Snapshots of modern mathematics
from Oberwolfach

№ 4/2019

Positive Scalar Curvature and Applications

Jonathan Rosenberg[1] • David Wraith

We introduce the idea of curvature, including how it developed historically, and focus on the scalar curvature of a manifold. A major current research topic involves understanding positive scalar curvature. We discuss why this is interesting and how it relates to general relativity.

1 The concept of curvature

A well-known limerick, attributed to Leo Moser and found in [3], says the following about the famous mathematician Paul Erdős:

> A conjecture both deep and profound
> Is whether a circle is round.
> In a paper of Erdős
> Written in Kurdish
> A counterexample is found.

Obviously this is a joke, but it has some real mathematical content, because there are two notions of curvature, that is, "the state of being round", one *extrinsic* and one *intrinsic*, and a circle has the first and not the second. Extrinsic curvature is a measure of how a geometric object embedded in space curves relative to that space. A circle in the plane is a perfect example – at every point

[1] Jonathan Rosenberg is supported by the US National Science Foundation, grant number DMS-1607162.

it is "curving" toward the center, and the curvature of a circle is the reciprocal $1/r$ of its radius, r. Circles of small radius turn more sharply, and have larger curvature, than circles of big radius. In fact, in the limit as the radius increases to infinity, a circle looks less and less "curvy" or "round" relative to the plane around it. Intrinsic curvature, on the other hand, relates to properties of the geometry on the object itself, without regard to the space around it. Imagine a bug who lives on a circle and cannot see anything outside of this one-dimensional world. From his point of view, the circle is "flat", in the sense that measurement of distances is just as on a straight (Euclidean) line. The only difference is that if he walks far enough, he comes back to his starting point. So from the point of view of this bug, the circle is not round.

The distinction between intrinsic and extrinsic curvature becomes more interesting in the case of two-dimensional surfaces. This distinction was first studied in detail by Carl Friedrich Gauss in the 1820s [4]. As an example, consider the surface in three-dimensional space given as the graph $z = f(x, y)$ of the function

$$f(x,y) = \frac{1}{2}\left(Ax^2 + Bxy + Cy^2\right),$$

where A,B and C are real numbers. To measure the extrinsic geometry of a surface in 3-dimensional space, we consider the matrix of second derivatives[2] of f. In this case it is given by

$$S = \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}.$$

Of particular interest are the quantities $H = \frac{1}{2}(A + C)$, called the *mean curvature*, and $K = AC - \frac{B^2}{4}$, called the *Gaussian curvature*. At stationary points of f, the curvature K is given as the determinant of S, whereas H is half of its trace. To give another instructive example, the Gaussian curvature of the sphere

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = R^2\}$$

of radius R is equal to $K = \frac{1}{R^2}$ (to see why, try calculating the partial derivatives by writing the top half of the sphere as the graph of a function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$). So the curvature of the sphere is inversely proportional to the square of its radius. Gauss's interesting discovery, which he called his *Theorema*

[2] If we have a function of one variable, which we can picture as a curve in the plane, the first derivative gives us the slope of the tangent vector at each point. The second derivative is then the rate of change of the slopes. For a surface, that is, for a function of two variables, we obtain a matrix for the second derivative because we must calculate the derivatives with respect to two directions. The matrix of derivatives evaluated at the origin is referred to as the *shape operator* of the surface.

Egregium, or “excellent theorem”, is that the Gaussian curvature is actually an *intrinsic* invariant, in that it measures the intrinsic geometry and not just the geometry of the object as it appears in 3-dimensional space. For example, if $K = 0$, the surface is *flat*, in the sense that it can locally be flattened onto a flat sheet of paper (see Figure 1), whereas if $K > 0$, the surface is called *positively curved*, and if $K < 0$, the surface is *negatively curved*. In these latter cases, it can be proved that it is impossible to “flatten” the surface without distortion. The positive curvature of the earth’s surface explains why any 2-dimensional map of a portion of the Earth is necessarily distorted, while the fact that the curvature is the reciprocal of the square of the radius of the Earth (which is on the order of 6000 km) explains why one could make the mistake of assuming that the Earth is flat.

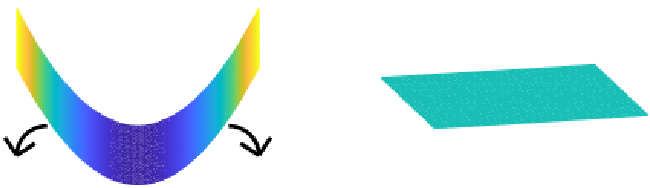

Figure 1: Flattening a surface with $K = 0$ (left) to a plane (right).

2 Manifolds and scalar curvature

The next major step after Gauss was taken by Riemann [11], who introduced two of the most fundamental notions in geometry, the notions of a *manifold* (of dimension n) and of a *Riemannian metric* (which he expressed in terms of what he called a *line element*), a way of measuring lengths of curves.

To introduce the concept of a manifold, let us begin by presenting the most elementary examples: the Euclidean spaces, of which there is one in each dimension. In dimension one we have the set of real numbers, which we can interpret as a line. In dimension two, we have the set of pairs of real numbers, $\{(x, y) \,|\, x, y \in \mathbb{R}\}$, or $\mathbb{R}^2$ for short, and by viewing pairs as coordinates we can identify $\mathbb{R}^2$ with the plane. In dimension three we have $\mathbb{R}^3 = \{(x, y, z) \,|\, x, y, z \in \mathbb{R}\}$, or in other words, the set of triples of real numbers, which we can identify with three-dimensional space. As we can consider collections of n real numbers (or *n-tuples*) for any positive integer n, we can define n-dimensional Euclidean space by $\mathbb{R}^n = \{(x_1, ..., x_n) \,|\, x_1, ..., x_n \in \mathbb{R}\}$, though if $n > 3$ this is not so easy to picture!

At this point it is worth commenting on the notion of dimension in mathematics. Although the idea of working in high dimensions might seem like science fiction, in fact the mathematical reality is much more mundane. The number of dimensions an object possesses is simply the number of parameters required to describe it. For instance, consider a pair of bugs living on a table-top. Each of the bugs can move to any point on the table (other than the point currently occupied by its friend), and we can describe the position of each bug by specifying a pair of coordinates. Thus to describe the positions of both bugs at any point in time requires four coordinates, and if we think about the set of all possible positions of the bugs, this is simply a subset of $\mathbb{R}^4$. If we want to record the position of the bugs as they change over time, then we need to introduce a time parameter, meaning that we are now working in $\mathbb{R}^5$. Indeed the movement of the bugs can then be described by a path through this five-dimensional space.

The idea of a manifold generalizes in a natural way the notion of a surface. Notice that any surface (for example the sphere) has the property that it *locally* looks like a region of the plane $\mathbb{R}^2$. Locally means that it is true for a small neighbourhood of each point, but not for the surface as a whole. More precisely, if we zoom in on any point, then what we see is essentially indistinguishable from a neighbourhood within $\mathbb{R}^2$. An n-dimensional manifold is the concept which results from an obvious generalization of this idea: an n-manifold M^n is an object which locally resembles $\mathbb{R}^n$, the n-dimensional Euclidean space. This “locally Euclidean” nature of manifolds allows us to generalize some of the natural notions of Euclidean space, such as coordinates: if we restrict attention to small enough pieces of M, our position is determined by the values of n *local coordinates* $x_1, x_2, \cdots, x_n$. With a little work we can generalize differential and integral calculus to a manifold setting. Indeed differentiation is a key tool for understanding the curvature of manifolds.

It is important to emphasize that the notion of a manifold is not just a generalization for generalization’s sake. Manifolds occur naturally throughout mathematics, science and beyond. For example, the set of points which our pair of bugs above can occupy forms a four-dimensional manifold. If these bugs happened instead to live on the surface of a ball, specifying the position of each bug would need three spatial coordinates, so the positions of both at any time can be recorded by an element of $\mathbb{R}^6$. The set of all possible positions of the two bugs is still a four-dimensional manifold (as the surface of the ball is two-dimensional, two of the dimensions are contributed by each bug), however now our manifold is naturally a subset of $\mathbb{R}^6$.

It is a theorem due to John Nash [10], a mathematician more famous for his contributions to game theory[3], that any manifold, having any degree of distortion, can be *embedded* into any Euclidean space of sufficiently high dimension in such a way that its shape is preserved. Embedding means that we can identify every point on the manifold with a point in the Euclidean space in such a way that the local properties are preserved. Moreover, in any one of these Euclidean spaces, we have infinitely many choices for this embedding. This suggests that if we want to investigate the geometry on our manifold, it would be be useful to have some way of doing this which avoids first having to choose some explicit embedding into an ambient space. In other words, we would like to find an intrinsic viewpoint from which to consider the geometric properties of manifolds. Riemann proposed a way of doing this, by introducing the concept that is now known as a Riemannian metric. Although we will not define this idea precisely, as it is somewhat technical, we will outline its key features.

In general a *metric* is a way of measuring the distance between any two points in a given space. For example, in the plane, the "usual" metric is defined by setting the distance between two points to be the length of the straight line segment that joins them. This is not the only option though, we could instead define the distance between two points to be the sum of the distance between the x-coordinates and the distance between the y-coordinates. This is known as the *taxi-cab* metric. On a surface we can measure lengths using a piece of string: given two points on the surface, imagine it is possible to run the string along the surface between the two points. (This is easier in theory than in practice!) Now pull the string taut, keeping it in contact with the surface, so it now occupies the shortest path between the two points. This shortest path is called a *geodesic*. In the plane $\mathbb{R}^2$ with the usual metric the geodesics are straight lines; on the surface of the sphere they are arcs of great circles. If we mark the string where it meets the two points, taking it off the surface and pulling it straight allows us to measure the distance with a ruler.

Basic plane geometry requires the notions of length and angles. We can think of an angle on a surface as being specified by two curves meeting at a point. For example, consider two lines of longitude on the earth meeting at the north pole. Of course the lines of longitude are really curves, but at any point such a curve has a *tangent line*, which is the straight line best fitting the curve at the point in question. At the north pole, all the possible tangent lines lie in the horizontal plane through the pole. Thus given any two lines of longitude, we obtain two tangent lines at the north pole belonging to the same plane (the *tangent plane* to the sphere at the pole), and we know how to measure angles in

[3] as depicted in the 2001 film 'A Beautiful Mind', see www.abeautifulmind.com

a plane. There is nothing special here about the sphere, lines of longitude, or the north pole: we can undertake angle measurements in exactly the same way at any point in any surface with any suitable pair of curves specifying an angle.

A Riemannian metric is a device which provides a manifold with notions of lengths and angles, and starting from this we can investigate any aspect of geometry. Thus a Riemannian metric provides a manifold with geometry, but crucially the converse is also true: any geometry expressed by a manifold corresponds to some Riemannian metric (for a given manifold, there may be more than one choice of Riemannian metric). So once a manifold has been equipped with a Riemannian metric, it makes sense to ask about areas or volumes for example, or to ask about curvature.

The Gaussian curvature, mentioned earlier, is the standard notion of curvature on a surface. It pre-dates Riemann's introduction of manifolds and Riemannian metrics. Although we all have an intuitive idea of curvature, it is important to know what the Gaussian curvature actually measures. We will provide one description below, but different (though equivalent) approaches are possible.

Everyone knows that the angles of a triangle add up to 180° degrees. However, as a general statement this popular phrase is simply not true! What is true is that the angles of a triangle in the plane add up to 180° degrees. But triangles can exist in spaces other than planes. Consider for example a sphere, or the surface of the earth. We can choose three distinct points on the sphere and join them with geodesics (remember that these are the shortest paths along the surface between these points). This results in what is called a *geodesic triangle* on our sphere. If we add the angles of this triangle, we will find that they add up to more than 180° degrees. Try drawing on a sphere the triangle that has one vertex at the north pole and the other two on the equator at a distance of one quarter of the circumference. You will find that the angles in this triangle are three right angles, so add up to 270°. Thus the angles of a triangle on a spherical surface are "fatter" than those of the corresponding triangle (that is, with the same side lengths) in the plane. On the other hand, if we were to repeat this exercise on a surface shaped like a saddle, the angles would add up to less than 180° degrees: the angles of a triangle on a saddle are "thinner" than the corresponding angles in the plane. An example is shown in Figure 2. Thus if α, β and γ denote the size of the angles in our geodesic triangle, the quantity $\alpha + \beta + \gamma - 180$, called the *angle excess*, detects the way in which the surface bends, with spherical surfaces having positive angle excess, planes having angle excess identically zero, and saddles with negative angle excess.

Although it detects curvature, the angle excess is not equivalent to the Gaussian curvature. The problem is that the angle excess will generally depend on the triangle you choose to look at, whereas what we would like is a way to

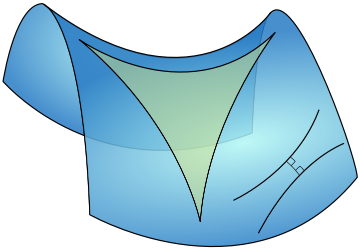

Figure 2: A triangle on a saddle-like surface, with angles that add up to less than 180°.

produce a number at each point of the surface which in some sense represents the curvature at that point. The idea behind the Gaussian curvature is as follows. Consider an infinite sequence of geodesic triangles containing some given point, which shrink down to the point, though never quite reach it. It turns out that if we divide the angle excess for each of these triangles by the area of the triangle, then the sequence of numbers we obtain always approaches some number, and this limiting number is precisely the Gaussian curvature at the point. Thus as suggested above, spheres are positively curved and saddles (like the surface $z = x^2 - y^2$) are negatively curved. In fact a surface at any point of positive Gaussian curvature will, to some extent, locally resemble a sphere, and will locally resemble a saddle in the case of negative Gaussian curvature.

Given that the notion of a manifold generalizes the notion of a surface, and that a Riemannian metric provides a manifold with geometry, it is natural to try and look for a generalization of the Gaussian curvature which works for all *Riemannian manifolds*, that is, manifolds equipped with a Riemannian metric. The basic idea is to reduce the problem back to dimension two. Consider an n-dimensional Riemannian manifold M. At a point x in M we have a "tangent space", which consists of all directions tangent to M at x. We can think of this as a copy of $\mathbb{R}^n$ with its origin "attached" to M at x, in the same way that we identified a horizontal plane as tangent to a sphere at the north pole in a previous example. Within this n-dimensional tangent space, there are infinitely many two-dimensional planes passing through the origin. Any one of these planes P can locally be integrated to a surface as follows: for $v \in P$ we follow the geodesic which starts at x with initial velocity v, and we follow this geodesic for some short time. Unifying these geodesics (over all possible starting velocities $v \in P$) then yields a small piece of surface through x whose tangent space at x is P. This surface has a Gaussian curvature at x. We define this Gaussian curvature to be the *sectional curvature* of M at x corresponding to the chosen plane.

The sectional curvature can therefore take many values at any point in a manifold, since there are infinitely many tangent planes at which we can evaluate it. This means that it is quite different from the Gaussian curvature, which as we have seen, is a function on a surface with each point having a single value. It is not unreasonable to hope to find a similar function on a Riemannian manifold, and in fact there is a straightforward way to do this. One merely takes the average of the sectional curvatures at each point (where we average over all 2-dimensional planes in the tangent space). The resulting function is called the *scalar curvature* of the manifold.

Being an average, one would expect much of the detailed curvature information present in the sectional curvature to be lost, and indeed this is the case. Nevertheless, there is still interesting geometric information remaining in the scalar curvature. For example, if $B(x, r)$ denotes a "ball" of radius r about a point x in a manifold M^n, by which we mean the set of all points in the manifold that are at most a distance r away from x, then the (n-dimensional) volume of $B(x, r)$ has an approximation for very small r given by the formula

$$\mathrm{vol}(B(x,r)) \approx r^n \omega(n)\Big(1 - \frac{n(n-1)}{6(n+2)} s(x) r^2\Big),$$

where $s(x)$ denotes the scalar curvature at x, and $\omega(n)$ is the volume of the standard n-dimensional ball of radius 1 in $\mathbb{R}^n$. For example, $\omega(2)$ is the area of a disc of radius 1 in the plane, so $\omega(2) = \pi$, and $\omega(3) = 4\pi/3$ is the volume of a sphere of radius 1. Thus the scalar curvature controls the volume of small balls in the manifold. To get a clearer idea of what this formula tells us, consider the case of the surface of a sphere of radius R again. Then, as already noted, $\omega(2) = \pi$ and the scalar curvature is just the Gaussian curvature, which here equals $1/R^2$, so we find

$$\mathrm{vol}(B(x,r)) \approx r^2 \pi \Big(1 - \frac{1}{12R^2} r^2\Big).$$

In other words, the larger the radius of the sphere, the closer the surface is to being flat, and the closer the area of a disc on the surface is to being equal to πr^2.

Whether the scalar curvature is positive or negative is of particular interest, though it is not as straightforward to interpret as the Gaussian curvature. It can be proved that any manifold of dimension at least three can be given a Riemannian metric with everywhere negative scalar curvature [8]. However the same cannot be said for positive scalar curvature: having positive scalar curvature imposes certain restrictions on a manifold. This raises the question of which manifolds admit metrics with positive scalar curvature. Although our understanding of this is well-developed, the picture at the time of writing is still some way from being complete.

Let us consider surfaces again. The usual round sphere (of any radius) has positive Gaussian curvature, as previously discussed. However the *torus*, which is what we call the surface of a donut, does not admit a Riemannian metric with everywhere positive Gaussian curvature, meaning that no matter how you embed this object into Euclidean space, there will always be points on the surface around which the torus appears to be flat or saddle-like. The same is true for a donut with two, three, or indeed any number of holes. We can generalize the notions of spheres and tori into higher dimensions. For example if n is any positive integer, the unit radius n-dimensional sphere is defined to be

$$S^n = \{(x_1, x_2, ..., x_{n+1}) \mid x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}.$$

Notice that using this notation the standard two-dimensional sphere is denoted S^2, and the circle S^1. For these objects in dimensions greater than two, the analogous comments apply when the Gaussian curvature is replaced by the scalar curvature.

If a manifold admits a Riemannian metric with positive scalar curvature (or if a surface admits positive Gaussian curvature), such a metric is not unique. That is, the manifold can express many different shapes with the positive curvature property. To see why, consider again an ordinary sphere. If we were to deform the shape very slightly (imagine again a balloon), then its curvature would change slightly. Provided this change is sufficiently small, the curvature will remain positive. Clearly, we can perform such small deformations in infinitely many different ways, demonstrating that in fact any manifold which admits a positive scalar curvature metric must admit infinitely many!

The next natural question to ask is what can be said about the collection of all positive scalar curvature metrics on any manifold for which this set is non-empty. It is not obvious that anything interesting can be said at all, but this set of metrics turns out to be a fascinating object in its own right, and is the focus of much current research. The ultimate aim is to have a good understanding of the topology of this set of metrics. One curious feature is that we can observe different phenomena in different dimensions, however we will postpone further discussion on this topic until Section 4.

3 Connections to general relativity

In the last section of the lecture in which he introduced the concepts of manifold and curvature [11], Riemann speculated that space itself might be a manifold and that its curvature might have physical significance. This prediction was borne out in a spectacular way in Albert Einstein's general theory of relativity [2].

Einstein's idea (which he got in part from Hendrik Lorentz, Hermann Minkowski, and Henri Poincaré) was to consider space and time as different "slices" through a single manifold called *space-time*. Thus if our physical universe is 3-dimensional, by adding time as another coordinate we get a 4-dimensional space-time. The geometry of this manifold is not exactly Riemannian, it is instead something called "pseudo-Riemannian". In many respects though, Riemannian geometry works as usual and one can define scalar curvature as before. Einstein's idea is that the curvature of space-time encodes what we think of as the "gravitational field". The basic equation of general relativity then says, under the simplifying assumption that one is looking at a portion of space-time where there is no matter (for example, in interstellar space), that space-time is what is now called an *Einstein manifold*, which means that the *Ricci curvature* is a constant. Here the Ricci curvature is an intermediate between the full curvature and the scalar curvature. It is calculated for a given direction in the manifold by taking the average over the sectional curvatures in two-dimensional slices which contain this direction, instead of averaging over all sectional curvatures.

This formulation uses "pseudo-Riemannian" geometry, and it would be nice to link it up with (ordinary) Riemannian geometry. We can do this if we assume that space-time splits as $M^3 \times \mathbb{R}$, where the first factor is "space" (with a genuine Riemannian metric, possibly time-dependent) and the second factor is "time".[4]

Now one can ask a natural question. Given a 3-manifold with a Riemannian metric, what geometric constraints does general relativity place on it so that it can be the "space" of space-time? This question (which one can generalize to arbitrary dimensions, though the connection with physics then becomes more obscure) led to what is called the *positive mass conjecture*. In many cases this is an actual theorem, the *positive mass theorem* [14, 15, 16], but generalizations of the theorem are still a subject of active research today.

The positive mass theorem basically concerns the following situation. Suppose M is a manifold with a Riemannian metric, and M is such that it "extends out to infinity". Also assume that out towards infinity, M is very close to a flat Euclidean space. (Imagine a physical situation where M is 3-dimensional, represents the "space" of space-time, and there is no matter out near infinity, so that space is not curved there.) Then from the equations of general relativity, one can infer from the metric of M what the mass had to be in the "curved" part of M. The positive mass theorem says that this mass is positive. While this might seem intuitively obvious (thinking of our own universe, or even solar system), it is not at all obvious mathematically.

[4] Indeed, the classical Schwarzschild and Robertson–Walker space-times are of this form.

4 Spaces of positive scalar curvature metrics

As we hinted above, much current research is about the question of what the space $\mathcal{R}^+(M)$ of Riemannian metrics of positive scalar curvature on a given manifold M looks like. The space of *all* Riemannian metrics, without curvature restrictions, is always *contractible*, that is, it can be continuously “squished” to a point. To make the discussion more concrete, let us investigate this property for spheres S^n in various dimensions n. It has been known for a long time that $\mathcal{R}^+(S^2)$ is contractible [13, Theorem 3.4], but that $\mathcal{R}^+(S^n)$ not only fails to be contractible but in fact has infinitely many “path components” if n is of the form $4k+3$ with $k \geq 1$ an integer ([5, Theorem 4.47] and [12, Theorem 2.6]). Let us consider what this means in more detail.

As noted above, the set of all Riemannian metrics on a manifold is contractible, so we can think of this as an infinite featureless “soup” of metrics, or, equivalently, geometries on our manifold. If we move around this soup in a continuous fashion, this is equivalent to continuously changing the geometry on our manifold. Suppose now we have two metrics which have positive scalar curvature. We can move through the soup of all metrics in many different ways from one of these geometries to the other. The question is: can we move from one to the other in such a way that all the intermediate geometries also have positive scalar curvature? On the two-dimensional sphere the answer is yes, but on S^{4k+3} with $k \geq 1$, the above result tells us the answer is no. In fact it is possible to choose infinitely many different positive scalar curvature metrics on S^{4k+3} such that no two of them can be joined by a path through positive scalar curvature metrics. This is what is meant by the statement that $\mathcal{R}^+(S^{4k+3})$ has infinitely many path components when $k \geq 1$.

Imagine we can view the soup of all metrics through special glasses which will highlight for us all metrics of positive scalar curvature. What we will see is a collection of “islands” of positive scalar curvature within the space of all metrics. Think of the soup now as minestrone! Each of these pieces floating in our soup is a path component of positive scalar curvature metrics. On S^{4k+3} we have infinitely many such pieces in our infinite bowl of soup. This, however, raises a further natural question: what can one say about these components of positive scalar curvature? What shape, or what features, do these pieces have? This turns out to be an even more delicate question, which we are only just starting to answer.

The first issue to consider is what kind of features we should look for. There are many ways in which one could interpret this question, but one could start by looking to identify “holes” in our space of positive scalar curvature metrics.

In order to illustrate our approach, let us consider the more straightforward problem of identifying such holes in the two-dimensional sphere S^2. If we draw any loop on this sphere, we can contract this loop down to a point without it having to leave the surface. Think of the loop as a stretched rubber band lying on the sphere, and then contract the rubber band. More formally, we can think of such a loop as a continuous map from the circle into the sphere, in other words, as a map $L\colon S^1 \to S^2$. We say that all such loops are "homotopic to a constant loop" to convey the contracting-a-rubber-band idea using more mathematical language. In greater detail, we say that two continuous maps from a space X to a space Y are homotopic if one can be deformed continuously into the other; a constant map is a map $C\colon X \to Y$ such that $C(x) = y_0$ for all $x \in X$ and some fixed $y_0 \in Y$, so a constant loop on S^2 is just a map $S^1 \to S^2$ for which the image of every point in S^1 is the same fixed point in S^2. The concept of maps being homotopic is very useful: there is, in general, a vast array of possible maps $X \to Y$, but if we choose to view two homotopic maps as equivalent (since they essentially contain the same information), then this means we can focus on a much smaller collection of maps. In particular, our observation that all loops on the sphere can be squished to a constant loop means that there is essentially only one loop on S^2 that we need consider, namely, a constant loop at some choice of "basepoint". It is not difficult to see that the same can be said for loops on any other sphere S^n with $n \geq 2$. Intuitively, we can think of this statement as saying that no loop on S^n for $n \geq 2$ encloses a two-dimensional hole, since if it did, we could not contract the loop over the hole.

There is, however, an obvious hole in the sphere. This is the three-dimensional hole enclosed by the sphere itself. We can capture this hole using the ideas above by saying that there are continuous maps $S^2 \to S^2$ (for example the "identity map" which maps each point of S^2 to itself) which cannot be squished to a point, or more formally, which are not homotopic to a constant map. Again, we can make the situation more tractable for ourselves by considering only homotopy classes (that is, considering only one representative from each set of homotopic maps) rather than all such maps.

Even though there are clearly no more holes to detect in S^2, there is nothing to stop us trying to extrapolate the above ideas by considering homotopy classes of maps $S^n \to S^2$ for $n \geq 3$. Perhaps not obviously, it turns out that this is an interesting thing to do, and it uncovers subtle aspects of the topology. This line of enquiry can be traced back to 1931, when Heinz Hopf discovered the existence of a map (now known as the "Hopf map") from S^3 to S^2 which is not homotopic to a constant map [7].

The computation of homotopy classes of maps from spheres into other spaces is usually difficult, but can be achieved in some important cases. One nice feature of such homotopy classes of maps is that they come ready-equipped with a natural algebraic structure: for a given sphere S^n and a given target space X, the set of homotopy classes of maps $S^n \to X$ form a *group*, which is a basic form of algebraic object, denoted by $\pi_n(X)$. This is called the n-dimensional homotopy group of X, and if $n = 1$, it is called the fundamental group of X. (We have omitted some technical details surrounding the definition of homotopy groups, however the above can be taken as accurate provided that X is path-connected, that is, that any two points in X can be joined by a continuous path.) Notice that we have argued above that the fundamental group of the sphere S^n is trivial (which means it consists of a single element) for $n \geq 2$.

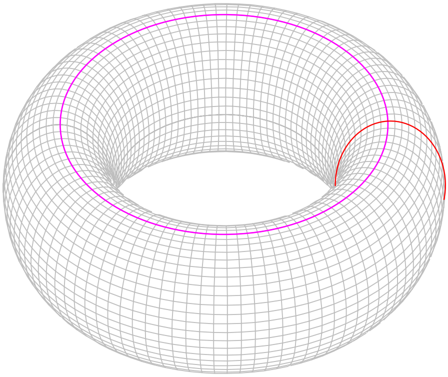

Figure 3: A torus, with two loops.

To provide a different example, consider a torus, that is, the boundary of a donut, or the inner-tube of a tyre. There is an obvious loop which runs once around the inner-most part of the torus. It is not difficult to see that this loop, in contrast to all loops on the sphere, is not homotopic to a constant loop. In fact, with a little more thought one can write down infinitely many more loops on the torus which are neither homotopic to a constant loop, nor homotopic to each other, see Figure 3. Thus the fundamental group of the torus is non-trivial, and we have found a tool which allows us to distinguish between the topology of spheres and tori.

The question is now: what can we say about the homotopy groups of spaces of positive scalar curvature metrics? Can these groups be non-trivial, and if so which ones, and how non-trivial? The simple answer to this question is yes, these groups can be non-trivial and in some cases highly non-trivial; however, our understanding of such phenomena is far from being complete. The most comprehensive answer to this question to date regards “spin” manifolds in dimension six and upwards: the homotopy groups $\pi_k(\mathcal{R}^+(M))$ are non-zero for infinitely many k [1]. For some of the π_k we know that they are non-trivial, for others, we even know that they are infinite, but for others we don't even know if they are trivial or not.

All in all, the space of positive scalar curvature metrics on a given manifold will typically have a highly complicated topological structure. The homotopy group results might suggest that the path components which make up this space take the form of a bizarre kind of Swiss cheese, with holes over an infinite range of dimensions. This is much more complicated than the minestrone analogy might lead us to believe! In fact, it is likely that the space of positive scalar curvature metrics is more complicated still than the study of homotopy groups is capable of revealing. Of course, this makes the result about the space of positive scalar curvature metrics on S^2 being contractible all the more striking.

To conclude this snapshot, let us return once more to the (apparently simpler) question of whether the space of positive scalar metrics within the space of all metrics is connected. The fact that for all $k \geq 1$ we observe the same kind of disconnectedness phenomenon on S^{4k+3} is not a coincidence; in fact there is some evidence that in high dimensions, the topology of $\mathcal{R}^+(S^n)$ should be periodic in n, with period 8.

It is then natural to ask: Is the space of positive scalar curvature metrics on the three-dimensional sphere S^3 also disconnected? The method of proof of the original theorem required techniques of high-dimensional topology, and does not work in dimension 3. In fact, somewhat paradoxically, manifold topology is actually much easier in dimensions 5 and up than in dimensions 3 and 4, which are most closely related to physical reality! But in a spectacular result from 2012, Marques [9] showed that the space $\mathcal{R}^+(S^3)$ is connected. The method of proof suggests that $\mathcal{R}^+(S^3)$ might be contractible.

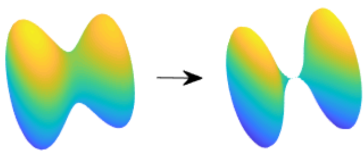

Figure 4: Picture of Ricci flow on a general metric of positive scalar curvature on S^3. The "dumbbell" on the left pinches off into two smaller spheres.

How can one approach the topology of $\mathcal{R}^+(S^3)$? The answer is by using two ingredients. First of all, we can analyze the special subset of metrics for which the sectional curvature is constantly equal to 1. Using results of [6], it can be deduced that this subset is contractible. The next ingredient uses a technique based on the *Ricci flow*, which is connected to the Ricci curvature that we already encountered in the previous section. Starting with a Riemannian metric $g(0)$ on S^3, we can let it evolve under the differential equation $\frac{dg(t)}{dt} = -2\mathrm{Ric}_{g(t)}$, where $\mathrm{Ric}_{g(t)}$ is the Ricci curvature of the (time-varying) metric $g(t)$, with both $\mathrm{Ric}_{g(t)}$ and $g(t)$ viewed as tensors (basically, as matrices). The effect of this flow is to "smooth out" the metric to make it more "sphere-like". It turns out that one can also show easily that under the Ricci flow, the scalar curvature can only increase. So the Ricci flow preserves the space $\mathcal{R}^+(S^3)$.

One might hope, therefore, that Ricci flow should contract $\mathcal{R}^+(S^3)$ down to the contractible space of metrics of constant curvature 1, which is contractible, and thus that $\mathcal{R}^+(S^3)$ itself is contractible. Unfortunately, this is too naive, since the Ricci flow for a general metric almost always develops singularities, and various "necks" will pinch off (see Figure 4). Nevertheless, there is some hope that with sufficient control of the singularities, one could get a modification of this idea to work.

Image credits

Figure 2 is in the public domain, Wikimedia Commons:

https://commons.wikimedia.org/wiki/File:Hyperbolic_triangle.svg

Figure 3 is in the public domain, Wikimedia Commons:

https://commons.wikimedia.org/wiki/File:Torus_cycles2.svg

All other images created by the authors.

References

[1] B. Botvinnik, J. Ebert, and O. Randal-Williams, *Infinite loop spaces and positive scalar curvature*, Invent. Math. **209** (2017), no. 3, 749–835.

[2] A. Einstein, *Zur algemeinen Relativitätstheorie (On the General Theory of Relativity)*, Sitzungsberichte Königlich Preuss. Akad. Wissen. (1915), http://einsteinpapers.press.princeton.edu/vol6-trans/110, available with translation in The Collected Papers of Albert Einstein, vol. 6, The Berlin Years: Writings, 1914–1917, document 21, Edited by Martin J. Klein, A. J. Kox, and Robert Schulman, Princeton Univ. Press, 1996.

[3] P. Erdős and G. Purdy, *Extremal problems in combinatorial geometry*, Handbook of Combinatorics, Vol. 1, Elsevier Sci. B. V., Amsterdam, 1995, pp. 809–874.

[4] K. F. Gauss, *General Investigations of Curved Surfaces of 1827 and 1825*, Princeton Univ. Press, 1902, http://www.gutenberg.org/ebooks/36856, visited on August 25, 2011, transl. by J. C. Morehead and A. M. Hiltebeitel.

[5] M. Gromov and H. B. Lawson, Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. (1983), no. 58, 83–196.

[6] A. E. Hatcher, *A proof of the Smale conjecture, Diff*$(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. (2) **117** (1983), no. 3, 553–607.

[7] H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann. **104** (1931), no. 1, 637–665.

[8] J. Lohkamp, *Metrics of negative Ricci curvature*, Annals of Mathematics **140** (1994), no. 3, 655–683.

[9] F. C. Marques, *Deforming three-manifolds with positive scalar curvature*, Ann. of Math. (2) **176** (2012), no. 2, 815–863.

[10] J. Nash, *The imbedding problem for Riemannian manifolds*, Annals of Mathematics **63** (1956), 20–63.

[11] B. Riemann, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, lecture delivered 1854, published 1868, http://www.emis.de/classics/Riemann/, visited on June 26, 2000, available with translation by W. K. Clifford.

[12] J. Rosenberg, *Manifolds of positive scalar curvature: a progress report*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 259–294.

[13] J. Rosenberg and S. Stolz, *Metrics of positive scalar curvature and connections with surgery*, Surveys on surgery theory, Vol. 2, Ann. of Math. Stud., vol. 149, Princeton Univ. Press, Princeton, NJ, 2001, pp. 353–386.

[14] R. Schoen and S. T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.

[15] ______, *Proof of the positive mass theorem. II*, Comm. Math. Phys. **79** (1981), no. 2, 231–260.

[16] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402.

Jonathan Rosenberg *is Ruth M. Davis Professor of Mathematics at the University of Maryland.*

David Wraith *is Professor of Mathematics at the National University Of Ireland, Maynooth.*

Mathematical subjects
Geometry and Topology

Connections to other fields
Physics

DOI
10.14760/SNAP-2019-004-EN

Snapshots of modern mathematics from Oberwolfach provide exciting insights into current mathematical research. They are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the interested public worldwide. All snapshots are published in cooperation with the IMAGINARY platform and can be found on www.imaginary.org/snapshots and on www.mfo.de/snapshots.

ISSN 2626-1995

Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

Director
Gerhard Huisken

Mathematisches Forschungsinstitut Oberwolfach

Member of Leibniz Association

IMAGINARY open mathematics