dim_K B = \sum_{\mathcal{O}\in\mathcal{O}(g)} m_\mathcal{O} n_\mathcal{O}^2$.
*(ii) B is a symmetric sperical biserial algebra.*

In particular, B is a tame algebra.

Let T be a triangulation of a surface S, $\vec{T}$ an orientation of triangles in T, $(Q(S,\vec{T}), f)$ the associated triangulation quiver, and $m_\bullet$ and $c_\bullet$ weight and parameter functions of $(Q(S,\vec{T}), f)$. Then $\Lambda(S, \vec{T}, m_\bullet, c_\bullet) = \Lambda(Q(S,\vec{T}), f, m_\bullet, c_\bullet)$ is said to be a *weighted surface algebra*, and $B(S, \vec{T}, m_\bullet, c_\bullet) = B(Q(S,\vec{T}), f, m_\bullet, c_\bullet)$ a *biserial weighted surface algebra*.

3. Exceptional weighted surface algebras

In this section we present several families of weighted surface algebras, which have exceptional properties, and explain also the assumptions 2.7, and show why some algebras must be excluded. For the examples we will use surface triangulations, but only to motivate the names for the algebras. The background is explained in [15] and we will not repeat this.

Example 3.1. We introduce *disc algebras*. Let T be the self-folded triangulation
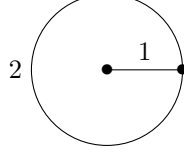
of the unit disc $D = D^2$ in $\mathbb{R}^2$, and $\vec{T}$ the canonical orientation of the edges of T. Then the associated triangulation quiver $(Q, f) = (Q(D, \vec{T}), f)$ is the quiver

with f-orbits $(\alpha\ \beta\ \gamma)$ and (σ). Then the g-orbits are $\mathcal{O}(\alpha) = (\alpha)$ and $\mathcal{O}(\beta) = (\beta\ \sigma\ \gamma)$. Let $c_\bullet : \mathcal{O}(g) \to K^*$ be a parameter function, and let $a = c_{\mathcal{O}(\alpha)}$ and $b = c_{\mathcal{O}(\beta)}$. We consider special cases of weight functions $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$, the first special case gives the disc algebras, and the second needs to be excluded.

(1) Assume that $m_{\mathcal{O}(\alpha)} = 3$ and $m_{\mathcal{O}(\beta)} = 1$. Then the associated weighted surface algebra $D(a, b) = \Lambda(D, \vec{T}, m_\bullet, c_\bullet)$ is given by the quiver Q and the relations:

$$\alpha\beta = b\beta\sigma, \quad \beta\gamma = a\alpha^2, \quad \gamma\alpha = b\sigma\gamma, \quad \sigma^2 = b\gamma\beta, \quad \alpha\beta\sigma = 0, \quad \beta\gamma\beta = 0,$$
$$\gamma\alpha^2 = 0, \quad \sigma^2\gamma = 0, \quad \alpha^2\beta = 0, \quad \beta\sigma^2 = 0, \quad \sigma\gamma\alpha = 0, \quad \gamma\beta\gamma = 0.$$

An algebra $D(a, b)$, with $a, b \in K^*$, is said to be a *disc algebra*. We note that the algebra $D(a, b)$ is isomorphic to the algebra $D(ab, 1)$. Indeed, there is an isomorphism of algebras $\varphi : D(ab, 1) \to D(a, b)$ given by $\varphi(\alpha) = \alpha$, $\varphi(\beta) = \beta$, $\varphi(\gamma) = b\gamma$, $\varphi(\sigma) = b\sigma$. For $\lambda \in K^*$, we set $D(\lambda) = D(\lambda, 1)$. A disc algebra $D(\lambda)$ with $\lambda \in K \setminus \{0, 1\}$ is said to be a *non-singular disc algebra*, and $D(1)$ the *singular disc algebra*.

(2) Assume that $m_{\mathcal{O}(\alpha)} = 2$ and $m_{\mathcal{O}(\beta)} = 1$. Then the associated weighted surface algebra $\Lambda = \Lambda(D, \vec{T}, m_\bullet, c_\bullet)$ is given by the quiver Q and the relations:

$$\gamma\alpha = b\sigma\gamma, \quad \alpha\beta = b\beta\sigma, \quad \beta\sigma^2 = 0, \quad \gamma\alpha^2 = 0, \quad \alpha\beta\sigma = 0,$$
$$\beta\gamma = a\alpha, \quad \sigma^2 = b\gamma\beta, \quad \sigma^2\gamma = 0, \quad \alpha^2\beta = 0, \quad \sigma\gamma\alpha = 0.$$

Then Λ is isomorphic to the algebra A given by the quiver

and the relations: $\beta\sigma = 0$, $\sigma\gamma = 0$, $\sigma^2 = \gamma\beta$. Hence Λ is a 7-dimensional symmetric representation-finite algebra of Dynkin type $\mathbb{A}_4$. Observe that we have $m_\beta n_\beta = 3$ and $m_{\bar\beta} n_{\bar\beta} = m_\alpha n_\alpha = 2$. Therefore, we do not consider such an algebra Λ, by the general assumption 2.7.

**Example 3.2.** We recall the *tetrahedral algebras* introduced in [15, Example 6.1]. Let T be the tetrahedron
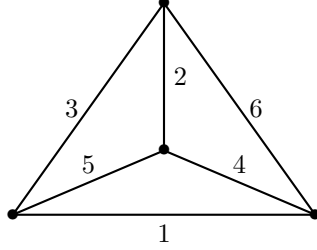
with the coherent orientation $\vec{T}$ of triangles: (1 5 4), (2 5 3), (2 6 4), (1 6 3). Then the associated triangulation quiver $(Q, f) = (Q(T, \vec{T}), f)$ is of the form
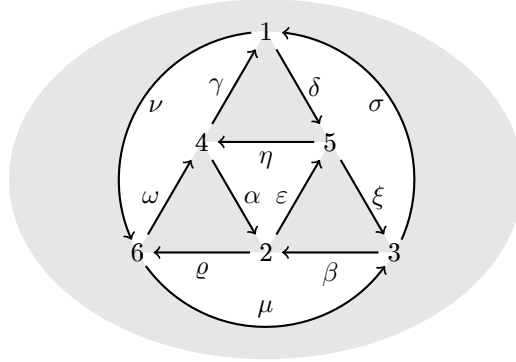
where f is the permutation of arrows of order 3 described by the shaded triangles. Then g is the permutation of arrows of Q of order 3 described by the four white triangles. Let $m_{\bullet} : \mathcal{O}(g) \to \mathbb{N}^*$ be the trivial weight function and $c_{\bullet} : \mathcal{O}(g) \to K^*$ an arbitrary parameter function. It was shown in [15, Section 6] that the weighted surface algebra $\Lambda(T, \vec{T}, m_{\bullet}, c_{\bullet})$ is isomorphic to the weighted triangulation algebra $\Lambda(\lambda) = \Lambda(Q, f, m_{\bullet}, c_{\bullet}^{\lambda})$, with $\lambda \in K^*$, and the parameter function $c_{\bullet}^{\lambda} : \mathcal{O}(g) \to K^*$ given by $c_{\mathcal{O}(\alpha)}^{\lambda} = \lambda$, $c_{\mathcal{O}(\beta)}^{\lambda} = 1$, $c_{\mathcal{O}(\gamma)}^{\lambda} = 1$, $c_{\mathcal{O}(\sigma)}^{\lambda} = 1$. Observe that $\Lambda(\lambda)$ is given by the quiver Q and the relations:

$$\gamma\delta = \lambda\alpha\varepsilon, \qquad \delta\eta = \nu\omega, \qquad \eta\gamma = \xi\sigma, \qquad \alpha\varrho = \gamma\nu, \qquad \varrho\omega = \lambda\varepsilon\eta, \qquad \omega\alpha = \mu\beta,$$

$$\beta\varepsilon = \sigma\delta, \qquad \varepsilon\xi = \varrho\mu, \qquad \xi\beta = \lambda\eta\alpha, \qquad \sigma\nu = \beta\varrho, \qquad \nu\mu = \delta\xi, \qquad \mu\sigma = \omega\gamma,$$

$$\theta f(\theta) g\big(f(\theta)\big) = 0 \qquad \text{and} \qquad \theta g(\theta) f\big(g(\theta)\big) = 0 \text{ for all arrows } \theta \in Q_1.$$

Moreover, by [15, Lemma 6.2], the algebra $\Lambda(\lambda)$ is isomorphic to the trivial extension algebra $\mathrm{T}(B(\lambda))$ of the algebra $B(\lambda)$ given by the quiver
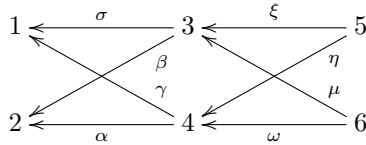
and the relations:

$$\eta\gamma = \xi\sigma, \qquad \xi\beta = \lambda\eta\alpha, \qquad \mu\sigma = \omega\gamma, \qquad \omega\alpha = \mu\beta.$$

We note that, for $\lambda \in K \setminus \{0,1\}$, $B(\lambda)$ is a tubular algebra of type $(2,2,2,2)$ in the sense of [31], and hence it is an algebra of polynomial growth. On the other hand, $B(1)$ is the tame minimal non-polynomial growth algebra (30) from [29]. Following [15], an algebra $\Lambda(\lambda)$ with $\lambda \in K^*$ is said to be a *tetrahedral algebra*. Further, an algebra $\Lambda(\lambda)$ with with $\lambda \in K \setminus \{0,1\}$ is called to be a *non-singular tetrahedral algebra*, while the algebra $\Lambda(1)$ the *singular tetrahedral algebra*.

There is a natural connection between the disc algebra $D(\lambda)$ and the tetrahedral algebra $\Lambda(\lambda)$, for any $\lambda \in K^*$. Namely, the cyclic group H of order 3 acts on $\Lambda(\lambda)$ by cyclic rotation of vertices and arrows of the quiver $Q = Q(T, \vec{T})$:
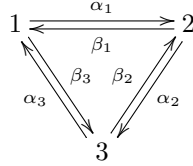
$$(1\ 6\ 3), \qquad (4\ 2\ 5), \qquad (\alpha\ \varepsilon\ \eta), \qquad (\beta\ \delta\ \omega), \qquad (\gamma\ \varrho\ \xi), \qquad (\sigma\ \nu\ \mu).$$

Then $D(\lambda)$ is the orbit algebra $\Lambda(\lambda)/H$.

Example 3.3. We introduce *triangle algebras*, and also describe an algebra which we must exclude. Let T be the triangulation
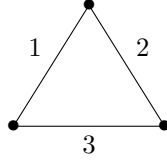
of the sphere S^2 in $\mathbb{R}^3$ given by two unfolded triangles and $\vec{T}$ the coherent orientation $(1\ 2\ 3)$ and $(2\ 1\ 3)$ of the triangles in T. Then the associated triangulation quiver $(Q, f) = (Q(S^2, \vec{T}), f)$ is of the form

with f-orbits $(\alpha_1\ \alpha_2\ \alpha_3)$ and $(\beta_1\ \beta_3\ \beta_2)$. Then $\mathcal{O}(g)$ consists of the three g-orbits $\mathcal{O}(\alpha_1) = (\alpha_1\ \beta_1)$, $\mathcal{O}(\alpha_2) = (\alpha_2\ \beta_2)$, $\mathcal{O}(\alpha_3) = (\alpha_3\ \beta_3)$. Let $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$ be the weight function with $m_{\mathcal{O}(\alpha_1)} = 2$, $m_{\mathcal{O}(\alpha_2)} = 2$, and $m_{\mathcal{O}(\alpha_3)} = 1$. Moreover, let $c_\bullet : \mathcal{O}(g) \to K^*$ be an arbitrary parameter function and $c_1 = c_{\mathcal{O}(\alpha_1)}$, $c_2 = c_{\mathcal{O}(\alpha_2)}$, $c_3 = c_{\mathcal{O}(\alpha_3)}$. Then the associated weighted surface algebra $T(c_1, c_2, c_3) = \Lambda(S^2, \vec{T}, m_\bullet, c_\bullet)$ is given by the quiver Q and the relations:

$$
\begin{aligned}
&\alpha_1\alpha_2 = c_3\beta_3, &\quad &\alpha_2\alpha_3 = c_1\beta_1\alpha_1\beta_1, &\quad &\alpha_3\alpha_1 = c_2\beta_2\alpha_2\beta_2, \\
&\beta_2\beta_1 = c_3\alpha_3, &\quad &\beta_1\beta_3 = c_2\alpha_2\beta_2\alpha_2, &\quad &\beta_3\beta_2 = c_1\alpha_1\beta_1\alpha_1, \\
&\alpha_2\alpha_3\beta_3 = 0, &\quad &\alpha_3\alpha_1\beta_1 = 0, &\quad &\beta_1\beta_3\alpha_3 = 0, &\quad &\beta_3\beta_2\alpha_2 = 0, \\
&\alpha_1\beta_1\beta_3 = 0, &\quad &\alpha_3\beta_3\beta_2 = 0, &\quad &\beta_2\alpha_2\alpha_3 = 0, &\quad &\beta_3\alpha_3\alpha_1 = 0.
\end{aligned}
$$

An algebra $T(c_1, c_2, c_3)$, with $c_1, c_2, c_3 \in K^*$, is said to be a *triangle algebra*. We note that the algebra $T(c_1, c_2, c_3)$ is isomorphic to the algebra $T(c_1c_2c_3^2, 1, 1)$. Indeed, there is an isomorphism of algebras $\varphi : T(c_1c_2c_3^2, 1, 1) \to T(c_1, c_2, c_3)$ given

by

$$\varphi(\alpha_1) = (c_2 c_3)^{-\frac{1}{4}} \alpha_1, \qquad \varphi(\alpha_2) = (c_2 c_3)^{\frac{1}{4}} \alpha_2, \qquad \varphi(\alpha_3) = c_3 \alpha_3,$$

$$\varphi(\beta_1) = (c_2 c_3)^{-\frac{1}{4}} \beta_1, \qquad \varphi(\beta_2) = (c_2 c_3)^{\frac{1}{4}} \beta_2, \qquad \varphi(\beta_3) = c_3 \beta_3.$$

For $\lambda \in K^*$, we set $T(\lambda) = T(\lambda, 1, 1)$. A triangle algebra $T(\lambda)$ with $\lambda \in K \setminus \{0, 1\}$ is said to be a *non-singular triangle algebra*, and $T(1)$ the *singular triangle algebra*.

The triangle algebra $T(\lambda)$ is isomorphic to the algebra $T(\lambda)^0$ given by the Gabriel quiver $Q_{T(\lambda)}$

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3$$

of $T(\lambda)$ and the induced relations:

$$
\begin{array}{lll}
\alpha_2 \beta_2 \beta_1 = \lambda \beta_1 \alpha_1 \beta_1, & \alpha_2 \beta_2 \beta_1 \alpha_1 \alpha_2 = 0, & \alpha_1 \beta_1 \alpha_1 \alpha_2 = 0, \\
\beta_2 \beta_1 \alpha_1 = \beta_2 \alpha_2 \beta_2, & \beta_2 \beta_1 \alpha_1 \beta_1 = 0, & \beta_2 \beta_1 \alpha_1 \alpha_2 \beta_1 = 0, \\
\beta_1 \alpha_1 \alpha_2 = \alpha_2 \beta_2 \alpha_2, & \beta_1 \alpha_1 \alpha_2 \beta_2 \beta_1 = 0, & \beta_2 \alpha_2 \beta_2 \beta_1 = 0, \\
\alpha_1 \alpha_2 \beta_2 = \lambda \alpha_1 \beta_1 \alpha_1, & \alpha_1 \alpha_2 \beta_2 \alpha_2 = 0, & \alpha_1 \alpha_2 \beta_2 \beta_1 \alpha_1 = 0.
\end{array}
$$

We also note that $T(1)^0$ is not a symmetric (even self-injective) algebra. Indeed, if $\lambda = 1$, then $\beta_1 \alpha_1 - \beta_2 \alpha_2$ and $\beta_1 \alpha_1 \beta_1 \alpha_1 = \beta_2 \alpha_2 \beta_2 \alpha_2$ are independent elements of the indecomposable projective module P_2 at the vertex 2, which are annihilated by the radical of $T(\lambda)^0$, and hence are in the socle of P_2. Therefore, $T(1) \cong T(1)^0$ is excluded here.

(2) It follows from our general assumption that if $m_{\alpha_3} n_{\alpha_3} = 2$ then $m_{\alpha_1} n_{\alpha_1} \geq 3$ and $m_{\alpha_2} n_{\alpha_2} \geq 3$, and hence $m_{\alpha_1} n_{\alpha_1} \geq 4$ and $m_{\alpha_2} n_{\alpha_2} \geq 4$. The reason for such restriction is as follows: if we would allow that two or three of the numbers $m_{\alpha_1} n_{\alpha_1}$, $m_{\alpha_2} n_{\alpha_2}$, $m_{\alpha_3} n_{\alpha_3}$, are equal to 2 then the associated triangulation algebra $\Lambda(Q, f, m_\bullet, c_\bullet)$ would be infinite dimensional.

Example 3.4. The following example will give another construction of some triangle algebras, as well it is related to algebras of quaternion type in [11]. Consider the triangulation T
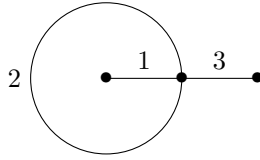
of the sphere S^2 in $\mathbb{R}^3$ given by two self-folded triangles, and the canonical orientation $\vec{T}$ of triangles of T. Then the associated triangulation quiver $(Q, f) = (Q(S^2, \vec{T}), f)$ is the quiver

$$\alpha \overset{\curvearrowright}{} 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \underset{\delta}{\overset{\sigma}{\rightleftarrows}} 3 \overset{\curvearrowleft}{} \eta$$

with f-orbits $(\alpha\ \beta\ \gamma)$ and $(\eta\ \delta\ \sigma)$. Then $\mathcal{O}(g)$ consists of the g-orbits: $\mathcal{O}(\alpha) = (\alpha)$, $\mathcal{O}(\beta) = (\beta\ \sigma\ \delta\ \gamma)$, $\mathcal{O}(\eta) = (\eta)$. Let $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$ be the weight function with $m_{\mathcal{O}(\alpha)} = 2$, $m_{\mathcal{O}(\beta)} = 1$ and $m_{\mathcal{O}(\eta)} = 2$. Moreover, let $c_\bullet : \mathcal{O}(g) \to K^*$ be an arbitrary parameter function. Write $a = c_{\mathcal{O}(\alpha)}$, $b = c_{\mathcal{O}(\beta)}$, $c = c_{\mathcal{O}(\eta)}$. Then

the associated weighted surface algebra $\Sigma(a,b,c) = \Lambda(S^2, \vec{T}, m_\bullet, c_\bullet)$ is given by the quiver Q and the relations:

$$\alpha\beta = b\beta\sigma\delta, \quad \beta\gamma = a\alpha, \quad \gamma\alpha = b\sigma\delta\gamma, \quad \eta\delta = b\delta\gamma\beta, \quad \delta\sigma = c\eta, \quad \sigma\eta = b\gamma\beta\sigma,$$

$$\alpha\beta\sigma = 0, \quad \gamma\alpha^2 = 0, \quad \eta\delta\gamma = 0, \quad \sigma\eta^2 = 0,$$

$$\alpha^2\beta = 0, \quad \beta\sigma\eta = 0, \quad \eta^2\delta = 0, \quad \delta\gamma\alpha = 0.$$

Observe now that there is an isomorphism of algebras $\psi : \Sigma(1, b(ac)^{\frac{1}{2}}, 1) \to \Sigma(a,b,c)$ given by

$$\psi(\alpha) = (ac)^{\frac{1}{2}}\alpha, \quad \psi(\beta) = a^{-\frac{1}{2}}\beta, \quad \psi(\gamma) = c^{\frac{1}{2}}\gamma, \quad \psi(\eta) = c\eta, \quad \psi(\delta) = \delta, \quad \psi(\sigma) = \sigma.$$

For $\lambda \in K^*$, we set $\Sigma(\lambda) = \Sigma(1, \lambda, 1)$. We observe that $\Sigma(\lambda)$ is isomorphic to the algebra $\Sigma(\lambda)^0$ given by the Gabriel quiver $Q_{\Sigma(\lambda)}$

$$1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \underset{\delta}{\overset{\sigma}{\rightleftarrows}} 3$$

of $\Sigma(\lambda)$ and the induced relations:

$$\beta\gamma\beta = \lambda\beta\sigma\delta, \quad \gamma\beta\gamma = \lambda\sigma\delta\gamma, \quad \delta\sigma\delta = \lambda\delta\gamma\beta, \quad \sigma\delta\sigma = \lambda\gamma\beta\sigma,$$

$$\beta\gamma\beta\sigma = 0, \quad \gamma\beta\gamma\beta\gamma = 0, \quad \delta\sigma\delta\gamma = 0, \quad \sigma\delta\sigma\delta\sigma = 0,$$

$$\beta\gamma\beta\gamma\beta = 0, \quad \delta\sigma\delta\sigma\delta = 0, \quad \delta\gamma\beta\gamma = 0.$$

We note that for any $\lambda \in K^*$, there is an isomorphism of algebras $\theta : T(\lambda)^0 \to \Sigma(-\lambda)^0$ given by $\theta(\beta) = -\beta$, $\theta(\gamma) = \gamma$, $\theta(\sigma) = \sigma$, $\theta(\delta) = \delta$.

We have the following lemma.

Lemma 3.5. *For each $\lambda \in K^*$, the algebras $T(\lambda^{-2})$ and $\Sigma(\lambda)$ are isomorphic.*

Proof. Fix $\lambda \in K^*$. Then there is an isomorphism of algebras $\varphi : T(\lambda^{-2})^0 \to \Sigma(\lambda)^0$ given by $\varphi(\alpha_1) = \lambda\beta$, $\varphi(\beta_1) = \gamma$, $\varphi(\alpha_2) = \sigma$, $\varphi(\beta_2) = \delta$. $\qquad\square$

In particular, we conclude that $\Sigma(1)^0 \cong T(1)^0$ and $\Sigma(-1)^0 \cong T(1)^0$ are not symmetric (even self-injective) algebras. Hence $\Sigma(1)$ and $\Sigma(-1)$ are excluded here.
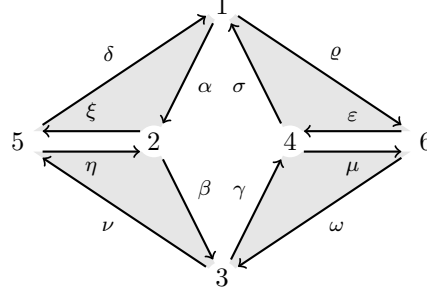
Example 3.6. We will now define *spherical algebras*. Consider the following triangulation $T(2)$ of the sphere S^2 in $\mathbb{R}^3$ (compare [17, Example 7.5])
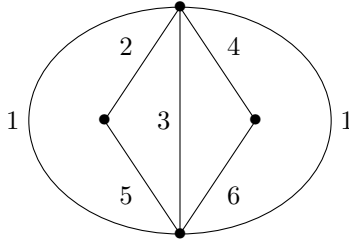
and $\overrightarrow{T(2)}$ is the coherent orientation of triangles in $T(2)$:

$$(1\ 2\ 5), \ (2\ 3\ 5), \ (3\ 4\ 6), \ (4\ 1\ 6).$$

Then the associated triangulation quiver $(Q, f) = (Q(S^2, \overrightarrow{T(2)}), f)$ is of the form

where the four shaded triangles denote the f-orbits. Then g has four orbits

$$\mathcal{O}(\alpha) = (\alpha \; \beta \; \gamma \; \sigma), \qquad \mathcal{O}(\varrho) = (\varrho \; \omega \; \nu \; \delta), \qquad \mathcal{O}(\xi) = (\xi \; \eta), \qquad \mathcal{O}(\mu) = (\mu \; \varepsilon).$$

Let $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$ be the weight function which takes all values 1. Moreover, let $c_\bullet : \mathcal{O}(g) \to K^*$ be an arbitrary parameter function and $a = c_{\mathcal{O}(\alpha)}$, $b = c_{\mathcal{O}(\varrho)}$, $c = c_{\mathcal{O}(\xi)}$, $d = c_{\mathcal{O}(\mu)}$. Then the associated weighted surface algebra $S(a, b, c, d) = \Lambda(S^2, \overrightarrow{T(2)}, m_\bullet^r, c_\bullet)$ is given by the quiver Q and the relations:
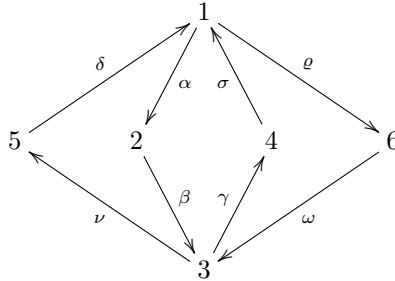
$$
\begin{array}{llllll}
\alpha\xi = b\varrho\omega\nu, & \xi\delta = a\beta\gamma\sigma, & \delta\alpha = c\eta, & \beta\nu = c\xi, & \nu\eta = a\gamma\sigma\alpha, \\
\eta\beta = b\delta\varrho\omega, & \gamma\mu = b\nu\delta\varrho, & \mu\omega = a\sigma\alpha\beta, & \omega\gamma = d\varepsilon, & \sigma\varrho = d\mu, \\
\varrho\varepsilon = a\alpha\beta\gamma, & \varepsilon\sigma = b\omega\nu\delta, & \alpha\xi\eta = 0, & \xi\delta\varrho = 0, & \nu\eta\xi = 0, & \eta\beta\gamma = 0, \\
\gamma\mu\varepsilon = 0, & \mu\omega\nu = 0, & \varrho\varepsilon\mu = 0, & \varepsilon\sigma\alpha = 0, & \beta\gamma\mu = 0, & \sigma\alpha\xi = 0, \\
\delta\varrho\varepsilon = 0, & \omega\nu\eta = 0, & \xi\eta\beta = 0, & \eta\xi\delta = 0, & \mu\varepsilon\sigma = 0, & \varepsilon\mu\omega = 0.
\end{array}
$$

An algebra $S(a, b, c, d)$ with $a, b, c, d \in K^*$ is said to be a *spherical algebra*. We observe now that the algebra $S(a, b, c, d)$ is isomorphic to the algebra $S(abcd, 1, 1, 1)$. Indeed, there is an isomorphism of algebras $\varphi : S(abcd, 1, 1, 1) \to S(a, b, c, d)$ given by

$$
\begin{array}{llll}
\varphi(\alpha) = \alpha, & \varphi(\beta) = \beta, & \varphi(\gamma) = \gamma, & \varphi(\sigma) = \sigma, \\
\varphi(\varrho) = (bc)^{\frac{1}{2}}\varrho, & \varphi(\omega) = (bc)^{\frac{1}{2}}\omega, & \varphi(\nu) = (bd)^{\frac{1}{2}}\nu, & \varphi(\delta) = (bd)^{\frac{1}{2}}\delta, \\
\varphi(\xi) = (bd)^{\frac{1}{2}}c\xi, & \varphi(\eta) = (bd)^{\frac{1}{2}}c\eta, & \varphi(\mu) = (bc)^{\frac{1}{2}}d\mu, & \varphi(\varepsilon) = (bc)^{\frac{1}{2}}d\varepsilon.
\end{array}
$$

For $\lambda \in K^*$, we set $S(\lambda) = S(\lambda, 1, 1, 1)$. A spherical algebra $S(\lambda)$ with $\lambda \in K \setminus \{0, 1\}$ is said to be a *non-singular spherical algebra*, and $S(1)$ the *singular spherical algebra*.

We observe now that a spherical algebra $S(\lambda)$ is isomorphic to the algebra $S(\lambda)^0$ given by the Gabriel quiver $Q_{S(\lambda)}$

of $S(\lambda)$ and the induced relations:

$$\alpha\beta\nu = \varrho\omega\nu, \qquad \beta\nu\delta = \lambda\beta\gamma\sigma, \qquad \nu\delta\alpha = \lambda\gamma\sigma\alpha, \qquad \delta\alpha\beta = \delta\varrho\omega,$$

$$\gamma\sigma\varrho = \nu\delta\varrho, \qquad \sigma\varrho\omega = \lambda\sigma\alpha\beta, \qquad \varrho\omega\gamma = \lambda\alpha\beta\gamma, \qquad \omega\gamma\sigma = \omega\nu\delta,$$

$$\alpha\beta\nu\delta\alpha = 0, \qquad \beta\nu\delta\varrho = 0, \qquad \nu\delta\alpha\beta\nu = 0, \qquad \delta\alpha\beta\gamma = 0,$$

$$\gamma\sigma\varrho\omega\gamma = 0, \qquad \sigma\varrho\omega\nu = 0, \qquad \varrho\omega\gamma\sigma\varrho = 0, \qquad \omega\gamma\sigma\alpha = 0,$$

$$\beta\gamma\sigma\varrho = 0, \qquad \sigma\alpha\beta\nu = 0, \qquad \delta\varrho\omega\gamma = 0, \qquad \omega\nu\delta\alpha = 0,$$

$$\beta\nu\delta\alpha\beta = 0, \qquad \delta\alpha\beta\nu\delta = 0, \qquad \sigma\varrho\omega\gamma\sigma = 0, \qquad \omega\gamma\sigma\varrho\omega = 0.$$

Moreover, a minimal set of relations defining $S(\lambda)^0$ is given by the above eight commutativity relations and the four zero relations:

$$\beta\nu\delta\varrho = 0, \qquad \delta\alpha\beta\gamma = 0, \qquad \sigma\varrho\omega\nu = 0, \qquad \omega\gamma\sigma\alpha = 0.$$

We also note that $S(1)^0$, and hence $S(1)$, is not a symmetric (even self-injective) algebra. Indeed, if $\lambda = 1$, then $\alpha\beta - \varrho\omega$ and $\alpha\beta\nu\delta = \delta\omega\gamma\sigma$ are independent elements of the indecomposable projective module P_1 at the vertex 1, which are annihilated by the radical of $S(1)^0$, and hence are in the socle of P_2. Therefore, we exclude $S(1)$.
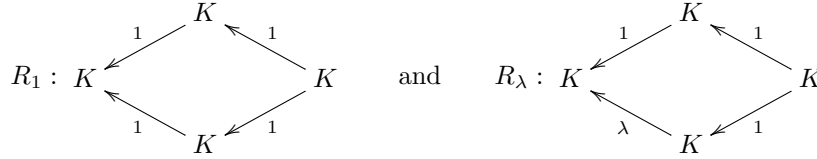
For each $\lambda \in K \setminus \{0, 1\}$, we denote by $C(\lambda)$ the K-algebra given by the quiver
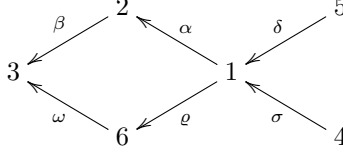
and the relations: $\delta\alpha\beta = \delta\varrho\omega$ and $\delta\varrho\omega = \lambda\sigma\alpha\beta$. We note that $C(\lambda)$ is the double one-point extension algebra of the path algebra $H = K\Delta$ of the quiver Δ
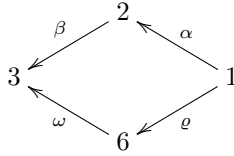
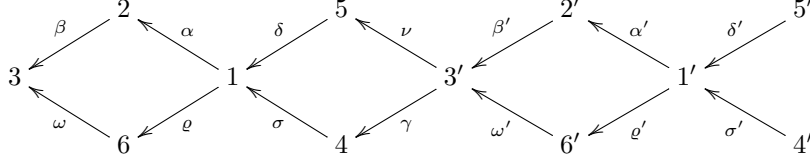of Euclidean type $\widetilde{\mathbb{A}}_3$ by two indecomposable modules

lying on the mouth of stable tubes of rank 1 in Γ_H. For $\lambda \in K \setminus \{0, 1\}$, the modules R_1 and R_λ are not isomorphic, and then $C(\lambda)$ is a tubular algebra of type $(2, 2, 2, 2)$ in the sense of [31], and consequently it is an algebra of polynomial growth. On the other hand, $C(1)$ is a tame algebra of non-polynomial growth (see [29]).

Lemma 3.7. *For any $\lambda \in K \setminus \{0, 1\}$, the algebras $S(\lambda)$ and $\mathrm{T}(C(\lambda))$ are isomorphic.*

Proof. By general theory (see [35]), the trivial extension algebra $\mathrm{T}(C(\lambda))$ is isomorphic to the orbit algebra $\widehat{C(\lambda)}/(\nu_{\widehat{C(\lambda)}})$ of the repetitive category $\widehat{C(\lambda)}$ of $C(\lambda)$

with respect to the infinite cyclic group $(\nu_{\widehat{C(\lambda)}})$ generated by the Nakayama auto-morphism $\nu_{\widehat{C(\lambda)}}$ of $\widehat{C(\lambda)}$. One checks directly that $\widehat{C(\lambda)}$ contains the full convex subcategory given by the quiver

and the relations:

$$\delta\alpha\beta = \delta\varrho\omega, \qquad \sigma\varrho\omega = \lambda\sigma\alpha\beta, \qquad \nu\delta\alpha = \lambda\gamma\sigma\alpha, \qquad \gamma\sigma\varrho = \nu\delta\varrho,$$
$$\beta'\nu\delta = \lambda\beta'\gamma\sigma, \qquad \omega'\gamma\delta = \omega'\nu\delta, \qquad \alpha'\beta'\nu = \varrho'\omega'\nu, \qquad \varrho'\omega'\gamma = \lambda\alpha'\beta'\gamma,$$
$$\beta'\nu\delta\varrho = 0, \qquad \omega'\gamma\delta\alpha = 0, \qquad \delta'\alpha'\beta'\gamma = 0, \qquad \sigma'\varrho'\omega'\nu = 0,$$

where $\nu_{\widehat{C(\lambda)}}(i) = i'$ for any vertex $i \in \{1,2,3,4,5,6\}$ and $\nu_{\widehat{C(\lambda)}}(\theta) = \theta'$ for any arrow $\theta \in \{\alpha, \beta, \omega, \varrho, \delta, \sigma\}$. We conclude that $\mathrm{T}(C(\lambda))$ is isomorphic to the algebra $S(\lambda)^0$, and hence to the spherical algebra $S(\lambda)$. $\qquad \square$

We also note that there is a natural action of the cyclic group G of order 2 on $S(\lambda)^0$ given by the cyclic rotation of vertices and arrows of the quiver $Q_{S(\lambda)}$:

$$(1\ 3), \qquad (2\ 4), \qquad (5\ 6), \qquad (\alpha\ \gamma), \qquad (\beta\ \sigma), \qquad (\varrho\ \nu), \qquad (\omega\ \delta).$$

Then the orbit algebra $S(\lambda)^0/G$ is isomorphic to the basic algebra $T(\lambda)^0$ of the triangle algebra $T(\lambda)$, for any $\lambda \in K \setminus \{0, 1\}$.

We describe now some special properties of the exceptional weighted surface algebras introduced above.

Proposition 3.8. *Let Λ be a non-singular algebra $D(\lambda)$, $\Lambda(\lambda)$, $T(\lambda)$, $S(\lambda)$, $\lambda \in K \setminus \{0, 1\}$. Then the following hold:*

(i) Λ is an algebra of polynomial growth.
(ii) The simple modules in $\mathrm{mod}\,\Lambda$ are periodic of period 4.
(iii) Λ is a periodic algebra of period 4.

Proof. (i) It follows from the above discussion that $\Lambda(\lambda) \cong \mathrm{T}(B(\lambda))$, $D(\lambda) \cong \Lambda(\lambda)/H$, $S(\lambda) \cong \mathrm{T}(C(\lambda))$, $T(\lambda) \cong S(\lambda)/G$, where $B(\lambda)$ and $C(\lambda)$ are tubular algebras of type $(2,2,2,2)$, and G and H are cyclic groups of orders 2 and 3, respectively. Then the fact that Λ is of polynomial growth follows from [34, Theorem].

(ii) It follows from general theory of self-injective algebras of type $(2,2,2,2)$ that all simple modules in $\mathrm{mod}\,\Lambda$ lie in stable tubes of rank 2 in Γ_Λ^s (see [28, Section 3] and [34, Section 3]). Since Λ is a symmetric algebra, we conclude that all simple modules in $\mathrm{mod}\,\Lambda$ are periodic of period 4.

(iii) It has been proved in [2, Proposition 7.1] that $D(\lambda)$ is a periodic algebra of period 4. Then, applying [9, Theorem 3.7], we concluded in [15, Proposition 5.8] that $\Lambda(\lambda)$ is a periodic algebra of period 4. Further, it follows from [24] (see also [31, 5.2(5)]) that the tubular algebras $B(\lambda)$ and $C(\lambda)$ are derived equivalent, and hence their trivial extension algebras $\mathrm{T}(B(\lambda))$ and $\mathrm{T}(C(\lambda))$ are derived equivalent, by [30, Theorem 3.1]. Then, since $\Lambda(\lambda) \cong \mathrm{T}(B(\lambda))$ is a periodic algebra of period 4, we conclude that $S(\lambda) \cong \mathrm{T}(C(\lambda))$ is also a periodic algebra of period 4 (see

[14, Theorem 2.9]). Finally, applying again [9, Theorem 3.7], we infer that $T(\lambda) \cong S(\lambda)/G$ is also a periodic algebra of period 4.   □

Proposition 3.9. *Let Λ be a singular algebra $D(1)$ or $\Lambda(1)$. Then the following hold:*

(i) Λ is a tame algebra of non-polynomial growth.
(ii) $\mathrm{mod}\,\Lambda$ does not have a simple periodic module.
(iii) Λ is not a periodic algebra.

Proof. (i) The fact that $\Lambda(1)$ is tame algebra of non-polynomial growth follows from [16, Theorem 2]. Applying arguments from [16, Section 5], we conclude similarly that the orbit algebra $D(1) \cong \Lambda(1)/H$ is also a tame algebra of non-polynomial growth. For $\Lambda = \Lambda(1)$, the statement (ii) follows from [15, Proposition 6.4]. Let $\Lambda = D(1)$. We note that for the indecomposable projective Λ-modules P_1 and P_2 at vertices 1 and 2, we have $\mathrm{rad}\,P_1/S_1 \cong \mathrm{rad}\,P_2/S_2$, and hence the simple modules S_1 and S_2 are non-periodic. Part (iii) follows from (ii) and general theory (see Theorem IV.11.19 of [36]).   □

4. Properties of general weighted surface algebras

We will first discuss the assumptions, and special cases, and then analyse positions of virtual arrows. We determine a basis of a weighted surface algebra. Then we prove that a weighted surface algebra is, other than the singular triangle, or spherical algebra, is symmetric.

Let $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ be a weighted triangulation algebra.

Remark 4.1. (i) We have excluded Example 3.1, part (2). Namely part (3) of Assumption 2.7 requires that $m_\beta n_\beta \geq 4$ since the arrow α of the example is a virtual loop.

(ii) In Example 3.3 we have that arrows α_3, β_3 are virtual and $m_\alpha n_\alpha = 4$ for any arrow $\alpha \neq \alpha_3, \beta_3$. If one would allow (say) $m_{\alpha_2} = 1$ then by Lemma 4.4 (below) the algebra would not be finite-dimensional. Therefore we exclude it, which is done by condition (2) of Assumption 2.7.

(iii) There are some special cases when certain parameters must be excluded: For the triangle algebra, we exclude 1. In Example 3.4, we must exclude parameters ± 1. For the spherical algebra, we also exclude 1. For these parameters the algebras are not symmetric, this will be proved in Proposition 4.9.

Remark 4.2. We analyse possible configurations near some virtual arrow. As we have already seen, conditions (2) and (3) of 2.7 show that it is not possible that $\alpha, \bar\alpha$ are both virtual, and using that g takes virtual arrows to virtual arrows, also $f^2(\alpha)$ and $f^2(\bar\alpha)$ cannot be both virtual.

(i) Assume $\bar\alpha$ is a virtual loop. Then by the above, no other $f^j(\alpha)$ or $f^j(\bar\alpha)$ is virtual. In fact, the g-cycle of α has length at least three. (If it has length three so that Q has two vertices then we have by condition (3) of 2.7, $m_\alpha \geq 2$.) The arrows $g(\alpha)$ and $g^{-1}(f^2(\bar\alpha))$ are therefore not virtual. We may have that $f(g(\alpha))$ is virtual. If it is a loop then Q is the quiver as in Example 3.4. Otherwise $|Q_0| \geq 5$.

(ii) Now assume $\bar\alpha$ is virtual but not a loop, then g has cycle $(\bar\alpha\ f^2(\alpha))$ and also $f^2(\alpha)$ is virtual. By conditions (2) and (3) of 2.7 no other $f^j(\alpha)$ or $f^j(\bar\alpha)$ is virtual.

Also none of these arrows can be a loop since otherwise Q would not be 2-regular. So Q has a subquiver
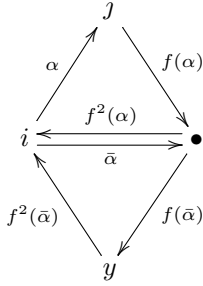
where j and y could be equal. If $j = y$ then $|Q_0| = 3$ and there are no further virtual arrow. If $j \neq y$ there may or may not be loops at j or/and y but they cannot be virtual, for example by part (i).

(iii) We note that if $\bar{\alpha}$ is a loop fixed by g, then in any case $n_\alpha \geq 3$.

We mention a few consequences.

Lemma 4.3. *Let $i \in Q_0$ and let $\alpha, \bar{\alpha}$ be the arrows starting at i.*

(i) Assume $f(\alpha)$ is virtual, then $\bar{\alpha}$ is not virtual.
(ii) If $f^2(\alpha)$ is virtual then $\bar{\alpha}$ is virtual and $g(f(\alpha)) = f(\bar{\alpha})$.
(iii) If $\alpha, \bar{\alpha}$ are double arrows then they are both not virtual.

Proof. (i) We know that $f^2(\alpha)$ is not virtual and then $g(f^2(\alpha)) = \bar{\alpha}$ also is not virtual.

(ii) Let $j = t(\bar{\alpha})$, then $g(f(\alpha), f(\bar{\alpha})$ start at j and also $f^2(\alpha)$ starts at j. Now, $f^2(\alpha)$ is virtual but $g(f(\alpha)), f(\bar{\alpha})$ are not virtual, so they must be equal.

(iii) Assume $\alpha, \bar{\alpha}$ are double arrows. Assume for a contradiction that $\bar{\alpha}$ (say) is virtual. Recall $\bar{\alpha} = g(f^2(\alpha))$, so g has a 2-cycle $(\bar{\alpha} \ f^2(\alpha))$. It follows that $f(\alpha)$ is a loop at $t(\alpha)$. It is necessarily fixed by g and we have a contradiction to 4.2 part (iii). □

The following, already announced, explains why condition (2) of Assumption 2.7 is necessary.

Lemma 4.4. *Suppose there exists a pair of virtual arrows α and $\bar{\alpha}$. Then Λ is not finite-dimensional.*

Proof. By 4.2 and 4.3, the arrows $\alpha, \bar{\alpha}$ are not loops or double arrows. Then using Remark 4.2(ii), we see that Q has a subquiver

$$1 \underset{\alpha}{\overset{f^2(\bar{\alpha})}{\rightleftarrows}} 2 \underset{f^2(\alpha)}{\overset{\bar{\alpha}}{\rightleftarrows}} 3 \ .$$

Since $g(f^2(\alpha)) = \bar{\alpha}$, it follows that $f(\alpha)$ is an arrow $1 \to 3$, and similarly $f(\bar{\alpha})$ is an arrow $3 \to 1$. The subquiver with vertices $1, 2, 3$ is 2-regular and hence it is equal to Q. This is the triangulation quiver as in Example 3.3, and we use the labelling from 3.3, and with this we have

$$g = (\alpha_1 \ \beta_1)(\alpha_2 \ \beta_2)(\alpha_3 \ \beta_3).$$

Assume (say) α_2, β_2 and α_3, β_3 are virtual. If α_1, β_1 are also virtual then we do not have any zero relations and the algebra is not finite-dimensional. Suppose now that α_1, β_1 are not virtual. Then the zero-relations are

$$\alpha_2\alpha_3\beta_3 = 0, \qquad\qquad \beta_3\beta_2\alpha_2 = 0,$$
$$\alpha_3\beta_3\beta_2 = 0, \qquad\qquad \beta_2\alpha_2\alpha_3 = 0.$$

Hence α_1, β_1 do not occur in a zero relation. We have other relations, in particular

$$\alpha_2\alpha_3 = c_{\alpha_1}(\beta_1\alpha_1)^{m-1}\beta_1, \quad \beta_3\beta_2 = c_{\alpha_1}(\alpha_1\beta_1)^{m-1}\alpha_1$$

(where $m = m_{\alpha_1}$), and one checks that they are consistent with the other relations, and do not cause a zero relation for $\beta_1\alpha_1$. It follows that the powers $(\beta_1\alpha_1)^r$ for $r = 1, 2, \ldots$ are linearly independent in Λ and the algebra is not finite-dimensional.

We note that directly imposing nilpotence relations for α_1, β_1 would not produce an algebra as we wish. Namely, suppose we add the relations

$$(\beta_1\alpha_1)^{m-1}\beta_1 = 0, \quad (\alpha_1\beta_1)^{m-1}\alpha_1 = 0.$$

Then the resulting algebra has Gabriel quiver consisting of one isolated vertex together with

$$1 \rightleftarrows 2$$

and the algebra is the product of an algebra of finite type with a 1-dimensional simple algebra. $\qquad\square$

Let $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ be a weighted surface algebra, and $I = I(Q, f, m_\bullet, c_\bullet)$. In order to study properties of Λ, and modules, we work towards specifying a suitable basis of the algebra Λ, defined in terms of cycles of g. In the following, we will as usual identify an element of KQ with its residue class in $\Lambda = KQ/I$. In addition to the elements A_α occuring in the definition, we will also use monomials of length $m_\alpha n_\alpha - 2(\geq 0)$. If α is an arrow, define A'_α by

$$\alpha A'_\alpha = A_\alpha.$$

If α is virtual then A'_α is the idempotent $e_{t(\alpha)}$.

Lemma 4.5. *Let α be an arrow in Q. Then the following hold:*

(i) $B_\alpha \operatorname{rad} \Lambda = 0$.
(ii) B_α *is non-zero.*
(iii) *If α is not virtual then $A_\alpha \operatorname{rad}^2 \Lambda = 0$.*

Proof. (i) We must show that $B_\alpha \bar\alpha = 0$ and $B_\alpha \alpha = 0$ in Λ. It follows from (i) and (iv) of Lemma 2.9 and the relations in Λ that

$$c_\alpha B_\alpha \bar\alpha = \alpha f(\alpha) f^2(\alpha) \bar\alpha.$$

If α is not virtual then $f(\alpha) f^2(\alpha) \bar\alpha = 0$ since $\bar\alpha = g(f^2(\alpha))$. Now assume α is virtual. Then

$$B_\alpha \bar\alpha = \begin{cases} \alpha^2 \bar\alpha & \alpha \text{ is a loop} \\ \alpha g(\alpha) \bar\alpha & \text{else.} \end{cases}$$

In the first case, $\alpha = g(\alpha)$ and $f(\alpha) = \bar\alpha$, and

$$\alpha^2 \bar\alpha = \alpha g(\alpha) f(g(\alpha)) = 0$$

noting that $f(\alpha) = \bar{\alpha}$ cannot be virtual (by the general assumption). In the second case, $g(\alpha) = f^2(\bar{\alpha})$ which is virtual, and then $f(\alpha)$ is not virtual since there would be otherwise two virtual arrows starting at the same vertex. Therefore

$$\alpha g(\alpha)\bar{\alpha} = \alpha f^2(\bar{\alpha})f^3(\bar{\alpha}) = 0.$$

Furthermore, by interchanging the roles of α and $\bar{\alpha}$ we obtain also $c_\alpha B_\alpha \alpha = c_{\bar{\alpha}} B_{\bar{\alpha}} \alpha = 0$.

(ii) This follows from the relations defining Λ.

(iii) We have $A_\alpha f^2(\bar{\alpha}) = B_\alpha$ which is in the socle by the previous. It remains to show that $A_\alpha g(f(\bar{\alpha})) = 0$. By the relations this is a non-zero scalar multiple of $\bar{\alpha}f(\bar{\alpha})g(f(\bar{\alpha}))$. Since $f^2(\bar{\alpha})$ is in the g-orbit of α, it is not virtual by the assumption. Hence by the relation (2) of the definition we have $\bar{\alpha}f(\bar{\alpha})g(f(\bar{\alpha})) = 0$ as required.
$\square$

Remark 4.6. We can motivate the zero relations of Definition 2.8, and also see that the relations give rise to zero conditions.

(i) Consider $\alpha f(\alpha)g(f(\alpha))$ when $f^2(\alpha)$ is virtual. Then also $g(f^2(\alpha))$ is virtual but $g(f^2(\alpha)) = \bar{\alpha}$, and then $g(f(\alpha)) = f(\bar{\alpha})$. So we have

$$\alpha f(\alpha)g(f(\alpha)) = c_{\bar{\alpha}}\bar{\alpha}g(f(\alpha)) = c_{\bar{\alpha}}\bar{\alpha}f(\bar{\alpha}) = c_{\bar{\alpha}}c_\alpha A_\alpha,$$

and A_α is non-zero, because B_α is non-zero. Since $\bar{\alpha}$ is virtual, α is not virtual (see Assumption 2.7). So $A_\alpha \operatorname{rad}^2 \Lambda = 0$, by Lemma 4.5.

(ii) In the original version, the relation $\alpha g(\alpha)f(g(\alpha)) = 0$ in Λ is a consequence of the definition. This is now not the case, and we must add but not always. Suppose $f(\alpha)$ is virtual. By relation (1) we have

$$\alpha g(\alpha)f(g(\alpha)) = \alpha c_{f(\alpha)}A_{f(\alpha)} = c_{f(\alpha)}c_{\bar{\alpha}}A_{\bar{\alpha}},$$

and $A_{\bar{\alpha}}$ is non-zero, because $B_{\bar{\alpha}}$ is non-zero. We know $\bar{\alpha}$ is not virtual (since $f(\alpha)$ is virtual, see Lemma 4.3). So again $A_{\bar{\alpha}} \operatorname{rad}^2 \Lambda = 0$.

Lemma 4.5 only shows that $\langle B_\alpha \rangle \subseteq \operatorname{soc} \Lambda$. We will now prove that equality holds. On the way, we see that for some of the algebras certain parameters need to be excluded.

Lemma 4.7.   (i)   Assume α starting at i is virtual. Then $\operatorname{soc}_2(e_i\Lambda)$ is generated by $A_{\bar{\alpha}}$. The module $e_i\Lambda$ has basis

$$\{e_i, \bar{\alpha}, \bar{\alpha}g(\bar{\alpha}), \ldots, \ A_{\bar{\alpha}}, B_{\bar{\alpha}}, \ \bar{\alpha}f(\bar{\alpha})\}.$$

(ii)   Assume $\alpha, \bar{\alpha}$ are not virtual and $\operatorname{soc}(e_i\Lambda) = \langle B_\alpha \rangle$. Then $e_i\Lambda$ has basis all proper initial submonomials of B_α and $B_{\bar{\alpha}}$ together with e_i and B_α.

Proof. (i) Let α be a virtual arrow starting at i, then $\bar{\alpha}$ is not virtual (see 2.7). Since virtual arrows are unions of g-cycles, no virtual arrow occurs as a factor of $B_{\bar{\alpha}}$. We express B_α in terms of arrows of the Gabriel quiver. If α is a loop then we get

$$B_\alpha = \alpha^2 = (c_\alpha^{-1}\bar{\alpha}f(\bar{\alpha}))^2.$$

Otherwise we get

$$B_\alpha = \alpha g(\alpha) = (c_\alpha^{-1}\bar{\alpha}f(\bar{\alpha}))(c_{g(\alpha)}^{-1}f(\alpha)f^2(\alpha)).$$

We claim that $e_i\Lambda$ has basis

$$\{e_i, \bar{\alpha}, \bar{\alpha}g(\bar{\alpha}), \ldots, \ A_{\bar{\alpha}}, B_{\bar{\alpha}}, \ \bar{\alpha}f(\bar{\alpha})\}.$$

Assume first α is a loop. Then

$$\bar{\alpha}f(\bar{\alpha})\bar{\alpha} = c_\alpha \alpha\bar{\alpha} = c_\alpha \alpha f(\alpha) = c_\alpha c_{\bar{\alpha}} A_{\bar{\alpha}}$$

and it follows that the set spans $e_i\Lambda$. Suppose α is not a loop then

$$\bar{\alpha}f(\bar{\alpha})f(\alpha) = c_\alpha \alpha f(\alpha) = c_\alpha c_{\bar{\alpha}} A_{\bar{\alpha}}$$

and the set spans $e_i\Lambda$. One checks that the set is linearly independent.

(ii) The set is a spanning set, by the relations. Using the assumption on the socle, one checks that it is linearly independent. □

We will now show that the condition on the socle in part (b) is satisfied except for those algebras which we have excluded anyway, in particular this will give us the required basis. For the proof, we use the following preparation.

Lemma 4.8. *Assume $\alpha, \bar{\alpha}$ start at vertex i and are both not virtual. Then A_α and $A_{\bar{\alpha}}$ are linearly independent in Λ.*

Proof. Assume (for a contradiction) that for some $t \in K$ we have

$$(*) \qquad\qquad A_\alpha + tA_{\bar{\alpha}} \in I.$$

We have $c_{\bar{\alpha}} A_{\bar{\alpha}} - \alpha f(\alpha) \in I$ and $c_\alpha A_\alpha - \bar{\alpha}f(\bar{\alpha}) \in I$. Note that arrows along $A_\alpha, A_{\bar{\alpha}}$ are not virtual. If an identity $(*)$ exists, and given that $\alpha, \bar{\alpha}$ are not virtual, it follows that at least one of $f(\alpha)$ or $f(\bar{\alpha})$ is virtual.

The last arrows of A_α and $A_{\bar{\alpha}}$ end at $t(f(\bar{\alpha}))$ and $t(f(\alpha))$, which must be the same (by $(*)$). Suppose (say) $f(\alpha)$ is virtual, then also $g(f(\alpha))$ is virtual. The arrows starting at $t(f(\alpha))$ are $f^2(\alpha)$ and $f^2(\bar{\alpha})$, and $g(f(\alpha)) \neq f^2(\alpha)$. So $g(f(\alpha)) = f^2(\bar{\alpha})$. That is, $f^2(\bar{\alpha})$ is virtual, and then also $g(f^2(\bar{\alpha}))$. But $g(f^2(\bar{\alpha})) = \alpha$ and α is not virtual by assumption, a contradiction. Similarly $f(\bar{\alpha})$ cannot be virtual. Hence there is no identity $(*)$. □

Proposition 4.9. *Assume $e_i\Lambda e_j$ has an element ζ with $\zeta J = 0$ but $\zeta \notin \langle B_\alpha \rangle$. Then Λ is isomorphic to the singular triangular algebra $T(1)$ or the singular spherical algebra $S(1)$. Conversely, $T(1)$ and $S(1)$ have this property.*

We split the proof into three parts, first a reduction, and then two lemmas.

For the reduction, if one of the arrows starting at i is virtual, then no such ζ exists, we can see this from the basis of $e_i\Lambda$. So assume the arrows α and $\bar{\alpha}$ starting at i are both not virtual. Then there are no virtual arrows occuring in $B_\alpha, B_{\bar{\alpha}}$ and we get a spanning set of $e_i\Lambda$ consisting of initial subwords of $B_\alpha, B_{\bar{\alpha}}$ (in particular the last arrows which are $f^2(\bar{\alpha})$ and $f^2(\alpha)$ are also not virtual). Then we can write ζ in terms of the spanning set as

$$\zeta = \zeta_1 + a\zeta_2 \in e_i\Lambda e_j$$

with $a \in K$ and ζ_1 a linear combination of paths along B_α and ζ_2 a linear combination of paths along $B_{\bar{\alpha}}$. Then the lowest terms of ζ_1, ζ_2 satisfy the same property, so we may assume that the ζ_i are monomials. They are then two paths from i to j, and are linearly independent in Λ.

Lemma 4.10. *Assume $\alpha, \bar{\alpha}$ are not virtual. Let $\zeta = \zeta_1 + a\zeta_2$ in $e_i\Lambda e_j$, such that $\zeta \operatorname{rad} \Lambda = 0$ but $\zeta \notin \langle B_\alpha \rangle$. Assume ζ_1 is a monomial along B_α and ζ_2 is a monomial along $B_{\bar{\alpha}}$. Then*

$$\zeta_1 = \alpha g(\alpha), \quad \zeta_2 = \bar{\alpha}g(\bar{\alpha}),$$

and moreover $f(\alpha)$ and $f(\bar{\alpha})$ are virtual.

Proof. Write δ_i for the last arrow in ζ_i. Let $\beta = g(\delta_1)$, this is not virtual and it starts at j. Then $\zeta_1\beta$ is an initial submonomial of B_α and it occurs in some relation. Therefore $\zeta_1\beta = A_\alpha$ or possibly B_α.

CLAIM: $\zeta_1\beta \neq B_\alpha$, that is $\zeta_1 \neq A_\alpha$. Assume $\zeta_1 = A_\alpha$, then $\beta = f^2(\bar{\alpha})$. We also have either $\beta = g(\delta_2)$ or $\beta = f(\delta_2)$ and we will show that both lead to contradictions.

(i) Suppose $\beta = g(\delta_2)$, then $\zeta_2\beta$ is a monomial along $B_{\bar{\alpha}}$, and occurs in a relation. So it is either $A_{\bar{\alpha}}$ or $B_{\bar{\alpha}}$ and then is B_α. But then $\beta = f^2(\alpha)$ and $f^2(\bar{\alpha}) = f^2(\alpha)$, a contradiction.

(ii) Suppose $\beta = f(\delta_2)$, then $\delta_2 = f(\bar{\alpha})$ and we have

$$\zeta_2\beta = \zeta_2' f(\bar{\alpha}) f^2(\bar{\alpha}) = \zeta_2' \cdot c_{g(\bar{\alpha})} A_{g(\bar{\alpha})}$$

and ζ_2' must be equal to $\bar{\alpha}$. However then we have $\zeta_2 = \bar{\alpha}\delta_2 = \bar{\alpha}f(\bar{\alpha})$ which is not a path along a g-cycle, a contradiction. So we must have that $\zeta_1\beta = A_\alpha$.

CLAIM: $g(\delta_2) = \bar{\beta}$ (and is not virtual). Assume not, then $g(\delta_2) = \beta$. This means that $\zeta_2\beta$ is an initial submonomial of $B_{\bar{\alpha}}$ and occurs in a relation. So it must be $A_{\bar{\alpha}}$ or $B_{\bar{\alpha}}$, and then by using the previous argument it must be $A_{\bar{\alpha}}$. But it is also a scalar multiple of A_α, and this contradicts Lemma 4.8. So $g(\delta_2) = \bar{\beta}$.

We can now use the previous argument again, and get that $\zeta_2\bar{\beta} = A_{\bar{\alpha}}$.

As well $\beta = f(\delta_2)$. Therefore

$$\zeta_2\beta = \zeta_2'\delta_2 f(\delta_2) = c_{\bar{\delta}_2} \zeta_2' A_{\bar{\delta}_2}.$$

We claim that $\bar{\delta}_2$ (and $\bar{\delta}_1$) are virtual.

Suppose $\bar{\delta}_2$ is not virtual, then $A_{\bar{\delta}_2}$ is in the second socle. It follows that $\zeta_2' = e_i$ and $\zeta_2 = \bar{\alpha}$, and $\beta = f(\bar{\alpha})$. This implies $f(\beta) = f^2(\bar{\alpha})$. On the other hand, β is the last arrow of A_α and therefore $g(\beta) = f^2(\bar{\alpha})$, that is $f(\beta) = g(\beta)$, a contradiction. The same reasoning using ζ_1 and $\bar{\beta} = f(\delta_1)$ shows that $\bar{\delta}_1$ is virtual.

Let $\eta = g^{-1}(\delta_2) = f^{-1}(\bar{\delta}_2)$. We have

$$(*) \qquad \zeta_2\beta = c_{\bar{\delta}}\zeta_2'\bar{\delta}_2 = c_{\bar{\delta}}\zeta_2''\eta f(\eta) = c_{\bar{\delta}_2} c_{\bar{\eta}}\zeta_2'' A_{\bar{\eta}}$$

and this is a scalar multiple of $A_\alpha = \zeta_1\beta$.

The arrow $\bar{\eta}$ is not virtual. If $\bar{\delta}_2$ is a virtual loop then g has cycle $(\eta \, \delta_2 \, \bar{\eta} \, \ldots)$ and $\bar{\eta}$ is not virtual. If $\bar{\delta}_2$ is virtual but not a loop, we see that f has cycle $(\eta \, \bar{\delta}_2 \, f^2(\eta))$ and $f^2(\eta)$ is not virtual. In this case $\bar{\eta} = g(f^2(\eta))$ and hence it is not virtual.

Now $A_{\bar{\eta}}$ is in the second socle, and therefore $\zeta_2'' = e_i$ and $\eta = g^{-1}(\delta_2) = \bar{\alpha}$, and $\delta_2 = g(\bar{\alpha})$. We have proved $\zeta_2 = \bar{\alpha}g(\bar{\alpha})$, and it follows that $f(\bar{\alpha}) = \bar{\delta}_2$ and we have proved that it is virtual.

The same reasoning, for ζ_1 and $\bar{\beta}$ shows $\zeta_1 = \alpha g(\alpha)$ and it follows that $f(\alpha) = \bar{\delta}_1$, so it is virtual. $\square$

Lemma 4.11. *Assume $\zeta = \zeta_1 + a\zeta_2$ with $\zeta J = 0$ but $\zeta \notin \langle B_\alpha \rangle$. Assume $\zeta_1 = \alpha g(\alpha)$ and $\zeta_2 = \bar{\alpha}g(\bar{\alpha})$, and moreover $f(\alpha)$ and $f(\bar{\alpha})$ are virtual, and $A_\alpha, A_{\bar{\alpha}}$ have length three. Then Λ is isomorphic to $T(1)$ or $S(1)$.*

Proof. Since A_α and $A_{\bar{\alpha}}$ have length 3, we have $m_\alpha n_\alpha = 4 = m_{\bar{\alpha}} n_{\bar{\alpha}}$.

CASE 1. Assume $i = j$. Then the arrows ending at i are $\{g(\alpha), g(\bar{\alpha})\} = \{f^2(\alpha), f^2(\bar{\alpha})\}$. So we have two cases to consider.

(a) Assume $g(\alpha) = f^2(\bar{\alpha})$ and $g(\bar{\alpha}) = f^2(\alpha)$. Let $x = t(\bar{\alpha})$ and $y = t(\alpha)$. Then $f(\bar{\alpha})$ is an arrow $x \to y$ and $f(\alpha)$ is an arrow $y \to x$. The full subquiver with vertices i, x, y is 2-regular and hence is equal to Q. Moreover we have $g = (\alpha \ f^2(\bar{\alpha}))(\bar{\alpha} \ f^2(\alpha))(f(\alpha) \ f(\bar{\alpha}))$ and hence $m_\alpha = 2$ and $m_{\bar{\alpha}} = 2$. Also $f = (\alpha \ f(\alpha) \ \delta_2)(\bar{\alpha} \ f(\bar{\alpha}) \ \delta_1)$, so these are precisely the data for the triangle algebra (Example 3.3), with the quiver
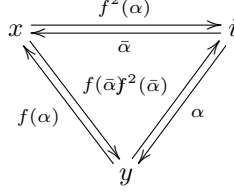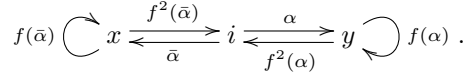

Moreover, it follows from Example 3.3 that we may take (up to algebra isomorphism) $c_{\bar{\alpha}} = \lambda$, $c_\alpha = 1$, $c_{f(\alpha)} = 1$, for some $\lambda \in K^*$. Since $\zeta J = 0$, we obtain the equalities

$$0 = \big(\alpha f^2(\bar{\alpha}) + a\bar{\alpha} f^2(\alpha)\big)\alpha = \alpha f^2(\bar{\alpha})\alpha + a\bar{\alpha} f^2(\alpha)\alpha = \bar{\alpha} f(\bar{\alpha}) + a\bar{\alpha} f(\bar{\alpha})$$
$$= (1 + a)\bar{\alpha} f(\bar{\alpha}),$$
$$0 = \big(\alpha f^2(\bar{\alpha}) + a\bar{\alpha} f^2(\alpha)\big)\bar{\alpha} = \alpha f^2(\bar{\alpha})\bar{\alpha} + a\bar{\alpha} f^2(\alpha)\bar{\alpha} = \alpha f(\alpha) + a\bar{\alpha} f^2(\alpha)\bar{\alpha}$$
$$= (1 + a\lambda^{-1})\alpha f(\alpha),$$

and hence $a = -1$ and $\lambda = 1$. In particular, we conclude that Λ is isomorphic to $T(1)$.

(b) Assume $g(\alpha) = f^2(\alpha)$ and $g(\bar{\alpha}) = f^2(\bar{\alpha})$. Then $f(\alpha)$ and $f(\bar{\alpha})$ are virtual loops and again Q has three vertices, and moreover $g = (\alpha \ f^2(\alpha) \ \bar{\alpha} \ f^2(\bar{\alpha}))(f(\alpha))(f(\bar{\alpha}))$ and $m_\alpha = 1$. These are the data which determine the algebra Σ introduced in Example 3.4, with the quiver

Moreover, it follows from Example 3.4 that we may take (up to algebra isomorphism) $c_\alpha = \lambda$, $c_{f(\alpha)} = 1$, $c_{f(\bar{\alpha})} = 1$, for some $\lambda \in K^*$, that is Λ is isomorphic to $\Sigma(\lambda)$. Since $\zeta J = 0$, we obtain the equalities

$$0 = \big(\alpha f^2(\alpha) + a\bar{\alpha} f^2(\bar{\alpha})\big)\alpha = \alpha f^2(\alpha)\alpha + a\bar{\alpha} f^2(\bar{\alpha})\alpha = \alpha f(\alpha) + a\lambda^{-1}\alpha f(\alpha)$$
$$= (1 + a\lambda^{-1})\alpha f(\alpha),$$
$$0 = \big(\alpha f^2(\alpha) + a\bar{\alpha} f^2(\bar{\alpha})\big)\bar{\alpha} = \alpha f^2(\alpha)\bar{\alpha} + a\bar{\alpha} f^2(\bar{\alpha})\bar{\alpha} = \lambda^{-1}\bar{\alpha} f(\bar{\alpha}) + a\bar{\alpha} f(\bar{\alpha})$$
$$= (\lambda^{-1} + a)\bar{\alpha} f(\bar{\alpha}),$$

and hence $a = -\lambda^{-1}$ and $\lambda^2 = 1$. In particular, we conclude that Λ is isomorphic to $\Sigma(1) \cong \Sigma(-1)$, and consequently to $T(1)$ (see Example 3.4).

CASE 2. Assume $i \neq j$. Note first that $\alpha, f^2(\alpha)$ and $\bar{\alpha}, f^2(\bar{\alpha})$ cannot be loops by 4.2 and 4.3.

We claim that arrows $\alpha, g(\alpha), f(\alpha)$ and $f^2(\alpha)$ are pairwise distinct. Clearly $f(\alpha)$ is different from the other three since it is the only one of these which is virtual. If $\alpha = g(\alpha)$ then α ends at j and $g(\alpha)$ is a loop, and α is a loop, a contradiction. Clearly $\alpha \neq f^2(\alpha)$ since otherwise $f(\alpha) = \alpha$. Finally $f^2(\alpha)$ ends at i and $g(\alpha)$ does

not end at i. We observe that $f(\alpha)$ is not a (virtual) loop. Otherwise we would have $f^2(\alpha) = g(\alpha)$ which we had excluded. Similarly $f(\bar\alpha)$ is not a loop. Similarly, we show that $\bar\alpha$, $g(\bar\alpha)$, $f(\bar\alpha)$ and $f^2(\bar\alpha)$ are pairwise distinct.

Let $x = t(f(\alpha))$ and $y = t(f(\bar\alpha))$. Since $m_\alpha n_\alpha = 4$ we have the arrow $g^2(\alpha) : j \to y$. Similarly $g^2(\bar\alpha) : j \to x$.

It follows that $g^2(\alpha) = f(g(\bar\alpha))$ and $g^2(\bar\alpha) = f(g(\alpha))$, and $f(g^2(\alpha)) = g(f(\bar\alpha))$ is virtual, and $f(g^2(\bar\alpha)) = g(f(\alpha))$ is virtual. By 4.2 and 4.3, the arrows $g^2(\alpha)$ and $g^2(\bar\alpha)$ are not loops. Hence Q is the quiver
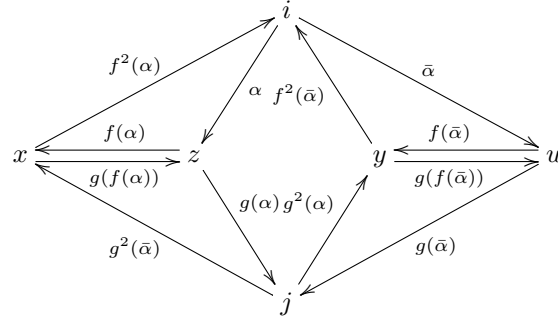
of a spherical algebra, with two pairs of virtual arrows, and in fact the data are those for the spherical algebra, namely $m_\alpha = 1$ and $m_{\bar\alpha} = 1$. Moreover, it follows from Example 3.6 that we may take (up to algebra isomorphism) $c_\alpha = \lambda$, $c_{\bar\alpha} = 1$, $c_{f(\alpha)} = 1$, $c_{f(\bar\alpha)} = 1$, for some $\lambda \in K^*$. Now, since $\zeta J = 0$, we obtain the equalities

$$0 = \big(\alpha g(\alpha) + a\bar\alpha g(\bar\alpha)\big)g^2(\alpha) = \alpha g(\alpha)g^2(\alpha) + a\bar\alpha g(\bar\alpha)g^2(\alpha)$$
$$= \lambda^{-1}\bar\alpha f(\bar\alpha) + a\bar\alpha f(\bar\alpha) = (\lambda^{-1} + a)\bar\alpha f(\bar\alpha),$$
$$0 = \big(\alpha g(\alpha) + a\bar\alpha g(\bar\alpha)\big)g^2(\bar\alpha) = \alpha g(\alpha)g^2(\bar\alpha) + a\bar\alpha g(\bar\alpha)g^2(\bar\alpha) = \alpha f(\alpha) + a\alpha f(\alpha)$$
$$= (1 + a)\alpha f(\alpha),$$

and hence $a = -1$ and $\lambda = 1$. In particular, we conclude that Λ is isomorphic to $S(1)$. $\qquad\square$

Since we exclude these algebras, Lemma 4.7 shows that we have a basis of $e_i\Lambda$ in terms of initial submonomials of B_α and $B_{\bar\alpha}$. Hence we get the expected formula for the dimension.

Corollary 4.12. *Let i be a vertex of Q and $\alpha, \bar\alpha$ the two arrows in Q with source i. Then $\dim_K e_i\Lambda = m_\alpha n_\alpha + m_{\bar\alpha} n_{\bar\alpha}$.*

Proposition 4.13. *Let (Q, f) be a triangulation quiver, $m_\bullet$ and $c_\bullet$ weight and parameter functions of (Q, f), and $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$. Then the following statements hold:*

(i) Λ is finite-dimensional with $\dim_K \Lambda = \sum_{\mathcal{O} \in \mathcal{O}(g)} m_{\mathcal{O}} n_{\mathcal{O}}^2$.

(ii) Λ is a symmetric algebra, except when Λ is the singular triangle, or spherical algebra.

Proof. Let $I = I(Q, f, m_\bullet, c_\bullet)$.

(i) It follows from Corollary 4.12 that, for each vertex i of Q, the indecomposable projective right Λ-module P_i at the vertex i has the dimension $\dim_K P_i = m_\alpha n_\alpha +$

$m_{\bar{\alpha}} n_{\bar{\alpha}}$, where $\alpha, \bar{\alpha}$ are the two arrows in Q with source i. Then we get

$$\dim_K \Lambda = \sum_{\mathcal{O} \in \mathcal{O}(g)} m_{\mathcal{O}} n_{\mathcal{O}}^2.$$

(ii) For each vertex $i \in Q_0$, we denote by $\mathcal{B}_i$ the basis of $e_i \Lambda$ consisting of e_i, all initial subwords of A_α and $A_{\bar{\alpha}}$, and $\omega_i = c_\alpha B_\alpha = c_{\bar{\alpha}} B_{\bar{\alpha}}$ (see Lemma 4.7 and Corollary 4.12). We know that ω_i generates the socle of $e_i \Lambda$. Then $\mathcal{B} = \bigcup_{i \in Q_0} \mathcal{B}_i$ is a K-linear basis of Λ. We defined a symmetrizing K-linear form $\varphi : \Lambda \to K$ which assigns to the coset $u + I$ of a path u in Q the element in K

$$\varphi(u + I) = \begin{cases} c_\alpha^{-1} & \text{if } u = \mathcal{B}_\alpha \text{ for an arrow } \alpha \in Q_1, \\ 0 & \text{if } u \in \mathcal{B} \text{ otherwise} \end{cases}$$

and extending linearly. $\qquad\square$

5. Periodicity of weighted surface algebras

In this section we will prove that every weighted surface algebra with at least two simple modules, not isomorphic to a disc, triangle, tetrahedral, spherical algebra, is a periodic algebra of period 4.

We recall briefly what we need from [15], for proofs and details we refer to [15]. Let $A = KQ/I$ be a bound quiver algebra, and let A^e be the enveloping algebra. Then the $(e_i \otimes e_j)A^e = A(e_i \otimes e_j)A$ for $i, j \in Q_0$, form a complete set of pairwise non-isomorphic indecomposable projective modules in $\mathrm{mod}\, A^e$ (see [36, Proposition IV.11.3]).

The following result by Happel [23, Lemma 1.5] describes the terms of a minimal projective resolution of A in $\mathrm{mod}\, A^e$.

Proposition 5.1. *Let $A = KQ/I$ be a bound quiver algebra, where Q is the Gabriel quiver of A. Then there is in $\mathrm{mod}\, A^e$ a minimal projective resolution of A of the form*

$$\cdots \to \mathbb{P}_n \xrightarrow{d_n} \mathbb{P}_{n-1} \to \cdots \to \mathbb{P}_1 \xrightarrow{d_1} \mathbb{P}_0 \xrightarrow{d_0} A \to 0,$$

where

$$\mathbb{P}_n = \bigoplus_{i,j \in Q_0} A(e_i \otimes e_j) A^{\dim_K \mathrm{Ext}_A^n(S_i, S_j)}$$

for any $n \in \mathbb{N}$.

We will need details for the first three differentials. We have

$$\mathbb{P}_0 = \bigoplus_{i \in Q_0} A e_i \otimes e_i A.$$

The homomorphism $d_0 : \mathbb{P}_0 \to A$ in $\mathrm{mod}\, A^e$ defined by $d_0(e_i \otimes e_i) = e_i$ for all $i \in Q_0$ is a minimal projective cover of A in $\mathrm{mod}\, A^e$. Recall that, for two vertices i and j in Q, the number of arrows from i to j in Q is equal to $\dim_K \mathrm{Ext}_A^1(S_i, S_j)$ (see [1, Lemma III.2.12]). Hence we have

$$\mathbb{P}_1 = \bigoplus_{\alpha \in Q_1} A e_{s(\alpha)} \otimes e_{t(\alpha)} A.$$

Then we have the following known fact (see [2, Lemma 3.3] for a proof).

Lemma 5.2. *Let $A = KQ/I$ be a bound quiver algebra, and $d_1 : \mathbb{P}_1 \to \mathbb{P}_0$ the homomorphism in $\operatorname{mod} A^e$ defined by*

$$d_1(e_{s(\alpha)} \otimes e_{t(\alpha)}) = \alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha$$

for any arrow α in Q. Then d_1 induces a minimal projective cover $d_1 : \mathbb{P}_1 \to \Omega^1_{A^e}(A)$ of $\Omega^1_{A^e}(A) = \operatorname{Ker} d_0$ in $\operatorname{mod} A^e$. In particular, we have $\Omega^2_{A^e}(A) \cong \operatorname{Ker} d_1$ in $\operatorname{mod} A^e$.

For the algebras A we will consider, the kernel $\Omega^2_{A^e}(A)$ of d will be generated, as an A-A-bimodule, by some elements of $\mathbb{P}_1$ associated to a set of relations generating the admissible ideal I. Recall that a relation in the path algebra KQ is an element of the form

$$\mu = \sum_{r=1}^{n} c_r \mu_r,$$

where $c_1, \ldots, c_r$ are non-zero elements of K and $\mu_r = \alpha_1^{(r)} \alpha_2^{(r)} \ldots \alpha_{m_r}^{(r)}$ are paths in Q of length $m_r \geq 2$, $r \in \{1, \ldots, n\}$, having a common source and a common target. The admissible ideal I can be generated by a finite set of relations in KQ (see [1, Corollary II.2.9]). In particular, the bound quiver algebra $A = KQ/I$ is given by the path algebra KQ and a finite number of identities $\sum_{r=1}^{n} c_r \mu_r = 0$ given by a finite set of generators of the ideal I. Consider the K-linear homomorphism $\varrho : KQ \to \mathbb{P}_1$ which assigns to a path $\alpha_1 \alpha_2 \ldots \alpha_m$ in Q the element

$$\varrho(\alpha_1 \alpha_2 \ldots \alpha_m) = \sum_{k=1}^{m} \alpha_1 \alpha_2 \ldots \alpha_{k-1} \otimes \alpha_{k+1} \ldots \alpha_m$$

in $\mathbb{P}_1$, where $\alpha_0 = e_{s(\alpha_1)}$ and $\alpha_{m+1} = e_{t(\alpha_m)}$. Observe that $\varrho(\alpha_1 \alpha_2 \ldots \alpha_m) \in e_{s(\alpha_1)} \mathbb{P}_1 e_{t(\alpha_m)}$. Then, for a relation $\mu = \sum_{r=1}^{n} c_r \mu_r$ in KQ lying in I, we have an element

$$\varrho(\mu) = \sum_{r=1}^{n} c_r \varrho(\mu_r) \in e_i \mathbb{P}_1 e_j,$$

where i is the common source and j is the common target of the paths $\mu_1, \ldots, \mu_r$. The following lemma shows that relations always produce elements in the kernel of d_1; the proof is straightforward.

Lemma 5.3. *Let $A = KQ/I$ be a bound quiver algebra and $d_1 : \mathbb{P}_1 \to \mathbb{P}_0$ the homomorphism in $\operatorname{mod} A^e$ defined in Lemma 5.2. Then for any relation μ in KQ lying in I, we have $d_1(\varrho(\mu)) = 0$.*

For an algebra $A = KQ/I$ in our context, we will define a family of relations $\mu^{(1)}, \ldots, \mu^{(q)}$ such that the associated elements $\varrho(\mu^{(1)}), \ldots, \varrho(\mu^{(q)})$ generate the A-A-bimodule $\Omega^2_{A^e}(A) = \operatorname{Ker} d_1$. In fact, using Lemma 5.3, we will show that

$$\mathbb{P}_2 = \bigoplus_{j=1}^{q} A e_{s(\mu^{(j)})} \otimes e_{t(\mu^{(j)})} A,$$

and the homomorphism $d_2 : \mathbb{P}_2 \to \mathbb{P}_1$ in $\operatorname{mod} A^e$ such that

$$d_2\big(e_{s(\mu^{(j)})} \otimes e_{t(\mu^{(j)})}\big) = \varrho(\mu^{(j)}),$$

for $j \in \{1, \ldots, q\}$, defines a projective cover of $\Omega^2_{A^e}(A)$ in $\operatorname{mod} A^e$. In particular, we have $\Omega^3_{A^e}(A) \cong \operatorname{Ker} d_2$ in $\operatorname{mod} A^e$. We will denote this homomorphism d_2 by R. The differential $S := d_3 : \mathbb{P}_3 \to \mathbb{P}_2$ will be defined later.

Now we fix a weighted surface algebra $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ for a triangulation quiver (Q, f) with at least two vertices, a weight function $m_\bullet$ and a parameter function $c_\bullet$. Moreover, we assume that Λ is not a (non-singular) tetrahedral, or disc, or triangle, or spherical algebra (they are already dealt with in Proposition 3.8).

We fix a vertex i of Q, and we show that the simple module S_i is periodic of period four. Let α and $\bar{\alpha}$ be the arrows starting at i.

Proposition 5.4. *Assume that the arrows $\alpha, \bar{\alpha}$ are not virtual. Then there is an exact sequence in $\mathrm{mod}\,\Lambda$*

$$0 \to S_i \to P_i \xrightarrow{\pi_3} P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))} \xrightarrow{\pi_2} P_{t(\alpha)} \oplus P_{t(\bar{\alpha})} \xrightarrow{\pi_1} P_i \to S_i \to 0,$$

which give rise to a minimal projective resolution of S_i in $\mathrm{mod}\,\Lambda$. In particular, S_i is a periodic module of period 4.

If the arrows α and $\bar{\alpha}$ are both not virtual then this is Proposition 7.1 of [15].

Now assume that $\bar{\alpha}$ is a virtual loop, then α is not virtual. Note that by Assumption 2.7 we have $m_\alpha n_\alpha \geq 4$. The quiver Q contains a subquiver

$$\bar{\alpha} \,\circlearrowright\, i \underset{f(\alpha)}{\overset{\alpha}{\rightleftarrows}} j$$

and f has a cycle $(\bar{\alpha}\ \alpha\ f(\alpha))$. Let γ be the other arrow starting at vertex j, and δ be the other arrow ending at j.

Lemma 5.5. *There is an exact sequence of Λ-modules*

$$0 \to \Omega^{-1}(S_i) \to P_j \to P_j \to \Omega(S_i) \to 0$$

which gives rise to a periodic minimal projective resolution of S_i in $\mathrm{mod}\Lambda$. In particular S_i is periodic of period 4.

Proof. We have $\Omega(S_i) = \alpha\Lambda$, and we take $\Omega^2(S_i)$ as

$$\Omega^2(S_i) = \{x \in e_j\Lambda \mid \alpha x = 0\}.$$

We have the following relations in Λ:
   (i) $\alpha f(\alpha) = c_{\bar{\alpha}}\bar{\alpha}$,
   (ii) $\bar{\alpha}\alpha = c_\alpha A_\alpha$.
Hence $c_\alpha A_\alpha = \bar{\alpha}\alpha = c_{\bar{\alpha}}^{-1}\alpha f(\alpha)\alpha$ and if we set

$$\varphi := f(\alpha)\alpha - c_{\bar{\alpha}}c_\alpha A'_\alpha$$

(where $\alpha A'_\alpha = A_\alpha$), then $\varphi\Lambda \subseteq \Omega^2(S_i)$. The module $\Omega^2(S_i)$ has dimension $m_\alpha n_\alpha - 1$. We will now show that $\varphi\Lambda$ has the same dimension which will give equality.
   (1) First we observe that $\varphi f(\alpha) = 0$. Namely

$$\varphi f(\alpha) = f(\alpha)\alpha f(\alpha) - c_\alpha c_{\bar{\alpha}}A'_\alpha f(\alpha) = f(\alpha)c_{\bar{\alpha}}\bar{\alpha} - c_\alpha c_{\bar{\alpha}}A_\gamma = 0$$

since $f(\alpha)\bar{\alpha} = c_\gamma A_\gamma$ and $c_\alpha = c_\gamma$. Hence $\varphi\,\mathrm{rad}\,\Lambda$ is generated by $\varphi\gamma$.
   (2) We show that $A'_\alpha\gamma$ lies in the socle. We have $\gamma = f(\delta)$ and A'_α has length ≥ 2. Therefore $A'_\alpha\gamma = A''_\alpha g^{-1}(\delta)\delta f(\delta)$. The product of the last three arrows is zero unless possibly $f(g^{-1}(\delta))$ is virtual, and if so then it lies in the second socle, by Lemma 4.5 (iii). Moreover, in this case, A''_α is in the radical, because Λ is not the triangle algebra considered in Example 3.4. Hence, in any case $A'_\alpha\gamma$ lies in the socle.

(3) We can now compute the dimension of $\varphi\Lambda$. By (1) and (2) the radical of $\varphi\Lambda$ is generated by $\varphi\gamma = f(\alpha)\alpha\gamma + u$, for an element u in the socle. Now $f(\alpha)\alpha\gamma$ is a monomial along $B_{f(\alpha)}$ which has length ≥ 4 and hence $f(\alpha)\alpha\gamma$ is not in the socle. It follows that $\varphi\Lambda$ has basis

$$\{\varphi, \ \varphi\gamma, \ \varphi\gamma g(\gamma) = f(\alpha)\alpha\gamma g(\gamma), \ldots, B_{f(\alpha)}\}$$

of size $m_\alpha n_\alpha - 1$. Hence $\varphi\Lambda = \Omega^2(S_i)$. Note that this is not simple.

(4) We identify $\Omega^3(S_i)$ with $\{x \in e_j\Lambda \mid \varphi x = 0\}$, which has dimension $m_\alpha n_\alpha - 1$. By (1) we know that this contains $f(\alpha)\Lambda$. Moreover $f(\alpha)\Lambda$ is isomorphic to $\Omega^{-1}(S_i)$ which has the same dimension. Hence we have $\Omega^3(S_i) \cong \Omega^{-1}(S_i)$ and S_i has period 4. $\qquad\square$

Now assume $\bar{\alpha}$ is virtual but not a loop. Then α is not virtual, and it cannot be a loop (see 4.2 and 4.3). We have the following diagram:
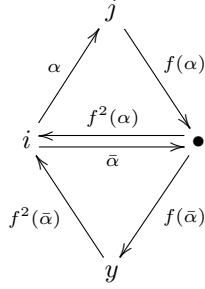
Lemma 5.6. *There is an exact sequence of Λ-modules*

$$0 \to \Omega^{-1}(S_i) \to P_y \to P_j \to \Omega(S_i) \to 0$$

which gives rise to a minimal projective resolution of period 4.

Proof. We identify $\Omega(S_i) = \alpha\Lambda$ and then $\Omega^2(S_i) = \{x \in e_j\Lambda \mid \alpha x = 0\}$. We have the following relations in Λ:

(i) $\alpha f(\alpha) = c_{\bar{\alpha}}\bar{\alpha}$,
(ii) $\bar{\alpha} f(\bar{\alpha}) = c_\alpha A_\alpha$.

Hence $c_\alpha A_\alpha = \bar{\alpha} f(\bar{\alpha}) = c_{\bar{\alpha}}^{-1}\alpha f(\alpha) f(\bar{\alpha})$ and if we set

$$\varphi := f(\alpha)f(\bar{\alpha}) - c_{\bar{\alpha}}c_\alpha A'_\alpha$$

(where $\alpha A'_\alpha = A_\alpha$), then $\varphi\Lambda \subseteq \Omega^2(S_i)$.

The module $\Omega^2(S_i)$ has dimension $m_{f(\alpha)}n_{f(\alpha)} - 1$. We want to show that $\varphi\Lambda$ has the same dimension. Assume first that $j \neq y$. Let $\gamma = g(f(\bar{\alpha}))$ and $\delta = f^{-1}(\gamma)$.

(1) First we observe that $\varphi f^2(\bar{\alpha}) = 0$: Namely

$$\varphi f^2(\bar{\alpha}) = f(\alpha)f(\bar{\alpha})f^2(\bar{\alpha}) - c_\alpha c_{\bar{\alpha}}A'_\alpha f^2(\bar{\alpha}).$$

The first term is

$$f(\alpha)c_{f^2(\alpha)}A_{f^2(\alpha)} = c_{f^2(\alpha)}f(\alpha)f^2(\alpha) = c_{f^2(\alpha)}c_{g(\alpha)}A_{g(\alpha)}$$

and the second term is the same since $A'_\alpha f^2(\bar{\alpha}) = A_{g(\alpha)}$ and $c_\alpha = c_{g(\alpha)}$, and $c_{f^2(\alpha)} = c_{\bar{\alpha}}$. So we get zero. Hence $\varphi \operatorname{rad}\Lambda$ is generated by $\varphi\gamma$.

(2) We analyse $A'_\alpha\gamma$ and compute the dimension of $\varphi\Lambda$.

(2a) Assume $m_\alpha n_\alpha = 3$. Then we claim that $A'_\alpha \gamma$ is in the second socle, and moreover $m_{f(\alpha)} n_{f(\alpha)} \geq 5$.

With this assumption, $A_\alpha = \alpha\delta$ where $\delta = g(\alpha)$, so $A'_\alpha \gamma = \delta\gamma = c_{f(\alpha)} A_{f(\alpha)}$ which is in the second socle. Consider the g-orbit of $f(\alpha)$, it is $(f(\alpha) \ f(\bar{\alpha}) \ \gamma \ \ldots)$ and the last arrow in this orbit is the arrow $g^{-1}(f(\alpha)) \neq \alpha$ ending at j.

We know that γ cannot end at j since then the f-cycle of $g(\alpha)$ would not have length three. Let $z = t(\gamma)$. Since $\gamma = f(g(\alpha))$, the arrow $f(\gamma)$ is an arrow $z \to j$ and this is then the last arrow in the g-cycle in question. This cannot be $g(\gamma)$ since it is $f(\gamma)$. At best, $g(\gamma)$ is a loop at z and then the g-cycle of $f(\alpha)$ has length 5, and in general it has length ≥ 5.

In this case $\varphi\gamma = f(\alpha)f(\bar{\alpha})\gamma + v$, for an element v in the second socle. We postmultiply by $g(\gamma)g^2(\gamma)$ and get

$$f(\alpha)f(\bar{\alpha})\gamma g(\gamma)g^2(\gamma) + 0$$

and this is a non-zero monomial along $B_{f(\alpha)}$. We deduce using also (1) that $\dim \varphi\Lambda = m_{f(\alpha)} n_{f(\alpha)} - 1$, and we are done in this case.

(2b) Assume $m_\alpha n_\alpha \geq 5$, then $A'_\alpha \gamma = A''_\alpha g^{-1}(\delta)\delta f(\delta) = 0$ since A''_α is in the square of the radical and the other factor is at least in the second socle.

It follows that the dimension of $\varphi\Lambda$ is equal to $m_{f(\alpha)} n_{f(\alpha)} - 1$ as required.

(2c) Assume $m_\alpha n_\alpha = 4$. Consider $A'_\alpha \gamma$.

In this case $A'_\alpha = g(\alpha)\delta$ and $A'_\alpha \gamma = g(\alpha)\delta f(\delta)$. This is zero if $f(g(\alpha))$ is not virtual. We are left with the case where $f(g(\alpha))$ is virtual, and then $g(\alpha)\delta f(\delta)$ is a scalar multiple of $g(\alpha)\bar{\delta} = c_{f(\alpha)} A_{f(\alpha)}$.

So $\varphi\gamma = f(\alpha)f(\bar{\alpha})\gamma - \lambda A_{f(\alpha)}$ for a non-zero scalar λ. Because Λ is not a spherical algebra, we have $m_{f(\alpha)} n_{f(\alpha)} \geq 5$, then it follows as before that $\dim \varphi\Lambda = m_{f(\alpha)} n_{f(\alpha)} - 1$.

(3) By (1) we know $\Omega^3(S_i)$ contains $f^2(\bar{\alpha})\Lambda$ and this is isomorphic to $\Omega^{-1}(S_i)$. One sees that they have the same dimension and hence are equal. This completes the proof in the case $i \neq j$.

Assume now that $i = j$, then Q has three vertices. Then g is the product of three 2-cycles, namely

$$g = \big(\alpha \ f^2(\bar{\alpha})\big)\big(f(\alpha) \ f(\bar{\alpha})\big)\big(\bar{\alpha} \ f^2(\alpha)\big).$$

By Assumption 2.7, we have $m_\alpha \geq 2$ and $m_{f(\alpha)} \geq 2$. Moreover, since Λ is not a triangle algebra, we have $m_\alpha + m_{f(\alpha)} \geq 5$. One sees similarly as before that $\varphi\Lambda$ has dimension $m_{f(\alpha)} n_{f(\alpha)} - 1$, and then it follows again that S_i has period four. $\qquad\square$

To identify the projective $\mathbb{P}_2$ of a minimal bimodule resolution, we need $\mathrm{Ext}^2_\Lambda(S_i, S_j)$ for simple modules S_i, S_j.

Lemma 5.7. *The dimension of $\mathrm{Ext}^2_\Lambda(S_i, S_j)$ is equal to the number of arrows $j \to i$ in the Gabriel quiver of Λ.*

Proof. This follows from the calculation of syzygies of the simple modules. $\qquad\square$

Now we construct the first steps of a minimal projective bimodule resolution of Λ. Then we will show that $\Omega^4_\Lambda(\Lambda) \cong \Lambda$ in $\mathrm{mod}\,\Lambda^e$. We shall use the notation introduced in earlier in this section. Recall the first few steps of a minimal projective resolution of Λ in $\mathrm{mod}\,\Lambda^e$,

$$\mathbb{P}_3 \xrightarrow{S} \mathbb{P}_2 \xrightarrow{R} \mathbb{P}_1 \xrightarrow{d} \mathbb{P}_0 \xrightarrow{d_0} \Lambda \to 0$$

where

$$\mathbb{P}_0 = \bigoplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda,$$

$$\mathbb{P}_1 = \bigoplus_{\alpha \in (Q_\Lambda)_1} \Lambda e_{s(\alpha)} \otimes e_{t(\alpha)} \Lambda,$$

the homomorphism d_0 is defined by $d_0(e_i \otimes e_i) = e_i$ for all $i \in Q_0$, and the homomorphism $d : \mathbb{P}_1 \to \mathbb{P}_0$ is defined by

$$d\big(e_{s(\alpha)} \otimes e_{t(\alpha)}\big) = \alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha$$

for any arrow α of the Gabriel quiver Q_Λ. In particular, we have $\Omega^1_\Lambda(\Lambda) = \mathrm{Ker}\, d_0$ and $\Omega^2_\Lambda(\Lambda) = \mathrm{Ker}\, d$.

We define now the homomorphism $R : \mathbb{P}_2 \to \mathbb{P}_1$. For each arrow α of Q which is not virtual, that is, it is an arrow of the Gabriel quiver Q_Λ, consider the element in KQ

$$\mu_\alpha := \bar{\alpha} f(\bar{\alpha}) - c_\alpha A_\alpha.$$

Since we work with the Gabriel quiver, we must make substitutions. Note first that since α is not virtual, no arrow in the g-cycle of α is virtual and therefore A_α is a path in Q_Λ. If $\bar{\alpha}$ is virtual then we substitute $\bar{\alpha} = \alpha f(\alpha)$ (using that $c_{\bar{\alpha}} = 1$ by assumption). Note that if $\bar{\alpha}$ is virtual then $f(\alpha)$ is not virtual (see 4.2 and 4.3). Similary we substitute $f(\alpha) = g(\alpha) f(g(\alpha))$ if $f(\alpha)$ is virtual. Recall that $\alpha, f(\alpha)$ cannot be both virtual.

Note also that $\mu_\alpha = e_{s(\bar{\alpha})} \mu_\alpha e_{t(f(\bar{\alpha}))}$. It follows from Proposition 5.1 and Lemma 5.7 that $\mathbb{P}_2$ is of the form

$$\mathbb{P}_2 = \bigoplus_{\alpha \in (Q_\Lambda)_1} \Lambda e_{s(\bar{\alpha})} \otimes e_{t(f(\bar{\alpha}))} \Lambda.$$

We define the homomorphism $R : \mathbb{P}_2 \to \mathbb{P}_1$ in $\mathrm{mod}\, \Lambda^e$ by

$$R\big(e_{s(\bar{\alpha})} \otimes e_{t(f(\bar{\alpha}))}\big) := \varrho(\mu_\alpha)$$

for any arrow α of the Gabriel quiver of Λ, where $\varrho : KQ \to \mathbb{P}_1$ is the K-linear homomorphism defined earlier this section.

It follows from Lemma 5.3 that $\mathrm{Im}\, R \subseteq \mathrm{Ker}\, d$.

Lemma 5.8. *The homomorphism $R : \mathbb{P}_2 \to \mathbb{P}_1$ induces a projective cover $\Omega^2_{\Lambda^e}(\Lambda)$ in $\mathrm{mod}\, \Lambda^e$. In particular, we have $\Omega^3_{\Lambda^e}(\Lambda) = \mathrm{Ker}\, R$.*

Proof. We know that $\mathrm{rad}\, \Lambda^e = \mathrm{rad}\, \Lambda^{\mathrm{op}} \otimes \Lambda + \Lambda^{\mathrm{op}} \otimes \mathrm{rad}\, \Lambda$ (see [36, Corollary IV.11.4]). It follows from the definition that the generators $\varrho(\mu_\alpha)$ of the image of R are elements of $\mathrm{rad}\, \mathbb{P}_1$ which are linearly independent in $\mathrm{rad}\, \mathbb{P}_1 / \mathrm{rad}^2 \mathbb{P}_1$, provided both $\alpha, \bar{\alpha}$ are not virtual. Suppose (say) $\bar{\alpha}$ is virtual, then we consider $S_i \otimes_\Lambda \varrho(\mu_\alpha)$. This is precisely the generator of the module $\Omega^2(S_i)$ as constructed in Lemmas 5.5 and 5.6. Therefore $\varrho(\mu_\alpha)$ is a generator for the image of R. We conclude that $\varrho(\mu_\alpha)$, $\alpha \in Q_1$, form a minimal set of generators of the right Λ^e-module $\Omega^2_{\Lambda^e}(\Lambda)$. Summing up, we obtain that $R : \mathbb{P}_2 \to \Omega^2_{\Lambda^e}(\Lambda)$ is a projective cover of $\Omega^2_{\Lambda^e}(\Lambda)$ in $\mathrm{mod}\, \Lambda^e$. $\square$

By Proposition 5.1, and the result that simple modules have Ω-period four, we have that $\mathbb{P}_3$ is of the form

$$\mathbb{P}_3 = \bigoplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda.$$

For each vertex $i \in Q_0$, we define an element in $\mathbb{P}_2$. If both arrows $\alpha, \bar{\alpha}$ starting at i are not virtual (so that also the arrows $f^2(\alpha), f^2(\bar{\alpha})$ ending at i are not virtual) then we define (as in [15])

$$\begin{aligned}
\psi_i &= \big(e_i \otimes e_{t(f(\alpha))}\big)f^2(\alpha) + \big(e_i \otimes e_{t(f(\bar{\alpha}))}\big)f^2(\bar{\alpha}) - \alpha\big(e_{t(\alpha)} \otimes e_i\big) - \bar{\alpha}\big(e_{t(\bar{\alpha})} \otimes e_i\big) \\
&= \big(e_{s(\alpha)} \otimes e_{t(f(\alpha))}\big)f^2(\alpha) + \big(e_{s(\bar{\alpha})} \otimes e_{t(f(\bar{\alpha}))}\big)f^2(\bar{\alpha}) - \alpha\big(e_{s(f(\alpha))} \otimes e_{t(f^2(\alpha))}\big) \\
&\quad - \bar{\alpha}\big(e_{s(f(\bar{\alpha}))} \otimes e_{t(f^2(\bar{\alpha}))}\big).
\end{aligned}$$

Suppose $\bar{\alpha}$ is virtual and then also $f^2(\alpha)$ is virtual, and $\alpha, f^2(\bar{\alpha})$ are not virtual. In this case, we take the same formula but omit the terms which have virtual arrows (and idempotents which do not occur in $\mathbb{P}_2$). That is we define

$$\psi_i = \big(e_i \otimes e_{t(f(\bar{\alpha}))}\big)f^2(\bar{\alpha}) - \alpha\big(e_{t(\alpha)} \otimes e_i\big).$$

Similarly, if α is virtual and then $f^2(\bar{\alpha})$ is virtual, and then $\bar{\alpha}, f^2(\alpha)$ are not virtual, we define

$$\psi_i = \big(e_i \otimes e_{t(f(\alpha))}\big)f^2(\alpha) - \bar{\alpha}\big(e_{t(\bar{\alpha})} \otimes e_i\big).$$

We define now a homomorphism $S : \mathbb{P}_3 \to \mathbb{P}_2$ in $\mathrm{mod}\,\Lambda^e$ by

$$S(e_i \otimes e_i) = \psi_i$$

for any vertex $i \in Q_0$.

Lemma 5.9. *The homomorphism $S : \mathbb{P}_3 \to \mathbb{P}_2$ induces a projective cover of $\Omega^3_{\Lambda^e}(\Lambda)$ in $\mathrm{mod}\,\Lambda^e$. In particular, we have $\Omega^4_{\Lambda^e}(\Lambda) = \mathrm{Ker}\,S$.*

Proof. We will prove first that $R(\psi_i) = 0$ for any $i \in Q_0$. Fix a vertex $i \in Q_0$. If both arrows starting at i are not virtual, this is identical with the calculation in [15], we will not repeat this (note that only arrows in g-orbits occur which are not virtual). Suppose now that α is not virtual and $\bar{\alpha}$ is virtual. Then we have, in $\mathbb{P}_1$, that

$$\begin{aligned}
R(\psi_i) &= \varrho(\mu_{\bar{\alpha}})f^2(\bar{\alpha}) - \alpha\varrho\big(\mu_{f(\alpha)}\big) \\
&= \varrho\big(\alpha f(\alpha)f(\bar{\alpha})\big)f^2(\bar{\alpha}) - \alpha\varrho\big(f(\alpha)f(\bar{\alpha})f^2(\bar{\alpha})\big) \\
&\quad + c_\alpha\big(-\varrho(A_\alpha)f^2(\bar{\alpha}) + \alpha\varrho(A_{g(\alpha)})\big) \\
&= e_i \otimes f(\alpha)f(\bar{\alpha})f^2(\bar{\alpha}) - \alpha f(\alpha)f(\bar{\alpha}) \otimes e_i + c_\alpha(-e_i \otimes A_{g(\alpha)} + A_\alpha \otimes e_i) \\
&= 0,
\end{aligned}$$

because $f^2(\bar{\alpha}) = g^{n_\alpha - 1}(\alpha)$, $f^2(\alpha) = g^{n_{\bar{\alpha}} - 1}(\bar{\alpha})$, $f(\alpha)f(\bar{\alpha})f^2(\bar{\alpha}) = f(\alpha)f^2(\alpha) = c_{g(\alpha)}A_{g(\alpha)}$ and $\alpha f(\alpha)f(\bar{\alpha}) = \bar{\alpha}f(\bar{\alpha}) = c_\alpha A_\alpha$. Similarly one shows that $R(\psi_i) = 0$ if α is virtual and $\bar{\alpha}$ not virtual. Hence $\mathrm{Im}\,S \subseteq \mathrm{Ker}\,R$. Further, it follows from the definition that the generators ψ_i, $i \in Q_0$, of the image of S are elements of $\mathrm{rad}\,\mathbb{P}_2$ which are linearly independent in $\mathrm{rad}\,\mathbb{P}_2/\mathrm{rad}^2\,\mathbb{P}_2$. We conclude from the form of $\mathbb{P}_2$ that these elements form a minimal set of generators of $\mathrm{Ker}\,R = \Omega^3_{\Lambda^e}(\Lambda)$. Hence $S : \mathbb{P}_3 \to \Omega^3_{\Lambda^e}(\Lambda)$ is a projective cover of $\Omega^3_{\Lambda^e}(\Lambda)$ in $\mathrm{mod}\,\Lambda^e$. $\qquad\square$

Theorem 5.10. *There is an isomorphism $\Omega_{\Lambda^e}^4(\Lambda) \cong \Lambda$ in $\mathrm{mod}\,\Lambda^e$. In particular, Λ is a periodic algebra of period 4.*

Proof. This is very similar to the proof of [13, Theorem 5.9]. Let φ be the symmetrizing K-linear form as defined in Proposition 4.13. Then, by general theory, we have the symmetrizing bilinear form $(-,-) : \Lambda \times \Lambda \to K$ such that $(x,y) = \varphi(xy)$ for any $x, y \in \Lambda$. Observe that, for any elements $x \in \mathcal{B}_i$ and $y \in \mathcal{B}$, we have

$$(x,y) = \text{ the coefficient of } \omega_i \text{ in } xy,$$

when xy is expressed as a linear combination of the elements of $e_i\mathcal{B}$ over K. Consider also the dual basis $\mathcal{B}^* = \{b^* \mid b \in \mathcal{B}\}$ of Λ such that $(b, c^*) = \delta_{bc}$ for $b, c \in \mathcal{B}$. Observe that, for $x \in e_i\mathcal{B}$ and $y \in \mathcal{B}$, the element (x,y) can only be non-zero if $y = ye_i$. In particular, if $b \in e_i\mathcal{B}e_j$ then $b^* \in e_j\mathcal{B}e_i$.

For each vertex $i \in Q_0$, we define the element of $\mathbb{P}_3$

$$\xi_i = \sum_{b\in\mathcal{B}_i} b \otimes b^*.$$

We note that ξ_i is independent of the basis of Λ (see [13, part (2a) on the page 119]). It follows from [13, part (2b) on the page 119] that, for any element $a \in e_i(\mathrm{rad}\,\Lambda)e_j \setminus e_i(\mathrm{rad}\,\Lambda)^2 e_j$, we have

$$a\xi_i = \xi_j a.$$

Consider now the homomorphism

$$\theta : \Lambda \to \mathbb{P}_3$$

in $\mathrm{mod}\,\Lambda^e$ such that $\theta(e_i) = \xi_i$ for any $i \in Q_0$. Then $\theta(1_\Lambda) = \sum_{i\in Q_0} \xi_i$, and consequently we have

$$a\Big(\sum_{i\in Q_0} \xi_i\Big) = \theta(a) = \Big(\sum_{i\in Q_0} \xi_i\Big)a$$

for any element $a \in \Lambda$. We claim that θ is a monomorphism. It is enough to show that θ is a monomorphism of right Λ-modules. We know that $\Lambda = \bigoplus_{i\in Q_0} e_i\Lambda$ and each $e_i\Lambda$ has simple socle generated by ω_i. For each $i \in Q_0$, we have

$$\theta(\omega_i) = \Big(\sum_{j\in Q_0} \xi_j\Big)\omega_i = \xi_i\omega_i = \sum_{b\in\mathcal{B}_i}(b \otimes b^*)\omega_i = \sum_{b\in\mathcal{B}_i} b \otimes b^*\omega_i = \omega_i \otimes \omega_i \neq 0.$$

Hence the claim follows. Our next aim is to show that $S(\xi_i) = 0$ for any $i \in Q_0$, or equivalently, that $\mathrm{Im}\,\theta \subseteq \mathrm{Ker}\,S = \Omega_{\Lambda^e}^4(\Lambda)$. Applying arguments from [13, part (3) on the pages 119 and 120], we obtain that

$$\sum_{b\in\mathcal{B}} b(a^r \otimes a^s)b^* = \sum_{b\in\mathcal{B}} b \otimes a^{r+s}b^*$$

for all integers $r, s \geq 0$ and any element $a = e_p a e_q$ in $\mathrm{rad}\,\Lambda$, with $p, q \in Q_0$. In particular, for each arrow α in Q_1, we have

$$\sum_{b\in\mathcal{B}} b\alpha \otimes b^* = \sum_{b\in\mathcal{B}} b \otimes \alpha b^*,$$

and hence

$$\sum_{b\in\mathcal{B}_i} b\alpha \otimes b^* = \sum_{b\in\mathcal{B}_i} b \otimes \alpha b^*.$$

for any $i \in Q_0$. We note that every arrow β in Q occurs once as a left factor of some ψ_j (with negative sign) and once a right factor of some ψ_k (with positive sign), because $\beta = f^2(\alpha)$ for a unique arrow α. Then, for any $i \in Q_0$, the following equalities hold

$$S(\xi_i) = \sum_{b \in \mathcal{B}_i} S(b \otimes b^*) = \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} S(be_j \otimes e_j b^*) = \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} b S(e_j \otimes e_j) b^*$$

$$= \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} b \psi_j b^* = \sum_{\alpha \in (Q_\Lambda)_1} \left[\sum_{b \in \mathcal{B}_i} -(b\alpha \otimes b^*) + \sum_{b \in \mathcal{B}_i} b \otimes \alpha b^* \right] = 0.$$

Hence, indeed $\operatorname{Im} \theta \subseteq \operatorname{Ker} S = \Omega^4_{\Lambda^e}(\Lambda)$, and we obtain a monomorphism $\theta : \Lambda \to \Omega^4_{\Lambda^e}(\Lambda)$ in $\operatorname{mod} \Lambda^e$.

Finally, it follows from Theorem 2.4 and Proposition 5.4 that $\Omega^4_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\sigma$ in $\operatorname{mod} \Lambda^e$ for some K-algebra automorphism σ of Λ. Then $\dim_K \Lambda = \dim_K \Omega^4_{\Lambda^e}(\Lambda)$, and consequently θ is an isomorphism. Therefore, we have $\Omega^4_{\Lambda^e}(\Lambda) \cong \Lambda$ in $\operatorname{mod} \Lambda^e$. Clearly, then Λ is a periodic algebra of period 4. $\qquad\square$

Corollary 5.11. *Let (Q, f) be a triangulation quiver with at least four vertices, let $m_\bullet$ and $c_\bullet$ be weight and parameter functions of (Q, f), and let $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ be the associated weighted triangulation algebra. Then the Cartan matrix C_Λ of Λ is singular.*

Proof. This follows from Theorems 2.5 and 5.10. $\qquad\square$

6. The representation type

The aim of this section is to prove Theorem 1.2. We start by describing the general strategy. We are given a weighted triangulation algebra $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ of dimension d. We aim to define an algebraic family of algebras $\Lambda(t)$ for $t \in K$ in the variety $\operatorname{alg}_d(K)$ such that for every arrow α of Q we have

  (i) $\alpha f(\alpha) = c_{\bar\alpha} t^{v(\alpha)} A_{\bar\alpha}$ with $v(\alpha)$ a natural number ≥ 1,
 (ii) $c_\alpha B_\alpha = c_{\bar\alpha} B_{\bar\alpha}$, and
(iii) the zero relations as in Definition 2.8 hold.

Then the algebra $A(0)$ is the biserial weighted triangulation algebra associated to Λ. Theorem 1.2 for Λ will follow if we can make sure that $A(t) \cong A(1)$ for any non-zero $t \in K$. We will define a map $\varphi_t : A(1) \to A(t)$ such that $\varphi_t(\alpha) = t^{u(\alpha)}\alpha \in A(t)$ where $u(\alpha) \geq 1$ is a natural number, and extend to products and linear combinations. This will define an algebra isomorphism $A(1) \to A(t)$ if and only if for all arrows α the following identity holds:

$$(\dagger) \qquad\qquad v(\alpha) + u(\alpha) + u(f(\alpha)) = u(A_{\bar\alpha})$$

where we define $u(\mu) = \sum_{i=1}^r u(\alpha_i)$ for a monomial $\mu = \alpha_1 \ldots \alpha_r$ in KQ. We deal first with some of the exceptions, in Proposition 6.6 it will be clear why these need to be treated separately.

Let (Q, f) be the triangulation quiver (as in Example 3.6)
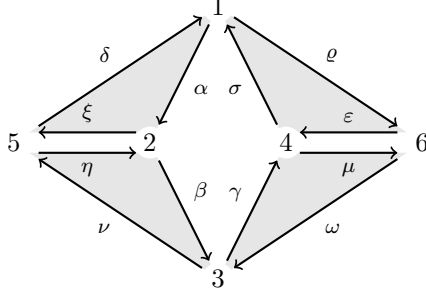
where f is the permutation of arrows of order 3 described by the shaded subquivers. Then g has four orbits

$$\mathcal{O}(\alpha) = (\alpha\ \beta\ \gamma\ \sigma), \qquad \mathcal{O}(\varrho) = (\varrho\ \omega\ \nu\ \delta), \qquad \mathcal{O}(\xi) = (\xi\ \eta), \qquad \mathcal{O}(\varepsilon) = (\varepsilon\ \mu).$$

Let $r \geq 2$ be a natural number and let $m_\bullet^r : \mathcal{O}(g) \to \mathbb{N}^*$ be the weight function given by $m_{\mathcal{O}(\alpha)} = m_{\mathcal{O}(\varrho)} = m_{\mathcal{O}(\xi)} = 1$ and $m_{\mathcal{O}(\varepsilon)} = r$. Moreover, let $c_\bullet : \mathcal{O}(g) \to K^*$ be an arbitrary parameter function. We consider the weighted triangulation algebra

$$S(r, c_\bullet) = \Lambda(Q, f, m_\bullet^r, c_\bullet).$$

Lemma 6.1. *The algebra $S(r, c_\bullet)$ degenerates to the biserial weighted triangulation algebra $B(Q, f, m_\bullet^r, c_\bullet)$. In particular, $S(r, c_\bullet)$ is a tame algebra.*

Proof. We write $a = c_{\mathcal{O}(\alpha)}$, $b = c_{\mathcal{O}(\varrho)}$, $c = c_{\mathcal{O}(\xi)}$, $d = c_{\mathcal{O}(\varepsilon)}$. For each $t \in K$, consider the algebra $A(t)$ given by the quiver Q and the relations:

$$\begin{aligned}
\alpha\xi &= bt^r \varrho\omega\nu, & \xi\delta &= at^r \beta\gamma\sigma, & \delta\alpha &= ct^r \eta, \\
\beta\nu &= ct^r \xi, & \nu\eta &= at^r \gamma\sigma\alpha, & \eta\beta &= bt^r \delta\varrho\omega, \\
\sigma\varrho &= dt^{3r-4}(\mu\varepsilon)^{r-1}\mu, & \varrho\varepsilon &= at^{3r-4}\alpha\beta\gamma, & \varepsilon\sigma &= bt^{3r-4}\omega\nu\delta, \\
\gamma\mu &= bt^{3r-4}\nu\delta\varrho, & \mu\omega &= at^{3r-4}\sigma\alpha\beta, & \omega\gamma &= dt^{3r-4}(\varepsilon\mu)^{r-1}\varepsilon, \\
a\alpha\beta\gamma\sigma &= b\varrho\omega\nu\delta, & a\beta\gamma\sigma\alpha &= c\xi\eta, & a\gamma\sigma\alpha\beta &= b\nu\delta\varrho\omega, \\
a\sigma\alpha\beta\gamma &= d(\mu\varepsilon)^r, & b\delta\varrho\omega\nu &= c\eta\xi, & b\omega\nu\delta\varrho &= d(\varepsilon\mu)^r,
\end{aligned}$$

$$\theta f(\theta) g\big(f(\theta)\big) = 0 \text{ for } \theta \in Q_1 \setminus \{\delta, \beta\},$$
$$\theta g(\theta) f\big(g(\theta)\big) = 0 \text{ for } \theta \in Q_1 \setminus \{\alpha, \nu\}.$$

Note that $3r - 4 \geq 1$ because $r \geq 2$. Then $A(t)$, $t \in K$, is an algebraic family in the variety $\mathrm{alg}_d(K)$, with $d = 4r + 36$. Observe also that $A(1) = S(r, c_\bullet)$ and $A(0) = B(Q, f, m_\bullet^r, c_\bullet)$. Fix $t \in K \setminus \{0\}$. Then there exists an isomorphism of K-algebras $\varphi_t : A(1) \to A(t)$ given by

$$\begin{aligned}
\varphi_t(\alpha) &= t^r \alpha, & \varphi_t(\beta) &= t^{2r} \beta, & \varphi_t(\gamma) &= t^r \gamma, & \varphi_t(\sigma) &= t^{4r} \sigma, \\
\varphi_t(\varrho) &= t^r \varrho, & \varphi_t(\omega) &= t^{4r} \omega, & \varphi_t(\nu) &= t^r \nu, & \varphi_t(\delta) &= t^{2r} \delta, \\
\varphi_t(\xi) &= t^{4r} \xi, & \varphi_t(\eta) &= t^{4r} \eta, & \varphi_t(\varepsilon) &= t^4 \varepsilon, & \varphi_t(\mu) &= t^4 \mu.
\end{aligned}$$

Therefore, applying Proposition 2.2, we conclude that $S(r, c_\bullet)$ degenerates to $B(Q, f, m_\bullet^r, c_\bullet)$, and $S(r, c_\bullet)$ is a tame algebra. $\qquad\square$

Let (Q, f) be the triangulation quiver

$$\alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \underset{\delta}{\overset{\sigma}{\rightleftarrows}} 3 \circlearrowright \eta$$

(as in Example 3.4) with f-orbits $(\alpha\ \beta\ \gamma)$ and $(\eta\ \delta\ \sigma)$. Then g has orbits

$$\mathcal{O}(\alpha) = (\alpha), \qquad \mathcal{O}(\beta) = (\beta\ \sigma\ \delta\ \gamma), \qquad \mathcal{O}(\eta) = (\eta).$$

Let $r \geq 3$ be a natural number and $m_\bullet^r : \mathcal{O}(g) \to \mathbb{N}^*$ the weight function given by $m_{\mathcal{O}(\alpha)}^r = 2$, $m_{\mathcal{O}(\beta)}^r = 1$ and $m_{\mathcal{O}(\eta)}^r = r$. Moreover, let $c_\bullet : \mathcal{O}(g) \to K^*$ be an arbitrary parameter function. We consider the weighted triangulation algebra

$$\Sigma(r, c_\bullet) = \Lambda(Q, f, m_\bullet^r, c_\bullet).$$

Lemma 6.2. $\Sigma(r, c_\bullet)$ *degenerates to the biserial weighted triangulation algebra* $B(Q, f, m_\bullet^r, c_\bullet)$. *In particular,* $\Sigma(r, c_\bullet)$ *is a tame algebra.*

Proof. We abbreviate $a = c_{\mathcal{O}(\alpha)}$, $b = c_{\mathcal{O}(\beta)}$ and $c = c_{\mathcal{O}(\eta)}$. We consider two cases.

(1) Assume $r = 3$. For each $t \in K$, we denote by $\Lambda(t)$ the algebra given by the quiver Q and the relations:

$$\alpha\beta = bt\beta\sigma\delta, \qquad \beta\gamma = at\alpha, \qquad \gamma\alpha = bt\sigma\delta\gamma,$$

$$\eta\delta = bt\delta\gamma\beta, \qquad a\alpha^2 = b(\beta\sigma\delta\gamma), \qquad b(\gamma\beta\sigma\delta) = b(\sigma\delta\gamma\beta),$$

$$b(\delta\gamma\beta\sigma) = c\eta^r, \qquad \delta\sigma = ct\eta^2, \qquad \sigma\eta = bt\gamma\beta\sigma,$$

$$\theta f(\theta)g(f(\theta)) = 0 \text{ for } \beta \neq \theta \in Q_1, \quad \theta g(\theta)f(g(\theta)) = 0 \text{ for } \gamma \neq \theta \in Q_1.$$

Then $\Lambda(t)$, $t \in K$, is an algebraic family in the variety $\mathrm{alg}_{21}(K)$, with $\Lambda(1) = \Sigma(3, c_\bullet)$, and $\Lambda(0) = B(Q, f, m_\bullet^3, c_\bullet)$. Moreover, for each $t \in K \setminus \{0\}$, there exists an isomorphism of K-algebras $\varphi_t : \Lambda(1) \to \Lambda(t)$ given by

$$\varphi_t(\alpha) = t^6\alpha, \qquad \varphi_t(\beta) = t^2\beta, \qquad \varphi_t(\gamma) = t^3\gamma,$$

$$\varphi_t(\eta) = t^4\eta, \qquad \varphi_t(\delta) = t^4\delta, \qquad \varphi_t(\sigma) = t^3\sigma.$$

Therefore, it follows from Proposition 2.2 that $\Sigma(3, c_\bullet)$ degenerates to $B(Q, f, m_\bullet^3, c_\bullet)$, and $\Sigma(3, c_\bullet)$ is tame.

(2) Assume $r \geq 4$. For each $t \in K$, we denote by $A(t)$ the algebra given by the quiver Q and the relations:

$$\alpha\beta = bt^r\beta\sigma\delta, \qquad \beta\gamma = at^r\alpha, \qquad \gamma\alpha = bt^r\sigma\delta\gamma,$$

$$\eta\delta = bt^{2r-6}\delta\gamma\beta, \qquad a\alpha^2 = b(\beta\sigma\delta\gamma), \qquad b(\gamma\beta\sigma\delta) = b(\sigma\delta\gamma\beta),$$

$$b(\delta\gamma\beta\sigma) = c\eta^r, \qquad \delta\sigma = ct^{2r-6}\eta^{r-1}, \qquad \sigma\eta = bt^{2r-6}\gamma\beta\sigma,$$

$$\theta f(\theta)g(f(\theta)) = 0 \text{ for } \beta \neq \theta \in Q_1, \quad \theta g(\theta)f(g(\theta)) = 0 \text{ for } \gamma \neq \theta \in Q_1.$$

We note that $2r - 6 \geq 1$, because $r \geq 4$. Then $A(t)$, $t \in K$, is an algebraic family in the variety $\mathrm{alg}_d(K)$, with $d = r + 18$. Observe also that $A(1) = \Sigma(r, c_\bullet)$ and $A(0) = B(Q, f, m_\bullet^r, c_\bullet)$. Further, for each $t \in K \setminus \{0\}$, there exists an isomorphism of K-algebras $\psi_t : A(1) \to A(t)$ given by

$$\psi_t(\alpha) = t^{3r}\alpha, \qquad \psi_t(\beta) = t^r\beta, \qquad \psi_t(\gamma) = t^r\gamma,$$

$$\psi_t(\eta) = t^6\eta, \qquad \psi_t(\delta) = t^{2r}\delta, \qquad \psi_t(\sigma) = t^{2r}\sigma.$$

Therefore, applying Proposition 2.2, we conclude that $\Sigma(r, c_\bullet)$ degenerates to $B(Q, f, m_\bullet^r, c_\bullet)$, and $\Sigma(r, c_\bullet)$ is tame. $\qquad\square$

Let (Q, f) be the triangulation quiver (as in Example 3.1)

$$\alpha \,\circlearrowright\, 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \,\circlearrowleft\, \sigma$$

with f-orbits $(\alpha \ \beta \ \gamma)$ and (σ), so that the g-orbits are (α) and $(\beta \ \sigma \ \gamma)$. We fix a natural number $r \geq 4$, and we take $m_\bullet$ to be the weight function $m_\alpha = r$ and $m_\beta = 1$. Moreover, for $c_\bullet$ we take an arbitrary parameter function. We consider the weighted triangulation algebra

$$\Omega(r, c_\bullet) = \Lambda(Q, f, m_\bullet^r, c_\bullet).$$

Lemma 6.3. *The algebra $\Omega(r, c_\bullet)$ degenerates to the biserial weighted triangulation algebra $B(Q, f, m_\bullet^r, c_\bullet)$. In particular, $\Omega(r, c_\bullet)$ is tame.*

Proof. We abbreviate $a = c_\alpha$ and $b = c_\beta$. We consider two cases.

(1) Assume $r = 4$. For each $t \in K$, we denote by $\Lambda(t)$ the algebra given by the quiver Q and the relations:

$$\sigma^2 = bt\gamma\beta, \qquad \gamma\alpha = bt\sigma\gamma, \qquad \alpha\beta = bt\beta\sigma, \qquad \beta\gamma = at\alpha^3,$$
$$a\alpha^4 = b(\beta\sigma\gamma), \qquad b(\gamma\beta\sigma) = b(\sigma\gamma\beta),$$
$$\theta f(\theta)g(f(\theta)) = 0 \ (\theta \in Q_1), \quad \theta g(\theta)f(g(\theta)) = 0 \ (\theta \in Q_1).$$

Then $\Lambda(t)$, $t \in K$, is an algebraic family in the variety $\mathrm{alg}_{13}(K)$, with $\Lambda(1) = \Sigma(4, c_\bullet)$ and $\Lambda(0) = B(Q, f, m_\bullet^4, c_\bullet)$. Moreover, for each $t \in K^*$, there exists an isomorphism of K-algebras $\varphi_t : \Lambda(1) \to \Lambda(t)$ given by

$$\varphi_t(\sigma) = t^5\sigma, \qquad \varphi_t(\gamma) = t^5\gamma, \qquad \varphi_t(\beta) = t^6\beta, \qquad \varphi_t(\alpha) = t^4\alpha.$$

Therefore, it follows from Proposition 2.2 that $\Omega(4, c_\bullet)$ degenerates to $B(Q, f, m_\bullet^4, c_\bullet)$, and $\Omega(4, c_\bullet)$ is a tame algebra.

(2) Assume $r \geq 5$. For each $t \in K$, we denote by $A(t)$ the algebra given by the quiver Q and the relations:

$$\sigma^2 = bt^r\gamma\beta, \qquad \gamma\alpha = bt^{r-4}\sigma\gamma, \qquad \alpha\beta = bt^{r-4}\beta\sigma,$$
$$\beta\gamma = at^{r-4}\alpha^{r-1}, \qquad a\alpha^r = b(\beta\sigma\gamma), \qquad b(\gamma\beta\sigma) = b(\sigma\gamma\beta),$$
$$\theta f(\theta)g(f(\theta)) = 0 \ (\theta \in Q_1), \qquad \theta g(\theta)f(g(\theta)) = 0 \ (\theta \in Q_1).$$

Then $A(t)$, $t \in K$, is an algebraic family in the variety $\mathrm{alg}_d(K)$, with $d = 9 + r$. Observe also that $A(1) = \Omega(r, c_\bullet)$ and $A(0) = B(Q, f, m_\bullet^r, c_\bullet)$. Further, for each $t \in K \setminus \{0\}$, there exists an isomorphism of K-algebras $\psi_t : A(1) \to A(t)$ given by

$$\psi_t(\sigma) = t^r\sigma, \qquad \psi_t(\gamma) = t^r\gamma, \qquad \psi_t(\beta) = t^{2r}\beta, \qquad \psi_t(\alpha) = t^4\alpha.$$

Therefore, applying Proposition 2.2, we conclude that $\Omega(r, c_\bullet)$ degenerates to $B(Q, f, m_\bullet^r, c_\bullet)$, and $\Omega(r, c_\bullet)$ is a tame algebra. $\qquad\square$

Let (Q, f) be the triangulation quiver
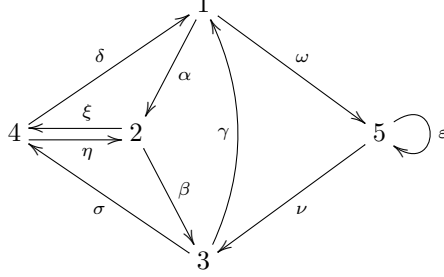
with f-orbits $(\alpha\ \xi\ \delta)$, $(\beta\ \sigma\ \eta)$, $(\gamma\ \omega\ \nu)$ and (ε). Then g has orbits

$$\mathcal{O}(\alpha) = (\alpha\ \beta\ \gamma), \qquad \mathcal{O}(\delta) = (\delta\ \omega\ \varepsilon\ \nu\ \sigma), \qquad \mathcal{O}(\xi) = (\xi\ \eta).$$

Let $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$ be the weight function given by $m_{\mathcal{O}(\alpha)} = m_{\mathcal{O}(\delta)} = m_{\mathcal{O}(\xi)} = 1$ and let $c_\bullet : \mathcal{O}(g) \to K^*$ be an arbitrary parameter function. We consider the weighted triangulation algebra

$$\Phi(c_\bullet) = \Lambda(Q, f, m_\bullet, c_\bullet).$$

Lemma 6.4. *The algebra $\Phi(c_\bullet)$ degenerates to the biserial weighted triangulation algebra $B(Q, f, m_\bullet, c_\bullet)$. In particular, $\Phi(c_\bullet)$ is a tame algebra.*

Proof. We write $a = c_{\mathcal{O}(\alpha)}$, $b = c_{\mathcal{O}(\delta)}$, $c = c_{\mathcal{O}(\xi)}$. For each $t \in K$, consider the algebra $A(t)$ given by the quiver Q and the relations:

$$\alpha\xi = bt\omega\varepsilon\nu\sigma, \qquad \xi\delta = at\beta\gamma, \qquad \sigma\eta = at\gamma\alpha, \qquad \omega\nu = at^2\alpha\beta,$$
$$\eta\beta = bt\delta\omega\varepsilon\nu, \qquad \delta\alpha = ct\eta, \qquad \beta\sigma = ct\xi, \qquad \nu\gamma = bt^2\varepsilon\nu\sigma\delta,$$
$$\gamma\omega = bt^2\sigma\delta\omega\varepsilon, \qquad\qquad \theta f(\theta)g\big(f(\theta)\big) = 0 \text{ for } \theta \in Q_1 \setminus \{\beta, \delta\},$$
$$\varepsilon^2 = bt^8\nu\sigma\delta\omega, \qquad\qquad \theta g(\theta)f\big(g(\theta)\big) = 0 \text{ for } \theta \in Q_1 \setminus \{\alpha, \sigma\},$$

and in addition

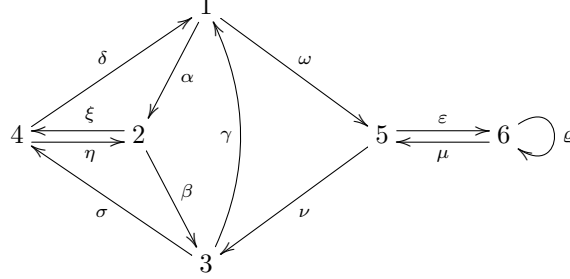$$a(\alpha\beta\gamma) = b(\omega\varepsilon\nu\sigma\delta), \qquad a(\beta\gamma\alpha) = c(\xi\eta), \qquad b(\sigma\delta\omega\varepsilon\nu) = a(\gamma\alpha\beta),$$
$$b(\delta\omega\varepsilon\nu\sigma) = c(\eta\xi), \qquad b(\varepsilon\nu\sigma\delta\omega) = b(\nu\sigma\delta\omega\varepsilon).$$

Then $A(t)$, $t \in K$, is an algebraic family in the variety $\text{alg}_{38}(K)$, with $A(1) = \Phi(c_\bullet)$ and $A(0) = B(Q, f, m_\bullet, c_\bullet)$. Moreover, for each $t \in K^*$, there exists an isomorphism of K-algebras $\varphi_t : A(1) \to A(t)$ given by

$$\varphi_t(\alpha) = t^5\alpha, \quad \varphi_t(\beta) = t^5\beta, \quad \varphi_t(\gamma) = t^{10}\gamma, \quad \varphi_t(\xi) = t^{10}\xi, \quad \varphi_t(\eta) = t^{10}\eta,$$
$$\varphi_t(\delta) = t^4\delta, \quad \varphi_t(\omega) = t^4\omega, \quad \varphi_t(\varepsilon) = t^4\varepsilon, \quad \varphi_t(\nu) = t^4\nu, \quad \varphi_t(\sigma) = t^4\sigma.$$

Therefore, it follows from Proposition 2.2 that $\Phi(c_\bullet)$ degenerates to $B(Q, f, m_\bullet, c_\bullet)$, and $\Phi(c_\bullet)$ is tame. $\qquad\square$

Let (Q, f) be the triangulation quiver

with f-orbits $(\alpha\ \xi\ \delta)$, $(\beta\ \sigma\ \eta)$, $(\gamma\ \omega\ \nu)$, $(\varepsilon\ \varrho\ \mu)$. Then g has orbits

$$\mathcal{O}(\alpha) = (\alpha\ \beta\ \gamma), \quad \mathcal{O}(\delta) = (\delta\ \omega\ \varepsilon\ \mu\ \nu\ \sigma), \quad \mathcal{O}(\xi) = (\xi\ \eta), \quad \mathcal{O}(\varrho) = (\varrho).$$

Let $r \geq 2$ be a natural number and $m_\bullet^r : \mathcal{O}(g) \to \mathbb{N}^*$ be the weight function given by $m_{\mathcal{O}(\alpha)}^r = m_{\mathcal{O}(\delta)}^r = m_{\mathcal{O}(\xi)}^r = 1$ and $m_{\mathcal{O}(\varrho)}^r = r$. Moreover, let $c_\bullet : \mathcal{O}(g) \to K^*$ be an arbitrary parameter function. We consider the weighted triangulation algebra

$$\Psi(r, c_\bullet) = \Lambda(Q, f, m_\bullet^r, c_\bullet).$$

Lemma 6.5. *The algebra $\Psi(r, c_\bullet)$ degenerates to the associated biserial weighted triangulation algebra $B(Q, f, m_\bullet^r, c_\bullet)$. In particular, $\Psi(r, c_\bullet)$ is a tame algebra.*

Proof. We write $a = c_{\mathcal{O}(\alpha)}$, $b = c_{\mathcal{O}(\delta)}$, $c = c_{\mathcal{O}(\xi)}$, $d = c_{\mathcal{O}(\varrho)}$. For each $t \in K$, consider the algebra $A(t)$ given by the quiver Q and the relations:

$$\alpha\xi = bt^r\omega\varepsilon\mu\nu\sigma, \qquad \xi\delta = at^r\beta\gamma, \qquad \delta\alpha = ct^r\eta,$$
$$\eta\beta = bt^r\delta\omega\varepsilon\mu\nu, \qquad \sigma\eta = at^r\gamma\alpha, \qquad \beta\sigma = ct^r\xi,$$
$$\gamma\omega = bt^{2r}\sigma\delta\omega\varepsilon\mu, \qquad \omega\nu = at^r\alpha\beta, \qquad \nu\gamma = bt^{2r}\varepsilon\mu\nu\sigma\delta,$$
$$\varepsilon\varrho = bt^{8r-12}\nu\sigma\delta\omega\varepsilon, \qquad \varrho\mu = bt^{8r-12}\mu\nu\sigma\delta\omega, \qquad \mu\varepsilon = dt^{8r-12}\varrho^{r-1},$$
$$\theta f(\theta) g\big(f(\theta)\big) = 0 \text{ for } \theta \in Q_1 \setminus \{\beta, \delta, \mu\},$$
$$\theta g(\theta) f\big(g(\theta)\big) = 0 \text{ for } \theta \in Q_1 \setminus \{\alpha, \sigma, \varepsilon\},$$
$$\mu\varepsilon\mu = 0, \quad \varepsilon\mu\varepsilon = 0, \quad \text{if } r \geq 3.$$

In addition we have the relations

$$a(\alpha\beta\gamma) = b(\omega\varepsilon\mu\nu\sigma\delta), \qquad a(\beta\gamma\alpha) = c(\xi\eta), \qquad b(\sigma\delta\omega\varepsilon\mu\nu) = a(\gamma\alpha\beta),$$
$$b(\delta\omega\varepsilon\mu\nu\sigma) = c(\eta\xi), \qquad b(\varepsilon\mu\nu\sigma\delta\omega) = b(\nu\sigma\delta\omega\varepsilon\mu), \quad b(\mu\nu\sigma\delta\omega\varepsilon) = d\varrho^r.$$

We note that $8r - 12 \geq 1$, because $r \geq 2$. Then $A(t)$, $t \in K$, is an algebraic family in the variety $\mathrm{alg}_n(K)$, with $n = r + 49$. Observe also that $A(1) = \Psi(r, c_\bullet)$ and $A(0) = B(Q, f, m_\bullet^r, c_\bullet)$. Further, for each $t \in K \setminus \{0\}$, there is an isomorphism of K-algebras $\varphi_t : A(1) \to A(t)$ given by

$$\varphi_t(\alpha) = t^{3r}\alpha, \qquad \varphi_t(\beta) = t^{3r}\beta, \qquad \varphi_t(\gamma) = t^{6r}\gamma, \qquad \varphi_t(\xi) = t^{6r}\xi,$$
$$\varphi_t(\eta) = t^{6r}\eta, \qquad \varphi_t(\varrho) = t^{12}\varrho, \qquad \varphi_t(\delta) = t^{2r}\delta, \qquad \varphi_t(\omega) = t^{2r}\omega,$$
$$\varphi_t(\varepsilon) = t^{2r}\varepsilon, \qquad \varphi_t(\mu) = t^{2r}\mu, \qquad \varphi_t(\nu) = t^{2r}\nu, \qquad \varphi_t(\sigma) = t^{2r}\sigma.$$

Therefore, it follows from Proposition 2.2 that $\Psi(r, c_\bullet)$ degenerates to $B(Q, f, m_\bullet^r, c_\bullet)$, and $\Psi(r, c_\bullet)$ is tame. $\qquad\square$

Towards the general case, let $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ be an arbitrary weighted triangulation algebra. We define

$$M := M_\Lambda = 2\operatorname{lcm}\{m_\mathcal{O} n_\mathcal{O} \mid \mathcal{O} \in \mathcal{O}(g)\},$$

$$q(\alpha) := m_\alpha n_\alpha, \ q : Q_1 \to \mathbb{N}^*,$$

$$v(\alpha) := M\left(1 - \frac{1}{q(\alpha)} - \frac{1}{q(f(\alpha))} - \frac{1}{q\big(f^2(\alpha)\big)}\right)$$

for any $\alpha \in Q_1$. We will see in Proposition 6.7 that as long as $v(\alpha) \geq 1$ for all arrows α, we can define algebraic family of algebras and show that Λ degenerates to the associated biserial triangulation algebra. First we determine quivers which have arrows α with $v(\alpha) \leq 0$.

Proposition 6.6. *Assume that $v(\alpha) \leq 0$ for some arrow $\alpha \in Q_1$. Then Λ is isomorphic to one of the algebras $D(\lambda)$, $T(\lambda)$, $S(\lambda)$, $\Lambda(\lambda)$, with $\lambda \in K^*$, $S(r, c_\bullet)$, with $r \geq 2$, $\Sigma(r, c_\bullet)$, with $r \geq 3$, $\Omega(r, c_\bullet)$, with $r \geq 4$, $\Phi(c_\bullet)$, or $\Psi(r, c_\bullet)$, with $r \geq 2$.*

Proof. We have $v(\alpha) \leq 0$ if and only if $\frac{1}{q(\alpha)} + \frac{1}{q(f(\alpha))} + \frac{1}{q(f^2(\alpha))} \geq 1$. That is, (up to rotation) $\big(q(\alpha), q(f(\alpha)), q(f^2(\alpha))\big)$ is one of the triples: $(2, 2, n)$, $n \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, $(2, 3, 6)$, $(2, 4, 4)$, and $(3, 3, 3)$.

(1) The case $(2, 2, n)$ does not occur: Suppose we have $\alpha \neq f(\alpha)$ and they are both virtual. Then $g(\alpha)$ and $f(\alpha)$ are virtual starting at the same vertex, which contradicts Assumption 2.7.

(2) We determine when $q(\alpha) = 2$ and $v(\alpha) \leq 0$ when α is a loop. Then (Q, f) has a subquiver of the form

$$\alpha \, \circlearrowright \, \bullet \underset{\gamma}{\overset{\beta}{\rightleftarrows}} \bullet$$

with f-orbit $(\alpha \ \beta \ \gamma)$ and $g(\alpha) = \alpha$. Moreover, $m_\alpha = 2$. Further, $\mathcal{O}(\beta) = \mathcal{O}(\gamma)$ has length at least 3. Hence we have $q(\beta) = m_\beta n_\beta = m_\gamma n_\gamma = q(\gamma) \geq 3$, and since $v(\alpha) \leq 0$, it is equal to 3 or 4, and then $m_\beta = 1$. By Assumption 2.7 we can only have $m_\beta n_\beta = 4$. Then (Q, f) is the quiver

$$\alpha \, \circlearrowright \, \bullet \underset{\gamma}{\overset{\beta}{\rightleftarrows}} \bullet \underset{\delta}{\overset{\sigma}{\rightleftarrows}} \bullet \, \circlearrowleft \, \eta$$

with f-orbits $(\alpha \ \beta \ \gamma)$ and $(\eta \ \delta \ \sigma)$, and g-orbits $\mathcal{O}(\alpha) = (\alpha)$, $\mathcal{O}(\beta) = (\beta \ \sigma \ \delta \ \gamma)$, $\mathcal{O}(\eta) = (\eta)$. Then Λ is isomorphic to one of the algebras $T(\lambda)$, for some $\lambda \in K^*$, or $\Sigma(r, c_\bullet)$ for some $r \geq 3$.

(3) We determine when $q(\xi) = 2$ and $v(\xi) \leq 0$ where ξ is not a loop. Then (Q, f) contains a subquiver of the form
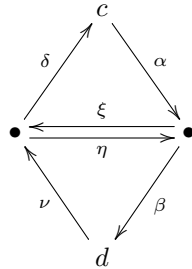
with f-orbits $(\xi\ \delta\ \alpha)$ and $(\beta\ \nu\ \eta)$ and g-orbit $(\xi\ \eta)$. Since $q(\xi) = 2$, we have that ξ, η are virtual arrows.

Assume first that $c = d$. Clearly, then we have g-orbits $(\alpha\ \beta)$ and $(\nu\ \delta)$. It follows from Assumption 2.7 that α, β, γ, δ are not virtual. Hence we have $m_\alpha n_\alpha = m_\beta n_\beta \geq 4$ and $m_\nu n_\nu = m_\delta n_\delta \geq 4$. But then $v(\xi) \leq 0$ implies $q(\alpha) = 4$ and $q(\delta) = 4$. Thus $m_\alpha n_\alpha = m_\beta n_\beta = 4$ and $m_\nu n_\nu = m_\delta n_\delta = 4$, and consequently Λ is isomorphic to $T(\lambda)$ for some $\lambda \in K^*$.

Assume now that $c \neq d$. Then the g-orbits $\mathcal{O}(\alpha) = \mathcal{O}(\beta)$ and $\mathcal{O}(\nu) = \mathcal{O}(\delta)$ have lengths at least 3.

Suppose (say) $m_\alpha n_\alpha = 3$, so that g as a cycle $(\alpha\ \beta\ \gamma)$ where γ is an arrow $d \mapsto c$. It follows from this that the g-orbit of δ has the form

$$\left(f^2(\gamma)\ \nu\ \delta\ f(\gamma)\ *\right)$$

of length ≥ 5. We assume $v(\xi) \leq 0$ and hence the length can only be 5 or 6. Suppose it has length 5, then $*$ is just a loop, $\varepsilon = g(f(\gamma))$, the quiver has five vertices and Λ is an algebra of the form $\Phi(c_\bullet)$.

Suppose the g-orbit of δ has length 6, say it is $(f^2(\gamma)\ \nu\ \delta\ f(\gamma)\ \varepsilon\ \mu)$, then we must have $f(\mu) = \varepsilon$, and $f(\varepsilon)$ is a loop. It follows that Q has six vertices and Λ is isomorphic to $\Psi(r, c_\bullet)$ for some $r \geq 2$.

Otherwise $q(\alpha) = m_\alpha n_\alpha \geq 4$ and $q(\delta) = m_\delta n_\delta \geq 4$. But then $v(\xi) \leq 0$ implies $q(\alpha) = 4$ and $q(\delta) = 4$, and therefore we have $m_\alpha = 1$, $n_\alpha = 4$, $m_\delta = 1$, $n_\delta = 4$. As well $m_\beta = 1$, $n_\beta = 4$, $m_\nu = 1$, $n_\nu = 4$. Summing up, we conclude that that (Q, f) is the triangulation quiver of the form
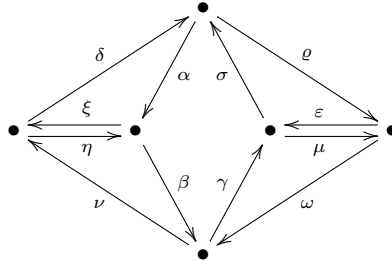
with f-orbits $(\alpha\ \xi\ \delta)$, $(\beta\ \nu\ \eta)$, $(\sigma\ \varrho\ \varepsilon)$, $(\gamma\ \mu\ \omega)$ and the g-orbits $\mathcal{O}(\xi) = (\xi\ \eta)$, $\mathcal{O}(\alpha) = (\alpha\ \beta\ \gamma\ \sigma)$, $\mathcal{O}(\delta) = (\delta\ \varrho\ \omega\ \nu)$, $\mathcal{O}(\varepsilon) = (\varepsilon\ \mu)$. Therefore, Λ is isomorphic to one of the algebras $S(\lambda)$, for some $\lambda \in K^*$, or $S(r, c_\bullet)$ for some $r \geq 2$.

(4) Assume now that there is a loop $\sigma \in Q_1$ with $f(\sigma) = \sigma$ and $v(\sigma) \leq 0$, and hence $q(\sigma) = 3$ and $v(\sigma) = 0$. Then (Q, f) is the triangulation quiver

$$\alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \circlearrowleft \sigma$$

with f-orbits $(\alpha\ \beta\ \gamma)$, (σ), and the g-orbits (α), $(\beta\ \sigma\ \gamma)$. Moreover, we have $m_\beta n_\beta = m_\gamma n_\gamma = m_\sigma n_\sigma = 3$ and $m_\alpha = m_\alpha n_\alpha = r$ for some $r \geq 3$ (by Assumption 2.7). Therefore, Λ is isomorphic to one of the algebras $D(\lambda)$, for some $\lambda \in K^*$, or $\Omega(r, c_\bullet)$ for some $r \geq 4$.

(5) The last case to consider is where $(\eta\ \gamma\ \delta)$ is an f-orbit in Q_1 of lenght 3 with $v(\eta) \leq 0$ and $q(\eta) \geq q(\gamma) \geq q(\delta) \geq 3$. Then $q(\eta) = q(\gamma) = q(\delta) = 3$. We

have two cases. Assume first that none of η, γ, δ is a loop. Then by some basic combinatorics, one sees that (Q, f) is the triangulation quiver
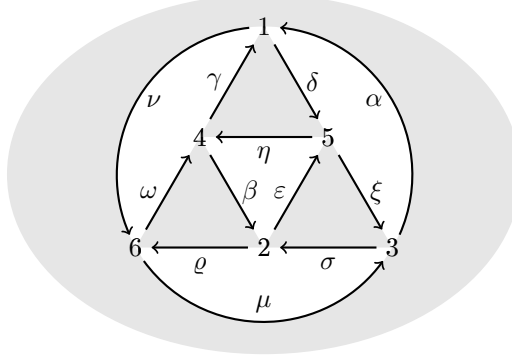
with f-orbits described by the shaded triangles and g-orbits described by the white triangles, and the weight function $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$ taking value 1. Then Λ is isomorphic to a tetrahedral algebra $\Lambda(\lambda)$, for some $\lambda \in K^*$.

This leaves the case, where one of the three arrows is a loop. We label the arrows now as α, β, γ and we assume α is a loop, and f has the cycle $(\alpha\ \beta\ \gamma)$ and we have $q(\alpha) = q(\beta) = q(\gamma) = 3$. Then $g(\beta)$ must be a loop, σ say, which then has to be fixed by f. Hence Q has two vertices, and the algebra be isomorphic to $D(\lambda)$ for some $\lambda \in K^*$. $\square$

Proposition 6.7. *Let $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ be a weighted triangulation algebra which is not isomorphic to one of the algebras $D(\lambda)$, $T(\lambda)$, $S(\lambda)$, $\Lambda(\lambda)$, with $\lambda \in K^*$, $S(r, c_\bullet)$ with $r \geq 2$, $\Sigma(r, c_\bullet)$ with $r \geq 3$, $\Omega(r, c_\bullet)$ with $r \geq 4$, $\Phi(c_\bullet)$, or $\Psi(r, c_\bullet)$, for some $r \geq 2$. Then Λ degenerates to the biserial weighted triangulation algebra $B = B(Q, f, m_\bullet, c_\bullet)$. In particular, Λ is tame.*

Proof. Let $M, q, v : Q_1 \to \mathbb{N}^*$ be the functions defined above. We set now $u(\alpha) := \frac{M}{q(\alpha)}$, this is an integer. It follows from Proposition 6.6 and the assumption that $v(\alpha) \geq 2$ for any arrow $\alpha \in Q_1$. For each $t \in K$, consider the algebra $\Lambda(t)$ given by the quiver Q and the relations:

- $\alpha f(\alpha) = c_{\bar\alpha} t^{v(\alpha)} A_{\bar\alpha}$ for any arrow $\alpha \in Q_1$,
- $c_\alpha B_\alpha = c_{\bar\alpha} B_{\bar\alpha}$ for any arrow $\alpha \in Q_1$,
- $\alpha f(\alpha) g(f(\alpha)) = 0$ for any arrow $\alpha \in Q_1$ with $f^{-1}(\alpha)$ not virtual,
- $\alpha g(\alpha) f(g(\alpha)) = 0$ for any arrow $\alpha \in Q_1$ with $f(\alpha)$ not virtual.

Then $\Lambda(\lambda)$, $\lambda \in K$, is an algebraic family in the variety $\mathrm{alg}_d(K)$, with $d = \sum_{\mathcal{O} \in \mathcal{O}(g)} m_{\mathcal{O}} n_{\mathcal{O}}^2$. Observe also that $\Lambda(1) = \Lambda$ and $\Lambda(0) = B$. Further, we claim that for any $t \in K^*$, we have an isomorphism of K-algebras $\varphi_t : \Lambda(1) \to \Lambda(t)$ given by $\varphi_t(\alpha) = t^{u(\alpha)}\alpha$ for any arrow $\alpha \in Q_1$, and extension to products. Indeed, it follows from definition of the function u that $\varphi_t(B_\sigma) = t^M B_\sigma$ for any arrow $\sigma \in Q_1$, and hence $\varphi_t(A_\sigma) = t^{M-u(f^2(\bar\sigma))} A_\sigma$, because $B_\sigma = A_\sigma g^{n_\sigma - 1}(\sigma)$ and $g^{n_\sigma - 1}(\sigma) = f^2(\bar\sigma)$. Then, for any arrow $\alpha \in Q_1$, we have the equalities

$$\varphi_t\big(\alpha f(\alpha)\big) = t^{u(\alpha)+u(f(\alpha))}\alpha f(\alpha) = t^{u(\alpha)+u(f(\alpha))} t^{v(\alpha)} c_{\bar\alpha} A_{\bar\alpha} = t^{M-u(f^2(\alpha))} c_{\bar\alpha} A_{\bar\alpha}$$
$$= \varphi_t(c_{\bar\alpha} A_{\bar\alpha}),$$

and hence φ_t is a well-defined isomorphism of K-algebras. Therefore, by Proposition 2.2, Λ degenerates to B, and Λ is a tame algebra. $\qquad\square$

We shall prove now that every weighted surface algebra non-isomorphic to a disc algebra, triangle algebra, tetrahedral algebra, or spherical algebra is of non-polynomial growth. We consider first two distinguished cases.
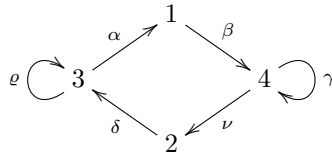
Example 6.8. Let T be the triangulation
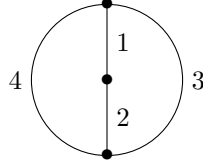
of the unit disc $D = D^2$ in $\mathbb{R}^2$ by two triangles and $\vec{T}$ the orientation $(1\ 3\ 2)$, $(2\ 4\ 1)$ of triangles in T. Then the triangulation quiver $(Q, f) = (Q(D, \vec{T}), f)$ is the quiver
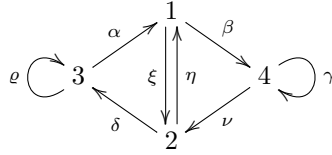
with f-orbits $(\alpha\ \xi\ \delta)$, $(\beta\ \nu\ \eta)$, (ϱ), (γ). Then g has two orbits, $\mathcal{O}(\alpha) = (\alpha\ \beta\ \gamma\ \nu\ \delta\ \varrho)$ and $\mathcal{O}(\xi) = (\xi\ \eta)$. Let $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$ be the trivial multiplicity function and $c_\bullet : \mathcal{O}(g) \to K^*$ an arbitrary parameter function. We write $c_{\mathcal{O}(\alpha)} = a$ and $c_{\mathcal{O}(\xi)} = b$. Then the weighted surface algebra $D(a,b)^{(2)} = \Lambda(D, \vec{T}, m_\bullet, c_\bullet)$ is given by the above quiver and the relations:

$$
\begin{array}{lllll}
\alpha\xi = a\varrho\alpha\beta\gamma\nu, & \xi\delta = a\beta\gamma\nu\delta\varrho, & \delta\alpha = b\eta, & \varrho^2 = a\alpha\beta\gamma\nu\delta, & \alpha\xi\eta = 0, \\
\nu\eta = a\gamma\nu\delta\varrho\alpha, & \eta\beta = a\delta\varrho\alpha\beta\gamma, & \beta\nu = b\xi, & \gamma^2 = a\nu\delta\varrho\alpha\beta, & \xi\eta\beta = 0, \\
\xi\delta\varrho = 0, & \nu\eta\xi = 0, & \eta\beta\gamma = 0, & \varrho^2\alpha = 0, & \gamma^2\nu = 0, \\
\varrho\alpha\xi = 0, & \eta\xi\delta = 0, & \gamma\nu\eta = 0, & \delta\varrho^2 = 0, & \beta\gamma^2 = 0.
\end{array}
$$

Observe that the algebra $D(a, b)^{(2)}$ is isomorphic to the algebra $D(ab, 1)^{(2)}$. Indeed, there is an isomorphism of algebras $\varphi : D(ab, 1)^{(2)} \to D(a, b)^{(2)}$ given by $\varphi(\alpha) = \alpha$, $\varphi(\xi) = b\xi$, $\varphi(\delta) = \delta$, $\varphi(\varrho) = \varrho$, $\varphi(\beta) = \beta$, $\varphi(\gamma) = \gamma$, $\varphi(\nu) = \nu$, $\varphi(\eta) = b\eta$. For $\lambda \in K^*$, we set $D(\lambda)^{(2)} = D(\lambda, 1)^{(2)}$. We see now that $D(\lambda, 1)^{(2)}$ is given by its Gabriel quiver $Q^{(2)}$

and the relations:

$$\alpha\beta\nu = \lambda\varrho\alpha\beta\gamma\nu, \qquad \beta\nu\delta = \lambda\beta\gamma\nu\delta\varrho, \qquad \varrho^2 = \lambda\alpha\beta\gamma\nu\delta,$$

$$\nu\delta\alpha = \lambda\gamma\nu\delta\varrho\alpha, \qquad \delta\alpha\beta = \lambda\delta\varrho\alpha\beta\gamma, \qquad \gamma^2 = \lambda\nu\delta\varrho\alpha\beta,$$

$$\alpha\beta\nu\delta\alpha = 0, \quad \beta\nu\delta\varrho = 0, \quad \nu\delta\alpha\beta\nu = 0, \quad \delta\alpha\beta\gamma = 0, \quad \varrho^2\alpha = 0, \quad \gamma^2\nu = 0,$$

$$\beta\nu\delta\alpha\beta = 0, \quad \varrho\alpha\beta\nu = 0, \quad \delta\alpha\beta\nu\delta = 0, \quad \gamma\nu\delta\alpha = 0, \quad \delta\varrho^2 = 0, \quad \beta\gamma^2 = 0.$$

We consider also the orbit algebra $D(a,b)^{(1)} = D(a,b)^{(2)}/H$ of $D(a,b)^{(2)}$ with respect to action of the cyclic group of order 2 on $D(a,b)^{(2)}$ given by the cyclic rotation of vertices and arrows of the quiver Q:

$$(1\ 2), \qquad (3\ 4), \qquad (\alpha\ \nu), \qquad (\beta\ \delta), \qquad (\xi\ \eta), \qquad (\gamma\ \varrho).$$

Then $D(a,b)^{(1)}$ is given by the triangulation quiver (Q', f') of the form

$$\xi \,\circlearrowleft\, 1 \underset{\alpha}{\overset{\beta}{\rightleftarrows}} 3 \,\circlearrowright\, \gamma$$

with f'-orbits $(\xi\ \beta\ \alpha)$ and (γ), and the relations:

$$\alpha\xi = a\gamma\alpha\beta\gamma\alpha, \qquad \xi\beta = a\beta\gamma\alpha\beta\gamma, \quad \beta\alpha = b\xi, \quad \gamma^2 = a\alpha\beta\gamma\alpha\beta,$$

$$\alpha\xi^2 = 0, \quad \xi\beta\gamma = 0, \quad \gamma^2\alpha = 0, \qquad \xi^2\beta = 0, \quad \gamma\alpha\xi = 0, \quad \beta\gamma^2 = 0.$$

We note that $D(a,b)^{(1)}$ is the weighted triangular algebra $\Lambda(Q', f', m'_\bullet, c'_\bullet)$, where the weight function $m'_\bullet : \mathcal{O}(g') \to \mathbb{N}^*$ is given by $m'_{\mathcal{O}(\alpha)} = 2 = m'_{\mathcal{O}(\xi)}$, and the parameter function $c'_\bullet : \mathcal{O}(g') \to K^*$ by $c'_{\mathcal{O}(\alpha)} = a$ and $c'_{\mathcal{O}(\xi)} = b$. Similarly as above, we conclude that $D(a,b)^{(1)}$ is isomorphic to the algebra $D(ab,1)^{(1)}$. For $\lambda \in K^*$, we set $D(\lambda)^{(1)} = D(\lambda,1)^{(1)}$. Further, the Gabriel quiver $Q^{(1)}$ of $D(\lambda)^{(1)}$ is the orbit quiver $Q^{(2)}/H$ of the Gabriel quiver $Q^{(2)}$ of $D(\lambda)^{(2)}$ with respect to the induced action of H, and is of the form

$$1 \underset{\alpha}{\overset{\beta}{\rightleftarrows}} 3 \,\circlearrowright\, \gamma \;.$$

Hence, $D(\lambda)^{(1)}$ is given by the quiver $Q^{(1)}$ and the relations:

$$\alpha\beta\alpha = \lambda\gamma\alpha\beta\gamma\alpha, \qquad \beta\alpha\beta = \lambda\beta\gamma\alpha\beta\gamma, \qquad \gamma^2 = \lambda\alpha\beta\gamma\alpha\beta,$$

$$\alpha\beta\alpha\beta\alpha = 0, \quad \beta\alpha\beta\gamma = 0, \quad \gamma^2\alpha = 0, \quad \beta\alpha\beta\alpha\beta = 0, \quad \gamma\alpha\beta\alpha = 0, \quad \beta\gamma^2 = 0.$$

Lemma 6.9. *For each $\lambda \in K^*$, the algebras $D(\lambda)^{(1)}$ and $D(\lambda)^{(2)}$ are of non-polynomial growth.*
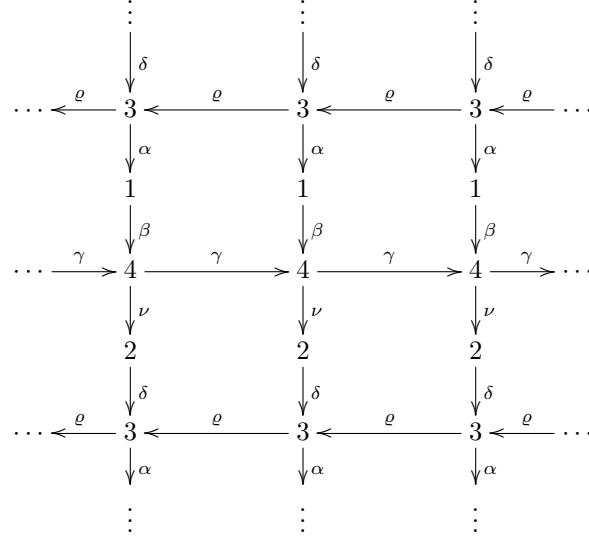
Proof. We fix $\lambda \in K^*$ and consider the quotient algebra $A^{(2)}$ of $D(\lambda)^{(2)}$ given by its Gabriel quiver $Q^{(2)}$ and the zero-relations:

$$\alpha\beta\nu = 0, \qquad \beta\nu\delta = 0, \qquad \nu\delta\alpha = 0, \qquad \delta\alpha\beta = 0, \qquad \varrho^2 = 0, \qquad \gamma^2 = 0,$$

$$\varrho\alpha\beta\gamma\nu = 0, \quad \alpha\beta\gamma\nu\delta = 0, \quad \beta\gamma\nu\delta\varrho = 0, \quad \gamma\nu\delta\varrho\alpha = 0, \quad \nu\delta\varrho\alpha\beta = 0, \quad \delta\varrho\alpha\beta\gamma = 0.$$
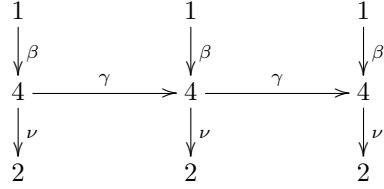
Then $A^{(2)}$ admits the Galois covering

$$F^{(2)} : R \to R/G^{(2)} = A^{(2)}$$

where R is the locally bounded category given by the infinite quiver

and the induced relations, and $G^{(2)}$ is the free abelian group of rank 2 generated by the obvious horizontal and vertical shifts. We observe now that R contains the full convex subcategory B given by the quiver

and the relation $\gamma^2 = 0$, which is a minimal non-polynomial growth algebra of type (1) in [29, Theorem 3.2]. Hence, by general theory (see [6, 7, 19]), $A^{(2)}$ is an algebra of non-polynomial growth.

Similarly, consider the quotient algebra $A^{(1)}$ of $D(\lambda)^{(1)}$ given by its Gabriel quiver $Q^{(1)}$ and the relations:

$$\alpha\beta\alpha = 0, \qquad \beta\alpha\beta = 0, \qquad \gamma\alpha\beta\gamma\alpha = 0, \qquad \alpha\beta\gamma\alpha\beta = 0, \qquad \beta\gamma\alpha\beta\gamma = 0.$$

Then $A^{(1)}$ admits the Galois covering

$$F^{(1)} : R \to R/G^{(1)} = A^{(1)}$$

where R is the locally bounded category considered above and $G^{(1)}$ is the free abelian group of rank 2 generated by the obvious horizontal and vertical shifts such that $G^{(2)}$ is a subgroup of $G^{(1)}$ and $G^{(1)}/G^{(2)}$ is the Klein group $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. Then we conclude as above that $A^{(1)}$ is of non-polynomial growth, and consequently $D(\lambda)^{(1)}$ is of non-polynomial growth.

The algebras $D(\lambda)^{(1)}$ and $D(\lambda)^{(2)}$, for $\lambda \in K^*$, are tame algebras (see Proposition 6.7). $\qquad\qquad\square$

Let (Q, f) be a triangulation quiver, $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$ a weight function, and $c_\bullet : \mathcal{O}(g) \to K^*$ a parameter function. We consider the quotient algebra

$$\Gamma(Q, f, m_\bullet, c_\bullet) = KQ/L(Q, f, m_\bullet, c_\bullet),$$

where $L(Q, f, m_\bullet, c_\bullet)$ is the ideal in the path algebra KQ of Q over K generated by the elements $\alpha f(\alpha)$ and A_α, for all arrows $\alpha \in Q_1$. Then $\Gamma(Q, f, m_\bullet, c_\bullet)$ is a string algebra, which we call the string algebra of the weighted triangulation algebra $\Lambda(Q, f, m_\bullet, c_\bullet)$. We note that it is the largest string quotient algebra of $\Lambda(Q, f, m_\bullet, c_\bullet)$, with respect to dimension, and the Gabriel quiver of $\Gamma(Q, f, m_\bullet, c_\bullet)$ is obtained from Q by removing all virtual arrows. Observe also that $\Gamma(Q, f, m_\bullet, c_\bullet)$ is a quotient algebra of the biserial weighted triangulation algebra $B(Q, f, m_\bullet, c_\bullet)$.

Theorem 6.10. *Let $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ be a weighted triangulation algebra which is not isomorphic to one of the algebras $D(\lambda)$, $D(\lambda)^{(1)}$, $D(\lambda)^{(2)}$, $T(\lambda)$, $S(\lambda)$, $\Lambda(\lambda)$, for $\lambda \in K^*$. Then $\Gamma = \Gamma(Q, f, m_\bullet, c_\bullet)$ is of non-polynomial growth. In particular, Λ is of non-polynomial growth.*

Proof. We have the presentation $\Gamma = KQ_\Gamma/I_\Gamma$, where Q_Γ is the Gabriel quiver of Γ and $I_\Gamma = L(Q, f, m_\bullet, c_\bullet) \cap KQ_\Gamma$. Observe that I_Γ is the ideal in the path algebra KQ_Γ generated by paths $\alpha f(\alpha)$ and A_α for all non-virtual arrows α in Q_1. By general theory, in order to prove that Γ is of non-polynomial growth, it is sufficient to indicate two primitive walks v and w of the bound quiver (Q_Γ, I_Γ) such that vw and wv are also primitive walks (see the proof of [33, Lemma 1]. We consider several cases.

(1) Assume $m_\alpha n_\alpha \geq 3$ for all $\alpha \in Q_1$. If $|Q_0| \geq 3$, then the required primitive walks are constructed in the proof of [15, Proposition 10.2]. If $|Q_0| = 2$, then (Q, f) is of the form

$$\alpha \circlearrowleft 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \circlearrowright \sigma$$

with f-orbits $(\alpha\ \beta\ \gamma)$ and (σ). Since Λ is not isomorphic to a disc algebra $D(\lambda)$, we have $m_\alpha \geq 4$ or $m_\beta \geq 2$. If $m_\alpha \geq 4$, we may take $v = \alpha\gamma^{-1}\sigma\beta^{-1}$ and $w = \alpha^2\gamma^{-1}\sigma\beta^{-1}$. For $m_\beta \geq 2$, we may take $v = \alpha\gamma^{-1}\sigma\beta^{-1}$ and $w = \alpha\gamma^{-1}\sigma\beta^{-1}\gamma^{-1}\sigma\beta^{-1}$.

We may then assume that $|Q_0| \geq 3$.

(2) Assume now that there is a virtual loop α in Q_1. Then (Q, f) contains a subquiver of the form

$$\alpha \circlearrowleft 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2$$

with f-orbit $(\alpha\ \beta\ \gamma)$, $\beta = g(\gamma)$, and $g(\beta) \neq g^{-1}(\gamma)$. In particular, we have $n_\beta \geq 4$. In the special case $m_\beta n_\beta = 4$, (Q, f) is the quiver

$$\alpha \circlearrowleft 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \underset{\delta}{\overset{\sigma}{\rightleftarrows}} 3 \circlearrowright \eta$$
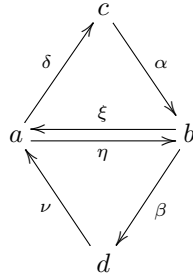
with f-orbits $(\alpha\ \beta\ \gamma)$ and $(\eta\ \delta\ \sigma)$, and g-orbits (α), $(\beta\ \sigma\ \delta\ \gamma)$, (η). Since Λ is not isomorphic to a triangle algebra $T(\lambda)$, we conclude that η is not virtual, and hence $m_\eta \geq 4$. Then we may take $v = \delta\beta^{-1}\gamma^{-1}\sigma$ and $w = \delta\beta^{-1}\gamma^{-1}\sigma\eta^{-1}$. We also note that, if (Q, f) is the above quiver and $m_\beta \geq 2$, then we may take $v = \delta\beta^{-1}\gamma^{-1}\sigma$

and $w = \delta\beta^{-1}\gamma^{-1}\sigma\delta\sigma$. Hence we may assume that $|Q_0| \geq 4$. Clearly, then $n_\beta \geq 5$, and hence $m_\beta n_\beta \geq 5$. Let $\sigma = g(\beta)$, $\delta = g^{-1}(\gamma)$, and $\xi = f(\sigma)$. Assume ξ is a virtual arrow. Then (Q, f) admits a subquiver of the form
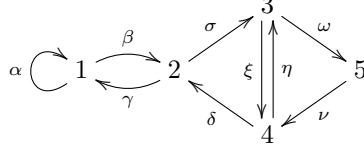
with f-orbits $(\alpha\ \beta\ \gamma)$, $(\sigma\ \xi\ \delta)$, $(\omega\ \nu\ \eta)$. Then we conclude that $n_\beta \geq 7$. Let u be the path $g^3(\beta)g^4(\beta)\dots g^{n_\beta-4}(\beta) = g(\omega)\dots g^{-1}(\nu)$ of length ≥ 1 from 5 to 5. Observe that $\gamma\beta$, $\sigma\omega$, $\nu\delta$ and u are non-zero paths in Γ. Then we may take the primitive walks $v = \nu\delta\beta^{-1}\gamma^{-1}\sigma\omega$ and $w = \nu\delta\beta^{-1}\gamma^{-1}\sigma\omega u^{-1}$. Finally, assume that the arrow ξ is not virtual, and so ξ is an arrow of Q_Γ. Let p be the path $g^2(\beta)g^3(\beta)\dots g^{n_\beta-3}(\beta) = g(\sigma)\dots g^{-1}(\delta)$ of length ≥ 1. We note that $\sigma p\delta$ is a non-zero path of Γ because it is of length $n_\beta - 2$. We may take the required primitive walks as follows $v = \delta\beta^{-1}\gamma^{-1}\sigma p$ and $w = \delta\beta^{-1}\gamma^{-1}\sigma p\xi^{-1}p$.

(3) Assume now that there is a pair ξ, η of virtual arrows in Q_1. Then (Q, f) contains a subquiver of the form
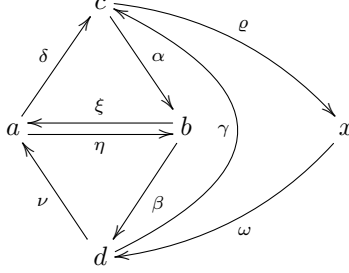
with f-orbits $(\alpha\ \xi\ \delta)$ and $(\beta\ \nu\ \eta)$. Consider first the case when $c = d$, so (Q, f) is of the form
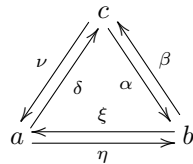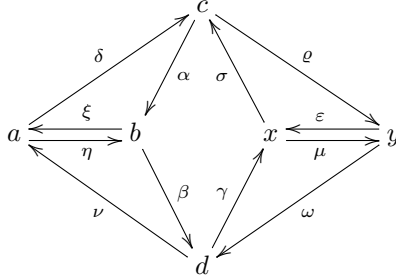
with g-orbits $(\xi\ \eta)$, $(\alpha\ \beta)$, $(\nu\ \delta)$. It follows from our general assumption that α, β, γ, δ are not virtual arrows, and hence $m_\alpha n_\alpha \geq 4$ and $m_\delta n_\delta \geq 4$. Moreover, because Λ is not isomorphic to a triangle algebra $T(\lambda)$, we may assume that $m_\alpha n_\alpha \geq 6$. Then $v = \alpha\beta\delta^{-1}\nu^{-1}$ and $w = \alpha\beta\alpha\beta\delta^{-1}\nu^{-1}$ is a required pair of primitive walks.

Assume now that $c \neq d$. Then $n_\alpha \geq 3$ and $n_\delta \geq 3$. We note that if $\mathcal{O}(\alpha) = \mathcal{O}(\delta)$, then $n_\alpha = n_\delta \geq 6$. Consider first the case when one of $m_\alpha n_\alpha$ or $m_\delta n_\delta$, say $m_\alpha n_\alpha$,

is equal to 3. Then (Q, f) contains a subquiver of the form

with f-orbits $(\alpha \ \xi \ \delta)$, $(\beta \ \nu \ \eta)$, $(\gamma \ \varrho \ \omega)$, and $n_\delta \geq 5$. We denote by u the path $g^2(\delta) \ldots g^{n_\delta-3}(\delta)$ of length ≥ 1 from x to x. Observe that $\nu\delta\varrho$, $\omega\nu\delta$ and u are non-zero paths in Γ. Then we may choose the pair of primitive walks $v = \nu\delta\gamma^{-1}$ and $w = \nu\delta\varrho p^{-1}\omega$. We consider now the case when $m_\alpha n_\alpha = 4 = m_\delta n_\delta$. Then $n_\alpha = 4 = n_\delta$, $m_\alpha = 1 = m_\delta$, and (Q, f) is the quiver of the form
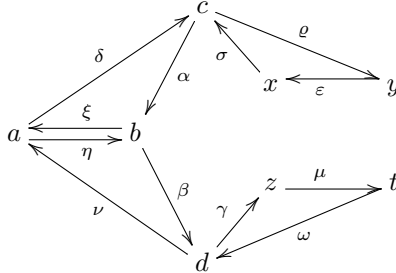
with f-orbits $(\alpha \ \xi \ \delta)$, $(\beta \ \nu \ \eta)$, $(\gamma \ \mu \ \omega)$, $(\sigma \ \varrho \ \varepsilon)$, and hence g-orbits $(\xi \ \eta)$, $(\alpha \ \beta \ \gamma \ \sigma)$, $(\delta \ \varrho \ \omega \ \nu)$, $(\varepsilon \ \mu)$. Since Λ is not isomorphic to a spherical algebra $S(\lambda)$, we have $m_\eta n_\eta \geq 4$. We note that $\nu\delta$ and $\gamma\sigma$ are non-zero paths in Γ. Then we may take a required pair of primitive walks as follows: $v = \gamma^{-1}\nu\delta\sigma^{-1}$ and $w = \mu\varepsilon\gamma^{-1}\nu\delta\sigma^{-1}$.

Assume now that $n_\alpha \geq 4$ and $n_\delta \geq 5$. We have two cases to consider . Let $\mathcal{O}(\alpha) \neq \mathcal{O}(\delta)$. We denote

$$\gamma = g(\beta), \quad \sigma = g^{-1}(\alpha), \quad \mu = f(\gamma), \quad \omega = f(\mu), \quad \varrho = f(\sigma), \quad \varepsilon = f(\varrho).$$
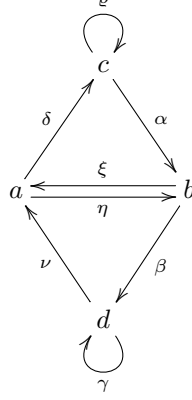
Then (Q, f) contains a subquiver

where $y \neq t$, and possibly $x = z$. Moreover, we have the path $q = g^2(\delta) \ldots g^{n_\delta-3}(\delta)$ of length ≥ 1 from y to t, and the subpath p of $\alpha g(\alpha) \ldots g^{n_\alpha-1}(\alpha)$ of length ≥ 0 from z to x. Then we may choose a required pair of primitive walks as follows: $v = \sigma\delta^{-1}\nu^{-1}\gamma p$ and $w = \sigma\delta^{-1}\nu^{-1}\gamma p\varepsilon^{-1}q\mu^{-1}p$.

Finally, assume that $\mathcal{O}(\alpha) = \mathcal{O}(\delta)$. Consider first the case $n_\alpha = 6$. Then (Q, f) is of the form

with $f(\varrho) = \varrho$, $f(\gamma) = \gamma$, and $\mathcal{O}(\alpha) = (\alpha\ \beta\ \gamma\ \nu\ \delta\ \varrho)$. Since Λ is not isomorphic to an algebra $D(\lambda)^{(2)}$, with $\lambda \in K^*$, we have also $m_\alpha \geq 2$. Then we may take a required pair of primitive walks as follows: $v = \alpha\beta\gamma^{-1}\nu\delta\varrho^{-1}$ and $w = \alpha\beta\gamma^{-1}\nu\delta\varrho^{-1}\alpha\beta\gamma\nu\delta\varrho$. Assume now that $n_\alpha \geq 7$. Then one of the arrows $\gamma = g(\beta)$ or $\varrho = g(\delta)$, say γ, is not fixed by f. Hence we have a triangle
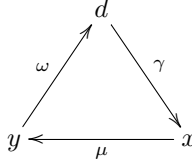
with $\mu = f(\gamma)$ and $\omega = f(\mu)$. Consider the subpaths of $\alpha g(\alpha) \ldots g^{n_\alpha-1}(\alpha)$: $p = g(\delta) \ldots g^{n_\alpha-1}(\alpha)$ and $q = g(\gamma) \ldots g^{-1}(\omega)$. Then we choose a required pair of primitive walks as follows: $v = \alpha\beta\omega^{-1}q^{-1}\gamma^{-1}\nu\delta p^{-1}$ and $w = \alpha\beta\gamma q\mu^{-1}q\omega\nu\delta p^{-1}$. $\qquad\square$

We have also the following consequence of Propositions 3.8 and 3.9, Lemma 6.9, and Theorem 6.10.

Theorem 6.11. *Let $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ be a weighted triangulation algebra. Then the following statements are equivalent:*

(i) Λ is of polynomial growth.
(ii) Λ is isomorphic to a non-singular disc, triangle, tertahedral, or spherical algebra.

Acknowledgements

References

[1] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.

[2] J. Białkowski, K. Erdmann, A. Skowroński, Periodicity of self-injective algebras of polynomial growth, J. Algebra 443 (2015) 200–269.

[3] I. Burban, O. Iyama, B. Keller, I. Reiten, Cluster tilting for one-dimensional hypersurface singularities, Adv. Math. 217 (2008) 2443–2484.

[4] M. C. R. Butler, C. M. Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras, Comm. Algebra 15 (1987) 145–179.

[5] W. Crawley-Boevey, Tameness of biserial algebras, Arch. Math. (Basel) 65 (1995) 399–407.

[6] P. Dowbor, A. Skowroński, On the representation type of locally bounded categories, Tsukuba J. Math. 10 (1986) 63–72.

[7] P. Dowbor, A. Skowroński, On Galois coverings of tame algebras, Arch. Math. (Basel) 44 (1985) 522–529.

[8] P. Dowbor, A. Skowroński, Galois coverings of representation-infinite algebras, Comment. Math. Helv. 62 (1987) 311–337.

[9] A. Dugas, Periodic resolutions and self-injective algebras of finite type, J. Pure Appl. Algebra 214 (2010) 990–1000.

[10] K. Erdmann, On the number of simple modules of certain tame blocks and algebras, Arch. Math. (Basel) 51 (1988) 34–38.

[11] K. Erdmann, Blocks of Tame Representation Type and Related Algebras, in: Lecture Notes in Math., vol. 1428, Springer-Verlag, Berlin-Heidelberg, 1990.

[12] K. Erdmann, A. Skowroński, On Auslander-Reiten components of blocks and self-injective biserial algebras, Trans. Amer. Math. Soc. 330 (1992) 165–189.

[13] K. Erdmann, A. Skowroński, The stable Calabi-Yau dimension of tame symmetric algebras, J. Math. Soc. Japan 58 (2006) 97–128.

[14] K. Erdmann, A. Skowroński, Periodic algebras, in: Trends in Representation Theory of Algebras and Related Topics, in: Europ. Math. Soc. Series Congress Reports, European Math. Soc., Zürich, 2008, pp. 201–251.

[15] K. Erdmann, A. Skowroński, Weighted surface algebras, J. Algebra 505 (2018) 490–558.

[16] K. Erdmann, A. Skowroński, Higher tetrahedral algebras, Algebr. Represent. Theor. 2018, in press, DOI 10.1007/s10468-018-9772-x.

[17] K. Erdmann, A. Skowroński, Algebras of generalized dihedral type, Nagoya Math. J., 2019, in press, DOI 10.1017/nmj.2019.1.

[18] K. Erdmann, A. Skowroński, From Brauer graph algebras to biserial weighted surface algebras, J. Algebr. Comb., 2018, in press, DOI 10.1007/s10801-018-0867-6.

[19] P. Gabriel, The universal cover of a representation-finite algebra, in: Representations of Algebras, in: Lecture Notes in Math., vol. 903, Springer-Verlag, Berlin-Heidelberg, 1981, pp. 68–105.

[20] C. Geiss, On degenerations of tame and wild algebras, Arch. Math. (Basel) 64 (1995) 11–16.

[21] E. L. Green, N. Snashall, Ø. Solberg, The Hochschild cohomology ring of a selfinjective algebra of finite representation type, Proc. Amer. Math. Soc. 131 (2003) 3387–3393.

[22] D. Happel, Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras, London Math. Soc. Lect. Note Ser., vol. 119, Cambridge University Press, Cambridge, 1988.

[23] D. Happel, Hochschild cohomology of finite-dimensional algebras, in: Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, in: Lecture Notes in Math., vol. 1404, Springer-Verlag, Berlin-Heidelberg, 1989, pp. 108–126.

[24] D. Happel, C. M. Ringel, The derived category of a tubular algebra, in: Representation Theory I, in: Lecture Notes in Math., vol. 1177, Springer-Verlag, Berlin-Heidelberg, 1986, pp. 156–180.

[25] T. Holm, Derived equivalence classification of algebras of dihedral, semidihedral, and quaternion type, J. Algebra 211 (1999) 159–205.

[26] H. Kraft, Geometric methods in representation theory, in: Representations of Algebras, in: Lecture Notes in Math., vol. 944, Springer-Verlag, Berlin-Heidelberg, 1982, pp. 180–258.

[27] S. Ladkani, From groups to clusters, in: Representation Theory – Current Trends and Perspectives, in: Europ. Math. Soc. Ser. Congr. Rep., European Math. Soc., Zürich, 2017, pp. 427–500.

[28] J. Nehring, A. Skowroński, Polynomial growth trivial extensions of simply connected algebras, Fund. Math. 132 (1989) 117–134.

[29] R. Nörenberg, A. Skowroński, Tame minimal non-polynomial growth simply connected algebras, Colloq. Math. 73 (1997) 301–330.
[30] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. 43 (1991) 37–48.
[31] C. M. Ringel, Tame Algebras and Integral Quadratic Forms, in: Lecture Notes in Math., vol. 1099, Springer-Verlag, Berlin-Heidelberg, 1984.
[32] D. Simson and A. Skowroński, Elements of the Representation Theory of Associative Algebras 3: Representation-Infinite Tilted Algebras, London Mathematical Society Student Texts, vol. 72, Cambridge University Press, Cambridge, 2007.
[33] A. Skowroński, Group algebras of polynomial growth, Manuscripta Math. 59 (1987) 499–516.
[34] A. Skowroński, Selfinjective algebras of polynomial growth, Math. Ann. 285 (1989) 177–199.
[35] A. Skowroński, Selfinjective algebras: finite and tame type, in: Trends in Representation Theory of Algebras and Related Topics, in: Contemp. Math., vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 169–238.
[36] A. Skowroński, K. Yamagata, Frobenius Algebras 1: Basic Representation Theory, European Mathematical Society Textbooks in Mathematics, European Math. Soc. Publ. House, Zürich, 2011.
[37] B. Wald, J. Waschbüsch, Tame biserial algebras, J. Algebra 95 (1985) 480–500.

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