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ON THE AUTONOMOUS METRIC ON GROUPS OF HAMILTONIAN DIFFEOMORPHISMS OF CLOSED HYPERBOLIC SURFACES

MICHAEL BRANDENBURSKY

ABSTRACT. Let $\mathbf{\Sigma}_g$ be a closed hyperbolic surface of genus g and let $\mathrm{Ham}(\mathbf{\Sigma}_g)$ be the group of Hamiltonian diffeomorphisms of $\mathbf{\Sigma}_g$. The most natural word metric on this group is the autonomous metric. It has many interesting properties, most important of which is the bi-invariance of this metric. In this work we show that $\mathrm{Ham}(\mathbf{\Sigma}_g)$ is unbounded with respect to this metric.

1. INTRODUCTION AND MAIN RESULT

Let $\mathbf{\Sigma}_g$ be a closed hyperbolic surface of genus g and let ω be an area form on $\mathbf{\Sigma}_g$. For every smooth normalized function $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$, i.e. H has a zero mean with respect to ω, there exists a unique vector field X_H which satisfies

$$dH(\cdot) = \omega(X_H, \cdot).$$

It is easy to see that X_H is tangent to the level sets of H. Let h be the time-one map of the flow h_t generated by X_H. The diffeomorphism h is area-preserving and every diffeomorphism arising in this way is called *autonomous*. Such a diffeomorphism is relatively easy to understand in terms of its generating function.

Denote by $\mathrm{Ham}(\mathbf{\Sigma}_g)$ the group of Hamiltonian diffeomorphisms of $\mathbf{\Sigma}_g$. It follows from results of Banyaga that every Hamiltonian diffeomorphism is a composition of finitely many autonomous diffeomorphisms [3]. We define the *autonomous norm* on $\mathrm{Ham}(\mathbf{\Sigma}_g)$ by

$$\|f\|_{\mathrm{Aut}} := \min\{m \in \mathbf{N} \mid f = h_1 \cdots h_m \text{ where each } h_i \text{ is autonomous}\}.$$

The associated metric is defined by $\mathbf{d}_{\mathrm{Aut}}(f,g) := \|fg^{-1}\|_{\mathrm{Aut}}$. Since the set of autonomous diffeomorphisms is invariant under conjugation, the autonomous metric is bi-invariant. Our main result is the following

Theorem 1. *The metric group* $(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathbf{d}_{\mathrm{Aut}})$ *is unbounded.*

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1.A. **Remarks.**

(1) The study of bi-invariant metrics on groups of geometric origin was initiated by Burago, Ivanov and Polterovich [10]. The autonomous metric is a particular case of such metrics. Moreover, it is the most natural word metric on the group of Hamiltonian diffeomorphisms of a symplectic manifold. It is particulary interesting in the two-dimensional case due to the following observation.

Let $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$ be a Morse function and let h be a Hamiltonian diffeomorphism generated by H. After cutting the surface $\mathbf{\Sigma}_g$ along critical level sets we are left with finite number of regions, so that each one of them is diffeomorphic to annulus. By Arnol'd-Liouville theorem [2], there exist angle-action symplectic coordinates on each one of these annuli, so that h rotates each point on a regular level curve with the same speed, i.e. the speed depends only on the level curve. It follows that a generic Hamiltonian diffeomorphism of $\mathbf{\Sigma}_g$ may be written as a finite composition of autonomous diffeomorphisms, such that each one of these diffeomorphisms is "almost everywhere rotation" in right coordinates, and hence relatively simple. Of course this decomposition is nor unique, neither canonical. However, it is plausible that it might be useful in dynamical systems.

(2) Let $\mathrm{Diff}(\mathbf{D}^2, \mathrm{area})$ be the group of smooth compactly supported area-preserving diffeomorphisms of the open unit disc $\mathbf{D}^2$ in the Euclidean plane. In [8] together with Kedra we showed that for every positive number $k \in \mathbf{N}$ there exists an injective homomorphism $\mathbf{Z}^k \to \mathrm{Diff}(\mathbf{D}^2, \mathrm{area})$ which is bi-Lipschitz with respect to the word metric on $\mathbf{Z}^k$ and the autonomous metric on $\mathrm{Diff}(\mathbf{D}^2, \mathrm{area})$.

(3) We must notice that Theorem 1 follows from a more general result, see [9, Theorem 3]. Nevertheless, we think that the proof given in this paper is easier than the one given in [9]. In a subsequent paper [9], which in particular generalizes results presented in this paper, we are going to show, using slightly different methods, that the above result holds for each group of Hamiltonian diffeomorphisms of closed orientable surface of a positive genus. We should mention that Gambaudo and Ghys showed that the diameter of the group of area-preserving diffeomorphisms of the 2-sphere equipped with the autonomous metric is infinite as well, see [12, Section 6.3].

2. Preliminaries

In this paper we consider only normalized functions $\mathbf{\Sigma}_g \to \mathbf{R}$.

2.A. **Quasi-morphisms.** A function $\psi\colon G \to \mathbf{R}$ from a group G to the reals is called a *quasi-morphism* if there exists a real number $A \geq 0$ such that

$$|\psi(gg') - \psi(g) - \psi(g')| \leq C$$

for all $g, g' \in G$. The infimum of such C's is called the *defect* of ψ and is denoted by D_ψ. If $\psi(g^n) = n\psi(g)$ for all $n \in \mathbf{Z}$ and $g \in G$ then ψ is called *homogeneous.* Any quasi-morphism ψ can be homogenized by setting

$$\overline{\psi}(g) := \lim_{p \to +\infty} \frac{\psi(g^p)}{p}.$$

The vector space of homogeneous quasi-morphisms on G is denoted by $Q(G)$. The space of homogeneous quasi-morphisms on G modulo the space of homomorphisms on G is denoted by $\widehat{Q}(G)$. For more information about quasi-morphisms and their connections to different brunches of mathematics, see [11].

2.B. **Polterovich construction.** Let $\mathbf{M}$ be an oriented closed Riemannian manifold equipped with a Riemannian volume form, and denote by $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$ the identity component of the group of volume preserving diffeomorphisms of $\mathbf{M}$. Let us describe a construction, due to L. Polterovich, of quasi-morphisms on $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$.

Let $z \in \mathbf{M}$. Suppose that the group $\pi_1(\mathbf{M}, z)$ has a trivial center and it admits a *non-trivial* homogeneous quasi-morphism

$$\psi\colon \pi_1(\mathbf{M}, z) \to \mathbf{R}.$$

For each $x \in \mathbf{M}$ let us choose a short geodesic path from x to z. In [18] Polterovich constructed the induced *non-trivial* homogeneous quasi-morphism $\overline{\Psi}$ on $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$ as follows:
For each $x \in \mathbf{M}$ and an isotopy $\{g_t\}_{t \in [0,1]}$ between Id and g, let g_x be a closed loop in $\mathbf{M}$ which is a concatenation of a geodesic path from z to x, the path $g_t(x)$ and a described above geodesic path from $g(x)$ to z. Denote by $[g_x]$ the corresponding element in $\pi_1(\mathbf{M}, z)$ and set

$$\Psi(g) := \int_{\mathbf{M}} \psi([g_x])\,\mathrm{vol} \qquad \overline{\Psi}(g) := \lim_{p \to \infty} \frac{1}{p} \int_{\mathbf{M}} \psi([(g^p)_x])\,\mathrm{vol}\,.$$

The maps Ψ and $\overline{\Psi}$ are well-defined quasi-morphisms because every diffeomorphism in $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$ is volume-preserving. They do not depend on a choice of the path $\{g_t\}$ because the evaluation map

$$\mathrm{ev}\colon \mathrm{Diff}_0(\mathbf{M}, \mathrm{vol}) \to \mathbf{M},$$

where $\mathrm{ev}(g) = g(z)$, induces a map between $\pi_1(\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol}), \mathrm{Id})$ and $\pi_1(\mathbf{M}, z)$ whose image lies in the center of $\pi_1(\mathbf{M}, z)$, which is trivial by our assumption. In addition, the quasi-morphism $\overline{\Psi}$ neither depends on the choice of a family of geodesic paths, nor on the choice of a base point z. For more details see [18]. We abuse the notation and sometimes write $\pi_1(\mathbf{M})$ instead of $\pi_1(\mathbf{M}, z)$.

Denote by $\mathfrak{Polt}_{\mathbf{M}}$ the linear map, induced by Polterovich construction, from $\widehat{Q}(\pi_1(\mathbf{M})) \to \widehat{Q}(\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol}))$. Since every homogeneous quasi-morphism on $\pi_1(\mathbf{M})$, which is not a homomorphism, defines a homogeneous quasi-morphism on $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$, which is also not a homomorphism, we obtain the following

Corollary 2.1. *The linear map* $\mathfrak{Polt}_{\mathbf{M}}\colon \widehat{Q}(\pi_1(\mathbf{M})) \to \widehat{Q}(\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol}))$ *is injective.*

Let $g > 1$. Since the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$ is simple [3, 4], the linear space $\widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$ coincides with $Q(\mathrm{Ham}(\mathbf{\Sigma}_g))$. We denote by $\mathfrak{Polt}_g$ the linear map $\widehat{Q}(\pi_1(\mathbf{\Sigma}_g)) \to \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$ induced by Polterovich construction.

Proposition 2.2. *The linear map* $\mathfrak{Polt}_g\colon \widehat{Q}(\pi_1(\mathbf{\Sigma}_g)) \to \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$ *is injective.*

Proof. Suppose that $\mathfrak{Polt}_g$ is not injective. Then there exists a quasi-morphism ψ in $Q(\pi_1(\mathbf{\Sigma}_g))$, which is not a homomorphism, such that the induced homogeneous quasi-morphism $\overline{\Psi}$ on $\mathrm{Diff}_0(\mathbf{\Sigma}_g, \omega)$ vanishes on the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$.

Lemma 2.3. *Let G be a group and G' its commutator subgroup. Then every homogeneous quasi-morphism on G that vanishes on G' is a homomorphism.*

Proof. Let $\varphi\colon G \to \mathbf{R}$ be a homogeneous quasi-morphism such that $\varphi(g') = 0$ for every $g' \in G'$. Since G' is a normal subgroup of G, the map $\widehat{\varphi}\colon G/G' \to \mathbf{R}$, given by $\widehat{\varphi}(gG') = \varphi(g)$, is a well-defined homogeneous quasi-morphism. The group G/G' is abelian and hence $\widehat{\varphi}$ is a homomorphism and so is φ. □

By a theorem of Banyaga [4] the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$ coincides with the commutator subgroup of $\mathrm{Diff}_0(\mathbf{\Sigma}_g, \omega)$. It follows from Lemma 2.3 that $\overline{\Psi}$ is a homomorphism, which contradicts the fact that the map

$$\mathfrak{Polt}_{\mathbf{\Sigma}_g}\colon \widehat{Q}(\pi_1(\mathbf{\Sigma}_g)) \to \widehat{Q}(\mathrm{Diff}_0(\mathbf{\Sigma}_g, \omega))$$

is injective by Corollary 2.1. □

3. Proofs

3.A. **Curves traced by Morse autonomous flows.** Let h_t be an autonomous flow generated by a Morse function $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$ and set $h := h_1$. Let $x \in \mathbf{\Sigma}_g$ which satisfies the following conditions:

- x is a regular point of H,
- x belongs to only one connected component, i.e. a simple closed curve in $\mathbf{\Sigma}_g$, of the set $H^{-1}(H(x))$.

Such a set of points in $\mathbf{\Sigma}_g$ is denoted by Reg_H. Note that the measure of $\mathbf{\Sigma}_g \setminus \mathrm{Reg}_H$ is zero. For each $x \in \mathrm{Reg}_H$ let

$$c_x\colon [0,1] \to \mathbf{\Sigma}_g$$

be an injective path (on $(0,1)$), such that $c_x(0) = c_x(1) = x$ and its image is a simple closed curve which is a connected component of $H^{-1}(H(x))$. For every $y_1, y_2 \in \mathbf{\Sigma}_g$ choose an injective map

$$s_{y_1y_2}\colon [0,1] \to \mathbf{\Sigma}_g$$

whose image is a short geodesic path from y_1 to y_2. Define

$$\gamma_x(t) := \begin{cases} s_{zx}(3t) & \text{for } t \in \left[0, \frac{1}{3}\right] \\ c_x(3t-1) & \text{for } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ s_{xz}(3t-2) & \text{for } t \in \left[\frac{2}{3}, 1\right] \end{cases}. \tag{1}$$

Denote by $[\gamma_x]$ the corresponding element in $\pi_1(\mathbf{\Sigma}_g, z)$. Let $x \in \mathrm{Reg}_H$ and let $[h_x]$ be an element in $\pi_1(\mathbf{\Sigma}_g, z)$ represented by a path which is a concatenation of paths s_{zx}, $h_t(x)$ and s_{xz}. Then for each $p \in \mathbf{N}$ the element $[h_x^p]$ can be written as a product

$$[h_x^p] = \alpha'_{p,x} \circ [\gamma_x]^{k_{h,p}} \circ \alpha''_{p,x}\,, \tag{2}$$

where $k_{h,p}$ is an integer which depends only h, p and x, and the word length of elements $\alpha'_{p,x}\,, \alpha''_{p,x}$ in $\pi_1(\mathbf{\Sigma}_g, z)$ is bounded by some constant K which is independent of h, x and p.

Denote by $\mathcal{MCG}_g^1$ the mapping class group of a surface $\mathbf{\Sigma}_g$ with one puncture z. Recall that there is a following short exact sequence due to Birman [6]

$$1 \to \pi_1(\mathbf{\Sigma}_g, z) \to \mathcal{MCG}_g^1 \to \mathcal{MCG}_g \to 1, \tag{3}$$

where $\mathcal{MCG}_g$ is the mapping class group of a surface $\mathbf{\Sigma}_g$. Hence we view $\pi_1(\mathbf{\Sigma}_g, z)$ as a normal subgroup of $\mathcal{MCG}_g^1$.

Proposition 3.1. *Let $g > 1$. There exists a finite set S_g of elements in $\mathcal{MCG}_g^1$, such that for every Morse function $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$ and every $x \in \mathrm{Reg}_H$ the loop $[\gamma_x] \in \pi_1(\mathbf{\Sigma}_g, z) < \mathcal{MCG}_g^1$ is conjugated to some element in S_g.*

Proof. Let $x \in \mathrm{Reg}_H$. If the loop $\gamma_x(t)$ is homotopically trivial in $\mathbf{\Sigma}_g$, then $[\gamma_x] = 1_{\mathcal{MCG}_g^1}$.

Suppose that $\gamma_x(t)$ is homotopically non-trivial in $\mathbf{\Sigma}_g$. We say that simple closed curves δ and δ' in $\mathbf{\Sigma}_g$ are equivalent $\delta \cong \delta'$, if there exists a homeomorphism $f\colon \mathbf{\Sigma}_g \to \mathbf{\Sigma}_g$ such that $f(\delta) = \delta'$. It follows from classification of surfaces that the set of equivalence classes $\mathcal{E}_g$ is finite. Let c_x be a simple closed curve defined in (1). Since $\mathbf{\Sigma}_g$ and c_x are oriented, the curve c_x splits in $\mathbf{\Sigma}_g \setminus \{x\}$ into two simple closed curves $c_{x,+}$ and $c_{x,-}$ which are homotopic in $\mathbf{\Sigma}_g$, see Figure 1.

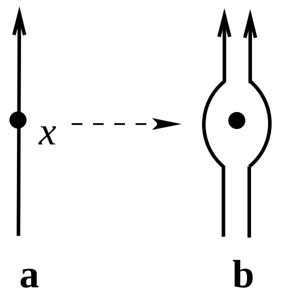

FIGURE 1. Part of the curve c_x is shown in Figure **a**. Its splitting into curves $c_{x,+}$ and $c_{x,-}$ is shown in Figure **b**. The left curve in Figure **b** is $c_{x,+}$ and the right curve is $c_{x,-}$.

The image of the element $[\gamma_x]$ in $\mathcal{MCG}_g^1$, under the Birman embedding (3) of $\pi_1(\mathbf{\Sigma}_g, z)$ into $\mathcal{MCG}_g^1$, is conjugated to $t_{c_{x,+}} \circ t_{c_{x,-}}^{-1}$, where $t_{c_{x,+}}$ and $t_{c_{x,-}}$ are Dehn twists in $\mathbf{\Sigma}_g \setminus \{x\}$ about curves $c_{x,+}$ and $c_{x,-}$ respectively.

Note that if $c_x \cong \delta$ then there exists a homeomorphism $f\colon \mathbf{\Sigma}_g \to \mathbf{\Sigma}_g$ such that $f(c_x) = \delta$, hence $f(c_{x,+}) = \delta_+$ and $f(c_{x,-}) = \delta_-$. We have

$$t_{\delta_+} = f \circ t_{c_{x,+}} \circ f^{-1} \qquad t_{\delta_-} = f \circ t_{c_{x,-}} \circ f^{-1}.$$

This yields

$$f \circ (t_{c_{x,+}} \circ t_{c_{x,-}}^{-1}) \circ f^{-1} = t_{\delta_+} \circ t_{\delta_-}^{-1}.$$

In other words, an element $[\gamma_x]$ is conjugated in $\mathcal{MCG}_g^1$ to some $t_{\delta_+} \circ t_{\delta_-}^{-1}$, where δ is a representative of an equivalence class in $\mathcal{E}_g$. Let $\{\delta_i\}_{i=1}^{\#\mathcal{E}_g}$ be a set of simple closed curves in $\mathbf{\Sigma}_g$, such that each equivalence class in $\mathcal{E}_g$ is represented by some δ_i. Let

$$S_g := \{t_{\delta_{1,+}} \circ t_{\delta_{1,-}}^{-1}, \ldots, t_{\delta_{\#\mathcal{E}_g,+}} \circ t_{\delta_{\#\mathcal{E}_g,-}}^{-1}\}. \tag{4}$$

It follows that $[\gamma_x]$ is conjugated to some element in S_g. Noting that the set S_g neither depend on H nor on x, we conclude the proof of the proposition. □

3.B. **Continuity of Polterovich quasi-morphisms.** The aim of this subsection is to prove the following technical result which will be used in the proof of Theorem 1.

Theorem 3.2. *Let $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$ and $\{H_k\}_{k=1}^{\infty}$ be a sequence of functions such that each $H_k\colon \mathbf{\Sigma}_g \to \mathbf{R}$ and $H_k \xrightarrow[k\to\infty]{} H$ in C^1-topology. Let h and h_k be the time-one maps of the Hamiltonian flows generated by H and H_k respectively. Then*

$$\lim_{k\to\infty} \overline{\Psi}(h_k) = \overline{\Psi}(h),$$

where $\overline{\Psi}$ is a homogeneous quasi-morphism on $\mathrm{Ham}(\mathbf{\Sigma}_g)$ induced by Polterovich construction.

Proof. The proof of this theorem is similar to the proof of Theorem 3.4 in [8]. For the reader convenience, we present the proof below.

At this point we recall a definition of the L^1-metric on the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$. It is defined as follows. Let

$$\mathcal{L}_1\{h_t\} := \int_0^1 \int_{\mathbf{\Sigma}_g} |\dot{h}_t(x)|\omega dt$$

be the L^1-length of a path $\{h_t\}_{t\in[0,1]} \in \mathrm{Ham}(\mathbf{\Sigma}_g)$, where $|\dot{h}_t(x)|$ denotes the length of the tangent vector $\dot{h}_t(x) \in T_x\mathbf{\Sigma}_g$ induced by the Riemannian metric. Observe that this length is right-invariant, that is, $\mathcal{L}_1\{h_t \circ f\} = \mathcal{L}_p\{h_t\}$ for any $f \in \mathrm{Ham}(\mathbf{\Sigma}_g)$. It defines a non-degenerate right-invariant metric on $\mathrm{Ham}(\mathbf{\Sigma}_g)$ by

$$\mathbf{d}_1(h_0, h_1) := \inf_{h_t} \mathcal{L}_1\{h_t\},$$

where the infimum is taken over all paths from h_0 to h_1. See Arnol'd-Khesin [1] for a detailed discussion. We set $\|h\|_1 := \mathbf{d}_1(\mathrm{Id}, h)$. In [7] the author proved the following

Theorem 3.3 ([7])**.** *Let $\mathbf{\Sigma}_g$ be a closed hyperbolic surface, and $\overline{\Psi}$ a homogeneous quasi-morphism on $\mathrm{Ham}(\mathbf{\Sigma}_g)$ induced by Polterovich construction. Then $\overline{\Psi}$ is Lipschitz with respect to the L^1-metric on the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$, i.e. there exists $C > 0$ such that $\forall h \in \mathrm{Ham}(\mathbf{\Sigma}_g)$*

$$\overline{\Psi}(h) \le C\|h\|_1.$$

Lemma 3.4. *Let $G\colon \mathbf{\Sigma}_g \to \mathbf{R}$ be smooth function. Then for any $\epsilon > 0$ and $p \in \mathbf{N}$ there exists $\delta_p > 0$, such that if G is δ_p-close to a smooth function $F\colon \mathbf{\Sigma}_g \to \mathbf{R}$ in C^1-topology, then*

$$\mathbf{d}_1(g^p, f^p) < \epsilon,$$

where g_t and f_t are the Hamiltonian flows generated by G and F, and g and h are time-one maps of these flows.

Proof. We replace $\mathbf{D}^2$ by $\mathbf{\Sigma}_g$ in the proof of Lemma 3.7 in [8]. Now the proof is identical to the proof of Lemma 3.7 in [8]. □

Proposition 3.5. *Let $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$. Then for any $\epsilon > 0$ there exists $\delta > 0$, such that if $F\colon \mathbf{\Sigma}_g \to \mathbf{R}$ is δ-close to H in C^1-topology then:*

$$\left|\overline{\Psi}(h) - \overline{\Psi}(f)\right| \leq \epsilon,$$

where h and f are time-one maps of flows generated by H and F.

Proof. Fix some $\epsilon > 0$. Let C be the constant which was defined in Theorem 3.3. Take $p \in \mathbf{N}$ such that $\frac{D_{\overline{\Psi}}+C}{p} < \epsilon$. It follows from Lemma 3.4 that there exists $\delta_p > 0$, such that if F is δ_p-close to H in C^1-topology, then $\mathbf{d}_1(f^p, h^p) < 1$. Thus we obtain

$$\left|\overline{\Psi}(f) - \overline{\Psi}(h)\right| = \frac{1}{p}\left|\overline{\Psi}(f^p) - \overline{\Psi}(h^p)\right| \leq \frac{D_{\overline{\Psi}} + \left|\overline{\Psi}(f^p h^{-p})\right|}{p}.$$

It follows from Theorem 3.3 that

$$\left|\overline{\Psi}(f^p h^{-p})\right| \leq C\mathbf{d}_1(Id, f^p h^{-p}) = C\mathbf{d}_1(f^p, h^p) < C.$$

Thus

$$\left|\overline{\Psi}(f) - \overline{\Psi}(h)\right| < \frac{D_{\overline{\Psi}} + C}{p} < \epsilon.$$

□

Proposition 3.5 concludes the proof of Theorem 3.2. □

3.C. **Proof of Theorem 1.** Recall that the group $\pi_1(\mathbf{\Sigma}_g)$ is a subgroup of $\mathcal{MCG}_g^1$. Denote by $Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ the space of homogeneous quasi-morphisms on $\pi_1(\mathbf{\Sigma}_g)$ so that:

- For each $\varphi \in Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ there exists $\widehat{\varphi} \in Q(\mathcal{MCG}_g^1)$ such that $\widehat{\varphi}|_{\pi_1(\mathbf{\Sigma}_g)} = \varphi$,
- each φ vanishes on the finite set S_g,

where S_g is the set defined in (4). The group $\pi_1(\mathbf{\Sigma}_g)$ contains a non-abelian free group, and thus is not virtually abelian. It is an infinite normal subgroup of $\mathcal{MCG}_g^1$ and hence is a non-reducible subgroup of $\mathcal{MCG}_g^1$, see [14, Corollary 7.13]. Now, by a result of Bestvina-Fujiwara [5, Theorem 12] we have the following

Corollary 3.6. *The space $Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ is infinite dimensional.*

Note that since every non-trivial quasi-morphism in $Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ vanishes on S_g, it can not be a homomorphism. Hence the space

$Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ may be viewed as a linear subspace of $\widehat{Q}(\pi_1(\mathbf{\Sigma}_g))$. Recall that by Corollary 2.1 the linear map

$$\mathfrak{Polt}_g \colon \widehat{Q}(\pi_1(\mathbf{\Sigma}_g)) \hookrightarrow \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$$

is injective. Hence the map

$$\mathfrak{Polt}_g \colon Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g) \hookrightarrow \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$$

is also injective.

Denote by $Q(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathrm{Aut})$ the space of homogeneous quasi-morphisms on $\mathrm{Ham}(\mathbf{\Sigma}_g)$ that vanish on the set $\mathrm{Aut} \subset \mathrm{Ham}(\mathbf{\Sigma}_g)$ of all autonomous diffeomorphisms. Since $Q(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathrm{Aut})$ contains no non-trivial homomorphisms, it is viewed as a linear subspace of $\widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$. Now we a ready to state and prove our key proposition.

Proposition 3.7. *The image of the map*

$$\mathfrak{Polt}_g \colon Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g) \hookrightarrow \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$$

lies in the linear space $Q(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathrm{Aut})$.

Proof. Let $\psi \in Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ and $h \in \mathrm{Ham}(\mathbf{\Sigma}_g)$ an autonomous diffeomorphism. We need to show that $\overline{\Psi}(h) = 0$, where $\overline{\Psi} = \mathfrak{Polt}_g(\psi)$. Since Morse functions on $\mathbf{\Sigma}_g$ form a dense subset in the set of all smooth functions in C^1-topology [16], by Theorem 3.2 it is enough to show that $\overline{\Psi}(h) = 0$, where h is a time-one map of the flow generated by some Morse function $H \colon \mathbf{\Sigma}_g \to \mathbf{R}$. Recall that by definition we have

$$\overline{\Psi}(h) = \int\limits_{\mathbf{\Sigma}_g} \lim_{p\to\infty} \frac{\psi([h_x^p])}{p} \omega \,.$$

Since the set Reg_H is of full measure in $\mathbf{\Sigma}_g$, it is enough to show that for each $x \in \mathrm{Reg}_H$ the following equality holds

$$\lim_{p\to\infty} \frac{|\psi([h_x^p])|}{p} = 0.$$

The group $\pi_1(\mathbf{\Sigma}_g)$ admits the following presentation

$$\pi_1(\mathbf{\Sigma}_g) = \langle \alpha_i, \beta_i |\ 1 \leq i \leq g, \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \rangle. \tag{5}$$

For every $\alpha \in \pi_1(\mathbf{\Sigma}_g)$ denote by $l(\alpha)$ the word length of α with respect to the set of generators given in (5). Since $\psi \in Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$, it is constant on conjugacy classes and thus $\psi(\alpha_i) = \psi(\beta_i) = 0$ for each $1 \leq i \leq g$. In addition, for every $\alpha \in \pi_1(\mathbf{\Sigma}_g)$ we have $|\psi(\alpha)| \leq D_\psi l(\alpha)$. It follows from (2) that for every $p \in \mathbf{N}$ and $x \in \mathrm{Reg}_H$ we have

$$[h_x^p] = \alpha'_{p,x} \circ [\gamma_x]^{k_{h,p}} \circ \alpha''_{p,x} \,,$$

where $k_{h,p}$ is an integer which depends only h, p and x, and $l(\alpha'_{p,x})$, $l(\alpha''_{p,x})$ are bounded by some constant $K > 0$ independent of h, x and p. Hence for every $p \in \mathbf{N}$ and $x \in \mathrm{Reg}_H$ we have

$$0 \leq \frac{|\psi([h_x^p])|}{p} \leq \frac{|\psi(\alpha'_{p,x})| + |k_{h,p}||\psi([\gamma_x])| + |\psi(\alpha''_{p,x})| + 2D_\psi}{p} \quad .$$

Since $\psi \in Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$, by definition it extends to a homogeneous quasi-morphism on $\mathcal{MCG}_g^1$ and vanishes on the set S_g. Hence by Proposition 3.1 we have $\psi([\gamma_x]) = 0$, and hence

$$0 \leq \frac{|\psi([h_x^p])|}{p} \leq \frac{2K \cdot D_\psi + 2D_\psi}{p} = \frac{2D_\psi(K+1)}{p} \quad .$$

By taking $p \to \infty$ we conclude the proof of the proposition. □

Since the map $\mathfrak{Polt}_g$ is injective and by Corollary 3.6 the linear space $Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ is infinite dimensional, an immediate consequence of Proposition 3.7 is the following

Corollary 3.8. *The space* $Q(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathrm{Aut})$ *is infinite dimensional.*

Remark 3.9. The existence of a non-trivial homogeneous quasi-morphism $\overline{\Psi}\colon \mathrm{Ham}(\mathbf{\Sigma}_g) \to \mathbf{R}$, which is trivial on the set $\mathrm{Aut} \subset \mathrm{Ham}(\mathbf{\Sigma}_g)$ of all autonomous diffeomorphisms, implies that the autonomous norm is unbounded. Indeed, for every $f \in \mathrm{Ham}(\mathbf{\Sigma}_g)$ we have that

$$|\overline{\Psi}(f)| = |\overline{\Psi}(h_1 \circ \ldots \circ h_m)| \leq mD_{\overline{\Psi}}$$

and hence for every natural number n we get $\|f^n\|_{\mathrm{Aut}} \geq \frac{|\overline{\Psi}(f)|}{D_{\overline{\Psi}}} n > 0$, provided $\overline{\Psi}(f) \neq 0$.

Remark 3.9 and Corollary 3.8 conclude the proof of the theorem. □

4. Comparison of bi-invariant metrics on $\mathrm{Ham}(\mathbf{\Sigma}_g)$

The most famous metric on the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$ is the Hofer metric, see [13, 15]. The associated norm is defined by

$$\|h\|_{\mathrm{Hofer}} := \inf_{H_t} \int_0^1 \max_{\mathbf{\Sigma}_g} H_t - \min_{\mathbf{\Sigma}_g} H_t dt,$$

where H_t is a normalized Hamiltonian function generating the Hamiltonian flow h_t from the identity to $h = h_1$. Let α_1 be a curve which represents a generator $[\alpha_1]$ of $\pi_1(\mathbf{\Sigma}_g)$ defined in (5). Let $h \in \mathrm{Ham}(\mathbf{\Sigma}_g)$ be a diffeomorphism generated by a function $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$ which satisfies the following conditions:

- there exists a positive number C such that $H|_{\alpha_1} > C$,
- and $H|_{\alpha_1}$ is not constant.

It implies that all powers of h are autonomous diffeomorphisms and hence $\|h^n\|_{\mathrm{Aut}} = 1$ for all $n \in \mathbf{Z}$. On the other hand, nH generates h^n and $nH|_{\alpha_1} > Cn$, hence by a result of L. Polterovich [17] we have $\|h^n\|_{\mathrm{Hofer}} \geq Cn$. On the other hand, for every $\epsilon > 0$, one can easily construct an autonomous diffeomorphism whose Hofer norm is less then ϵ This shows that the identity homomorphism between the autonomous metric and the Hofer metric is not Lipschitz in neither direction.

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References

[1] Arnol'd V., Khesin B.: *Topological methods in hydrodynamics,* Applied Mathematical Sciences 125, (1998).

[2] Arnol'd V.: *Mathematical methods of classical mechanics,* translated from the 1974 Russian edition by K. Vogtmann and A. Weinstein, Graduate Texts in Math., Vol. 60, Springer-Verlag, New York and Berlin, 1978.

[3] Banyaga A.: *Sur la structure du groupe des diffeomorphismes qui preservent une forme symplectique,* (French) Comment. Math. Helv. 53 (1978), no. 2, 174–227.

[4] Banyaga A.: *The structure of classical diffeomorphism groups,* volume 400 of Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1997.

[5] Bestvina M., Fujiwara K.: *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. **6** (2002), 69–89.

[6] Birman J.: *Mapping class groups and their relationship to braid groups,* Comm. Pure Appl. Math. **22** (1969), 213–238.

[7] Brandenbursky M.: *Quasi-morphisms and L^p-metrics on groups of volume-preserving diffeomorphisms,* Journal of Topology and Analysis Vol. 4, No. 2 (2012) 255–270.

[8] Brandenbursky M., Kedra J.: *On the autonomous metric on the group of area-preserving diffeomorphisms of the 2-disc*, Algebraic & Geometric Topology, **13** (2013), 795–816.

[9] Brandenbursky M.: *Bi-invariant metrics and quasi-morphisms on groups of Hamiltonian diffeomorphisms of surfaces,* in preparation.

[10] Burago D., Ivanov S., Polterovich L.: *Conjugation-invariant norms on groups of geometric origin,* Advanced Studies in Pure Math. **52** Groups of Diffeomorphisms (2008) 221–250.

[11] Calegari D.: *scl*, MSJ Memoirs 20, Mathematical Society of Japan (2009).

[12] Gambaudo J.M., Ghys E.: *Commutators and diffeomorphisms of surfaces,* Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1591–1617.

[13] Hofer H.: *On the topological properties of symplectic maps,* Proc. Roy. Soc. Edinburgh Sect. A 115 (1990), no. 1-2, 25–38.

[14] Ivanov N.: *Subgroups of Teichmüller modular groups,* Translations of Mathematical Monographs, Volume 115. American Mathematical Society, Providence, RI, 1992.
[15] Lalonde F., McDuff D.: *The geometry of symplectic energy,* Ann. of Math. (2) 141 (1995), no. 2, 349–371.
[16] Milnor J. W.: *Lectures on the h-cobordism theorem,* notes by L. Siebenmann and J. Sondow, Princeton University Press, Princeton, NJ, (1965).
[17] Polterovich L: *Hofer's diameter and Lagrangian intersections,* Int. Math. Research Notes 4 (1998), 217–223.
[18] Polterovich L: *Floer homology, dynamics and groups,* in Morse theoretic methods in nonlinear analysis and in symplectic topology, 417–438, Springer 2006.

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